Matrices: Foundations and Applications

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Sommario

Matrices are fundamental mathematical constructs used to represent and solve complex problems across various scientific and engineering disciplines. They consist of a rectangular array of numbers, symbols, or expressions arranged in rows and columns, enabling operations such as addition, subtraction, and multiplication. Serving as the foundation of linear algebra, matrices are indispensable for handling linear transformations, solving systems of linear equations, and performing vector operations. Their applications span diverse fields, including computer graphics, quantum mechanics, and machine learning. Understanding matrices enhances computational skills and supports significant technological advancements and theoretical developments.

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1 Introduction

A matrix is a rectangular array of numbers arranged in rows and columns. The general form of an $m \times n$ matrix A is:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where a_{ij} represents the element in the ith row and jth column. For example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

This is a 3×3 matrix, which is a square matrix since it has the same number of rows and columns.

1.1 Matrix Notation and Terminology

- A matrix with m rows and n columns is called an $m \times n$ matrix
- A matrix with only one row is called a *row vector*
- A matrix with only one column is called a *column vector*
- A matrix with the same number of rows and columns ($m = n$) is called a *square matrix*
- The elements $a_{11}, a_{22}, \ldots, a_{nn}$ form the *main diagonal* of the matrix

1.2 Matrices and Systems of Linear Equations

Matrices are particularly useful for representing systems of linear equations. For instance, the system:

$$\begin{aligned} x + 2y + 3z &= 10 \\ 4x + 5y + 6z &= 20 \\ 7x + 8y + 9z &= 30 \end{aligned}$$

can be represented in matrix form as:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

or more compactly as $AX = B$, where A is the coefficient matrix, X is the variable vector, and B is the constant vector.

2 Matrix Operations

2.1 Addition and Subtraction

Matrices can only be added or subtracted if they have the same dimensions. The result is obtained by adding or subtracting corresponding elements.

Definition 2.1 *For matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same dimensions $m \times n$:*

$$A + B = [a_{ij} + b_{ij}] \tag{1}$$

$$A - B = [a_{ij} - b_{ij}] \tag{2}$$

Example 2.1 *Given:*

$$A = \begin{bmatrix} 1 & -7 & 5 \\ 0 & 3 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 0 & 3 \\ 11 & -1 & -7 \end{bmatrix}$$

Addition:

$$A + B = \begin{bmatrix} 1+5 & -7+0 & 5+3 \\ 0+11 & 3+(-1) & -10+(-7) \end{bmatrix} = \begin{bmatrix} 6 & -7 & 8 \\ 11 & 2 & -17 \end{bmatrix}$$

Subtraction:

$$A - B = \begin{bmatrix} 1-5 & -7-0 & 5-3 \\ 0-11 & 3-(-1) & -10-(-7) \end{bmatrix} = \begin{bmatrix} -4 & -7 & 2 \\ -11 & 4 & -3 \end{bmatrix}$$

2.2 Scalar Multiplication

A scalar multiplied by a matrix results in each element of the matrix being multiplied by that scalar.

Definition 2.2 *For a scalar c and a matrix $A = [a_{ij}]$:*

$$cA = [c \cdot a_{ij}]$$

Example 2.2

$$3 \times \begin{bmatrix} 7 & 5 & -10 \\ 3 & 8 & 0 \end{bmatrix} = \begin{bmatrix} 3 \cdot 7 & 3 \cdot 5 & 3 \cdot (-10) \\ 3 \cdot 3 & 3 \cdot 8 & 3 \cdot 0 \end{bmatrix} = \begin{bmatrix} 21 & 15 & -30 \\ 9 & 24 & 0 \end{bmatrix}$$

2.3 Matrix Multiplication

Definition 2.3 *For an $m \times n$ matrix A and an $n \times p$ matrix B, their product $C = AB$ is an $m \times p$ matrix where:*

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

Example 2.3

$$\begin{bmatrix} 2 & -2 \\ 5 & 3 \end{bmatrix} \times \begin{bmatrix} -1 & 4 \\ 7 & -6 \end{bmatrix} = \begin{bmatrix} 2 \cdot (-1) + (-2) \cdot 7 & 2 \cdot 4 + (-2) \cdot (-6) \\ 5 \cdot (-1) + 3 \cdot 7 & 5 \cdot 4 + 3 \cdot (-6) \end{bmatrix} = \begin{bmatrix} -16 & 20 \\ 16 & -2 \end{bmatrix}$$

Important Note

Matrix multiplication is not commutative in general: $AB \neq BA$. The order of multiplication matters.

