Complete Guide to Integration Techniques(Gemini Made)

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Contents

1 Basic Integration

1.1 Indefinite Integrals

Here are some basic indefinite integrals:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (\text{for } n \neq -1) \tag{1.1}$$

$$\int \frac{1}{x} \, dx = \ln|x| + C \tag{1.2}$$

$$\int e^x \, dx = e^x + C \tag{1.3}$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C \quad (\text{for } a > 0, a \neq 1) \tag{1.4}$$

$$\int \sin(x) \, dx = -\cos(x) + C \tag{1.5}$$

$$\int \cos(x) \, dx = \sin(x) + C \tag{1.6}$$

$$\int \sec^2(x) \, dx = \tan(x) + C \tag{1.7}$$

$$\int \csc^2(x) \, dx = -\cot(x) + C \tag{1.8}$$

$$\int \sec(x)\tan(x) \, dx = \sec(x) + C \tag{1.9}$$

$$\int \csc(x)\cot(x) \, dx = -\csc(x) + C \tag{1.10}$$

1.2 Definite Integrals

For a definite integral of a continuous function $f(x)$ from a to b, we evaluate:

$$\int_a^b f(x) \, dx = F(b) - F(a) \tag{1.11}$$

where $F(x)$ is any antiderivative of $f(x)$ (i.e., $F'(x) = f(x)$). This is part of the Fundamental Theorem of Calculus.

1.3 Example: Evaluating a Definite Integral

Let's compute $\int_0^1 \cos(\pi x) \, dx$:

To compute this integral, we first find the antiderivative of $\cos(\pi x)$:

$$\int \cos(\pi x) \, dx = \frac{\sin(\pi x)}{\pi} + C \tag{1.12}$$

Now, we evaluate this antiderivative from 0 to 1:

$$\int_0^1 \cos(\pi x) \, dx = \left[\frac{\sin(\pi x)}{\pi}\right]_0^1 \tag{1.13}$$

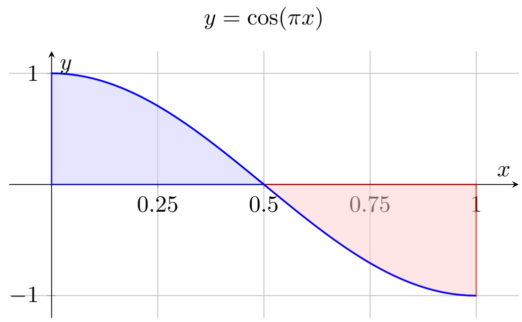
$$= \frac{\sin(\pi)}{\pi} - \frac{\sin(0)}{\pi} \tag{1.14}$$

$$= \frac{0}{\pi} - \frac{0}{\pi} = 0 \tag{1.15}$$

Thus, the integral evaluates to:

$$\int_0^1 \cos(\pi x)\, dx = 0$$

The definite integral represents the net signed area between the curve and the x-axis. In this case, the value of 0 indicates that the area above the x-axis (from $x = 0$ to $x = 0.5$) is equal to the area below the x-axis (from $x = 0.5$ to $x = 1$), resulting in a net cancellation.



2 Integration Techniques

2.1 Integration by Substitution (u-substitution)

Integration by substitution is based on reversing the chain rule for derivatives. If we have an integral of the form $\int f(g(x))g'(x)\,dx$, we can let $u = g(x)$, which implies $du = g'(x)\,dx$. The integral then transforms to:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du \tag{2.1}$$

After integrating with respect to u, we substitute $g(x)$ back in for u.

Example 2.1.1. $\int (3x^2 + 2x)e^{(x^3+x^2)}\,dx$ Let $u = x^3 + x^2$, then $du = (3x^2 + 2x)\,dx$.

$$\int (3x^2 + 2x)e^{(x^3+x^2)}\,dx = \int e^u\,du \tag{2.2}$$
$$= e^u + C \tag{2.3}$$
$$= e^{x^3+x^2} + C \tag{2.4}$$

$$\int (3x^2 + 2x)e^{(x^3+x^2)}\,dx = e^{x^3+x^2} + C$$

Example 2.1.2. $\int \frac{1}{x\ln x}\,dx$ Let $u = \ln x$, then $du = \frac{1}{x}\,dx$.

