Section 15.2: Double Integrals Over More General Regions (cont.)

Now we consider $\iint_D f(x,y)\,dA$ where D is not a rectangle.

I.) D is a vertically simple region:

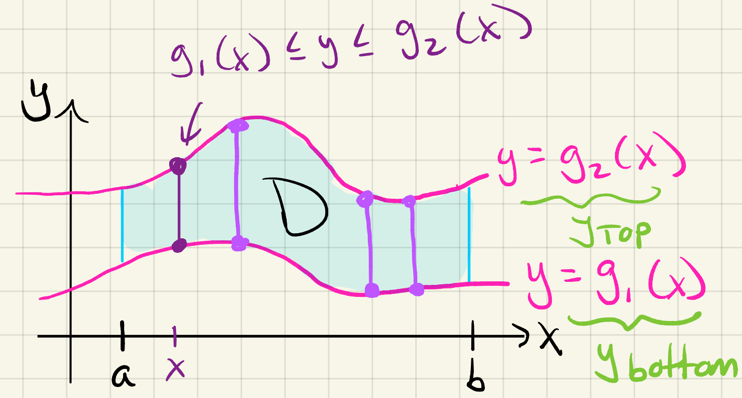
$$D: a \le x \le b, \quad \underbrace{g_1(x)}_{y_{bottom}} \le y \le \underbrace{g_2(x)}_{y_{Top}}$$

Then,

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

1

2



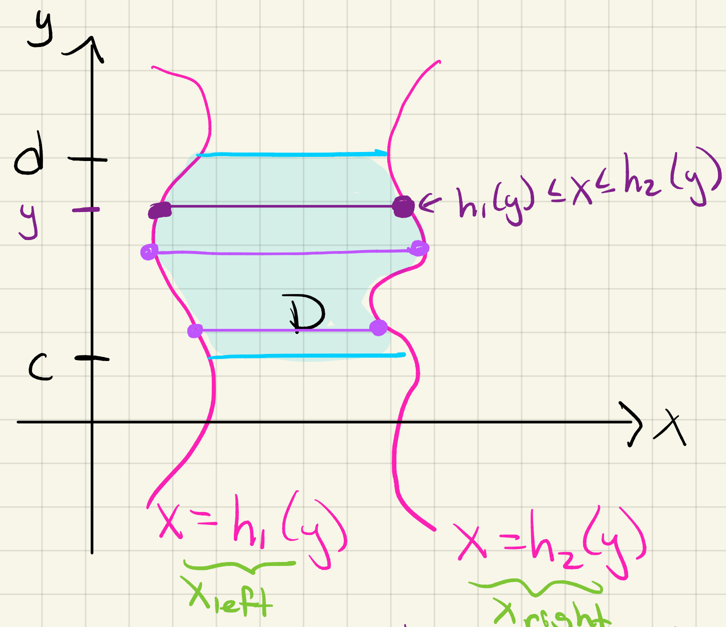
Fix x & integrate with respect to y first

II.) D is a horizontally simple region:

$$D: c \le y \le d, \quad \underbrace{h_1(y)}_{x_{left}} \le x \le \underbrace{h_2(y)}_{x_{right}}$$

Then,

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

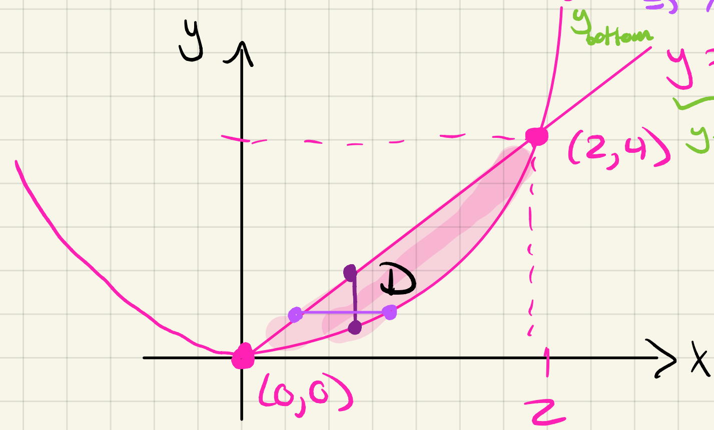


Fix y & integrate with respect to x first.

