Integrals and Differential Equations

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Sommario

This document provides a comprehensive introduction to integrals and differential equations, drawing from various mathematical resources. It explains fundamental concepts, solution techniques, and applications, complemented by visual representations and worked examples to enhance understanding. Topics covered include separable differential equations, Euler's method, and slope fields, with emphasis on both theoretical foundations and practical problem-solving approaches.

Indice

1 Introduction to Differential Equations

A differential equation is an equation that relates an unknown function to its derivatives. The order of a differential equation is the highest derivative that appears in the equation.

Definition 1.1 (Differential Equation)**.** A differential equation is a mathematical equation that relates a function with its derivatives. In general, it can be written as:

$$F(x, y, y', y'', \dots, y^{(n)}) = 0 \tag{1}$$

where $y^{(n)}$ represents the n-th derivative of y with respect to x.

1.1 Notation for Differential Equations

There are several equivalent notations for expressing differential equations:

- Using prime notation: $y'' + 2y' = 3y$
- Using function notation: $F''(x) + 2F'(x) = 3F(x)$
- Using Leibniz notation: $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 3y$

All these forms represent the same differential equation.

2 Solving Linear Differential Equations

2.1 Example: Second-Order Linear Equation

Consider the differential equation:

$$y'' + 2y' = 3y \tag{2}$$

2.1.1 Verifying Solutions

Let's verify if $y_1(x) = e^{2x}$ is a solution:

$$\begin{aligned} y_1(x) &= e^{2x} \\ y_1' &= 2e^{2x} \\ y_1'' &= 4e^{2x} \end{aligned}$$

Substituting into the original equation:

$$\begin{aligned} y_1'' + 2y_1' - 3y_1 &= 4e^{2x} + 2(2e^{2x}) - 3e^{2x} \\ &= 4e^{2x} + 4e^{2x} - 3e^{2x} \\ &= 5e^{2x} \neq 0 \end{aligned}$$

Therefore, $y_1(x) = e^{2x}$ is not a solution.

Let's verify if $y_2(x) = e^{-x}$ is a solution:

$$\begin{aligned} y_2(x) &= e^{-x} \\ y_2' &= -e^{-x} \\ y_2'' &= e^{-x} \end{aligned}$$

Substituting:

$$\begin{aligned} y_2'' + 2y_2' - 3y_2 &= e^{-x} + 2(-e^{-x}) - 3e^{-x} \\ &= e^{-x} - 2e^{-x} - 3e^{-x} \\ &= -4e^{-x} \neq 0 \end{aligned}$$

Therefore, $y_2(x) = e^{-x}$ is also not a solution.

2.1.2 Finding the Correct Solution

Let's try $y_3(x) = e^{-3x}$:

$$\begin{aligned} y_3(x) &= e^{-3x} \\ y_3' &= -3e^{-3x} \\ y_3'' &= 9e^{-3x} \end{aligned}$$

Substituting:

$$\begin{aligned} y_3'' + 2y_3' - 3y_3 &= 9e^{-3x} + 2(-3e^{-3x}) - 3e^{-3x} \\ &= 9e^{-3x} - 6e^{-3x} - 3e^{-3x} \\ &= 0 \end{aligned}$$

Therefore, $y_3(x) = e^{-3x}$ is a solution.

2.1.3 Characteristic Equation Method

For linear differential equations with constant coefficients, we can use the characteristic equation.

For the equation $y'' + 2y' - 3y = 0$, the characteristic equation is:

$$\begin{aligned} \lambda^2 + 2\lambda - 3 &= 0 \\ (\lambda + 3)(\lambda - 1) &= 0 \\ \lambda = -3 \quad \text{or} \quad \lambda &= 1 \end{aligned}$$

Thus, the general solution is:

$$y(x) = C_1 e^{-3x} + C_2 e^{x} \tag{3}$$

where C_1 and C_2 are arbitrary constants determined by initial conditions.

