Announcements:

- There are no classes on Monday, 2/17.
- Tuesday, 2/18 follows a Monday schedule, so we have lecture on Tuesday, 2/18.
  - There are no Tuesday recitations (2/18)

Section 16.2 : Line Integrals (cont.)

Scalar Line Integrals:

Thm 1: Computing a scalar line integral

Let $\vec{r}(t)$ be a parameterization that directly traverses a curve C for $a \le t \le b$. If $f(x,y,z)$ and $\vec{r}(t)$ are continuous, then

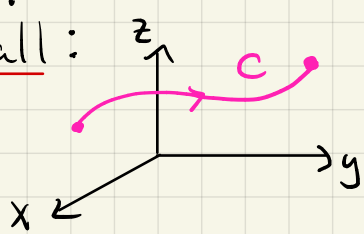
$$\int_C \underbrace{f(x,y,z)}_{\text{integrand is scalar function}} ds = \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt \quad \text{(similiar in 2D)}$$

curve/path

- $ds = \|\vec{r}\,'(t)\|\,dt$ is called the arc length differential
- The value of the scalar line integral does not depend on the parameterization of C used as long as C is only traced once from $t=a$ to $t=b$.

May wish to review sections 13.1 – 13.3 in the textbook:

recall:



C: curve in $\mathbb{R}^2$ or $\mathbb{R}^3$

$\vec{r}(t)$: parameterization for C for $a \le t \le b$

in $\mathbb{R}^3$, $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ (similiar in 2D)

$$\Rightarrow \|\vec{r}\,'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

Arc length: $s(t) = \int_a^t \|\vec{r}\,'(u)\|\,du$

$\dfrac{ds}{dt} = s'(t) = \|\vec{r}\,'(t)\|$: speed

$ds = s'(t)\,dt$

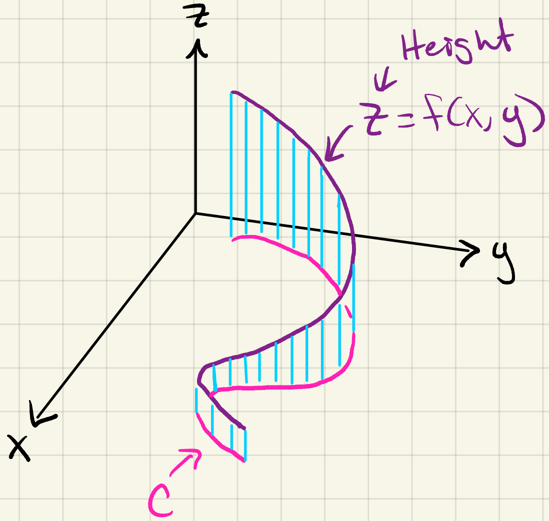
$ds = \|r'(t)\|\,dt$

Notes:

1.) $\int_C 1\, ds = \text{length}(C)$

"curtain"

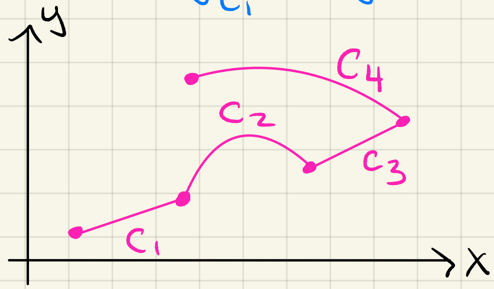
2.) If $f(x,y) \geq 0$, $\int_C f(x,y)\,ds$ represents the area of one side of the ribbon-like structure whose base is C and whose height above the point (x,y) is $f(x,y)$.



3.) If C is piece-wise smooth, i.e., a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$. Then

$$\int_C f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds + \ldots + \int_{C_n} f(x,y,z)\,ds$$ similiar in 2D

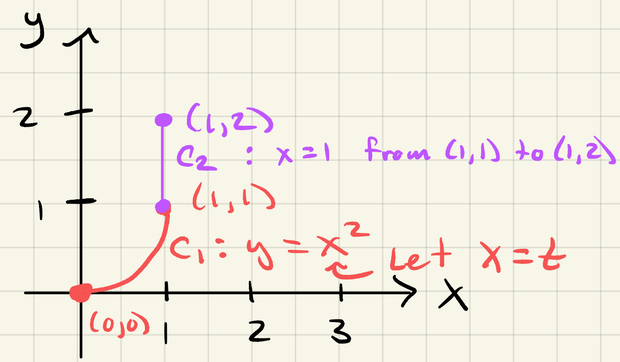
Ex:



$C = C_1 + C_2 + C_3 + C_4$

4.) Applications: Find total mass of a wire in the shape of C with mass density f. Find total charge along C with charge density f.

