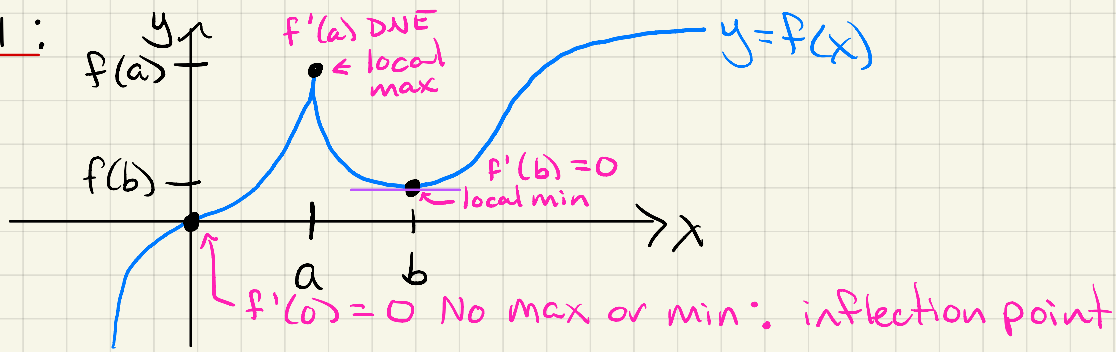
Announcements:

- Exam 1 is on Wednesday 1/29 (next week) during Test Block.
  - Exam 1 information has been posted on LMS in the "Exam Info" tab. Read carefully!
  - Additional Exam 1 review problems are also available in the "Exam Info" tab on LMS. Filled Slides with the worked Solutions will <u>not</u> be provided.

Section 14.7: Optimization in Several Variables

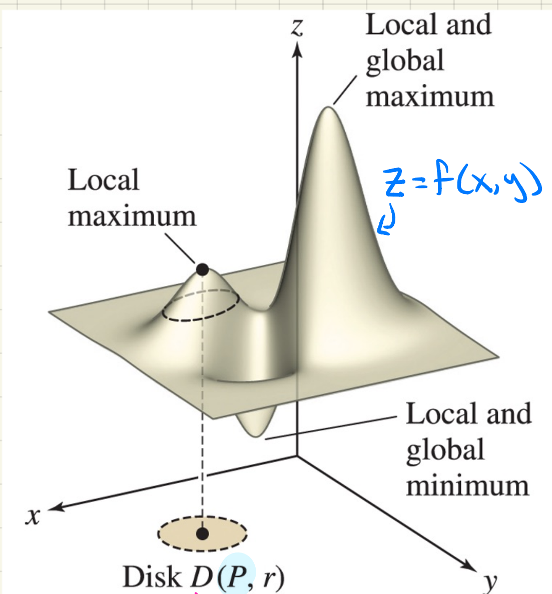
Calc 1:



Fermat's Theorem: If f has a local max or min at c, then $\underbrace{f'(c)=0 \text{ or } f'(c) \text{ DNE}}_{c \text{ is a critical point of } f}$.

Def: A function $f(x,y)$ has an absolute max (absolute min) at (a,b) if $f(x,y) \le f(a,b)$ ($f(x,y) \ge f(a,b)$) for all (x,y) in the domain of f.

Def: A function $f(x,y)$ has a local max at (a,b) if $f(x,y) \le f(a,b)$ for all (x,y) in some open disk centered at (a,b). $f(x,y)$ has a local min at (a,b) if $f(x,y) \ge f(a,b)$ for all (x,y) in some open disk centered at (a,b).



↳ open disk centered at $P=(a,b)$ with radius $r>0$.

Note: For any unit vector $\vec{u}$, what can we say about $D_{\vec{u}} f(P)$?

↖ $P=(a,b)$

$$D_{\vec{u}} f(P) = 0$$

$$\Rightarrow \nabla f(P) \cdot \vec{u} = 0 \quad \text{for } \underline{\underline{\text{all}}} \ \vec{u}$$

$$\therefore \ \nabla f(P) = \vec{0} \quad \text{so } f_x(P) = f_y(P) = 0$$

"Therefore"

Def: A point $P=(a,b)$ in the domain of $f(x,y)$ is a critical point of f if:

- $f_x(P) = f_y(P) = 0$
- or either $f_x(P)$, $f_y(P)$ does not exist.

Theorem (Fermat's Theorem): If $f(x,y)$ has a local max or local min at $P=(a,b)$, then (a,b) is a critical point of $f(x,y)$.

Note: If f has a local max/min at $(a,b) \Rightarrow (a,b)$ is a critical point of f. The converse is not true! That is, if (a,b) is a critical point of f, it is possible that f does not have either a local max or min at (a,b).

