# Announcements:

- Exam 2 is on Wednesday, 2/26 during Test Block (next week).
  - Exam 2 information and additional exam review problems will be posted on LMS later today.

- The UTA's are offering a review session on Tuesday 2/25 from 4-6 PM in Amos Eaton 214. This is <u>optional</u>.
  - There will be no UTA office hours on Wednesday 2/26.

- We will be starting Matrix Algebra content on Monday 2/24. (This is the start of Exam 3 material - not on Exam 2)

## Section 16.3: Conservative Vector Fields (cont.)

If $C$ is a closed curve, we often refer to the line integral of any vector field $\vec{F}$ around $C$ as the circulation of $\vec{F}$ around $C$ denoted:

notation: $C$ is closed $\rightarrow \oint_C \vec{F} \cdot d\vec{r}$.

If $\vec{r}(t), a \le t \le b$ is the parameterization of the closed curve $C$, then $\vec{r}(a) = \vec{r}(b)$.

Recall: A vector field $\vec{F}$ is called conservative if it is equal to the gradient of some scalar function $f$, i.e., $\vec{F} = \nabla f$ for some function $f$. Such an $f$ is called a potential function of $\vec{F}$.

Let $C$ be parameterized by $\vec{r}(t), a \le t \le b$ with $\underbrace{\vec{r}(a)}_{\text{start}} = P$ and $\underbrace{\vec{r}(b)}_{\text{end}} = Q$. Let $\vec{F}$ be a conservative vector field, then $\vec{F} = \nabla f$.

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_a^b \underbrace{\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)}\, dt$$

$$\overset{\text{chain rule}}{=} \int_a^b \underbrace{\frac{d}{dt}\left[ f(\vec{r}(t)) \right]}\, dt$$

$$\overset{\text{FTC}}{=} f(\vec{r}(t)) \Big|_{t=a}^{t=b}$$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

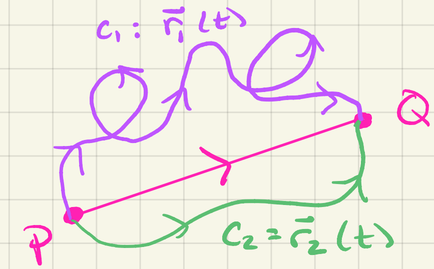
$$= f(Q) - f(P)$$

Thus, a line integral over a conservative vector field depends only on the end points $P$ and $Q$, not the actual path from $P$ to $Q$!

## Fundamental Theorem for Conservative Vector Fields:

Assume that $\vec{F} = \nabla f$ on a domain $D$, (i.e, $\vec{F}$ is conservative)

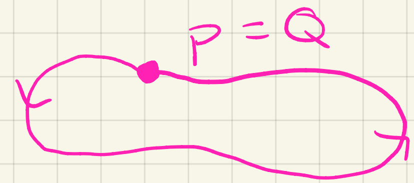
1.) If $\vec{r}(t)$, $a \leq t \leq b$, is a path along a curve $C$ from $P$ to $Q$ in $D$, then




$$\int_C \vec{F} \cdot d\vec{r} = f(Q) - f(P).$$

We say that $\vec{F}$ is path independent.

2.) If $C$ is a closed curve, i.e., $P = Q$, then




$$\oint_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} = 0$$

Recall: If $\vec{F}$ is conservative, then $\text{curl}(\vec{F}) = \vec{0}$ or equivalently,

$$\underbrace{\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}}_{\vec{F} \text{ in } \mathbb{R}^2\text{, this is the only cross-partial condition}}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z} \quad \text{(cross-partials condition)}$$

The converse is only true sometimes.

- $\vec{F}$ needs to have a simply connected domain (no holes).

**Theorem Existence of a potential function**: Let $\vec{F}$ be a vector field on a simply connected domain $D$. If $\vec{F}$ satisfies the cross-partials condition ($\text{curl}(\vec{F}) = \vec{0}$), then $\vec{F}$ is conservative.

