HPMPC regerence guide

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Chapter 1

Introduction

HPMPC – A library for High-Performance implementation of solvers for MPC.

HPMPC has been developed with the aim of providing extremely fast building blocks to implement algorithms for Model Predictive Control (MPC). This has been achieved by carefully implementing the linear algebra routines using high-performance computing techniques with a focus on small-medium scale problems, typical of MPC applications.

The current version of the library contains Interior-Point Method (IPM) and Alternating Direction Methods of Multipliers (ADMM) solvers for the linear MPC (LMPC) problem with box constraints, and a Riccati-based solver for the unconstrained MPC problem, that is used as a routine in the IPM. A basic API is provided for these solvers, while lower level interfaces provide access to kernels, linear-algebra routines and MPC solvers. The library is self-contained, not requiring any other library beside the standard C library.

The code is highly-optimized for a number of common architectures, plus a reference version in plain C code (C99 standard). The code has been developed on Linux machines and tested also on MAC machines. The code is intended to be compiled using `gcc` as a compiler (the code for some architecture makes use of extended asm inline assembly); it can be compiled using `clang` as well, while it may need some hacking to work with other compilers. In any case, the time-critical routines have been carefully implemented in assembly exploiting architecture-specific features, so there would be no practical advantage in compiling the code with more performing (e.g. commercial) compilers.

The library is released under the LGPL version 3 licence, and it can be used as a basis for the development of higher-level solvers for MPC or other applications requiring extremely fast solvers for small-medium scale problems.

Chapter 2

Problems definition

In this chapter the unconstrained and constrained MPC problems are presented.

2.1 Unconstrained MPC Problem

The unconstrained MPC problem is the equality constrained quadratic program

$$
\begin{aligned}
\min_{u_n, x_n} \quad & \phi = \sum_{n=0}^{N-1} \varphi_n(x_n, u_n) + \varphi_N(x_N) \\
s.t. \quad & x_{n+1} = A_n x_n + B_n u_n + b_n \\
& x_0 = \hat{x}_0
\end{aligned}
\tag{2.1}
$$

where $n \in \{0, 1, \dots, N-1\}$ and

$$
\begin{aligned}
\varphi_n(x_n, u_n) &= \begin{bmatrix} u_n \\ x_n \\ 1 \end{bmatrix}^T \begin{bmatrix} R_n & S_n & r_n \\ S_n^T & Q_n & q_n \\ r_n^T & q_n^T & \rho_n \end{bmatrix} \begin{bmatrix} u_n \\ x_n \\ 1 \end{bmatrix} = \mathcal{X}_n^T \mathcal{Q}_n \mathcal{X}_n \\
\varphi_N(x_N) &= \begin{bmatrix} u_N \\ x_N \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & Q_N & q_N \\ 0 & q_N^T & \rho_N \end{bmatrix} \begin{bmatrix} u_N \\ x_N \\ 1 \end{bmatrix} = \mathcal{X}_N^T \mathcal{Q}_N \mathcal{X}_N
\end{aligned}
\tag{2.2}
$$

All matrices in this formulation can in general be dense and time variant. The matrices $\mathcal{Q}_n$ have to be symmetric and positive semi-definite, while the R_n matrices have to be symmetric and strictly positive definite.

2.2 Linear MPC problem

The linear MPC problem with box and general polytopic constraints is the quadratic program

$$
\begin{aligned}
\min_{u_n, x_n} \quad & \phi = \sum_{n=0}^{N-1} \varphi_n(x_n, u_n) + \varphi_N(x_N) \\
s.t. \quad & x_{n+1} = A_n x_n + B_n u_n + b_n \\
& x_0 = \hat{x}_0 \\
& u_n^l \le u_n \le u_n^u \\
& x_n^l \le x_n \le x_n^u \\
& d_n^l \le C_n x_n + D_n u_n \le d_n^u \\
& d_N^l \le C_N x_n \le d_N^u
\end{aligned}
\tag{2.3}
$$

where $n \in \{0, 1, \dots, N-1\}$, $\varphi_n(x_n, u_n)$ and $\varphi_N(x_N)$ are defined as in (2.2). Again, all matrices can in general be dense and time variant.

