Alternate Coordinate Systems (Bases) Exercises

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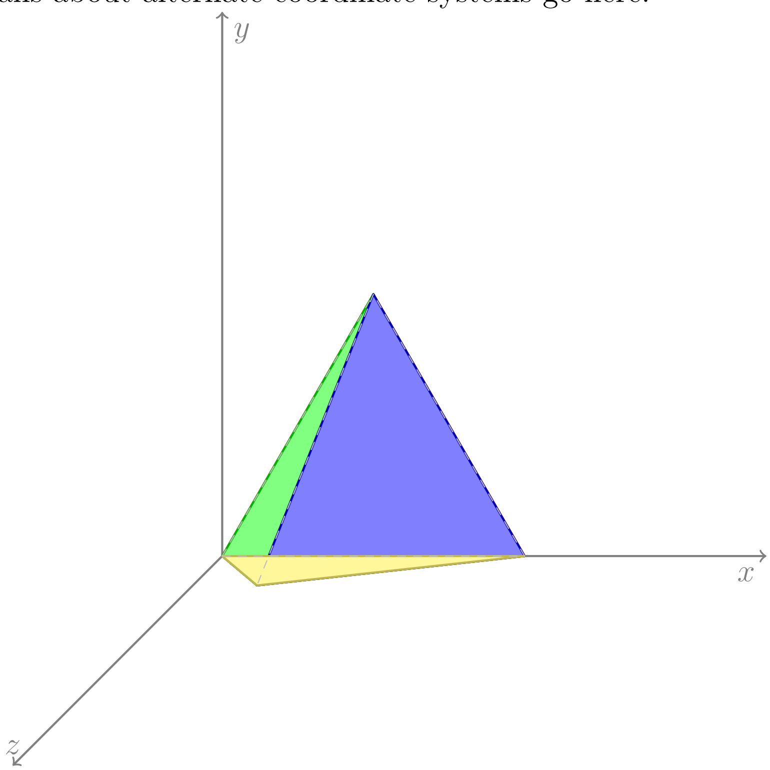
1 Introduction

This section provides an introduction to the document.

2 Alternate Coordinate Systems

Details about alternate coordinate systems go here.



Linear Algebra: Projections and Subspaces

Vector Space V The vector space V is defined as:

$$V = \{\boldsymbol{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}.$$

This can also be written as:

$$V = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

Matrix Representation of V The matrix A whose columns span V is:

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Projection Formula The projection of $\boldsymbol{x} \in \mathbb{R}^3$ onto V is given by:

$$\mathrm{Proj}_V \boldsymbol{x} = A(A^T A)^{-1} A^T \boldsymbol{x}.$$

Orthogonal Decomposition of $\boldsymbol{x}$ Every vector $\boldsymbol{x} \in \mathbb{R}^3$ can be decomposed as:

$$\boldsymbol{x} = \boldsymbol{x}^V + \boldsymbol{x}^W,$$

where:

$$\boldsymbol{x}^V \in V \quad \text{and} \quad \boldsymbol{x}^W \in V^\perp.$$

$$\boldsymbol{x}^V = \mathrm{Proj}_V \boldsymbol{x}, \quad \boldsymbol{x}^W = \boldsymbol{x} - \mathrm{Proj}_V \boldsymbol{x}.$$

Projection Matrix Derivation To compute the projection matrix, we start with:

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

First, calculate $A^T A$:

$$A^T A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Next, compute $(A^T A)^{-1}$:

$$(A^T A)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

The projection matrix is then:

$$P = A(A^T A)^{-1} A^T.$$

Substitute the values of A and $(A^T A)^{-1}$:

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

Performing the matrix multiplication step-by-step, we obtain:

$$P = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Projection Matrix B and Complement C The projection matrix onto V is:

$$B = \text{Proj}_V = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

The projection matrix onto the orthogonal complement $V^\perp$ is:

$$C = I_3 - B,$$

where I_3 is the 3×3 identity matrix:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Substituting for B, we get:

$$C = I_3 - \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Projection onto $V^\perp$ The matrix C simplifies to:

$$C = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Thus:

$$\text{Proj}_{V^\perp}\boldsymbol{x} = C\boldsymbol{x}.$$

Verification of Orthogonality To verify orthogonality:

$$B + C = I_3,$$

and:

$$BC = 0, \quad CB = 0.$$

Summary of Results - The projection of $\boldsymbol{x}$ onto V:

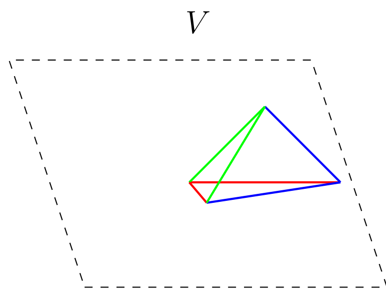
$$\text{Proj}_V\boldsymbol{x} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\boldsymbol{x}.$$

- The projection of $\boldsymbol{x}$ onto $V^\perp$:

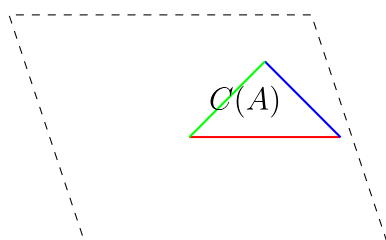
$$\text{Proj}_{V^\perp}\boldsymbol{x} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\boldsymbol{x}.$$

- The decomposition of $\boldsymbol{x}$:

$$\boldsymbol{x} = \boldsymbol{x}^V + \boldsymbol{x}^W, \quad \boldsymbol{x}^V = \mathrm{Proj}_V \boldsymbol{x}, \quad \boldsymbol{x}^W = \mathrm{Proj}_{V^\perp} \boldsymbol{x}.$$



$$||\vec{x} - \mathrm{Proj}_v \vec{x}|| \leq ||\vec{x} - \vec{v}||$$
$$\vec{b} = \mathrm{Proj}_v \vec{x} - \vec{v}, \quad \vec{b} \in V$$
$$||\vec{x} - \vec{v}||^2 = ||\vec{b} - \vec{a}||^2$$
$$= (\vec{b} + \vec{a}) \cdot (\vec{b} + \vec{a})$$
$$= \vec{b} \cdot \vec{b} + 2\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{a}$$
$$= ||\vec{b}||^2 = ||\vec{x} - \vec{v}||^2 \geq ||\vec{a}||^2$$
$$||\vec{x} - \vec{v}||^2 \leq ||\vec{a}||^2$$
$$||\vec{x} - \vec{v}|| \geq ||\vec{a}||$$
$$||\vec{x} - \vec{v}|| \geq ||\vec{x} - \mathrm{Proj}_v \vec{x}||$$



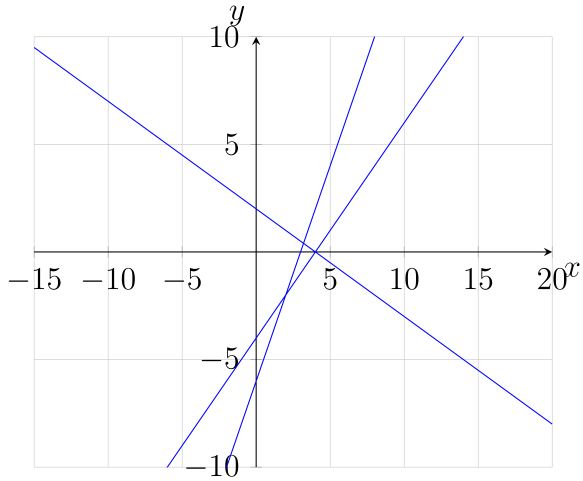
Consider the equation $A\vec{x} = \vec{b}$, where $\vec{x} \in \mathbb{R}^k$ and $\vec{b} \in \mathbb{R}^2$. (1)

Note: $\vec{b}$ is not in the column space of A. (2)

Let $\vec{x}^*$ be such that $A\vec{x}^*$ is as close as possible to $\vec{b}$. (3)

Then, the error is $||\vec{b} - A\vec{x}^*|| = || \begin{bmatrix} b_1 - V_1 \\ b_2 - V_2 \\ \vdots \end{bmatrix} ||^2$. (4)

Thus, $\vec{x}^* = \mathrm{Proj}_{C(A)} \vec{b}$ [1] (5)



$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$ When solving the system $A\vec{x} = \vec{b}$, we find that there is no solution.

