Announcements:

- The last day to add a class or drop a class (without being issued a W grade) is Monday 1/20.
  - This includes submitting the Authorization Form to get switched into one of the half coures (2011 or 2012) if you already have official credit on your transcript for half of the course.

Section 14.4: Differentiability, Tangent Planes, & Linear Approximation (cont.)

Linear Approximation:

If $f(x,y)$ is differentiable at (a,b), we may use the tangent plane to approximate the value of f for values of (x,y) near (a,b).

The **Linearization** of f centered at (a,b) is:

$$L(x,y) = f(a,b) + f_x(a,b)\underbrace{(x-a)}_{\Delta x} + f_y(a,b)\underbrace{(y-b)}_{\Delta y}$$

used to approximate $f(x,y)$ near (a,b).

$f(x,y) \approx L(x,y)$ for (x,y) near (a,b).

We can write $x = a + \Delta x$ and $y = b + \Delta y$ so the **linear approximation** is,

- $f(\underbrace{a+\Delta x}_{\text{"x"}}, \underbrace{b+\Delta y}_{\text{"y"}}) \approx f(a,b) + f_x(a,b)\Delta x + f_y(a,b)\Delta y$

- $\underbrace{\Delta f}_{\text{actual change in function value}} = f(x,y) - f(a,b) \approx f_x(a,b)\Delta x + f_y(a,b)\Delta y$

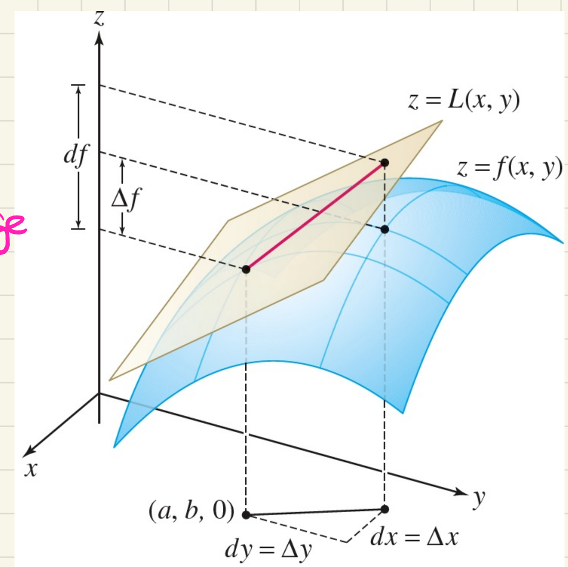
The **differential of f**, df, is defined as:

$$df = f_x(x,y)\,dx + f_y(x,y)\,dy$$

where $\Delta x = dx$ and $\Delta y = dy$.

$$\Delta f \approx df$$

df is the change in the height of the tangent plane.



Example: Use linearization to approximate $\frac{3.03}{1.99}$.

i.e., Use the linearization of $f(x,y) = \frac{x}{y}$ cenered at $(a,b) = (3,2)$ to estimate $f(3.03, 1.99)$.

$$L(x,y) = f(3,2) + f_x(3,2)(x-3) + f_y(3,2)(y-2)$$

$$f(3,2) = \frac{3}{2}$$

$$f(x,y) = \frac{1}{y}\cdot x \quad \Rightarrow f_x(x,y) = \frac{1}{y} \quad \Rightarrow f_x(3,2) = \frac{1}{2}$$

$$f(x,y) = x y^{-1} \quad \Rightarrow f_y(x,y) = -xy^{-2} = \frac{-x}{y^2} \quad \Rightarrow f_y(3,2) = \frac{-3}{4}$$

$$\therefore L(x,y) = \frac{3}{2} + \frac{1}{2}(x-3) - \frac{3}{4}(y-2)$$

$$\begin{aligned} f(3.03, 1.99) \approx L(3.03, 1.99) &= \frac{3}{2} + \frac{1}{2}(3.03-3) - \frac{3}{4}(1.99-2) \\ &= 1.5 + 0.5(\underbrace{0.03}_{\Delta x}) - 0.75(\underbrace{-0.01}_{\Delta y}) \\ &= 1.5 + 0.015 + 0.0075 \\ &= 1.5225 \end{aligned}$$

$$\therefore f(3.03, 1.99) = \frac{3.03}{1.99} \approx 1.5225$$

using calculator $f(3.03, 1.99) = 1.52261$ (rounded)

If we want to approximate $\Delta f \approx df = f_x(3,2)\underbrace{dx}_{\Delta x} + f_y(3,2)\underbrace{dy}_{\Delta y}$

$= 0.0225$

Extends to 3 or more variables...