Chapter 3

Compilation and installation of the library

The code has been developed on Linux machines, using the `gcc` compiler. It has been also tested on MAC machines using the `clang` compiler. The library is mainly written in C code, with time-critical parts (kernels) written using intrinsics or extended asm inline assembly: from here the need for the `gcc` or `clang` compilers.

Time-critical routines are carefully optimized by hand for a number of architectures, using assembly to explicitly exploit e.g. the number of registers and SIMD instruction sets specific to each architecture.

The `gcc` compiler is found already installed on most Linux distributions.

Redarding utilities, `make` is used to automate the compilation of the code, and `ar` to build the static library: both of them are found installed on all Linux distributions. Alternatively, the library can be built using `CMake`.

3.1 Target architectures

Once the code has been downloaded, the first step is the editing of the configuration file `Makefile.rule`. In general, the only part that needs to be edited is the `TARGET`, used to choose architeture-specific code and flags. In the following, the abbreviation ISA stands for Instruciton Set Architecture.

Currently supported targets are

X64_AVX2 this a recent x86_64 processor supporting AVX2 and FMA3 ISAs (e.g. Intel Haswell, Intel Broadwell or Intel Skylake micro-architectures). The 64-bit version of the OS is required. At the moment, these micro-architectures are supported, but only the double-precision version of the library if fully-optimized.

X64_AVX this is a x86_64 processor supporting the AVX ISA (e.g. Intel Sandy Bridge and Intel Ivy Bridge; AMD Bulldozer or more recent micro-architectures). The 64-bit version of the OS is required.

X64_SSE3 this is a x86_64 processor supporting the SSE3 ISA (e.g. Intel Pentium 4 Prescott, Intel Core and Intel Nehalem; AMD Athlon 64 revision E or more recent micro-architectures). The 64-bit version of the OS is required. The code is not fully optimized yet.

CORTEX_A57 this is a processor implementing the ARMv8A architecture with VFPv4 and NEONv2 ISAs, code optimized for ARM Cortex A57. At the moment the architecture is supported using reference code, and only the `dgemm` and `sgemm` routines are optimized.

CORTEX_A15 this is a processor implementing the ARMv7A architecture with VFPv4 and NEONv2 ISAs, code optimized for ARM Cortex A15.

CORTEX_A9 this is a processor implementing the ARMv7A architecture with VFPv3 and NEON ISAs, code optimized for ARM Cortex A9.

CORTEX_A7 this is a processor implementing the ARMv7A architecture with VFPv4 and NEONv2 ISAs, code optimized for ARM Cortex A7.

C99_4X4 this version is written entirely in C code and works with all machines, even if performing worse that machine-specific code. The code works better on a machine with at least 32 scalar FP registers.

More architectures are supported in the older version 0.1 of the library, that can still be downloaded from github.

The supported ISAs can be easily found by googling the processor name, or (on Linux machines) by typing

```
less \proc\cpuinfo
```

on a terminal, and looking among the flags. In any case, even if the code is compiled for the wrong architecutre, an `Illegal instruction` error will be raised at run time, so this can be immediately discovered running a test problem.

3.2 Compilation and installation process

Once the architecture has been chosen, the static library can be build by typing

```
make static_library
```

on a terminal. The dynamic library can be build by typing

```
make shared_library
```

on a terminal.

The macro `PREFIX` in the configuration file `Makefile.rule` contains the installation directory (the default one is `/opt`). The static library and the headers can be installed by typing

```
sudo make install_static
```

on a terminal. The dynamic library and the headers can be installed by typing

```
sudo make install_shared
```

on a terminal.

The libraries and the headers can be uninstalled by typing

```
sudo make uninstall
```

on a terminal.

Chapter 4

Running test problems

4.1 Running Octave test problems

A number of Octave test problems is vailable in the folder `interfaces/octave`.

A `mex` interface is used as a wrapper around the solvers. Octave test problems call the `mex` wrappers.

4.1.1 Linear MPC problem example

Here the Octave interface of the IPM solver is used to solve a linear MPC problem. The test problem is `interfaces/octave/test_ip_hard.m`.

The wrapper to the IPM solver is contained in the file `interfaces/octave/HPMPC_ip_hard.c`. The interface to solve hard-constrained linear MPC problems looks like

```
HPMPC_ip_hard(kk, k_max, mu0, tol, N, nx, nu, nb, ng, ngN, time_invariant,
A, B, b, Q, QN, R, S, q, qN, r, lb, ub, C, D, lg, ug, CN, lgN, ugN, x, u,
infos, compute_res, inf_norm_res, compute_mult, mult_pi, mult_lam, mult_t);
```

where

`kk` Output: the number of iterations for the IPM to converge.

