Vectors: Fundamental Mathematical Tools

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Sommario

Vectors are fundamental mathematical tools, indispensable in a wide range of applications that extend into computer science and beyond. In mathematics, vectors are used to represent quantities that have both magnitude and direction, such as force and velocity, making them essential for modeling and analyzing real-world phenomena. They are a cornerstone of linear algebra, where they facilitate operations like addition, scaling, and dot products, which are crucial for solving systems of linear equations. Vectors provide a framework for understanding geometric transformations, such as rotations and translations, which have direct applications in fields like computer graphics and machine learning. Additionally, in physics, vectors are used to model motion and forces, allowing for realistic simulations and analysis. Their mathematical rigor and versatility make vectors a vital component in both theoretical and applied mathematics, driving innovation and understanding across various scientific disciplines.

Indice

1 Introduction to Vectors

1.1 Vectors and Scalars

In mathematics and physics, quantities are classified as either scalars or vectors:

Definition 1.1 (Scalar) *A scalar is a quantity that has only magnitude (size). Examples include temperature, mass, and time.*

Definition 1.2 (Vector) *A vector is a quantity that has both magnitude (size) and direction. Examples include velocity, force, and displacement.*

The fundamental difference between scalars and vectors is that vectors contain directional information, which is crucial for accurately representing many physical phenomena.

1.2 Vector Representation

Vectors can be represented in various ways:

- Geometrically, as arrows in space
- Algebraically, as ordered pairs or tuples of numbers
- Using unit vectors in coordinate systems

