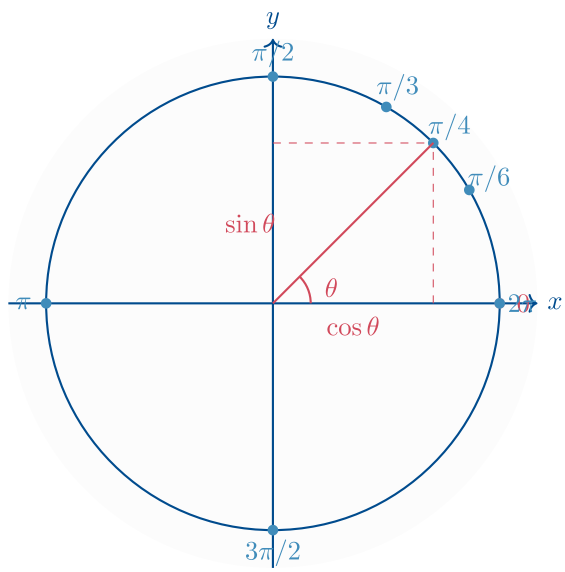

Trigonometry

A Mathematical Foundation

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1 Introduction to Trigonometry

Historical Context

Trigonometry originated in the 3rd century BC with early astronomical studies. The word "trigonometry" comes from the Greek words "trigonon" (triangle) and "metron" (measure).

Trigonometry is a branch of mathematics that studies relationships between the angles and sides of triangles. It is crucial for solving problems in fields like engineering, physics, astronomy, and architecture. The fundamental concepts of trigonometry serve as the foundation for many advanced mathematical and scientific theories.

This document explores the core principles of trigonometry, beginning with the unit circle and extending to applications in various geometric contexts.

1.1 The Unit Circle: A Fundamental Framework

The unit circle is a circle with radius 1 centered at the origin of a coordinate plane. It provides a geometric interpretation of trigonometric functions and establishes the relationship between angles and coordinates.

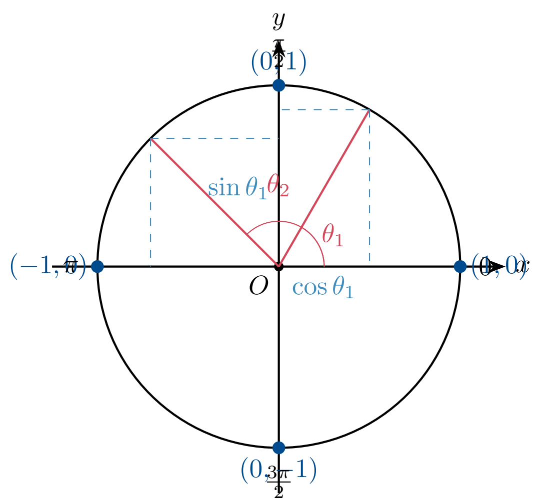

Figure 1: The Unit Circle with Key Angles and Coordinates

On the unit circle, angles are measured in radians, where one complete revolution equals 2π radians. The coordinates of any point on the unit circle correspond directly to the cosine and sine values of the angle formed with the positive x-axis:

Definition 1.1 (Sine and Cosine on the Unit Circle)**.** For any angle θ in standard position on the unit circle, the coordinates of the point where the terminal side of the angle intersects the unit circle are:

$$x = \cos\theta \tag{1}$$
$$y = \sin\theta \tag{2}$$

2 Trigonometric Functions and Their Properties

2.1 Primary Trigonometric Functions

The six primary trigonometric functions are defined as follows:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \tag{3}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} \tag{4}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x} \tag{5}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{r}{y} \tag{6}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x} \tag{7}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{x}{y} \tag{8}$$