`k_max` Input: maximum number of iterations in the IPM.

`mu0` Input: initial value of the parameter μ in the IPM. If equal or smaller than 0, it is fixed in the IPM as the largest absolute value of the elements of the cost function matrices and vectors.

`tol` Input: tolerance on the duality measure for convergence.

`N` Input: horizon lenght, where 0 is the fist stage, 1 to N-1 are the middle stages and N is the last stage.

`nx` Input: number of states per stage.

`nu` Input: number of inputs per stage.

`nb` Input: number of box constraints per stage, meaning that the first `nb` element of the variables vector at each stage are bounded, where the variables vector at the generic stage n is $v_n = \begin{bmatrix} u_n^T & x_n^T \end{bmatrix}^T$.

`ng` Input: number of general constraints on stages 0 to N-1.

`ngN` Input: number of general constraints at the last stage N.

`time_invariant` Input: flag to choose between time-invariant (1) and time-variant (0) interfaces.

`A` Input: corresponds to the A_n matrix in (2.3). If `time_invariant` is 1, then `A` is a $n_x \times n_x$ matrix; if `time_invariant` is 0, then `A` is a $n_x \times N n_x$ matrix, corresponding to $\begin{bmatrix} A_0 & A_1 & \dots & A_{N-1} \end{bmatrix}$.

`B` Input: corresponds to the B_n matrix in (2.3). If `time_invariant` is 1, then `B` is a $n_x \times n_u$ matrix; if `time_invariant` is 0, then `B` is a $n_x \times N n_u$ matrix, corresponding to $\begin{bmatrix} B_0 & B_1 & \dots & B_{N-1} \end{bmatrix}$.

`b` Input: corresponds to the b_n vector in (2.3). If `time_invariant` is 1, then `b` is a $n_x \times 1$ matrix; if `time_invariant` is 0, then `b` is a $n_x \times N$ matrix, corresponding to $\begin{bmatrix} b_0 & b_1 & \dots & b_{N-1} \end{bmatrix}$.

`Q` Input: corresponds to the Q_n matrix in (2.2). If `time_invariant` is 1, then `Q` is a $n_x \times n_x$ matrix; if `time_invariant` is 0, then `Q` is a $n_x \times N n_x$ matrix, corresponding to $\begin{bmatrix} Q_0 & Q_1 & \dots & Q_{N-1} \end{bmatrix}$.

`QN` Input: corresponds to the Q_N matrix in (2.2). `QN` is a $n_x \times n_x$ matrix.

`R` Input: corresponds to the R_n matrix in (2.2). If `time_invariant` is 1, then `R` is a $n_u \times n_u$ matrix; if `time_invariant` is 0, then `R` is a $n_u \times N n_u$ matrix, corresponding to $\begin{bmatrix} R_0 & R_1 & \dots & R_{N-1} \end{bmatrix}$.

`S` Input: corresponds to the S_n matrix in (2.2). If `time_invariant` is 1, then `S` is a $n_u \times n_x$ matrix; if `time_invariant` is 0, then `S` is a $n_u \times N n_x$ matrix, corresponding to $\begin{bmatrix} S_0 & S_1 & \dots & S_{N-1} \end{bmatrix}$.

`q` Input: corresponds to the q_n vector in (2.2). If `time_invariant` is 1, then `q` is a $n_x \times 1$ matrix; if `time_invariant` is 0, then `q` is a $n_x \times N$ matrix, corresponding to $\begin{bmatrix} q_0 & q_1 & \dots & q_{N-1} \end{bmatrix}$.

`qN` Input: corresponds to the q_N vector in (2.2). `qN` is a $n_x \times 1$ matrix.

`r` Input: corresponds to the r_n vector in (2.2). If `time_invariant` is 1, then `r` is a $n_u \times 1$ matrix; if `time_invariant` is 0, then `r` is a $n_u \times N$ matrix, corresponding to $\begin{bmatrix} r_0 & r_1 & \dots & r_{N-1} \end{bmatrix}$.

