Sheffield talk: Higher Galois Theory

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HIGHER ALGEBRA

WHAT ARE SPECTRA?

DEFINITION 1: HOMOLOGY THEORY

Let An_* be the ∞-category of pointed spaces (or 'animae'). A **spectrum** is a functor $F : \mathrm{An}_* \to \mathrm{An}_*$ such that

(H1) F is **finitary**; F commutes with filtered colimits

(H2) F is **reduced**; $F(\mathrm{pt}) = \mathrm{pt}$, and

(H3) F is **excisive**; F sends homotopy pushouts to homotopy pullbacks.

We denote by $\mathrm{Sp} \subseteq \mathrm{Hom}(\mathrm{An}_*, \mathrm{An}_*)$ the sub-category of spectra.

Typical examples of spectra are the Eilenberg-MacLane spectra HA that represent homology with coefficients in a commutative ring A so that $HA(X) = H^*(X; A)$, the K-theory spectrum (representing K-theory), and the sphere spectrum $\mathbb{S}$ representing stable homotopy groups $\mathbb{S}(X) = \pi_*^s(X)$. All of these are examples of *ring-spectra*, which will be our focus in these notes.

First, we discuss the connection between spaces and spectra, to do this we want to use the **Goodwillie differential**, or **Quillen's functor** which is a way of forcing functors to be excisive, or 'linear'. This differential will allow us to easily come us with examples of spectra, since defining finitary and reduced functors isn't so difficult, but finding ones which are excisive becomes quite a bit more challenging - we can then just use the Goodwillie differential to force things to be excisive.

DEFINITION 2: QUILLEN

Let $E : \mathrm{An} \to \mathrm{An}$ be a reduced, finitary functor, we can obtain a spectrum DF as

$$QF := \operatorname*{colim}_{n\in\mathbb{N}} \Omega^n F(\Sigma^n -).$$

We call Q **Quillen's functor**. We have that QF is excisive, in particular, it is a spectrum.

THEOREM 3

Let $X \in \mathrm{An}_*$ be a pointed space, we have an adjunction

$$\Sigma^\infty : \mathrm{An} \leftrightarrows \mathrm{Sp} : \Omega^\infty$$

given by

$$(\Sigma^\infty X)(Y) = \operatorname*{colim}_{n\in\mathbb{N}} \Omega^n \Sigma^n (X \wedge Y) \qquad \text{and} \qquad \Omega^\infty E = E(S^0).$$

DEFINITION 4: SPHERE SPECTRUM

The **sphere spectrum** is given by $\mathbb{S} := \Sigma^\infty(S^0)$.

DEFINITION 5: SMASH PRODUCT OF SPECTRA

Let $E, F \in \mathrm{Sp}$ be spectra, the **Day convolution** (or 'smash product') of E and F is given by

$$(E \otimes_{\mathbb{S}} F)(X) = Q\left(\operatorname*{colim}_{A\wedge B\to X} E(A) \wedge F(B)\right),$$

which endows Sp with a symmetric monoidal structure with unit $\mathbb{S}$.

We also have a relative version of the tensor product [Lur17, §4.4.2], generalising the classical case in which $A \otimes_B C$ is the coequilizer of $A \otimes B \otimes C \rightrightarrows A \otimes C$ where the two arrows are given by the right and left action of B on A and C.

DEFINITION 6: RELATIVE TENSOR PRODUCT

Let $E, E', F \in \mathrm{Sp}$ be spectra, the **relative tensor product** is the geometric realisation in Sp of the diagram

$$E \otimes_F E' := |\, E \otimes_{\mathbb{S}} F^n \otimes_{\mathbb{S}} E' \,|\,.$$

DEFINITION 7: STABLE HOMOTOPY GROUPS

Let $E \in \mathrm{Sp}$ be a spectrum, its **(stable) homotopy groups** are given by $\pi_n E = \mathrm{colim}_{k\in\mathbb{N}}\, \pi_{n+k} E_k$.

If $\Sigma^\infty X$ is a suspension spectrum then the adjunction $\Sigma^\infty \dashv \Omega^\infty$ of theorem 3 gives

$$\pi_n(\Sigma^\infty X) = \mathrm{colim}_{n\in\mathbb{N}}\, \pi_{n+k}\Sigma^n X,$$

which agrees with the classical notion of stable homotopy group given by the Hurewicz theorem.

THEOREM 8: BROWN REPRESENTABILITY

An **(additive) Steenrod Cohomology theory** is a pair $(E^\bullet, \sigma)$ where $E^\bullet : \mathrm{An}_*^{\mathrm{op}} \to \mathrm{gr}_{\mathbb{Z}}\mathrm{Ab}$ is a functor and $\sigma = (\sigma_n)_{n\in\mathbb{N}}$ is a collection of equivalences $\sigma_n : E^{n+1}\Sigma \xrightarrow{\sim} E^n$, such that

(SC1) $E^\bullet$ takes (small) coproducts to products, and

(SC2) $E^\bullet$ takes cofibre sequences $A \xhookrightarrow{i} X \xrightarrow{j} \mathrm{Cone}(i)$ to exact sequences $E(\mathrm{Cone}(i)) \xrightarrow{j^*} E(X) \xrightarrow{i^*} A$.

In this case we have that E is represented by a spectrum, that is; there exists a spectrum $E_\bullet$ such that

$$E^n(X) = \pi_0 \,\mathrm{Hom}(X, E_n).$$

Classically, spectra were defined using Ω-spectra, which will remain a convenient way of us defining spectra in some cases. An **Ω-spectrum** is a collection $(E_i, \delta_i)_{i\in\mathbb{N}}$ where each $E_i \in \mathrm{An}_*$ is a pointed space and we have homotopy equivalences $\delta_i : E_i \xrightarrow{\sim} \Omega E_{i+1}$, we can then define the category of spectra as the colimit of the sequence $\cdots \xrightarrow{\Omega} \mathrm{An}_* \xrightarrow{\Omega} \mathrm{An}_*$ where $\Omega X \equiv * \times_X *$ is the loop functor. A warning should be made here that a map $(E_i, \delta_i)_{i\in\mathbb{N}} \to (E'_i, \delta'_i)_{i\in\mathbb{N}}$ of Ω-spectra need only consist of a *cofinal* set of maps $(f_i : E_i \to E'_i)_{i\in S}$ for some $S \subseteq \mathbb{N}$.

EXAMPLES OF SPECTRA

EXAMPLE 9: ORDINARY HOMOLOGY

Let A be a commutative ring, then the Eilenberg-MacLane space $K(A,n)$ is the unique space such that

$$\pi_k K(A,n) = \begin{cases} A & \text{if } k = n \\ 0 & \text{otherwise.} \end{cases}$$

Noting that, for stable homotopy groups, we have $\pi_k \Omega K(A,n+1) = \pi_{k+1} K(A,n+1) = \pi_k K(A,n)$ we get weak equivalences $K(A,n) \simeq \Omega K(A,n+1)$. As in the proof of Brown representability, this then defines a spectrum HA from the cohomology theory $X \mapsto \pi_0 \operatorname{Hom}(X, K(A,n)) \simeq H^n(X,A)$.

EXAMPLE 10: K-THEORY

We can obtain complex K-theory as a cohomology theory by letting $K^0_{\mathbb{C}}(X)$ be the group completion of $(\mathrm{Vect}_{\mathbb{C}}(X), \otimes)$ and defining spaces $K^{-n}(X) \equiv K(\Sigma^n X)$. This gives a spectrum KU via Brown representability such that $\pi_{2k} KU = \mathbb{Z}$ and $\pi_{2k+1} KU = 0$ for all k.