2.4 Identity Matrix

Definition 2.4 *The identity matrix I_n is an $n \times n$ square matrix with 1's on the main diagonal and 0's elsewhere:*

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

For any $n \times n$ matrix A, we have $AI_n = I_nA = A$.

Example 2.4 *For $n = 2$:*

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For $n = 3$:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.5 Transpose of a Matrix

Definition 2.5 *The transpose of an $m \times n$ matrix A, denoted by A^T, is an $n \times m$ matrix formed by exchanging the rows and columns of A:*

$$(A^T)_{ij} = A_{ji}$$

Example 2.5 *If:*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Then:

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

2.6 Properties of Matrix Operations

Property	**Equation**
Associative property of multiplication	$(AB)C = A(BC)$
Distributive properties	$A(B + C) = AB + AC$ $(A + B)C = AC + BC$
Scalar multiplication	$c(AB) = (cA)B = A(cB)$ $c(A + B) = cA + cB$ $(c + d)A = cA + dA$
Transpose properties	$(A + B)^T = A^T + B^T$ $(AB)^T = B^T A^T$ $(A^T)^T = A$ $(cA)^T = cA^T$
Identity properties	$AI_n = I_m A = A$ (for A of size $m \times n$) $I_n^T = I_n$
Zero matrix properties	$A + 0 = 0 + A = A$ $A \cdot 0 = 0 \cdot A = 0$

3 Special Types of Matrices

3.1 Diagonal Matrix

A diagonal matrix has non-zero elements only on the main diagonal.

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

Diagonal matrices have several useful properties:

- The product of two diagonal matrices is diagonal
- The inverse of a diagonal matrix (if it exists) is also diagonal with elements $1/d_i$
- Diagonal matrices are easy to raise to powers: D^k has elements d_i^k on the diagonal

3.2 Symmetric and Skew-Symmetric Matrices

Definition 3.1 *A square matrix A is symmetric if $A = A^T$, meaning $a_{ij} = a_{ji}$ for all i, j.*

Example 3.1

$$S = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Definition 3.2 *A square matrix A is skew-symmetric if $A = -A^T$, meaning $a_{ij} = -a_{ji}$ for all i, j. This implies that $a_{ii} = 0$ for all i.*

Example 3.2

$$K = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix}$$

3.3 Triangular Matrices

Definition 3.3 *An upper triangular matrix has all elements below the main diagonal equal to zero.*

Example 3.3

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Definition 3.4 *A lower triangular matrix has all elements above the main diagonal equal to zero.*

Example 3.4

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

3.4 Orthogonal Matrix

Definition 3.5 *A square matrix Q is orthogonal if $Q^T Q = QQ^T = I$, or equivalently, $Q^T = Q^{-1}$.*

Orthogonal matrices represent rotations and reflections in Euclidean space and preserve length and angles.

Example 3.5 *The rotation matrix in 2D:*

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

4 Determinants

The determinant is a scalar value that can be calculated from a square matrix and has important geometric and algebraic interpretations.

4.1 Definition and Basic Properties

Definition 4.1 *The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is:*

$$\det(A) = |A| = ad - bc$$

For a 3×3 matrix:

$$\begin{aligned} \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ei - fh) - b(di - fg) + c(dh - eg) \end{aligned}$$

For larger matrices, determinants can be computed using cofactor expansion, which recursively applies the formula for smaller determinants.

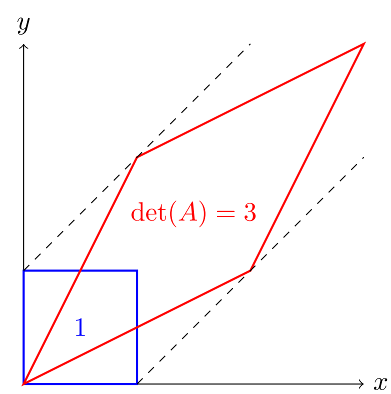
4.2 Properties of Determinants

1. $\det(I_n) = 1$
2. $\det(AB) = \det(A) \cdot \det(B)$
3. $\det(A^T) = \det(A)$
4. $\det(cA) = c^n \det(A)$ for an $n \times n$ matrix A
5. If A has a row or column of zeros, then $\det(A) = 0$
6. If A has two identical rows or columns, then $\det(A) = 0$
7. The determinant changes sign when two rows or columns are interchanged
8. Adding a multiple of one row (or column) to another doesn't change the determinant

4.3 Geometric Interpretation

The determinant of a matrix represents the scaling factor of the linear transformation defined by the matrix on volumes:

- For a 2×2 matrix, the determinant equals the area of the parallelogram formed by the transformed unit vectors
- For a 3×3 matrix, the determinant equals the volume of the parallelepiped formed by the transformed unit vectors



5 Matrix Inverse

Definition 5.1 *The inverse of a square matrix A, denoted as A^{-1}, is a matrix that satisfies:*

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

Not all matrices have inverses. A matrix is invertible (or nonsingular) if and only if its determinant is non-zero.

