Announcements:

- Exam 1 is this Wednesday 1/29 during Test Block.
  - Exam 1 information has been posted on LMS in the "Exam Info" tab. Read carefully!
  - Additional Exam 1 review problems are also available in the "Exam Info" tab on LMS. Filled Slides with the worked Solutions will <u>not</u> be provided.
- The UTA's are offering a review session on Tuesday 1/28 from 4-6 PM in Amos Eaton 214. This is <u>optional</u>.
  - There will be no UTA office hours on Wednesday 1/29.

Section 14.7: Optimization in Several Variables (Cont.)

Theorem (Existence and Location of Global Extrema):
Let $f(x,y)$ be a continuous function on a closed, bounded domain D in $\mathbb{R}^2$. Then

1.) $f(x,y)$ takes on both a global maximum and a global minimum value on D.

2.) The extreme values occur either at critical points in the interior of D or at points on the boundary of D.

This is also called the extreme value theorem for functions of 2 variables.

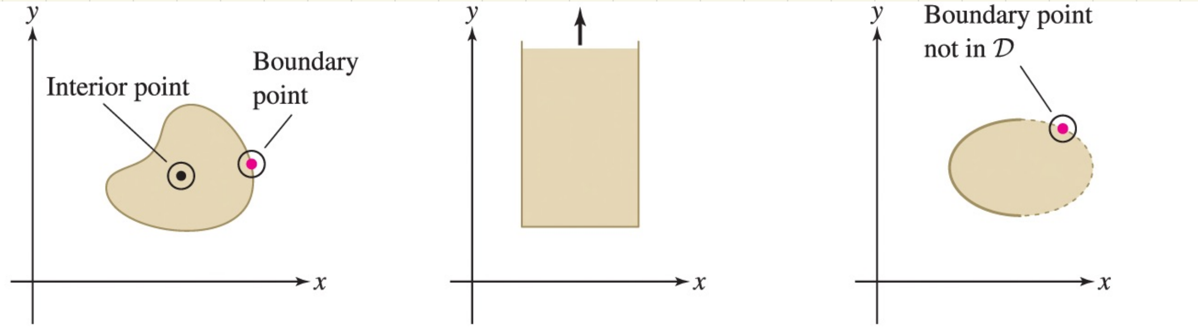


Fig. 11 (A) This domain is bounded and closed (contains all boundary points). (B) An unbounded domain (contains points arbitrarily far from the origin). (C) A nonclosed domain (contains some but not all boundary points).

To find the Absolute Max and Min:

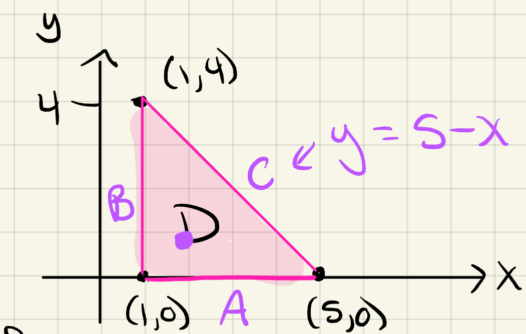
1.) Find the values of f at the critical points of f in D.

2.) Find the extreme values of f on the boundary of D.

3.) The largest from 1 & 2 is the absolute max & the smallest is the absolute min.

global

Example: Find the absolute max & min values of $f(x,y) = 3 + xy - x - 2y$ on the domain D which is the closed region with vertices $(1,0)$, $(5,0)$, $(1,4)$.



1.) $\begin{cases} f_x = y - 1 = 0 & \Rightarrow y = 1 \\ f_y = x - 2 = 0 & \Rightarrow x = 2 \end{cases}$ $\therefore$ critical point: $(2,1)$

$f(2,1) = 3 + \cancel{2} - \cancel{2} - 2 = 1$

2.) A: $\underbrace{y=0,\ 1 \le x \le 5}_{\text{defines line segment for "A"}}$: $f(x,0) = \underbrace{3 - x}_{\text{This is the function on the line segment "A"}}$ is a decreasing function

max is $f(1,0) = 3 - 1 = 2$ & min is $f(5,0) = 3 - 5 = -2$

B: $x = 1,\ 0 \le y \le 4$: $f(1,y) = 3 + y - 1 - 2y = 2 - y$ is a decreasing function

max is $\underbrace{f(1,0) = 2}_{\text{already found}}$ & min is $f(1,4) = 2 - 4 = -2$

C: $\underbrace{y = 5 - x,\ 1 \le x \le 5}_{\text{defines line segment for "c"}}$: $f(x, 5-x) = 3 + x(5-x) - x - 2(5-x) = -x^2 + 6x - 7$

Need to find the max & min values of $f(x, 5-x) = -x^2 + 6x - 7$ on $1 \le x \le 5$. Solve using calc 1.