EXAMPLE 11: THOM SPECTRA AND MU

Complex cobordism gives rise to a spectrum MU via the **Thom construction**, as in [And+13]. Let R be an $\mathbb{E}_\infty$-ring, and let $\mathrm{Line}_R \subseteq \mathrm{Mod}_R$ be the ∞-groupoid of rank one R-modules, i.e. R-modules L with a specified isomorphism $L \xrightarrow{\sim} R$. We say an **R-module bundle** on a space X is a functor $X^{\mathrm{op}} \to \mathrm{Mod}_R$, and a **$R$-line bundle** is a map $X^{\mathrm{op}} \to \mathrm{Line}_R$. The **Thom spectrum** Mf associated to a line bundle $f : X^{\mathrm{op}} \to \mathrm{Line}_R$ is given by

$$Mf = \operatorname{colim} \left(X^{\mathrm{op}} \xrightarrow{f} \mathrm{Line}_R \hookrightarrow \mathrm{Mod}_R \right).$$

Choosing, in particular, $R = \mathbb{S}$ the sphere spectrum and $X = BU$, with map $f : BU^{\mathrm{op}} \to \mathrm{Line}_{\mathbb{S}} \simeq BGL_1\mathbb{S}$ given by the complex J-homomorphism we obtain the **complex cobordism spectrum**

$$MU = \operatorname{colim}(BU \to BGL_1\mathbb{S} \hookrightarrow \mathrm{Mod}_{\mathbb{S}} \simeq \mathrm{Sp}).$$

RING SPECTRA

DEFINITION 12: RING SPECTRA

A spectrum R is a **commutative ring spectrum** if it is a commutative monoid object in the category of spectra.

This definition can be made precise using the language of operads, where we'd say that R is a commutative ring spectrum if it is an $\mathbb{E}_\infty$-algebra in Sp. In essence, we want homotopy coherent and 'stable' multiplication and identity maps

$$R \otimes R \to R, \qquad \mathbb{S} \to R.$$

Following on from example 9 we have that HA is actually a ring spectrum via the *Steenrod squares* and other stable homology operations. We can't use the cup product $H^m(X;A) \otimes H^l(X;A) \to H^{m+l}(X;A)$ from ordinary homology since it's not stable; if X is an n-dimensional manifold and $\alpha \in H^n(X;A)$ then $\alpha \cup \beta = 0$ for any $\beta \in H^m(X;A)$ with $m \geq 1$, so no elements survive in the stable homology groups $H^n_{st}(X;A) = \operatorname{colim}_m H^{m+n}(X; B^m A)$. The Steenrod squares are a fix to this, they are a *stabilisation* of the cup product; we consider the morphisms $Q^s : \Sigma^{2s(p-1)}(H\mathbb{F}_p \otimes E) \to H\mathbb{F}_p \otimes E$ as in [Bar09, §1.13] given by

$$Q^s(x) = \begin{cases} 0 & \text{if } 2s < |x| \in \pi_*(H\mathbb{F}_p \otimes E) \\ x^p & \text{if } 2s = |x| \in \pi_*(H\mathbb{F}_p \otimes E). \end{cases}$$

In order for E to be an $\mathbb{E}_\infty$-ring there is a hierarchy of formulas Q^s must satisfy such as the Cartan formulas, Adem relations, Nishida relations, etc. Typically we encounter $\mathbb{E}_\infty$-structures in two ways, either because we have constructed an $\mathbb{E}_\infty$-ring out of other $\mathbb{E}_\infty$-rings, and so the structure is inherited, or we can find an $\mathbb{E}_\infty$-structure using some kind of obstruction theory. See [PV21; Maz18] for a description of Goerss-Hopkins obstruction theory, and see [Bar09, §1.13] for a more complete list of stable operations that $\mathbb{E}_\infty$-rings enjoy.

MODULES OF RING SPECTRA

Let R be a commutative ring, then R is an Ab-enriched category with one object $*$ and $\mathrm{Hom}_R(*,*) = R$, if $r, s \in R$ then $rs = r \circ s$ and $r + s$ is given by the Ab-enrichment. An R-module is then an additive functor $M : R \to \mathrm{Ab}$, i.e. M sends the unique object $* \in R$ to an abelian group $M := M(*)$ and any element $r \in R$ is sent to $M(r) : M \to M$, giving an R-action on M.

If we take seriously the notion that Sp is the natural $(\infty, 1)$-categorical enhancement of the 1-category Ab of abelian groups then, given an $\mathbb{E}_\infty$-ring R, we can simply define Mod_R as the category of Sp-enriched functors $\underline{\mathrm{Hom}}(R, \mathrm{Sp})$. We obtain a symmetric monoidal structure $\otimes_R$ on Mod_R by taking the Day convolution of the E_∞-structure on R and the smash product $\otimes_{\mathrm{Sp}}$ on Sp, which is related to definition 6 via the following theorem.

THEOREM 13

Given an $\mathbb{E}_\infty$-ring R, the Day convolution $\otimes_R$ of the $\mathbb{E}_\infty$-structure on R and the smash product $\otimes_{\mathrm{Sp}}$ has resolution given by the geometric realisation of the two-sided Bar resolution

$$E \otimes_R E' = |\, E \otimes_{\mathbb{S}} R^n \otimes_{\mathbb{S}} E' \,| \,.$$

This gives a symmetric-monoidal structure on $\mathrm{Mod}_R = \underline{\mathrm{Hom}}(R, \mathrm{Sp})$.

THEOREM 14: FLAT MORPHISMS [BAR09, THM 1.27]

The following are equivalent for a morphism $f : R \to S$ of connective $\mathbb{E}_\infty$-rings

(F1) $S \in \mathrm{Mod}_R$ can be written as a filtered colimit of free R-modules

(F2) for all discrete $M \in \mathrm{Mod}_R$ we have $M \otimes_R S$ is discrete

(F3) the functor $f^* = - \otimes_R S : \mathrm{Mod}_R \to \mathrm{Mod}_S$ is left t-exact,

(F4) *(i)* the morphism $\pi_0 f : \pi_0 R \to \pi_0 S$ is a flat morphism of commutative rings, and

(ii) the natural map

$$\pi_* R \otimes_{\pi_0 R} \pi_0 S \to \pi_* S$$

is an isomorphism.

If any (hence all) of the above conditions are satisfied, we call f a **flat morphism**. If we have furthermore that $f^* \equiv - \otimes_R S : \mathrm{Mod}_S \to \mathrm{Mod}_R$ is conservative, we call f **faithfully flat**.

DEFINITION 15: ÉTALE R-ALGEBRA

Let R be an $\mathbb{E}_\infty$-ring, an R-algebra S is **Étale** if

(É1) $\pi_0 S$ is an étale $\pi_0 R$-algebra, and

(É2) $\pi_0 S \otimes_{\pi_0 R} \pi_* R \to \pi_* S$ is an isomorphism.

THEOREM 16: [LUR17, P. 7.5.4.2]

The $(\infty, 1)$-category of étale R-algebras for R an $\mathbb{E}_\infty$-ring is equivalent to the $(1, 1)$-category of étale $\pi_0 R$-algebras.

Nilpotence and Descent

This section draws heavily on the following works; [Mat14, Part 3, 4], [MNN17, Part 1], and [Mor23, Section 3].

Given a faithfully flat map $A \xrightarrow{f} B$ of discrete commutative rings we have the diagram

$$A \longrightarrow B \rightrightarrows B \otimes_A B \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} B \otimes_A B \otimes_A B$$

given by tensoring with f repeatedly. Classical descent theory is the statement that the 2-category of (discrete) A-modules Mod^0_A is given as the 2-categorical limit

$$\mathrm{Mod}^0_A \cong \lim \left(\mathrm{Mod}^0_B \rightrightarrows \mathrm{Mod}^0_{B\otimes_A B} \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \mathrm{Mod}^0_{B\otimes_A B\otimes_A B} \right)$$

We can prove this by using the Barr-Beck theorem, i.e. by showing that if f is faithfully flat then then the tensor-forget adjunction $\mathrm{Mod}_A \rightleftarrows \mathrm{Mod}_B$ is comonadic.