5.1 Calculating the Inverse of a 2×2 Matrix

For a 2×2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If $\det(A) = ad - bc \neq 0$, then:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 5.1

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

$\det(A) = 3 \cdot 2 - 1 \cdot 2 = 6 - 2 = 4$

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

Verification:

$$AA^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 3 \cdot \frac{1}{2} + 1 \cdot (-\frac{1}{2}) & 3 \cdot (-\frac{1}{4}) + 1 \cdot \frac{3}{4} \\ 2 \cdot \frac{1}{2} + 2 \cdot (-\frac{1}{2}) & 2 \cdot (-\frac{1}{4}) + 2 \cdot \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

5.2 Calculating the Inverse of Larger Matrices

For larger matrices, the inverse can be calculated in several ways:

1. Using the adjugate (classical adjoint) matrix: $A^{-1} = \frac{1}{\det(A)}(A)$
2. Using elementary row operations to transform $[A|I]$ into $[I|A^{-1}]$
3. Using numerical methods such as Gaussian elimination

Algorithm 1 Calculating Matrix Inverse using Row Operations

Create the augmented matrix $[A|I]$
Perform row operations to transform A to I
The right side will become A^{-1}

5.3 Properties of Matrix Inverses

1. $(A^{-1})^{-1} = A$
2. $(A^T)^{-1} = (A^{-1})^T$
3. $(AB)^{-1} = B^{-1}A^{-1}$
4. $(A^n)^{-1} = (A^{-1})^n$

5.4 Applications of Matrix Inverses

Matrix inverses are essential for solving systems of linear equations. The system $AX = B$ has the solution $X = A^{-1}B$ when A is invertible.

Example 5.2 *Solve the system:*

$$\begin{cases} 3x + y = 7 \\ 2x + 2y = 8 \end{cases}$$

Matrix form: $\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

Using the inverse from the previous example:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \cdot 7 + (-\frac{1}{4}) \cdot 8 \\ (-\frac{1}{2}) \cdot 7 + \frac{3}{4} \cdot 8 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} - \frac{8}{4} \\ -\frac{7}{2} + \frac{24}{4} \end{bmatrix} = \begin{bmatrix} \frac{7}{2} - \frac{2}{1} \\ -\frac{7}{2} + \frac{6}{1} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{5}{2} \end{bmatrix}$$

Therefore, $x = \frac{3}{2}$ *and* $y = \frac{5}{2}$.

6 Transforming Vectors Using Matrices

Matrices can be used to represent linear transformations of vectors. When a matrix multiplies a vector, it transforms the vector according to the linear mapping defined by the matrix.

6.1 Basic Transformations

1. **Scaling**: $S = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$
2. **Rotation** (counter-clockwise by angle θ): $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
3. **Reflection** (about y-axis): $M = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
4. **Shear** (in x-direction): $H = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

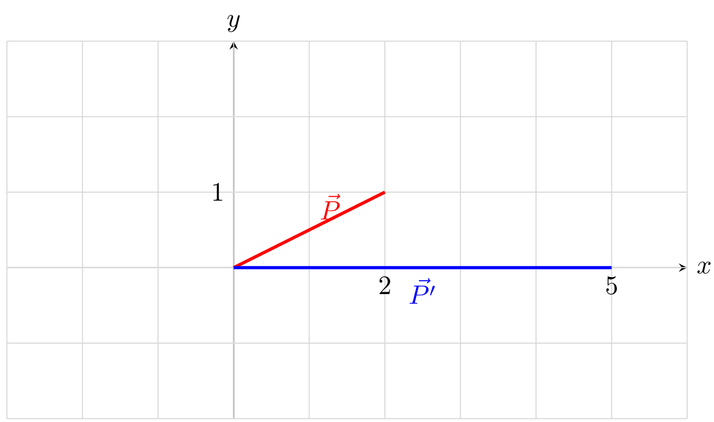
6.2 Example: Vector Transformation

Given:

$$\vec{P} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{T} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}.$$

The transformation produces:

$$\vec{P'} = \vec{T} \cdot \vec{P} = \begin{bmatrix} 2 \cdot 2 + 1 \cdot 1 \\ -1 \cdot 2 + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}.$$



The matrix $\vec{T}$ can be understood as a combination of rotation and scaling. In this case, it rotates the vector and changes its length.

