Integrals 3

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Contents

1 Integration Practice Problems

1.1 Basic Indefinite Integrals

1. $\int x^2\,dx = \frac{x^3}{3} + C$

2. $\int e^x\,dx = e^x + C$

3. $\int \sin(x)\,dx = -\cos(x) + C$

4. $\int \frac{1}{x}\,dx = \ln|x| + C$

5. $\int (3x^2 + 2x + 1)\,dx = x^3 + x^2 + x + C$

1.2 Definite Integrals

6. $\int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1^4}{4} - \frac{0^4}{4} = \frac{1}{4}$

7. $\int_0^\pi \cos(x)\,dx = [\sin(x)]_0^\pi = \sin(\pi) - \sin(0) = 0 - 0 = 0$

8. $\int_1^e \frac{1}{x}\,dx = [\ln|x|]_1^e = \ln(e) - \ln(1) = 1 - 0 = 1$

9. $\int_{-1}^1 x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^1 = \frac{1^3}{3} - \frac{(-1)^3}{3} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$

10. $\int_0^2 (x^2 + 1)\,dx = \left[\frac{x^3}{3} + x\right]_0^2 = \left(\frac{2^3}{3} + 2\right) - \left(\frac{0^3}{3} + 0\right) = \frac{8}{3} + 2 = \frac{14}{3}$

1.3 More Advanced Integrals

11. $\int xe^x\,dx = xe^x - e^x + C$ (using integration by parts)

12. $\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx = \lim_{b\to\infty} \left[-e^{-x}\right]_0^b = \lim_{b\to\infty} (-e^{-b} - (-e^0)) = 0 - (-1) = 1$

13. $\int_0^1 \sqrt{1 - x^2}\,dx = \frac{\pi}{4}$ (This represents the area of a quarter of the unit circle)

2 Improper Integrals

2.1 Evaluating Improper Integrals with Infinite Limits

Example 2.1. Evaluate $\int_1^\infty \frac{1}{x^2}\,dx$

Solution. Using the definition of an improper integral with an infinite upper limit:

$$\begin{aligned}
\int_1^\infty \frac{1}{x^2}\,dx &= \lim_{n\to\infty} \int_1^n \frac{1}{x^2}\,dx \\
&= \lim_{n\to\infty} \int_1^n x^{-2}\,dx \\
&= \lim_{n\to\infty} \left[-\frac{x^{-1}}{1}\right]_1^n \\
&= \lim_{n\to\infty} \left[-\frac{1}{x}\right]_1^n \\
&= \lim_{n\to\infty} \left(-\frac{1}{n} - \left(-\frac{1}{1}\right)\right) \\
&= \lim_{n\to\infty} \left(-\frac{1}{n} + 1\right) \\
&= 0 + 1 = 1
\end{aligned}$$

The integral converges to 1. □

2.2 Evaluating Integrals with Symmetric Limits

Example 2.2. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{250}{25+x^2}\,dx$

Solution. We can approach this by splitting the integral:

$$\int_{-\infty}^{\infty} \frac{250}{25+x^2}\,dx = \int_{-\infty}^{0} \frac{250}{25+x^2}\,dx + \int_{0}^{\infty} \frac{250}{25+x^2}\,dx$$

Since the function $f(x) = \frac{250}{25+x^2}$ is even (i.e., $f(-x) = f(x)$), the integral is symmetric:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{250}{25+x^2}\,dx &= 2\int_{0}^{\infty} \frac{250}{25+x^2}\,dx \\
&= 500\int_{0}^{\infty} \frac{1}{25+x^2}\,dx
\end{aligned}$$

This is related to the arctangent integral $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \arctan(\frac{x}{a})$. Here $a = 5$.

$$
\begin{aligned}
500 \int_0^\infty \frac{1}{5^2 + x^2}\, dx &= 500 \lim_{b \to \infty} \int_0^b \frac{1}{5^2 + x^2}\, dx \\
&= 500 \lim_{b \to \infty} \left[\frac{1}{5} \arctan\left(\frac{x}{5} \right) \right]_0^b \\
&= 500 \lim_{b \to \infty} \left(\frac{1}{5} \arctan\left(\frac{b}{5} \right) - \frac{1}{5} \arctan\left(\frac{0}{5} \right) \right) \\
&= 500 \left(\frac{1}{5} \cdot \frac{\pi}{2} - \frac{1}{5} \cdot 0 \right) \\
&= 500 \cdot \frac{\pi}{10} \\
&= 50\pi
\end{aligned}
$$

