Studying Functions: Domain and Range

Simone Capodivento

November 23, 2025

Contents

1 Abstract

Understanding how to study a function begins with identifying its **domain** and **range**. These concepts help determine where a function is defined and what output values it can produce.

2 This is a function

$$\text{Input } \rightarrow \boxed{\text{Function}} \rightarrow \text{Output}$$

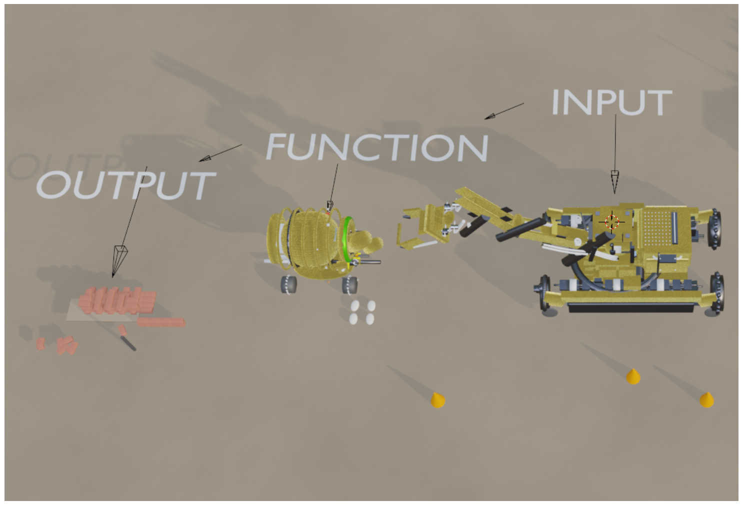


Figure 1: The excavator is the input, the cement mixer it works as function and the brick is the outup.

So you have to think that as a box that recive an input ad gives back an output.

Definition of function it has only one output. when you have two or more outputs it is a relation not a function.

3 Domain of a Function

If you have a $F(x)$ and you want to understand which is the $f(-1) =?$ you must see where the coordinates in y are at $x = -1$.

Starting from a graphed $f(x)$, you just need to see where it encloses the y-axis.

everything beside The **domain** of a function is the set of all possible input values (usually denoted as x) for which the function is defined. To find the domain, we look for any restrictions on the input values, such as:

- Values that make the denominator zero (in rational functions).
- Values that make the argument of a square root negative.
- Values that make a logarithm undefined or complex.
- Other context-specific restrictions.

4 Range of a Function

Set of y. The **range** of a function is the set of all possible output values (usually denoted as y). To find the range, we can:

- Analyze the function's behavior as x approaches specific values (e.g., using limits).
- Identify maximum or minimum values.
- Examine the function's graph.
- Use algebraic manipulation to solve for y.

5 Example 1: Analyze the Function $f(x) = \frac{(x^2-1)}{3x}$

To find the **domain**:

- The numerator $x^2 - 1$ is defined for all real x.
- The denominator $3x$ must not be zero, so $x \neq 0$.

Thus, the domain is:

$$D = \mathbb{R} \setminus \{0\}$$

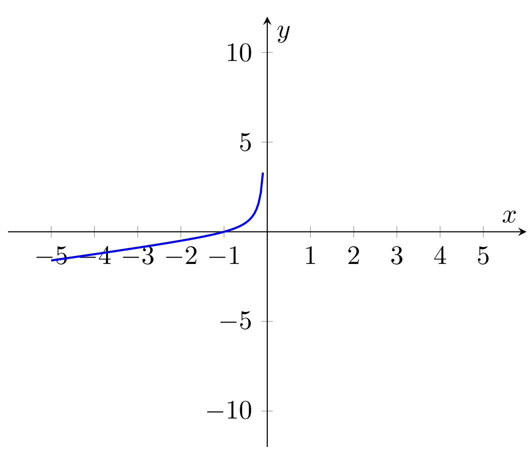
Asymptotic Behavior

Let's examine the limits near the vertical asymptote:

$$\lim_{x\to 0^+} \frac{(x^2-1)}{3x} = -\infty, \quad \lim_{x\to 0^-} \frac{(x^2-1)}{3x} = +\infty$$

So, the function has a vertical asymptote at $x = 0$.

