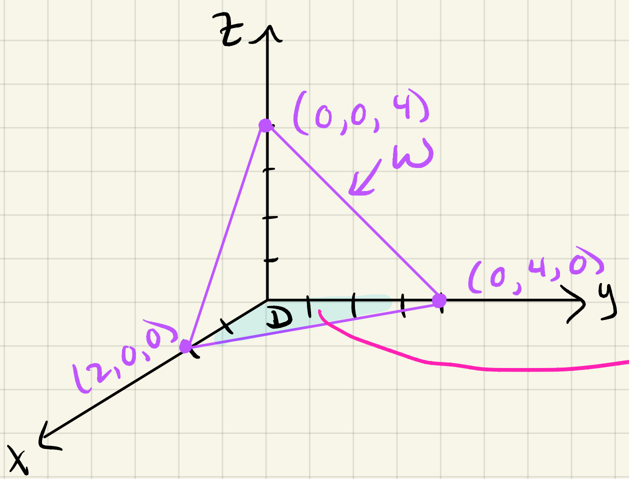
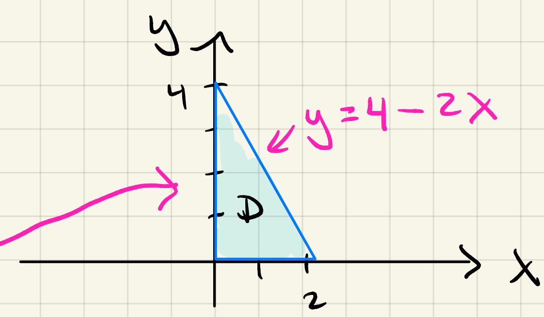
Section 15.3: Triple Integrals (cont)

Example: Evaluate $\iiint_W x \, dV$ ($f(x,y,z) = x$), where W is the tetrahedron bounded by the four planes $\underbrace{x=0}_{yz\text{-plane}}$, $\underbrace{y=0}_{xz\text{-plane}}$, $\underbrace{z=0}_{xy\text{-plane}}$, $\underbrace{2x+y+z=4}_{z = 4-2x-y \text{ (plane)}}$. "z-simple"

Sketch W:



Sketch D:



W:

$0 \le z \le 4-2x-y$

D is vertically simple $\left\{ D: \begin{array}{l} 0 \le x \le 2 \\ 0 \le y \le 4-2x \end{array} \right.$

D also horizontally simple

$D: \left\{ \begin{array}{l} 0 \le y \le 4 \\ 0 \le x \le \frac{y-4}{-2} \end{array} \right.$

W is a tetrahedron in first octant under the plane $z = 4-2x-y$

Find the intercepts: (plane hits the axes)

y-intercept: $x=0$: $z = 4-y$; $z=0 \Rightarrow 0 = 4-y \Rightarrow y=4$

x-intercept: $y=0=z$: $z = 4-2x \Rightarrow 0 = 4-2x \Rightarrow x=2$

z-intercept: $x=y=0$: $z=4$

$$\iiint_W x \, dV = \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} x \, dz \, dy \, dx = \int_0^2 \int_0^{4-2x} xz \Big|_{z=0}^{z=4-2x-y} dy \, dx$$

① ② ③

$$= \int_0^2 \int_0^{4-2x} \underbrace{x(4-2x-y)}_{4x-2x^2-xy} \, dy \, dx = \ldots = 8/3 \quad \text{(15.2 integral)}$$

Section 16.1 : Vector Fields

Def : Let D be a subset of $\mathbb{R}^2$. A vector field in $\mathbb{R}^2$ is a function $\vec{F}$ that assigns to each point (x,y) in D a vector $\vec{F}(x,y)$.

We can write $\vec{F}(x,y) = \langle F_1(x,y), F_2(x,y) \rangle = F_1(x,y)\hat{\imath} + F_2(x,y)\hat{\jmath}$

F_1 and F_2 are called component functions.