Fundamental Relationships

Key trigonometric identities:

$$\sin^2\theta + \cos^2\theta = 1 \tag{9}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{10}$$

$$1 + \tan^2\theta = \sec^2\theta \tag{11}$$

$$1 + \cot^2\theta = \csc^2\theta \tag{12}$$

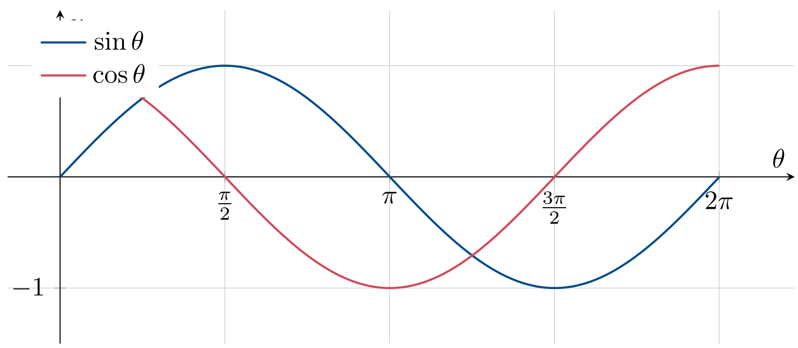

Figure 2: Sine and Cosine Functions Over One Period

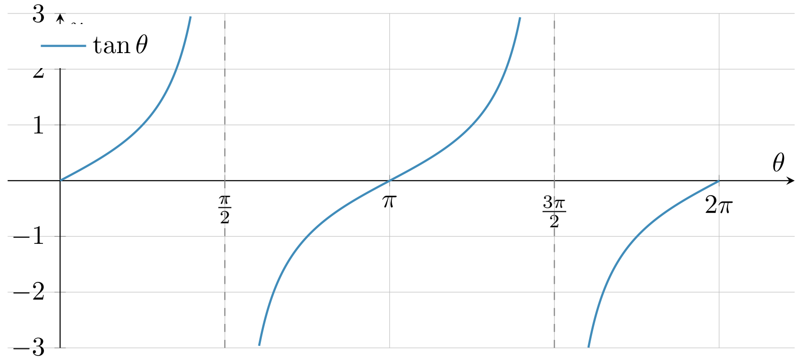

Figure 3: Tangent Function with Asymptotes at $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$

3 Radians and Degree Conversions

Angles in trigonometry can be measured in both degrees and radians. Radians are the standard unit in advanced mathematics due to their natural relationship with the unit circle.

Definition 3.1 (Radian)**.** One radian is the angle subtended at the center of a circle by an arc whose length equals the radius of the circle.

3.1 Conversion Between Degrees and Radians

To convert between degrees and radians, use these formulas:

$$\text{Radians} = \text{Degrees} \times \frac{\pi}{180} \tag{13}$$

$$\text{Degrees} = \text{Radians} \times \frac{180}{\pi} \tag{14}$$

Table 1: Common Angle Conversions

Degrees	**Radians (Exact)**	**Radians (Decimal)**	**Position on Unit Circle**
$0°$	0	0	$(1, 0)$
$30°$	$\frac{\pi}{6}$	≈ 0.524	$(\frac{\sqrt{3}}{2}, \frac{1}{2})$
$45°$	$\frac{\pi}{4}$	≈ 0.785	$(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$
$60°$	$\frac{\pi}{3}$	≈ 1.047	$(\frac{1}{2}, \frac{\sqrt{3}}{2})$
$90°$	$\frac{\pi}{2}$	≈ 1.571	$(0, 1)$
$180°$	π	≈ 3.142	$(-1, 0)$
$270°$	$\frac{3\pi}{2}$	≈ 4.712	$(0, -1)$
$360°$	2π	≈ 6.283	$(1, 0)$

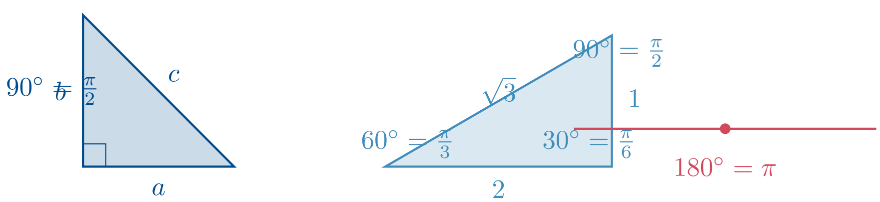

Figure 4: Important Angles in both Degrees and Radians

4 Visualizing the Unit Circle

The relationship between trigonometric functions and the unit circle provides powerful geometric insights.