Examples: Evaluate

1.) $\iint_D (x^2+y)\,dA$, where D is the region bounded by $y=x^2$ & $y=2x$.

Sketch the region D:



Vertically simple and it is also horizontally simple.

As vertically simple region:

$$D: \quad 0 \le x \le 2$$
$$\underbrace{x^2}_{y\ bottom} \le y \le \underbrace{2x}_{y\ Top}$$

$$\iint_D (x^2+y)\,dA = \int_0^2 \underbrace{\int_{x^2}^{2x} (x^2+y)\,dy}_{\text{fix } x \text{ \& integrate wrt } y \text{ first.}}\,dx$$

$$= \int_0^2 \underbrace{\left(x^2y+\frac{y^2}{2}\right)\Big|_{y=x^2}^{y=2x}}_{\text{Apply FTC}}\,dx$$

$$= \int_0^2 \left[\left(x^2(2x)+\frac{(2x)^2}{2}\right)-\left(x^2(x^2)+\frac{(x^2)^2}{2}\right)\right]dx \quad \text{(simplify)}$$

$$= \int_0^2 \left(2x^3+2x^2-\frac{3}{2}x^4\right)dx \quad \text{(calc 1 integral ... use power rule)}$$

$$= \ldots = 56/15$$

Find the points of intersection

$$x^2 = 2x$$
$$\Rightarrow x^2-2x=0$$
$$\Rightarrow x(x-2)=0 \quad \therefore x=0,\ x=2$$

As horizonally simple region: $D: \ 0 \le y \le 4$, $\ \frac{y}{2} \le x \le \sqrt{y}$

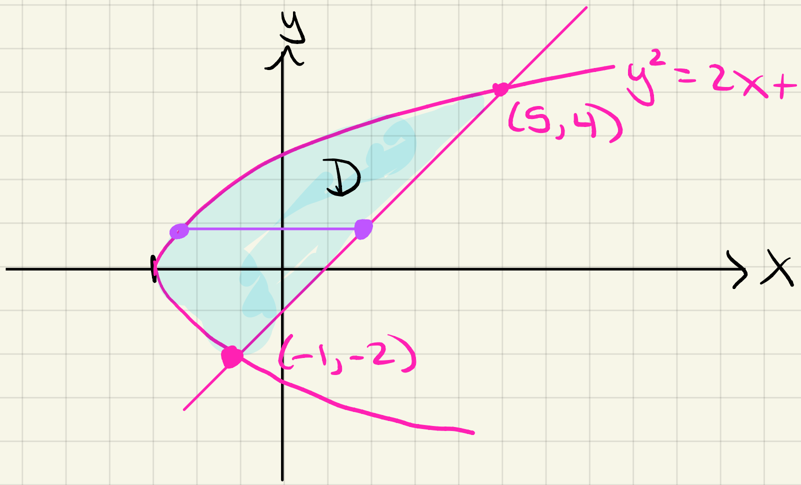
$$\Rightarrow \iint_D (x^2+y)\,dA = \int_0^4 \int_{y/2}^{\sqrt{y}} (x^2+y)\,dx\,dy = \ldots = \frac{56}{15}$$

2.) $\iint_D xy\,dA$, where D is the region bounded by $y = x-1$ & $y^2 = 2x+6$

Sketched below.

$y = x-1 \Rightarrow \underbrace{x = y+1}_{x_{right}}$

$y^2 = 2x+6 \Rightarrow \underbrace{x = \frac{y^2-6}{2}}_{x_{left}}$



D is $\underline{\underline{\text{not}}}$ vertically simple

D is horizontally simple!