If $f(x,y,z)$ is differentiable near (a,b,c), then

$$f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(\underbrace{x-a}_{\Delta x}) + f_y(a,b,c)(\underbrace{y-b}_{\Delta y}) + f_z(a,b,c)(\underbrace{z-c}_{\Delta z})$$

approximately equals

and

$$\Delta f \approx f_x(a,b,c)\Delta x + f_y(a,b,c)\Delta y + f_z(a,b,c)\Delta z$$

Section 14.5: The Gradient and Directional Derivatives

Def: The Gradient of a function $f(x,y)$ at a point $P(a,b)$ is the vector

$$\nabla f_P = \nabla f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle$$

For a function $f(x,y,z)$ of three variables and point $P(a,b,c)$

$$\nabla f_P = \nabla f(a,b,c) = \langle f_x(a,b,c), f_y(a,b,c), f_z(a,b,c) \rangle.$$

- The point P may be omitted, and the gradient of f is the vector of partial derivatives of f.

In the two and three variable cases, we can write

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \quad \text{and} \quad \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

respectively.

$$= \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$$

- The gradient ∇f assigns a vector ∇f_P to each point P in the domain of f.
- Extends to n variables: for $f(x_1, x_2, \dots x_n)$, $\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots \frac{\partial f}{\partial x_n} \right\rangle$.

Note: ∇ : del or nabla is an operator.

In $\mathbb{R}^2$: $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$

In $\mathbb{R}^3$: $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

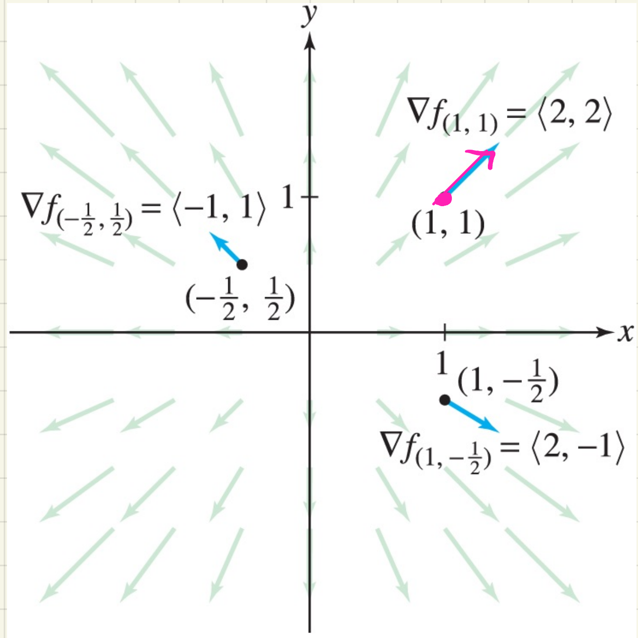
$\vdots$

Examples: Find the gradient.

1.) $f(x,y) = x^2 + y^2$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 2x, 2y \rangle$$

Note: $\nabla f(1,1) = \langle 2x, 2y \rangle \Big|_{(1,1)} = \langle 2, 2 \rangle$



2.) $f(x,y,z) = x^2 + 3y^3 z$

$f_x(x,y,z) = 2x$

$f_y(x,y,z) = 9y^2 z$

$f_z(x,y,z) = 3y^3$

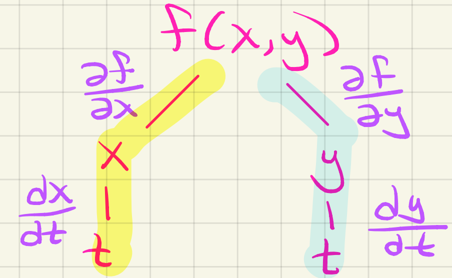
$$\nabla f = \langle 2x, 9y^2 z, 3y^3 \rangle$$

The Chain Rule for Paths :