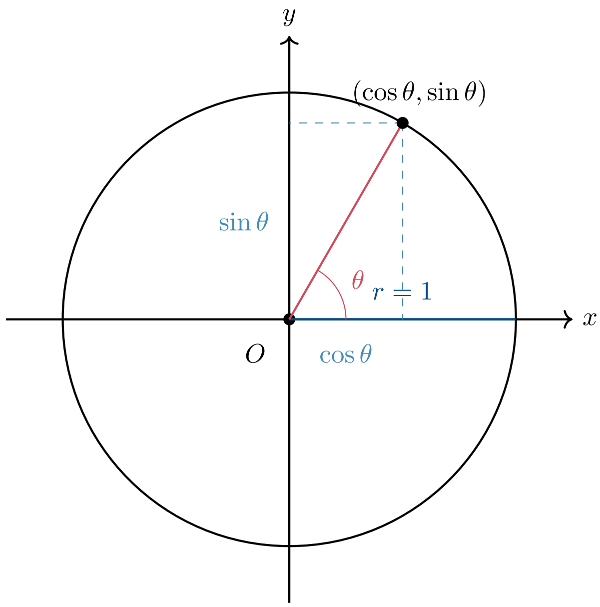

Figure 5: Unit Circle with Sine and Cosine Projections

Example 4.1 (Finding Trigonometric Values)**.** Consider an angle $\theta = \frac{\pi}{4} = 45°$. The coordinates of the point on the unit circle are:

$$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \approx 0.7071 \tag{15}$$

$$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \approx 0.7071 \tag{16}$$

Therefore, $\tan\frac{\pi}{4} = \frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$

5 Trigonometric Identities

5.1 Pythagorean Identities

The fundamental Pythagorean identity, derived from the unit circle, states:

$$\sin^2\theta + \cos^2\theta = 1 \tag{17}$$

From this, we can derive two additional identities:

$$1 + \tan^2\theta = \sec^2\theta \tag{18}$$

$$1 + \cot^2\theta = \csc^2\theta \tag{19}$$

5.2 Angle Addition Formulas

For any angles α and β:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{20}$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{21}$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \tag{22}$$

Theorem 5.1 (Double Angle Formulas)**.** For any angle θ:

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{23}$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta \tag{24}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} \tag{25}$$

6 Applications of Trigonometry

Trigonometry has numerous applications across various fields:

- **Physics**: Describing wave motion, oscillations, and rotational dynamics
- **Engineering**: Structural analysis, electrical circuit analysis, and signal processing
- **Astronomy**: Calculating planetary positions and navigational coordinates
- **Architecture**: Designing stable structures and calculating load distributions
- **Computer Graphics**: Rendering 3D objects and animations

Example 6.1 (Wave Motion)**.** The displacement y of a point in a simple harmonic motion can be described by:

$$y = A\sin(2\pi ft + \phi) \tag{26}$$

where A is the amplitude, f is the frequency, t is time, and ϕ is the phase shift.

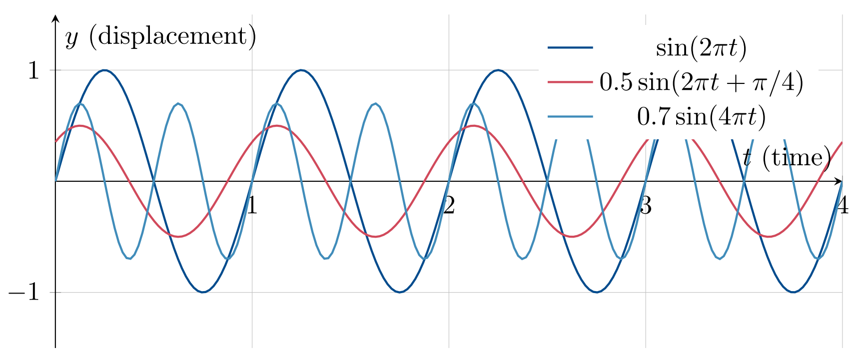

Figure 6: Wave Functions with Different Parameters