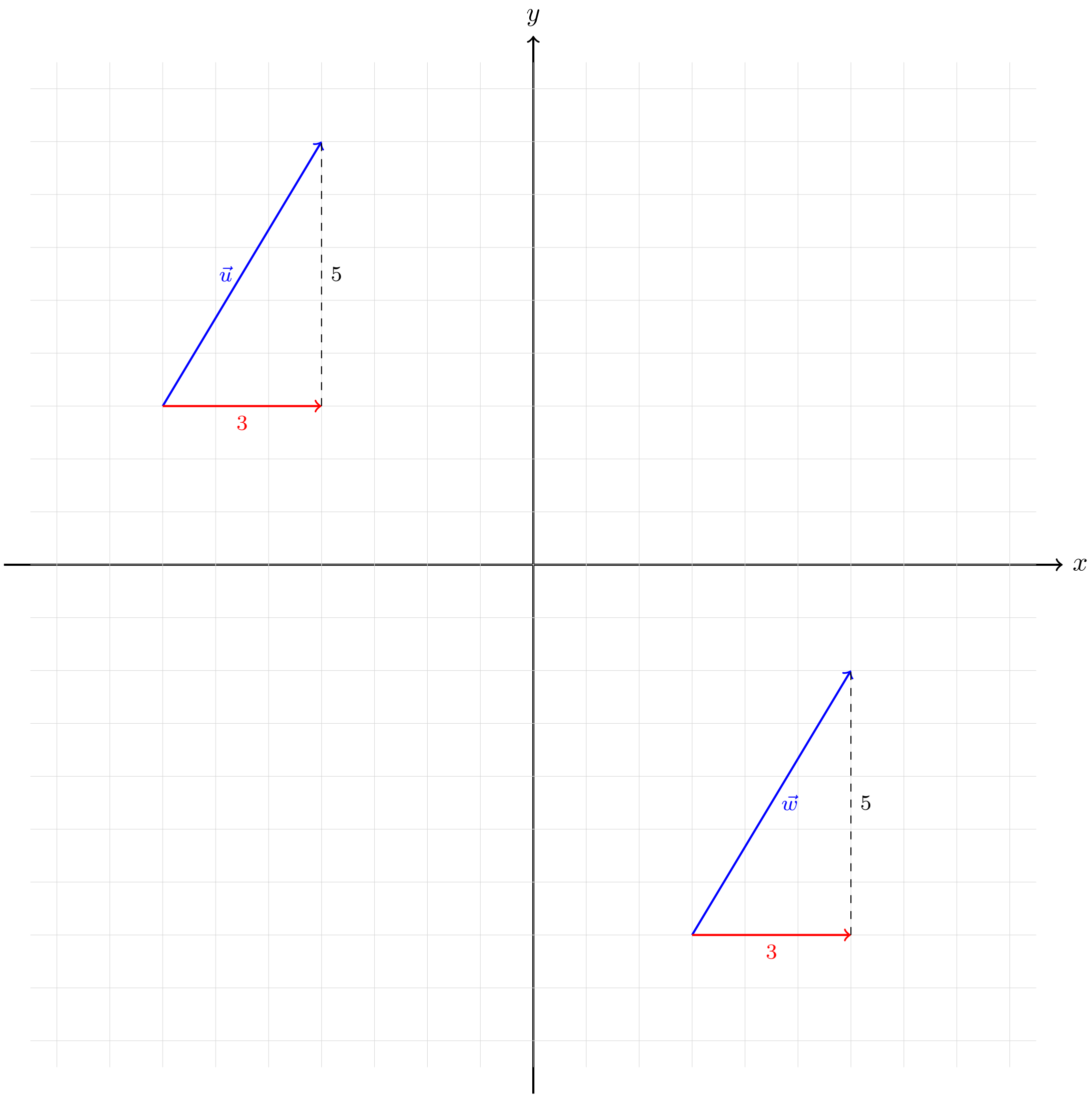

Figura 1: Two vectors $\vec{u}$ and $\vec{w}$ shown with their component breakdown. Both vectors have horizontal components of 3 units and vertical components of 5 units.

The graph depicts a two-dimensional Cartesian coordinate system with both horizontal and vertical axes ranging from -10 to 10. Light gray grid lines fill the background, providing a reference for plotting. Two vectors, $\vec{u}$ and $\vec{w}$, are prominently displayed in blue. Vector $\vec{u}$ extends from point (-7, 3) to (-4, 8) and is accompanied by two supporting components: a red horizontal vector of length 3 and a dashed vertical line of length 5. Similarly, vector $\vec{w}$ spans from (3, -7) to (6, -2), also paired with a red horizontal vector of length 3 and a dashed vertical line of length 5, illustrating the breakdown of each vector into its horizontal and vertical components.

1.3 Vector Magnitude

The magnitude of a vector $\vec{v}$ is denoted by $\|\vec{v}\|$ and represents the length of the vector. For a two-dimensional vector $\vec{v} = (v_x, v_y)$, the magnitude is calculated using the Pythagorean theorem:

$$\|\vec{v}\| = \sqrt{v_x^2 + v_y^2} = \sqrt{(\Delta x)^2 + (\Delta y)^2} \tag{1}$$

For the vectors shown in Figure 1, the magnitude would be:

$$\|\vec{u}\| = \|\vec{w}\| = \sqrt{3^2 + 5^2} = \sqrt{9 + 25} = \sqrt{34} \approx 5.83 \tag{2}$$

2 Vector Operations

2.1 Vector Components

When working with vectors in a Cartesian coordinate system, we often need to determine the components along each axis.

