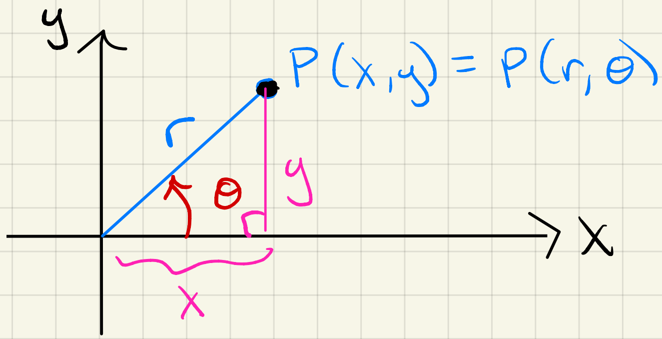
Section 15.4 : Integration in Polar Coordinates

Recall: (calc 2)



Polar to Rectangular

$x = r\cos\theta$

$y = r\sin\theta$

$\cos\theta = \frac{adj}{hyp} = \frac{x}{r}$

$\sin\theta = \frac{opp}{hyp} = \frac{y}{r}$

Rectangular to Polar

$x^2+y^2 = r^2 \Rightarrow r = \pm\sqrt{x^2+y^2}$

$\tan\theta = \frac{y}{x}$, $x \neq 0$. only deal with $r>0$

(signs of y and x tell us which quadrant θ is in.)

• Review sections 11.3 (Polar cordinates) and 7.2 (Trig Integrals)

What would be analogous to integrating over a rectangle in Polar Coordinates?

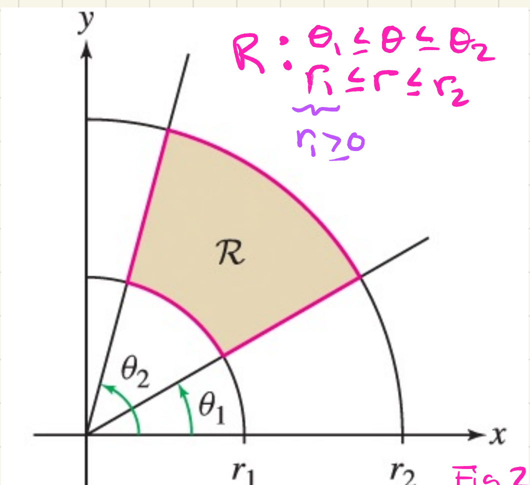


Fig 2

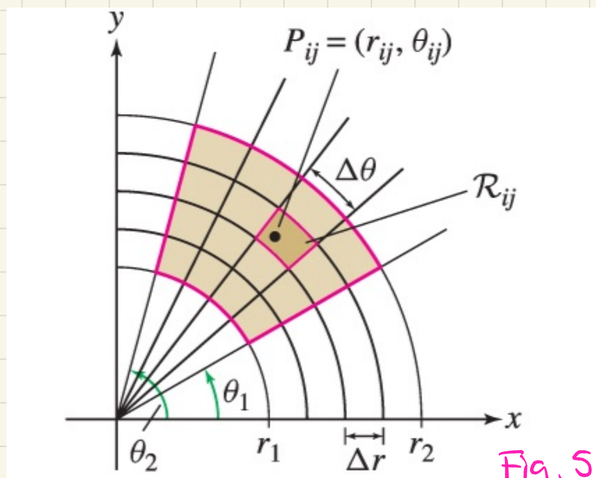


Fig. 5

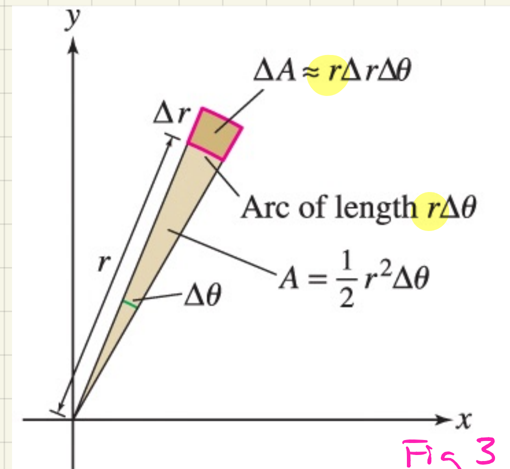
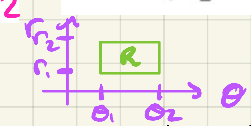


