Probability & Statistics

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7 maggio 2025

Indice

1 Introduction

This is the introduction.

1.1 Motivation Here we explain the motivation for the study of probability and statistics.

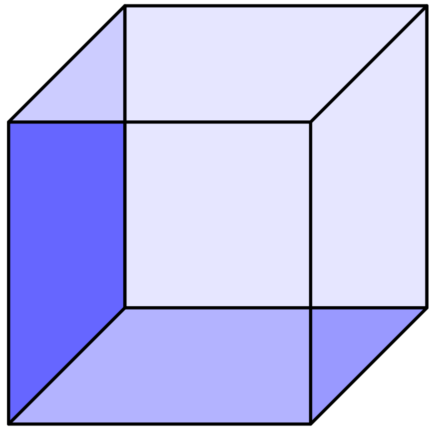
1.1.1 Applications Probability and statistics have numerous applications in various fields.

2 Theory

This section contains the theoretical background.

2.1 Basic Concepts In this subsection, we introduce some basic concepts.

2.1.1 Random Variables A random variable is a key concept in probability theory.



Probability Experiment

Flipping a Coin

$$\text{Flip a coin:} \quad P(H) = ?$$
$$P(H) = \frac{1}{2} = 50\%$$

Rolling a Die

$$P(1) = \frac{1}{6},$$
$$P(1 \text{ or } 6) = \frac{2}{6} = \frac{1}{3},$$
$$P(2 \text{ and } 3) = 0.$$

Probability of Selecting a Yellow Marble

$$P(\text{yellow marble}) = \frac{\text{number satisfying my condition}}{\text{total possibilities}}$$

$$P(\text{yellow, yellow}) = \frac{3}{8} \cdot \frac{2}{7} = \frac{6}{56} = \frac{3}{28}.$$

Rolling a Die (Again) Sample space: $\{1, 2, 3, 4, 5, 6\}$.

$$P(\text{rolling } \leq 2) = \frac{2}{6} = \frac{1}{3},$$
$$P(\text{rolling } \geq 3) = \frac{4}{6} = \frac{2}{3}.$$

Probability Extremes

$$P(\text{rolling } 7) = \frac{0}{6} = 0 \quad \text{(Impossible)},$$
$$P(\text{rolling } 1-6) = \frac{6}{6} = 1 \quad \text{(Certain)}.$$

P(sunrise)=0.999 P(gofer writes a great novel)