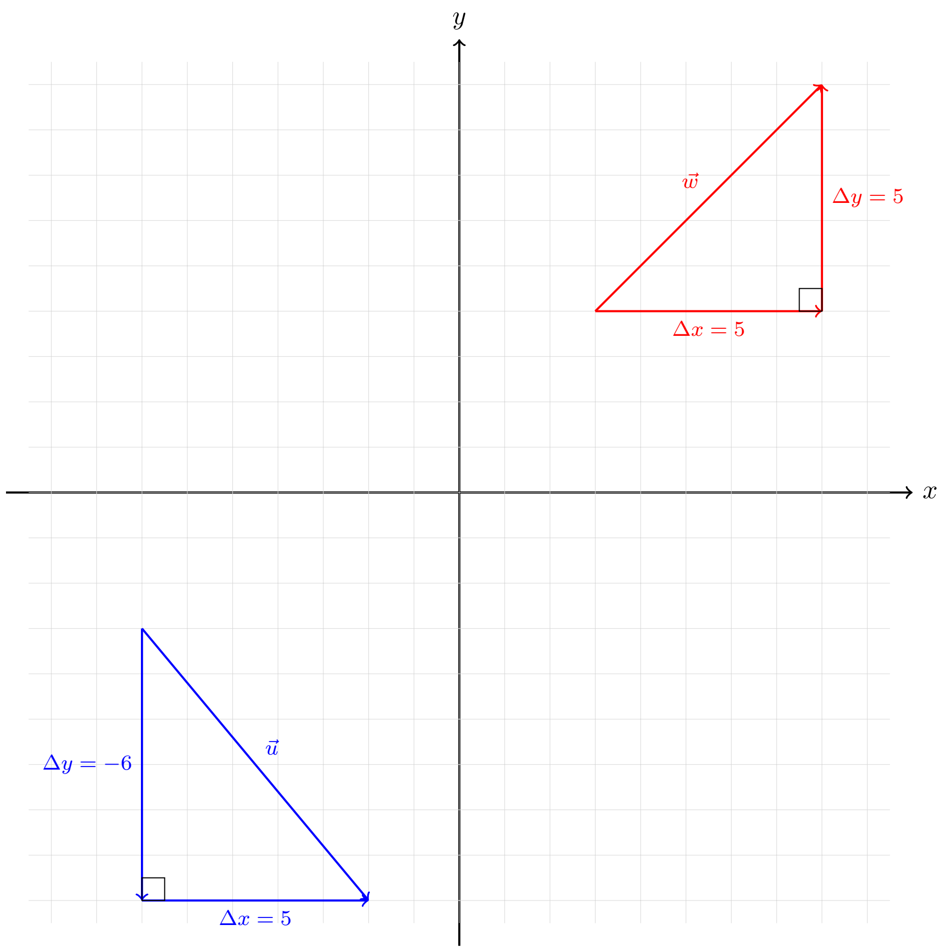

Figura 2: Representation of vectors with their respective components. Vector $\vec{u}$ (blue) has components $\Delta x = 5$ and $\Delta y = -6$, while vector $\vec{w}$ (red) has components $\Delta x = 5$ and $\Delta y = 5$.

The graph depicts two vectors, each distinguished by color. The blue vector $\vec{u}$ begins at (-7, -3) and extends to (-2, -9), with components $\Delta x = 5$ and $\Delta y = -6$. The red vector $\vec{w}$ starts at (3, 4) and extends to (8, 9), with components $\Delta x = 5$ and $\Delta y = 5$. Both form right triangles with their component vectors, illustrating how vectors can be decomposed into horizontal and vertical components.

2.2 Finding the Components of a Vector

For a vector from point A to point B, the components are calculated as follows:

$$\vec{AB} = (B_x - A_x, B_y - A_y) \tag{3}$$

Example 1. Finding vector components Given points $A(4, 4)$ and $B(-7, -8)$, the vector $\vec{AB}$ is calculated as:

$$\vec{AB} = (B_x - A_x, B_y - A_y) \tag{4}$$
$$= (-7 - 4, -8 - 4) \tag{5}$$
$$= (-11, -12) \tag{6}$$

