Announcements:

- The last day to add a class or drop a class (without being issued a W grade) is Monday 1/20.
  - This includes submitting the Authorization Form to get switched into one of the half coures (2011 or 2012) if you already have official credit on your transcript for half of the course.

- No classes Monday 1/20. Tuesday follows a regular Tuesday Schedule.

# Section 14.5: The Gradient and Directional Derivatives (cont.)

**Theorem (Computing the Directional Derivative):** If $f$ is differentiable at $P$ and $\vec{u}$ is a unit vector (so $\|\vec{u}\| = 1$), then the directional derivative in the direction of $\vec{u}$ is given by

$$\boxed{D_{\vec{u}} f(P) = \nabla f_P \cdot \vec{u}} = \frac{d}{dt} f(\vec{r}(t)) \Big|_{t=0} \quad \text{(chain rule for paths)}$$

$\vec{u}$ must be a unit vector.

Where $\vec{r}(t) = \underbrace{\vec{P}} + t\vec{u}$, is a line through $P$ in direction of $\vec{u}$. position vector: vector from origin to point $P$

Interpretation: $D_{\vec{u}} f$ is the "slope" of $f$ in the direction of $\vec{u}$.

Thus:
- $D_{\vec{u}} f < 0 \Rightarrow$ "downhill" in direction of $\vec{u}$.
- $D_{\vec{u}} f > 0 \Rightarrow$ "uphill" in direction of $\vec{u}$.

2.) Find the directional derivative of $f(x,y,z) = x^2 + 3y^3 z$ in the direction $\vec{v} = \langle 1, -2, 2 \rangle$ at the point $P(2,1,3)$.

Last lecture we found: $\nabla f = \langle 2x,\ 9y^2 z,\ 3y^3 \rangle$
(Example 2 of gradients)

$$\vec{u} = \vec{e}_{\vec{v}} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 1, -2, 2 \rangle}{\sqrt{1^2 + (-2)^2 + 2^2}} = \langle 1/3,\ -2/3,\ 2/3 \rangle$$

$$\nabla f_P = \nabla f(2,1,3) = \langle 2(2),\ 9(1)^2(3),\ 3(1)^3 \rangle = \langle 4, 27, 3 \rangle$$

or $\nabla f(P)$

$$D_{\vec{u}} f(P) = \nabla f_P \cdot \vec{u} = \langle 4, 27, 3 \rangle \cdot \langle 1/3,\ -2/3,\ 2/3 \rangle$$

$$= \frac{4}{3} - \frac{54}{3} + \frac{6}{3}$$

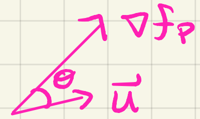
$$= -\frac{44}{3}$$

"Downhill" direction

**Question:** In what direction is the rate of change the largest?

**Recall:** $\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos(\theta)$ where $0 \leq \theta \leq \pi$ is the angle between non-zero vectors $\vec{v}$ and $\vec{w}$. (see section 12.3)

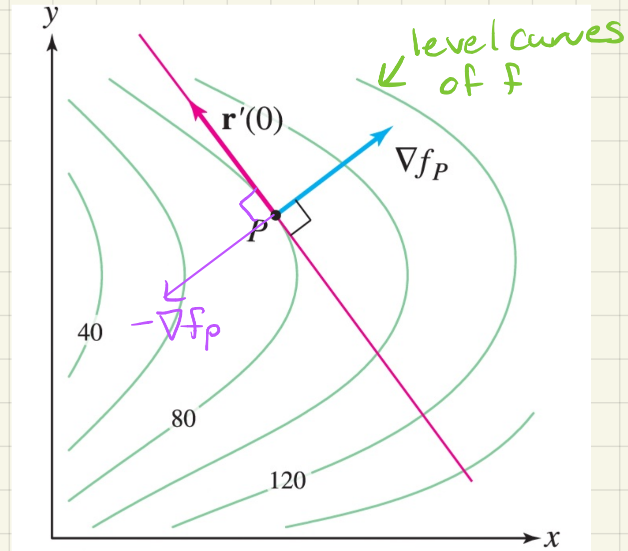
Assume that $\nabla f_P \neq \vec{0}$. Let $\vec{u}$ be a unit vector making an angle $\theta$ with $\nabla f_P$ where $0 \leq \theta \leq \pi$. Then,




$$D_{\vec{u}} f(P) = \nabla f_P \cdot \vec{u}$$
$$= \|\nabla f_P\| \underbrace{\|\vec{u}\|}_{=1} \cos\theta$$
$$= \|\nabla f_P\| \underbrace{\cos\theta}_{-1 \leq \cos\theta \leq 1}$$

$$\therefore \quad -\|\nabla f_P\| \leq D_{\vec{u}} f(P) \leq \|\nabla f_P\|$$

$\theta = \pi$, $\cos(\pi) = -1$ $\qquad$ $\theta = 0$, $\cos(0) = 1$




- $\nabla f_P$ points in the direction of fastest rate of increase of $f$ at $P$. (direction of steepest ascent) and this maximum rate of increase is $\|\nabla f_P\|$.