We can extend this to $(\infty,1)$-categories by considering the **cobar construction** $CB^\bullet(f)$, which is the following (augmented) cosimplicial diagram

$$A \longrightarrow B \rightrightarrows B \otimes_A B \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} B \otimes_A B \otimes_A B \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots$$

with $\mathrm{CB}^n(f) = B^{\otimes_A n}$. With certain nice conditions on f one might hope to reconstruct A as the limit of the cosimplicial cobar construction $A \cong \lim \mathrm{CB}^\bullet(f)$. This can be done, for instance, along the complexification map $KO \to KU$, or along the map $L_{E_n}\mathbb{S} \to E_n$ from the E_n-localised sphere spectrum to Morava E-theory E_n. The nilpotence theorem by Devinatz-Hopkins-Smith states that the map $L_{E_n}\mathbb{S} \to E_n$ is *descendable*[1] in the sense that E_n generates the category Sp_{E_n} of E_n-local spectra as a thick $\otimes$-ideal.

Theorem 17: Barr-Beck-Lurie

An adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ between two $(\infty,1)$-categories is comonadic if and only if

(i) F is conservative

(ii) If $X^\bullet : \Delta \to \mathcal{C}$ is a cosimplicial object in $\mathcal{C}$ such that $F(X^\bullet)$ splits in $\mathcal{D}$, then $\mathrm{Tot}(X^\bullet)$ exists and $F(\mathrm{Tot}(X^\bullet)) = \mathrm{Tot}(F(X^\bullet))$.

Definition 18: Descendable Objects

Let $(\mathcal{C}, \otimes, 1)$ be a stable presentable symmetric-monoidal $(\infty,1)$-category. An algebra $A \in \mathrm{CAlg}(\mathcal{C})$ **admits descent** or is **descendable** if it generates $\mathcal{C}$ as a thick $\otimes$-ideal.

Theorem 19: [Mat14, Prop 3.22]

Let $A \in \mathrm{CAlg}(\mathcal{C})$ be a descendable object in a stable presentable symmetric-monoidal $(\infty,1)$-category $(\mathcal{C}, \otimes, 1)$, then the tensor-forget adjunction

$$- \otimes A : \mathrm{Mod}_A \rightleftarrows \mathcal{C} : \underline{\mathrm{Hom}}(1, -)$$

is comonadic, and gives an equivalence

$$\mathcal{C} \xrightarrow{\sim} \mathrm{Tot}\left(\mathrm{Mod}_A \rightrightarrows \mathrm{Mod}_{A\otimes A} \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \ldots\right).$$

A module A admits descent if and only if the thick $\otimes$-ideal $\mathcal{I}_A$ of A-zero maps (i.e. maps $f : B \to C$ such that $f \otimes \mathrm{id}_A \simeq 0$) is nilpotent, $\mathcal{I}_A^n = 0$ for some n.

[1] See [Mat14, §3].

Bousfield Localisation

See [Lur10, Lecture 20] and [Bar09].

Definition 20: Acyclic Modules

Let R be an $\mathbb{E}_\infty$-ring and M an R-module. We say an R-module N is **M-acyclic** if $M \otimes_R N \simeq 0$.

Definition 21: Local Module

Let R be an $\mathbb{E}_\infty$-ring and M an R-module. We say an R-module P is **M-local** if $[N, P] \simeq 0$ for any M-acyclic R-module N. The denote the subcategory of M-local R-modules by $L_M\mathrm{Mod}_R \subseteq \mathrm{Mod}_R$.

Definition 22: Bousfield Localisation

The inclusion $L_M\mathrm{Mod}_R \hookrightarrow \mathrm{Mod}_R$ has a left adjoint $L_M : \mathrm{Mod}_R \to L_M\mathrm{Mod}_R$ called **Bousfield localisation**.

Definition 23: Smashing Localisation

We say a Bousfield localisation $L_M : \mathrm{Mod}_R \to L_M\mathrm{Mod}_R$ is **smashing** if $L_M \to - \otimes_R L_M R$ is an equivalence. In this case we have $L_M\mathrm{Mod}_R = \mathrm{Mod}_{L_M R}$.

If a Bousfield localisation is smashing then we have an equivalence $\mathrm{Mod}_{L_M R} \simeq L_M\mathrm{Mod}_R$ so that $\operatorname{Spec} L_M R \to \operatorname{Spec} R$ is a Zariski open immersion. More surprisingly we have an equivalence between smashing localisations and Zariski opens;

Theorem 24

A localisation $L_M : \mathrm{Mod}_R \to L_M\mathrm{Mod}_R$ is smashing if and only if $\operatorname{Spec} L_M R \to \operatorname{Spec} R$ is a Zariski open immersion.

THE CHROMATIC STORY

There are a number of references for chromatic homotopy theory, the story we present here follows on principally from [Hop99; Lur10; Rav03; Bar09; Gre23]. The classical references for studying Morava K-theory are [Mar99] and [DH04], and more recent developments in chromatic homotopy theory can be seen in [BSY22] and [Mor23].

The Atiyah-Hirzebruch spectral sequence $E_2^{p,q}(X) = H^p(X, E^q(*)) \Rightarrow H^{p+q}(X)$ gives us a method of computing the E-cohomology $E^\bullet(X)$ of a space X. If $X = \mathbb{C}P^\infty$ then degeneration of this sequence at page two (such is the case for *complex oriented* cohomology theories) gives, in particular, an isomorphism $E^\bullet(\mathbb{C}P^\infty) \simeq E^\bullet(*)[\![c_1^E]\!]$ where $c_1^E \in E^2(\mathbb{C}P^\infty)$ is called the (universal) **E-Chern class**. Given a line bundle $f : X \to \mathbb{C}P^\infty$ then we get a Chern class $c_1^E(X) \equiv f^* c_1^E$.

The projections $\pi_1, \pi_2 : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ give us three line bundles on $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$

$$\mathcal{L}_1 = \pi_1^* \mathcal{O}(1) \qquad \mathcal{L}_1 = \pi_2^* \mathcal{O}(1) \quad \text{and} \quad \mathcal{L}_3 = \mathcal{L}_1 \otimes \mathcal{L}_2$$

and we note that $\mathcal{L}_3 = \pi^* \mathcal{O}(1)$ for some $\pi : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$. Applying $E^\bullet$ to the identity $\mathcal{L}_3 = \mathcal{L}_1 \otimes \mathcal{L}_2$ and noting that $E^\bullet(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \simeq E^\bullet(*)[\![\pi_1^* c_1, \pi_2^* c_1]\!]$ then gives that $\pi^* c_1^E = f(\pi_1^* c_1^E, \pi_2^* c_1^E)$ for some $f \in E^\bullet(*)[\![\pi_1^* c_1, \pi_2^* c_1]\!]$.

We know very little about f, but as it must respect operations on line bundles we know that $f(u, v) = f(v, u)$ for any classes u and v, corresponding to the fact that the tensor product of line bundles is symmetric, and we know similarly that there must be a unit, and associativity relations. This leads to the following definition.

DEFINITION 25: FORMAL GROUP LAW

A **formal group law** (R, f) over a ring R is a power series $f(x, y) \in R$ such that

(a) $f(x, y) = f(y, x)$

(b) $f(x, 0) = f(0, x) = x$

(c) $f(f(x, y), z) = f(x, f(y, z))$.

