Integrali

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Sommario

This document introduces the concept of integrals by utilizing various online resources and additional references. It includes visual representations and examples to aid comprehension.

Indice

Elenco delle figure

1 Definizione Integrali

Evaluation of the integral of a function

Antiderivative, or indefinite integral, is the inverse operation of differentiation.

$$\int 2x\,dx = x^2 + c$$

$$\frac{d}{dx}\left[x^{n+1} + c\right] = x^n, \quad \int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$$

$$\int x^5\,dx = \frac{x^{5+1}}{5+1} + c = \frac{x^6}{6} + c$$

$$\int 5x^{-2}\,dx = 5\int x^{-2}\,dx = 5\left(-\frac{1}{x}\right) + c = -\frac{5}{x} + c$$

$$\int x^{-7}\,dx = -\frac{1}{6}x^{-6} + c$$

When encountering a more complex integral, break it into simpler terms:

$$\int \left(7x^3 - 5\sqrt{x} + \frac{18\sqrt{x}}{x^3} + x^{-40}\right) dx$$

Breakdown:

$$\int 7x^3\,dx - \int 5\sqrt{x}\,dx + \int \frac{18\sqrt{x}}{x^3}\,dx + \int x^{-40}\,dx$$

Each term is integrated separately:

$$\int 7x^3\,dx = \frac{7x^4}{4}, \quad \int 5\sqrt{x}\,dx = \int 5x^{1/2}\,dx = \frac{5x^{3/2}}{3/2} = \frac{10x^{3/2}}{3}$$

$$\int \frac{18\sqrt{x}}{x^3}\,dx = \int 18x^{-5/2}\,dx = -\frac{36}{3}x^{-3/2} = -12x^{-3/2}, \quad \int x^{-40}\,dx = \frac{x^{-39}}{-39} = -\frac{x^{-39}}{39}$$

Combine the results:

$$\int \left(7x^3 - 5\sqrt{x} + \frac{18\sqrt{x}}{x^3} + x^{-40}\right) dx = \frac{7x^4}{4} - \frac{10x^{3/2}}{3} - 12x^{-3/2} - \frac{x^{-39}}{39} + c$$

Note: c is a constant of integration.
[1]

2 Visually Determining Antiderivative

The relationship between a function $f(x)$ and its antiderivative $F(x)$ is defined as:

$$F'(x) = f(x)$$

$$xyf(x) = 2x^2;^2 + C$$

$$\int 2x\,dx = \frac{2x^2}{2} = x^2 + C$$

Example: Antiderivative of $f(x) = x^2$

$$\int x^2\,dx$$

$$xy^2;$$

$$\int x^2\,dx = \frac{x^3}{3} + C$$

Example: Antiderivative of $\frac{1}{x}$

$$\int \frac{1}{x}\,dx = \log|x| + C, \quad x \neq 0$$

[1]The general rule for integration: $\int f'(x)\,dx = f(x) + c$, where c is a constant.

$$\frac{d}{dx}\log|x| = \frac{1}{x}$$

Solving for $f(x)$ Given Specific Conditions Given: $f'(x) = \frac{24}{x^3}$

$$f(x) = \int 24x^{-3}\,dx = 24 \cdot \frac{x^{-2}}{-2} + C = -\frac{12}{x^2} + C$$

If $f'(-1) = 3$:

$$f'(-1) = -\frac{12}{(-1)^2} + C = 3 \quad \Rightarrow \quad C = 15$$

Thus:

$$f(x) = -\frac{12}{x^2} + 15$$

Given: $f'(x) = 5e^x$ and $f(7) = 40 + 5e^7$

$$f(x) = \int 5e^x\,dx = 5e^x + C$$

Substitute $x = 7$:

$$f(7) = 5e^7 + C = 40 + 5e^7 \quad \Rightarrow \quad C = 40$$

Final function:

$$f(x) = 5e^x + 40$$

At $x = 0$:

$$f(0) = 5e^0 + 40 = 5 + 40 = 45$$

2.1 Definite Integrals (Limits): $\int_a^b$

Examples of Definite Integrals:
1.
$$\int_5^2 h(x)\,dx$$
2.
$$\int_4^0 g(x)\,dx$$
3. Special Case:
$$\int_3^3 \sqrt{9-x^2}\,dx = 0$$
4. Area of a Semicircle:
$$\int_{-3}^3 \sqrt{9-x^2}\,dx = \frac{\pi 3^2}{2} = \frac{9\pi}{2}$$
Key Properties of Definite Integrals: 1. **Scaling by a Constant:**
$$\int_a^b c \cdot f(x)\,dx = c\int_a^b f(x)\,dx$$
2. **Reversing the Limits:**
$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$
3. **Zero-Length Interval:**
$$\int_a^a f(x)\,dx = 0$$
4. **Additivity Over Intervals:**
$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

Visualizing Definite Integrals: The definite integral $\int_a^b f(x)\,dx$ represents the net area between the curve $f(x)$ and the x-axis, computed from $x = a$ to $x = b$.

$$xy^2;^2$$

This graph shows the integral of $f(x) = x^2$ from $x = 0$ to $x = 4$, with the shaded region representing the computed area.

