Announcements:

- Quiz 1 will be given during recitation next week. Check the Quiz Schedule in the Homework & Quizzes tab on LMS.
- You must include your section number in all emails sent about this course.
- Office hours have been updated on the syllabus and the office hours tab on LMS.

Section 14.3: Partial Derivatives (cont.)

- The **partial derivatives** are the rates of change with respect to each variable seperately.

Notation: Two-variable functions $z = f(x,y)$

Partial derivative with respect to x: f_x, $f_x(x,y)$, z_x, $\frac{\partial f}{\partial x}$, $\frac{\partial z}{\partial x}$

- To find, treat y as a constant and differentiate with respect to x.

Partial derivative with respect to y: f_y, $f_y(x,y)$, z_y, $\frac{\partial f}{\partial y}$, $\frac{\partial z}{\partial y}$

- To find, treat x as a constant and differentiate with respect to y.

Example: Compute f_x and f_y. (From last lecture)

3.) $f(x,y) = x^2 e^{xy}$

Treat y as constant

$$f_x = \frac{\partial}{\partial x}\underbrace{(x^2 e^{xy})}_{\text{product rule}} = \frac{\partial}{\partial x}(x^2)e^{xy} + x^2 \underbrace{\frac{\partial}{\partial x}(e^{xy})}_{\text{chain rule}}$$

$$= 2x e^{xy} + x^2 e^{xy} y = (2x + x^2 y)e^{xy}$$

Treat x as constant

$$f_y = \frac{\partial}{\partial y}(x^2 e^{xy}) = x^2 \frac{\partial}{\partial y}(\underbrace{e^{xy}}_{\text{chain rule}}) = x^2(x e^{xy}) = x^3 e^{xy}$$

Higher-Order Partial Derivatives:

$$f(x,y)$$

$\frac{\partial}{\partial x}(f)$ → $f_x(x,y)$ $\qquad$ $\frac{\partial}{\partial y}(f)$ → $f_y(x,y)$

1st-order partials: $f_x(x,y)$ $\qquad$ $f_y(x,y)$

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$ $\qquad$ $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ $\qquad$ $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ $\qquad$ $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$

2nd-order partials: $f_{xx}(x,y)$ $\qquad$ $(f_x)_y = f_{xy}(x,y)$ $\qquad$ $(f_y)_x = f_{yx}(x,y)$ $\qquad$ $f_{yy}(x,y)$

3rd-order partials: f_{xxx} $\quad$ f_{xxy} $\quad$ f_{xyx} $\quad$ f_{xyy} $\quad$ f_{yxx} $\quad$ f_{yxy} $\quad$ f_{yyx} $\quad$ f_{yyy}

The order in **mixed partials** does not matter (as long as they are continuous)

which we will have in this course!

Clairaut's Thm: $\boxed{f_{xy} = f_{yx}}$

* Also true for higher-order partial derivatives: $f_{xyy} = f_{yxy} = f_{yyx}$

Other Notation:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{xy} = \frac{\partial^2 f}{\partial y\,\partial x}, \quad f_{yx} = \frac{\partial^2 f}{\partial x\,\partial y}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

Example: Find all second order partial derivatives for $f(x,y) = x^2 e^{xy}$.

Note: Found $f_x = (2x + x^2 y) e^{xy}$ and $f_y = x^3 e^{xy}$ in example 3.

$$\underbrace{f_{xx}}_{\frac{\partial^2 f}{\partial x^2}} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left[\underbrace{(2x + x^2 y)\, e^{xy}}_{\text{product rule}}\right] = (2 + 2xy)e^{xy} + (2x + x^2 y) y e^{xy}$$

← can stop here!

$$= (2 + 4xy + x^2 y^2) e^{xy}$$

$$\underbrace{f_{xy}}_{\frac{\partial^2 f}{\partial y \partial x}} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left[\underbrace{(2x + x^2 y)\, e^{xy}}_{\text{product rule}}\right] = x^2 e^{xy} + (2x + x^2 y) x e^{xy}$$

$$= (3x^2 + x^3 y) e^{xy}$$

equal ✓

$$\underbrace{f_{yx}}_{\frac{\partial^2 f}{\partial x \partial y}} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left[\underbrace{x^3 e^{xy}}_{\text{product rule}}\right] = 3x^2 e^{xy} + x^3 y e^{xy}$$

$$= (3x^2 + x^3 y) e^{xy}$$

$$\underbrace{f_{yy}}_{\frac{\partial^2 f}{\partial y^2}} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\left[x^3 e^{xy}\right] = x^3 (x e^{xy}) = x^4 e^{xy}$$