A morphism of $\phi : (R, f) \to (R, f')$ formal group laws over R is a power series $\phi \in R[\![x]\!]$ such that $\phi(0) = 0$ and $\phi(f(x, y)) = f'(\phi(x), \phi(y))$. Sometimes we write $x +_f y := f(x, y)$.

Given a morphism $\varphi : R \to S$ of rings and a formal group law $f = \sum_{i,j} a_{ij} x^i y^j$ over R, we obtain a formal group law $\varphi^* f := \sum_{i,j} f(a_{ij}) x^i y^j$ over S. We actually have a universal formal group law given by the Lazard ring.

DEFINITION 26: LAZARD

Let R be a ring and $f \in R[\![x, y]\!]$ a formal group law. The **Lazard ring** L given by

$$L = \mathbb{Z}[a_{ij}]/I$$

where I is the ideal generated by the relations $a_{ij} = a_{ji}$, $a_{01} = a_{10} = 1$, $\forall i : a_{i0} = 0$ and

$$\ell(x, \ell(y, z)) = \ell(\ell(x, y), z) \text{ where } \ell(x, y) = \sum_{ij} a_{ij} x^i y^j \in L[\![x, y]\!].$$

THEOREM 27: LAZARD [LUR10, LECTURES 2&3]

The pair (L, ℓ) in definition 26 is universal among formal group laws; any formal group law (R, f) is given by a morphism $\varphi : L \to R$ such that $\ell = \varphi^* f$. In other words, the functor

$$\mathrm{CRing} \to \mathcal{S}\mathrm{et}, \quad R \mapsto \mathrm{FGL}(R)$$

sending a ring R to its set of formal group laws is corepresented by (L, ℓ). We can extend this to graded rings by letting $\deg(a_{ij}) = 2(i + j)$.

EXAMPLE 28

Let R be a ring.

(i) The **additive formal group law** on R is given by $\mathbb{G}_a(x,y) = x + y$.

(ii) The **multiplicative formal group law** on R is given by $\mathbb{G}_m(x,y) = x + y - xy$.

(iii) If $g \in R[\![t]\!]$ is a power series then we get a formal group law $f(x,y) = g^{-1}(g(x) + g(y))$ isomorphic to $\mathbb{G}_a$.

It can be shown that $\mathbb{G}_a$ and $\mathbb{G}_m$ are isomorphic if and only if the **logarithm**; $\phi(x) = -\log(1-x) = \sum_{i=1}^{\infty} \frac{x^n}{n} \in R[\![x]\!]$ exists, hence if $n \in R$ is invertible for all n, i.e. if and only if R is a $\mathbb{Q}$-algebra.

THEOREM 29: QUILLEN

Complex cobordism has homotopy groups given by the Lazard ring $\pi_* MU = L$.

In other words, giving a formal group law $(R_\bullet, f)$ over a graded ring $R_\bullet$ is equivalent to giving a map $\pi_* MU \to R_\bullet$. We can then ask when the functor

$$E(R,f) : \mathrm{An} \to \mathrm{An}, \qquad X \mapsto R_\bullet \otimes_{MU(*)} MU(X) \tag{1}$$

defines a spectrum, i.e. when is the functor $R_\bullet \otimes_{MU(*)} -$ exact. Landweber gave sufficient conditions on $R_\bullet$ for this to be the case.

DEFINITION 30: HEIGHT

Let $(R_\bullet, f)$ be a formal group law over a graded ring $R_\bullet$. Let p be a prime number, and $v_k(p) \in R_\bullet$ the coefficient of x^{p^k} in

$$[p]_f(x) := \underbrace{x +_f \cdots +_f x}_{p \text{ times}},$$

The **height** of f is n if $v_k = 0$ for all $k < n$ and v_n is invertible.

THEOREM 31: LANDWEBER EXACT FUNCTOR

If $v_0(p), \ldots, v_k(p)$ is such that $v_i(p)$ is not a zero-divisor in $R/(v_0, \ldots, v_{i-1})$ for all k, $i < k$ and primes p, then we say $(R_\bullet, f)$ is **Landweber exact**. In this case $E(R,f)$ in equation 1 forms a spectrum.

DEFINITION 32

Let k be a perfect field of characteristic p. An **infinitesimal thickening** (A, ϕ) of k is a commutative ring A along with a surjection $\phi : A \twoheadrightarrow k$ such that

- the kernel $\ker \phi$ is nilpotent, $(\ker \phi)^N = 0$ for some large N, and
- the quotient $(\ker \phi)^n / (\ker \phi)^{n+1}$ is a finite-dimensional k-vectorspace.

DEFINITION 33: DEFORMATION

Let k be a perfect field of characteristic p and (k, f) a formal group law over k. A **deformation** $(A, \phi, \tilde{f})$ of (k, f) over an infinitesimal thickening (A, ϕ) consists of a formal group law $\tilde{f}$ over A such that the image of $\tilde{f}$ under the induced map $\mathrm{FGL}(A) \to \mathrm{FGL}(k)$ is f. Two deformations are isomorphic if they differ by a formal power series $g(t) \in A[\![t]\!]$ with $g(t) = t \mod \ker \phi$. We let $\mathrm{Def}(f)$ be the set of isomorphism classes of f.

It turns out that the groupoid $\mathrm{Def}(f)$ is actually discrete, so we only need to consider it as a set. We now look at a particularly nice way of constructing a deformation. Recall that the **Frobenius** endomorphism on $\mathbb{F}_p$ is the endomorphism $\mathrm{frob}_p : \mathbb{F}_p \to \mathbb{F}_p, x \mapsto x^p$, which we can lift to an arbitrary $\mathbb{F}_p$-algebra A by $\mathrm{frob}_p^A : A \to A, a \mapsto a^p$. This gives, for example, the Adams operations ψ_p on K-theory.

DEFINITION 34: Λ-RING

Let R be a commutative ring equipped with an endomorphism $F_R : R \to R$. We call R a **(p-typical) Λ-ring** if the lift of F_R to the $\mathbb{F}_p$-algebra $A = R \otimes_{\mathbb{Z}} \mathbb{F}_p$ gives the Frobenius morphism frob_p^A. These assemble into a category $\Lambda\mathrm{Ring}_p$.

DEFINITION 35: WITT VECTORS

The (p-typical) **Witt vectors** $W_p : \mathrm{CRing} \to \Lambda\mathrm{Ring}_p$ of a ring R is given by the right adjoint to the forgetful functor $\mathrm{forget} : \Lambda\mathrm{Ring} \to \mathrm{CRing}, (R, F_R) \mapsto R$ i.e.

$$\mathrm{forget} : \Lambda\mathrm{Ring} \rightleftarrows \mathrm{CRing} : W_p.$$

For example, the Witt vectors of $\mathbb{F}_p$ are the p-adic integers $W_p(\mathbb{F}_p) = \mathbb{Z}_p$. We can explicitly construct $W_p(R)$ as follows, as a set we have that $W_p(R) = R[x_0, x_1, \dots]$, and we give $W_p(R)$ the unique ring structure such that the maps $w : W_p(R) \to R[x_0, x_1, \dots]$ given by

$$(x_0, x_1, \dots) \mapsto \left(x_0, x_0^p + px_1, \dots, \sum_{i=0}^{k} p^i x_i^{p^{k-i}}, \dots \right)$$

are ring homomorphisms. Here we have given $R[x_0, x_1, \dots]$ the unique[2] ring structure with addition and multiplication given by polynomials with integer coefficients that are independent of R, and the projections $W_p(R) \to R[x_i]$ are ring homomorphisms. See [Bor15] for details.

DEFINITION 36: LUBIN-TATE FORMAL GROUP

Let k be a perfect field, the ring $R(k,p) = W_p(k)[\![v_1, \dots, v_{n-1}]\!]$ is called the (height n) **Lubin-Tate ring**.