Similiarly, for a vector field in $\mathbb{R}^3$:

$$\vec{F}(x,y,z) = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) \rangle$$
$$= F_1(x,y,z)\hat{\imath} + F_2(x,y,z)\hat{\jmath} + F_3(x,y,z)\hat{k}$$

In general, a vector field is a function whose domain is a set of points in $\mathbb{R}^n$ and whose range is a set of vectors in $\mathbb{R}^n$.

* We will focus on vector fields in $\mathbb{R}^2$ & $\mathbb{R}^3$. *

Examples:

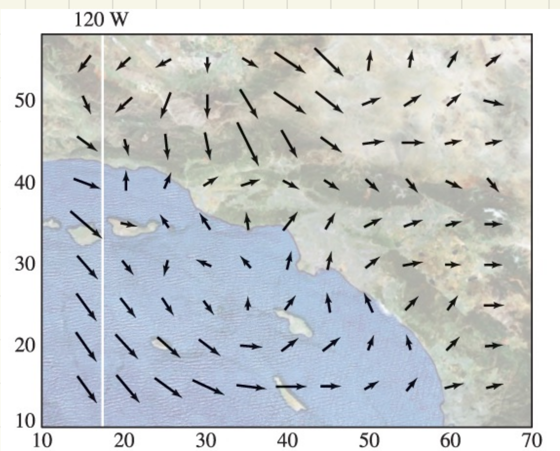


Figure 1

Air velocity vectors that indicate wind speed and direction.

Other examples in Physics:
- Velocity of fluid at any point
- gravitational force field
- electric fields

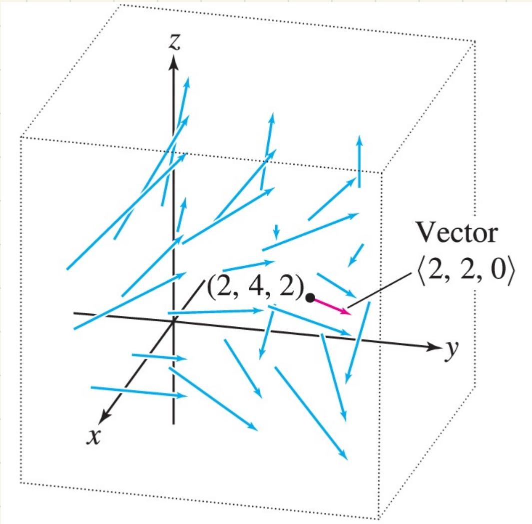


Figure 2

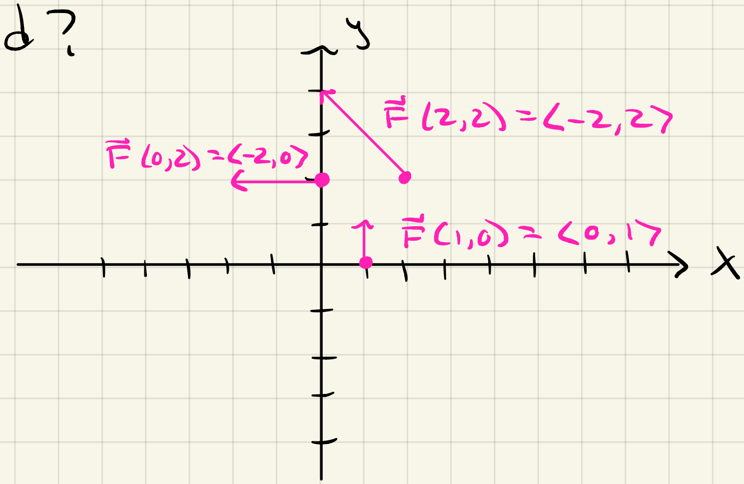
What is the graph of a vector field?