3-Variable Function:

Example: $f(x,y,z) = yz\ln(xy)$. Find the indicated partial derivatives.

$$f_x(x,y,z) = yz\frac{1}{x\cancel{y}}\cdot\cancel{y} = \frac{yz}{x}$$

fix y & z (treated as constant)

$$f_y(x,y,z) = \frac{\partial}{\partial y}\Big[\underbrace{(yz)\ln(xy)}_{\text{product rule}}\Big] = z\ln(xy) + \cancel{y}z\frac{1}{\cancel{x}\cancel{y}}\cdot\cancel{x} = z\ln(xy)+z$$

fix x & z

$$= z(\ln(xy)+1)$$

$$f_z(x,y,z) = y\ln(xy)$$

fix x & y

Find f_{yzx}.

$$f_{yz}(x,y,z) = \frac{\partial}{\partial z}\big[z(\ln(xy)+1)\big] = \ln(xy)+1$$

$$f_{yzx}(x,y,z) = \frac{\partial}{\partial x}\big[\ln(xy)+1\big] = \frac{1}{x\cancel{y}}\cdot\cancel{y} = \frac{1}{x}$$

equal ✓

Find f_{zxy}.

$$f_{zx}(x,y,z) = \frac{\partial}{\partial x}\big[y\ln(xy)\big] = y\cdot\frac{1}{x\cancel{y}}\cdot\cancel{y} = \frac{y}{x} = \frac{1}{x}\cdot y$$

$$f_{zxy}(x,y,z) = \frac{1}{x}$$

Section 14.4: Differentiability, Tangent Planes, & Linear Approximation

Some of you may not have worked much with vectors before, or you may need a review. Chapter 12 of our textbook covers vectors.

In particular, you may want to review:

- Section 12.3 – Dot product
- Section 12.4 – Cross product
- Section 12.5 – Planes

Equations of Planes: (From 12.5)

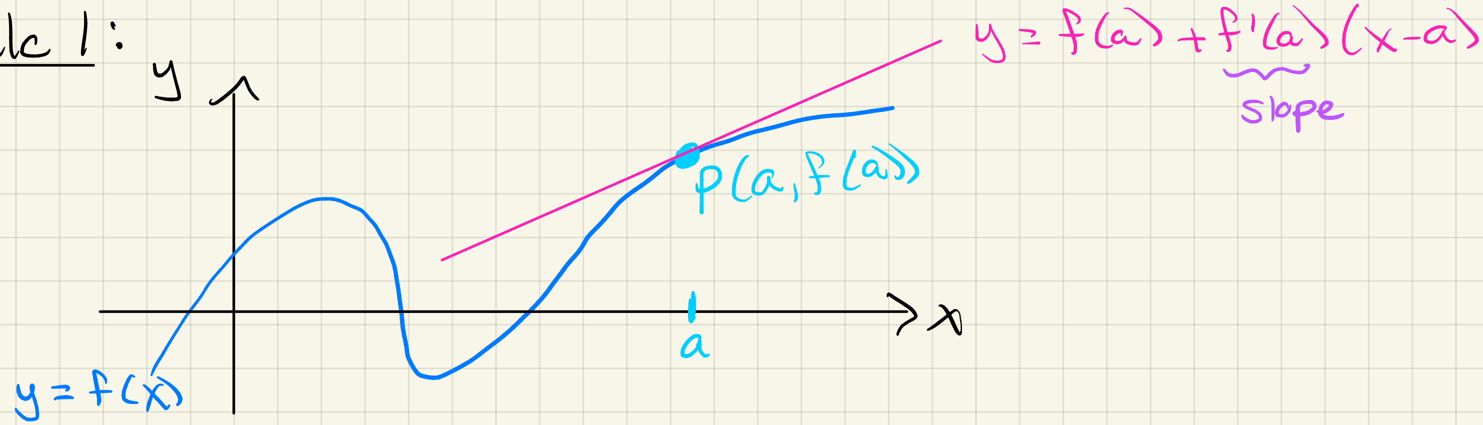
The plane through the point $P(x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c \rangle$ is:

Vector form: $\vec{n} \cdot \langle x, y, z \rangle = d$ or $\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

Scalar forms: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ or $ax + y + cz = d$

where $d = ax_0 + by_0 + cz_0$.