The Lubin-Tate ring $R(k,p)$ is equipped with a homomorphism $\pi : R(k,p) \to k$ with kernel $\ker(\pi) = (p, v_1, \dots, v_{n-1})$, i.e. $R(k,p)$ is an infinitesimal thickening of k, in fact it is a universal deformation. Let (k, f) be a formal group law over k which, by theorem 27, is classified by a map $\varphi_f : L \to k$ from the Lazard ring (L, ℓ). Finding a deformation $\tilde{f}$ of (k, f) is then equivalent to finding a map $\varphi_{\tilde{f}} : L \to R(k,p)$ such that $\varphi_f = \pi\varphi_{\tilde{f}}$.

THEOREM 37: LUBIN TATE

Let (k, f) be a perfect field k of characteristic p and f a height n formal group law over k. Any lift of f to a formal group law $\tilde{f}$ over $W(k)[\![v_1, \dots . v_{n-1}]\!]$ is *universal*; any infinitesimal thickening A of k gives a bijection

$$\mathrm{Hom}_{/k}(W(k)[\![v_1, \dots . v_{n-1}]\!], A) \xrightarrow{\sim} \mathrm{Def}(A).$$

A choice of universal deformation $(W_p(k)[\![v_1, \dots, v_{n-1}]\!], \tilde{f})$ of a formal group law (k, f) gives a Landweber exact ring, and hence defines a spectrum called Morava E-theory. Before we give the definition of this spectrum, we will briefly give a construction of a formal group law Γ over the ring $W_p(k)[\![v_1, \dots, v_{n-1}]\!]$ of Witt vectors called the **Honda formal group law**. This choice of formal group law turns out to be particularly nice for computations in chromatic homotopy theory; a different choice of formal group law would give a spectrum equivalent to Morava E-theory, but with a different $\mathbb{E}_\infty$-ring structure.

DEFINITION 38: MORAVA E-THEORY

Let (k, f) be a perfect field k of characteristic p and f a height n formal group law over k. We define the spectrum $E_n(k, f)$, called **Morava E-theory**, as $E(R, f)$ with $R = W_p(k)[\![v_1, \dots, v_{n-1}]\!]$.

Often we omit the k, f factors and denote it $E_n := E_n(k, f)$. Morava E-theory has homotopy groups $\pi_* E_n = W_p(k)[\![v_1, \dots, v_{n-1}]\!][\beta^{\pm 1}]$ where β is the Bott element.

[2] It is a theorem of Witt that such a ring structure exists and is unique.

DEFINITION 39: MORAVA K-THEORY [LUR10, LECTURE 24]

For a fixed prime p and $n \in \mathbb{N}$, we define the **Morava K-theory** spectrum $K(n)_\bullet$ as the (unique) associative spectrum which is

- complex oriented[a]; i.e. the image of the morphism $S^2 \hookrightarrow \mathbb{C}P^\infty$, $K(n)_2(\mathbb{C}P^\infty) \to K(n)_2(S^2)$, is surjective,
- the formal group law corresponding to $K(n)$ has height exactly n,
- has homotopy groups $\pi_* K(n) \cong \mathbb{F}_p[v_n^{\pm 1}]$ with $\deg(v_n) = 2(p^n - 1)$.

[a] We actually need to take *periodic* $K(n)$ for this, which is the wedge sum of $p^n - 1$ copies.

Let M_{FG}^n be the stack of formal group laws of height *exactly* n, so that a height n formal group law over R is given by a map $\operatorname{Spec} R \to M_{FG}$. There is a (unique up to isomorphism) height n formal group law Γ_n on the algebraic closure $\overline{\mathbb{F}}_p$ called the **Honda formal group law**, and the map $\operatorname{Spec} \overline{\mathbb{F}}_p \to M_{FG}$ is faithfully flat, giving us a cover of M_{FG}^n. We can then try to understand M_{FG} via the stack

$$\mathbb{G}_n \equiv \operatorname{Spec} B = \operatorname{Spec} \overline{\mathbb{F}}_p \times_{M_{FG}} \operatorname{Spec} \overline{\mathbb{F}}_p.$$

Since $\overline{\mathbb{F}}_p$ is algebraically closed and B is a filtered colimit $B = \operatorname{colim}_{\alpha \in \Lambda} k_\alpha$ of finite étale extensions k_α of $\overline{\mathbb{F}}_p$ (hence all the $k_\alpha = \overline{\mathbb{F}}_p$) we have that B is a filtered colimit of finite sets. Thus $\operatorname{Spec} B$ is a pro-finite set.

To give a k-point $\operatorname{Spec} k \to \mathbb{G}_n$ of $\mathbb{G}_n$, where k is an algebraic closure of $\mathbb{F}_p$, is equivalent to giving an isomorphism class of maps $B \to k$, i.e. to giving

- a pair $\eta, \eta' : \overline{\mathbb{F}}_p \to k$
- an isomorphism between the formal group laws $\eta(\Gamma_n) \xrightarrow{\sim} \eta'(\Gamma_n)$.

Fixing $k = \overline{\mathbb{F}}_p$ and $\eta' = \mathrm{id}_{\overline{\mathbb{F}}_p}$ we see this is just the data of an automorphism η of $\overline{\mathbb{F}}_p$ over $\mathbb{F}_p$ (i.e. an element of the absolute Galois group $\operatorname{Gal}(\mathbb{F}_p)$) along with an automorphism of formal groups $\eta(\Gamma_n) \xrightarrow{\sim} \Gamma_n$. In other words, we have that

$$\mathbb{G}_n \cong \operatorname{Aut}(\Gamma_n) \rtimes \operatorname{Gal}(\overline{\mathbb{F}}_p).$$

THEOREM 40: [LUR10, LECTURE 19, PROP 1]

We have that $M_{FG}^n \cong \operatorname{Spec} \overline{\mathbb{F}}_p / \mathbb{G}_n$, and the inclusion $M_{FG}^n \hookrightarrow M_{FG}$ corresponds (under Lubin-Tate) to the Bousfield localisation $L_{K(n)}$.

The goal of this lecture is to understand the following theorem.

THEOREM 41: MORAVA STABILISER

The n^{th} Morava stabiliser group $\mathbb{G}_n := \operatorname{Aut}(\Gamma_n) \rtimes \operatorname{Gal}(\mathbb{F}_{p^n} : \mathbb{F}_p)$ is the Galois group for the étale cover $L_{K(n)}\mathbb{S} \to E_n$.

GROTHENDIECK'S GALOIS THEORY

FINITE FIELDS

DEFINITION 42: GALOIS EXTENSION

Let $k \subseteq K$ be a field extension, and $G \leq \mathrm{Aut}_k(K)$ be such that $K^G = k$. Then $k \subseteq K$ is ***G*-Galois** if it is finite, normal and separable.

This definition of separability clearly favours a Galois theory of fields, so our goal is to find a notion of separable extension that is more categorical in nature so as to be applicable to other settings. This is achieved by the following definition (found in [Ric00]).

THEOREM 43

Let $k \subseteq K$ be a field extension and let $G \leq \mathrm{Aut}_k(K)$, then $k \subseteq K$ is G-Galois if and only if

(GI) K has G-fixed points $K^G = k$.

(GII) The map

$$K \otimes_k K \to \prod_{g \in G} K, \quad u \otimes v \mapsto \big(u\,(g \cdot v)\big)_{g \in G}$$

is a G-equivariant isomorphism, where $G = \mathrm{Aut}_k(K)$, the automorphism group of K over k, acts on $K \otimes_k K$ with $g \cdot (w \otimes z) = w \otimes (g \cdot z)$ and G acts on $\prod_{g \in G} K$ by $g \cdot (\alpha_h)_{h \in G} = (\alpha_{hg})_{h \in G}$.