- $-\nabla f_P$ points in the direction of fastest rate of decrease of $f$ at $P$. (direction of steepest descent) and this maximum rate of decrease is $-\|\nabla f_P\|$.

- $\nabla f_P$ is normal $\underbrace{\text{(orthogonal)}}_{\perp}$ to the level curve (or level surface) of $f$ at $P$.

Example: Given $f(x,y,z) = x + \frac{y}{z}$ and point $P(4,3,-1)$.

a.) Find the maximum rate of increase of $f$ at $P$.

$$f(x,y,z) = x + y \cdot \frac{1}{z} = x + y \cdot z^{-1}$$

$$\nabla f = \langle 1, \frac{1}{z}, -yz^{-2} \rangle = \langle 1, \frac{1}{z}, \frac{-y}{z^2} \rangle$$

$$\nabla f_P = \nabla f(4,3,-1) = \langle 1, -1, -3 \rangle$$

$$\|\nabla f_P\| = \sqrt{1^2 + (-1)^2 + (-3)^2} = \sqrt{1+1+9}$$

The maximum rate of increase of $f$ at $P$ is $\underline{\quad \sqrt{11} \quad}$.

b.) Find a unit vector in the direction of the maximum rate of increase (steepest ascent) at $P$.

$$\frac{\nabla f_P}{\|\nabla f_P\|} = \frac{\langle 1, -1, -3 \rangle}{\sqrt{11}} = \langle 1/\sqrt{11}, -1/\sqrt{11}, -3/\sqrt{11} \rangle$$

## Theorem (Gradient as a Normal Vector):

Let $P=(a,b,c)$ be a point on the surface given by $F(x,y,z)=K$, for constant $K$, and assume $\nabla F_P \neq \vec{0}$. Then $\nabla F_P$ is normal to the tangent plane to the surface at $P$.

We can write the equation of the tangent plane to the surface at $P$ as

$$\boxed{F_x(a,b,c)(x-a)+F_y(a,b,c)(y-b)+F_z(a,b,c)(z-c)=0}$$

or: $\underbrace{\nabla F_P}_{\vec{n}} \cdot \langle x-a, y-b, z-c\rangle = 0$

Note: If $z=f(x,y)$, then taking $F(x,y,z)=f(x,y)-z=0$ will get same equation of tangent plane as in sec 14.4. Since $F_x=f_x$, $F_y=f_y$, $F_z=-1$ and $c=f(a,b)$, we have

$$f_x(a,b)(x-a)+f_y(a,b)(y-b)-1(z-f(a,b))=0$$

**Example:** Find an equation of the tangent plane to $\underbrace{x^2+2y^2+5z^2=8}_{F(x,y,z)=K}$ at $(1,1,-1)$.

$$\nabla F = \langle F_x, F_y, F_z \rangle = \langle 2x, 4y, 10z \rangle$$

$$\nabla F_p = \nabla F(1,1,-1) = \langle 2, 4, -10 \rangle$$ normal vector to tangent plane at P.

Tangent plane: $2(x-1)+4(y-1)-10(z+1) =$

# Section 14.6: Multivariable calculus Chain Rules

In one dimension (calc I): $h(t) = f(x(t))$

$$\Rightarrow \frac{dh}{dt} = \frac{df}{dx}\frac{dx}{dt} = f'(x(t))\,x'(t)$$

For functions of more than one variable, there are several options.

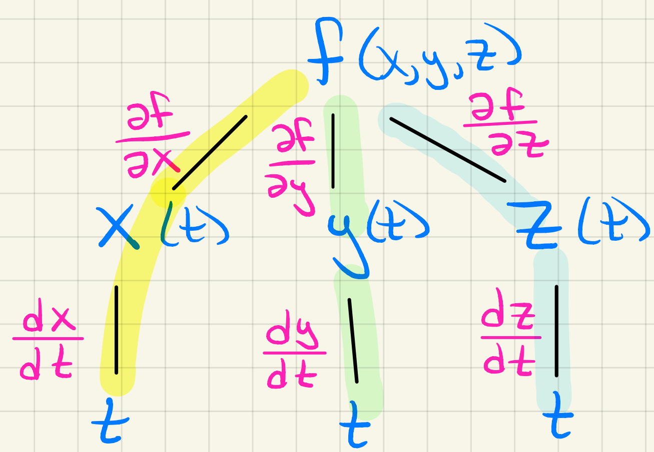
**Chain Rule for Paths (sec. 14.5)**: If $f$ and $\vec{r}(t)$ are differentiable, then

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

Writing $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$:

$$\frac{d}{dt} f(x(t), y(t), z(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Evaluate at $\langle x(t), y(t), z(t) \rangle$



Now, let's consider more general composite functions.