Recall from Calc I:



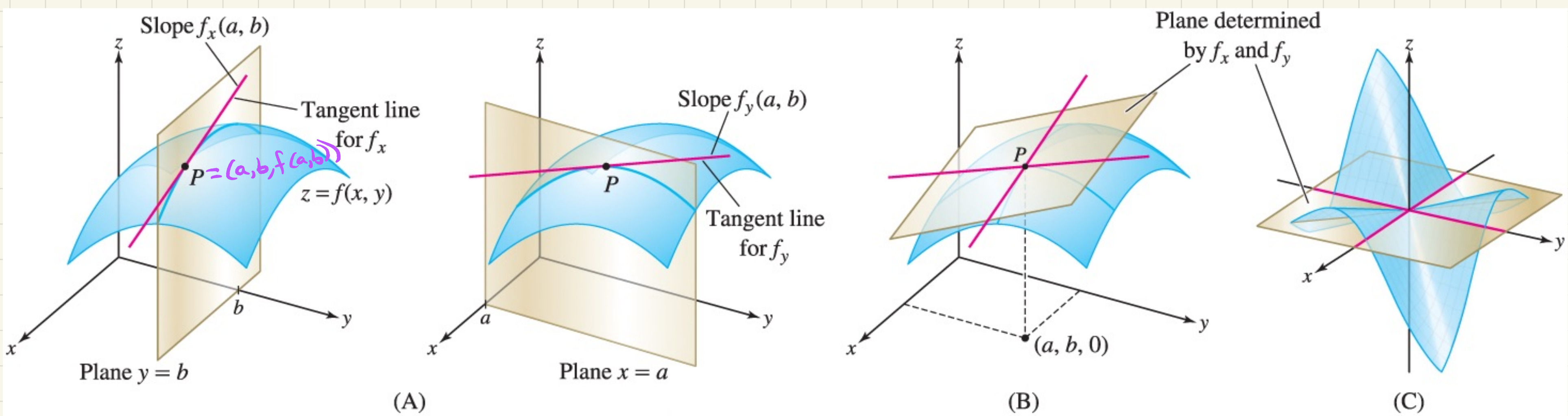
Tangent line to graph $y = f(x)$ at $P(a, f(a))$:

$y - f(a) = f'(a)(x-a)$ or $y = f(a) + f'(a)(x-a)$

The tangent line gives the linear approximation to $y = f(x)$ for $x \approx a$.

Thus, $f(x) \approx f(a) + f'(a)(x-a)$ when x is near a.

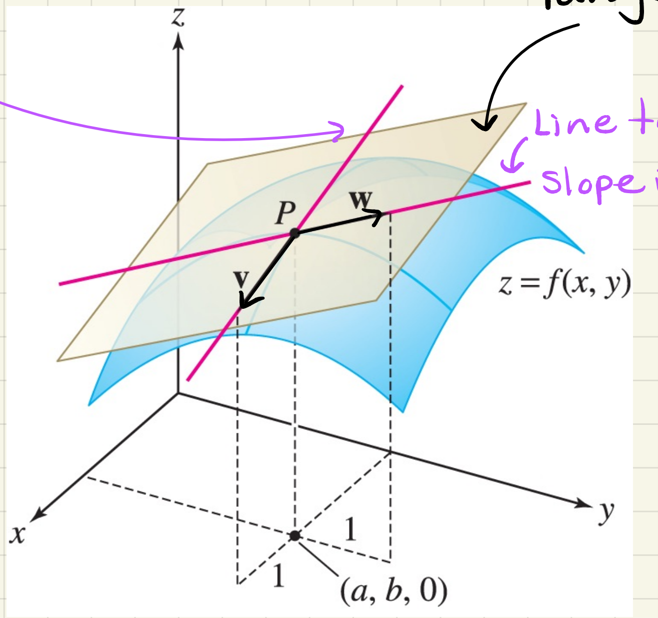
Tangent Planes, & Linear Approximation:



Line tangent to trace curve $(x, b, f(x,b))$

Slope is $f_x(a,b)$

Tangent Plane at $P = (a, b, f(a,b))$



Line tangent to the trace curve $(a, y, f(a,y))$

slope is $f_y(a,b)$

We need two non-parallel vectors in the plane: $\vec{v}, \vec{w}$:

$\vec{v}$: Tangent to the surface and lies in the plane $y = b$.