To find y limits of integration find points of intersection

$$\frac{y^2-6}{2} = y+1$$

$$y^2-6 = 2y+2$$

$$y^2-2y-8 = 0$$

$$(y-4)(y+2) = 0$$

$$\Rightarrow y = 4,\ y = -2$$

Find D as a horizontally simple region:

$$D: \begin{array}{l} -2 \le y \le 4 \\ \underbrace{\frac{y^2-6}{2}}_{\substack{x_{left} \\ h_1(y)}} \le x \le \underbrace{y+1}_{\substack{x_{right} \\ h_2(y)}} \end{array}$$

Set up integral & evaluate:

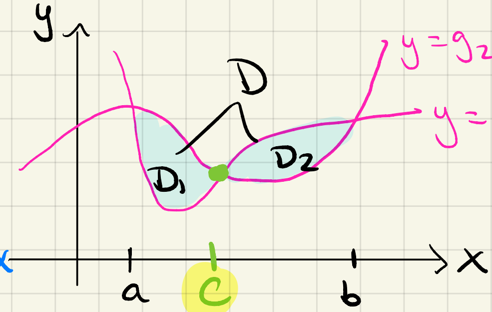
$$\iint_D xy\,dA = \int_{-2}^{4} \int_{\frac{y^2-6}{2}}^{y+1} xy\,dx\,dy$$

$$= \int_{-2}^{4} \underbrace{\frac{1}{2}x^2 y \Big|_{x=\frac{y^2-6}{2}}^{x=y+1}}_{\text{Apply FTC}} dy$$

$$= \int_{-2}^{4} \underbrace{\left[\frac{1}{2}(y+1)^2 y - \frac{1}{2}\left(\frac{y^2-6}{2}\right)^2 y\right]}_{\text{expand \& simplify then integrate again}} dy$$

$$= \ldots = 36$$

What if D is not a simple region?



$\ast$ not a simple region.

• Split into a sum of simple regions!

$$\iint_D f(x,y)\,dA = \int_a^{c}\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx + \int_c^{b}\int_{g_2(x)}^{g_1(x)} f(x,y)\,dy\,dx$$

• Double integrals over more general regions D satisfy the same linearity properties (Thm 2, sec. 15.1) as rectangular regions R.

<u>Theorem 3</u>: Let $f(x,y)$ and $g(x,y)$ be integrable functions on D.

(a) If $f(x,y) \le g(x,y)$ for all $(x,y) \in D$, then

$$\iint_D f(x,y)\,dA \le \iint_D g(x,y)\,dA$$

(b) If $m \le f(x,y) \le M$ for all $(x,y) \in D$, then $\quad m$ & M constants.

$$m \cdot \text{Area}(D) \le \iint_D f(x,y)\,dA \le M \cdot \text{Area}(D)$$

<u>Decomposing the Domain into Smaller Domains</u>:

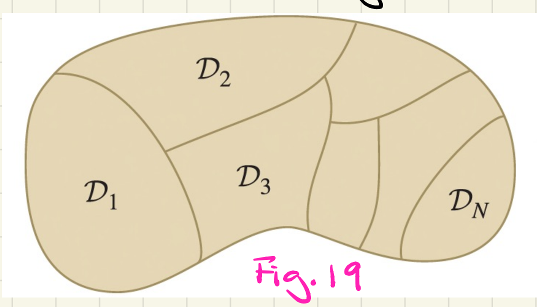


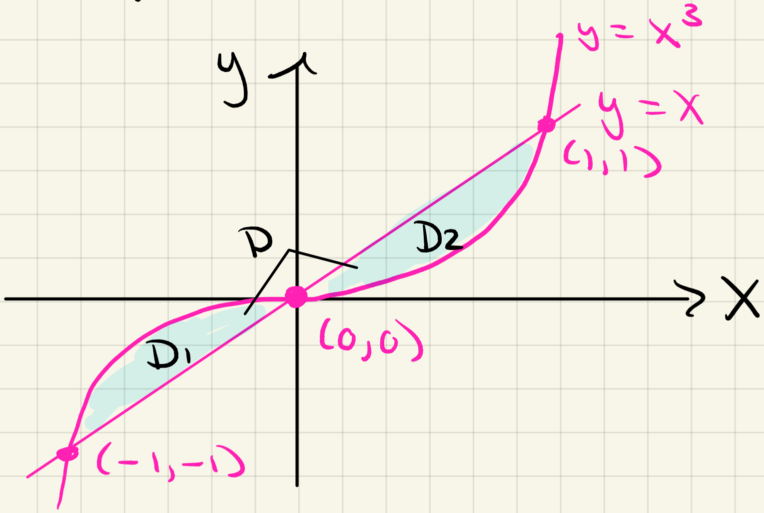
Fig. 19

Double integrals are additive with respect to the domain: If D is the union of domains $D_1, D_2, \ldots, D_N$ that do not overlap except possibly on boundary curves (fig. 19), then

$$\boxed{\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \cdots + \iint_{D_N} f(x,y)\,dA}$$

Example: Evaluate $\iint_D xy\,dA$ where D: region between $y=x^3$ and $y=x$

Sketch D



Split into sum of two vertically simple regions:

D_1 : $-1 \le x \le 0$

$\underbrace{x}_{y_{bottom}} \le y \le \underbrace{x^3}_{y_{Top}}$

D_2 : $0 \le x \le 1$

$\underbrace{x^3}_{y_{bottom} \text{ for } D_2} \le y \le \underbrace{x}_{y_{Top} \text{ for } D_2}$

To find the x-values of D_1 and D_2

Find points of intersection

$$x^3 = x$$

$$\Rightarrow x^3 - x = 0$$

$$\Rightarrow x(x^2-1) = 0$$

$$\Rightarrow x(x+1)(x-1) = 0$$

$$\Rightarrow x=0,\ x=-1,\ x=1$$

$$\iint_D xy\,dA = \iint_{D_1} xy\,dA + \iint_{D_2} xy\,dA$$

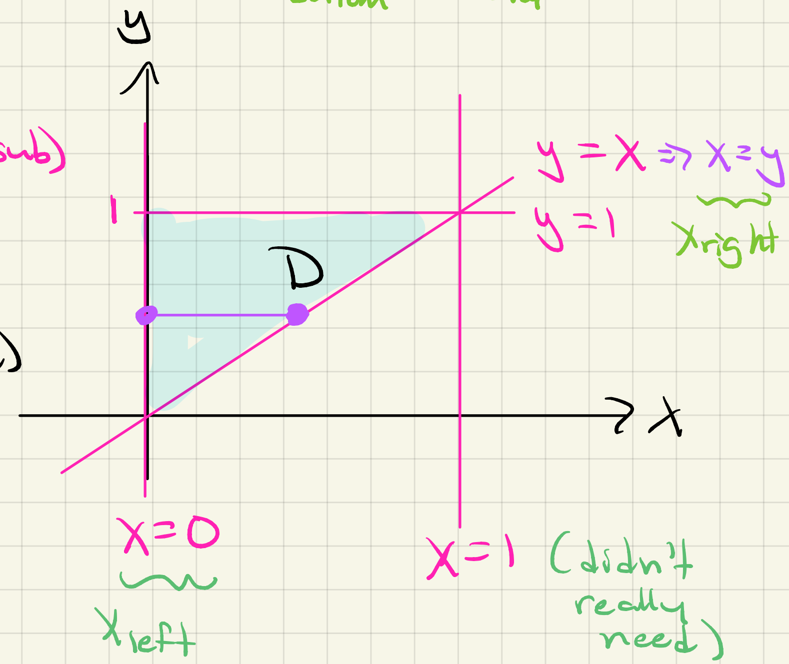
$$= \underbrace{\int_{-1}^{0}\int_{x}^{x^3} xy\,dy\,dx}_{} + \underbrace{\int_{0}^{1}\int_{x^3}^{x} xy\,dy\,dx}_{}$$