**Example:** a.) Show that $\vec{F} = \langle \underbrace{3x^2 + 2y^2}_{F_1}, \underbrace{4xy + 3}_{F_2} \rangle$ is conservative, then find a potential function for $\vec{F}$. b.) Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $C$ is given by $\vec{r}(t) = \langle \cos t, \sin t \rangle$, $\underbrace{0}_{a} \leq t \leq \underbrace{\pi/2}_{b}$.

a) Domain of $\vec{F}$ is $\mathbb{R}^2$, which is simply connected ✓

Check cross-partials condition: $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ (in $\mathbb{R}^2$)

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left[3x^2 + 2y^2\right] = 4y$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left[4xy + 3\right] = 4y$$

$= ✓$ $\therefore$ $\vec{F}$ is conservative.

Now, find $f(x,y)$ such that $\nabla f = \vec{F} \Rightarrow \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle \underbrace{3x^2 + 2y^2}_{F_1}, \underbrace{4xy + 3}_{F_2} \rangle$:

$$f(x,y) = \int F_1(x,y)\, dx = \int (3x^2 + 2y^2)\, dx = x^3 + 2xy^2 + g(y)$$

$$f(x,y) = \int F_2(x,y)\, dy = \int (4xy + 3)\, dy = 2xy^2 + 3y + h(x)$$

These need to be equal!

$$f(x,y) = x^3 + 2xy^2 + g(y) = 2xy^2 + 3y + h(x)$$

$$f(x,y) = 2xy^2 + x^3 + 3y + C$$ (can add any constant $C$ and have a valid potential function).

verify correct potential function! Show $\nabla f = \vec{F}$

b.) $\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = f(Q) - f(P)$

$= f(0,1) - f(1,0)$

$= (2(0)(1)^2 + 0^3 + 3(1)) - (2(1)(0)^2 + 1^3 + 3(0))$

$= 3 - 1$

$= 2$

$Q = \vec{r}(\pi/2) = \langle \cos(\pi/2), \sin(\pi/2) \rangle = (0,1)$

$P = \vec{r}(0) = \langle \cos(0), \sin(0) \rangle = (1,0)$

Note: Same result calculating the line integral of $\vec{F}$ along $C$ directly where $\vec{F} = \langle 3x^2 + 2y^2, 4xy + 3 \rangle$, $\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le \pi/2$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt \quad \text{(Sec. 16.2)}$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \underbrace{\langle 3\cos^2 t + 2\sin^2 t, 4\cos t \sin t + 3 \rangle}_{\vec{F}(\vec{r}(t))} \cdot \underbrace{\langle -\sin t, \cos t \rangle}_{\vec{r}\,'(t)} dt$$

$$= \int_0^{\pi/2} (-3\cos^2 t \sin t - 2\sin^3 t + 4\cos^2 t \sin t + 3\cos t)\, dt$$

$$= \int_0^{\pi/2} (\cos^2 t \sin t - 2\sin^3 t + 3\cos t)\, dt$$

$$= \int_0^{\pi/2} \cos^2 t \sin t\, dt - 2\int_0^{\pi/2} \underbrace{\sin^3 t}\, dt + 3\int_0^{\pi/2} \cos t\, dt$$

$u = \cos t$, $du = -\sin t\, dt$

rewrite as $(1-\cos^2 t)\sin t$ (Sec 7.3)
then $u = \cos t$, $du = -\sin t\, dt$

$$= -\frac{1}{3}\cos^3 t \Big|_0^{\pi/2} - 2\left(\frac{1}{3}\cos^3 t - \cos t\right)\Big|_0^{\pi/2} + 3\sin t \Big|_0^{\pi/2}$$

$$= \ldots = \frac{1}{3} - \frac{4}{3} + 3 = 2$$

This method is typically more work if $\vec{F}$ is conservative and the integral may be very difficult to solve!