`lb` Input: corresponds to the $\begin{bmatrix} (u_n^l)^T & (x_n^l)^T \end{bmatrix}^T$ vector in (2.2). If `time_invariant` is 1, then `lb` is a $n_b \times 1$ matrix; if `time_invariant` is 0, then `lb` is a $n_b \times (N+1)$ matrix, corresponding to $\begin{bmatrix} u_0^l & u_1^l & \dots & u_{N-1}^l & * \\ * & x_1^l & \dots & x_{N-1}^l & x_N^l \end{bmatrix}$.

`ub` Input: corresponds to the $\begin{bmatrix} (u_n^u)^T & (x_n^u)^T \end{bmatrix}^T$ vector in (2.2). If `time_invariant` is 1, then `ub` is a $n_b \times 1$ matrix; if `time_invariant` is 0, then `ub` is a $n_b \times (N+1)$ matrix, corresponding to $\begin{bmatrix} u_0^u & u_1^u & \dots & u_{N-1}^u & * \\ * & x_1^u & \dots & x_{N-1}^u & x_N^u \end{bmatrix}$.

`C` Input: corresponds to the C_n matrix in (2.3). If `time_invariant` is 1, then `C` is a $n_g \times n_x$ matrix; if `time_invariant` is 0, then `C` is a $n_g \times N n_x$ matrix, corresponding to $\begin{bmatrix} C_0 & C_1 & \dots & C_{N-1} \end{bmatrix}$.

`D` Input: corresponds to the D_n matrix in (2.3). If `time_invariant` is 1, then `D` is a $n_g \times n_u$ matrix; if `time_invariant` is 0, then `D` is a $n_g \times N n_u$ matrix, corresponding to $\begin{bmatrix} D_0 & D_1 & \dots & D_{N-1} \end{bmatrix}$.

`lg` Input: corresponds to the d_n^l matrix in (2.3). If `time_invariant` is 1, then `lg` is a $n_g \times 1$ matrix; if `time_invariant` is 0, then `lg` is a $n_g \times N$ matrix, corresponding to $\begin{bmatrix} d_0^l & d_1^l & \dots & d_{N-1}^l \end{bmatrix}$.

`ug` Input: corresponds to the d_n^u matrix in (2.3). If `time_invariant` is 1, then `ug` is a $n_g \times 1$ matrix; if `time_invariant` is 0, then `ug` is a $n_g \times N$ matrix, corresponding to $\begin{bmatrix} d_0^u & d_1^u & \dots & d_{N-1}^u \end{bmatrix}$.

`CN` Input: corresponds to the C_N matrix in (2.3). `CN` is a $n_{gN} \times n_x$ matrix.

`lgN` Input: corresponds to the d_N^l matrix in (2.3). `lgN` is a $n_{gN} \times 1$ matrix.

`ugN` Input: corresponds to the d_N^u matrix in (2.3). `ugN` is a $n_{gN} \times 1$ matrix.

`x` It is a matrix of size $n_x \times N + 1$. Input: the first column correspond to $\hat{x}_0$ (2.3). Output: the columns from 2 to N+1 return the state vector at the solution, corresponding to stages 1 to N respectively.

`u` It is a matrix of size $n_u \times N$. Output: the columns from 1 to N return the input at the solution, corresponding to stages 1 to N respectively.

`infos` It is a matrix of size $kk \times 5$. Output: informations about the IPM covergence at each IPM iteration: σ, α_{aff}, μ_{aff}, α, μ.

`compute_res` Input: flags that enables (1) or disables (0) the comutation of the residuals.

`inf_norm_res` It is a vector of size 4. Output: it returns $\begin{bmatrix} ||r_q||_\infty & ||r_b||_\infty & ||r_d||_\infty & \mu \end{bmatrix}^T$.

`compute_mult` Input: flags that enables (1) or disables (0) the comutation of the Lagrangian multipliers.

`mult_pi` Matrix of size $n_x \times N + 1$. Output: it returns the Lagrangian multipliers of the equality constraints.

`mult_lam` Vector of size $2(n_b + n_g)N + 2(n_b + n_{gN})$. Output: it returns the Lagrangian multipliers of the inequality constraints.

`mult_t` Vector of size $2(n_b + n_g)N + 2(n_b + n_{gN})$. Output: it returns the slack variables associated with Lagrangian multipliers of the inequality constraints.

