Variance, Random Variables, and Probability Distributions

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Abstract: This document explores key concepts in probability and statistics, including variance, random variables, and common probability distributions such as the binomial and Poisson distributions.

Contents

1 Variance and Random Variables

1.1 Key Properties

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$$

Example: Let $X = \#$ hours a random person slept yesterday, $Y = \#$ hours the same person was awake yesterday.

$$\text{Var}(X) = 4, \quad \text{Var}(Y) = 16$$

$$\text{Var}(X+Y) \neq \text{Var}(X) + \text{Var}(Y)$$

Since $X + Y = 24$ hours (not independent).

1.2 Independent Random Variables

$$E(X+Y) = E(X) + E(Y)$$

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$$

For independent random variables:

$$\text{Var}(Y) = E[(Y - E(Y))^2]$$

Let $Z = X + Y$, where X and Y are independent. Then:

$$E(Z) = E(X) + E(Y)$$

$$\text{Var}(Z) = \text{Var}(X) + \text{Var}(Y)$$

$$\mu_Z = \mu_X + \mu_Y$$

$$\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2$$

Important Note: If Y is a random variable, then $E(-Y) = -E(Y)$.

2 Probability Example: Comparing Heights

Suppose: - Men have a mean height of 178 cm with a standard deviation of 8 cm. - Women have a mean height of 170 cm with a standard deviation of 6 cm. - The male and female heights are each normally distributed.

Find: Probability that a woman is taller than a man.

$$Z = \frac{X - Y - \mu}{\sigma}$$

$$\mu_X = 178, \quad \sigma_X^2 = 64$$
$$\mu_Y = 170, \quad \sigma_Y^2 = 36$$
$$\mu_{X-Y} = \mu_X - \mu_Y = 178 - 170 = 8$$
$$\sigma_{X-Y}^2 = \sigma_X^2 + \sigma_Y^2 = 64 + 36 = 100$$
$$\sigma_{X-Y} = \sqrt{100} = 10$$

The distribution of $X - Y$ is $N(\mu = 8, \sigma = 10)$.

$$P(X - Y < 0) = P(Z < \frac{0-8}{10}) = P(Z < -0.8)$$

Using the standard normal table:

$$P(Z < -0.8) \approx 0.2119$$

3 Binomial Distribution

For n flips of a coin:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Example with $n = 5$, $p = 0.5$:

$$P(X = 2) = \binom{5}{2} (0.5)^2 (0.5)^3 = 10 \cdot (0.03125) = 0.3125$$

The binomial coefficients are given by:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

4 Poisson Distribution

The Poisson distribution is given by:

$$P(X = k) = \frac{x^k e^{-x}}{k!}$$

For example:

$$x = 3, \quad k = 2$$

$$P(X = 2) = \frac{3^2 e^{-3}}{2!} = \frac{9e^{-3}}{2}$$

Probability of making a free throw:

$$\text{Probability of scoring} = 70\% \text{ or } 0.7$$
$$\text{Probability of missing} = 30\% \text{ or } 0.3$$

Binomial Probability Formula:

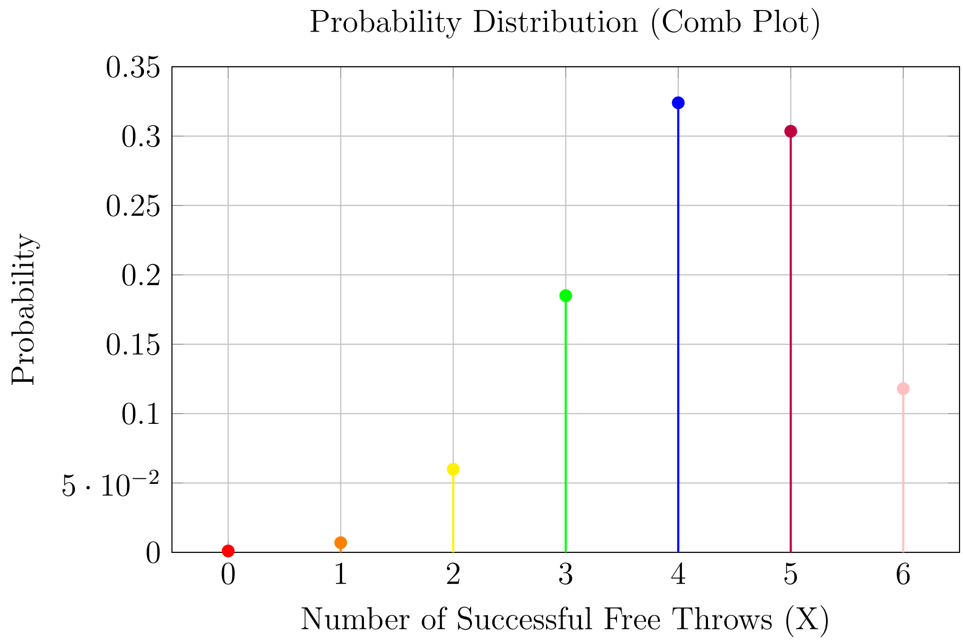
$$P(\text{Exactly } k \text{ scores in } n \text{ attempts}) = \binom{n}{k} f^k (1-f)^{n-k}$$