Note: If $f_x(a,b) = 0 = f_y(a,b)$, then $\nabla f(a,b) = \vec{0}$ and the tangent plane of f at (a,b) is:

$$z = f(a,b) + \underbrace{f_x(a,b)}_{=0}(x-a) + \underbrace{f_y(a,b)}_{=0}(y-b)$$

$$\Rightarrow \quad z = \underbrace{f(a,b)}_{\text{constant (a \#)}} \quad : \text{Horizontal Tangent plane.}$$

Example: Find all critical points of $f(x,y) = x e^{-x^2-y^2}$.

$f_x = e^{-x^2-y^2} + x e^{-x^2-y^2}(-2x) = (1-2x^2)e^{-x^2-y^2}$

$f_y = x e^{-x^2-y^2}(-2y) = -2xy\underbrace{e^{-x^2-y^2}}_{>0}$

f_x and f_y are defined for all x & y

$$\begin{cases} f_x = 0: & (1-2x^2)\underbrace{e^{-x^2-y^2}}_{>0 \text{ (never 0!)}} = 0 \Rightarrow 1-2x^2 = 0 \Rightarrow 2x^2 = 1 \\ f_y = 0: & -2xy\underbrace{e^{-x^2-y^2}}_{>0 \text{ (never 0!)}} = 0 \Rightarrow -2xy = 0 \Rightarrow xy = 0 \end{cases}$$

$x^2 = 1/2$

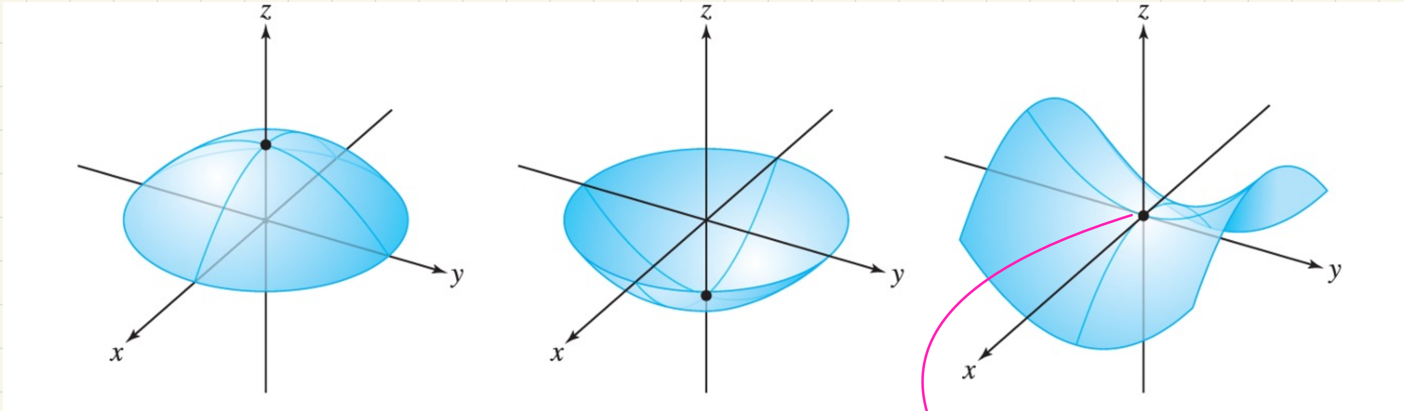
$x = \pm 1/\sqrt{2}$

We need both $f_x = 0$ and $f_y = 0$

$\Rightarrow y = 0$ ($x = 0$ would not satisfy $f_x = 0$)

Critical points: $(1/\sqrt{2}, 0)$, $(-1/\sqrt{2}, 0)$

Note: A function may not have a local max or local min at a critical point (a,b).



(A) Local maximum (B) Local minimum (C) Saddle

Local max in x-direction
Local min in y-direction
$\therefore$ No overall local extremum.

The Second Derivative Test is used to classify f for each critical point (a,b) of a twice-differentiable function $f(x,y)$ as a local max, local min, or saddle. This test relies on the sign of the discriminant D evaluated at the critical point (a,b).

Def: Suppose $f(x,y)$ is a twice-differentiable function. The **discriminant** is defined as

$$D(x,y) = \begin{array}{c|cc|} x & f_{xx} & f_{xy} \\ y & f_{yx} & f_{yy} \\ & x & y \end{array} = f_{xx}f_{yy} - f_{xy}\underbrace{f_{yx}}_{f_{yx}=f_{xy}} = f_{xx}f_{yy} - (f_{xy})^2$$

Theorem (Second Derivative Test): Suppose $P=(a,b)$ is a critical point of $f(x,y)$ such that $f_x(a,b)=f_y(a,b)=0$. Assume that f_{xx}, f_{yy}, f_{xy} are continuous near P. Then

1.) If $D(a,b)>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local min.

2.) If $D(a,b)>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local max.

3.) If $D(a,b)<0$, then f has a saddle point at (a,b).