7 Transforming Polygons Using Matrices

We can transform entire shapes by applying a transformation matrix to each vertex of the polygon.

7.1 Example: Polygon Transformation

Given points:

$$\vec{P_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{P_2} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \quad \vec{P_3} = vecP_3 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Transformation matrix:

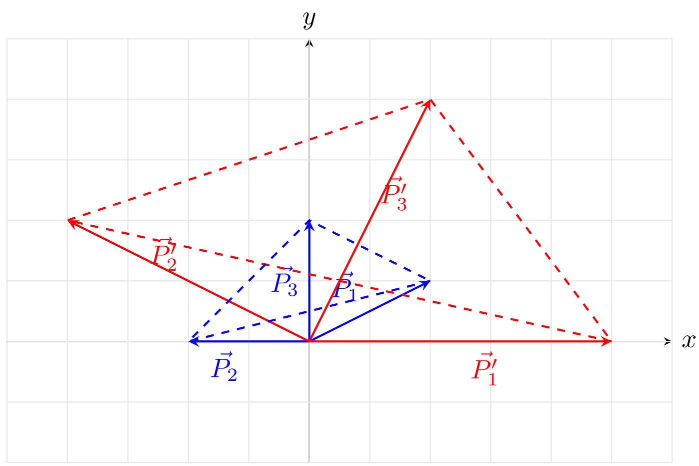
$$\vec{T} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}.$$

Transformed points:

$$\vec{P}_1' = \vec{T}\vec{P}_1 = \begin{bmatrix} 2 \cdot 2 + 1 \cdot 1 \\ -1 \cdot 2 + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$\vec{P}_2' = \vec{T}\vec{P}_2 = \begin{bmatrix} 2 \cdot (-2) + 1 \cdot 0 \\ -1 \cdot (-2) + 2 \cdot 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

$$\vec{P}_3' = \vec{T}\vec{P}_3 = \begin{bmatrix} 2 \cdot 0 + 1 \cdot 2 \\ -1 \cdot 0 + 2 \cdot 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$



This transformation has rotated, scaled, and skewed the original triangle.

8 Eigenvalues and Eigenvectors

Definition 8.1 *An eigenvector of a square matrix A is a non-zero vector $\vec{v}$ such that when A multiplies $\vec{v}$, the result is a scalar multiple of $\vec{v}$:*

$$A\vec{v} = \lambda\vec{v}$$

where λ is a scalar called the eigenvalue corresponding to $\vec{v}$.

8.1 Finding Eigenvalues and Eigenvectors

To find the eigenvalues and eigenvectors of a matrix A:

1. Solve the characteristic equation: $\det(A - \lambda I) = 0$ for eigenvalues λ
2. For each eigenvalue λ, solve $(A - \lambda I)\vec{v} = \vec{0}$ to find the corresponding eigenvector(s)

Example 8.1 *Find the eigenvalues and eigenvectors of:*

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Step 1: Find the eigenvalues by solving $\det(A - \lambda I) = 0$

$$\det \begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} = 0$$

$$(3-\lambda)^2 - 1 = 0$$

$$(3-\lambda)^2 = 1$$

$$3 - \lambda = \pm 1$$

$$\lambda = 3 \pm 1 = 2 \text{ or } 4$$

Step 2: Find the eigenvectors
For $\lambda = 2$:

$$(A - 2I)\vec{v} = \begin{bmatrix} 3-2 & 1 \\ 1 & 3-2 \end{bmatrix} \vec{v} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{v} = \vec{0}$$