Alternatively, using the substitution $x = 5 \tan\theta$ as in the original document:

$$
\begin{aligned}
x = 5\tan\theta &\implies dx = 5\sec^2\theta\, d\theta \\
25 + x^2 &= 25 + 25\tan^2\theta = 25(1 + \tan^2\theta) = 25\sec^2\theta
\end{aligned}
$$

Limits: As $x \to 0$, $\tan\theta \to 0 \implies \theta \to 0$. As $x \to \infty$, $\tan\theta \to \infty \implies \theta \to \frac{\pi}{2}$.

$$
\begin{aligned}
2 \int_0^\infty \frac{250}{25 + x^2}\, dx &= 2 \int_0^{\pi/2} \frac{250}{25\sec^2\theta} \cdot (5\sec^2\theta\, d\theta) \\
&= 2 \int_0^{\pi/2} \frac{250 \cdot 5}{25}\, d\theta \\
&= 2 \int_0^{\pi/2} 50\, d\theta \\
&= 2 \cdot [50\theta]_0^{\pi/2} \\
&= 2 \cdot \left(50 \cdot \frac{\pi}{2} - 50 \cdot 0 \right) \\
&= 2 \cdot 25\pi = 50\pi
\end{aligned}
$$

□

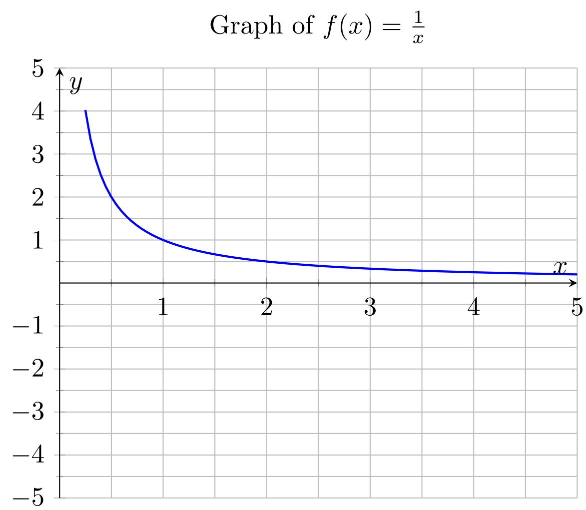

Figure 1: Graph of $f(x) = \frac{1}{x}$

3 Integration by Parts

3.1 The Product Rule and Integration by Parts Formula

The product rule for differentiation states:

$$\frac{d}{dx}\big[f(x)g(x)\big] = f'(x)g(x) + f(x)g'(x)$$

Taking the integral of both sides with respect to x:

$$\int \frac{d}{dx}\big[f(x)g(x)\big]\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

The left-hand side simplifies by the Fundamental Theorem of Calculus:

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

Rearranging to isolate $\int f(x)g'(x)\,dx$:

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx}$$

This is the integration by parts formula. Using the common substitutions $u = f(x)$ (so $du = f'(x)dx$) and $dv = g'(x)dx$ (so $v = g(x)$), it is written as:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

3.2 Examples of Integration by Parts

Example 3.1. Evaluate $\displaystyle\int x\cos x\,dx$

Solution. Let:

$$\begin{aligned} u &= x & \qquad dv &= \cos x\,dx \\ du &= 1\,dx & \qquad v &= \sin x \end{aligned}$$

Substituting into the formula $\int u\,dv = uv - \int v\,du$:

$$\begin{aligned} \int x\cos x\,dx &= (x)(\sin x) - \int (\sin x)(1\,dx) \\ &= x\sin x - \int \sin x\,dx \\ &= x\sin x - (-\cos x) + C \\ &= x\sin x + \cos x + C \end{aligned}$$