Ex: $\vec{F}(x,y) = \langle \underbrace{-y}_{F_1}, \underbrace{x}_{F_2} \rangle$

$\vec{F}(1,0) = \langle 0, 1 \rangle$

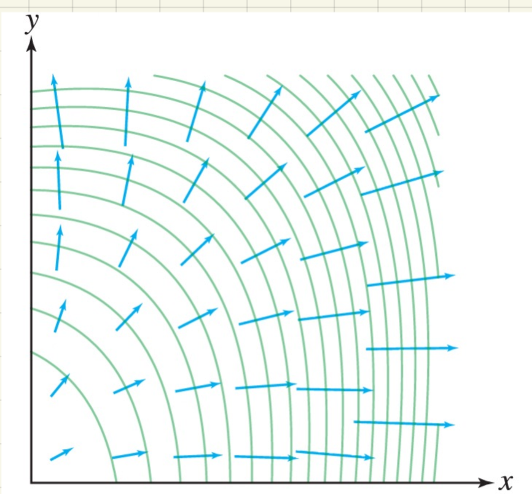
$\vec{F}(2,2) = \langle -2, 2 \rangle$

$\vec{F}(0,2) = \langle -2, 0 \rangle$



Def: A vector field $\vec{F}$ is called conservative if it is equal to the gradient of some scalar function f, i.e., $\vec{F} = \nabla f$ for some function f. Such an f is called a potential function of $\vec{F}$.

For a scalar function $f(x,y,z)$, $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$ is called its gradient vector field.



A conservative vector field is the gradient of a potential function $\therefore$ is orthogonal to the potential functions level curves.

Is there a way to tell whether a vector field is conservative?

We need to intoduce new operations.

Recall: The del operator ∇, which is the vector of derivative operators: ← called "Del"

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \quad (\text{in } \mathbb{R}^3)$$

Operations on Vector Fields:

Def: Let $\vec{F} = \langle F_1, F_2, F_3 \rangle$ be a vector field. We define

$F_1(x,y,z)$ $F_2(x,y,z)$ $F_3(x,y,z)$

- the divergence of $\vec{F}$ is:

apply operator to corresponding component and add together.

$$\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle F_1, F_2, F_3 \rangle = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\frac{\partial}{\partial x}(F_1) + \frac{\partial}{\partial y}(F_2) + \frac{\partial}{\partial z}(F_3)$$

- the curl of $\vec{F}$ is:

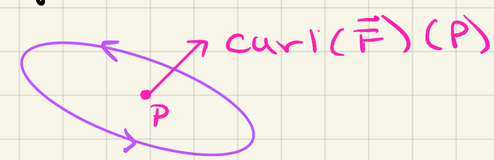
$$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \end{array}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right)\hat{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)\hat{k}$$

$$= \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right), \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

Interpretation: Flow of a liquid.

- Divergence: Tendancy to collect or disperse at a point (sink/source)
- Curl: How much the liquid is swirling (magnitude)
  axis of "rotation" (Direction)



Note:

	Operation	Input	Output
∇f	Gradient	Scalar function f	Vector field
$\nabla \cdot \vec{F}$	Divergence	vector field $\vec{F}$	Scalar function
$\nabla \times \vec{F}$	curl	vector field $\vec{F}$	vector field

Example: Let $\vec{F} = \langle \underbrace{xz}_{F_1(x,y,z)}, \underbrace{xyz}_{F_2(x,y,z)}, \underbrace{-y^2}_{F_3(x,y,z)} \rangle$. Calculate $\text{div}(\vec{F})$ and $\text{curl}(\vec{F})$.

$$\text{div}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2)$$

$$= z + xz + 0$$

$$\boxed{\text{div}(\vec{F}) = z + xz}$$

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz)\right)\hat{i} - \left(\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz)\right)\hat{j} + \left(\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz)\right)\hat{k}$$

$$= (-2y - xy)\hat{i} - (0 - x)\hat{j} + (yz - 0)\hat{k}$$

$$\boxed{\text{curl}(\vec{F}) = \langle -2y - xy, x, yz \rangle}$$

THEOREM 1

Curl of a Conservative Vector Field

1. In $\mathbf{R}^2$, if the vector field $\mathbf{F} = \langle F_1, F_2 \rangle$ is conservative, then

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

2. In $\mathbf{R}^3$, if the vector field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ is conservative, then

$$\mathrm{curl}\,(\mathbf{F}) = \mathbf{0}, \quad \text{or equivalently,} \quad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$$

can still use cross product:

$$\vec{F}(x,y) = \langle F_1(x,y), F_2(x,y) \rangle$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1(x,y) & F_2(x,y) & 0 \end{vmatrix}$$

$$= (0-0)\hat{i} - (0-0)\hat{j} + \underbrace{\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)}_{=0}\hat{k}$$