Integrate & evaluate and sum answers

$$= \ldots = 1/8$$

Example: Evaluate $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$ Consider D: $\underset{y_{bottom}}{0 \le x \le 1}$ (vertically simple)
$\underbrace{x}_{y_{bottom}} \le y \le \underbrace{1}_{y_{top}}$

not an integral we can solve as written
(integrand not elementary function, can't use u-sub)



Rewrite the problem by changing the order of integration. (originally set up as vertically simple)

Write D as a horizontally simple region and integrate wrt x first!

D: $0 \le y \le 1$
$0 \le x \le y$ (horizontally simple)

$$\int_{x=0}^{1} \int_{y=x}^{1} \sin(y^2)\,dy\,dx = \int_{y=0}^{1} \int_{x=0}^{y} \sin(y^2)\,dx\,dy$$

$$= \int_0^1 \sin(y^2)\,x \Big|_{x=0}^{x=y} dy$$

$$= \int_0^1 \left(y\sin(y^2) - \cancel{0}\right) dy$$

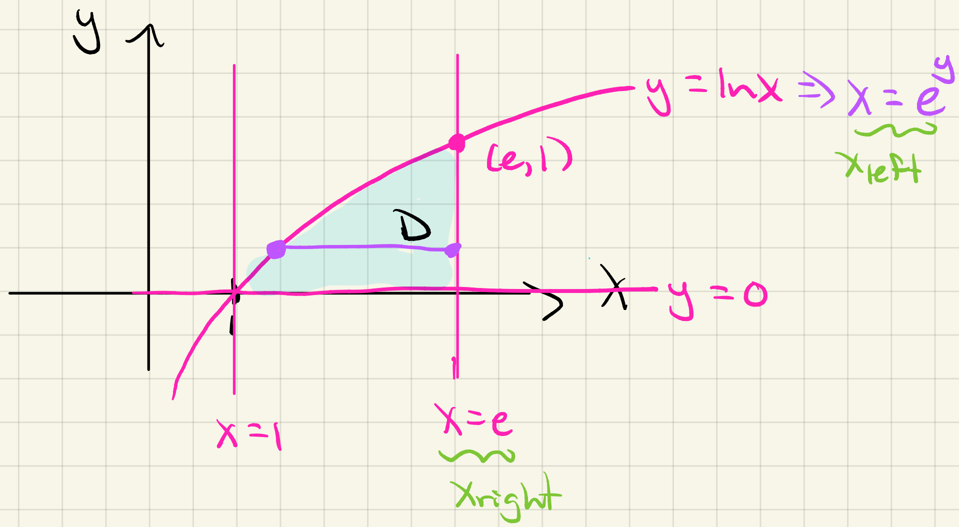
$$= \int_0^1 y\sin(y^2)\,dy \longrightarrow \begin{cases} u = y^2 \\ du = 2y\,dy \\ \frac{1}{2}du = y\,dy \end{cases} \quad \begin{matrix} y=0: u = 0^2 = 0 \\ y = 1: u = 1^2 = 1 \end{matrix}$$

$$= \int_{u=0}^{u=1} \frac{1}{2}\sin(u)\,du$$

$$= -\frac{1}{2}\cos(u)\Big|_{u=0}^{u=1} = -\frac{1}{2}\Big(\cos(1) - \underbrace{1}_{\cos(0)}\Big)$$

Example: $\displaystyle\int_1^e \int_0^{\ln x} f(x,y)\,dy\,dx$. Change the order of integration.