Probability Calculations

Examples

$$P(\text{sunrise}) = 0.99999$$

$$P(\text{gofer writes a great novel}) = 0.000001$$

Picking Red Balls

$$P(\text{pick red}) = \frac{50}{100} = \frac{1}{2}$$

$$\text{After 10 experiments: } 7, 3$$

$$\text{After 10,000 experiments: } 8000, 2000$$

$$\text{Experimental probability: } \frac{8000}{10,000} = 80\%$$

Theoretical Probabilities

$$P(H) = \frac{1}{2} \quad \text{(coin flip: heads or tails)}$$

$$\text{Dice rolls: } 1, 2, 3, 4, 5, 6$$

$$P(\geq 3) = \frac{4}{6} = \frac{2}{3}$$

$$\text{Expected value: } 2 + 4 + 5 + 3 + 2 = 16$$

$$\text{Game 17: } P(\text{score } \geq 36) = \frac{5}{16}$$

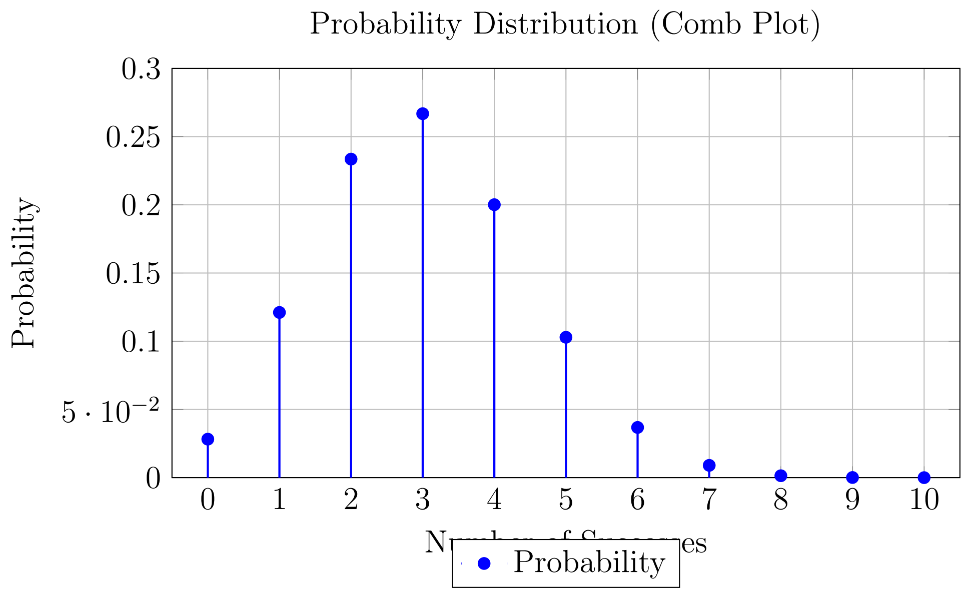
Data and Histogram Visualization

- Exercise data:
  - 17 turtles
  - 12 coconuts
  - 7 watches
  - 9 tigers

Histogram Data Visualization

Bucket	Count
0-9	6
10-19	3
20-29	5
30-39	1
40-49	2
50-59	2

Histogram for Probability Distribution



Probability and Data Analysis

Ball and Spin Calculations

- Total balls: 32
- Sum of $5 + 8 + 4 + 3 = 20$
- Number of piles with size 60 or more: $8 + 4 + 3 = 15$

Spin Probabilities

$$1 \text{ spin: } P(\text{elephant}) = \frac{4}{7}$$

$$20 \times \frac{4}{7} = 120 \text{ times}$$

Line Data Analysis

Time: 4:00 PM

Line Size	Times Observed	Probability
0	24	$\frac{24}{50} = 0.48 = 48\%$
1	18	$\frac{18}{50} = 0.36 = 36\%$
2	8	$\frac{8}{50} = 0.16 = 16\%$

Visits and Probabilities

Visiting 500 times:

$$500 \times \frac{8}{50} = 80 \text{ times with size 2}$$

Sample Space: Coin Flip

$$\text{Sample Space: } \{H, T\}$$

Compound Sample Space: Flavors and Sizes

- Flavors: Chocolate, Strawberry, Vanilla
- Sizes: Small (S), Medium (M), Large (L)

Sample Space and Compound Sample Space

Sample Space: Coin Flip

$$\text{Sample Space: } \{H, T\}$$

Flavors and Sizes

- **Flavors:** Chocolate, Strawberry, Vanilla
- **Sizes:** Small, Medium, Large

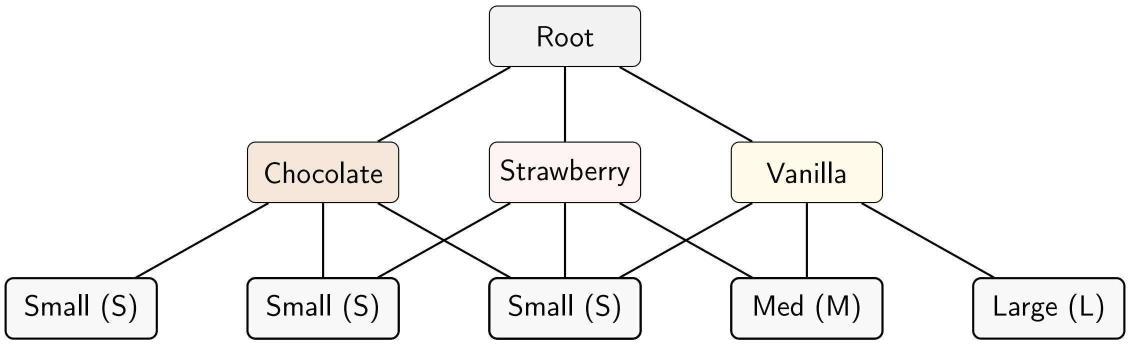


Figura 1: Hierarchical flavor tree with size options

Tree Diagram: Compound Sample Space

Size	S	C	V
S	0.85	0.72	0.68
M	0.78	0.90	0.82
L	0.70	0.85	0.95

Tabella 1: My Table Caption

Compound Sample Space Table

Notation for Sample Space

$$\{S, M, L\}, \quad \{C, S, V\}$$

3 Bayes' Theorem

Bayes' Theorem provides a mathematical framework for updating the probability of a hypothesis H given new evidence E. It is expressed as:

$$P(H|E) = \frac{P(E|H) \cdot P(H)}{P(E)}$$

Key Concepts

- **Prior Probability ($P(H)$):** The initial probability of the hypothesis before considering the evidence.
- **Conditional Probability ($P(E|H)$):** The probability of observing the evidence assuming the hypothesis is true.
- **Posterior Probability ($P(H|E)$):** The updated probability of the hypothesis after accounting for the evidence.
- **Evidence Probability ($P(E)$):** The overall probability of the evidence occurring.

Example Calculation Given:

- $P(E|H) = 0.95$: Probability of observing E if H is true.
- $P(H) = 0.00001$: Prior probability of the hypothesis.
- $P(E) = 0.01$: Total probability of observing the evidence.

Applying Bayes' Theorem:

$$P(H|E) = \frac{P(E|H) \cdot P(H)}{P(E)}$$

$$P(H|E) = \frac{0.95 \cdot 0.00001}{0.01} = 0.000095$$

Insights

The posterior probability $P(H|E)$ is very low (0.00095), despite the high likelihood of E given H. This demonstrates the **base rate fallacy**: a rare prior probability ($P(H)$) can dominate the posterior, even when $P(E|H)$ is large.

Key Takeaway Conditional probability requires careful consideration of prior probabilities and the likelihood of evidence. Bayes' Theorem emphasizes the importance of balancing these factors to avoid misinterpretation of probabilities.

4 Random Variables

A random variable is a numerical outcome of a random experiment. Below are examples of random variables:

Examples 1. Coin Flip** Let X be a random variable defined as:

$$X = \begin{cases} 1, & \text{if the result is heads,} \\ 0, & \text{if the result is tails.} \end{cases}$$

Rolling 7 Dice Let Y represent the sum of the outcomes of rolling 7 fair dice. We are interested in the following probabilities:

$$P(Y \leq 30), \quad P(Y \text{ is even}).$$

System of Equations Given two variables x and y, their relationship can be described by the following equations:

$$x + 5 = 6,$$

$$y = x + 7.$$

4.1 ex p 37

Anush is playing a carnival game that involves shooting 2 free-throws. The table below displays the probability distribution of X, the number of shots that Anush makes in a set of 2 attempts, along with some summary statistics.

$X = \#$ of makes

X	0	1	2
$P(X)$	0.16	0.48	0.36

$$\mu_X = 1.2 \quad \sigma_X \approx 0.69$$

If the game costs Anush \$15 to play and he wins \$10 per shot he makes, what are the mean and standard deviation of his net gain from playing the game, N?

4.2 Example: Probability Calculations (p. 32)

$$\begin{align} P(H) &= \frac{1}{2} \quad \text{(Probability of heads)} \tag{1} \\ P(T) &= \frac{1}{2} \quad \text{(Probability of tails)} \tag{2} \\ & \tag{3} \end{align}$$

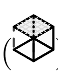
The possible outcomes when flipping a coin twice are:

$$\begin{matrix} H & H \\ H & T \\ T & H \\ T & T \end{matrix} \quad \text{(Total outcomes: 4)}$$

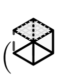
For specific probabilities:

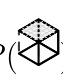
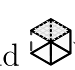
$$\begin{align} P(HH) &= P(H_1) \cdot P(H_2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}, \tag{4} \\ P(THT) &= P(T_1) \cdot P(H_1) \cdot P(T_2) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}. \tag{5} \end{align}$$

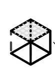
4.3 Example from Page 33

$P($$) = ?$

$$P(Y) = \frac{12}{29}$$

$P($ and $Y) = \frac{5}{29}$

$P(Y$ or $) = P(Y) + P($$) - P(Y$ and $)$

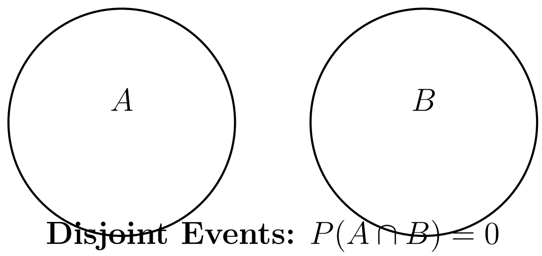
$P(Y$ or $) = \frac{12}{29} + \frac{13}{29} - \frac{5}{29} = \frac{20}{29}$