Result: $\displaystyle\int x\cos x\,dx = x\sin x + \cos x + C$ □

Example 3.2. Evaluate $\int \ln x \, dx$

Solution. Let:

$$u = \ln x \qquad dv = 1\,dx$$
$$du = \frac{1}{x}\,dx \qquad v = x$$

Substituting into the formula $\int u\,dv = uv - \int v\,du$:

$$\begin{aligned}\int \ln x \, dx &= (\ln x)(x) - \int (x)\left(\frac{1}{x}\,dx\right) \\ &= x \ln x - \int 1\,dx \\ &= x \ln x - x + C\end{aligned}$$

Result: $\int \ln x \, dx = x \ln x - x + C$ □

Example 3.3. Evaluate $\int x^2 e^x \, dx$

Solution. We need to apply integration by parts twice. First application: Let:

$$u_1 = x^2 \qquad dv_1 = e^x\,dx$$
$$du_1 = 2x\,dx \qquad v_1 = e^x$$

Substituting into the formula:

$$\begin{aligned}\int x^2 e^x \, dx &= (x^2)(e^x) - \int (e^x)(2x\,dx) \\ &= x^2 e^x - 2\int x e^x \, dx\end{aligned}$$

Now we need to evaluate $\int x e^x \, dx$. Second application: Let:

$$u_2 = x \qquad dv_2 = e^x\,dx$$
$$du_2 = 1\,dx \qquad v_2 = e^x$$

Substituting into the formula:

$$\begin{aligned}\int x e^x \, dx &= (x)(e^x) - \int (e^x)(1\,dx) \\ &= x e^x - \int e^x \, dx \\ &= x e^x - e^x\end{aligned}$$

Returning to the original integral:

$$\begin{aligned}\int x^2 e^x \, dx &= x^2 e^x - 2 \int x e^x \, dx \\ &= x^2 e^x - 2(x e^x - e^x) + C \\ &= x^2 e^x - 2x e^x + 2e^x + C\end{aligned}$$

Result: $\displaystyle\int x^2 e^x \, dx = x^2 e^x - 2x e^x + 2e^x + C$ □

Example 3.4. Evaluate $\displaystyle\int (3x^2 + 2x) e^{x^3 + x^2} \, dx$

Solution. This integral is best solved using u-substitution, not integration by parts. Let $u = x^3 + x^2$. Then the differential is $du = (3x^2 + 2x) \, dx$. Notice that the integrand is exactly $e^u \, du$.

$$\begin{aligned}\int (3x^2 + 2x) e^{x^3 + x^2} \, dx &= \int e^u \, du \\ &= e^u + C\end{aligned}$$

Substituting back $u = x^3 + x^2$:

$$\int (3x^2 + 2x) e^{x^3 + x^2} \, dx = e^{x^3 + x^2} + C$$

Result: $\displaystyle\int (3x^2 + 2x) e^{x^3 + x^2} \, dx = e^{x^3 + x^2} + C$ □

4 Substitution Methods

4.1 Basic Logarithmic Substitution

Example 4.1. Evaluate $\displaystyle\int \frac{1}{x \ln x} \, dx$

Solution. We use the substitution $u = \ln x$.

$$\begin{aligned}u &= \ln x \\ du &= \frac{1}{x} \, dx\end{aligned}$$

This transforms our integral:

$$\begin{aligned}\int \frac{1}{x \ln x} \, dx &= \int \frac{1}{\ln x} \cdot \frac{1}{x} \, dx \\ &= \int \frac{1}{u} \, du \\ &= \ln |u| + C\end{aligned}$$

Substituting back $u = \ln x$:

$$\ln|\ln x| + C$$

Result: $\displaystyle\int \frac{1}{x \ln x}\, dx = \ln|\ln x| + C$ $\square$

4.2 Exponential Substitution

Example 4.2. Evaluate $\displaystyle\int \frac{2^{\ln x}}{x}\, dx$

Solution. We use the substitution $u = \ln x$.

$$\begin{aligned} u &= \ln x \\ du &= \frac{1}{x}\, dx \end{aligned}$$

This transforms our integral:

$$\begin{aligned} \int \frac{2^{\ln x}}{x}\, dx &= \int 2^{\ln x} \cdot \frac{1}{x}\, dx \\ &= \int 2^u\, du \\ &= \frac{2^u}{\ln 2} + C \quad \left(\text{since } \int a^u du = \frac{a^u}{\ln a} + C\right) \end{aligned}$$