We won't prove this, but instead give an example which exemplifies the similarity between this notion of Galois extension and the previously given one.

EXAMPLE 44

We show that $\mathbb{R} \subseteq \mathbb{C}$ is a Galois extension. As an $\mathbb{R}$-vectorspace we have that $\dim_{\mathbb{R}} \mathbb{C} = 2$, which is certainly finite, and normality is clear by noting that $\mathbb{C}$ is the splitting field of $x^2 + 1$.

To show separability we need to see that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \prod_{\mathbb{Z}/2\mathbb{Z}} \mathbb{C}$ is an isomorphism, where $G = \mathbb{Z}/2\mathbb{Z} = \{0, \tau\}$ acts on $\mathbb{C}$ (over $\mathbb{R}$) by complex conjugation $\tau(z) = \overline{z}$. The image of $w \otimes z \in \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is $(wz, w\overline{z})$. This map is injective since if $w \otimes z \mapsto (0,0)$ then we have $wv = 0$ hence either $w = 0$ or $z = 0$, so that $w \otimes z = 0$. This map is also surjective, to see this note that

$$\begin{aligned}
\frac{(1 \otimes 1) - (i \otimes i)}{2} &\mapsto (1, 0), \\
\frac{(1 \otimes 1) + (i \otimes i)}{2} &\mapsto (0, 1), \\
\frac{(i \otimes 1) + (1 \otimes i)}{2} &\mapsto (i, 0), \\
\frac{(i \otimes 1) - (1 \otimes i)}{2} &\mapsto (0, i),
\end{aligned}$$

and so we have that, if $(a + bi, c + di) \in \prod_{\mathbb{Z}/2\mathbb{Z}} \mathbb{C}$ then

$$\frac{(1 \otimes 1) - (i \otimes i)}{2}\,a + \frac{(1 \otimes 1) + (i \otimes i)}{2}\,c + \frac{(i \otimes 1) + (1 \otimes i)}{2}\,b + \frac{(i \otimes 1) - (1 \otimes i)}{2}\,d \mapsto (a + bi, c + di).$$

Thus the map $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \prod_{\mathbb{Z}/2\mathbb{Z}} \mathbb{C}$ is an isomorphism, and hence $\mathbb{R} \subseteq \mathbb{C}$ is a separable field extension.

Now that we've defined Galois extensions we can state the main theorem of Galois theory. First, recall

DEFINITION 45: ÉTALE ALGEBRA

An algebra K over k is **étale**[a] if K is isomorphic to a finite product of finite separable extensions of k.

[a] This is the same as saying that $\mathrm{Spec}\, K \to \mathrm{Spec}\, k$ is an étale morphism of schemes.

THEOREM 46: GALOIS CORRESPONDENCE

Let $k \subseteq K$ be a G-Galois extension, then there is an equivalence of categories given by the adjunction

$$\mathrm{Hom}_k(-,K) : \mathrm{Alg}_k^{\text{fin, ét}} \to G \text{ - } \mathcal{S}\mathrm{et}^{\text{fin}}$$

between the category of finite étale algebras over k, and the category of finite G-sets.

The (left) action of $g \in G \leq \mathrm{Aut}_k(K)$ on $\phi \in \mathrm{Hom}_k(A,K)$ is given by composition $g \cdot \phi = g \circ \phi$, so putting K into this functor we get the G-set $\mathrm{Hom}_k(K,K)$ of automorphisms of K over k, recovering the classical Galois group. We call $\mathrm{Hom}_k(-,K)$ the *fibre functor*. The inverse functor takes a finite G-set X and gives the free finite étale k-algebra with generators given by X.

EXAMPLE 47

Lets look at the Galois correspondence for the G-Galois extension $\mathbb{R} \subseteq \mathbb{C}$ as in example 44 with $G = \mathbb{Z}/2\mathbb{Z} = \{1, \tau\}$. Theorem 46 states that there is an equivalence

$$\mathrm{Hom}_{\mathbb{R}}(-,\mathbb{C}) : \mathrm{Alg}_{\mathbb{R}}^{\text{fin, ét}} \to (\mathbb{Z}/2\mathbb{Z}) \text{ - } \mathcal{S}\mathrm{et}^{\text{fin}}.$$

Let $A = \prod_{i=1}^n L_i$ be an étale $\mathbb{R}$-algebra, with each L_i separable extensions of $\mathbb{R}$. Note that if $f \in \mathrm{Hom}_{\mathbb{R}}\left(\prod_{i=1}^n L_i, \mathbb{C}\right)$ has $f(L_i) \neq 0$ then $L_i \hookrightarrow \mathbb{C}$ since any non-zero map of fields is injective, thus $\dim_{\mathbb{R}} L_i \leq 2$ for all i. Furthermore $L_i \times L_j$ cannot inject into $\mathbb{C}$ (since $\mathbb{C}$ has no zero divisors) so we have that f must map at most one L_i injectively into $\mathbb{C}$. Combining this with $|\mathrm{Hom}_{\mathbb{R}}(L_i,\mathbb{C})| = \dim_{\mathbb{R}} L_i$ we get that $|\mathrm{Hom}_{\mathbb{R}}(A,\mathbb{C})| = \sum_{i=1}^n \dim_{\mathbb{R}} L_i$ is a finite set. The action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathrm{Hom}_{\mathbb{R}}(A,\mathbb{C})$ is given simply by sending $f : A \to \mathbb{C}$ to the map $A \to \mathbb{C}$ given by $z \mapsto \overline{f(z)}$.

Conversely, suppose given a finite $(\mathbb{Z}/2\mathbb{Z})$-set X with $|X| = n$ for some n, then for each $x \in X$ we let

$$L_x = \begin{cases} \mathbb{R} & \text{if } \tau \cdot x = x \\ \mathbb{C} & \text{if } \tau \cdot x \neq x \end{cases}$$

and so $A = \prod_{x \in X} L_x$ is an étale $\mathbb{R}$-algebra. For each $x \in X$ we get a map $A \to \mathbb{C}$ by composing the projection $A \to L_x$ with the inclusion $L_x \hookrightarrow \mathbb{C}$.

We may recover the classical Galois theory of sub-field extensions and normal subgroups by demanding that our étale algebras A over k are connected (i.e. $\mathrm{Spec}\, A \to \mathrm{Spec}\, k$ is connected) and that the action of G is transitive. In example 47 note that $G = \mathbb{Z}/2\mathbb{Z}$ acts transitively on a finite set X if and only if $|X| = 1, 2$, whence X corresponds to the étale algebra $\mathbb{R}$ (if $|X| = 1$) with trivial action, or $\mathbb{C}$ (if $|X| = 2$) with $(\mathbb{Z}/2\mathbb{Z})$-action $i \mapsto -i$.

Galois Categories

Grothendieck originally proved the fundamental theorem of Galois theory in SGA1.

Definition 48: Galoisian 1-Category

We call a category $\mathcal{C}$ **Galoisian** and a functor $F:\mathcal{C}\to\mathcal{S}\mathrm{et}^{\mathrm{fin}}$ a **fibre functor** if

(G1) $\mathcal{C}$ has finite limits and colimits,

(G2) Every arrow $f:X\to Y$ in $\mathcal{C}$ splits as a composition $f=up$ where $p:X\twoheadrightarrow Z$ is a strict epimorphism[a] and $u:Z\hookrightarrow Z\sqcup Z'=Y$ is a coprojection and a monomorphism.

(G3) F is conservative (refelcts isomorphisms) and exact (preserves finite limits and colimits).

[a]i.e. p is the colimit of the diagram of all parallel pairs $g,h:X'\rightrightarrows X$ such that $pg=ph$.