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{\ln x}\cdot\frac{1}{x}\,dx \tag{2.5}$$
$$= \int \frac{1}{u}\,du \tag{2.6}$$
$$= \ln|u| + C \tag{2.7}$$
$$= \ln|\ln x| + C \tag{2.8}$$

$$\int \frac{1}{x\ln x}\,dx = \ln|\ln x| + C$$

Example 2.1.3. $\int \frac{2^{\ln x}}{x}\,dx$ Let $u = \ln x$, then $du = \frac{1}{x}\,dx$.

$$\int \frac{2^{\ln x}}{x}\,dx = \int 2^u\,du \tag{2.9}$$
$$= \frac{2^u}{\ln 2} + C \tag{2.10}$$
$$= \frac{2^{\ln x}}{\ln 2} + C \tag{2.11}$$

$$\int \frac{2^{\ln x}}{x}\,dx = \frac{2^{\ln x}}{\ln 2} + C$$

Example 2.1.4. $\int (x+3)(x-1)^5\,dx$ Let $u = x-1$, then $x = u+1$ and $dx = du$. Therefore, $x+3 = (u+1)+3 = u+4$.

$$\int (x+3)(x-1)^5\,dx = \int (u+4)u^5\,du \tag{2.12}$$

$$= \int (u^6 + 4u^5)\,du \tag{2.13}$$

$$= \frac{u^7}{7} + \frac{4u^6}{6} + C \tag{2.14}$$

$$= \frac{(x-1)^7}{7} + \frac{2(x-1)^6}{3} + C \tag{2.15}$$

$$\int (x+3)(x-1)^5\,dx = \frac{(x-1)^7}{7} + \frac{2(x-1)^6}{3} + C$$

Example 2.1.5. $\int x^2 2^{x^3}\,dx$ Let $u = x^3$, then $du = 3x^2\,dx$, which implies $x^2\,dx = \frac{du}{3}$.

$$\int x^2 2^{x^3}\,dx = \int 2^u \cdot \frac{1}{3}\,du \tag{2.16}$$

$$= \frac{1}{3}\int 2^u\,du \tag{2.17}$$

$$= \frac{1}{3} \cdot \frac{2^u}{\ln 2} + C \tag{2.18}$$

$$= \frac{2^{x^3}}{3\ln 2} + C \tag{2.19}$$

$$\int x^2 2^{x^3}\,dx = \frac{2^{x^3}}{3\ln 2} + C$$

2.2 Nested Substitutions

For complex integrals, we sometimes need to apply multiple substitutions, or a substitution within a substitution.

Example 2.2.1. Evaluate $\frac{1}{5}\int \frac{\cos(5x)}{e^{\sin(5x)}}\,dx$ Let the integral be $I = \frac{1}{5}\int \cos(5x)e^{-\sin(5x)}\,dx$. First, consider the inner part: $J = \int \cos(5x)e^{-\sin(5x)}\,dx$. Let $u = \sin(5x)$, then $du = 5\cos(5x)\,dx$, so $\cos(5x)\,dx = \frac{1}{5}\,du$.

$$J = \int e^{-u}\left(\frac{1}{5}\,du\right) = \frac{1}{5}\int e^{-u}\,du \tag{2.20}$$

Now, for $\int e^{-u}\,du$, let $w = -u$, then $dw = -du$.

$$\frac{1}{5}\int e^{-u}\,du = \frac{1}{5}\int e^{w}(-dw) \tag{2.21}$$

$$= -\frac{1}{5}\int e^{w}\,dw \tag{2.22}$$

$$= -\frac{1}{5}e^{w} + C_0 \tag{2.23}$$

$$= -\frac{1}{5}e^{-u} + C_0 \tag{2.24}$$

$$= -\frac{1}{5}e^{-\sin(5x)} + C_0 \tag{2.25}$$

So, $J = -\frac{1}{5}e^{-\sin(5x)} + C_0$. The original problem was $I = \frac{1}{5}J$:

$$I = \frac{1}{5}\left(-\frac{1}{5}e^{-\sin(5x)} + C_0\right) = -\frac{1}{25}e^{-\sin(5x)} + \frac{C_0}{5} \tag{2.26}$$