Graph of the Function



6 Notation

Closed Interval Example

$$-3 \leq x \leq 2 \xrightarrow{\text{fancier}} [-3, 2] \xrightarrow{\text{fancier}} \{x \in \mathbb{R} \mid -3 \leq x \leq 2\} \xrightarrow{\text{fancier}} \{x \in [-3, 2]\}$$

Open Interval Example

$$-1 < x < 4 \xrightarrow{\text{fancier}} \{x \in \mathbb{R} \mid -1 < x < 4\} \xrightarrow{\text{fancier}} \{x \in (-1, 4)\}$$

If one bound is strict ($<$) and the other is inclusive ($\leq$), the interval is written as:

$$(-4, -1]$$

Excluding a Value

$$\begin{aligned} x \in \mathbb{R},\ x \neq 1 &\xrightarrow{\text{fancier}} \{x \in \mathbb{R} \mid x < 1 \text{ or } x > 1\} \\ &\xrightarrow{\text{fancier}} \{x \in (-\infty, 1) \cup (1, \infty)\} \end{aligned}$$

7 Example 2: Analyze the Function $f(x) = \frac{|x-1|}{x+2}$

Domain

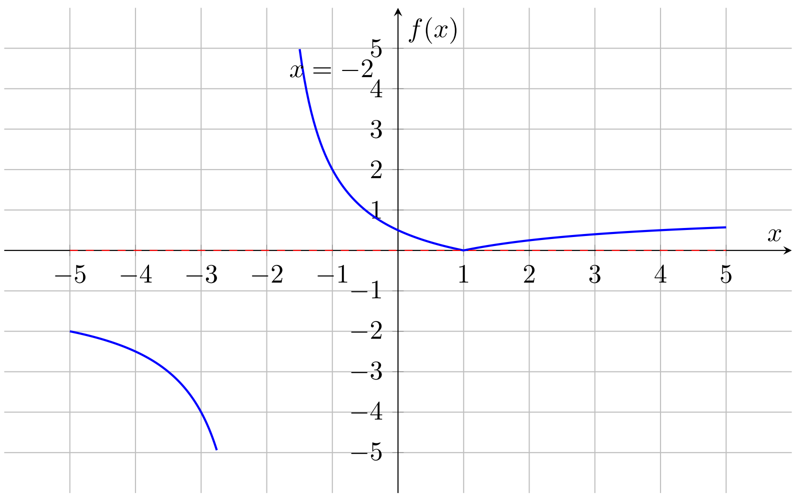
The denominator $x + 2$ cannot be zero, so $x \neq -2$:

$$D = \mathbb{R} \setminus \{-2\}$$

Range

The absolute value in the numerator ensures the output is always non-negative. As $x \to -2^{\pm}$, the function approaches $+\infty$, so:

$$R = [0, +\infty)$$



There is a vertical asymptote at

$$x = -2 \tag{1}$$

Asymptotes

Vertical Asymptotes:

$$\lim_{x \to a^-} f(x) = \pm\infty \tag{2}$$

$$\lim_{x \to a^+} f(x) = \pm\infty \tag{3}$$

where $x = a$ is a vertical asymptote.

Horizontal Asymptotes:

$$\lim_{x \to +\infty} f(x) = L \tag{4}$$

$$\lim_{x \to -\infty} f(x) = L \tag{5}$$

where $y = L$ is a horizontal asymptote.

Oblique/Slant Asymptotes: If $\lim_{x \to \pm\infty} \frac{f(x)}{x} = m$ and $\lim_{x \to \pm\infty} [f(x) - mx] = b$, then $y = mx + b$ is an oblique asymptote.

7.1 Symmetries

$\boxed{\text{Symbol: } S_n}$

We compute the first and second derivatives to understand the rate of change of the function — whether it is increasing or decreasing — and to analyze its concavity.

How to solve this problem:

$$-8 \cdot f(0) + 4 \cdot g(-8) \quad \text{given that} \quad g(x) = 2$$

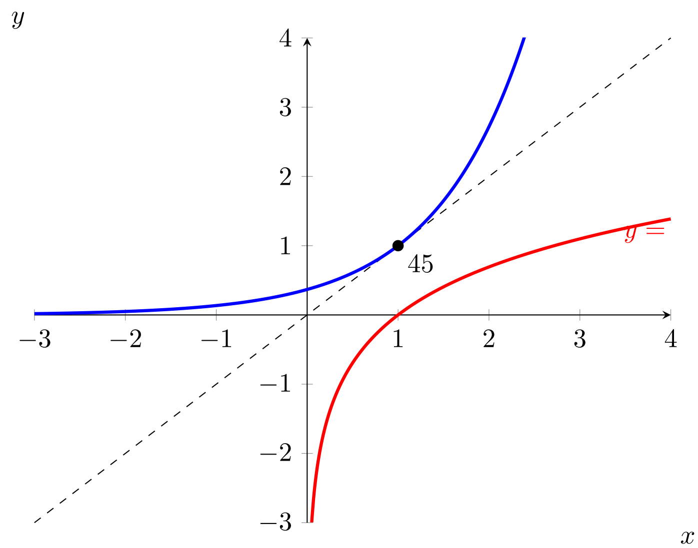
8 C.E. (Conditions of Existence)