Time-invariant case

The test problem is the mass-spring system, that has been widely used to test linear MPC solvers thanks to its scalability properties.

The problem size is $N = 30$, $n_x = 8$, $n_u = 3$. Therefore there are 4 masses, and a force acts on the the first 3 of them. Each mass is equal to 1 Kg, and the constant of each spring is equal to 1 N/m.

The forces are bounded between -0.5 and 0.5. The position of the masses is bounded between -4.0 and 4.0. The velocity of the masses is not constrainted. A terminal constraint imposes that the position and the velocity of all masses must be equal to 0 (i.e. the masses must be at resting position at stage N). Since the bounds are time-invariant, the terminal constraints must be handled as general polytopic constraints at the last stage N.

Therefore, the number of bounds is $n_b = n_u + \frac{n_x}{2} = 7$. Since only the first $\frac{n_x}{2}$ states are bounded, the states are ordered such that the first $\frac{n_x}{2}$ states corresponds to the masses position. The number of general polytopic constraints at stages 0 to N-1 is $n_g = 0$. The number of general polytopic constraints at the last stage N is $n_{gN} = 8$.

The cost function matrices and vectors are initialized as: $Q = I$, $QN = I$, $R = 2I$, $S = 0$, $q = 0$, $qN = 0$, $r = 0$.

The initial state is

About the choice of the IPM parameters, the maximum number of iterations is fixed to 20. The tolerance in the duality measure is fixed to 10^{-8}. Since the largest absolute value of the elements in the cost function expression is 2, it holds $\mu_0 = 2$. Both residuals and Lagrangian multipliers are computed.

The code is runned on a Linux machine with processor Intel core i7 3520M, and 8 GB of DDR3 RAM in dual-channel configuration, running at 1600 MHz (maximum bandwidth of 25.6 GB/s). The HPMPC library is compiled for the AVX target.

The IPM solver returns after 8 iterations, and the solution time averaged over 1000 calls to the IPM solver is $3.14 \cdot 10^{-4}$ seconds.

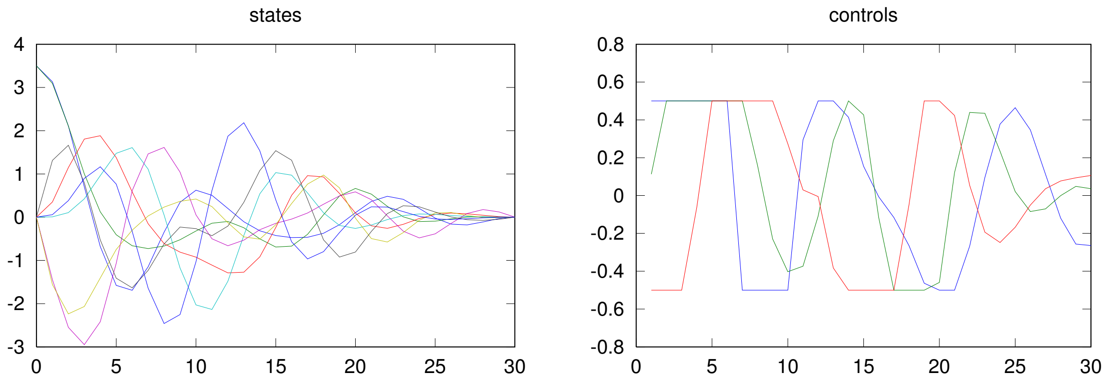
The `infos` about the IPM iterations are summarized in Table 4.1

The absolute value of the residuals is

$$\begin{bmatrix} ||r_q||_\infty \\ ||r_b||_\infty \\ ||r_d||_\infty \\ \mu \end{bmatrix} = \begin{bmatrix} 1.8806 \cdot 10^{-06} \\ 3.0816 \cdot 10^{-10} \\ 1.0004 \cdot 10^{-11} \\ 9.9658 \cdot 10^{-10} \end{bmatrix}$$

Table 4.1: Content of the `infors` matrix.