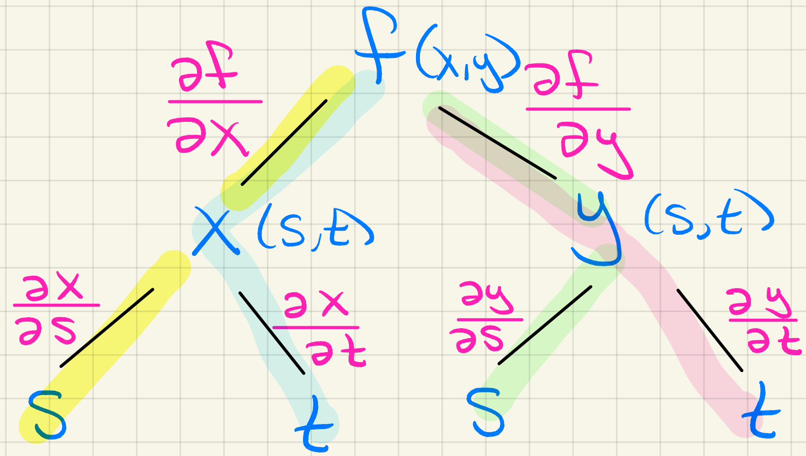
Let $f(x,y)$ be a differentiable function of $x$ and $y$ where $x = x(s,t)$ and $y = y(s,t)$ are differentiable functions of $s$ and $t$. Then,

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

Evaluated at $(x(s,t), y(s,t))$

and

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$




Could use other notation: $f_s = f_x x_s + f_y y_s$ & $f_t = f_x x_t + f_y y_t$

If we let $\vec{R}(s,t) = \langle x(s,t), y(s,t) \rangle$, then

$$\frac{\partial f}{\partial s} = \nabla f(\vec{R}(s,t)) \cdot \underbrace{\left\langle \frac{\partial x}{\partial s}, \frac{\partial y}{\partial s} \right\rangle}_{\text{"}\partial \vec{R}/\partial s\text{"}}$$

and

$$\frac{\partial f}{\partial t} = \nabla f(\vec{R}(s,t)) \cdot \left\langle \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t} \right\rangle$$

This extends to more variables.

General version of the Chain Rule: Let $f(x_1, \dots, x_n)$ be a differentiable function of $n$ variables. Suppose that each of the variables $x_1, \dots, x_n$ is a differentiable function of $m$ variables $t_1, \dots, t_m$. Then, for $k = 1, \dots, m$,

$$\boxed{\frac{\partial f}{\partial t_k} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_k} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_k} + \dots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_k}}$$

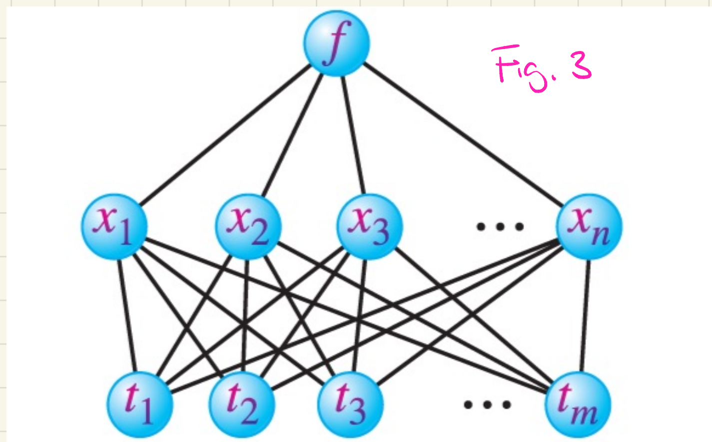



Fig. 3

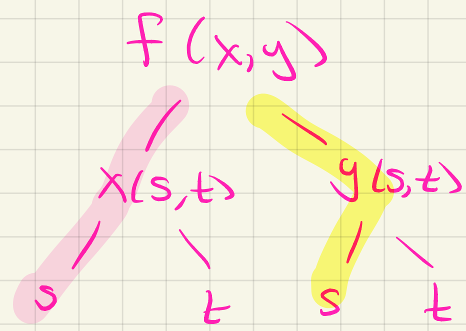
Note: Let $\vec{R}(t_1, \dots, t_m) = \langle x_1(t_1, \dots, t_m), \dots, x_n(t_1, \dots, t_m) \rangle$

$$\frac{\partial f}{\partial t_k} = \underbrace{\left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle}_{\text{evaluated at } \vec{R}(t)} \cdot \left\langle \frac{\partial x_1}{\partial t_k}, \frac{\partial x_2}{\partial t_k}, \dots, \frac{\partial x_n}{\partial t_k} \right\rangle$$

$$= \nabla f(\vec{R}(t_1, \dots, t_m)) \cdot \left\langle \frac{\partial x_1}{\partial t_k}, \frac{\partial x_2}{\partial t_k}, \dots, \frac{\partial x_n}{\partial t_k} \right\rangle$$

Example: Given $f(x,y) = e^x \sin y$ where $x(s,t) = st^2$ and $y(s,t) = s^2 t$.

Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.




$$\underbrace{\frac{\partial f}{\partial s}}_{f_s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= (e^x \sin y)(t^2) + (e^x \cos y)(2st)$$

Evaluated at $(x(s,t), y(s,t))$

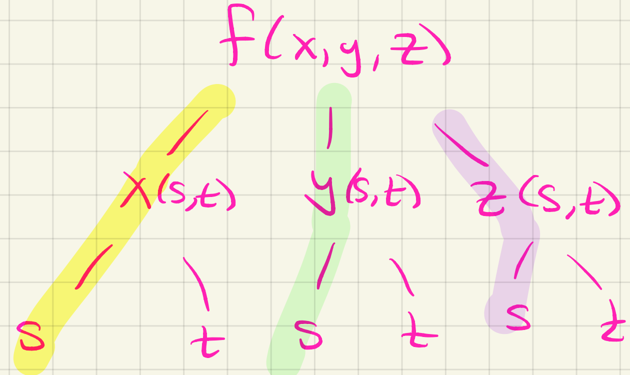
$$= \left(e^{st^2} \sin(s^2 t)\right) t^2 + \left(e^{st^2} \cos(s^2 t)\right)(2st)$$

only a function of $s$ and $t$.

$$\underbrace{\frac{\partial f}{\partial t}}_{f_t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} = \left(e^{st^2} \sin(s^2 t)\right)(2st) + \left(e^{st^2} \cos(s^2 t)\right)(s^2)$$

Same as above

__Example__: Given $f(x,y,z) = xy + yz + xz$ where $x(s,t) = st$, $y(s,t) = e^{st}$, and $z(s,t) = t^2$. Find $\frac{\partial f}{\partial s}$ at $(s,t) = (0,1)$.




$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\, \frac{\partial z}{\partial s}$$

$$= (y+z)(t) + (x+z)(te^{st}) + \cancel{(y+x)(0)}$$

When $(s,t) = (0,1)$: $x(0,1) = 0(1) = 0$, $y(0,1) = e^{0\cdot 1} = 1$, $z(0,1) = 1^2 = 1$

$$\left.\frac{\partial f}{\partial s}\right|_{\underbrace{(s,t)=(0,1)}_{\text{evaluated at}}} = (\underbrace{1}_{y(0,1)} + \underbrace{1}_{z(0,1)})(1) + (\underbrace{0}_{x(0,1)} + \underbrace{1}_{z(0,1)})(1 \cdot e^{0\cdot 1}) \quad = 2 + 1(1) = 3$$

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Could Alternatively plug in $x(s,t)$, $y(s,t)$, $z(s,t)$ first then evaluate

$$\frac{\partial f}{\partial s} = (e^{st} + t^2)(t) + (st + t^2)(te^{st})$$

$$\left.\frac{\partial f}{\partial s}\right|_{(s,t)=(0,1)} = (e^0 + 1^2)(1) + (0 + 1^2)(1e^0) = (1+1)(1) + 1(1) = 3$$

# Implicit Differentiation:

Assume that $F(x,y,z(x,y)) = 0$ implicitly defines $z$ as a differentiable function of $x$ and $y$. We use the Chain Rule to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$\frac{\partial}{\partial x}\left[F(x,y,z(x,y))\right] = \frac{\partial}{\partial x}[0]$$

$$\frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{1} + \cancel{\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}}\;\underbrace{}_{=0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \Rightarrow F_x + F_z\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \boxed{\underbrace{\frac{\partial z}{\partial x}}_{z_x} = -\frac{F_x}{F_z}} \quad \text{if } F_z \neq 0$$

Similiarly,

$$\boxed{\underbrace{\frac{\partial z}{\partial y}}_{z_y} = -\frac{F_y}{F_z}} \quad \text{if } F_z \neq 0$$

Example: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $\underbrace{x^3+y^3+z^3+6xyz=1}$.

$F(x,y,z)=x^3+y^3+z^3+6xyz-1=0$

$F_x = 3x^2+6yz$

$F_y = 3y^2+6xz$

$F_z = 3z^2+6xy$

$$z_x = \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{(3x^2+6yz)}{3z^2+6xy}$$

$$z_y = \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{(3y^2+6xz)}{3z^2+6xy}$$