A fibre functor $F:\mathcal{C}\to\mathcal{S}\mathrm{et}^{\mathrm{fin}}$ can be reconstructed by its value on the *finite* objects $\mathcal{C}^{\mathrm{fin}}:=\mathcal{C}\times_{\mathcal{S}\mathrm{et}}\mathcal{S}\mathrm{et}^{\mathrm{fin}}$ via left Kan extension along the inclusion $\iota:\mathcal{C}^{\mathrm{fin}}\hookrightarrow\mathcal{C}$

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{fin}} \arrow[r, "F|_{\mathcal{C}^{\mathrm{fin}}}"] \arrow[d, hook, "\iota"'] & \mathcal{S}\mathrm{et}^{\mathrm{fin}} \\
\mathcal{C} \arrow[ur, "F"'] &
\end{tikzcd}$$

Writing the Kan extension as a colimit gives

$$\iota_! F|_{\mathcal{C}^{\mathrm{fin}}}(b)=\operatorname*{colim}_{\iota X\to b}F|_{\mathcal{C}^{\mathrm{fin}}}(X)=F\left(\operatorname*{colim}_{\iota X\to b}X\right)=F(b).$$

since (by axiom G3) F preserves filtered colimits in $\mathcal{C}^{\mathrm{fin}}$.

This isn't really a proof now, is it? Why does F preserve filiered colimits of finite objects?

Definition 49: Galois Fundamental Group

Let $\mathcal{C}$ be a Galoisable category with fibre functor F, the **Galois group** of $\mathcal{C}$ at F is given by

$$\hat{\pi}_1^{\text{ét}}(\mathcal{C},F):=\mathrm{Aut}(F).$$

Definition 50: Galoisian ∞-Category [Mat14, Def 5.15]

An $(\infty,1)$-category $\mathcal{C}$ is **Galoisian** if

(G1') $\mathcal{C}$ admits finite limits and coproducts, and any map $x\to\emptyset$ to the initial object $\emptyset$ is an isomorphism (we say $\emptyset$ is 'empty'),

(G2') Coproducts are disjoint[a] and distributive[b] in $\mathcal{C}$

(G3') Every object $x\in\mathcal{C}$ is *locally elementary*; there exists an $x'=\coprod_{i=1}^n x_i\in\mathcal{C}$ such that

(i) the adjunction $-\circ t:\mathcal{C}_{/x'}\rightleftarrows\mathcal{C}:t^*$ (given by composition/pullback with the terminal map $t:x'\to *$) is comonadic.

(ii) the maps $x\times x_i\to x_i$ decompose as the fold map $x\times x_i\simeq\coprod_{s\in S_i}x_i\to x_i$ for some finite set S.

[a]The pushout of $\emptyset\to x$ and $\emptyset\to y$ is $x\sqcup y$.
[b]i.e. $f:x\to y$ inducing $\mathcal{C}_{/y}\to\mathcal{C}_{/x}$ commutes with coproducts.

GALOIS GROUPOIDS

Given a Galoisable category $\mathcal{C}$, we obtain a profinite groupoid by considering the groupoid

$$\pi_{\leq 1}\mathcal{C} = \underline{\mathrm{Hom}}^{\mathrm{fib}}(\mathcal{C}, \mathcal{S}\mathrm{et}^{\mathrm{fin}})^{\simeq}$$

of fibre functors (i.e. exact conservative functors $F : \mathcal{C} \to \mathcal{S}\mathrm{et}^{\mathrm{fin}}$) and their automorphisms. This profinite groupoid is to definition 49 as the fundamental groupoid is to the fundamental group in topology. To see that this groupoid is profinite we recall that any fibre functor F can be written as the left Kan extension $\iota_! F|_{\mathcal{C}^{\mathrm{fin}}}$ of F restricted to $\mathcal{C}^{\mathrm{fin}} = \mathcal{C} \times_{\mathcal{S}\mathrm{et}} \mathcal{S}\mathrm{et}^{\mathrm{fin}}$ along the inclusion $\iota : \mathcal{C}^{\mathrm{fin}} \hookrightarrow \mathcal{C}$. Thus any $F \in \pi_{\leq 1}\mathcal{C}$ is uniquely determined by its restriction $\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{fin}}, \mathcal{S}\mathrm{et}^{\mathrm{fin}})$, the latter being a finite groupoid since for any $F : \mathcal{C}^{\mathrm{fin}} \to \mathcal{S}\mathrm{et}^{\mathrm{fin}}$ we have there are finitely many automorphisms of each finite set $F(c)$ for $c \in \mathcal{C}^{\mathrm{fin}}$, and as F is conservative there are finitely many automorphisms of any c.

Conversely, given a pro-finite groupoid $G = \lim_{\alpha\in\Lambda} G_\alpha$ we claim that we have a Galoisable category $\mathcal{C} = \mathrm{colim}_{\alpha\in\Lambda} \underline{\mathrm{Hom}}(G_\alpha, \mathcal{S}\mathrm{et}^{\mathrm{fin}})$. Axiom G2 is trivially satisfied, since every morphism is an isomorphism, which is equivalent to being a strict epimorphism and a monomorphism. Giving a fibre functor $F : \mathcal{C} = \mathrm{colim}_{\alpha\in\Lambda} \underline{\mathrm{Hom}}(G_\alpha, \mathcal{S}\mathrm{et}^{\mathrm{fin}}) \to \mathcal{S}\mathrm{et}^{\mathrm{fin}}$ is equivalent to giving $F_\alpha : \underline{\mathrm{Hom}}(G_\alpha, \mathcal{S}\mathrm{et}^{\mathrm{fin}}) \to \mathcal{S}\mathrm{et}^{\mathrm{fin}}$ for each $\alpha \in \Lambda$ compatibly. Every object $x \in G_\alpha$ gives rise to such a fibre functor $\mathrm{fib}_x : \underline{\mathrm{Hom}}(G_\alpha, \mathcal{S}\mathrm{et}^{\mathrm{fin}}) \to \mathcal{S}\mathrm{et}^{\mathrm{fin}}$ given by $\mathrm{fib}_x(F_\alpha) = F_\alpha(x)$, and it can be checked that the F_α are exact and conservative.

How to show co/completeness, and do we *show* that the F_α are exact/conservative or should we actually define $\mathcal{C} = \mathrm{colim}_{\alpha\in\Lambda} \underline{\mathrm{Hom}}^{\mathrm{fib}}(G_\alpha, \mathcal{S}\mathrm{et}^{\mathrm{fin}})$ as fiber functors, not ?

We have thus given two sides of the following equivalence.

THEOREM 51: [MAT14, PROP 5.36]

There is an equivalence of 2-categories

$$\mathrm{Pro}(\mathrm{Grpd}_{\mathrm{fin}})^{\mathrm{op}} \to \mathrm{GalCat}$$

between the opposite category of profinite groupoids and the category of Galois categories.

EXAMPLE 52

Let $k \subseteq K$ be a Galois extension and $F = \mathrm{Hom}_k(-, K) : \mathrm{Alg}_k^{\text{ét}} \to \mathcal{S}\mathrm{et}$, then the Galois group is

$$\begin{aligned}\hat{\pi}_1^{\text{ét}}(\mathcal{C}, F) &= \mathrm{Aut}(F)\\ &= \mathrm{Hom}(\mathrm{Hom}_k(-, K), \mathrm{Hom}_k(-, K))\\ &= \mathrm{Hom}_k(K, K)\\ &= \mathrm{Aut}_k(K)\end{aligned}$$

by the Yoneda lemma.