iteration	σ	α_{aff}	μ_{aff}	α	μ
1	$2.7514 \cdot 10^{-01}$	$3.2839 \cdot 10^{-01}$	$1.3008 \cdot 10^{+00}$	$3.9179 \cdot 10^{-01}$	$1.5862 \cdot 10^{+00}$
2	$7.7808 \cdot 10^{-02}$	$5.4011 \cdot 10^{-01}$	$6.7718 \cdot 10^{-01}$	$7.1498 \cdot 10^{-01}$	$6.2662 \cdot 10^{-01}$
3	$4.3610 \cdot 10^{-03}$	$8.3084 \cdot 10^{-01}$	$1.0238 \cdot 10^{-01}$	$9.1943 \cdot 10^{-01}$	$6.2629 \cdot 10^{-02}$
4	$2.2303 \cdot 10^{-02}$	$7.3060 \cdot 10^{-01}$	$1.7629 \cdot 10^{-02}$	$8.8868 \cdot 10^{-01}$	$8.7287 \cdot 10^{-03}$
5	$8.6078 \cdot 10^{-03}$	$8.1046 \cdot 10^{-01}$	$1.7889 \cdot 10^{-03}$	$9.3176 \cdot 10^{-01}$	$7.4587 \cdot 10^{-04}$
6	$1.8652 \cdot 10^{-03}$	$8.9358 \cdot 10^{-01}$	$9.1813 \cdot 10^{-05}$	$9.8345 \cdot 10^{-01}$	$2.0746 \cdot 10^{-05}$
7	$3.1456 \cdot 10^{-05}$	$9.8589 \cdot 10^{-01}$	$6.5489 \cdot 10^{-07}$	$9.9763 \cdot 10^{-01}$	$1.9884 \cdot 10^{-07}$
8	$3.7490 \cdot 10^{-07}$	$9.9926 \cdot 10^{-01}$	$1.4338 \cdot 10^{-09}$	$1.0000 \cdot 10^{+00}$	$9.9658 \cdot 10^{-10}$



(a) States (b) Inputs

Figure 4.1: Solution.

Time-variant case

The time-invariant interface is more general than the time-invariant interface, and therefore it can handle the time-invariant case as well. The drawback is that it needs to pack a different matrix at each stage, since all matrices and vectors can potentially be time-variant.

The matrix `A` is built using the `repmat` command in Octave, as

```
A = repmat(A0, 1, N);
```

where `A0` corresponds to the `A` matrix in the time-invariant case.

Similarly are built the other state space and cost function matrices and vectors.

The returned results are identical to the time-invariant case.

4.2 Running C test problems

A number of test problems is available in the folder `test_problems`, and it is possible to choose between them by editing the file `test_problems/Makefile`. Notice that test problems are not part of the library. The chosen test problem is compiled by typing

```
make test_problem
```

on a terminal opened on the main HPMPC folder, and it is run by typing

```
make run
```

on a terminal opened on the main HPMPC folder.

Chapter 5

High-level API

In this chapter the high-level API is presented. It is intended to give accesso to the higher level routines in the library, namely the Riccati solver for the unconstrained problem (2.1), and the IPM solver for the MPC problem (2.3).

For performance purposes, internally the library makes use of a packed matrix format, and routines are provided to convert to and from this packed format and standard column- and row-major orders.

In this high-level API, all matrices are passed using standard column- or row-major orders, and wrappers take care of the conversions. For the best performance, users may consider to skip the wrappers and directly call the solvers using the packed matrix format (low-level API).

Chapter 6

References

The HPMPC library is the result of a long research path that led to a novel way to implement solvers for MPC, specially tailored to small-medium scale problems. The library has undergone several revisions before being published, and some of the steps along this research path are documented in the following papers.

- G. Frison, H.H.B. Sørensen, B. Dammann, J.B. Jørgensen, *High-Performance Small-Scale Solvers for Linear Model Predictive Control*, in proceedings of 13th European Control Conference, Strasbourg (France), 2014.
- G. Frison, L.E. Sokoler, J.B. Jørgensen, *A Family of High-Performance Solvers for Linear Model Predictive Control*, in proceedings of 19th IFAC World Congress, Cape Town (South Africa), 2014.
- G. Frison, D.K.M. Kufualor, L. Imsland, J.B. Jørgensen, *Efficient Implementation of Solvers for Linear Model Predictive Control on Embedded Devices*, in proceedings of IEEE Multi-conference on Systems and Control, Antibes (France), 2014.
- G. Frison, J.B. Jørgensen, *MPC Related Computational Capabilities of ARMv7A Processors*, in proceedings of 14th European Control Conference, Linz (Austria), 2015.