where:

$$f = \text{probability of making a free throw} = 0.7$$
$$n = 6 \text{ (total attempts)}$$

Probability Distribution for X (number of made free throws in 6 attempts):

$$P(X = 0) = \binom{6}{0}(0.7)^0(0.3)^6 \approx 0.001 \approx 0.1\%$$
$$P(X = 1) = \binom{6}{1}(0.7)^1(0.3)^5 \approx 0.007 \approx 0.7\%$$
$$P(X = 2) = \binom{6}{2}(0.7)^2(0.3)^4 \approx 0.06 \approx 6\%$$
$$P(X = 3) = \binom{6}{3}(0.7)^3(0.3)^3 \approx 0.185 \approx 18.5\%$$
$$P(X = 4) = \binom{6}{4}(0.7)^4(0.3)^2 \approx 0.324 \approx 32.4\%$$
$$P(X = 5) = \binom{6}{5}(0.7)^5(0.3)^1 \approx 0.3035 \approx 30.35\%$$
$$P(X = 6) = \binom{6}{6}(0.7)^6(0.3)^0 \approx 0.118 \approx 11.8\%$$



Probability Calculations:

$$X = \text{Number of successful free throws in 7 trials with } p = 0.35$$
$$P(X = 4) = \text{Binompdf}(7, 0.35, 4) \approx 0.14$$
$$P(X \leq 4) = \text{Binomcdf}(7, 0.35, 4) \approx 0.94$$

Bernoulli Distribution:
Expected Value and Standard Deviation:

$$\mu = P, \quad \sigma^2 = P(1 - P), \quad \sigma = \sqrt{P(1 - P)}$$

Let X represent the number of successes after n trials, where P is the probability of success. The expected value is:

$$E(X) = nP$$

For $n = 10$ and $P = 0.3$, we have:

$$E(X) = 10 \times 0.3 = 3$$

The variance of X is:

$$\text{Var}(X) = n \times P(1 - P) = 10 \times 0.3 \times 0.7 = 2.1$$

Since Y is a Bernoulli random variable, $E(Y) = P$ and $\text{Var}(Y) = P(1 - P)$.

The expected value of the sum of n independent Bernoulli random variables is:

$$E(X) = nE(Y) = n \times P$$

Let x be the number of defective chips in a sample of 500 chips.

x follows a **Binomial distribution** with:

$$P(\text{defective chips}) = 0.02$$

The expected value (mean) of x is:

$$\mu = E(x) = n \times p = 500 \times 0.02 = 10$$

The standard deviation of x is:

$$\text{Standard deviation} = \sqrt{\text{Var}(x)} = \sqrt{n \times p \times (1-p)} = \sqrt{500 \times 0.02 \times 0.98} = \sqrt{9.8} \approx 3.13$$

4.1 Geometric Random Variables

A **Geometric Random Variable** represents the number of trials required to achieve the first success in independent Bernoulli trials, each with success probability p. **1. Number of 6's after 12 rolls of a fair die:**

- $X \sim \text{Binomial}(n = 12, p = \frac{1}{6})$
- Trial outcome: Success (6) or failure (not 6)
- Fixed number of trials ($n = 12$)
- Same probability of success for each trial ($P(\text{success}) = \frac{1}{6}$)

2. Number of rolls until the first 6:

- $Y \sim \text{Geometric}(p = \frac{1}{6})$
- Trial outcome: Success (6) or failure (not 6)
- Independent trials with the same probability of success
- Minimum $Y = 1$ (first roll can be a 6)
- Maximum $Y = \infty$ (theoretically infinite trials to get a 6)