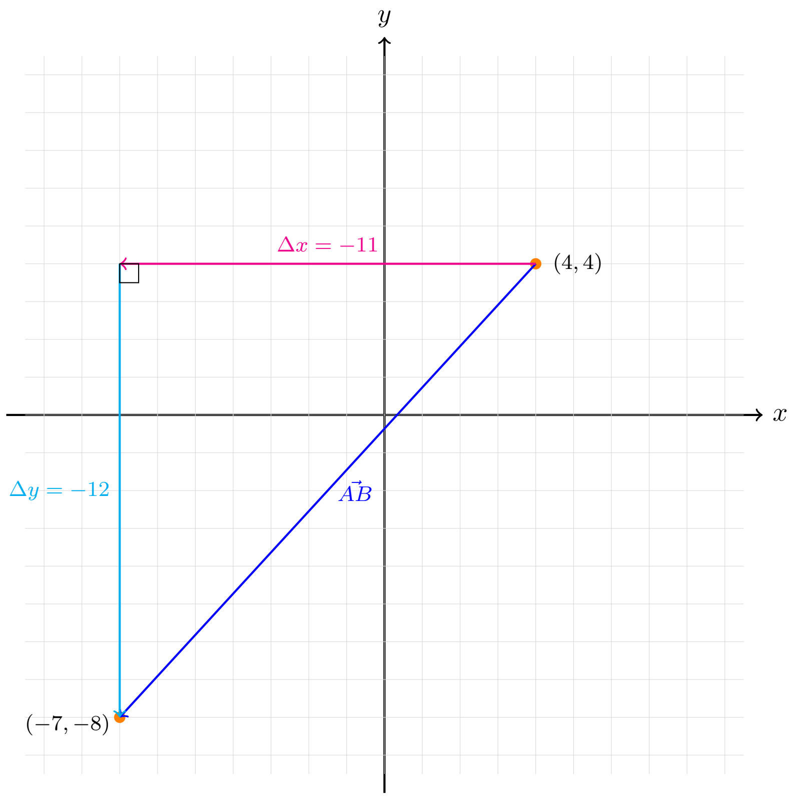

Figura 3: Vector $\vec{AB}$ from point $A(4, 4)$ to point $B(-7, -8)$ with components $\Delta x = -11$ and $\Delta y = -12$.

The magnitude of vector $\vec{AB}$ is:

$$\|\vec{AB}\| = \sqrt{(-11)^2 + (-12)^2} = \sqrt{121 + 144} = \sqrt{265} \approx 16.28 \tag{7}$$

2.3 Scalar Multiplication

When a vector is multiplied by a scalar, its magnitude changes proportionally, and:

- If the scalar is positive, the direction remains unchanged
- If the scalar is negative, the direction is reversed

For a vector $\vec{v} = (v_x, v_y)$ and a scalar k:

$$k\vec{v} = (kv_x, kv_y) \tag{8}$$

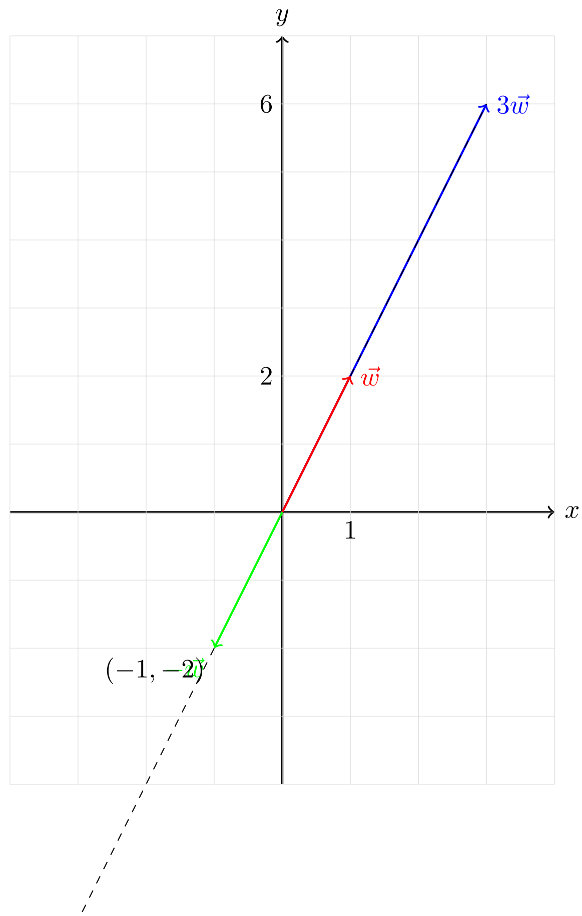

Figura 4: Scalar multiplication of vector $\vec{w} = (1, 2)$, showing $\vec{w}$, $3\vec{w} = (3, 6)$, and $-\vec{w} = (-1, -2)$.

$$\vec{w} = (1, 2) \tag{9}$$

$$3\vec{w} = 3 \times (1, 2) = (3, 6) \tag{10}$$

$$-\vec{w} = -1 \times (1, 2) = (-1, -2) \tag{11}$$

2.4 Vector Addition

Vector addition follows the parallelogram law: when two vectors are represented as adjacent sides of a parallelogram, their sum is the diagonal from their common point.