Fig 3

R is a polar rectangle :



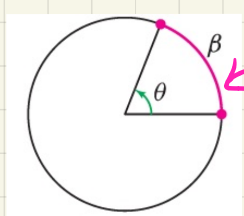
$\Delta A = r\Delta\theta\Delta r$

$dA = r\,dr\,d\theta$

OR

$dA = r\,d\theta\,d\theta$

Note:



what is the length of β ?

$(\text{length of } \beta) = \underbrace{\frac{\theta}{\cancel{2}\pi}}_{\text{fraction of } 2\pi \text{ covered by } \theta} \cdot \underbrace{\cancel{2}\pi r}_{\text{circumference of circle}} = \theta r$

Double Integral over Polar Rectangle:

For a continuous function f on the Polar rectangle domain

$R: \theta_1 \le \theta \le \theta_2, \; r_1 \le r \le r_2$ (assume $r_1 \ge 0$)

$$\boxed{\iint_R f(x,y)\,dA = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

(in either order)

Note: Fubini's Theorem applies and the iterated integral (in either order) is equal.

Theorem 1 - Double Integral in Polar Coordinates:

For a continuous function f on the domain

$D: \theta_1 \le \theta \le \theta_2, \; r_1(\theta) \le r \le r_2(\theta)$

$$\boxed{\iint_D f(x,y)\,dA = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta_1)}^{r_2(\theta_2)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

Here, the region D is radially simple.

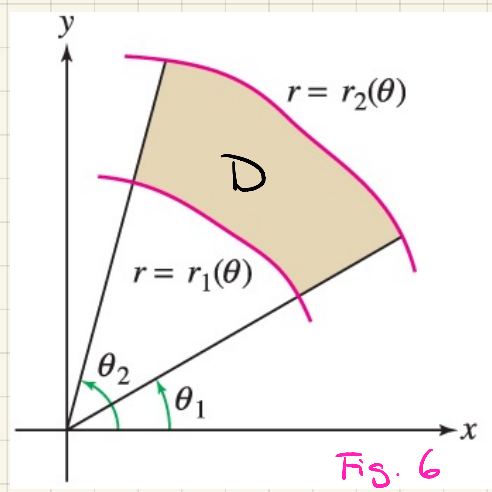


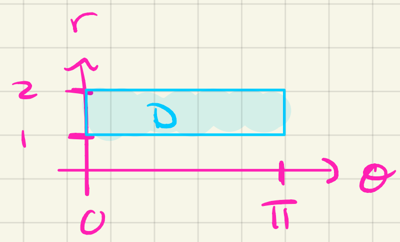
Fig. 6

Examples:

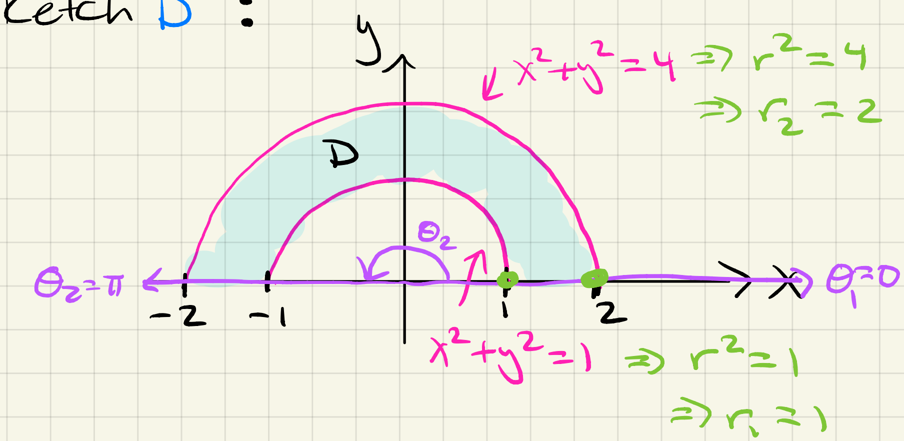
1.) Evaluate $\iint_D \underbrace{(3x+4y^2)}_{f(x,y)=3x+4y^2}\,dA$, where D is the region in the upper half-plane $(y \geq 0)$ bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Write D in polar coordinates (will be polar rectangle)