3 First-Order Differential Equations

3.1 Real-World Applications

Differential equations can model various physical phenomena:

Physics Application: Motion

When an object moves under the influence of a force, its position S and speed v are related by:

$$v = \frac{dS}{dt}$$
$$a = \frac{dv}{dt} = \frac{d^2S}{dt^2}$$

3.2 Separable Differential Equations

A first-order differential equation is separable if it can be written in the form:

$$\frac{dy}{dx} = g(x)h(y) \tag{4}$$

Example 3.1 (Separable Equation)**.** Consider the equation:

$$\frac{dy}{dx} = \frac{4y}{x}$$

Separating variables:

$$\frac{dy}{y} = \frac{4\,dx}{x}$$
$$\int \frac{dy}{y} = \int \frac{4\,dx}{x}$$
$$\ln|y| = 4\ln|x| + C$$
$$\ln|y| = \ln|x^4| + C$$
$$y = Kx^4$$

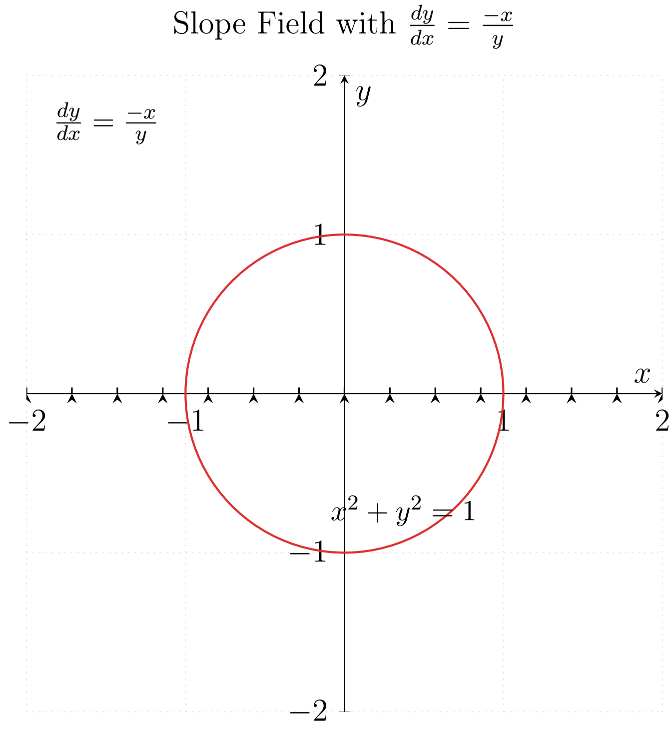
where $K = \pm e^C$ is an arbitrary constant.

4 Slope Fields and Graphical Solutions

4.1 Definition of Slope Field

A slope field (or direction field) is a graphical representation of a first-order differential equation. It consists of short line segments whose slopes are given by the differential equation at specific points in the xy-plane.

4.2 Slope Field for $\frac{dy}{dx} = \frac{-x}{y}$

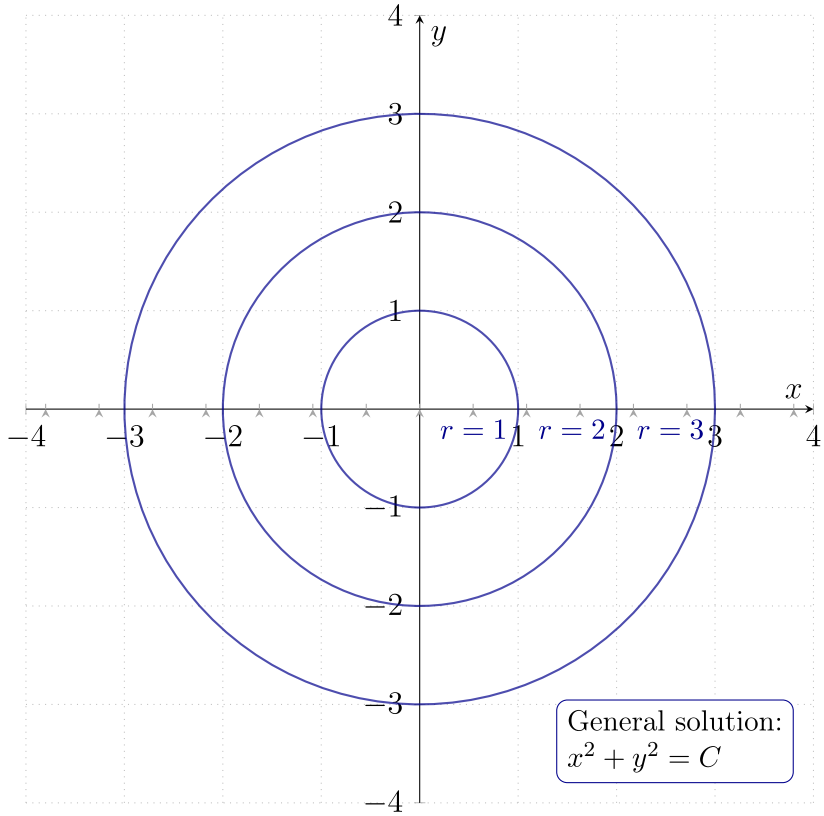


4.3 Values Table for the Slope Field

The table below shows the value of $\frac{dy}{dx}$ at various points:

x	y	$\frac{dy}{dx} = \frac{-x}{y}$	**Interpretation**
0	1	0	Horizontal tangent
1	1	-1	Slope of -1
1	0	Undefined	Vertical tangent
1	-1	1	Slope of 1
-1	-1	-1	Slope of -1
-1	0	Undefined	Vertical tangent
-1	1	1	Slope of 1

4.4 Extended Slope Field with Solution Curves



Remark 4.1. The differential equation $\frac{dy}{dx} = \frac{-x}{y}$ can be rewritten as $y\,dy = -x\,dx$. Integrating both sides gives $\frac{y^2}{2} = -\frac{x^2}{2} + C$, or equivalently, $x^2 + y^2 = 2C$. This explains why the solution curves are circles centered at the origin.

5 Numerical Methods: Euler's Method

Euler's method is a first-order numerical procedure for solving ordinary differential equations with a given initial value.

5.1 Algorithm Description

Given a differential equation $\frac{dy}{dx} = f(x, y)$ with initial condition $y(x_0) = y_0$, Euler's method approximates the solution using the following recursive formula:

$$y_{n+1} = y_n + h \cdot f(x_n, y_n) \tag{5}$$

where $h = \Delta x$ is the step size, and (x_n, y_n) is the n-th approximation point.

5.2 Example Application

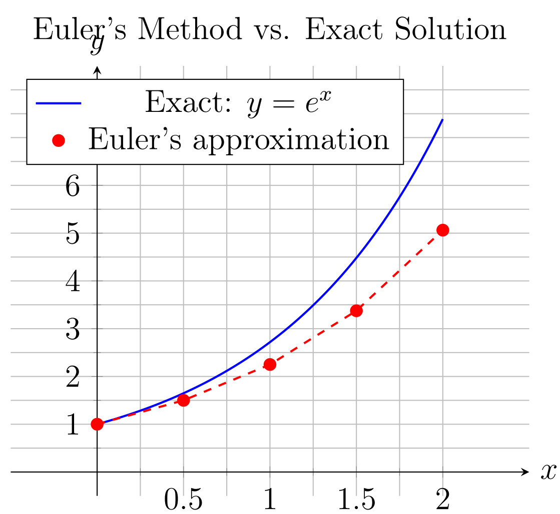
Consider the differential equation:

$$\frac{dy}{dx} = y, \quad y(0) = 1$$

The exact solution to this equation is $y(x) = e^x$.
Using Euler's method with a step size of $\Delta x = 0.5$, we can approximate the solution:

Step	x_n	y_n	$f(x_n, y_n) = y_n$	$y_{n+1} = y_n + h \cdot f(x_n, y_n)$
0	0	1	1	$1 + 0.5 \cdot 1 = 1.5$
1	0.5	1.5	1.5	$1.5 + 0.5 \cdot 1.5 = 2.25$
2	1.0	2.25	2.25	$2.25 + 0.5 \cdot 2.25 = 3.375$
3	1.5	3.375	3.375	$3.375 + 0.5 \cdot 3.375 = 5.0625$
4	2.0	5.0625	5.0625	$5.0625 + 0.5 \cdot 5.0625 = 7.59375$

5.3 Comparison with Exact Solution



Remark 5.1. Notice that Euler's method provides an approximation that increasingly deviates from the exact solution as x increases. This is because Euler's method has a local truncation error of order $O(h^2)$, which accumulates with each step.