We study the **conditions of existence (C.E.)** because they define the constraints that a radical, an algebraic function, an equation, an inequality, or a function must satisfy to have meaning within the set of real numbers $\mathbb{R}$.

$$y = x^2 + \frac{1}{x^2} \tag{6}$$

$$x \neq 0, \quad \forall x \in \mathbb{R} \tag{7}$$

9 Inverse functions



exercises write a formula for f(a) in terms of a

f(a)=-3a+6b=a+4b (8)

solution 6b-4b=a+3a b=2a

you have to solve a but keep the other letter incognito.

10 Properties of functions list in a nutshell

- Domain
- Symmetry
- Increasing/decreasing behavior

- Maximum and minimum
- Asymptotes
- Continuity
- Periodicity
- Injective
- Surjective
- Bijective

Complete Analysis of $f(x) = -x^4 + 6x^2 - 5$

Domain

Domain: $\mathbb{R}$ (all real numbers)
The function is a polynomial, so it's defined for all real values of x.

Symmetry

Even function (symmetric about the y-axis)
Check: $f(-x) = -(-x)^4 + 6(-x)^2 - 5 = -x^4 + 6x^2 - 5 = f(x)$
Since $f(-x) = f(x)$, the function has even symmetry.

Increasing/Decreasing Behavior

First derivative: $f'(x) = -4x^3 + 12x = -4x(x^2 - 3) = -4x(x - \sqrt{3})(x + \sqrt{3})$
Critical points: $x = -\sqrt{3}, x = 0, x = \sqrt{3}$
Sign analysis of $f'(x)$:

- For $x \in (-\infty, -\sqrt{3})$: $f'(x) > 0$ (increasing)
- For $x \in (-\sqrt{3}, 0)$: $f'(x) < 0$ (decreasing)
- For $x \in (0, \sqrt{3})$: $f'(x) < 0$ (decreasing)
- For $x \in (\sqrt{3}, \infty)$: $f'(x) > 0$ (increasing)

Maximum and Minimum

Second derivative: $f''(x) = -12x^2 + 12 = 12(1 - x^2)$
Analysis at critical points:

- At $x = -\sqrt{3}$: $f''(-\sqrt{3}) = -24 < 0 \rightarrow$ **Local maximum** at $(-\sqrt{3}, 4)$
- At $x = 0$: $f''(0) = 12 > 0 \rightarrow$ **Local minimum** at $(0, -5)$
- At $x = \sqrt{3}$: $f''(\sqrt{3}) = -24 < 0 \rightarrow$ **Local maximum** at $(\sqrt{3}, 4)$

Global behavior:

- **Global maximum:** $y = 4$ (at $x = \pm\sqrt{3}$)
- **Global minimum:** Does not exist (function approaches $-\infty$ as $x \rightarrow \pm\infty$)

Asymptotes

No asymptotes
Since this is a polynomial function:

- No vertical asymptotes (function is continuous everywhere)
- No horizontal asymptotes (degree ≥ 1)
- No oblique asymptotes

Continuity

Continuous everywhere on $\mathbb{R}$

Polynomial functions are continuous at every point in their domain.

Periodicity

Not periodic

The function has no repeating pattern. As a polynomial of degree 4, it cannot be periodic.

Injective (One-to-One)

Not injective

The function fails the horizontal line test. For example:

- $f(-2) = f(2) = -16 + 24 - 5 = 3$
- $f(-\sqrt{3}) = f(\sqrt{3}) = 4$

Since different x-values can produce the same y-value, the function is not injective.

Surjective (Onto)

Not surjective (when considering $f : \mathbb{R} \to \mathbb{R}$)

Range: $(-\infty, 4]$

Since the function has a global maximum of $y = 4$ and approaches $-\infty$ as $x \to \pm\infty$, it cannot reach any y-value greater than 4. Therefore, it's not surjective onto $\mathbb{R}$.

However, $f : \mathbb{R} \to (-\infty, 4]$ would be surjective.

Bijective

Not bijective

Since the function is neither injective nor surjective (when considering $f : \mathbb{R} \to \mathbb{R}$), it cannot be bijective.

Summary: This is an even, continuous polynomial function with two local maxima at $(\pm\sqrt{3}, 4)$, one local minimum at $(0, -5)$, and no asymptotes. It's neither injective, surjective (onto $\mathbb{R}$), nor bijective.

Graph