4.) If $D(a,b)=0$, the test is inconclusive. (See example 5 in textbook)

Summary:

Sign of $D(a,b)$	Sign of $f_{xx}(a,b)$ (or $f_{yy}(a,b)$)	Type of critical point
+	+	Local Min $\cup$
+	—	Local Max $\cap$
—	(Any)	Saddle
0	(Any)	Inconclusive

Note: If $D(a,b)>0$, then $f_{xx}(a,b)$ and $f_{yy}(a,b)$ must have the same sign, so the sign of $f_{yy}(a,b)$ also determines whether $f(a,b)$ is a local min or local max when $D(a,b)>0$.

$$D(a,b)>0: \underbrace{f_{xx}(a,b)\,f_{yy}(a,b)}_{+:\text{ same sign}}-\underbrace{[f_{xy}(a,b)]^2}_{+}>0 \ \ \& \ \ f_{xx}(a,b)\,f_{yy}(a,b)>[f_{xy}(a,b)]^2$$

Example: Apply the second derivative test to classify the critical points of $f(x,y) = x e^{-x^2-y^2}$.

From the previous example: $f_x = (1-2x^2)e^{-x^2-y^2}$ and $f_y = -2xy\, e^{-x^2-y^2}$

critical points: $P_1 = (1/\sqrt{2}, 0)$, $P_2 = (-1/\sqrt{2}, 0)$

$$f_{xx} = -4xe^{-x^2-y^2} + (1-2x^2)(-2x)e^{-x^2-y^2} = (4x^3-6x)e^{-x^2-y^2}$$

$$f_{xy} = (1-2x^2)(-2y)\,e^{-x^2-y^2}$$

$$f_{yy} = -2x\,e^{-x^2-y^2} - 2xy(-2y)e^{-x^2-y^2} = -2x(1-2y^2)e^{-x^2-y^2}$$

Since these derivatives are "messy" evaluate f_{xx}, f_{yy} & f_{xy} at critical points, then find D.

$$f_{xx}(P_1) = \frac{-4}{\sqrt{2}}\underbrace{e^{-1/2}}_{>0}, \quad f_{xy}(P_1) = 0, \quad f_{yy}(P_1) = \frac{-2}{\sqrt{2}}\underbrace{e^{-1/2}}_{>0}$$

$$f_{xx}(P_2) = \frac{4}{\sqrt{2}}e^{-1/2}, \quad f_{xy}(P_2) = 0, \quad f_{yy}(P_2) = \frac{2}{\sqrt{2}}e^{-1/2}$$

Apply 2nd derivative test:

$$D(P_1) = \underbrace{f_{xx}(P_1)\, f_{yy}(P_1)}_{(-)(-)=+} - [f_{xy}(P_1)]^2 > 0 \qquad \Big(\underbrace{\frac{-4}{\sqrt{2}}e^{-1/2}}_{(-)}\Big)\Big(\underbrace{-\frac{2}{\sqrt{2}}e^{-1/2}}_{(-)}\Big) - 0^2$$

$D(P_1) > 0$ and $\underbrace{f_{xx}(P_1) < 0}_{\text{or } f_{yy}(P_1) < 0}$ $\therefore$ local max at $P_1 = (1/\sqrt{2}, 0)$

The local max is $f(P_1)$.

$$D(P_2) = \underbrace{f_{xx}(P_2)\, f_{yy}(P_2)}_{(+)(+)=+} - [\underbrace{f_{xy}(P_2)}_{0}]^2 > 0 \quad \text{and} \quad \underbrace{f_{xx}(P_2) > 0}_{\text{or } f_{yy}(P_2) > 0}$$

$\therefore$ local min at $P_2 = (-1/\sqrt{2}, 0)$. The local min is $f(P_2)$

Example: Find and classify all critical points of $f(x,y) = 2x^2 - 4xy + y^3 + 2$.

$$\begin{cases} f_x = 4x - 4y = 0 & (1) \\ f_y = -4x + 3y^2 = 0 & (2) \end{cases}$$

Solve this system of equations for the critical points

From (1): $4x - 4y = 0 \Rightarrow 4x = 4y \Rightarrow x = y$

Substitute into (2): $\underbrace{-4y + 3y^2 = 0}_{\text{or } -4x+3x^2=0\ldots} \Rightarrow y(4-3y) = 0$

$\Rightarrow y = 0, \ y = 4/3$

critical points: $(0,0)$, $(4/3, 4/3)$

To classify, first find $D(x,y)$:

$$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 4 & -4 \\ -4 & 6y \end{vmatrix} = 4(6y) - (-4)^2$$

$\therefore D(x,y) = 24y - 16$

note $f_{xx} = 4$ for all (x,y)

Apply 2nd derivative test:

$D(0,0) = 24(0) - 16 = -16 < 0 \quad \therefore$ saddle point at $(0,0)$

$D(4/3, 4/3) = \cancel{24}^{8}(4/\cancel{3}) - 16 = 16 > 0$ and $f_{xx}(4/3, 4/3) = 4 > 0$

$\therefore$ local min at $(4/3, 4/3)$.