D: $0 \leq \theta \leq \pi$
$1 \leq r \leq 2$



Sketch D:



$$\iint_D \underbrace{(3x+4y^2)}_{f(x,y)}\,dA = \int_0^{\pi}\int_1^2 \underbrace{\left(3r\cos\theta + 4(r\sin\theta)^2\right)}_{f(r,\theta)} r\,dr\,d\theta$$

"distribute r in"

$$= \int_0^{\pi}\int_1^2 \left(3r^2\cos\theta + 4r^3\sin^2\theta\right) dr\,d\theta$$

$$= \int_0^{\pi} \left(r^3\cos\theta + r^4\sin^2\theta\right)\Big|_{r=1}^{r=2} d\theta$$

$$= \int_0^{\pi} \left[\left(8\cos\theta + 16\sin^2\theta\right) - \left(\cos\theta + \sin^2\theta\right)\right] d\theta$$

$$= \int_0^{\pi} \left(7\cos\theta + 15\sin^2\theta\right) d\theta$$

(from 7.2 trig integral)

Recall: Double-angle Identities

$\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$

$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$

$$= \int_0^{\pi} \left(7\cos\theta + \frac{15}{2}\left(1-\cos(2\theta)\right)\right) d\theta$$

$$= \int_0^{\pi} \left(7\cos\theta + \frac{15}{2} - \frac{15}{2}\cos(2\theta)\right) d\theta$$

$$= \left(7\sin\theta + \frac{15}{2}\theta - \frac{15}{4}\sin(2\theta)\right)\Bigg|_{\theta=0}^{\pi}$$

$$= \left(7\cancelto{0}{\sin(\pi)} + \frac{15}{2}(\pi) - \frac{15}{4}\cancelto{0}{\sin(2\pi)}\right) - \left(7\cancelto{0}{\sin(0)} + \frac{15}{2}(0) - \frac{15}{4}\cancelto{0}{\sin(0)}\right)$$

$$= \frac{15\pi}{2}$$

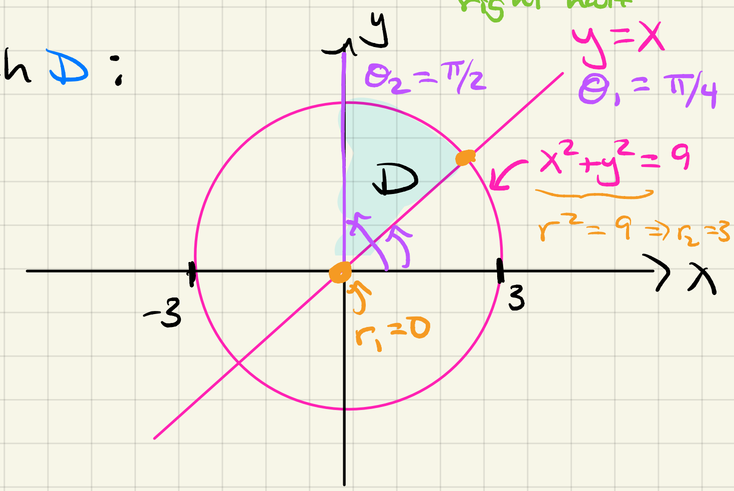
2.) $\iint_D x \, dA$, where D is the part of the disk $x^2+y^2=9$ with $\underbrace{x \geq 0}_{\text{right half}}$, $y \geq x$.