However, we use the equation:

$$A^T A\vec{a}^* = A^T\vec{b} = \frac{3\sqrt{35}}{7}$$

to explore further.
Now, consider solving for x in the matrix equation:

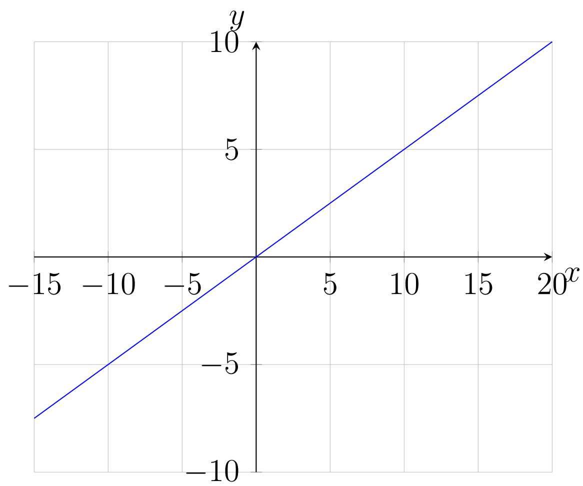
$$\begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 9 \\ 4 \end{bmatrix} \vec{x^*} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$$

Expanding the equation, we have:

$$\begin{bmatrix} 6+1 \\ 1+6 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$$

Perform Gaussian elimination to simplify:

$$\begin{bmatrix} 1 & 6 \\ 6 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 6 & | & 4 \\ 0 & -35 & | & -15 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & \frac{10}{7} \\ 0 & 1 & | & \frac{3}{7} \end{bmatrix} \vec{x^*} \begin{bmatrix} \frac{10}{7} \\ \frac{3}{7} \end{bmatrix}$$



Linear Equation of a Line We begin with the linear equation of a line:

$$y = mx + b \tag{6}$$

Given the specific linear equation:

$$y = \frac{2}{5}x + \frac{2}{5} \tag{7}$$

We define the function $F(x)$ as follows:

$$F(x) = mx + b = y \tag{8}$$

Function Evaluation Let's evaluate $F(x)$ at several points:

$$F(-1) = -m + b = 0, \tag{9}$$
$$F(0) = b = 1, \tag{10}$$
$$F(1) = m + b = 2, \tag{11}$$
$$F(2) = 2m + b = 1. \tag{12}$$

Matrix Representation Consider the matrix form of the linear equations:

$$\begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} x^* = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \tag{13}$$

$$
\begin{aligned}
6m^* + 2b^* &= 4 && (14)\\
-2m^* - 4b^* &= -4 && (15)\\
\Rightarrow \quad -2m^* - 4b^* &= -4 && (16)\\
\Rightarrow \quad m^* &= \frac{2}{5} && (17)\\
& && (18)\\
2m^* + 4b^* &= 4 && (19)\\
\Rightarrow \quad b^* &= \frac{4}{5} && (20)
\end{aligned}
$$

$$x\vec{x}^* = \begin{bmatrix} \frac{2}{5} \\ \frac{4}{5} \end{bmatrix} \text{ Ex pag 24}$$