2.2

Let's add a point

$\int_{c_{2\;3}}^b f(x)\,dx \qquad (a \leq b)(\leq c)$

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[2] Absolute value
[3] remember $n^0 = n^1$

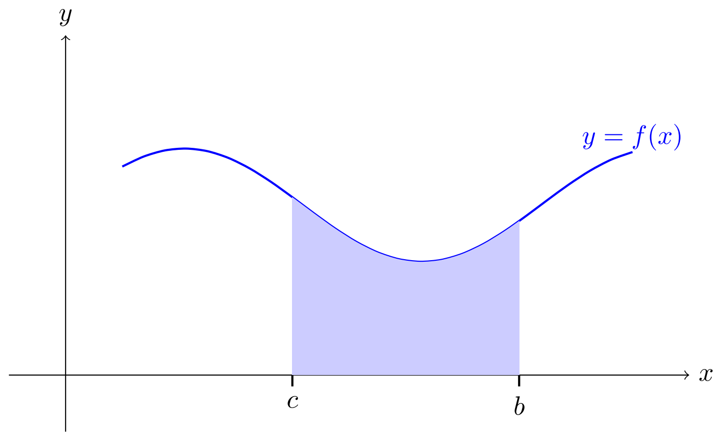


Figura 1: $\int_c^b f(x)\,dx$

3 Definite integral

In this section, we will explore the concept of the definite integral of a shifted function. Specifically, we will investigate the impact of shifting a function on its integral.

Example

$$\int_a^b f(x)\,dx = 5$$

Property: Translation of Function Argument

$$\int_a^b f(x-c)\,dx = \int_a^b f(x)\,dx$$

Let us consider the implications of changing the bounds of an integral. If the standard delta is b-a, then it becomes a-b.

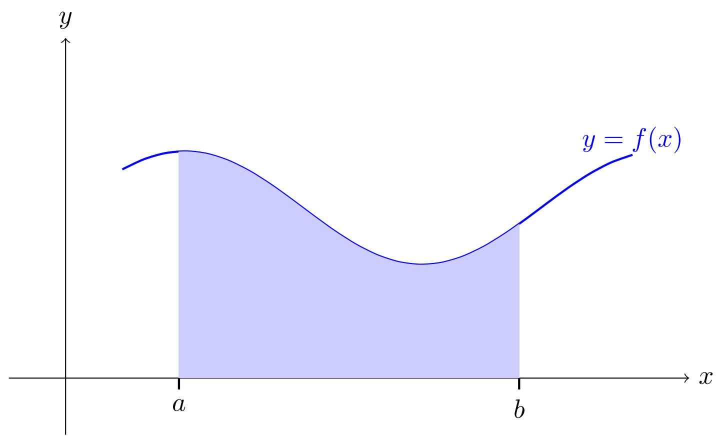


Figura 2: $\int_a^b f(x)\,dx$

Formula explanation

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x, \quad \text{where} \quad \Delta x = \frac{b-a}{n}$$

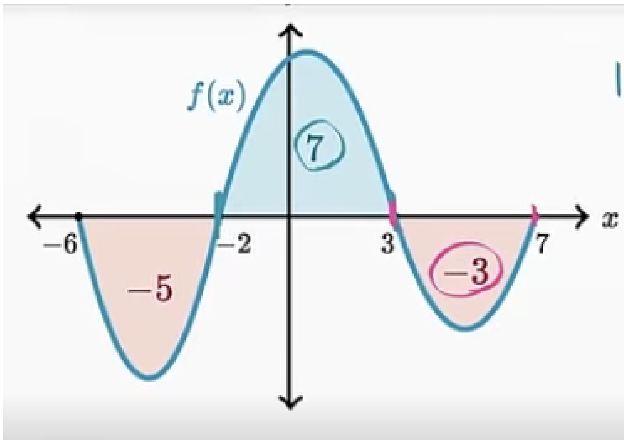
h 

Figura 3: $\int_{-2}^{3} 2f(x)\,dx + \int_{3}^{7} 3f(x)\,dx$=

then simply become the opposite so the result is -$\int_a^b f(x)\,dx$

Another example

let evaluate this

$$\int_{-2}^{3} 2f(x)\,dx + \int_{3}^{7} 3f(x)\,dx$$

Step 1: Factor Out the Constants Using the property of definite integrals, where $\int_a^b c{\cdot}f(x)\,dx = c{\cdot}\int_a^b f(x)\,dx$, we can rewrite the expression as:

$$2\int_{-2}^{3} f(x)\,dx + 3\int_{3}^{7} f(x)\,dx$$

Final Expression

This simplifies the integrals while keeping the constants factored out.