- In general, the converse ($\mathrm{curl}(\vec{F}) = \vec{0} \Rightarrow \vec{F}$ is conservative) is not true, but we will see conditions under which it is true in Section 16.3.
- To show $\vec{F}$ is <u>not</u> conservative, show $\mathrm{curl}(\vec{F}) \neq \vec{0}$.

Example: Show that $\vec{F} = \langle \underbrace{y}_{F_1}, \underbrace{z}_{F_2}, \underbrace{x}_{F_3} \rangle$ is not conservative.

Show $\text{curl}(\vec{F}) \neq \vec{0}$

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \overset{+}{\hat{i}} & \overset{-}{\hat{j}} & \overset{+}{\hat{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = (0-1)\hat{i} - (1-0)\hat{j} + (0-1)\hat{k}$$

$$= \langle -1, -1, -1 \rangle \neq \underbrace{\langle 0, 0, 0 \rangle}_{\vec{0}} \quad \therefore \text{ Not conservative!}$$

not conservative vector field, won't be able to find a potential function.

Example: Find a potential function for $\vec{F} = \langle 2x\sin(y), x^2\cos(y) \rangle$ by inspection. (i.e, "Guess" a function $f(x,y)$ such that $\nabla f = \vec{F}$)

$f(x,y) = x^2\sin(y) + C$ potential function is unique up to a constant added on (not needed).

Check: $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle \underbrace{2x\sin(y)}_{F_1}, \underbrace{x^2\cos(y)}_{F_2} \rangle = \vec{F}$ ✓

Since $\vec{F}$ is conservative from Thm 1: $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$

$2x\cos(y) = 2x\cos(y)$ ✓

Section 16.2 : Line Integrals

Scalar Line Integrals:

Thm 1: Computing a scalar line integral

Let $\vec{r}(t)$ be a parameterization that directly traverses a curve C for $a \le t \le b$. If $f(x,y,z)$ and $\vec{r}(t)$ are continuous, then

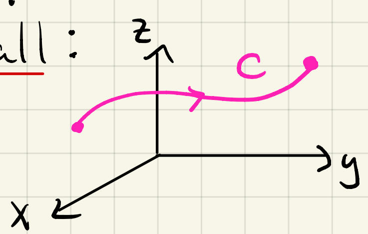
$$\int_C \underbrace{f(x,y,z)}_{\text{integrand is scalar function}} ds = \int_a^b f(\vec{r}(t)) \|\vec{r}\,'(t)\| \, dt \quad \text{(similiar in 2D)}$$

curve/path

- $ds = \|\vec{r}\,'(t)\| \, dt$ is called the arc length differential
- The value of the scalar line integral does not depend on the parameterization of C used as long as C is only traced once from $t=a$ to $t=b$.

May wish to review sections 13.1 – 13.3 in the textbook:

recall:



C: curve in $\mathbb{R}^2$ or $\mathbb{R}^3$

$\vec{r}(t)$: parameterization for C for $a \le t \le b$

in $\mathbb{R}^3$, $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ (similiar in 2D)

$$\Rightarrow \|\vec{r}\,'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

Arc length: $s(t) = \int_a^t \|\vec{r}\,'(u)\| \, du$

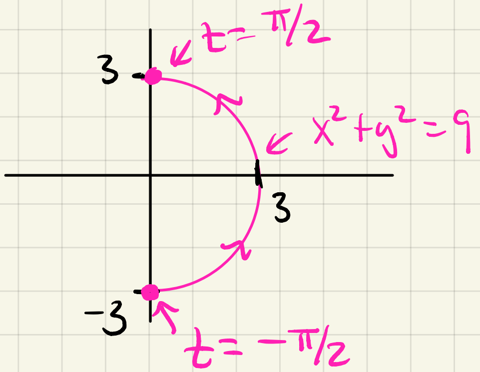
$$\frac{ds}{dt} = s'(t) = \|\vec{r}\,'(t)\| : \text{speed}$$