For vectors $\vec{a} = (a_x, a_y)$ and $\vec{b} = (b_x, b_y)$:

$$\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y) \tag{12}$$

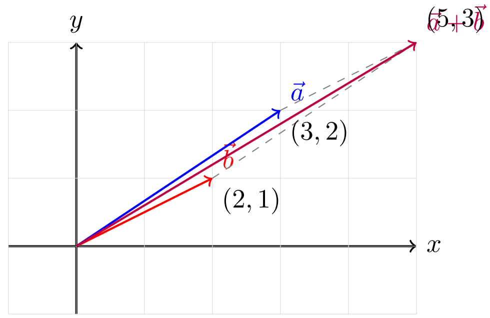

Figura 5: Vector addition illustrated with the parallelogram law. Adding $\vec{a} = (3, 2)$ and $\vec{b} = (2, 1)$ yields $\vec{a} + \vec{b} = (5, 3)$.

Property 2.1 (Commutative Property) *Vector addition is commutative:* $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

Example 2. Vector Addition For vectors $\vec{a} = (6, -2)$ and $\vec{b} = (-4, 4)$:

$$\vec{a} + \vec{b} = (6, -2) + (-4, 4) \tag{13}$$
$$= (6 + (-4), -2 + 4) \tag{14}$$
$$= (2, 2) \tag{15}$$

Similarly:

$$\vec{b} + \vec{a} = (-4, 4) + (6, -2) \tag{16}$$
$$= (-4 + 6, 4 + (-2)) \tag{17}$$
$$= (2, 2) \tag{18}$$

This confirms the commutative property of vector addition.

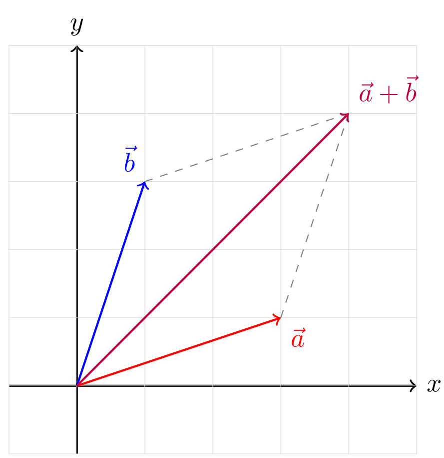

Figura 6: Another example of vector addition showing $\vec{a} + \vec{b} = (4, 4)$.

3 Unit Vectors

3.1 Definition and Properties

A unit vector is a vector with a magnitude of 1. It's often used to indicate direction without regard to magnitude.

Definition 3.1 (Unit Vector) *For any non-zero vector $\vec{v}$, the corresponding unit vector $\hat{v}$ is given by:*

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|} = \left(\frac{v_x}{\|\vec{v}\|}, \frac{v_y}{\|\vec{v}\|} \right) \tag{19}$$

Example 3. Unit Vector Calculation For vector $\vec{a} = (3, 4)$:

$$\|\vec{a}\| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5 \tag{20}$$

$$\hat{a} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{1}{5}(3, 4) = \left(\frac{3}{5}, \frac{4}{5} \right) \tag{21}$$

To verify this is indeed a unit vector:

$$\|\hat{a}\| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} \tag{22}$$

$$= \sqrt{\frac{9}{25} + \frac{16}{25}} \tag{23}$$

$$= \sqrt{\frac{25}{25}} \tag{24}$$

$$= 1 \tag{25}$$

3.2 Standard Unit Vectors

In two-dimensional Cartesian coordinates, we commonly use two standard unit vectors:

$$\hat{\imath} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \hat{\jmath} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{26}$$