Let $g(x) = -x^2 + 6x - 7$

$g'(x) = -2x + 6 = 0 \Rightarrow x = 3$

$g(1) = -1 + 6 - 7 = -2$, $g(3) = -9 + 18 - 7 = 2$, $g(5) = -25 + 30 - 7 = -2$

$\| \quad f(1, 5-1) = f(1,4)$ $\quad \| \quad f(3, 5-3) = f(3,2)$ $\quad \| \quad f(5, 5-5) = f(5,0)$

max is $f(3,2) = 2$, min is $f(1,4) = f(5,0) = -2$

The absolute max value of f is 2.

The absolute min value of f is -2.

Section 14.8: Lagrange Multipliers – Optimizing with a Constraint

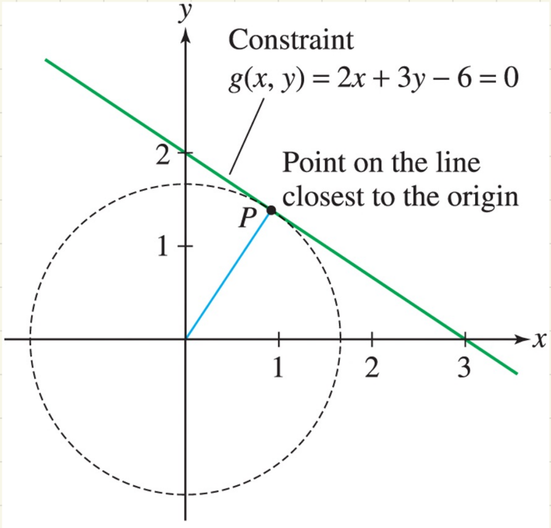
Problem: Find the min (or max) value of $f(x,y)$ subject to the constraint $g(x,y) = 0$.

For example: Find the minimum value of

$$f(x,y) = \sqrt{x^2+y^2}$$

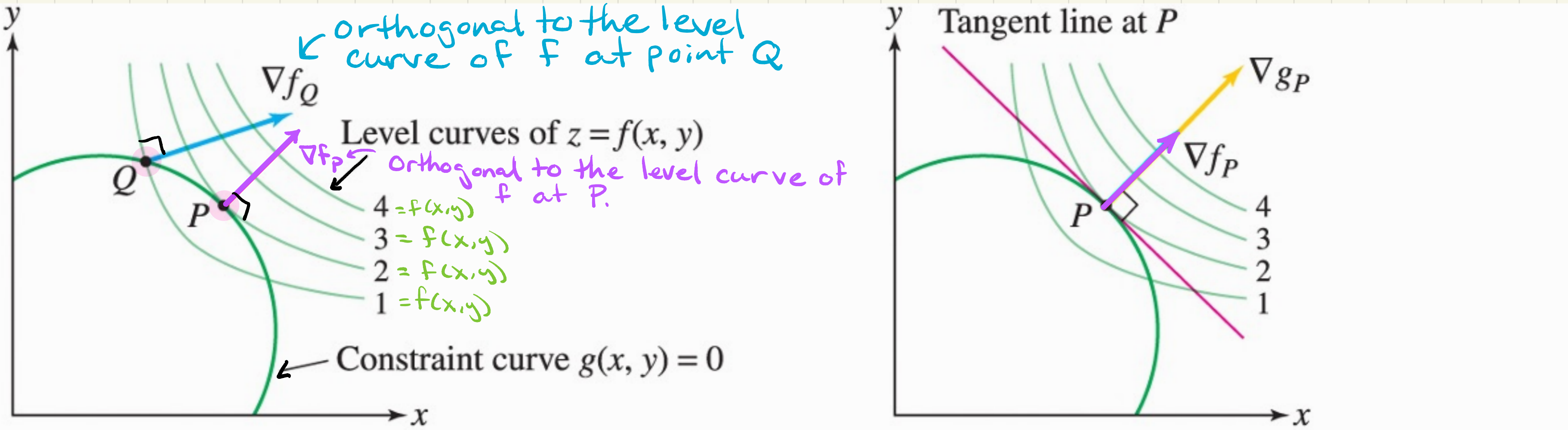
Note: $f(x,y)$ represents the distance of a point (x,y) from the origin.

subject to $\underbrace{2x+3y-6}_{g(x,y)} = 0$ (constraint)



Need to find the minimum value of $f(x,y)$ among all points that lie on the line $2x+3y-6=0 \Rightarrow y = 2 - \frac{2}{3}x$.