Let $C = C_0/5$.

$$\frac{1}{5}\int \frac{\cos(5x)}{e^{\sin(5x)}}\,dx = -\frac{1}{25}e^{-\sin(5x)} + C$$

2.3 Integration by Parts

Integration by parts is derived from the product rule for derivatives: $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$. Integrating both sides with respect to x and rearranging gives:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \tag{2.27}$$

The formula is commonly written by letting $u = f(x)$ (so $du = f'(x)\,dx$) and $dv = g'(x)\,dx$ (so $v = g(x)$):

$$\int u\,dv = uv - \int v\,du$$

The key is to choose u and dv such that $\int v\,du$ is easier to integrate than the original integral. A common mnemonic for choosing u is LIATE (Logarithmic, Inverse trigonometric, Algebraic, Trigonometric, Exponential).

Example 2.3.1. $\int x\cos x\,dx$ Let:

- $u = x \quad\Rightarrow\quad du = dx$
- $dv = \cos x\,dx \quad\Rightarrow\quad v = \sin x$

Applying the formula:

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx \tag{2.28}$$
$$= x\sin x - (-\cos x) + C \tag{2.29}$$
$$= x\sin x + \cos x + C \tag{2.30}$$

$$\int x\cos x\,dx = x\sin x + \cos x + C$$

Example 2.3.2. $\int \ln x\,dx$ Let:

- $u = \ln x \quad\Rightarrow\quad du = \frac{1}{x}\,dx$
- $dv = dx \quad\Rightarrow\quad v = x$

Applying the formula:

$$\int \ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}\,dx \tag{2.31}$$
$$= x\ln x - \int 1\,dx \tag{2.32}$$
$$= x\ln x - x + C \tag{2.33}$$

$$\int \ln x\,dx = x\ln x - x + C$$

Example 2.3.3. $\int x^2 e^x \, dx$ This requires repeated application of integration by parts.

First application: Let $u_1 = x^2 \Rightarrow du_1 = 2x\,dx$ and $dv_1 = e^x\,dx \Rightarrow v_1 = e^x$.

$$\int x^2 e^x \, dx = x^2 e^x - \int 2xe^x \, dx \tag{2.34}$$

For the new integral $\int 2xe^x\,dx$, apply integration by parts again: Let $u_2 = 2x \Rightarrow du_2 = 2\,dx$ and $dv_2 = e^x\,dx \Rightarrow v_2 = e^x$.

$$\int 2xe^x \, dx = 2xe^x - \int 2e^x \, dx \tag{2.35}$$

$$= 2xe^x - 2e^x + C_0 \tag{2.36}$$

Substituting eq. (2.36) back into eq. (2.34):

$$\int x^2 e^x \, dx = x^2 e^x - (2xe^x - 2e^x + C_0) + C_1 \tag{2.37}$$

$$= x^2 e^x - 2xe^x + 2e^x - C_0 + C_1 \tag{2.38}$$

$$= e^x(x^2 - 2x + 2) + C \quad \text{(where } C = C_1 - C_0\text{)} \tag{2.39}$$

$$\int x^2 e^x \, dx = e^x(x^2 - 2x + 2) + C$$

2.4 The Reverse Chain Rule (Direct Substitution)

The reverse chain rule is essentially a way of spotting a u-substitution directly. If an integrand is of the form $g'(f(x))f'(x)$, we recall that the derivative of $g(f(x))$ is $g'(f(x))f'(x)$. Thus, the integration formula is:

$$\int g'(f(x))f'(x)\,dx = g(f(x)) + C \tag{2.40}$$

This is particularly useful when $f'(x)$ is clearly present (or a constant multiple of it).