EXAMPLE 53: TOPOLOGICAL SPACES

Recall that a *cover* of topological spaces is a map $p : Y \to X$ such that any $x \in X$ has a neighbourhood U for which $p^{-1}(U) = \bigcup_i V_i$ for disjoint $V_i \cong U$. This forms a category Cov_X. An *automorphism* of a cover $Y \to X$ is just an automorphism of Y over X, and forms a group $\mathrm{Aut}_X(Y)$, and we say $Y \to X$ is *Galois* if $X \cong Y/\mathrm{Aut}_X(Y)$.

Given a Galois cover $Y \to X$ and a basepoint $x \in X$ we get a functor

$$\mathrm{fib}_x : \mathrm{Cov}_X \to \mathcal{S}\mathrm{et}, \quad (Y \to X) \mapsto Y_x := Y \times_X \{x\}.$$

and we can note that $\pi_1(X, x)$ acts on Y_x via *monodromy*; given a loop γ in X and a $y \in Y_x$, γ lifts uniquely to a path $\tilde{\gamma} : [0,1] \to Y$ with endpoint $\tilde{\gamma}(0) = y$, the monodromy action of γ on y is $\gamma \cdot y := \tilde{\gamma}(1)$. This defines a Galois category if we restrict to *finite* covers, and theorem 51 gives that $\hat{\pi}_1(X, x)$-sets are equivalent to finite covers of X. For non-finite covers we have a universal cover, which is to say, fib_x is representable!

EXAMPLE 54: SCHEMES

Let X be a scheme, we have an analogous category of covers given by the finite étale maps $Y \to X$, which defines a category $\mathrm{Sch}^{\text{fin, ét}}_{/X}$. Given finite étale $Y \to X$ and a geometric point $x : \mathrm{Spec}(k) \to X$ (i.e. k is algebraically closed) we have an action of $\mathrm{Aut}_X(Y)$ on $Y_x := Y \times_X \mathrm{Spec}(k)$ given by base-changing along the automorphism. Thus, given a geometric point $x : \mathrm{Spec}(k) \to X$ we have a functor

$$\mathrm{fib}_x : \mathrm{Sch}^{\text{fin, ét}}_{/X} \to \mathcal{S}\mathrm{et}, \quad (Y \to X) \mapsto Y_x := Y \times_X \mathrm{Spec}(k),$$

which by theorem 51 gives an equivalence between finite étale covers of X and $\mathrm{Aut}(\mathrm{fib}_x)$, we actually use this to define the *étale fundamental group* $\hat{\pi}_1^{\text{ét}}(X, x) := \mathrm{Aut}(\mathrm{fib}_x)$. A nice consequence of this for X quasicompact and geometrically integral[a] is the exact sequence of profinite groups

$$\hat{\pi}_1^{\text{ét}}(X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, K, x) \hookrightarrow \hat{\pi}_1^{\text{ét}}(X, x) \twoheadrightarrow \mathrm{Aut}_k(K)$$

where $k \subseteq K$ is a separable closure in some algebraic closure $\overline{K}$ and geometric point $x : \mathrm{Spec}\, \overline{K} \to X$. Furthermore, in nice cases, if X is a nice $\mathbb{C}$-scheme and we equip the set of $\mathbb{C}$-points $X(\mathbb{C})$ with the analytic topology $X(\mathbb{C})^{\mathrm{an}}$ then we actually have that $\hat{\pi}_1^{\text{ét}}(X, x)$ is the profinite completion of $\pi_1(X(\mathbb{C})^{\mathrm{an}}, x)$.

[a] That is, if $X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, K$ is integral for any algebraic closure $k \subseteq K$.

GALOIS THEORY FOR SPECTRA

This section is based heavily on the following works; [Win05], [Mat14], [Rog05] and [Ric00]. We will say E is a G-spectrum for a finite group G if G acts on $\pi_*(E) := \operatorname{colim}_n \pi_{n+k}(E(S^n))$.

DEFINITION 55: GALOIS EXTENSION [ROG05]

Let R be ring spectrum. We say an object $S \in \mathrm{CAlg}(\mathrm{Mod}_R)$ with G-action (for G a finite group) is a **G-Galois extension** of R if

(G1) $S^{hG} = R$, and

(G2) $\Theta : S \otimes_R S \to \prod_{g \in G} S$ is an isomorphism.

A G-Galois extension is **faithful** if $- \otimes_R S$ is a conservative functor.

EXAMPLE 56

Let $k \subseteq K$ be a Galois extension of fields, then the Eilenberg-MacLane spectra $Hk \to HK$ are a Galois extension of ring spectra. This is easily seen by noting that $k \hookrightarrow K$ is flat, so $\mathrm{Tor}_s^k(K, K) = 0$ for $s \neq 0$, and so the homotopy fixed point spectral sequence

$$E^2_{s,t} = H^{-s}(G; \pi_t HK) = \mathrm{Ext}^{-s,-t}(k, K) \Rightarrow \pi_{s+t}((HK)^{hG})$$

and the Künneth spectral sequence

$$E^2_{s,t} = \mathrm{Tor}^k_{s,t}(K, K) \Rightarrow \pi_{s+t}(HK \otimes_{Hk} HK)$$

both collapse at $s = t = 0$. In other words $(HK)^{hG} = H(K^G) = Hk$ and $HK \otimes_{Hk} HK = H(K \otimes_k K)$.

EXAMPLE 57

Let KO and KU denote the real and complex K-theory spectra, respectively, the complexification map $KO \to KU$ is a $(\mathbb{Z}/2\mathbb{Z})$-Galois extension.

DEFINITION 58: WEAK COVER [MAT14, DEF 6.2]

Let $(\mathcal{R}, \otimes, 1)$ be a 2Ring (i.e. a stable, presentable symmetric-monoidal $(\infty, 1)-$category). An object $A \in \mathrm{CAlg}(\mathcal{R})$ is a **weak**[a] **finite cover** if there exists an $A' \in \mathrm{CAlg}(\mathcal{R})$ such that

(C1) $- \otimes A'$ is conservative and commutes with all limits, and

(C2) $A \otimes A' \simeq \prod_{s \in S} A'$ in $\mathrm{CAlg}(\mathrm{Mod}_{A'})$ for some finite set S.

[a] We call the cover **strong** if instead of (C1) A' admits descent (i.e. generates $\mathcal{R}$ as a thick $\otimes$-ideal).

Theorem 51 allows us to assign, to any Galois category $\mathcal{C}$, a profinite groupoid $\hat{\pi}^{\text{ét}}_{\leq 1}(\mathcal{C})$, which we will call the *profinite Galois fundamental groupoid* of $\mathcal{C}$. We define the Galois groupoid of a ring spectrum R to be that of $\mathrm{Mod}(R)$.Recall that R is the unit in $\mathrm{Mod}(R)$, if R is a compact generator of $\mathrm{Mod}(R)$ then we have that $S \in \mathrm{Mod}(R)$ is Galois if[3] it is *descendable* (i.e. if it generates $\mathrm{Mod}(R)$ as a thick tensor ideal[4]).

THEOREM 59: $K(n)$-LOCAL STBLE HOMOTOPY THEORY [MAT14, THM 10.9]

The absolute Galois group of $L_{K(n)}\mathrm{Sp}$ is the extended Morava stabilizer group $\mathbb{G}_n$, i.e. the group of pairs (σ, ϕ) where $\sigma \in \mathrm{Aut}_{\mathbb{F}_p}(\overline{\mathbb{F}}_p)$ and $\phi : f \to \sigma^* f$ is an isomorphism of formal groups.

[3] Combining prop 6.13 and cor 6.16 of [Mat14]

[4] i.e. if the thick tensor ideal $\{A \xrightarrow{f} B \in \mathrm{Mod}(R) : f \otimes \mathrm{id}_A \simeq 0\}$ of A-zero maps is nilpotent, [Mat14, prop 3.27].

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