6 Solving Separable Differential Equations

6.1 Example: Equation with Exponential Term

Given the equation:

$$\frac{dy}{dx} = \frac{-x}{ye^{x^2}}$$

Step 1: Separate the variables.

$$y\,dy = -x\,e^{-x^2}\,dx$$

Step 2: Integrate both sides.

$$\int y\,dy = \int -x\,e^{-x^2}\,dx$$

Step 3: Evaluate the left-hand side.

$$\frac{y^2}{2} + C_1$$

Step 4: Evaluate the right-hand side using substitution $u = -x^2$, $du = -2x\,dx$.

$$\begin{aligned}\int -x\,e^{-x^2}\,dx &= \frac{1}{2}\int e^u\,du \\ &= \frac{1}{2}e^u + C_2 \\ &= \frac{1}{2}e^{-x^2} + C_2\end{aligned}$$

Step 5: Combine results.

$$\frac{y^2}{2} = \frac{1}{2}e^{-x^2} + C$$

where $C = C_2 - C_1$.
Step 6: Multiply through by 2 and solve for y.

$$y^2 = e^{-x^2} + 2C$$

Step 7: Apply the initial condition $y(0) = 1$.

$$\begin{aligned}1^2 &= e^{-0^2} + 2C \\ 1 &= 1 + 2C \\ C &= 0\end{aligned}$$

Step 8: Write the final solution.

$$y^2 = e^{-x^2}$$

Taking the square root (assuming $y > 0$ based on the initial condition):

$$y = e^{-\frac{x^2}{2}}$$

Graphical Representation

The solution $y = e^{-\frac{x^2}{2}}$ represents a bell-shaped curve that achieves its maximum value of 1 at $x = 0$ and approaches 0 as $|x|$ increases.

6.2 General Approach for Separable Equations

For a separable equation $\frac{dy}{dx} = g(x)h(y)$:

1. Rewrite as $\frac{dy}{h(y)} = g(x)\,dx$ 2. Integrate both sides: $\int \frac{dy}{h(y)} = \int g(x)\,dx$ 3. Solve for y in terms of x 4. Apply initial conditions if available

7 Additional Differential Equation Examples

7.1 Example: Nonlinear First-Order Equation

Consider the differential equation:

$$\frac{dg}{dx} = 2g^2$$

Step 1: Separate the variables.

$$\frac{dg}{g^2} = 2\,dx$$

Step 2: Integrate both sides.

$$\int \frac{1}{g^2}\,dg = \int 2\,dx$$

Step 3: Evaluate the integrals.

$$\begin{aligned} -\frac{1}{g} &= 2x + C \\ \frac{1}{g} &= -(2x + C) \\ g &= \frac{-1}{2x + C} \end{aligned}$$

Step 4: Simplify the expression.

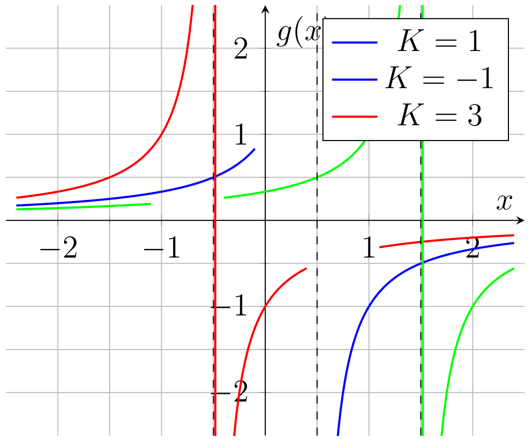
$$g(x) = \frac{1}{-2x - C} = \frac{1}{-2x + K}$$

where $K = -C$ is an arbitrary constant.

Remark 7.1. The solution has a vertical asymptote at $x = \frac{K}{2}$, indicating that the solution blows up at this point. This is a common feature of nonlinear differential equations and represents the phenomenon known as "finite-time blow-up."

7.2 Visual Representation of the Solution

Solutions to $\frac{dg}{dx} = 2g^2$



8 The Derivative as a Limit

The derivative of a function can be defined as the limit:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{6}$$

This definition forms the foundation for both differential and integral calculus, connecting the concepts of rate of change and accumulation.

9 Conclusion

This document has explored various aspects of differential equations, from their basic definitions to solution techniques for specific types of equations. We've seen:

- Different forms of differential equations and their notation
- Methods for solving linear differential equations
- Techniques for separable equations
- Graphical representations through slope fields
- Numerical approximation using Euler's method

Understanding differential equations is crucial for modeling diverse phenomena in physics, engineering, economics, and many other fields. The techniques presented here provide a foundation for approaching more complex differential equations and systems.

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