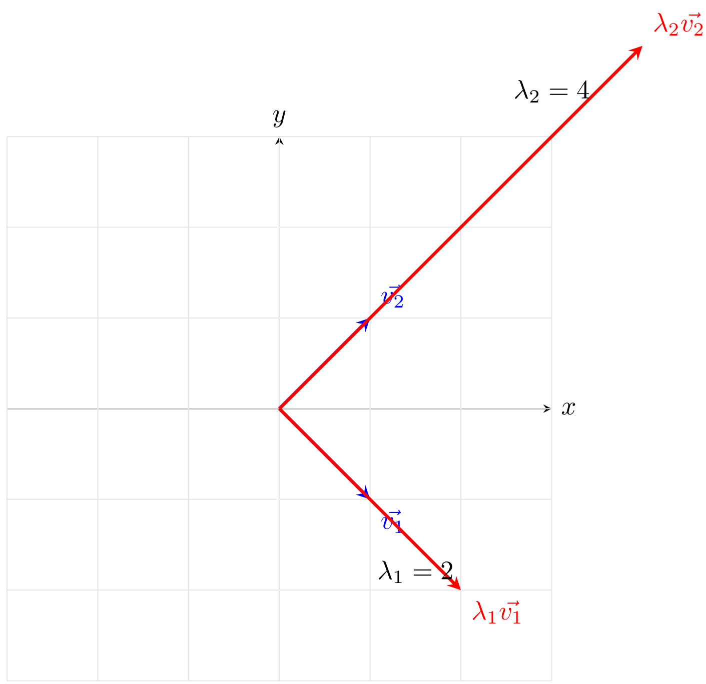
Solving $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, *we get* $v_1 + v_2 = 0$, *so* $v_2 = -v_1$.
Let $v_1 = 1$, *then* $v_2 = -1$. *So an eigenvector is* $\vec{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
For $\lambda = 4$:

$$(A - 4I)\vec{v} = \begin{bmatrix} 3-4 & 1 \\ 1 & 3-4 \end{bmatrix} \vec{v} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \vec{v} = \vec{0}$$

Solving $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, *we get* $-v_1 + v_2 = 0$, *so* $v_2 = v_1$.
Let $v_1 = 1$, *then* $v_2 = 1$. *So an eigenvector is* $\vec{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

8.2 Geometric Interpretation

Eigenvectors represent directions that are stretched or compressed by the linear transformation, without changing their direction. The corresponding eigenvalue represents the factor by which the vector is stretched or compressed.



8.3 Properties and Applications of Eigenvalues and Eigenvectors

1. The trace of a matrix (sum of diagonal elements) equals the sum of its eigenvalues
2. The determinant of a matrix equals the product of its eigenvalues

3. A matrix is invertible if and only if all its eigenvalues are non-zero

4. Eigenvalues and eigenvectors are crucial in:

   - Principal Component Analysis (PCA) in statistics and machine learning
   - Solving systems of differential equations
   - Analyzing stability of dynamic systems
   - Quantum mechanics (where eigenvalues represent observable quantities)
   - Google's PageRank algorithm (which uses the dominant eigenvector of a web matrix)

9 Matrix Rank and Nullity

9.1 Matrix Rank

Definition 9.1 *The rank of a matrix is the dimension of the vector space generated by its columns (or equivalently, by its rows). It equals the number of linearly independent columns (or rows) of the matrix.*

Example 9.1 *Consider the matrix:*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 5 & 8 \end{bmatrix}$$

The second row is twice the first row, and the third row is the sum of the first and second rows. Therefore, only one row is linearly independent, so $\mathit{rank}(A) = 1$.

Example 9.2 *Consider the matrix:*

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

The first two rows are linearly independent, while the third row is all zeros. Therefore, $\mathit{rank}(B) = 2$.

9.2 Methods for Finding Rank

1. Use elementary row operations to transform the matrix to row echelon form. The rank equals the number of non-zero rows.

2. Compute the determinants of all square submatrices. The rank is the size of the largest submatrix with a non-zero determinant.

9.3 Properties of Matrix Rank

1. $0 \leq \text{rank}(A) \leq \min(m, n)$ for an $m \times n$ matrix A

2. $\text{rank}(A) = \text{rank}(A^T)$

3. $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$

4. $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$

5. A square matrix is invertible if and only if its rank equals its size

9.4 Nullity and the Rank-Nullity Theorem

Definition 9.2 *The nullity of an* $m \times n$ *matrix* A *is the dimension of its null space (or kernel), which is the set of all vectors* $\vec{x}$ *such that* $A\vec{x} = \vec{0}$.

Theorem 9.1 (Rank-Nullity Theorem) *For an* $m \times n$ *matrix* A*:*

$$\mathit{rank}(A) + \mathit{nullity}(A) = n$$

This theorem relates the number of linearly independent columns to the dimension of the solution space of the homogeneous system $A\vec{x} = \vec{0}$.