The ray $y=x$ has polar equation:

$$r\sin\theta = r\cos\theta \Rightarrow \tan\theta = 1$$

$$\Rightarrow \theta = \pi/4 \text{ (in first quadrant)}$$

Sketch D:



$$D: \left.\begin{array}{l} \pi/4 \leq \theta \leq \pi/2 \\ 0 \leq r \leq 3 \end{array}\right\} \text{ (polar rectangle)}$$

$$\iint_D x \, dA = \int_{\pi/4}^{\pi/2}\int_0^3 \underbrace{r\cos\theta \; r}_{r^2\cos\theta \,=\, g(r)h(\theta)} \, dr\, d\theta = \int_0^3 \int_{\pi/4}^{\pi/2} r\cos\theta \, r \, d\theta\, dr$$

$r^2\cos\theta = g(r)\,h(\theta)$! special case for rectangular regions 15.1

$$= \left(\int_{\pi/4}^{\pi/2} \cos\theta \, d\theta\right)\left(\int_0^3 r^2\, dr\right)$$

$$= \ldots = 9\left(1 - 1/\sqrt{2}\right)$$

3.) Evaluate the iterated integral $\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{\sqrt{1+x^2+y^2}}\, dx\, dy$ by first converting to Polar coordinates.

Find D in polar coordinates:

Given D: $0 \le y \le \sqrt{2}$

$y \le x \le \sqrt{4-y^2}$

$x = \sqrt{4-y^2}$ (right semi circle of radius 2)

$\Rightarrow x^2 = 4 - y^2$

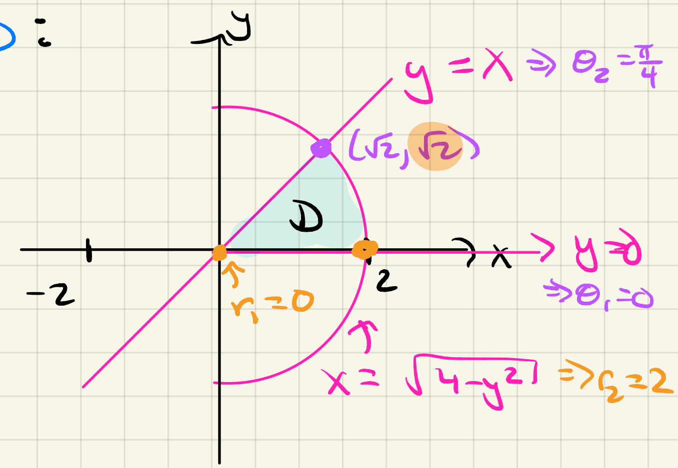
$\Rightarrow x^2 + y^2 = 4 \Rightarrow r = 2$

As a polar rectangle:

D: $0 \le \theta \le \pi/4$

$0 \le r \le 2$

Sketch D:



Point of intersection:

$y = \sqrt{4-y^2}$

$y^2 = 4 - y^2$

$2y^2 = 4 \Rightarrow y^2 = 2$

$\underbrace{y = + \sqrt{2}}_{\text{didn't need to find!}}$

$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{\sqrt{1+x^2+y^2}}\, dx\, dy = \int_0^{\pi/4} \int_0^2 \frac{1}{\sqrt{1+r^2}}\, r\, dr\, d\theta$$