Let $f(x,y)$ be a differentiable function of x and y where $x = x(t)$ and $y = y(t)$ are both differentiable functions of t. Then, f is a differentiable function of t and

$$\frac{df}{dt} = \underbrace{\frac{\partial f}{\partial x}}\frac{dx}{dt} + \underbrace{\frac{\partial f}{\partial y}}\frac{dy}{dt}$$

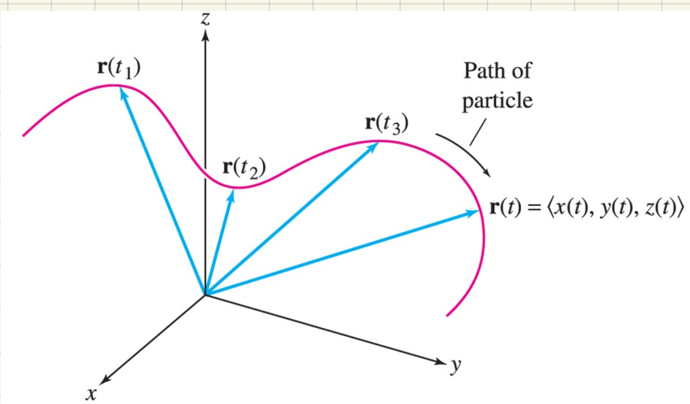
Evaluated at $\langle x(t), y(t) \rangle$



In the above, let $\vec{r}(t) = \langle x(t), y(t) \rangle$ and $f(x,y) = f(\vec{r}(t))$. Then,

$$\frac{df}{dt} = \frac{d}{dt} f(\vec{r}(t)) = \Big\langle \underbrace{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}}_{\text{evaluated at } \vec{r}(t)} \Big\rangle \cdot \langle x'(t), y'(t) \rangle = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

Dot product (see section 12.3)



Recall: $\vec{r}(t)$ can be described as a vector-valued function or path describing the movement of a particle.

See sections 13.1, 13.2 of our calculus textbook if you have not seen this before, or for a review.

The chain rule for paths: If f and $\vec{r}(t)$ are differentiable, then

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

extends to more variables...

Example: Suppose $f(x,y,z) = x^2 + 4yz^3$ and $\vec{r}(t) = \langle \underbrace{\sin t}_{x(t)}, \underbrace{\cos t}_{y(t)}, \underbrace{1+6t}_{z(t)} \rangle$.
Find $\frac{d}{dt} f(\vec{r}(t))$.

$$\nabla f = \langle 2x, 4z^3, 12yz^2 \rangle$$

$$\nabla f(\vec{r}(t)) = \langle 2\sin t, 4(1+6t)^3, 12\cos t\,(1+6t)^2 \rangle$$

$$\vec{r}\,'(t) = \langle \cos t, -\sin t, 6 \rangle$$

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

$$= \langle 2\sin t, 4(1+6t)^3, 12\cos t\,(1+6t)^2 \rangle \cdot \langle \cos t, -\sin t, 6 \rangle$$

$$= \underbrace{2\sin t\cos t - 4(1+6t)^3\sin t + 72\cos t\,(1+6t)^2}_{\text{function of } t \text{ only!}}$$

Directional Derivatives:

Consider $f = f(x,y)$.

f_x: rate of change in the x-direction
(in the direction of $\hat{\imath} = \langle 1,0\rangle$)

f_y: rate of change in the y-direction
(in the direction of $\hat{\jmath} = \langle 0,1\rangle$)