the result is $14 - 9 = 5$
[4] [5]

Integration Properties

$$\int kf(x)\,dx = k\int f(x)\,dx$$

Example: Given:

$$\int_{-1}^{3} 3f(x)\,dx - 2g(x)\,dx$$

Step 1: Factor Constants: Using the property $\int kf(x)\,dx = k\int f(x)\,dx$, rewrite as:

$$\int_{-1}^{3} 3f(x)\,dx - \int_{-1}^{3} 2g(x)\,dx$$

[4] when we have a defined integral from a to a the result will always be 0 $\int_a^a f(x)\,dx = 0$

[5] $\int_7^4 f(x)\,dx = -\int_4^7 f(x)\,dx$

Step 2: Combine: Combine into a single expression:

$$3\int_{-1}^{3} f(x)\,dx - 2\int_{-1}^{3} g(x)\,dx$$

Result: The given result is:

$$-16$$

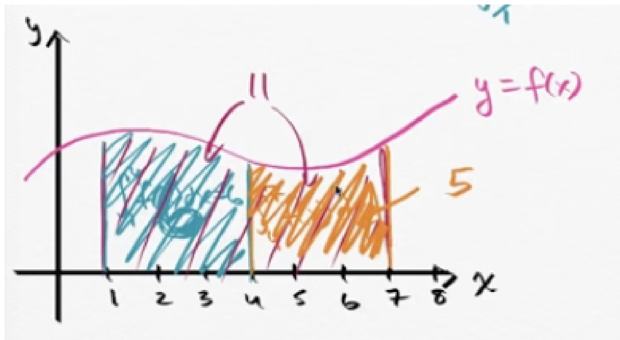


Figura 4: break integrals

[6]

Sum/Difference:

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

Constant multiple:

$$\int_a^b k \cdot f(x)dx = k\int_a^b f(x)dx$$

Reverse interval:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Zero-length interval:

$$\int_a^a f(x)dx = 0$$

Adding intervals:

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

[6] if $\int_{-5}^{-5} f(t)\,dt$ the result will be always 0

4 Riemann approximation Δx

Approximate the area under a curve using rectangles, we employ the method of Riemann sums. This involves dividing the interval into smaller subintervals and forming rectangles to estimate the area. The general formula for the Riemann sum is:

$$R = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

So now you can easily understand that dividing in rectangles evaluate the area one by one.

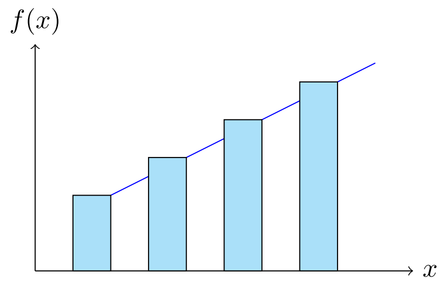


Figura 5: Approx with Rectangles.

Using this as an example you can see that the curve is made by rectangles and other shapes.

4.1 Trapezoidal sums

calculate the area of all and sum, at the end you will have the area under the curve.
Δx

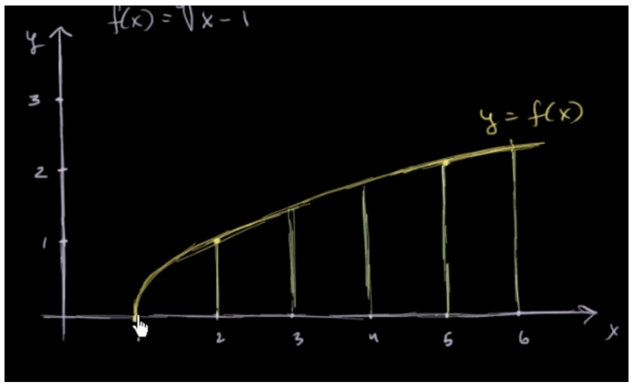


Figura 6: $\int_a^b f(x)\,dx$

The Riemann sum is a rigorous definition of an integral, approximating the area under a curve by partitioning the domain into subintervals, summing the areas of rectangles, and taking the limit as the partitions increase indefinitely. This concept is fundamental in calculus and integration.