Example: a) Show that $\vec{F} = \langle \cos z, 2y, -x\sin z \rangle$ is conservative, then find a potential function $f$ for $\vec{F}$. b) Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $C$ is any curve from $\underbrace{(2,1,0)}_{P}$ to $\underbrace{(1,-1,\pi/2)}_{Q}$.

a) Domain of $\vec{F}$ is $\mathbb{R}^3$ which is simply connected, if $\text{curl}(\vec{F}) = \vec{0}$, then $\vec{F}$ is conservative.

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{array}{ccc} + & - & + \end{array}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \cos z & 2y & -x\sin z \end{vmatrix} = \langle 0-0, -\underbrace{(-\sin z + (+\sin z))}_{0}, 0-0 \rangle$$

$$= \vec{0}$$

$\therefore \vec{F}$ is conservative

Now, find $f(x,y,z)$ such that $\nabla f = \vec{F} = \langle \underbrace{\cos z}_{F_1}, \underbrace{2y}_{F_2}, \underbrace{-x\sin z}_{F_3} \rangle$

$$f(x,y,z) = \int F_1(x,y,z)\,dx = \int \cos z\,dx = x\cos z + g_1(y,z)$$

$$f(x,y,z) = \int F_2(x,y,z)\,dy = \int 2y\,dy = y^2 + g_2(x,z)$$

$$f(x,y,z) = \int F_3(x,y,z)\,dz = \int -x\sin z\,dz = x\cos z + g_3(x,y)$$

These all need to be the same!

$$\therefore f(x,y,z) = x\cos z + y^2$$

b) $\int_C \vec{F}\cdot d\vec{r} = f(Q) - f(P)$

$$= f(1,-1,\pi/2) - f(2,1,0)$$

$$= (1\cdot \cancel{\cos(\pi/2)}^{\,0} + (-1)^2) - (2\cancel{\cos(0)}^{\,1} + 1^2)$$

$$= 1 - (2+1)$$

$$= -2$$

# Section 17.1: Green's Theorem

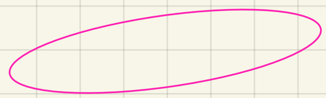
**Recall:** $\oint_C \vec{F} \cdot d\vec{r}$ denotes the line integral of a vector field $\vec{F}$ on the closed curve $C$. Also called the circulation of $\vec{F}$ around $C$.

**Notation:** If $D$ is a region in the $xy$-plane, $\partial D$ denotes the boundary of $D$.

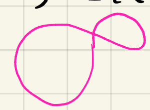
$C = \partial D$

**Some definitions:**

- A simple curve does not intersect itself, except possibly at the endpoints.

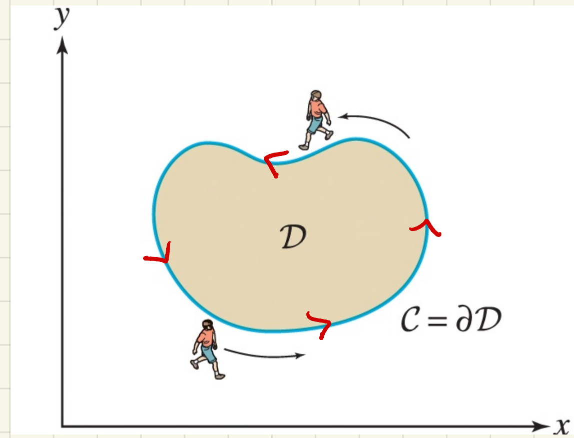


: Simple



: Not simple

- If $D$ is a region in the $xy$-plane, the boundary curve $C = \partial D$ is positively-oriented if it is traced counterclockwise, i.e., as you travel along the boundary $C = \partial D$, the region $D$ is always on your left.