What if we want the Global (or Absolute) max and min?

Theorem (Existence and Location of Global Extrema):

Let $f(x,y)$ be a continuous function on a closed, bounded domain D in $\mathbb{R}^2$. Then

1.) $f(x,y)$ takes on both a global maximum and a global minimum value on D.

2.) The extreme values occur either at critical points in the interior of D or at points on the boundary of D.

This is also called the extreme value theorem for functions of 2 variables.

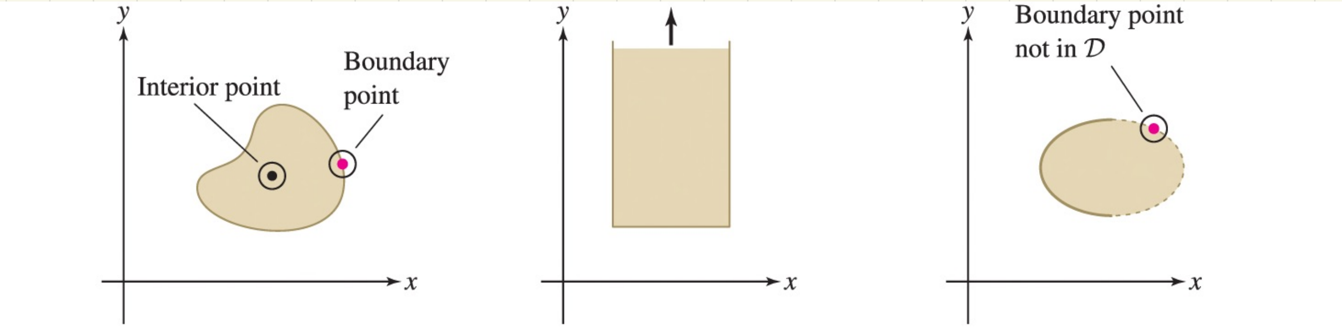


Fig. 11 (A) This domain is bounded and closed (contains all boundary points). (B) An unbounded domain (contains points arbitrarily far from the origin). (C) A nonclosed domain (contains some but not all boundary points).

To find the Absolute Max and Min:

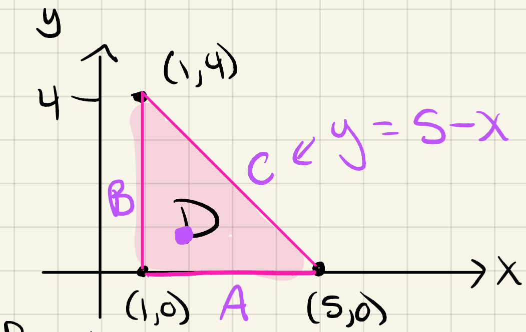
1.) Find the values of f at the critical points of f in D.

2.) Find the extreme values of f on the boundary of D.

3.) The largest from 1 & 2 is the absolute max & the smallest is the absolute min.

global

Example: Find the absolute max & min values of $f(x,y) = 3 + xy - x - 2y$ on the domain D which is the closed region with vertices $(1,0)$, $(5,0)$, $(1,4)$.



1.) $\begin{cases} f_x = y - 1 = 0 & \Rightarrow y = 1 \\ f_y = x - 2 = 0 & \Rightarrow x = 2 \end{cases}$ $\therefore$ critical point: $(2,1)$

$f(2,1) = 3 + \cancel{2} - \cancel{2} - 2 = 1$

2.) A: $\underbrace{y = 0,\ 1 \le x \le 5}_{\text{defines line segment for "A"}}$: $f(x,0) = \overbrace{3 - x}^{\text{This is the function on the line segment "A"}}$ is a decreasing function

max is $f(1,0) = 3 - 1 = 2$ & min is $f(5,0) = 3 - 5 = -2$

B: $x = 1$, $0 \le y \le 4$: $f(1,y) = 3 + y - 1 - 2y = 2 - y$ is a decreasing function

max is $\underbrace{f(1,0) = 2}_{\text{already found}}$ & min is $f(1,4) = 2 - 4 = -2$

C: $\underbrace{y = 5 - x,\ 1 \le x \le 5}_{\text{defines line segment for "c"}}$: $f(x, 5-x) = 3 + x(5-x) - x - 2(5-x) = -x^2 + 6x - 7$

$*$ Try to find the max & min values of $f(x, 5-x) = -x^2 + 6x - 7$ on the closed interval $1 \le x \le 5$ for next class. $*$

Hint: Let $g(x) = -x^2 + 6x - 7$ and use calc 1.