Definite Integral: $\int_a^b f(x)\,dx$

Example:

$$\lim_{n\to\infty}\sum_{i=1}^{n} e^{\frac{3i}{n}}\frac{3}{n} = \int_0^3 e^x\,dx$$

—

What If the Limits Are Reversed?
If the limits of the integral are reversed, we can use the property:

$$\int_0^{-2} f(t)\,dt = -\int_{-2}^{0} f(t)\,dt$$

This property allows us to compute integrals even when the lower limit is larger than the upper limit by switching the limits and introducing a negative sign.

—

The Fundamental Theorem of Calculus and Accumulation Functions

Fundamental Theorem of Calculus (Part 1): If $F(x)$ is the antiderivative of $f(x)$, then:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

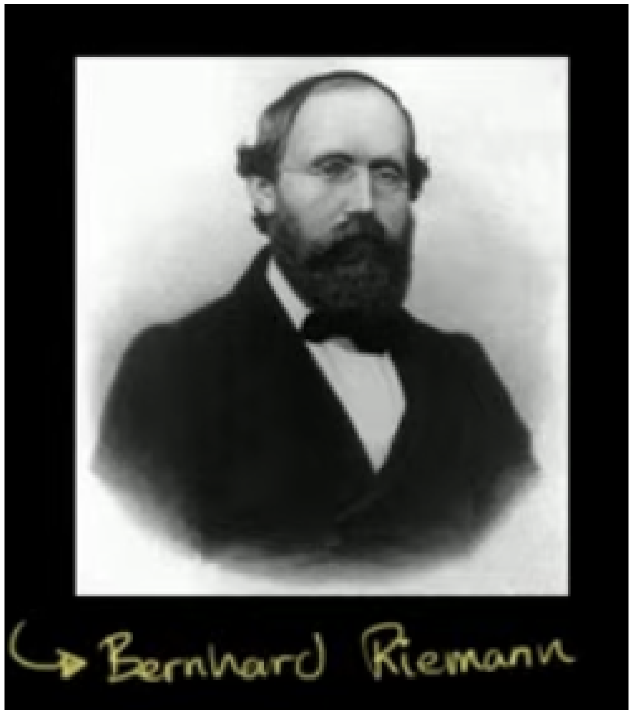


Figura 7: Riemann

This shows that the derivative of an accumulation function $F(x) = \int_a^x f(t)\,dt$ is simply the original function $f(x)$.

—

Notes: - The accumulation function $\int_a^x f(t)\,dt$ gives the "area under the curve" of $f(t)$ from $t = a$ to $t = x$. - Differentiating this function with respect to x recovers the original function $f(x)$, as shown by the fundamental theorem. So every f has an antiderivative F(x) that a connect between derivative and integrals.

The Second Fundamental Theorem of Calculus

If $f(x)$ is continuous on $[a, x]$, the definite integral can be expressed as:

$$F(x) = \int_a^x f(t)\,dt$$

The second fundamental theorem states:

$$F'(x) = f(x)$$

—

Example:

Let:

$$g(x) = \frac{d}{dx}\int_{19}^{x} \sqrt[3]{t}\,dt$$

By the second fundamental theorem:

$$g'(x) = \sqrt[3]{x}$$

To evaluate $g'(27)$:

$$g'(27) = \sqrt[3]{27} = 3$$

—

Finding Derivatives with the Fundamental Theorem (Lower Bound Involves x):

If x is the lower bound, we adjust using the property of reversing integral limits:

$$\frac{d}{dx}\int_x^3 \sqrt{|\cos t|}\,dt = \frac{d}{dx}\left(-\int_3^x \sqrt{|\cos t|}\,dt\right)$$

Differentiating:

$$\frac{d}{dx}\int_x^3 \sqrt{|\cos t|}\,dt = -\sqrt{|\cos x|}$$

—

Properties of Definite Integrals:

1. **Zero Width:**

$$\int_a^a f(x)\,dx = F(a) - F(a) = 0$$

2. **Reversing Limits:**

$$-\int_a^b f(x)\,dt = -(F(b) - F(a)) = F(a) - F(b) = \int_b^a f(x)\,dt$$

—

Riemann Sum Connection:

The definite integral can also be defined using a Riemann sum:

$$\int_a^b f(t)\,dt = \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)\Delta t, \quad \text{where} \quad \Delta t = \frac{b-a}{n}$$

—

Net Change Theorem:

For $S(x)$, the antiderivative of $f(x)$, the net change over an interval is given by:

$$S(b) - S(a) = \int_a^b f(x)\,dx$$