**Green's Theorem:** Let $D$ be a domain whose boundary $\partial D$, is a positively-oriented, simple closed curve in the plane. If $\vec{F} = \langle F_1, F_2 \rangle$, where $F_1$ and $F_2$ have continuous partial derivatives, then

$$\underbrace{\oint_{\partial D} \vec{F} \cdot d\vec{r} = \oint_{\partial D} F_1\,dx + F_2\,dy}_{\text{Two notations for vector line integral}} = \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA.$$

**Example:** Verify Green's Theorem by evaluating $\oint_C (x-y)\,dx + (x+y)\,dy$ using two methods: (a) directly and (b) using Green's Theorem, where $C$ is the circle with center at the origin with radius 2 oriented counterclockwise.

(a) $C: x^2+y^2=4$ : $\vec{r}(t) = \langle \underbrace{2\cos t}_{x(t)}, \underbrace{2\sin t}_{y(t)} \rangle$, $0 \le t \le 2\pi$

$$x(t) = 2\cos t \Rightarrow \frac{dx}{dt} = -2\sin t, \quad y(t) = 2\sin t \Rightarrow \frac{dy}{dt} = 2\cos t$$

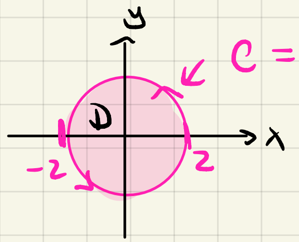
$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C (x-y)\,dx + (x+y)\,dy = \int_0^{2\pi} \Big[\underbrace{(2\cos t - 2\sin t)}_{\vec{F}_1(\vec{r}(t))}\underbrace{(-2\sin t)}_{\frac{dx}{dt}} + \underbrace{(2\cos t + 2\sin t)}_{F_2(\vec{r}(t))}\underbrace{(2\cos t)}_{dy/dt}\Big]dt$$

$$= \int_0^{2\pi} \left[\cancel{-4\cos t\sin t} + \underbrace{4\sin^2 t + 4\cos^2 t}_{4} + \cancel{4\cos t\sin t}\right]dt$$

$$= \int_0^{2\pi} 4\,dt = 4t\Big|_0^{2\pi} = 8\pi$$

same ✓

(b)




$D$: Disk of radius 2

Polar rectangle $D$: $0 \le \theta \le 2\pi$, $0 \le r \le 2$

$$\vec{F} = \langle F_1, F_2 \rangle = \langle x-y, x+y \rangle$$

Integrand: $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = \dfrac{\partial}{\partial x}(x+y) - \dfrac{\partial}{\partial y}(x-y)$
$= 1-(-1) = 2$

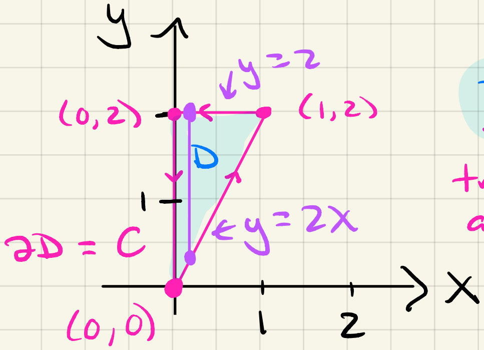
Apply Green's Thm

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C \underbrace{(x-y)}_{F_1}\,dx + \underbrace{(x+y)}_{F_2}\,dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dA = \iint_D 2\,dA = 2\,\text{Area}(D)$$

$$= 2\pi(2)^2 = 8\pi$$

OR $= \displaystyle\int_0^{2\pi}\int_0^2 2\,r\,dr\,d\theta = \dots = 8\pi$

**Example:** Calculate $\oint_C y^2\,dx + x^2\,dy$, where $C$ is the triangle with vertices $(0,0)$, $(1,2)$, $(0,2)$ oriented counterclockwise.