Substituting back $u = \ln x$:

$$\frac{2^{\ln x}}{\ln 2} + C$$

We can also use the property $a^{\ln b} = b^{\ln a}$, so $2^{\ln x} = x^{\ln 2}$.

$$\frac{x^{\ln 2}}{\ln 2} + C$$

Result: $\displaystyle\int \frac{2^{\ln x}}{x}\, dx = \frac{2^{\ln x}}{\ln 2} + C = \frac{x^{\ln 2}}{\ln 2} + C$ $\square$

4.3 Polynomial Substitution

Example 4.3. Evaluate $\displaystyle\int (x+3)(x-1)^5\, dx$

Solution. We use the substitution $u = x - 1$.

$$\begin{aligned} u &= x - 1 \implies x = u + 1 \\ du &= dx \end{aligned}$$

Also, $x+3=(u+1)+3=u+4$. This transforms our integral:

$$\begin{aligned}\int (x+3)(x-1)^5\,dx &= \int (u+4)u^5\,du \\ &= \int (u^6+4u^5)\,du \\ &= \frac{u^7}{7}+4\frac{u^6}{6}+C \\ &= \frac{u^7}{7}+\frac{2u^6}{3}+C\end{aligned}$$

Substituting back $u=x-1$:

$$\frac{(x-1)^7}{7}+\frac{2(x-1)^6}{3}+C$$

Result: $\int (x+3)(x-1)^5\,dx = \dfrac{(x-1)^7}{7}+\dfrac{2(x-1)^6}{3}+C$ □

4.4 Composite Function Substitution

Example 4.4. Evaluate $\int x^2 2^{x^3}\,dx$

Solution. We use the substitution $u=x^3$.

$$\begin{aligned}u &= x^3 \\ du &= 3x^2\,dx \implies x^2\,dx = \frac{du}{3}\end{aligned}$$

This transforms our integral:

$$\begin{aligned}\int x^2 2^{x^3}\,dx &= \int 2^{x^3}(x^2\,dx) \\ &= \int 2^u\cdot\frac{du}{3} \\ &= \frac{1}{3}\int 2^u\,du \\ &= \frac{1}{3}\cdot\frac{2^u}{\ln 2}+C\end{aligned}$$

Substituting back $u=x^3$:

$$\frac{2^{x^3}}{3\ln 2}+C$$

Result: $\int x^2 2^{x^3}\,dx = \dfrac{2^{x^3}}{3\ln 2}+C$ □

4.5 Constant Logarithm Integration

Example 4.5. Evaluate $\int x^2 \ln 2^3 \, dx$

Solution. First, simplify the logarithm. $\ln 2^3 = 3 \ln 2$. This is a constant.

$$\begin{aligned}
\int x^2 \ln 2^3 \, dx &= \int x^2 (3 \ln 2) \, dx \\
&= 3 \ln 2 \int x^2 \, dx \quad \text{(Constant factor pulled out)} \\
&= 3 \ln 2 \cdot \left(\frac{x^3}{3} \right) + C \\
&= (x^3)(\ln 2) + C
\end{aligned}$$

Result: $\int x^2 \ln 2^3 \, dx = x^3 \ln 2 + C$ □

5 Nested Substitutions

When dealing with nested substitutions, we substitute step-by-step and then reverse the substitutions carefully in the reverse order.

Example 5.1. Evaluate $\frac{1}{5} \int \frac{\cos(5x)}{e^{\sin(5x)}} \, dx$

Solution. We can rewrite the integral as $\frac{1}{5} \int e^{-\sin(5x)} \cos(5x) \, dx$. Let's use nested substitutions.