The method of Lagrange Multipliers is a procedure for solving optimization problems with constraints.



(A) f increases as we move to the right along the constraint curve.

Fig. 2

(B) The local maximum of f on the constraint curve occurs where ∇f_P and ∇g_P are parallel.

Conclusion: At points that lead to the max/min value of $f(x,y)$, subject to the constraint curve $g(x,y) = 0$, the level curve of f and the constraint curve g are tangent.

∇g_P is orthogonal to the constraint curve at P and thus also orthogonal to the level curve of f at P.

scalar: lambda

$\therefore$ ∇f_P and ∇g_P are parallel: $\nabla f_P = \lambda \nabla g_P$

Lagrange Multiplier

Theorem (Lagrange Multipliers):

Assume that $f(x,y)$ and $g(x,y)$ are differentiable functions. If $f(x,y)$ has a local min or local max on the constraint curve $g(x,y)=0$ at $P=(a,b)$, and if $\nabla g_P \neq \vec{0}$, then there is a scalar λ such that

$$\nabla f_P = \lambda \nabla g_P \quad \text{(Lagrange Condition)}$$

Written in terms of components:

$$\begin{cases} f_x(a,b) = \lambda g_x(a,b) \\ f_y(a,b) = \lambda g_y(a,b) \end{cases} \quad \text{(Lagrange Equations)}$$

Def: A point $P=(a,b)$ satisfying the Lagrange Equations/condition is called a critical point for the optimization problem with constraint and $f(a,b)$ is called a critical value.

General Procedure: To find the extreme values of $f(x,y)$ subject to $g(x,y)=0$:

1.) Solve the system

This system has 3 equations and 3 unknowns

$$* \begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) & \text{Lagrange condition} \\ g(x,y) = 0 & \text{constraint} \end{cases}$$

Extends to more variables and more constraints!

to find critical points.

2.) Calculate the critical values by evaluating f at all critical points. The largest critical value is the max and the smallest critical value is the min.

Example: Find the minimum and maximum values of $f(x,y) = x^2 - y^2$ on the circle $\underbrace{x^2+y^2=1}$.

Constraint: $g(x,y) = x^2+y^2-1 = 0$

Need to solve: $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{cases} \Rightarrow \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \end{cases}$

(1) $2x = \lambda(2x) \Rightarrow 2x - \lambda(2x) = 0 \Rightarrow 2x(1-\lambda) = 0 \Rightarrow x = 0$ **OR** $\lambda = 1$

(2) $-2y = \lambda(2y) \Rightarrow -2y - \lambda(2y) = 0 \Rightarrow -2y(1+\lambda) = 0 \Rightarrow y = 0$ **OR** $\lambda = -1$

(3) $x^2+y^2 = 1$ (or $x^2+y^2-1=0$)

Critical points must satisfy all 3 equations:

- $\underbrace{x=0}_{\text{Eq 1} ✓}$, $\underbrace{y=0}_{\text{Eq 2} ✓}$: check (3): $0^2+0^2 = 0 \neq 1$ (3) fails $\therefore$ (0,0) is not a critical point
- $\underbrace{x=0}_{\text{Eq 1} ✓}$, $\underbrace{\lambda=-1}_{\text{Eq 2} ✓}$: from (3): $x^2+y^2=1 \Rightarrow 0^2+y^2=1 \Rightarrow y^2=1 \Rightarrow y=\pm 1$

  cp's: $(0,1)$, $(0,-1)$
- $\underbrace{\lambda=1}_{\text{Eq 1} ✓}$, $\underbrace{y=0}_{\text{Eq 2}}$: from (3): $x^2+y^2=1 \Rightarrow x^2+0^2=1 \Rightarrow x^2=1 \Rightarrow x=\pm 1$

  Critical points (cp's): $(1,0)$, $(-1,0)$
- $\underbrace{\lambda=1}_{\text{eq 1}}$, $\underbrace{\lambda=-1}_{\text{eq 2}}$ Not possible! λ can't be both 1 and -1!