$\|\vec{u}\| = 1$

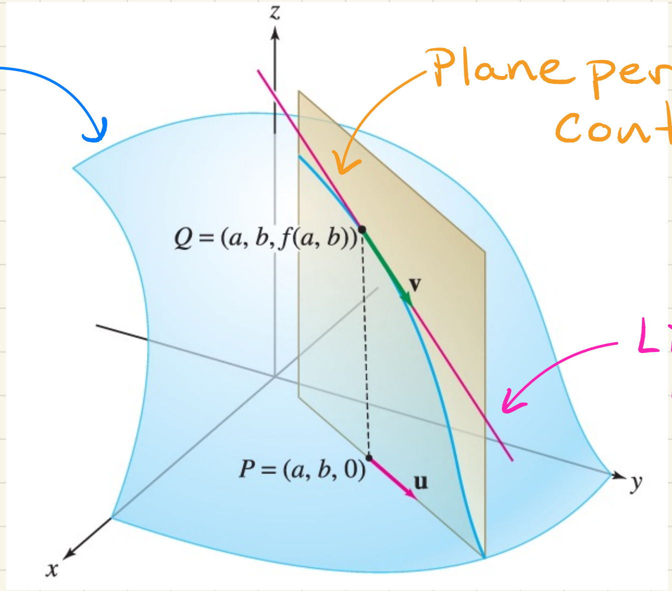
In general, for a unit vector $\vec{u} = \langle h, k\rangle$, we define the <u>directional derivative</u> of $f(x,y)$ at the point $P = (a,b)$ in the direction of the unit vector $\vec{u}$ is the limit (assuming it exists)

$$D_{\vec{u}} f(a,b) = \lim_{t\to 0} \frac{f(a+ht, b+kt) - f(a,b)}{t}$$

In practice, we can use the chain rule for paths to compute directional derivatives.

Consider a line through (a,b) in the direction of a unit vector $\vec{u} = \langle h, k \rangle$: $\vec{r}(t) = \langle a,b \rangle + t\vec{u} = \langle a+ht, b+kt \rangle$

$z = f(x,y)$



Plane perpendicular to xy-plane containing $\vec{r}(t)$.

Line tangent to the curve obtained when we intersect the surface with the plane at $(a,b,f(a,b))$

Slope $= D_{\vec{u}} f(a,b)$

Interpretation: $D_{\vec{u}} f$ is the "slope" of f in the direction of $\vec{u}$.

Thus:
- $D_{\vec{u}} f < 0 \Rightarrow$ "downhill" in direction of $\vec{u}$.
- $D_{\vec{u}} f > 0 \Rightarrow$ "uphill" in direction of $\vec{u}$.

The directional derivative of $z = f(x,y)$ at (a,b) in the direction of the unit vector $\vec{u}$ is

$$D_{\vec{u}} f(a,b) = \underbrace{\frac{d}{dt} f(\vec{r}(t))}_{\text{chain rule for paths}}\Big|_{t=0} = \nabla f(a,b) \cdot \vec{u}$$

where $\vec{r}(t) = \langle a,b \rangle + t\vec{u}$. (Assuming f is differentiable.

extends to more variables.

Theorem (Computing the Directional Derivative): If f is differentiable at P and $\vec{u}$ is a unit vector (so $\|\vec{u}\| = 1$), then the directional derivative in the direction of $\vec{u}$ is given by

$$\boxed{D_{\vec{u}} f(P) = \nabla f_P \cdot \vec{u}}$$

Note: For $f(x,y)$, $P = (a,b)$, and unit vector $\vec{u} = \langle h, k \rangle$:

$$D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u} = f_x(a,b)h + f_y(a,b)k$$

- If $\vec{u} = \langle 1, 0 \rangle = \hat{\imath}$: $D_{\hat{\imath}} f(a,b) = f_x(a,b)$
- If $\vec{u} = \langle 0, 1 \rangle = \hat{\jmath}$: $D_{\hat{\jmath}} f(a,b) = f_y(a,b)$

Examples:

1.) Find the directional derivative of $f(x,y) = e^{xy-y^2}$ in the direction of $\vec{v} = \underbrace{\langle 12, -5 \rangle}_{\|\vec{v}\| \neq 1}$ at the point $P(2,2)$.

; not a unit vector!

Find the unit vector in direction of $\vec{v}$:

$$\vec{u} = \vec{e}_{\vec{v}} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{\underbrace{12^2+(-5)^2}_{144+25=169}}}\langle 12, -5 \rangle = \langle 12/13, -5/13 \rangle$$

$$\nabla f = \langle f_x(x,y), f_y(x,y) \rangle = \langle e^{xy-y^2}(y), e^{xy-y^2}(x-2y) \rangle$$

$$\nabla f(2,2) = \langle \underbrace{e^{4-4}}_{e^0=1}(2), e^{4-4}(2-4) \rangle = \langle 2, -2 \rangle$$

$$D_{\vec{u}} f(2,2) = \nabla f(2,2) \cdot \vec{u}$$

$$= \langle 2, -2 \rangle \cdot \langle 12/13, -5/13 \rangle$$

$$= \frac{24}{13} + \frac{10}{13}$$

$$= \frac{34}{13}$$

2.) Find the directional derivative of $f(x,y,z) = x^2 + 3y^3 z$ in the direction $\vec{v} = \langle 1, -2, 2 \rangle$ at the point $P(2,1,3)$.