$u = 1 + r^2$

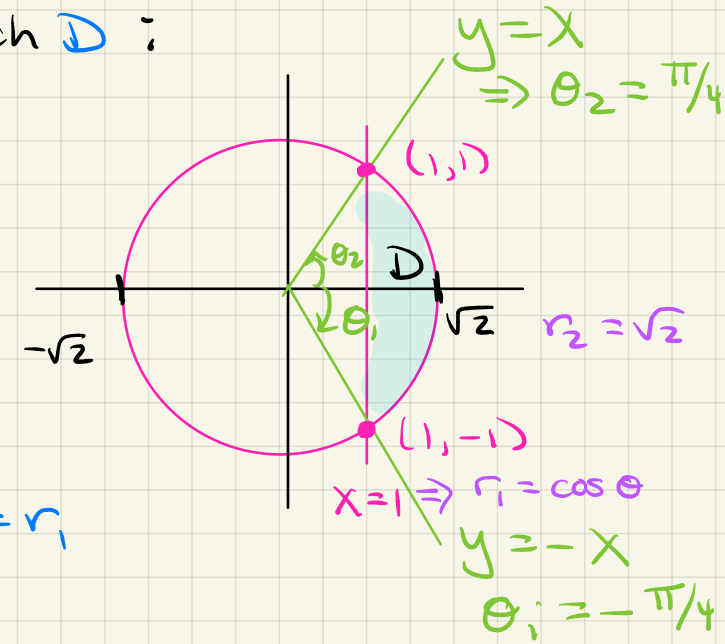
$du = 2r\, dr$

$\vdots$

$$= \ldots = (5^{1/2} - 1)\, \pi/4$$

4.) Let D be the region in the xy-plane given by $x^2+y^2 \le 2$, $x \ge 1$. Evaluate $\iint_D \frac{1}{(x^2+y^2)^2}\,dA$ using polar coordinates.

Sketch D:



Points of intersection: $(x=1)$

$$x^2+y^2=2 \quad \& \quad (x=1)$$

$$1+y^2=2 \Rightarrow y^2=1 \Rightarrow y=\pm 1$$

To get the bounds for r:

$$x=1 \Rightarrow r\cos\theta = 1 \Rightarrow r = \frac{1}{\cos\theta} = \sec\theta = r_1$$

$$D: \left.\begin{array}{l} -\pi/4 \le \theta \le \pi/4 \\ \sec\theta \le r \le \sqrt{2} \end{array}\right\} \text{ radially simple}$$

$$\iint_D \frac{1}{(x^2+y^2)^2}\,dA = \int_{-\pi/4}^{\pi/4} \int_{\sec\theta}^{\sqrt{2}} \underbrace{\frac{1}{(r^2)^2}}_{r^{-3}}\, r\,dr\,d\theta$$