$D:\ \begin{cases} 0 \le x \le 1 \\ 2x \le y \le 2 \end{cases}$ vertically simple region

triangle interior and boundary

As horizontally simple region

$D:\ \begin{array}{l} 0 \le y \le 2 \\ 0 \le x \le y/2 \end{array}$

$$\vec{F} = \langle F_1, F_2 \rangle = \langle y^2, x^2 \rangle$$

Integrand: $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = \dfrac{\partial}{\partial x}(x^2) - \dfrac{\partial}{\partial y}(y^2)$

$= 2x - 2y$

Apply Green's Thm:

$$\underset{\text{“}\partial D\text{”}}{\oint_C} \underbrace{y^2}_{F_1} dx + \underbrace{x^2}_{F_2} dy = \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$

$$= \int_0^1 \int_{2x}^2 (2x - 2y)\, dy\, dx \qquad \text{(15.2 integral)}$$

$$= \int_0^1 (2xy - y^2)\Big|_{y=2x}^{2} dx$$

$$= \vdots$$

$$= -2$$

Note: To evaluate the line integral directly (16.2) we need 3 line integrals!

## Note:

- If $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$, then $\oint_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D 1 \, dA = \text{Area}(D)$

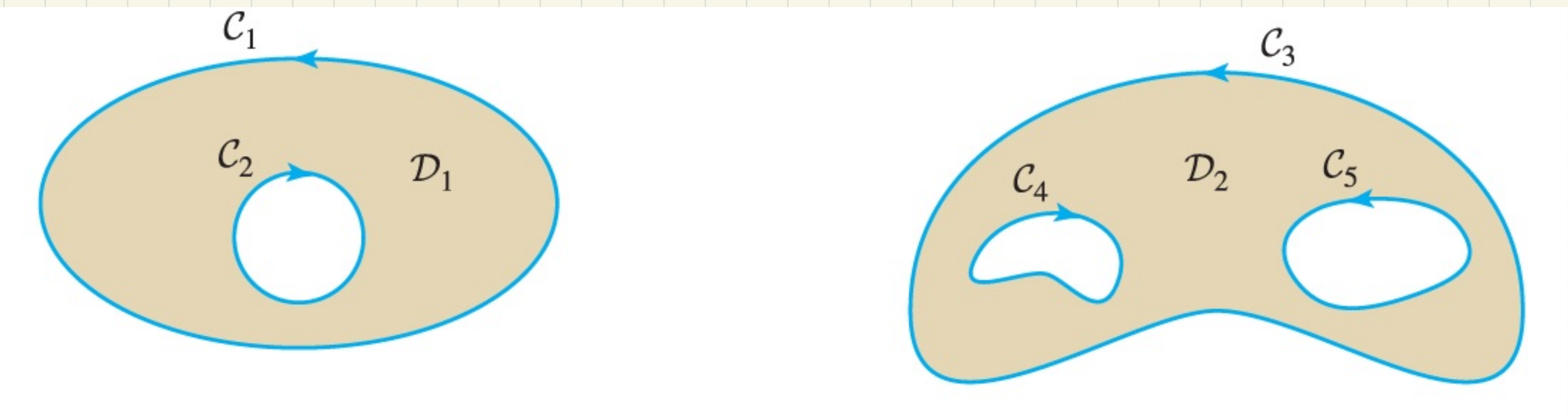
- Other formulas for calculating the area of $D$ enclosed by $C$:

$$\text{area enclosed by } C = \underbrace{\oint_C x\,dy}_{\vec{F} = \langle 0, x \rangle} = \underbrace{\oint_C -y\,dx}_{\vec{F} = \langle -y, 0 \rangle} = \underbrace{\frac{1}{2}\oint_C x\,dy - y\,dx}_{\vec{F} = \langle -\frac{1}{2}y, \frac{1}{2}x \rangle}$$

- See Example 3 in textbook for computing the area of an ellipse.

## More General Form of Green's Theorem:

- Domain $D$ whose boundary consists of more than one simple closed curve
  - Positively-oriented if the region $D$ lies to the left as the boundary curve $\partial D$ is traversed.