$ds = s'(t) \, dt$

$ds = \|r'(t)\| \, dt$

Example: Evaluate $\int_C \underbrace{(x^2+y^2)}_{f(x,y)}\,ds$, where C is the right half of the circle $x^2+y^2=9$.

Recall circle of radius r centered at $(0,0)$: $\vec{r}(t)=\langle r\cos t, r\sin t\rangle$, $0 \le t \le 2\pi$



$$C:\ \vec{r}(t)=\langle \underbrace{3\cos t}_{x(t)}, \underbrace{3\sin t}_{y(t)}\rangle,\quad \underbrace{-\pi/2}_{a} \le t \le \underbrace{\pi/2}_{b}$$

$$\vec{r}\,'(t)=\langle -3\sin t, 3\cos t\rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{9\sin^2 t + 9\cos^2 t} = \sqrt{9} = 3$$

$$\int_C \underbrace{(x^2+y^2)}_{f(x,y)}\,ds = \int_{-\pi/2}^{\pi/2} \underbrace{\left[(3\cos t)^2+(3\sin t)^2\right]}_{f(\vec{r}(t))}\ \underbrace{3\,dt}_{\|\vec{r}\,'(t)\|\,dt}$$

$$= \int_{-\pi/2}^{\pi/2} 9\cdot 3\,dt$$

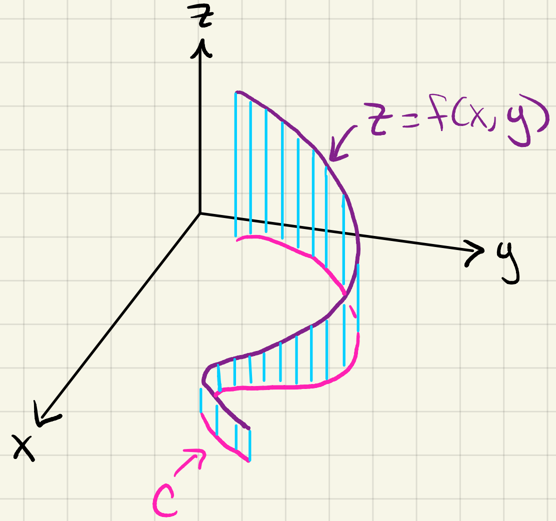
$$= 27t\,\Big|_{-\pi/2}^{\pi/2}$$

$$= 27\pi$$

Notes:

1.) $\int_C 1\, ds = \text{length}(C)$

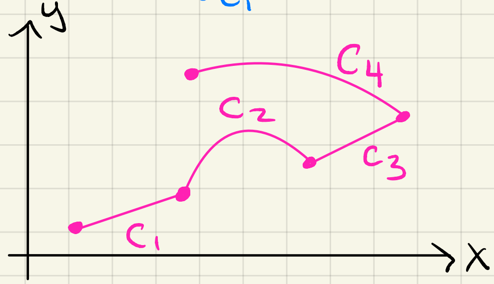
2.) If $f(x,y) \geq 0$, $\int_C f(x,y)\,ds$ represents the area of one side of the ribbon-like structure whose base is C and whose height above the point (x,y) is $f(x,y)$. "curtain"



3.) If C is piece-wise smooth, i.e., a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$. Then

$$\int_C f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds + \ldots + \int_{C_n} f(x,y,z)\,ds$$ similiar in 2D

Ex:



4.) Applications: Find total mass of a wire in the shape of C with mass density f. Find total charge along C with charge density f.

Example: Evaluate $\int_C 2x\, ds$ where C consists of the arc of the parabola $y=x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment from $(1,1)$ to $(1,2)$.