Example: Evaluate $\int_C 2x\,ds$ where C consists of the arc of the parabola $y=x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment from $(1,1)$ to $(1,2)$.



$$C = C_1 + C_2$$

C_1: $y=x^2$ from $(0,0)$ to $(1,1)$

C_2: $x=1$ from $(1,1)$ to $(1,2)$

$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds$$

$f(x,y)=2x$

$$C_1 : \vec{r}_1(t) = \langle x_1(t), y_1(t)\rangle = \langle t, t^2\rangle, \quad 0 \le t \le 1$$

since $x=t$

$$\vec{r}_1{}'(t) = \langle 1, 2t\rangle \Rightarrow \|\vec{r}_1{}'(t)\| = \sqrt{1^2+(2t)^2} = \sqrt{1+4t^2}$$

$$C_2 : \vec{r}_2(t) = \langle x_2(t), y_2(t)\rangle = \langle 1, t\rangle, \quad 1 \le t \le 2$$

since $x=1$ on line segment

$$\vec{r}_2{}'(t) = \langle 0,1\rangle \Rightarrow \|\vec{r}_2{}'(t)\| = \sqrt{0^2+1^2} = 1$$

$$\int_C 2x\,ds = \int_0^1 f(\vec{r}_1(t))\|\vec{r}_1{}'(t)\|\,dt + \int_1^2 f(\vec{r}_2(t))\|\vec{r}_2{}'(t)\|\,dt$$

$$= \int_0^1 \underbrace{2t\sqrt{1+4t^2}}\,dt + \int_1^2 2(1)(1)\,dt$$

$u = 1+4t^2$

$du = 8t\,dt \ldots$

$$= \frac{1}{\cancel{4}\,2}\cdot\frac{2}{3}(1+4t^2)^{3/2}\Big|_0^1 + 2t\Big|_1^2 = \frac{1}{6}(5^{3/2}-1)+2$$

Example: Calculate the total mass of a metal tube in the helical shape $\vec{r}(t) = \langle \cos t, \sin t, t^2 \rangle$ (distance in centimeters) for $0 \le t \le 2\pi$ if the mass density is $\rho(x,y,z) = \sqrt{z}$ g/cm.

Total mass : $M = \int_C \rho(x,y,z)\, ds = \int_C \sqrt{z}\, ds$

$$C : \vec{r}(t) = \langle \underbrace{\cos t}_{x(t)}, \underbrace{\sin t}_{y(t)}, \underbrace{t^2}_{z(t)} \rangle, \quad 0 \le t \le 2\pi$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 2t \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{\underbrace{\sin^2 t + \cos^2 t}_{=1} + (2t)^2} = \sqrt{1+4t^2}$$

$$M = \int_C \sqrt{z}\, ds = \int_0^{2\pi} \underbrace{\sqrt{t^2}}_{\rho(\vec{r}(t))} \underbrace{\sqrt{1+4t^2}}_{\|\vec{r}\,'(t)\|}\, dt$$

$$= \int_0^{2\pi} t\sqrt{1+4t^2}\, dt$$

$u = 1+4t^2$ $\quad$ $t=0 : u = 1$

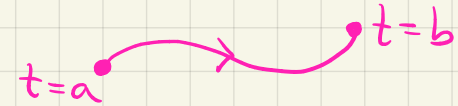
$du = 8t\, dt$ $\quad$ $t = 2\pi : u = 1+16\pi^2$

$\frac{1}{8} du = t\, dt$

$$= \frac{1}{8} \int_1^{1+16\pi^2} u^{1/2}\, du$$

$$= \frac{1}{12}\left[(1+16\pi^2)^{3/2} - 1 \right] \text{ g}$$

Vector Line Integrals:



Let $\vec{F}$ be a continuous vector field defined on a smooth, oriented curve C given by $\vec{r}(t)$, $a \le t \le b$. Then the line integral of $\vec{F}$ along C is:

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_C \vec{F} \cdot \underbrace{\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}}_{\vec{T}} \underbrace{\|\vec{r}\,'(t)\|\,dt}_{ds} = \int_C \vec{F} \cdot \vec{T}\,ds$$

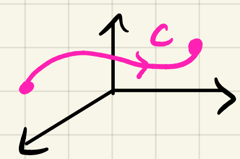
dot product dot product

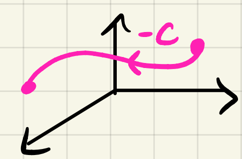
$\vec{T}$: unit tangent vector (13.4)

In the last integral, $\vec{T}$ is the unit vector to the curve C.

Notes:

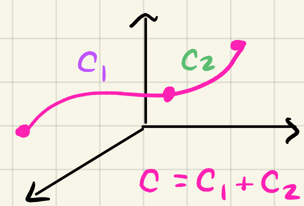
1.) Reversing the orientation: $\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$





2.) If C is piece-wise smooth, i.e., a union of a finite number of smooth curves $C_1, C_2, \dots, C_n$. Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} + \dots + \int_{C_n} \vec{F} \cdot d\vec{r}$$



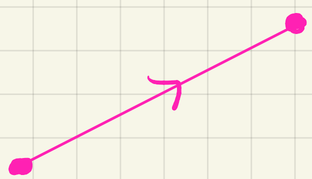
3.) Application:

Work done by $\vec{F}$ in moving a particle along C: $W = \int_C \vec{F} \cdot d\vec{r}$

Work performed against $\vec{F} = -\int_C \vec{F} \cdot d\vec{r}$

Example: Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = \langle yz, -z, y\rangle$ and C is the line segment from $\underbrace{(-1,1,2)}_{P}$ to $\underbrace{(2,4,3)}_{Q}$.

recall the vector parameterization for a line through a point P (Sec 12.2) or a line segment from P to Q.



$$\vec{r}(t) = \underbrace{\langle x_1, y_1, z_1\rangle}_{\vec{P}\text{ : position vector}} + t\,\overbrace{\langle x_2-x_1, y_2-y_1, z_2-z_1\rangle}^{\overrightarrow{PQ}},\quad 0\le t\le 1$$

$$= \langle x_1 + (x_2-x_1)t,\ y_1 + (y_2-y_1)t,\ z_1 + (z_2-z_1)t\rangle,\quad 0\le t\le 1$$

$$C:\ \vec{r}(t) = \langle -1,1,2\rangle + t\langle 2-(-1), 4-1, 3-2\rangle,\quad 0\le t\le 1$$

$$= \langle \underbrace{-1+3t}_{x(t)}, \underbrace{1+3t}_{y(t)}, \underbrace{2+t}_{z(t)}\rangle,\quad 0\le t\le 1$$

$$\vec{r}\,'(t) = \langle 3,3,1\rangle$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^1 \langle \underbrace{(1+3t)}_{y}\underbrace{(2+t)}_{z}, \underbrace{-(2+t)}_{-z}, \underbrace{(1+3t)}_{y}\rangle \cdot \langle 3,3,1\rangle\,dt$$

$$= \int_0^1 \left[3(1+3t)(2+t) - 3(2+t) + (1+3t)\right] dt$$

$$= \int_0^1 (9t^2 + 21t + 1)\,dt = 29/2$$

Another Notation: If $\vec{F} = \langle F_1, F_2, F_3 \rangle$ and $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$, then

$\vec{r}'(t) = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \Big(\underbrace{F_1(\vec{r}(t)) \frac{dx}{dt} + F_2(\vec{r}(t)) \frac{dy}{dt} + F_3(\vec{r}(t)) \frac{dz}{dt}}_{\text{expansion of dot product}} \Big) dt$$

$$= \underbrace{\int_C F_1\,dx + F_2\,dy + F_3\,dz}_{\text{alternate notation}} \quad \text{(Similiar in 2D)}$$

Example: Evaluate $\int_C \underbrace{y^2}_{F_1} dx + \underbrace{x}_{F_2} dy$ where C is the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$.