Sketch D :



$$D: \begin{array}{l} 1 \le x \le e \\ 0 \le y \le \ln x \end{array}$$ (vertically simple)

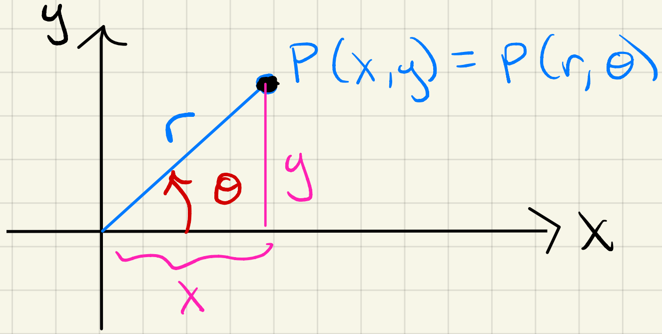
Rewrite D as horizontally simple

$$D: \begin{array}{l} 0 \le y \le 1 \\ e^y \le x \le e \end{array}$$

$$\int_1^e \int_0^{\ln x} f(x,y)\,dy\,dx = \boxed{\int_0^1 \int_{e^y}^{e} f(x,y)\,dx\,dy}$$

Section 15.4 : Integration in Polar Coordinates

Recall: (calc 2)



Polar to Rectangular

$x = r\cos\theta$

$y = r\sin\theta$

$\cos\theta = \frac{adj}{hyp} = \frac{x}{r}$

$\sin\theta = \frac{opp}{hyp} = \frac{y}{r}$

Rectangular to Polar

$x^2 + y^2 = r^2$

$\tan\theta = \frac{y}{x}, \; x \neq 0.$

(signs of y and x tell us which quadrant θ is in.)

- Review sections 11.3 (Polar cordinates) and 7.2 (Trig Integrals)

What would be analogous to integrating over a rectangle in Polar Coordinates?

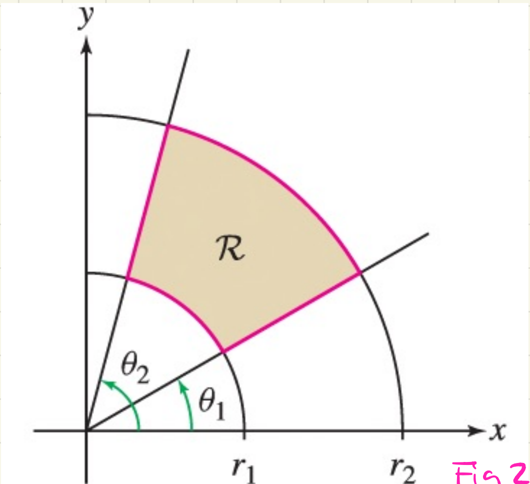


Fig 2

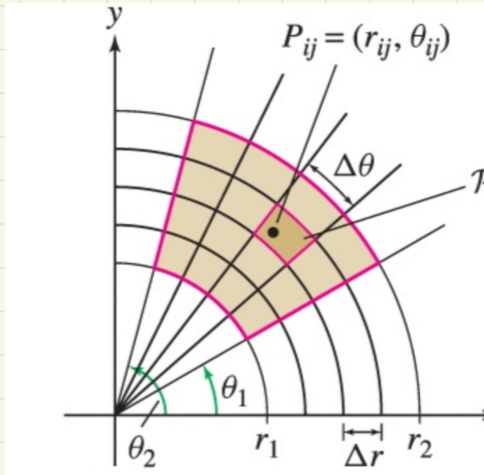


Fig. 5

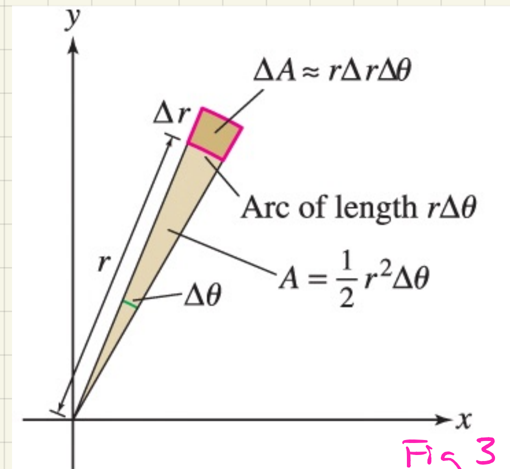
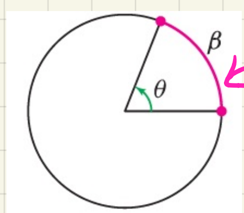