Any vector can be expressed as a linear combination of these unit vectors:

$$\vec{v} = v_x \hat{\imath} + v_y \hat{\jmath} \tag{27}$$

Example 4. Vector Representation with Unit Vectors Express $\vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ in terms of standard unit vectors:

$$\vec{v} = 2\hat{\imath} + 3\hat{\jmath} \tag{28}$$

Now, if we add another vector $\vec{b} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} = -1\hat{\imath} + 4\hat{\jmath}$ to $\vec{v}$:

$$\vec{v} + \vec{b} = (2\hat{\imath} + 3\hat{\jmath}) + (-1\hat{\imath} + 4\hat{\jmath}) \tag{29}$$

$$= (2 - 1)\hat{\imath} + (3 + 4)\hat{\jmath} \tag{30}$$

$$= 1\hat{\imath} + 7\hat{\jmath} \tag{31}$$

$$= \begin{pmatrix} 1 \\ 7 \end{pmatrix} \tag{32}$$

4 Vector Direction and Angle

The direction of a vector can be specified by the angle it makes with the positive x-axis.

4.1 Finding Vector Direction

For a vector $\vec{v} = (v_x, v_y)$, the angle θ it makes with the positive x-axis is:

$$\theta = \arctan\left(\frac{v_y}{v_x}\right) \tag{33}$$

Note: This formula gives the correct angle in the first and fourth quadrants. For vectors in the second and third quadrants, adjust by adding π (180°).

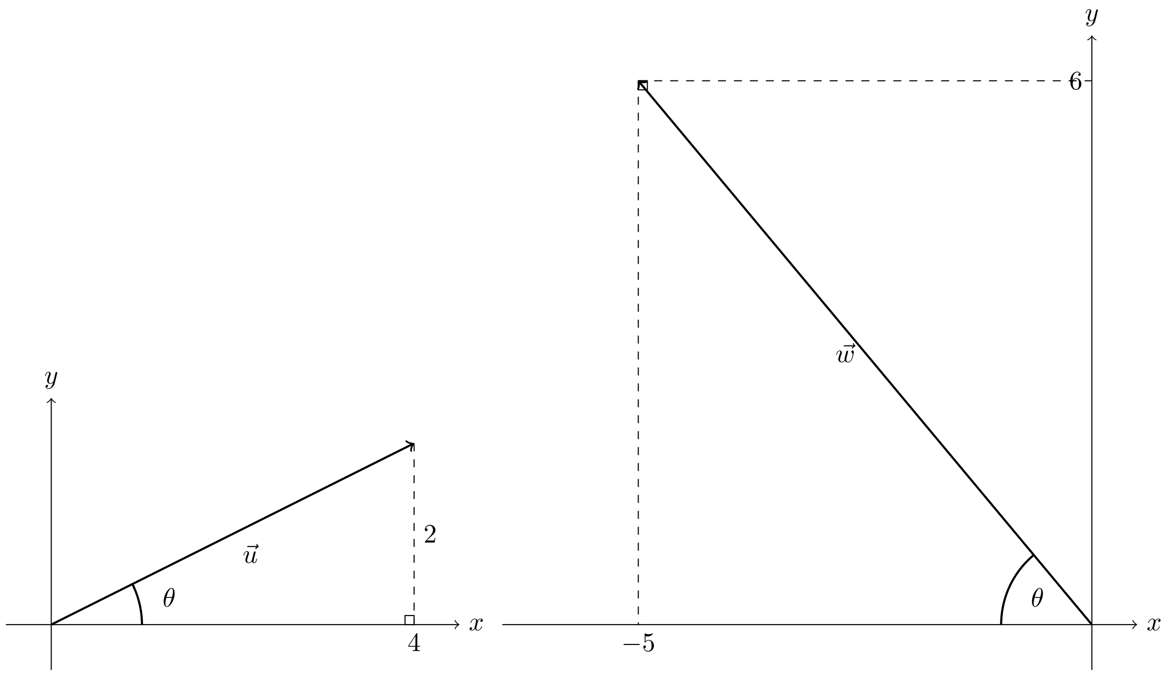

Figura 7: Vectors $\vec{u} = (4, 2)$ and $\vec{w} = (-5, 6)$ with their respective angles to the positive x-axis.