- move 1 unit in positive x-direction
- move 0 units in y-direction
- move $f_x(a,b)$ units in z-direction

(y-fixed)

$\therefore \quad \vec{v} = \langle 1, 0, f_x(a,b) \rangle$ direction vector of tangent line to trace curve $(x, b, f(x,b))$

$\vec{w}$: Tangent to the surface and lies in the plane $x = a$.

- move 0 units in x-direction
- move 1 unit in positive y-direction
- move $f_y(a,b)$ units in z-direction

(x-fixed)

$\therefore \quad \vec{w} = \langle 0, 1, f_y(a,b) \rangle$ direction vector of tangent line to trace curve $(a, y, f(a,y))$

To find the plane containing both lines in this figure, find the normal vector $\vec{n} = \vec{w} \times \vec{v}$:

$$\vec{n} = \vec{w} \times \vec{v} = \begin{array}{ccc} + & - & + \\ \end{array}$$
$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & f_y(a,b) \\ 1 & 0 & f_x(a,b) \end{vmatrix} = \langle f_x(a,b), f_y(a,b), -1 \rangle$$

* if $\vec{v} \times \vec{w}$, then $\vec{n} = \langle -f_x(a,b), -f_y(a,b), 1 \rangle$
(points in opposite direction)

Recall: A plane through $P(x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c \rangle$ is:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Our tangent plane passes through $\overbrace{(a, b, f(a,b))}^{P}$ and has normal vector $\vec{n} = \langle f_x(a,b), f_y(a,b), -1 \rangle$:

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) - 1(z - f(a,b)) = 0$$

OR $$\boxed{z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

is the Equation of the Tangent Plane at (a, b).

Note: may be asked for tangent plane at $(a, b, f(a,b))$, the point on the surface. So you already know $f(a,b)$!

Examples:

1.) Find the equation of the tangent plane to $f(x,y) = x^2y - 4xy^3$ at the point $\underbrace{(1,-1)}_{(a,b)}$.

$f(1,-1) = 1^2(-1) - 4(1)(-1)^3 = 3$

$f_x(x,y) = 2xy - 4y^3 \Rightarrow f_x(1,-1) = 2(1)(-1) - 4(-1)^3 = 2$

$f_y(x,y) = x^2 - 12xy^2 \Rightarrow f_y(1,-1) = 1^2 - 12(1)(-1)^2 = -11$

Tangent plane at $(1,-1)$: $\boxed{z = 3 + 2(x-1) - 11(y+1)}$

or $z = -10 + 2x - 11y$

2.) Find the point(s) on the graph of $z = 3x^2 - 4y^2$ at which $\vec{n} = \langle 3, 2, 2 \rangle$ is normal to the tangent plane.

Recall: $\vec{n} = \langle f_x(a,b), f_y(a,b), -1 \rangle$ is the normal vector to the tangent plane at (a,b)

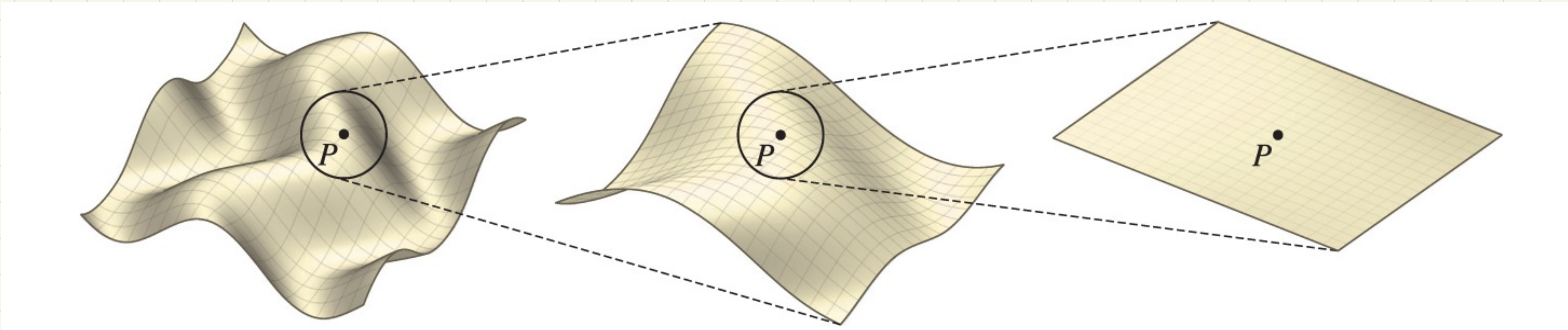
Notice $\vec{n} = \langle -3/2, -1, -1 \rangle$ is also normal to tangent plane