Evaluate f at critical points:

cp's	$f(x,y) = x^2-y^2$
$(1,0)$	$f(1,0) = 1-0 = 1$
$(-1,0)$	$f(-1,0) = 1-0 = 1$
$(0,1)$	$f(0,1) = 0-1 = -1$
$(0,-1)$	$f(0,-1) = 0-1 = -1$

Max value of f on g is: 1

Min value of f on g is: -1

Lagrange Multipliers in 3 variables:

Example: Find the minimum and maximum values of $f(x,y,z) = xy + 2z$

Subject to $\underbrace{z^2 = 36 - x^2 - y^2}$.

constraint: $g(x,y,z) = x^2 + y^2 + z^2 - 36 = 0$

(or $g(x,y,z) = 36 - x^2 - y^2 - z^2 = 0$)

Need to solve: $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y,z) = 0 \end{cases} \Rightarrow \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \end{cases}$

(1) $y = \lambda(2x)$

(2) $x = \lambda(2y)$

(3) $2 = \lambda(2z) \Rightarrow 1 = \lambda z \therefore \lambda \neq 0$ and $z \neq 0$

(4) $x^2 + y^2 + z^2 = 36$

(or $x^2 + y^2 + z^2 - 36 = 0$)

Multiply (1) by y: $y^2 = 2\lambda xy$

Multiply (2) by x: $\underbrace{x^2 = 2\lambda xy}_{\text{Same RHS!}}$

$\Rightarrow x^2 = y^2 \Rightarrow \underbrace{x = \pm y}_{\text{2 cases}}$

(I) $x = y$

(II) $x = -y$

(I) $x = y$: From (1) $y = \lambda(2x) \Rightarrow y = 2\lambda y \Rightarrow y - 2\lambda y = 0$

(could also use (2))

$\Rightarrow y(1 - 2\lambda) = 0$

$\Rightarrow y = 0$ or $\lambda = 1/2$

• $x=0, y=0$: From (4) $x^2+y^2+z^2=36 \Rightarrow 0^2+0^2+z^2=36 \Rightarrow z^2=36$
$\Rightarrow z=\pm 6$

CP's: $(0,0,6)$, $(0,0,-6)$

• $x=y$, $\lambda=1/2$: From (3) $1=\lambda z \Rightarrow 1=(1/2)z \Rightarrow \underbrace{z=2}_{\text{sub into (4)}}$

From (4) $x^2+y^2+z^2=36$

$\Rightarrow x^2+x^2+2^2=36 \Rightarrow 2x^2=32$

$\Rightarrow x^2=16$

$\Rightarrow x=\pm 4$

CP's: $(4,4,2)$, $(-4,-4,2)$

(II) $x=-y$: From (1) $y=\lambda(2x) \Rightarrow y=-2\lambda y \Rightarrow y+2\lambda y=0$

$\Rightarrow y(1+2\lambda)=0$

$\Rightarrow \underbrace{y=0}_{(x=0)}$ OR $\lambda=-1/2$

already considered!

only one new case

• $x=-y$, $\lambda=-1/2$: From (3): $1=\lambda z$

$\Rightarrow 1=(-1/2)z \Rightarrow \underbrace{z=-2}_{\text{sub into (4)}}$

From (4): $(-y)^2+y^2+(-2)^2=36 \Rightarrow 2y^2=32$

$\Rightarrow y^2=16$

$\Rightarrow y=\pm 4$

$\therefore$ CP's: $(-4,4,-2)$, $(4,-4,-2)$

Critical point	$f(x,y,z) = xy + z^2$
$(0,0,6)$	$f(0,0,6) = 12$
$(0,0,-6)$	$f(0,0,-6) = -12$
$(4,4,2)$	$f(4,4,2) = 16+4 = 20$
$(-4,-4,2)$	$f(-4,-4,2) = 16+4 = 20$
$(-4,4,-2)$	$f(-4,4,-2) = -16-4 = -20$
$(4,-4,-2)$	$f(4,-4,-2) = -20$

$$\boxed{\begin{aligned} \max &= 20 \\ \min &= -20 \end{aligned}}$$

at $(4,4,2)$, $(-4,-4,2)$

at $(-4,4,-2)$, $(4,-4,-2)$