Example 5. Calculating Vector Direction For vector $\vec{u} = (3, 4)$:

$$\tan\theta = \frac{4}{3} \tag{34}$$

$$\theta = \arctan\left(\frac{4}{3}\right) \approx 53.1^\circ \tag{35}$$

For vector $\vec{w} = (-5, 6)$:

$$\tan\theta' = \frac{6}{-5} = -\frac{6}{5} \tag{36}$$

$$\theta' = \arctan\left(-\frac{6}{5}\right) \approx -50.2^\circ \tag{37}$$

Since $\vec{w}$ is in the second quadrant, we adjust:

$$\theta = \theta' + 180^\circ \approx -50.2^\circ + 180^\circ \approx 129.8^\circ \tag{38}$$

5 Vectors in Polar Form

Vectors can also be represented in polar form, using magnitude and direction:

$$\vec{v} = r\angle\theta = (r\cos\theta, r\sin\theta) \tag{39}$$

where $r = \|\vec{v}\|$ is the magnitude and θ is the angle with the positive x-axis.

Example 6. Polar to Cartesian Conversion For a vector with magnitude 4 and direction 50°:

$$x = 4\cos(50^\circ) \approx 2.57 \tag{40}$$
$$y = 4\sin(50^\circ) \approx 3.06 \tag{41}$$
$$\vec{v} = (2.57, 3.06) \tag{42}$$

Example 7. Cartesian to Polar Conversion For vector $\vec{a} = (-1, -4)$:

$$\|\vec{a}\| = \sqrt{1^2 + 4^2} = \sqrt{17} \approx 4.12 \tag{43}$$
$$\tan\theta = \frac{-4}{-1} = 4 \tag{44}$$
$$\theta = \arctan(4) \approx 76^\circ \tag{45}$$

Since $\vec{a}$ is in the third quadrant, we adjust:

$$\theta_{actual} = 76^\circ + 180^\circ = 256^\circ \quad \text{or} \quad \theta_{actual} = 76^\circ - 180^\circ = -104^\circ \tag{46}$$

6 Vector Applications

6.1 Force and Displacement Problems

Vectors are particularly useful in solving problems involving force and displacement.

Example 8. Force Addition Three forces act on an object:

$$\vec{a} = 4\hat{\imath} \quad \text{(4 KN in positive x-direction)} \tag{47}$$
$$\vec{b} = -1\hat{\imath} + 4\hat{\jmath} \quad (\text{magnitude } \sqrt{17} \text{ KN}) \tag{48}$$
$$\vec{c} = -3\hat{\imath} - 3\hat{\jmath} \quad (\text{magnitude } \sqrt{18} \text{ KN}) \tag{49}$$

The resultant force is:

$$\vec{F}_R = \vec{a} + \vec{b} + \vec{c} \tag{50}$$
$$= 4\hat{\imath} + (-1\hat{\imath} + 4\hat{\jmath}) + (-3\hat{\imath} - 3\hat{\jmath}) \tag{51}$$
$$= (4 - 1 - 3)\hat{\imath} + (0 + 4 - 3)\hat{\jmath} \tag{52}$$
$$= 0\hat{\imath} + 1\hat{\jmath} \tag{53}$$
$$= (0, 1) \tag{54}$$

The magnitude of the resultant force is $\|\vec{F}_R\| = 1$ KN, acting in the positive y-direction.

Example 9. Displacement Problem Keita left camp three days ago on a journey into the jungle. The three days of his journey can be described by displacement vectors $\vec{d_1}$,