Fig 3

Note:



what is the length of β?

$$(\text{length of } \beta) = \underbrace{\frac{\theta}{2\pi}}_{\substack{\text{fraction of } 2\pi \\ \text{covered by } \theta}} \cdot \underbrace{2\pi r}_{\text{circumference of circle}} = \theta r$$

Double Integral over Polar Rectangle:

For a continuous function f on the Polar rectangle domain

$R: \theta_1 \le \theta \le \theta_2, \; r_1 \le r \le r_2$ (assume $r_1 \ge 0$)

$$\boxed{\iint_R f(x,y)\,dA = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

Note: Fubini's Theorem applies and the iterated integral (in either order) is equal.

Theorem 1 - Double Integral in Polar Coordinates:

For a continuous function f on the domain

$D: \theta_1 \le \theta \le \theta_2, \; r_1(\theta) \le r \le r_2(\theta)$

$$\boxed{\iint_D f(x,y)\,dA = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta_1)}^{r_2(\theta_2)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

Here, the region D is radially simple.

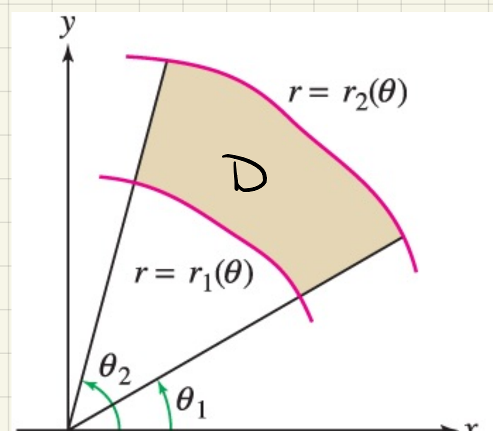
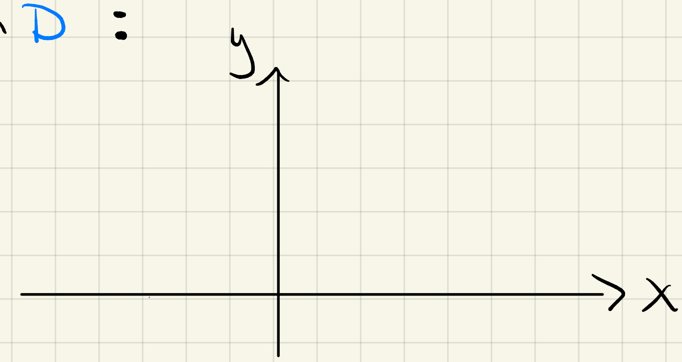


Fig. 6

Examples:

1.) Evaluate $\iint_D (3x+4y^2)\,dA$, where D is the region in the upper half-plane $(y \geq 0)$ bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Sketch D :



Recall: Double-angle Identities

$$\sin^2\theta = \frac{1}{2}\left(1-\cos(2\theta)\right)$$

$$\cos^2\theta = \frac{1}{2}\left(1+\cos(2\theta)\right)$$

2.) $\iint_D x \, dA$, where D is the part of the disk $x^2+y^2=9$ with $x \geq 0, y \geq x$.

3.) Evaluate the iterated integral $\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{\sqrt{1+x^2+y^2}} \, dx \, dy$ by first converting to Polar coordinates.

4.) Let D be the region in the xy-plane given by $x^2+y^2 \leq 2$, $x \geq 1$.
Evaluate $\iint_D \frac{1}{(x^2+y^2)^2}\, dA$ using polar coordinates.