First substitution: Let $u = \sin(5x)$.

$$\begin{aligned}
u &= \sin(5x) \\
du &= \cos(5x) \cdot 5 \, dx \implies \cos(5x) \, dx = \frac{du}{5}
\end{aligned}$$

This transforms our integral:

$$\begin{aligned}
\frac{1}{5} \int e^{-\sin(5x)} \cos(5x) \, dx &= \frac{1}{5} \int e^{-u} \cdot \frac{du}{5} \\
&= \frac{1}{25} \int e^{-u} \, du
\end{aligned}$$

Second substitution (or direct integration): Let $w = -u$.

$$\begin{aligned}
w &= -u \\
dw &= -du \implies du = -dw
\end{aligned}$$

Continuing:

$$\begin{aligned}\frac{1}{25}\int e^{-u}\,du &= \frac{1}{25}\int e^{w}\cdot(-dw)\\ &= -\frac{1}{25}\int e^{w}\,dw\\ &= -\frac{1}{25}e^{w}+C\end{aligned}$$

Reversing the substitutions: First reverse $w = -u$:

$$-\frac{1}{25}e^{-u}+C$$

Then reverse $u = \sin(5x)$:

$$-\frac{1}{25}e^{-\sin(5x)}+C$$

Result: $\frac{1}{5}\int \frac{\cos(5x)}{e^{\sin(5x)}}\,dx = -\frac{1}{25}e^{-\sin(5x)}+C$ □

6 Reverse Chain Rule (Substitution)

The method often called "reverse chain rule" is essentially u-substitution, recognizing the pattern $\int g'(f(x))f'(x)\,dx = g(f(x)) + C$.

6.1 General Form

Example 6.1. Evaluate $\displaystyle\int (\sin x)^2 \cos x\,dx$

Solution. Let $u = \sin x$. Then $du = \cos x\,dx$. This fits the pattern $\int u^2 du$.

$$\begin{aligned}\int (\sin x)^2\cos x\,dx &= \int u^2\,du\\ &= \frac{u^3}{3}+C\end{aligned}$$

Substituting back $u = \sin x$:

$$\frac{(\sin x)^3}{3}+C$$

Result: $\displaystyle\int (\sin x)^2\cos x\,dx = \frac{(\sin x)^3}{3}+C$ □

Example 6.2. Evaluate $\displaystyle\int \frac{x}{2}\sin(2x^2+2)\,dx$

Solution. Let $u = 2x^2 + 2$. Then $du = 4x\,dx$. We have an $x\,dx$ term in the integral. We can write $x\,dx = \frac{du}{4}$.

$$\begin{aligned}
\int \frac{x}{2}\sin(2x^2+2)\,dx &= \frac{1}{2}\int \sin(2x^2+2)\cdot x\,dx \\
&= \frac{1}{2}\int \sin(u)\cdot\frac{du}{4} \\
&= \frac{1}{8}\int \sin(u)\,du \\
&= \frac{1}{8}(-\cos(u)) + C \\
&= -\frac{1}{8}\cos(u) + C
\end{aligned}$$

Substituting back $u = 2x^2 + 2$:

$$-\frac{1}{8}\cos(2x^2+2) + C$$

Result: $\displaystyle\int \frac{x}{2}\sin(2x^2+2)\,dx = -\frac{1}{8}\cos(2x^2+2) + C$ □

6.2 Logarithmic Form

A common pattern is $\int \frac{f'(x)}{f(x)}\,dx$, which results from substituting $u = f(x)$.

$$u = f(x) \implies du = f'(x)\,dx$$

$$\int \frac{f'(x)}{f(x)}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|f(x)| + C$$

Example 6.3. Evaluate $\displaystyle\int \tan x\,dx$

Solution. First, rewrite $\tan x = \frac{\sin x}{\cos x}$.

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

Let $u = \cos x$. Then $f'(x) = -\sin x$. So $du = -\sin x\,dx$. We have $\sin x\,dx = -du$.

$$\begin{aligned}
\int \frac{\sin x}{\cos x}\,dx &= \int \frac{1}{\cos x}(\sin x\,dx) \\
&= \int \frac{1}{u}(-du) \\
&= -\int \frac{1}{u}\,du \\
&= -\ln|u| + C
\end{aligned}$$

Substituting back $u = \cos x$:

$$-\ln|\cos x| + C$$

This can also be written using logarithm properties:

$$\ln|(\cos x)^{-1}| + C = \ln|\sec x| + C$$

Result: $\int \tan x \, dx = -\ln|\cos x| + C = \ln|\sec x| + C$ $\square$

7 Partial Fractions

Example 7.1. Evaluate $\int \frac{x^2 + x - 5}{x^2 - 1} \, dx$

Solution. The degree of the numerator is equal to the degree of the denominator, so we perform polynomial long division first. $\begin{array}{r} 1 \\ x^2 - 1\overline{)x^2 + x - 5} \\ -(x^2 \quad - 1) \\ \hline x - 4 \end{array}$ So,

$\dfrac{x^2 + x - 5}{x^2 - 1} = 1 + \dfrac{x - 4}{x^2 - 1}$.