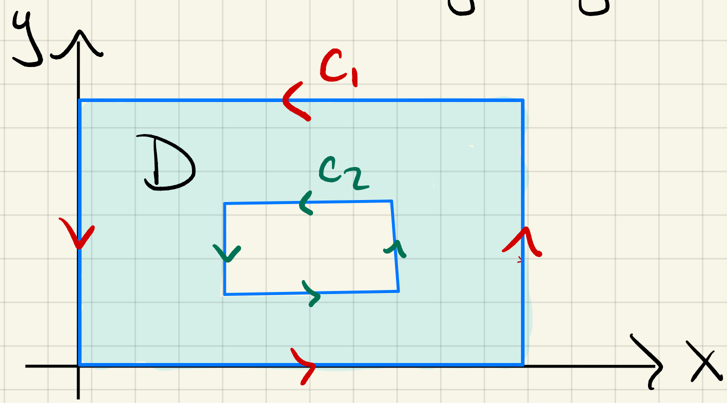



(A) Oriented boundary of $D_1$ is $C_1 + C_2$.

(B) Oriented boundary of $D_2$ is $C_3 + C_4 - C_5$.

(Fig. 12)

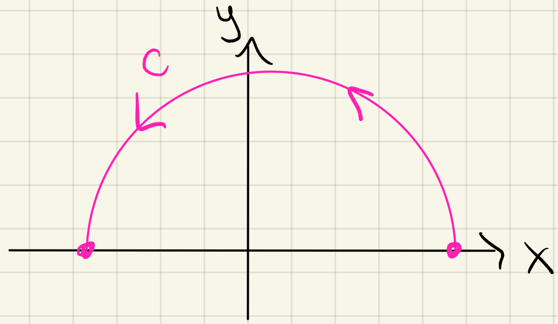
Example: Consider the following region.




If $\oint_{C_1} \vec{F} \cdot d\vec{r} = 4$ and $\iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = 1$, calculate $\oint_{C_2} \vec{F} \cdot d\vec{r}$.

# Green's Theorem for a curve that is not closed:

- If $C$ is positively oriented, but not closed, add a segment $C_1$ to form a closed curve. Let $C$ and $C_1$ form the boundary of $D$, so $\partial D = C + C_1$. Then,

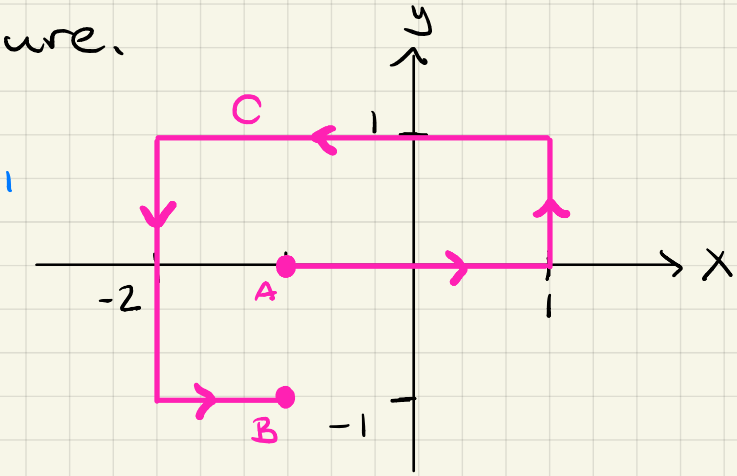



$$\int_C \vec{F}\cdot d\vec{r} + \int_{C_1} \vec{F}\cdot d\vec{r} = \oint_{\partial D} \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA - \int_{C_1} \vec{F}\cdot d\vec{r}$$

Note: This can be used to simplify a problem where calculating the line integral directly is more difficult.

**Example:** Compute the line integral of $\vec{F} = \langle x^3, 4x \rangle$ along the path from $A = (-1, 0)$ to $B = (-1, -1)$ in the given figure.




**Note:** To save work, can add a line segment $C_1$ from $B$ to $A$ to use Green's Theorem.