Alternate Coordinate Systems and Linear Algebra

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Sommario

In mathematics, alternate coordinate systems, or bases, are foundational concepts used to represent the position of points within a space. Unlike the standard Cartesian coordinates, alternate bases allow for the expression of vectors through different sets of linearly independent vectors, forming bases in vector spaces. This flexibility is essential in simplifying complex problems, enabling transformations, and facilitating computations in various fields like linear algebra, engineering, and physics. For example, polar and spherical coordinates provide significant advantages for analyzing radial symmetry. The choice of coordinate system can lead to more efficient solutions and deeper insights into spatial relationships and transformations.

Key insights include:

- Representation of points through linearly independent vectors
- Transformation between coordinate systems
- Applications in solving spatial and computational problems

Indice

1 Linear Subspaces and Basis of $\mathbb{R}^n$

$$V \subseteq \mathbb{R}^n, \quad \dim(V) = k$$

$$V + V^{\perp} = \mathbb{R}^n, \quad \dim(V) + \dim(V^{\perp}) = n$$

1.1 Definition of Orthogonal Subspace

$$V^{\perp} = \{\vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{v} = 0 \text{ for every } \vec{v} \in V\}.$$

1.2 Vector Properties

$$\vec{a} \cdot \vec{v} = 0, \quad \vec{b} \cdot \vec{v} = 0$$
$$(\vec{a} + \vec{b}) \cdot \vec{v} = \vec{a} \cdot \vec{v} + \vec{b} \cdot \vec{v} = 0$$
$$c \cdot \vec{a} \cdot \vec{v} = 0$$

1.3 Null Space and Orthogonality

$$N(A) - c(A^T)^{\perp}$$

$$A = \begin{bmatrix} 1^T \\ 2^T \\ n_m \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix}, \quad \vec{r}^T \cdot \vec{x} = 0$$

1.4 Subspace Basis

Assume $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in V$,

$$\vec{v}_i \neq 0, \quad \|\vec{v}_i\| = 1, \quad \forall \vec{v}_i \in V$$

Subspace V:

$$\text{basis of } V = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$$

$$\text{basis of } V^{\perp} = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_{n-k}\}$$

$$V \cap V^{\perp} = \{\vec{0}\}$$

For $\vec{x} \in V$:

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k \quad (c_i \in \mathbb{R})$$

For $\vec{x} \in V^{\perp}$:

$$\vec{x} \cdot \vec{v}_i = 0, \quad \forall i = 1, 2, \ldots, k$$

1.5 Uniqueness Check

Assume:

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = 0$$
$$c_1 = c_2 = \cdots = c_k = 0$$

V is a subspace of $\mathbb{R}^n$, and $V^{\perp}$ is a subspace of $\mathbb{R}^n$.

1.6 Null Space Relation

$$(A\vec{x} = \vec{0}) \implies (\vec{x} \in V^{\perp})$$

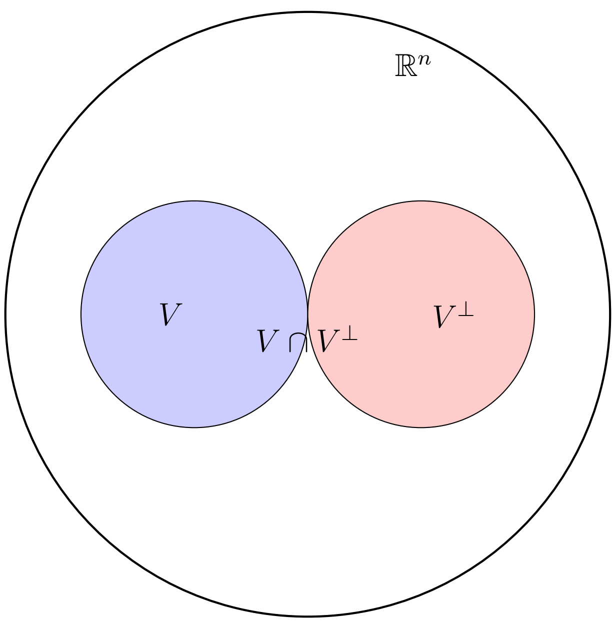
Basis of V:

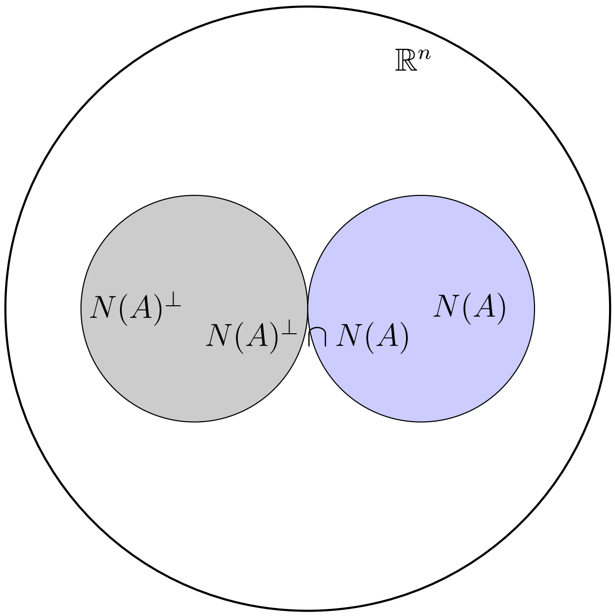
$$\dim(V) = k, \quad \dim(V^{\perp}) = n - k$$

[1]

[1] Perpendicular: $A \perp B$
Parallel: $A \parallel B$
Not Parallel: $A \nparallel B$
Intersection: $A^T, A \cap B$
Appartenance: $\in$
Real numbers: $\mathbb{R}$

$$
\begin{aligned}
\vec{V}^{\perp} &= \{\vec{x} \subseteq \mathbb{R}^n \mid \vec{x} \cdot \vec{v} = 0 \quad \text{for every} \quad \vec{v} \subseteq V\} \\
(\vec{V}^{\perp})^{\perp} &= \{\vec{x} \subseteq \mathbb{R}^n \mid \vec{x} \cdot \vec{w} = 0 \quad \text{for every} \quad \vec{w} \subseteq V\} \\
&\vec{x} \subseteq (\vec{V}^{\perp})^{\perp} \\
&\vec{x} = \vec{v} + \vec{w} \quad \text{where} \quad \vec{v} \subseteq V \quad \text{and} \quad \vec{w} \subseteq \vec{V}^{\perp} \\
\vec{x} \cdot \vec{w} = 0 \quad &\Rightarrow \quad (\vec{V} + \vec{w}) - \vec{w} = \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w} = ||\vec{w}||^2 \\
||\vec{w}||^2 = 0 &\implies \vec{w} = \vec{0} \\
&\implies \vec{x} = \vec{v} \implies \vec{x} \subseteq V \\
&\vec{v} \subseteq V \quad \text{with} \quad \vec{v} = \vec{w} + \vec{x} \quad \text{where} \quad \vec{w} \subseteq \vec{V}^{\perp} + \vec{x} \subseteq (\vec{V}^{\perp})^{\perp} \\
\vec{v} \cdot \vec{w} = 0 &\implies \vec{w} + \vec{x} \cdot \vec{w} = \vec{w} \cdot \vec{w} + \vec{x} \cdot \vec{w} = ||\vec{w}||^2 \\
&\implies \vec{w} = 0 \implies \vec{v} = \vec{x} \\
&\text{if} \quad \vec{v} \subseteq V, \quad \text{where} \quad \vec{v} \subseteq (\vec{V}^{\perp})^{\perp} \\
&\text{if} \quad \vec{x} \subseteq (\vec{V}^{\perp})^{\perp}, \quad \text{where} \quad \vec{x} \subseteq V.
\end{aligned}
$$





$$
\begin{aligned}
A\vec{v} &= \vec{0} \\
&\vec{x} \text{ is a solution to } A\vec{x} = \vec{b}, \quad \vec{x} \in \mathbb{R}^n, \quad \vec{x} = \vec{r}_0 + \vec{n}_0 \quad \text{where} \quad \vec{f}_0 \subseteq c(A^T) + \vec{n}_0 \subseteq N(A) \\
&\vec{r}_0 = \vec{x} - \vec{n}_0, \quad A\vec{r}_0 = A(\vec{x} - \vec{n}_0), \quad A\vec{r}_0 = A\vec{n}_0 = \vec{b} \\
\implies &\vec{r}_n \text{ is a solution to } A\vec{v} = \vec{b} \\
&\vec{r}_1 \subseteq c(A^T) \text{ is a solution to } A\vec{x} = \vec{b} \\
(\vec{r}_1 - \vec{r}_0) &\subseteq c(A), \quad A(\vec{r}_1 - \vec{r}_0) = A\vec{r}_1 - A\vec{r}_0 = \vec{b} - \vec{b} = \vec{0} \\
\implies (\vec{r}_1 - \vec{r}_0) &\subseteq N(A) \\
\vec{r}_1 - \vec{r}_0 &= \vec{0} \implies \vec{r}_1 = \vec{r}_0 \\
&\vec{b} \subseteq c(A) \implies \text{unique member } \vec{r}_n \subseteq c(A^T) \text{ such that } \vec{r}_n \text{ is a solution to } A\vec{x} = \vec{b}
\end{aligned}
$$

Any solution $A\vec{x} = \vec{b}$ can be written as a combination $\vec{x} = \vec{r}_n +$

$$
\begin{aligned}
&\vec{n}_0, \\
||\vec{x}||^2 &= ||\vec{r}_n + \vec{n}_n||^2 \\
&= ||\vec{r}_n||^2 + ||\vec{n}_n||^2, \\
||\vec{x}||^2 &\geq ||\vec{r}_n||^2, \\
||\vec{x}|| &\geq ||\vec{r}_n||, \\
\vec{b} &\subseteq c(A) \implies \exists \text{ a unique } \vec{r}_n \subseteq c(A^T) \\
&\quad \text{such that } \vec{r}_n \text{ is a solution to } A\vec{x} = \vec{b},
\end{aligned}
$$

No solution can have a smaller length than $\vec{r}_n$ (the solution with minimal length).

Linear Algebra Notes

Matrix Operations and Subspaces

Given:

$$A = \begin{bmatrix} 1 & 6 & -3 \\ 2 & 3 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 6 \\ 19 \end{bmatrix}$$

1. Solve $A\vec{x} = \vec{b}$:

$$\begin{bmatrix} 1 & 6 & -3 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 19 \end{bmatrix}$$

   Use row reduction:

$$\begin{bmatrix} 1 & 6 & -3 & 6 \\ 2 & 3 & 1 & 19 \end{bmatrix} \to \begin{bmatrix} 1 & 6 & -3 & 6 \\ 0 & -9 & 7 & 7 \end{bmatrix}.$$

   Solving:

$$x_3 = t, \quad x_2 = -\frac{2}{3} - \frac{7}{9}t, \quad x_1 = \frac{53}{9} + \frac{11}{9}t.$$

   General solution:

$$x = \begin{bmatrix} \frac{53}{9} \\ -\frac{7}{9} \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{11}{9} \\ -\frac{7}{9} \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Subspaces

- Row Space of A: span $\left(\begin{bmatrix} 1 & 6 & -3 \end{bmatrix}\right)$ - Column Space of A: span $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix}\right)$ - Null Space of A: span $\left(\begin{bmatrix} -11 \\ 7 \\ -9 \end{bmatrix}\right)$

Projections

- Projection of B onto Col(A):

$$\text{Proj}_{\text{Col}(A)}(B) = A(A^T A)^{-1} A^T B.$$

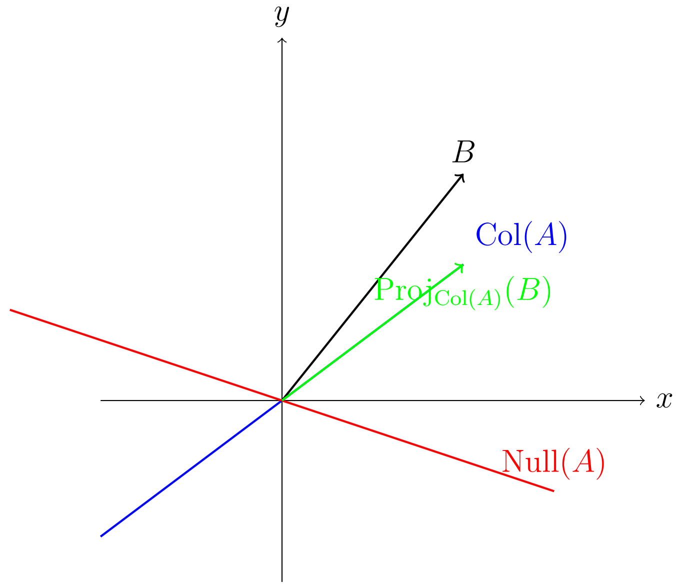
  Calculation:

$$A^T A = \begin{bmatrix} 1 & 2 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 1 & 6 & -3 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 12 \\ 12 & 45 \end{bmatrix}.$$

  Solve $(A^T A)^{-1} A^T B$ for the projection.

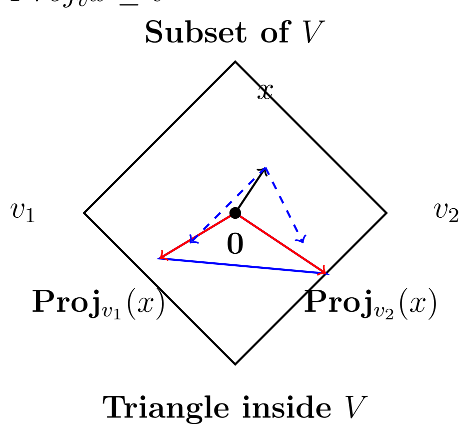
Graphs

Below is the graphical representation of the subspaces:



2 Projection onto a subspace

$\boxed{Proj_v\vec{x} = \vec{v}}$ $Proj_{v^\perp}\vec{x} = \vec{w}$
$Proj_\perp\vec{x}$ is the vector v in L such that $\vec{x} - \vec{v}$ is ortogonal $\vec{w}$to everything in L
$x - Proj_\perp$is orthogonal of L
$\vec{x} = \vec{v}\vec{w}$
$\mathbb{R}^3 Proj\vec{x} =$ some unique vector in v such that$\vec{x}$ is orthogonal of any member of v
$\vec{x} - Proj_v\vec{x} \subseteq v^\perp$



Projection and Linear Transformations

Projection Formula

$$\text{Proj}_V\,\vec{x} = A\hat{y}, \quad \hat{y} \in \mathbb{R}^k$$

If A is linearly independent (L.I.), then A^TA is invertible. This gives:

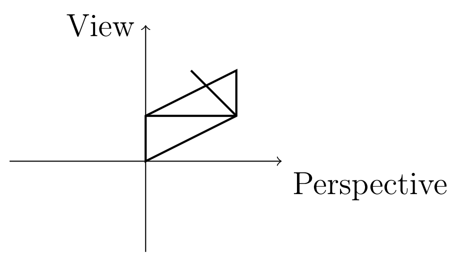
$$\hat{y} = (A^TA)^{-1}A^T\vec{x}, \quad \text{so that} \quad \text{Proj}_V\,\vec{x} = A(A^TA)^{-1}A^T\vec{x}.$$

Matrix Representation

$$\text{Proj}_{\text{Range}(A)}(\vec{x}) = A(A^TA)^{-1}A^T\vec{x}.$$

Some applications:

- Linear transformations.
- 3D graphics programming.
- Point of view projections.



Projection in $\mathbb{R}^4$

Given $V = \text{span}\{\vec{v}_1\} \subset \mathbb{R}^4$,

$$\text{Proj}_V(\vec{x}) = A(A^TA)^{-1}A^T\vec{x},$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A^TA = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}.$$

Computing the Projection

1. Compute $(A^T A)^{-1}$:

$$(A^T A)^{-1} = \frac{1}{3}\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}.$$

2. Compute the full projection matrix:

$$A(A^T A)^{-1}A^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

3. Apply to $\vec{x} \in \mathbb{R}^4$:

$$\text{Proj}_V(\vec{x}) = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

For example:

$$\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 4 \end{bmatrix}, \quad \text{Proj}_V(\vec{x}) = \begin{bmatrix} 3 \\ 3 \\ 4 \\ 0 \end{bmatrix}.$$

Thus,

$$\text{Proj}_V : \mathbb{R}^4 \to \mathbb{R}^4.$$