Let $y = t$

$$\vec{F} = \langle F_1, F_2 \rangle = \langle y^2, x \rangle$$

$$C: \vec{r}(t) = \langle x(t), y(t) \rangle = \langle 4 - t^2, t \rangle, \quad -3 \le t \le 2$$

$$x(t) = 4 - t^2 \Rightarrow \frac{dx}{dt} = -2t$$

$$y(t) = t \Rightarrow \frac{dy}{dt} = 1$$

Integrand: (with dt)

$$y^2\,dx + x\,dy = \left[y^2 \left(\frac{dx}{dt} \right) + x \left(\frac{dy}{dt} \right) \right] dt$$

$$= \left[t^2(-2t) + (4 - t^2)(1) \right] dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C y^2\,dx + x\,dy = \int_{-3}^{2} (-2t^3 + 4 - t^2)\,dt$$

$$= 245/6$$

Section 16.3: Conservative Vector Fields

If C is a closed curve, we often refer to the line integral of any vector field $\vec{F}$ around C as the circulation of $\vec{F}$ around C denoted:

notation: C is closed $\rightarrow$ $\oint_C \vec{F}\cdot d\vec{r}$.

If $\vec{r}(t)$, $a \le t \le b$ is the parameterization of the closed curve C, then $\vec{r}(a) = \vec{r}(b)$.

Recall: A vector field $\vec{F}$ is called conservative if it is equal to the gradient of some scalar function f, i.e., $\vec{F} = \nabla f$ for some function f. Such an f is called a potential function of $\vec{F}$.

Let C be parameterized by $\vec{r}(t)$, $a \le t \le b$ with $\underbrace{\vec{r}(a) = P}_{\text{start}}$ and $\underbrace{\vec{r}(b) = Q}_{\text{end}}$. Let $\vec{F}$ be a conservative vector field, then $\vec{F} = \nabla f$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt = \int_a^b \underbrace{\nabla f(\vec{r}(t))\cdot \vec{r}\,'(t)}\,dt$$

$$\overset{\text{Chain Rule}}{=} \int_a^b \underbrace{\frac{d}{dt}\left[f(\vec{r}(t))\right]}\,dt$$

$$\overset{\text{FTC}}{=} f(\vec{r}(t))\Big|_{t=a}^{t=b}$$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

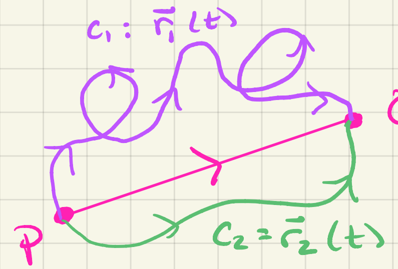
$$= f(Q) - f(P)$$

Thus, a line integral over a conservative vector field depends only on the the end points P and Q, not the actual path from P to Q!

Fundamental Theorem for Conservative Vector Fields:

Assume that $\vec{F} = \nabla f$ on a domain D, (i.e, $\vec{F}$ is conservative)

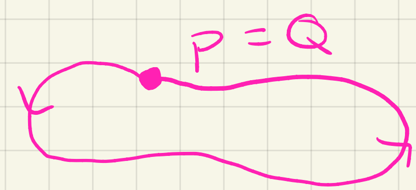
1.) If $\vec{r}(t)$, $a \le t \le b$, is a path along a curve C from P to Q in D, then



$$\int_C \vec{F} \cdot d\vec{r} = f(Q) - f(P).$$

We say that $\vec{F}$ is path independent.

2.) If C is a closed curve, i.e., $P = Q$, then



$$\int_C \vec{F} \cdot d\vec{r} = 0$$

Recall: If $\vec{F}$ is conservative, then $\text{curl}(\vec{F}) = \vec{0}$ or equivalently,