Note: Found gradient earlier in lecture (Ex 2 of gradients).

Try on your own for next class!

Question: In what direction is the rate of change the largest?

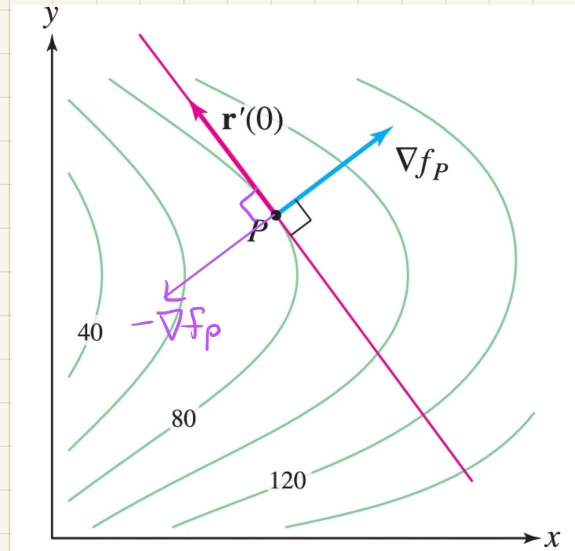
Recall: $\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos(\theta)$ where $0 \le \theta \le \pi$ is the angle between non-zero vectors $\vec{v}$ and $\vec{w}$. (see section 12.3)

Assume that $\nabla f_P \neq \vec{0}$. Let $\vec{u}$ be a unit vector making an angle θ with ∇f_P where $0 \le \theta \le \pi$. Then,

$$D_{\vec{u}} f(P) = \nabla f_P \cdot \vec{u}$$

$$= \|\nabla f_P\| \underbrace{\|\vec{u}\|}_{=1} \cos\theta$$

$$= \|\nabla f_P\| \underbrace{\cos\theta}_{-1 \le \cos\theta \le 1}$$



$$\therefore \quad \underset{\theta = \pi}{-\|\nabla f_P\|} \le D_{\vec{u}} f(P) \le \underset{\theta = 0}{\|\nabla f_P\|}$$

- ∇f_P points in the direction of fastest rate of increase of f at P. (direction of steepest ascent) and this maximum rate of increase is $\|\nabla f_P\|$.

- $-\nabla f_P$ points in the direction of fastest rate of decrease of f at P. (direction of steepest descent) and this maximum rate of decrease is $-\|\nabla f_P\|$.

- ∇f_P is normal (orthogonal) to the level curve (or level surface) of f at P.

Example: Given $f(x,y,z) = x + \frac{y}{z}$ and point $P(4,3,-1)$.

a.) Find the maximum rate of increase of f at P.

The maximum rate of increase of f at P is __________.

b.) Find a unit vector in the direction of the maximum rate of increase (steepest ascent) at P.

Theorem (Gradient as a Normal Vector):

Let $P=(a,b,c)$ be a point on the surface given by $F(x,y,z)=K$, for constant K, and assume $\nabla F_P \neq \vec{0}$. Then ∇F_P is normal to the tangent plane to the surface at P.

We can write the equation of the tangent plane to the surface at P as

$$\boxed{F_x(a,b,c)(x-a)+F_y(a,b,c)(y-b)+F_z(a,b,c)(z-c)=0}$$

Note: If $z=f(x,y)$, then taking $F(x,y,z)=f(x,y)-z=0$ will get same equation of tangent plane as in sec 14.4. Since $F_x=f_x$, $F_y=f_y$, $F_z=-1$ and $c=f(a,b)$, we have

$$f_x(a,b)(x-a)+f_y(a,b)(y-b)-1(z-f(a,b))=0$$

Example: Find an equation of the tangent plane to $x^2 + 2y^2 + 5z^2 = 8$ at $(1, 1, -1)$.