9.5 Applications of Rank

1. Determining if a system of linear equations has solutions:
   - If $\text{rank}(A) = \text{rank}([A|b])$, the system is consistent (has at least one solution)
   - If $\text{rank}(A) < \text{rank}([A|b])$, the system is inconsistent (has no solution)
2. Number of solutions:
   - If $\text{rank}(A) = \text{rank}([A|b]) = n$ (the number of variables), there is a unique solution
   - If $\text{rank}(A) = \text{rank}([A|b]) < n$, there are infinitely many solutions

10 Matrix Decompositions

Matrix decompositions (or factorizations) express a matrix as a product of simpler matrices, which can simplify calculations and reveal underlying structure.

10.1 LU Decomposition

LU decomposition factors a matrix A into the product of a lower triangular matrix L and an upper triangular matrix U:

$$A = LU$$

Example 10.1

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 8 \\ 6 & 5 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

LU decomposition is useful for:

- Solving systems of linear equations efficiently
- Computing the determinant ($\det(A) = \det(L) \cdot \det(U) = 1 \cdot \prod_i u_{ii}$)
- Finding the inverse of a matrix

10.2 QR Decomposition

QR decomposition factors a matrix A into the product of an orthogonal matrix Q (where $Q^T Q = I$) and an upper triangular matrix R:

$$A = QR$$

QR decomposition is particularly useful for:

- Solving least squares problems
- Computing eigenvalues (QR algorithm)
- Orthogonalizing a set of vectors (Gram-Schmidt process)

10.3 Singular Value Decomposition (SVD)

SVD decomposes a matrix A into the product:

$$A = U \Sigma V^T$$

where:

- U is an $m \times m$ orthogonal matrix
- Σ is an $m \times n$ diagonal matrix with non-negative real numbers on the diagonal (singular values)
- V is an $n \times n$ orthogonal matrix

SVD has numerous applications:

- Image compression
- Noise reduction
- Pseudoinverse computation
- Principal Component Analysis
- Recommender systems
- Machine learning algorithms

11 Matrix Visualization

Here is a visual representation of a matrix, emphasizing its structure:

$$A = \begin{array}{c} \\ \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Col 1} & \text{Col 2} & \text{Col 3} \end{array} \\ \left\{ \begin{array}{|c|c|c|} \hline \boxed{1} & \boxed{2} & \boxed{3} \\ \hline \boxed{4} & \boxed{5} & \boxed{6} \\ \hline \boxed{7} & \boxed{8} & \boxed{9} \\ \hline \end{array} \right\} \end{array}$$

12 Applications of Matrices

12.1 Computer Graphics

Matrices are fundamental in computer graphics for transformations such as:

- Translation
- Rotation
- Scaling
- Shearing
- Projection (3D to 2D)

In 3D graphics, homogeneous coordinates are used with 4×4 transformation matrices:

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where the 3×3 submatrix $[r_{ij}]$ represents rotation and scaling, and (t_x, t_y, t_z) represents translation.

12.2 Systems of Linear Equations

A system of linear equations can be written in matrix form as $AX = B$:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The solutions to this system depend on the properties of matrix A:

- If A is invertible, there is a unique solution: $X = A^{-1}B$
- If A is not invertible, there may be infinitely many solutions or no solution

12.3 Markov Chains

Markov chains model stochastic processes where future states depend only on the current state, not on past states. The transition matrix P contains probabilities of moving from one state to another:

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

where p_{ij} is the probability of moving from state i to state j, and each row sums to 1.

The state vector after k steps is given by $v^{(k)} = v^{(0)}P^k$, where $v^{(0)}$ is the initial state.

12.4 Machine Learning and Data Science

Matrices are essential in various machine learning algorithms:

- In Principal Component Analysis (PCA), eigenvalues and eigenvectors of the covariance matrix reveal principal components
- In neural networks, weight matrices connect layers of neurons
- In recommendation systems, user preference matrices predict ratings
- In computer vision, convolution matrices detect features in images

12.5 Quantum Mechanics

In quantum mechanics, Hermitian matrices represent observable quantities (like energy or momentum), with eigenvalues corresponding to possible measurement outcomes and eigenvectors representing the corresponding quantum states.

12.6 Economics

In input-output analysis, matrices model interdependencies between different sectors of an economy, showing how outputs from one sector become inputs for others.

12.7 Network Analysis

Adjacency matrices represent networks or graphs, with entries indicating connections between nodes:

$$A_{ij} = \begin{cases} 1 & \text{if nodes } i \text{ and } j \text{ are connected} \\ 0 & \text{otherwise} \end{cases}$$