must have $f_x(a,b) = -3/2$, $f_y(a,b) = -1$

Now, $f(x,y) = 3x^2 - 4y^2 \Rightarrow f_x(x,y) = 6x$, $f_y(x,y) = -8y$

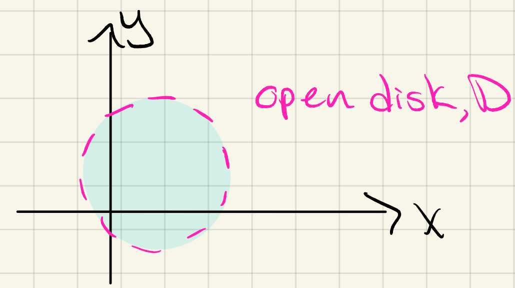
Thus, $6x = -3/2$, $-8y = -1$ $\Rightarrow$ $x = -1/4$, $y = 1/8$ $\therefore$ only point is $\underbrace{(-1/4, 1/8)}_{(a,b)}$

Note: For $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$ to be tangent to the surface $z = f(x,y)$ at $P(a,b,f(a,b))$, $f(x,y)$ must be locally linear.



We say $f(x,y)$ is differentiable at (a,b) if it is locally linear at $(a,b,f(a,b))$.

Theorem 1: (Criterion for Differentiability)
If $f_x(x,y)$ and $f_y(x,y)$ exist and are continuous on an open disk D, then $f(x,y)$ is differentiable on D.



Linear Approximation : Will continue here next class. Review this slide before lecture!

If $f(x,y)$ is differentiable at (a,b), we may use the tangent plane to approximate the value of f for values of (x,y) near (a,b).

The Linearization of f centered at (a,b) is:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$f(x,y) \approx L(x,y)$ for (x,y) near (a,b).

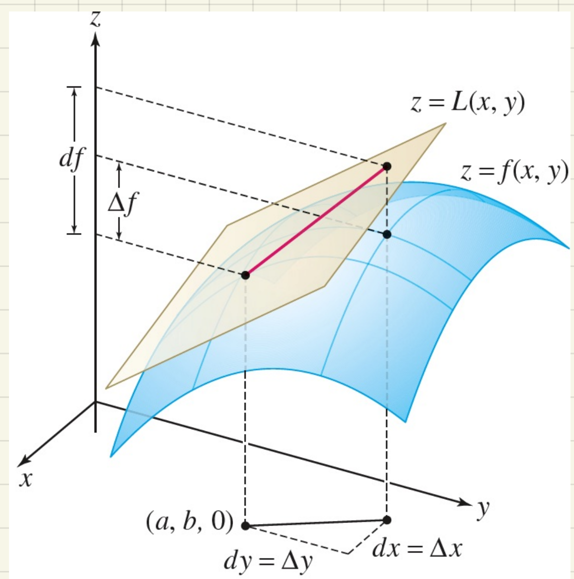
We can write $x = a + \Delta x$ and $y = b + \Delta y$ so the linear approximation is,

- $f(a+\Delta x, b+\Delta y) \approx f(a,b) + f_x(a,b)\Delta x + f_y(a,b)\Delta y$
- $\Delta f = f(x,y) - f(a,b) \approx f_x(a,b)\Delta x + f_y(a,b)\Delta y$

The differential of f, df, is defined as:

$$df = f_x(x,y)\,dx + f_y(x,y)\,dy$$

where $\Delta x = dx$ and $\Delta y = dy$.



Example: Use linearization to approximate $\frac{3.03}{1.99}$.

i.e., Use the linearization of $f(x,y) = \frac{x}{y}$ cenered at $(a,b) = (3,2)$.

Extends to 3 or more variables...

If $f(x,y,z)$ is differentiable near (a,b,c), then

$$f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

and

$$\Delta f \approx f_x(a,b,c)\Delta x + f_y(a,b,c)\Delta y + f_z(a,b,c)\Delta z$$