The general formula for the probability of the union of two events is:

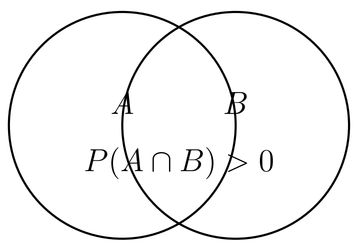
$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

4.4 Visual Representations

Diagrams of Events



[h] **Disjoint Events:** $P(A \cap B) = 0$



Intersecting Events

Exercise 1, Page 34

Dependent Probability

$$P(M \cap S) = P(M) \cdot P(S|M)$$

$$P(S|M) = \frac{P(M \cap S)}{P(M)}$$

Example:

$$P(\text{1st given 2nd}) = \frac{5}{6} \cdot \frac{3}{5} = \frac{25}{36}$$

Random Variables

Discrete Example Definition:

$$X = \begin{cases} 1 \text{ Head}, \\ 0 \text{ Tails}. \end{cases}$$

Zoo Example: Random animals selected at Orleans Zoo:

$$\text{Years: } \boxed{1985}, \boxed{1992}, \boxed{2001}$$

Mass of the animal:

$$5000\,\text{kg}$$

Continuous Example

$$Y = \text{mass of random animals (value in intervals, e.g., 123.5)}$$

Game Example A game with 3 gems and a 30% chance of winning:

$$\text{Win 1 gem: 30\% chance.}$$

Probability and Random Variables Notes

Random Variables

$$Z = \#\text{ of ants born tomorrow in the universe.}$$

X = Exact winning time (in seconds) for Men's 100m Dash, 2016 Olympics, rounded to the nearest hundredths.

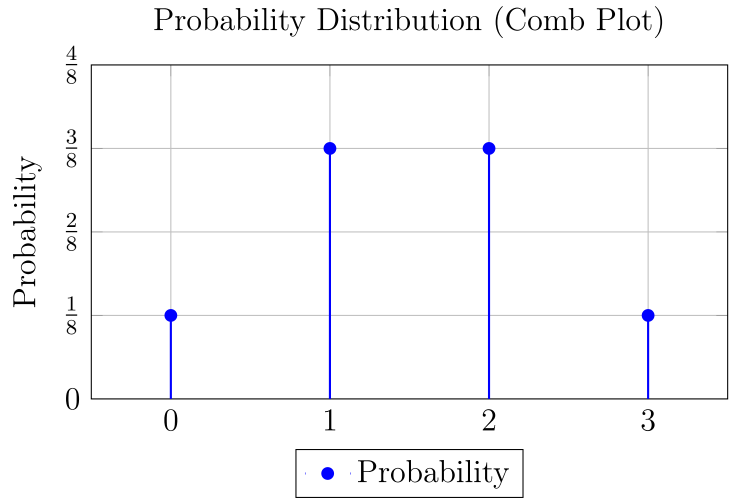
Discrete Random Variable:

$$X = \#\text{ of heads after 3 flips of a fair coin.}$$

Outcomes	X
HHT, HTH, THH, TTH, HTT, THT, TTT, HHH	0, 1, 2, 3

$$P(X=0) = \frac{1}{8}, \quad P(X=1) = \frac{3}{8}, \quad P(X=2) = \frac{3}{8}, \quad P(X=3) = \frac{1}{8}.$$

Histogram for Probability Distribution



Discrete Probability Distribution

$$P(X = x) \quad \text{discrete probability distribution function.}$$

Example of a discrete distribution:

$$\text{Value for } x = 0, 1, 2, 3$$

Graph: Probability vs x (bar heights correspond to probabilities).