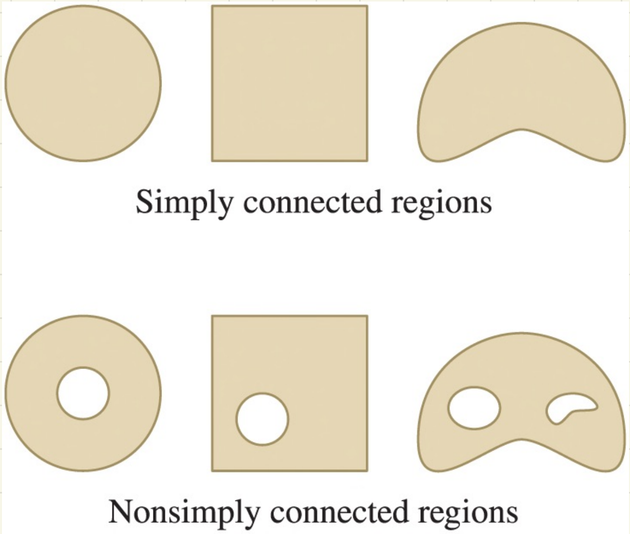
$$\underbrace{\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}}_{\vec{F} \text{ in } \mathbb{R}^2\text{, this is the only cross-partial condition}}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z} \quad \text{(cross-partials condition)}$$

The converse is only true sometimes.

- $\vec{F}$ needs to have a simply connected domain.

In $\mathbb{R}^2$, a domain is simply connected if it doesn't have any "holes"

Examples in $\mathbb{R}^2$: $\mathbb{R}^2$ itself is simply connected



Similiar in $\mathbb{R}^3$: no holes passing through D: boxes, spheres, $\mathbb{R}^3$ itself.

Theorem Existence of a potential function: Let $\vec{F}$ be a vector field on a simply connected domain D. If $\vec{F}$ satisfies the cross-partials condition ($\text{curl}(\vec{F}) = \vec{0}$), then $\vec{F}$ is conservative.

Example: a.) Show that $\vec{F} = \langle \underbrace{3x^2+2y^2}_{F_1}, \underbrace{4xy+3}_{F_2} \rangle$ is conservative, then find a potential function for $\vec{F}$. b.) Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where C is given by $\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le \pi/2$.

a) Domain of $\vec{F}$ is $\mathbb{R}^2$, which is simply connected ✓

Check cross-partials condition: $\dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x}$ (in $\mathbb{R}^2$)

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left[3x^2+2y^2\right] = 4y$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left[4xy+3\right] = 4y$$

$=$ ✓ $\quad \therefore\ \vec{F}$ is conservative.

Now, find f such that $\nabla f = \vec{F} \Rightarrow \left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y} \right\rangle = \langle 3x^2+2y^2, 4xy+3 \rangle$

Note: Same result calculating the line integral of $\vec{F}$ along C directly where $\vec{F} = \langle 3x^2 + 2y^2, 4xy + 3 \rangle$, $\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le \pi/2$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt \quad \text{(Sec. 16.2)}$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \langle 3\cos^2 t + 2\sin^2 t, 4\cos t \sin t + 3 \rangle \cdot \langle -\sin t, \cos t \rangle\, dt$$

$$= \int_0^{\pi/2} (-3\cos^2 t \sin t - 2\sin^3 t + 4\cos^2 t \sin t + 3\cos t)\, dt$$

$$= \int_0^{\pi/2} (\cos^2 t \sin t - 2\sin^3 t + 3\cos t)\, dt$$

$$= \int_0^{\pi/2} \cos^2 t \sin t\, dt - 2\int_0^{\pi/2} \sin^3 t\, dt + 3\int_0^{\pi/2} \cos t\, dt$$

$u = \cos t$, $du = -\sin t\, dt$

rewrite as $(1-\cos^2 t)\sin t$ (Sec 7.3) then $u = \cos t$, $du = -\sin t\, dt$

$$= -\frac{1}{3}\cos^3 t \Big|_0^{\pi/2} - 2\left(\frac{1}{3}\cos^3 t - \cos t\right)\Big|_0^{\pi/2} + 3\sin t \Big|_0^{\pi/2}$$

$$= \dots = \frac{1}{3} - \frac{4}{3} + 3 = 2$$

This method is typically more work if $\vec{F}$ is conservative and the integral may be very difficult to solve!

Example: a) Show that $\vec{F} = \langle \cos z, 2y, -x\sin z \rangle$ is conservative, then find a potential function f for $\vec{F}$. b) Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where C is any curve from $(2,1,0)$ to $(1,-1,\pi/2)$.

a)