Example 2.4.1. $\int (\sin x)^2 \cos x\,dx$ Let $f(x) = \sin x$, then $f'(x) = \cos x$. The integrand is of the form $(f(x))^2 f'(x)$. If we let $g'(u) = u^2$, then $g(u) = \frac{u^3}{3}$.

$$\int (\sin x)^2 \cos x\,dx = \int (f(x))^2 f'(x)\,dx \tag{2.41}$$

$$= g(f(x)) + C \tag{2.42}$$

$$= \frac{(\sin x)^3}{3} + C \tag{2.43}$$

Alternatively, using u-substitution: let $u = \sin x$, $du = \cos x\,dx$. $\int u^2\,du = \frac{u^3}{3} + C = \frac{\sin^3 x}{3} + C$.

$$\int (\sin x)^2 \cos x\,dx = \frac{(\sin x)^3}{3} + C$$

Example 2.4.2. $\int \frac{x}{2}\sin(2x^2+2)\,dx$ Let $f(x) = 2x^2 + 2$, then $f'(x) = 4x$. The integrand involves $\sin(f(x))$. We have $\frac{x}{2}$, and we need $f'(x) = 4x$. Note that $\frac{x}{2} = \frac{1}{8}(4x) = \frac{1}{8}f'(x)$.

$$\int \frac{x}{2}\sin(2x^2+2)\,dx = \int \frac{1}{8}f'(x)\sin(f(x))\,dx \tag{2.44}$$

$$= \frac{1}{8}\int \sin(f(x))f'(x)\,dx \tag{2.45}$$

If we let $g'(u) = \sin(u)$, then $g(u) = -\cos(u)$.

$$\frac{1}{8}\int \sin(f(x))f'(x)\,dx = \frac{1}{8}g(f(x)) + C \tag{2.46}$$

$$= \frac{1}{8}(-\cos(f(x))) + C \tag{2.47}$$

$$= -\frac{\cos(2x^2+2)}{8} + C \tag{2.48}$$

Alternatively, using u-substitution: let $u = 2x^2 + 2$, $du = 4x\,dx \Rightarrow \frac{1}{4}du = x\,dx$. $\int \frac{1}{2}\sin(u)\frac{1}{4}du = \frac{1}{8}\int \sin u\,du = -\frac{1}{8}\cos u + C = -\frac{\cos(2x^2+2)}{8} + C$.

$$\int \frac{x}{2}\sin(2x^2+2)\,dx = -\frac{\cos(2x^2+2)}{8} + C$$

3 Trigonometric Integrals

3.1 Common Integrals involving $\sin^2 x, \cos^2 x, \tan^2 x$

These integrals are typically solved using trigonometric identities:

- For $\int \sin^2 x\,dx$ and $\int \cos^2 x\,dx$, use power-reduction formulas: $\sin^2 x = \frac{1-\cos(2x)}{2}$ and $\cos^2 x = \frac{1+\cos(2x)}{2}$.
- For $\int \tan^2 x\,dx$, use the identity $\tan^2 x = \sec^2 x - 1$.

Results:

$$\int \sin^2 x\,dx = \int \frac{1-\cos(2x)}{2}\,dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C \tag{3.1}$$
$$\int \cos^2 x\,dx = \int \frac{1+\cos(2x)}{2}\,dx = \frac{x}{2} + \frac{\sin(2x)}{4} + C \tag{3.2}$$
$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C \tag{3.3}$$
$$\int \sec^2 x\,dx = \tan x + C \quad \text{(already basic)} \tag{3.4}$$

3.2 Products of Sines and Cosines

Integrals of the form $\int \sin(ax)\cos(bx)\,dx$, $\int \sin(ax)\sin(bx)\,dx$, $\int \cos(ax)\cos(bx)\,dx$ are solved using product-to-sum identities:

- $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)]$
- $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$
- $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$

(Note: The order of $\alpha+\beta$ and $\alpha-\beta$ terms can vary, ensure the identity used is correct.)

Example 3.2.1. $\int \sin(3x)\cos(5x)\,dx$ Using $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$ with $\alpha = 3x, \beta = 5x$:

$$\int \sin(3x)\cos(5x)\,dx = \int \frac{1}{2}[\sin(3x+5x) + \sin(3x-5x)]\,dx \tag{3.5}$$
$$= \int \frac{1}{2}[\sin(8x) + \sin(-2x)]\,dx \tag{3.6}$$
$$= \int \frac{1}{2}[\sin(8x) - \sin(2x)]\,dx \quad (\text{since } \sin(-A) = -\sin A) \tag{3.7}$$
$$= \frac{1}{2}\left[-\frac{\cos(8x)}{8} - \left(-\frac{\cos(2x)}{2}\right)\right] + C \tag{3.8}$$
$$= -\frac{\cos(8x)}{16} + \frac{\cos(2x)}{4} + C \tag{3.9}$$

$$\int \sin(3x)\cos(5x)\,dx = -\frac{\cos(8x)}{16} + \frac{\cos(2x)}{4} + C$$

3.3 Powers of Sines and Cosines ($\int \sin^m x \cos^n x \, dx$)

Strategy depends on whether m or n (or both) are odd or even.