Probability Table Example

$$\text{Probabilities: 0.2, 0.5, 0.3} \quad \text{(Total: 100\%).}$$

Must satisfy:

$$P(4) = 1 - 0.2 - 0.16 - 0.128 = 0.512$$

Another Example:

$$P(4) = 1 - 0.2 - 0.16 - 0.1 - 0.128 = 0.512.$$

Expected Value Example: Number of Workouts in a Week

$$x = \# \text{ of workouts in a week (discrete).}$$

x	$P(x)$
0	0.1
1	0.15
2	0.4
3	0.25
4	0.1

$$E(X) = \mu_x = (0 \cdot 0.1) + (1 \cdot 0.15) + (2 \cdot 0.4) + (3 \cdot 0.25) + (4 \cdot 0.1) = \boxed{2.1}.$$

4.5 Exercise 1, Page 36

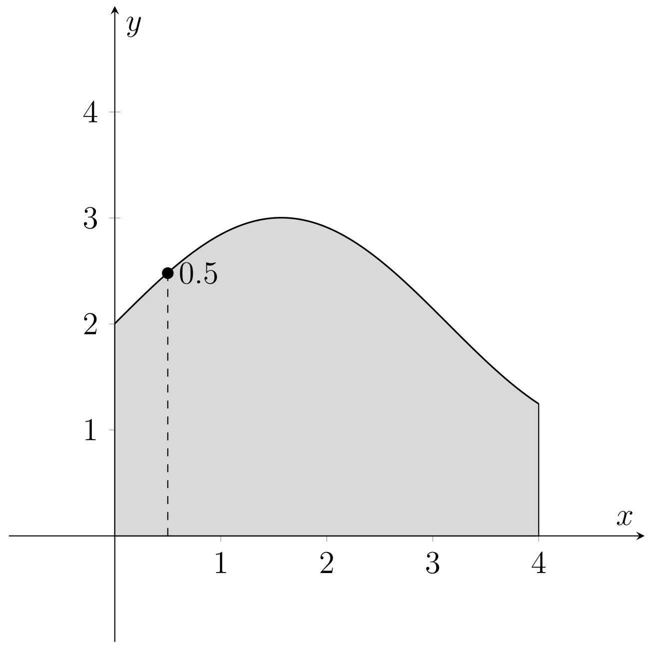
Discrete Functions and Properties:

$$\begin{aligned} f(x) &= 1, \\ f(x) &= (1.9 < x < 2.1) \cdot f(x), \\ p(x-2) &= 1, \\ p(y) &= 2(1-y), \\ p'(x) &= -2, \\ Y &= 2. \end{aligned}$$

Integrals:

$$\begin{aligned} &\int_{1.9}^{2.1} f(x)\,dx \quad \text{(Undefined integral)}, \\ &\int_{0}^{\infty} f(x)\,dx = 1. \end{aligned}$$

Graph:



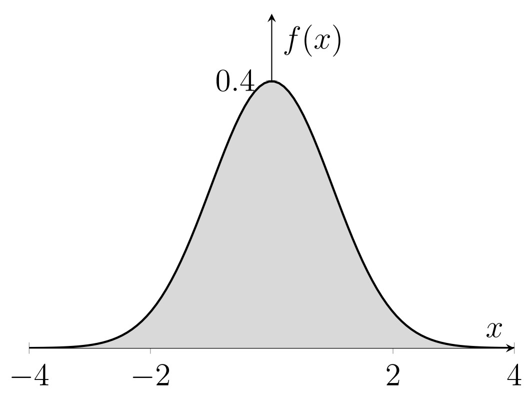
Binary Variable:

$$x = \begin{cases} 1 & \text{head,} \\ 0 & \text{tail.} \end{cases}$$

4.6 Exercise 2, Page 36

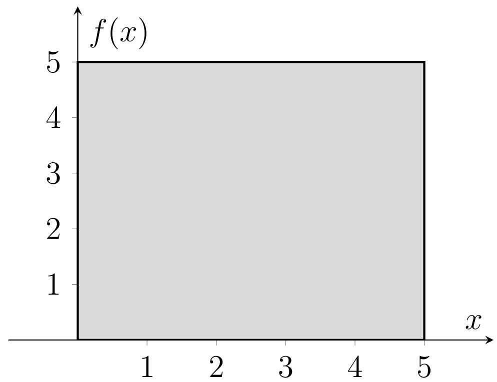
Graphs:

1. **Normal Distribution:**



2. **Rectangular Function:**

Rectangular Function



Calculations:

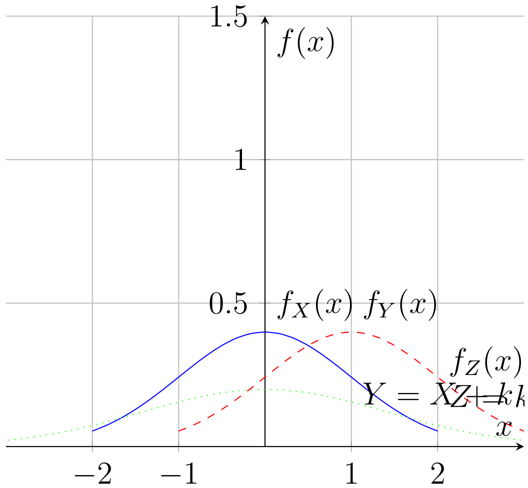
- $0.25 \times 3 = 0.75$
- $P(x < 4) = 0.75$
- Result: ≈ 0.16

4.7 Exercise 3, Page 36

Equations:

$$
\begin{aligned}
Y &= X + k, \\
Z &= kX, \\
\mu_Y &= \mu_X + k, \\
\mu_Z &= k\mu_X, \\
\sigma_Y^2 &= \sigma_X^2, \\
\sigma_Z^2 &= k^2\sigma_X^2.
\end{aligned}
$$

Graph:



4.8 Exercise 1, Page 37

Equations:

$$\begin{aligned}
N &= 120 - N, \\
N &= 10x - 15, \\
M &= 10 - 15 = -3, \\
P(N) &= 10 - 16 = 6.9.
\end{aligned}$$

Item	Value
N	$10x - 15$
M	$10 - 15 = -5$
$P(N)$	$10 - 16 = 0.48$
	0.36
N	1006.9

Tabella 2: Table of Calculations

Table:

4.9 Exercise 2, Page 37

Definitions:

$$\begin{aligned}
X &= \text{Number of dogs}, \\
Y &= \text{Number of cats}.
\end{aligned}$$

Expected Values:

$$E(X) = \mu_X = 3,$$
$$E(Y) = \mu_Y = 4.$$

Using Properties of Expectation:

$$E(X+Y) = \mu_{X+Y} = 7,$$
$$E(Y-X) = 1.$$

4.10 Exercise 3, Page 37

Details:

- X = cereal in box
- Y = weight of cereal in bowl
- $E(X) = 1$ oz
- $E(Y) = 4$ oz
- Standard deviation $\sigma_X = 0.8$ oz
- Standard deviation $\sigma_Y = 0.6$ oz
- $15 \leq X \leq 17$
- $3 \leq Y \leq 5$
- $E(X+Y) = E(X) + E(Y) = 5$ oz
- Assuming X and Y are independent:

$$\sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2} = 1 \text{ oz}$$

- $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$
- $\boxed{18} \leq X + Y \leq \boxed{22}$
- $\text{Var}(X-Y)$
- $10 \leq X - Y \leq 14$

$$\mathrm{Var}(x+y) = \mathrm{Var}(x) + \mathrm{Var}(y)$$

Let:

$$x = \text{number of hours a random person slept yesterday},$$
$$y = \text{number of hours the same person was awake yesterday}.$$

$$\mathrm{Var}(x) = 4\,(\mathrm{hrs})^2, \quad \sigma_x = 2\,\mathrm{hrs}.$$
$$\mathrm{Var}(y) = 4\,(\mathrm{hrs})^2, \quad \sigma_y = 2\,\mathrm{hrs}.$$

Since:

$$x + y = 24\,\mathrm{hrs},$$

x and y are not independent.

Independent Random Variables If x and y are independent:

$$E(x) = \mu_x, \quad E(y) = \mu_y,$$
$$\mathrm{Var}(x) = E\big[(x-\mu_x)^2\big] = \sigma_x^2,$$
$$\mathrm{Var}(y) = E\big[(y-\mu_y)^2\big] = \sigma_y^2.$$