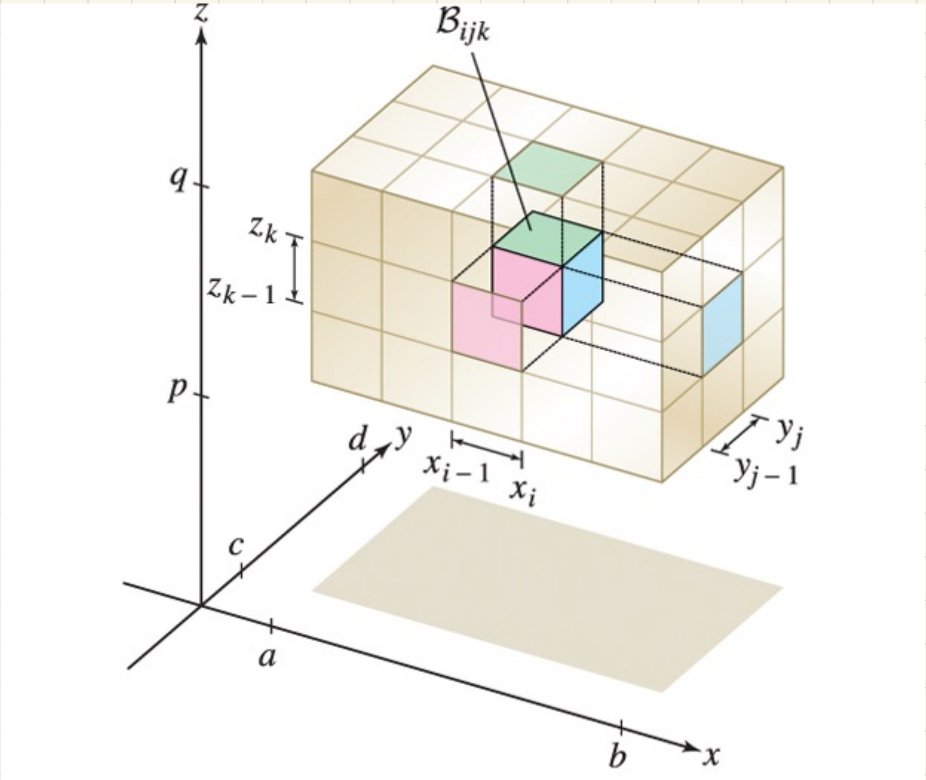
$$= \dots = \frac{1}{4}$$

Section 15.3 : Triple Integrals

Theorem 1 – Fubini's Theorem for Triple Integrals: The triple integral of a continuous function $f(x,y,z)$ over a box $B = \underbrace{[a,b]}_{a \le x \le b} \times \underbrace{[c,d]}_{c \le y \le d} \times \underbrace{[p,q]}_{p \le z \le q}$ is equal to the iterated integral:

$$\boxed{\iiint_B f(x,y,z)\, dV = \int_{x=a}^{b} \int_{y=c}^{d} \int_{z=p}^{q} f(x,y,z)\, dz\, dy\, dx}$$

Furthermore, the iterated integral may be evaluated in any order.



* there are 6 different orders for integrating over a box.

dV : is a volume element

Divide B into sub-boxes and take limit of a triple Reimann sum.

Example: $\iiint_B \underbrace{(xy+z^2)}_{f(x,y,z)}\,dV$, where $B = \underbrace{[-2,2]}_{x \text{ bounds}} \times \underbrace{[0,1]}_{y \text{ bounds}} \times \underbrace{[0,2]}_{z \text{ bounds}}$

Can integrate in any order!

$$\iiint_B (xy+z^2)\,dV = \int_0^2 \int_0^1 \int_{-2}^2 (xy+z^2)\,dx\,dy\,dz$$

Fix y & z and integrate wrt x first.

$$= \int_0^2 \int_0^1 \underbrace{\left(\tfrac{1}{2}x^2y + z^2x\right)}_{FTC}\Big|_{x=-2}^{x=2}\,dy\,dz$$

$$= \int_0^2 \int_0^1 \left(\left(\cancel{2y} + 2z^2\right) - \left(\cancel{2y} - 2z^2\right)\right) dy\,dz$$

$$= \int_0^2 \int_0^1 4z^2\,dy\,dz$$ ← (This is now a 15.1 integral of two variables and a rectangular domain).

$$= \int_0^2 4z^2y\Big|_{y=0}^{y=1}\,dz$$

$$= \int_0^2 4z^2\,dz$$

$$= \frac{4}{3}z^3\Big|_0^2 = \frac{32}{3}$$

Theorem 2: The triple integral of a continuous function f over the region

$$W = \{(x,y,z) : (x,y) \in D \text{ and } z_1(x,y) \le z \le z_2(x,y)\}$$ "z-simple"

is equal to the iterated integral

$$\iiint_W f(x,y,z)\,dV = \iint_D \left(\int_{z=z_1(x,y)}^{z=z_2(x,y)} f(x,y,z)\,dz \right) dA$$

The domain D is the projection of W onto the xy-plane.