- **If m is odd:** Save one $\sin x$ factor, convert remaining $\sin^2 x$ terms to $1 - \cos^2 x$, then use $u = \cos x$.
- **If n is odd:** Save one $\cos x$ factor, convert remaining $\cos^2 x$ terms to $1 - \sin^2 x$, then use $u = \sin x$.
- **If both m and n are even:** Use power-reduction formulas $\sin^2 x = \frac{1-\cos(2x)}{2}$ and $\cos^2 x = \frac{1+\cos(2x)}{2}$ repeatedly.

(An example for one of these cases would be beneficial here).

3.4 Powers of Tangents and Secants ($\int \tan^m x \sec^n x \, dx$)

Strategy depends on the powers m and n.

- **If n is even ($n = 2k, k \geq 1$):** Save a $\sec^2 x$ factor, convert remaining $\sec^{n-2} x$ terms to $(1 + \tan^2 x)^{(k-1)}$, then use $u = \tan x$.
- **If m is odd ($m = 2k+1, k \geq 0$) and $n \geq 1$:** Save a $\sec x \tan x$ factor, convert remaining $\tan^{m-1} x$ terms to $(\sec^2 x - 1)^k$, then use $u = \sec x$.
- **Other cases:** Can be more complex. May involve converting everything to sines and cosines, or using reduction formulas. $\int \tan x \, dx = \ln|\sec x| + C$, $\int \sec x \, dx = \ln|\sec x + \tan x| + C$.

(An example would be good here).

3.5 Trigonometric Substitution

Used for integrals containing expressions like $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$.

1. For $\sqrt{a^2 - x^2}$: Let $x = a \sin\theta$, then $dx = a \cos\theta \, d\theta$. $\sqrt{a^2 - x^2} = a\cos\theta$.
2. For $\sqrt{a^2 + x^2}$: Let $x = a \tan\theta$, then $dx = a \sec^2\theta \, d\theta$. $\sqrt{a^2 + x^2} = a\sec\theta$.
3. For $\sqrt{x^2 - a^2}$: Let $x = a \sec\theta$, then $dx = a \sec\theta \tan\theta \, d\theta$. $\sqrt{x^2 - a^2} = a\tan\theta$.

After integration, a reference triangle is used to convert back to x.

Example 3.5.1. $\int \frac{1}{x^2\sqrt{4-x^2}} \, dx$ This contains $\sqrt{a^2 - x^2}$ with $a = 2$. Let $x = 2\sin\theta$. Then $dx = 2\cos\theta \, d\theta$. And $\sqrt{4 - x^2} = \sqrt{4 - 4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2\cos\theta$ (assuming $2\cos\theta > 0$). Also $x^2 = 4\sin^2\theta$.

$$\int \frac{1}{x^2\sqrt{4-x^2}} \, dx = \int \frac{1}{(4\sin^2\theta)(2\cos\theta)}(2\cos\theta \, d\theta) \tag{3.10}$$

$$= \int \frac{1}{4\sin^2\theta} \, d\theta \tag{3.11}$$

$$= \frac{1}{4}\int \csc^2\theta \, d\theta \tag{3.12}$$

$$= -\frac{1}{4}\cot\theta + C \tag{3.13}$$

Now convert back to x. Since $x = 2\sin\theta$, $\sin\theta = x/2$. Construct a right triangle where the opposite side is x and hypotenuse is 2. The adjacent side is $\sqrt{2^2 - x^2} = \sqrt{4 - x^2}$. So, $\cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{\sqrt{4-x^2}}{x}$.

$$-\frac{1}{4}\cot\theta + C = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C \tag{3.14}$$

$$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = -\frac{\sqrt{4-x^2}}{4x} + C$$

4 Rational Functions

Integration of rational functions $\frac{P(x)}{Q(x)}$ often involves partial fraction decomposition.

4.1 Preliminary Step: Long Division

If the degree of the numerator $P(x)$ is greater than or equal to the degree of the denominator $Q(x)$, perform polynomial long division first. This will result in:

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where $S(x)$ is a polynomial (easy to integrate) and $\frac{R(x)}{Q(x)}$ is a proper rational function (degree of $R(x)$ is less than degree of $Q(x)$).

4.2 Partial Fractions Decomposition

For a proper rational function $\frac{R(x)}{Q(x)}$, decompose it into simpler fractions. The form of the decomposition depends on the factors of $Q(x)$:

1. **Distinct Linear Factors:** If $Q(x) = (a_1x + b_1)\dots(a_kx + b_k)$, then

   $$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \cdots + \frac{A_k}{a_kx + b_k}$$

2. **Repeated Linear Factors:** If $(ax + b)^k$ is a factor of $Q(x)$, the decomposition includes

   $$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_k}{(ax + b)^k}$$

3. **Irreducible Quadratic Factors:** If $(ax^2 + bx + c)^k$ with $b^2 - 4ac < 0$ is a factor of $Q(x)$, the decomposition includes

   $$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$$

   (Integrals of terms with quadratic denominators often lead to arctangents or logarithms after completing the square and substitution.)

Example 4.2.1. $\int \frac{2x+1}{x^2-1}\,dx$ (Distinct Linear Factors) The degree of the numerator (1) is less than the degree of the denominator (2), so no long division is needed. First, factor the denominator: $x^2 - 1 = (x - 1)(x + 1)$. Decompose $\frac{2x+1}{x^2-1}$:

$$\frac{2x + 1}{(x - 1)(x + 1)} = \frac{A}{x - 1} + \frac{B}{x + 1} \tag{4.1}$$

Multiply both sides by $(x - 1)(x + 1)$:

$$\begin{aligned} 2x + 1 &= A(x + 1) + B(x - 1) && (4.2)\\ &= Ax + A + Bx - B && (4.3)\\ &= (A + B)x + (A - B) && (4.4) \end{aligned}$$

Comparing coefficients of powers of x:

- Coefficient of x: $A + B = 2$
- Constant term: $A - B = 1$

Solving this system: Adding the two equations gives $2A = 3 \Rightarrow A = \frac{3}{2}$. Substituting A into $A + B = 2$ gives $\frac{3}{2} + B = 2 \Rightarrow B = 2 - \frac{3}{2} = \frac{1}{2}$. So:

$$\frac{2x+1}{x^2-1} = \frac{3/2}{x-1} + \frac{1/2}{x+1} \tag{4.5}$$

Now integrate:

$$\begin{align}
\int \frac{2x+1}{x^2-1}\,dx &= \int \left(\frac{3/2}{x-1} + \frac{1/2}{x+1}\right) dx \tag{4.6}\\
&= \frac{3}{2}\int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx \tag{4.7}\\
&= \frac{3}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C \tag{4.8}
\end{align}$$

$$\int \frac{2x+1}{x^2-1}\,dx = \frac{3}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C$$

5 Integration Using Tables

For many standard or complex integrals, tables of integrals can be a very efficient tool. These tables list pre-computed antiderivatives for various forms. It's important to correctly match the integrand to the form in the table and apply any necessary substitutions.

5.1 Exponential and Logarithmic Functions

Some common table entries include:

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C \tag{5.1}$$

$$\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx \quad \text{(Reduction formula)} \tag{5.2}$$

$$\int \ln(ax)\,dx = x\ln(ax) - x + C \tag{5.3}$$

$$\int x^n \ln(x)\,dx = \frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right) + C, \quad (n \neq -1) \tag{5.4}$$

5.2 Trigonometric Functions

Some common table entries (beyond basic ones):

$$\int \tan x\,dx = \ln|\sec x| + C = -\ln|\cos x| + C \tag{5.5}$$

$$\int \cot x\,dx = \ln|\sin x| + C \tag{5.6}$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C \tag{5.7}$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + C \tag{5.8}$$

When using tables, pay close attention to the conditions (e.g., values of n, a) under which the formula is valid.

6 Improper Integrals

Improper integrals involve integrating over an infinite interval or integrating a function with an infinite discontinuity within the interval.

6.1 Type 1: Infinite Intervals of Integration

1. If $\int_a^t f(x)\,dx$ exists for every $t \geq a$:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

2. If $\int_t^b f(x)\,dx$ exists for every $t \leq b$:

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

3. If both $\int_{-\infty}^{c} f(x)\,dx$ and $\int_{c}^{\infty} f(x)\,dx$ are convergent:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

If the limit exists and is finite, the improper integral converges; otherwise, it diverges.

Example 6.1.1. Evaluate $\int_1^\infty \frac{1}{x^2}\,dx$

$$\begin{aligned}
\int_1^\infty \frac{1}{x^2}\,dx &= \lim_{t\to\infty} \int_1^t x^{-2}\,dx && (6.1)\\
&= \lim_{t\to\infty} \left[\frac{x^{-1}}{-1}\right]_1^t && (6.2)\\
&= \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t && (6.3)\\
&= \lim_{t\to\infty} \left(-\frac{1}{t} - \left(-\frac{1}{1}\right)\right) && (6.4)\\
&= \lim_{t\to\infty} \left(-\frac{1}{t} + 1\right) && (6.5)\\
&= 0 + 1 = 1 && (6.6)
\end{aligned}$$

The integral converges to 1.

$$\int_1^\infty \frac{1}{x^2}\,dx = 1$$

6.2 Type 2: Discontinuous Integrands

1. If f is continuous on $[a, b)$ and discontinuous at b:

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

2. If f is continuous on $(a, b]$ and discontinuous at a:

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

3. If f has a discontinuity at c where $a < c < b$:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

(provided both integrals on the right converge).

If the limit exists and is finite, the integral converges; otherwise, it diverges.

Example 6.2.1. Evaluate $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ The function $f(x) = \frac{1}{\sqrt{x}}$ is discontinuous at $x = 0$.

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^1 x^{-1/2}\,dx \tag{6.7}$$

$$= \lim_{t\to 0^+} \left[\frac{x^{1/2}}{1/2}\right]_t^1 \tag{6.8}$$

$$= \lim_{t\to 0^+} \left[2\sqrt{x}\right]_t^1 \tag{6.9}$$

$$= \lim_{t\to 0^+} (2\sqrt{1} - 2\sqrt{t}) \tag{6.10}$$

$$= 2 - 2\sqrt{0} = 2 \tag{6.11}$$

The integral converges to 2.

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$$

7 Numerical Integration

When analytical integration (finding an exact antiderivative) is difficult or impossible, numerical methods are used to approximate the value of a definite integral.

7.1 The Trapezoidal Rule

The trapezoidal rule approximates the definite integral $\int_a^b f(x)\,dx$ by dividing the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$, and approximating the area under the curve in each subinterval by a trapezoid. The formula is:

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right] \tag{7.1}$$

where $x_i = a + i\Delta x$. This can be written as:

$$T_n = \frac{b-a}{2n}\left[f(a) + f(b) + 2\sum_{i=1}^{n-1} f(a + i\Delta x)\right]$$

7.2 Simpson's Rule

Simpson's rule provides a more accurate approximation by using parabolas to approximate the curve over pairs of subintervals. It requires n to be an even number of subintervals. The formula is:

$$\begin{split}\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots\right. \\ \left. \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]\end{split} \tag{7.2}$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$. This can be written as:

$$S_n = \frac{b-a}{3n}\left[f(a) + f(b) + 4\sum_{i=1}^{n/2} f(a + (2i-1)\Delta x) + 2\sum_{i=1}^{n/2-1} f(a + 2i\Delta x)\right]$$

Example 7.2.1. Using Simpson's rule to approximate $\int_0^1 e^{-x^2}dx$ with $n = 4$ Here $a = 0, b = 1, n = 4$. So $\Delta x = \frac{1-0}{4} = 0.25$. Our points are $x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1$. The function values are (approximated):

$$\begin{aligned}
f(x_0) &= e^{-(0)^2} = 1 \\
f(x_1) &= e^{-(0.25)^2} \approx 0.939413 \\
f(x_2) &= e^{-(0.5)^2} \approx 0.778801 \\
f(x_3) &= e^{-(0.75)^2} \approx 0.569783 \\
f(x_4) &= e^{-(1)^2} \approx 0.367879
\end{aligned}$$