Now we apply partial fractions to the remainder term $\frac{x-4}{x^2-1}$. Factor the denominator: $x^2 - 1 = (x + 1)(x - 1)$.

$$\frac{x - 4}{(x + 1)(x - 1)} = \frac{A}{x + 1} + \frac{B}{x - 1}$$

Multiply by the common denominator $(x + 1)(x - 1)$:

$$x - 4 = A(x - 1) + B(x + 1)$$

To find A and B, we can use the cover-up method or compare coefficients. Method 1: Cover-up Set $x = 1$: $1 - 4 = A(0) + B(1 + 1) \implies -3 = 2B \implies B = -\frac{3}{2}$. Set $x = -1$: $-1 - 4 = A(-1 - 1) + B(0) \implies -5 = -2A \implies A = \frac{5}{2}$.

Method 2: Compare coefficients $x - 4 = Ax - A + Bx + B = (A + B)x + (-A + B)$ Comparing coefficients of x: $A + B = 1$ Comparing constant terms: $-A + B = -4$ Adding the two equations: $2B = -3 \implies B = -\frac{3}{2}$. Substituting B into the first equation: $A - \frac{3}{2} = 1 \implies A = 1 + \frac{3}{2} = \frac{5}{2}$.

So the decomposition is $\frac{5/2}{x+1} - \frac{3/2}{x-1}$. Now integrate the full expression:

$$\begin{aligned} \int \frac{x^2 + x - 5}{x^2 - 1} \, dx &= \int \left(1 + \frac{5/2}{x + 1} - \frac{3/2}{x - 1}\right) dx \\ &= \int 1 \, dx + \frac{5}{2} \int \frac{1}{x + 1} \, dx - \frac{3}{2} \int \frac{1}{x - 1} \, dx \\ &= x + \frac{5}{2} \ln|x + 1| - \frac{3}{2} \ln|x - 1| + C \end{aligned}$$

Result: $\int \frac{x^2 + x - 5}{x^2 - 1} \, dx = x + \frac{5}{2} \ln|x + 1| - \frac{3}{2} \ln|x - 1| + C$ $\square$

8 Trigonometric Substitution

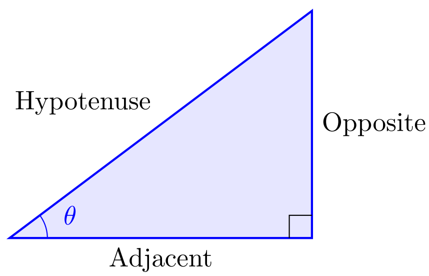

Figure 2: Right triangle illustrating sides relative to θ for trigonometric substitutions.

8.1 Form $\sqrt{a^2 - x^2}$: Inverse Sine Substitution

Example 8.1. Evaluate $\displaystyle\int \frac{1}{\sqrt{4 - x^2}}\, dx$

Solution. This integral has the form $\sqrt{a^2 - x^2}$ with $a = 2$. We use the substitution $x = a \sin\theta = 2\sin\theta$.

$$\begin{aligned} x &= 2\sin\theta \implies \sin\theta = \frac{x}{2} \\ dx &= 2\cos\theta\, d\theta \\ \sqrt{4 - x^2} &= \sqrt{4 - (2\sin\theta)^2} = \sqrt{4 - 4\sin^2\theta} \\ &= \sqrt{4(1 - \sin^2\theta)} = \sqrt{4\cos^2\theta} \\ &= 2|\cos\theta| \end{aligned}$$

Assuming θ is chosen such that $\cos\theta \geq 0$ (e.g., $-\pi/2 \leq \theta \leq \pi/2$, which corresponds to the range of arcsin), we have $\sqrt{4 - x^2} = 2\cos\theta$. This transforms our integral:

$$\begin{aligned} \int \frac{1}{\sqrt{4 - x^2}}\, dx &= \int \frac{1}{2\cos\theta} \cdot (2\cos\theta\, d\theta) \\ &= \int 1\, d\theta \\ &= \theta + C \end{aligned}$$

Substituting back using $\theta = \arcsin\left(\frac{x}{2}\right)$:

$$\arcsin\left(\frac{x}{2}\right) + C$$

Result: $\displaystyle\int \frac{1}{\sqrt{4 - x^2}}\, dx = \arcsin\left(\frac{x}{2}\right) + C$ $\square$

Example 8.2. Evaluate $\int \frac{1}{\sqrt{8-2x^2}}\,dx$

Solution. First, factor out the constant from the square root:

$$\sqrt{8-2x^2} = \sqrt{2(4-x^2)} = \sqrt{2}\sqrt{4-x^2}$$

The integral becomes:

$$\begin{aligned}\int \frac{1}{\sqrt{8-2x^2}}\,dx &= \int \frac{1}{\sqrt{2}\sqrt{4-x^2}}\,dx \\ &= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{4-x^2}}\,dx\end{aligned}$$

Using the result from the previous example:

$$\frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{4-x^2}}\,dx = \frac{1}{\sqrt{2}} \arcsin\left(\frac{x}{2}\right) + C$$

Result: $\int \frac{1}{\sqrt{8-2x^2}}\,dx = \frac{1}{\sqrt{2}} \arcsin\left(\frac{x}{2}\right) + C$ □

8.2 Form $\sqrt{x^2-a^2}$: Secant Substitution

Example 8.3. Evaluate $\int \frac{1}{\sqrt{3x^2-1}}\,dx$

Solution. Rewrite the term inside the square root to match the pattern $\sqrt{(kx)^2-a^2}$: $\sqrt{(\sqrt{3}x)^2-1^2}$. This suggests the substitution involving secant. Let $\sqrt{3}x = 1\sec\theta$.

$$\begin{aligned}\sqrt{3}x = \sec\theta &\implies x = \frac{1}{\sqrt{3}}\sec\theta \\ dx &= \frac{1}{\sqrt{3}}\sec\theta\tan\theta\,d\theta \\ \sqrt{3x^2-1} &= \sqrt{(\sqrt{3}x)^2-1} = \sqrt{\sec^2\theta-1} \\ &= \sqrt{\tan^2\theta} = |\tan\theta|\end{aligned}$$

Assuming θ is chosen such that $\tan\theta \geq 0$ (e.g., $0 \leq \theta < \pi/2$ or $\pi \leq \theta < 3\pi/2$), we have $\sqrt{3x^2-1} = \tan\theta$. This transforms our integral:

$$\begin{aligned}\int \frac{1}{\sqrt{3x^2-1}}\,dx &= \int \frac{1}{\tan\theta} \cdot \left(\frac{1}{\sqrt{3}}\sec\theta\tan\theta\,d\theta\right) \\ &= \frac{1}{\sqrt{3}} \int \sec\theta\,d\theta \\ &= \frac{1}{\sqrt{3}} \ln|\sec\theta + \tan\theta| + C\end{aligned}$$

Now, substitute back to x. We know $\sec\theta = \sqrt{3}x$. From $\sec^2\theta = \tan^2\theta + 1$, we have $\tan^2\theta = \sec^2\theta - 1 = (\sqrt{3}x)^2 - 1 = 3x^2 - 1$. So, $\tan\theta = \sqrt{3x^2 - 1}$ (consistent with our assumption $\tan\theta \geq 0$). Substituting these into the result:

$$\frac{1}{\sqrt{3}} \ln|\sqrt{3}x + \sqrt{3x^2 - 1}| + C$$

Result: $\int \frac{1}{\sqrt{3x^2 - 1}}\, dx = \frac{1}{\sqrt{3}} \ln|\sqrt{3}x + \sqrt{3x^2 - 1}| + C$ □