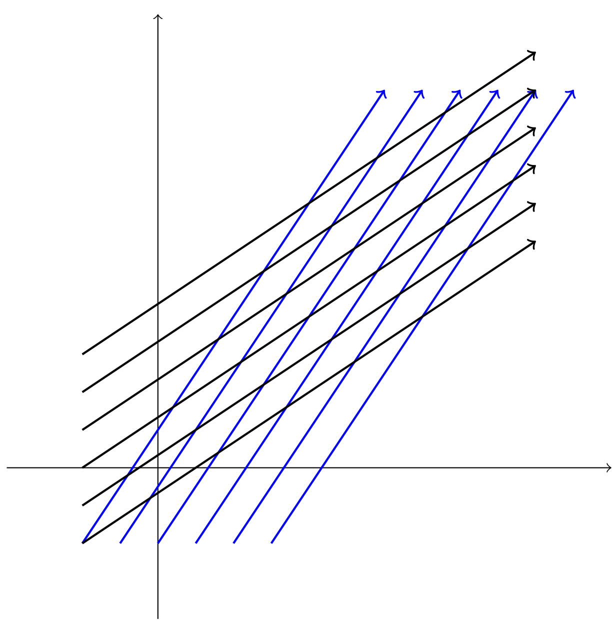
$$V \text{ is a subspace of } \mathbb{R}^n$$

$$[\vec{a}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$B = \text{ basis for } \mathbb{R}^2$$



$$3\vec{v}_1 + 2\vec{v}_2 = \begin{bmatrix} 8 \\ 7 \end{bmatrix} = \vec{a} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}$$

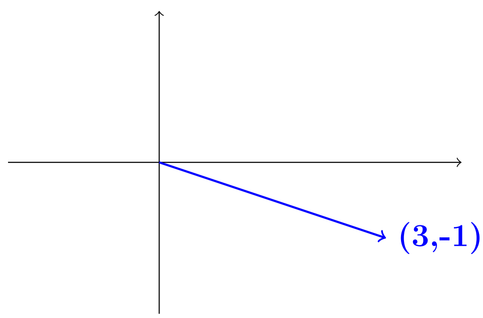
$$[\vec{a}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = x\vec{e}_1 + y\vec{e}_2$$

$$S = \text{standard basis for } \mathbb{R}^2$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_s = \begin{bmatrix} x \\ y \end{bmatrix}$$



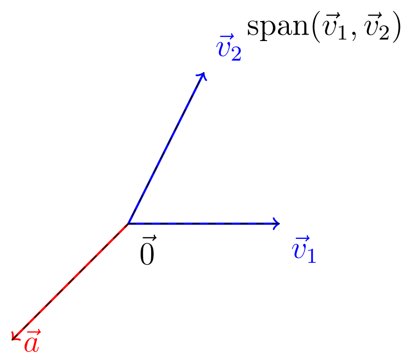
$$B = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k]$$

$$[\vec{a}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} \implies \vec{a} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$$

Graph goes here

$$C = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k]$$

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \vec{a} \implies \boxed{c\left[\vec{a}\right]_B} = \boxed{\vec{a}} \quad \text{(this is the change of basis matrix)}$$



$$\vec{a} \in \mathbb{R}^3$$

$$[\vec{a}]_B = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$

$$[\vec{d}]_B = \begin{bmatrix} 7 \\ -4 \end{bmatrix} \quad \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ 14 \\ 17 \end{bmatrix} = \vec{a}$$

$$\vec{d} = \begin{bmatrix} 8 \\ -6 \\ 2 \end{bmatrix} \quad \vec{d} \in \text{span}\{\vec{v}_1, \vec{v}_2\}$$

$$C([\vec{d}]_B) = \vec{d}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} \times$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \\ 2 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 2 & 8 \\ 2 & 0 & -6 \\ 3 & 1 & 2 \end{bmatrix}$$

Row reduce:

$$\begin{bmatrix}1 & 2 & 8\\0 & -4 & -22\\0 & -5 & -22\end{bmatrix} \to \begin{bmatrix}1 & 2 & 8\\0 & 1 & 11\\0 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & -3\\0 & 1 & 11\\0 & 0 & 0\end{bmatrix} \quad \implies \quad c_1 = -3,\ c_2 = 11$$

$$[\vec{d}]_B = \begin{bmatrix}-3\\11\end{bmatrix} \quad \implies \quad \vec{d} = -3\vec{v}_1 + 11\vec{v}_2$$

ex 2

$$B = \{\vec{v}_1, \vec{v}_2, \dots\}$$

Assumptions:

- C is invertible
- C is square
- B is a basis for $\mathbb{R}^n$

Claim:
If C is invertible, then Span of $B = \mathbb{R}^n$
Proof:
1. **Linear Independence of B:**
Since B is a basis for $\mathbb{R}^n$, it is linearly independent.
2. **Span of $B = \mathbb{R}^n$:**
Let v be an arbitrary vector in $\mathbb{R}^n$. Since C is invertible, there exists a unique vector u such that $Cu = v$.
Now, consider the linear combination of vectors in B:

$$u_1 b_1 + u_2 b_2 + \cdots + u_n b_n = u$$

where u_i are scalars and b_i are the basis vectors in B.
Applying C to both sides:

$$C(u_1 b_1 + u_2 b_2 + \cdots + u_n b_n) = Cu$$

Using the linearity of matrix multiplication:

$$u_1 Cb_1 + u_2 Cb_2 + \cdots + u_n Cb_n = v$$

This shows that any vector v in $\mathbb{R}^n$ can be expressed as a linear combination of the vectors in B. Therefore, Span of $B = \mathbb{R}^n$.

Conclusion:

If C is invertible, then Span of $B = \mathbb{R}^n$.

ex 3

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \qquad |C| = -5 \qquad C^{-1} = -\frac{1}{5} \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} 7 \\ 2 \end{bmatrix} \qquad [\vec{a}]_B = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

$$\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad [\vec{w}]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{w} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

B is a basis for $\mathbb{R}^2$

$[\vec{v}]_B = D[\vec{v}]$

$[\vec{v}]_B = C\vec{v}$

$[\vec{v}]_B = D[\vec{v}] = C\vec{v} = CA[C^{-1}]_B$

$D = CAC^{-1}$

$D = C^{-1}AC$

$$\begin{tikzcd}
 & T & D \\
\text{Standard Basis} \arrow[ur, "A"] \arrow[rr, leftrightarrow, "C"', "C^{-1}"'] & & \text{Basis } B \arrow[ul, "D"']
\end{tikzcd}$$

$$B = \{\vec{v}_1, \vec{v}_2\} \quad \text{is a basis for } \mathbb{R}^2 \tag{21}$$

$$T : \mathbb{R}^2 \to \mathbb{R}^2, \quad T(\vec{x}) = A\vec{x} \tag{22}$$

$$\text{A is the transformation matrix for } T \text{ with respect to the standard basis.} \tag{23}$$

$$[\vec{v}]_B = C\vec{v} \quad \text{C is invertible.} \tag{24}$$

$$[\vec{v}]_B = D[\vec{v}] \tag{25}$$

$$\text{D is the transformation matrix for } T \text{ with respect to the basis } B. \tag{26}$$

$$[\vec{v}]_B = D[\vec{v}] = C\vec{v} = CA[C^{-1}\vec{v}]_B \qquad D = CAC^{-1} \tag{27}$$

$$\tag{28}$$

$$D \text{ is the transformation matrix for } T \text{ w.r.t. } B\text{, and } C \text{ is the change-of-basis matrix for } B. \tag{29}$$

$$\text{A is the transformation matrix for } T \text{ with respect to the standard basis.} \tag{30}$$
$$D = C^{-1}AC \tag{31}$$

$$C \xrightarrow{T} [T(x)]_B$$

ex 5

$$\begin{aligned}
&T : \mathbb{R}^2 \to \mathbb{R}^2, \quad T(\vec{x}) = A\vec{x} \\
&\text{Standard coordinates} \\
&x \mapsto [x]_c \\
&[x]_B \mapsto [x]_c \\
&[x]_B = D[x]_B \\
&D = CAC^{-1} \\
&D = C^{-1}ACE \\
&A = CDC^{-1}
\end{aligned}$$

$$\begin{tikzcd}
& T \arrow[dr, "C^{-1}"] & \\
{[x]_c} \arrow[ur, "C"] \arrow[rr, bend right, "D"'] & & {[x]_B}
\end{tikzcd}$$