Using Simpson's rule ($n = 4$):

$$\int_0^1 e^{-x^2} dx \approx S_4 = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)] \tag{7.3}$$
$$\approx \frac{0.25}{3}[1 + 4(0.939413) + 2(0.778801) + 4(0.569783) + 0.367879] \tag{7.4}$$
$$\approx \frac{0.25}{3}[1 + 3.757652 + 1.557602 + 2.279132 + 0.367879] \tag{7.5}$$
$$\approx \frac{0.25}{3}[8.962265] \tag{7.6}$$
$$\approx 0.0833333 \times 8.962265 \approx 0.746855 \tag{7.7}$$

The document previously had 0.7469, which is consistent with rounding.

$$\int_0^1 e^{-x^2} dx \approx 0.74686 \text{ (using Simpson's rule with } n = 4)$$

8 Applications of Integration

8.1 Area Between Curves

The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is:

$$A = \int_a^b [f(x) - g(x)]\, dx \tag{8.1}$$

If the condition $f(x) \geq g(x)$ is not guaranteed, then $A = \int_a^b |f(x) - g(x)|\, dx$. One may need to find points of intersection to determine a, b or intervals where $f(x) \geq g(x)$ or $g(x) \geq f(x)$.

8.2 Volume of Revolution: Disk/Washer Method

The volume of a solid obtained by rotating the region under the curve $y = f(x)$ (where $f(x) \geq 0$) from $x = a$ to $x = b$ around the x-axis is given by the disk method:

$$V = \pi \int_a^b [f(x)]^2\, dx \quad \text{(Disk Method, around x-axis)} \tag{8.2}$$

If rotating the region between two curves $y = f(x)$ and $y = g(x)$ (with $f(x) \geq g(x) \geq 0$) around the x-axis, the washer method is used:

$$V = \pi \int_a^b ([f(x)]^2 - [g(x)]^2)\, dx \quad \text{(Washer Method, around x-axis)} \tag{8.3}$$

Similar formulas exist for rotation around the y-axis (integrating with respect to y).

8.3 Volume of Revolution: Cylindrical Shell Method

When rotating a region bounded by $y = f(x)$, $x = a$, $x = b$, and the x-axis around the y-axis, the cylindrical shell method is often useful:

$$V = 2\pi \int_a^b x f(x)\, dx \quad \text{(Shell Method, around y-axis, region under } f(x)\text{)} \tag{8.4}$$

Here, x is the radius of a cylindrical shell and $f(x)$ is its height. If rotating the region between $y = f(x)$ and $y = g(x)$ (with $f(x) \geq g(x)$) around the y-axis:

$$V = 2\pi \int_a^b x[f(x) - g(x)]\, dx \quad \text{(Shell Method, around y-axis)} \tag{8.5}$$

Similar formulas exist for rotation around the x-axis (integrating with respect to y).

8.4 Arc Length

The arc length L of a smooth curve $y = f(x)$ from $x = a$ to $x = b$ is given by:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \tag{8.6}$$

If the curve is given parametrically as $x = x(t), y = y(t)$ for $t_1 \leq t \leq t_2$:

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{8.7}$$

8.5 Surface Area of Revolution

The area S of the surface generated by revolving the curve $y = f(x)$, $a \leq x \leq b$, about the x-axis (assuming $f(x) \geq 0$) is:

$$S = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx \tag{8.8}$$

If revolving about the y-axis (for a curve $x = g(y)$, $c \leq y \leq d$, with $g(y) \geq 0$):

$$S = 2\pi \int_c^d g(y)\sqrt{1 + [g'(y)]^2}\, dy \tag{8.9}$$

Or, if revolving $y = f(x)$, $a \leq x \leq b$, about the y-axis (assuming $x \geq 0$ over the interval):

$$S = 2\pi \int_a^b x\sqrt{1 + [f'(x)]^2}\, dx \tag{8.10}$$