Define $z = x + y$:

$$E(z) = E(x+y) = E(x) + E(y),$$
$$\mu_z = \mu_x + \mu_y.$$

Define $a = x - y$:

$$E(a) = E(x-y) = E(x) - E(y),$$
$$\mu_a = \mu_x - \mu_y.$$

Variance of Sums and Differences

$$\mathrm{Var}(z) = \mathrm{Var}(x+y) = \mathrm{Var}(x) + \mathrm{Var}(y),$$
$$\sigma_z^2 = \sigma_x^2 + \sigma_y^2.$$

$$\mathrm{Var}(a) = \mathrm{Var}(x-y) = \mathrm{Var}(x) + \mathrm{Var}(-y),$$
$$\sigma_a^2 = \sigma_x^2 + \sigma_y^2.$$

Note that:

$$\mathrm{Var}(-y) = E\big[(-y - E(-y))^2\big],$$

and:

$$E(-y) = -E(y).$$

Therefore:

$$\mathrm{Var}(-y) = \mathrm{Var}(y),$$
$$\sigma_a^2 = \sigma_x^2 + \sigma_y^2.$$

Finally:

$$\mu_{x-y} = \mu_x - \mu_y.$$

Probability and Statistics Notes

Basic Definitions Let x be the number of hours a random person slept yesterday, and Y be the number of hours a random person was awake yesterday.

$$\text{Var}(x) = \sigma_x^2 \quad \text{Var}(Y) = \sigma_Y^2$$

Given:

$$G = 2 \text{ hours} \quad \sigma_Y = 2 \text{ hours}$$

Independence If X and Y are independent random variables:

$$E[X+Y] = E[X] + E[Y]$$

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$$

Expected Value and Variance

$$E[X] = \mu_X \quad E[Y] = \mu_Y$$

$$\text{Var}(X) = E[(X-\mu_X)^2]$$

$$\text{Var}(Y) = E[(Y-\mu_Y)^2]$$

Height Distribution The mean height of men is 178 cm with a standard deviation of 8 cm. The mean height of women is 170 cm with a standard deviation of 6 cm. Heights are normally distributed and independent.

Probability Calculation What is the probability that a randomly selected woman is taller than a randomly selected man?

Let H_m be the height of a random man and H_w be the height of a random woman.

$$H_m \sim N(178, 8^2) \quad H_w \sim N(170, 6^2)$$

We want to find $P(H_w > H_m)$.

Binomial Distribution Consider flipping a coin 5 times. The possible outcomes are:

$$2^5 = 32$$

The probability of getting exactly 2 heads in 5 flips is given by the binomial distribution:

$$P(X=2) = \binom{5}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 = \frac{10}{32} = \frac{5}{16}$$

Normal Distribution The normal distribution is characterized by the bell curve:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

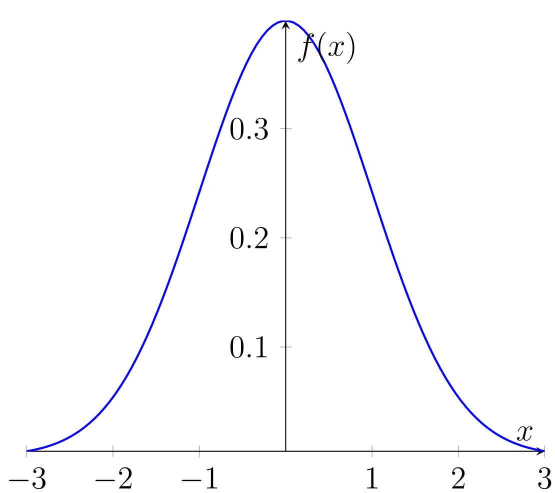
Probability of Scoring Given a probability of scoring of 70% and missing of 30%, the probability of exactly 2 scores in 6 attempts is:

$$P(X = 2) = \binom{6}{2} (0.7)^2 (0.3)^4$$

Calculating:

$$P(X = 2) = 15 \times 0.49 \times 0.0081 = 0.059535$$

$$\begin{aligned}
&E[X] = \mu \\
&Var(X) = \sigma^2 \\
&E[Y] = \mu_Y \\
&E[X + Y] = E[X] + E[Y] = \mu + \mu_Y \\
&Var(X + Y) = Var(X) + Var(Y) \\
&E[XY] = E[X]E[Y] \\
&E[X^2] = Var(X) + (E[X])^2 \\
&Var(X) = E[X^2] - (E[X])^2 \\
&f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\
&P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k} \\
&E[X] = np \\
&Var(X) = np(1 - p)
\end{aligned}$$



Binomial Distribution