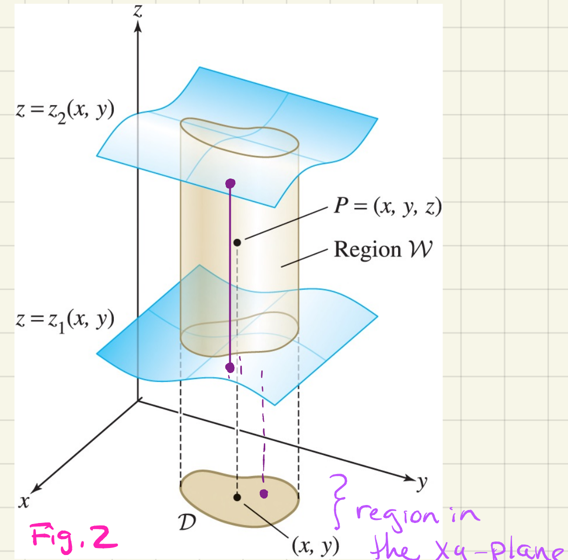


Fig. 2

z-simple region

The volume V of a region W is:

$$\boxed{V = \iiint_W 1\,dV}$$

in 15.1 $V = \iint_D f(x,y)\,dA$

$\text{Area}(D) = \iint_D 1\,dA$

Similiar definitions for triple integrals over x-simple or y-simple regions.

X-simple:

$$\iiint_W f(x,y,z)\,dV = \iint_D \left(\int_{x=x_1(y,z)}^{x=x_2(y,z)} f(x,y,z)\,dx \right) dA$$

y-simple:

$$\iiint_W f(x,y,z)\,dV = \iint_D \left(\int_{y=y_1(x,z)}^{y=y_2(x,z)} f(x,y,z)\,dy \right) dA$$

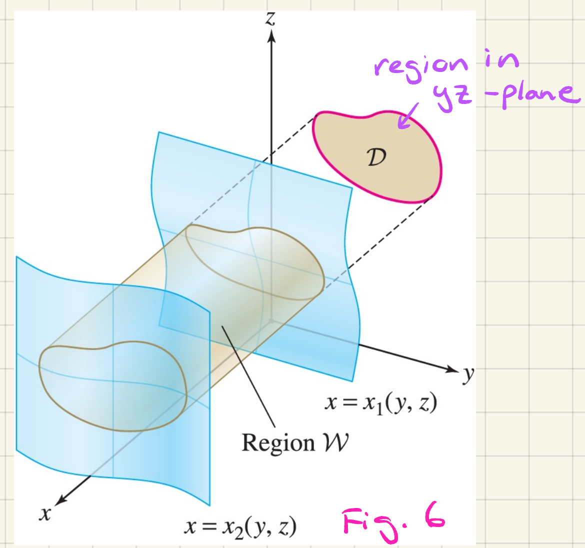


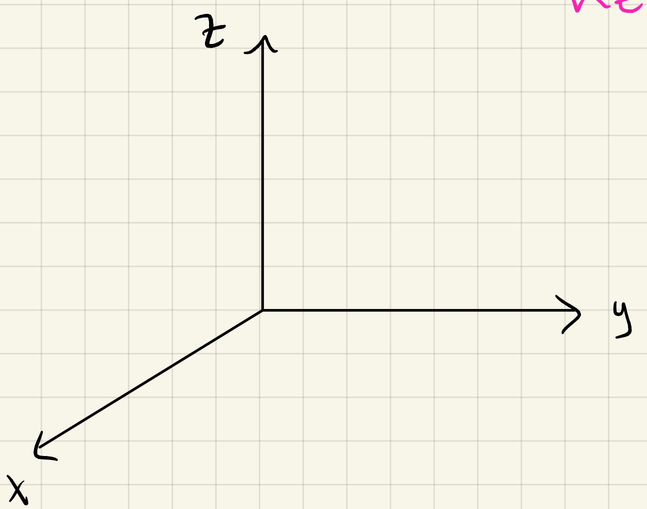
Fig. 6

X-simple region

Example: Evaluate $\iiint_W x\,dV$, where W is the tetrahedron bounded by the four planes $x=0$, $y=0$, $z=0$, $2x+y+z=4$.

Try to sketch W & D for next time.

Sketch W:



Sketch D: