Section 14.1 : Functions of Two or More Variables

Single variable : $y = f(x)$ domain is a subset of $\mathbb{R}$, range is a subset of $\mathbb{R}$.

Ex: $f(x) = x$ $(y = x)$; $f : \mathbb{R} \to \mathbb{R}$

Two variables : $z = f(x,y)$ domain is a subet of $\mathbb{R}^2$ (xy-plane), range is a subset of $\mathbb{R}$.

Ex: $z = f(x,y) = x^2 + y^2$; $f : \mathbb{R}^2 \to \mathbb{R}$

Three variables: $w = f(x,y,z)$ domain is a subset of $\mathbb{R}^3$ (3D-space), range is a subset of $\mathbb{R}$.

⋮

extends to n variables.

- The set of $(x_1, x_2, \dots, x_n)$ on which $f(x_1, x_2, \dots, x_n)$ can be defined is the domain of $f(x_1, x_2, \dots, x_n)$.

  such that, could also use "|"

  $$D_f = \{ (x_1, x_2, \dots, x_n) \in \mathbb{R}^n ; \underbrace{f(x_1, x_2, \dots, x_n) \text{ is defined}}_{\text{"rule"}} \}$$

  domain is a subset of $\mathbb{R}^n$

- The set of points that can be obtained by the values of the function $f(x_1, x_2, \dots, x_n)$ is called the range of f.

  $$R_f = \{ f(x_1, x_2, \dots, x_n) ; (x_1, x_2, \dots, x_n) \in D_f \}$$

  range is a subset of $\mathbb{R}$.

Examples: Find the domain and range.

1.) $f(x,y) = \sqrt{\underbrace{x^2+y^2}}$

$x^2+y^2 \geq 0$ always true!

$D_f = \mathbb{R}^2$

$R_f = [0, \infty)$

2.) $z = g(x,y) = \sqrt{\underbrace{4-x^2-y^2}}$

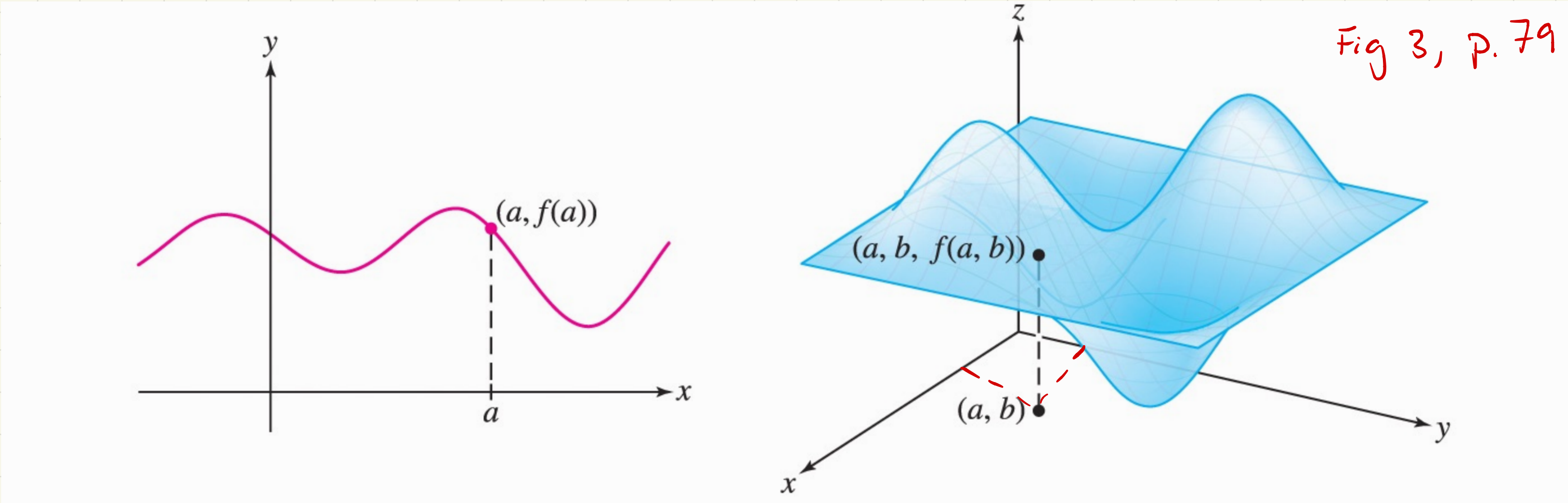
$4-x^2-y^2 \geq 0 \Rightarrow 4 \geq x^2+y^2$

$D_g = \{(x,y) \in \mathbb{R}^2 ; x^2+y^2 \leq 4\}$ This is the circle of radius 2 centered at the origin & its interior.

$R_g = [0, 2] = \{z ; 0 \leq z \leq 2\}$

Graph of $z = f(x,y)$: The set of all points $(x, y, \underbrace{f(x,y)}_{z})$ in space for all (x,y) in the domain of f, is the graph of f.

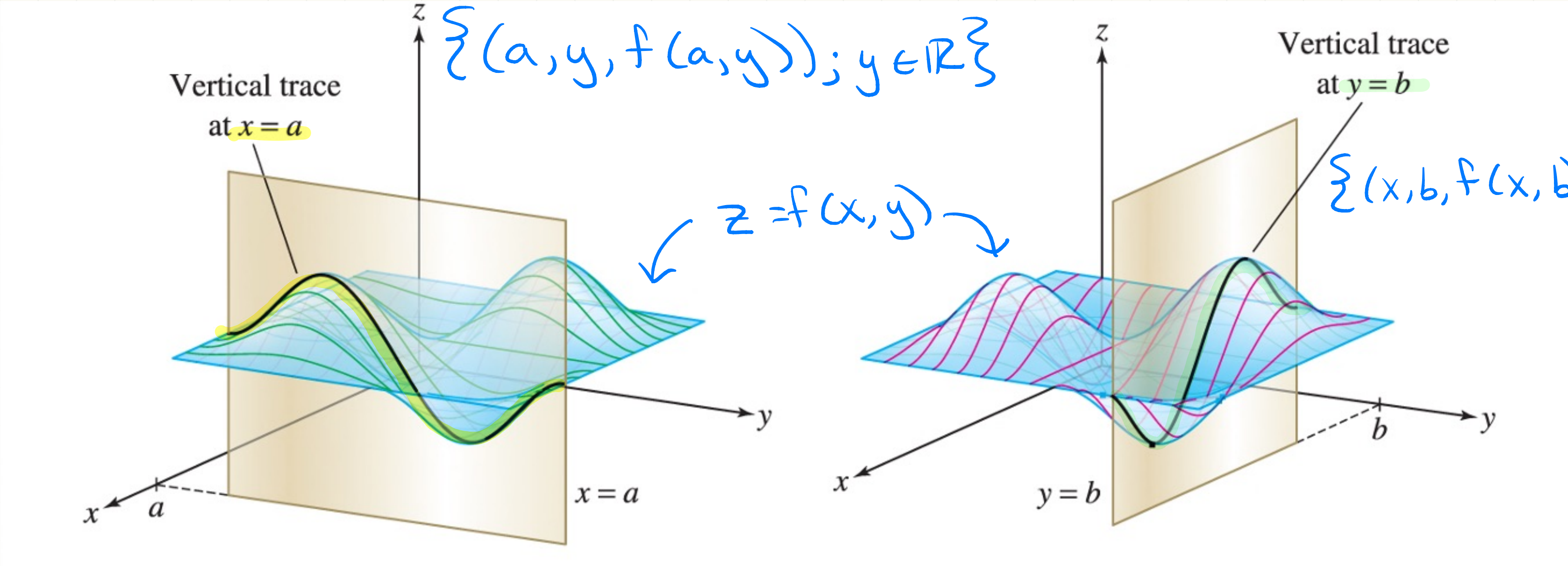
Also called the Surface $z = f(x,y)$.



(A) Graph of $y = f(x)$ (B) Graph of $z = f(x, y)$

Graphing functions of two variables is usually very difficult by hand. To help us analyze the graph of a function $f(x,y)$ we can use:

- Vertical traces intersection with planes $\{x = a\}$ or $\{y = b\}$
- Horizontal traces intersection with horizontal planes $f(x,y) = c$
- level curves / contour maps

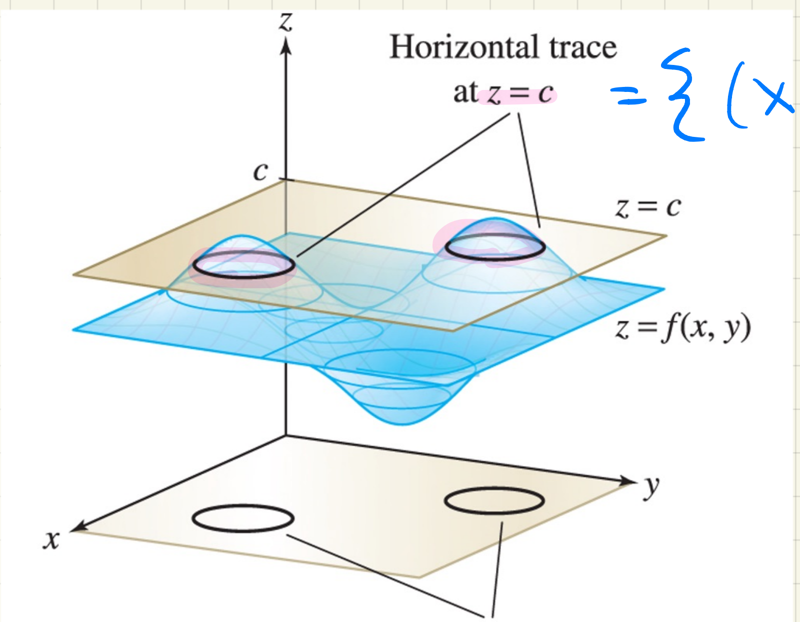


$\{(a, y, f(a,y)); y \in \mathbb{R}\}$

$z = f(x,y)$

$\{(x, b, f(x,b)); x \in \mathbb{R}\}$

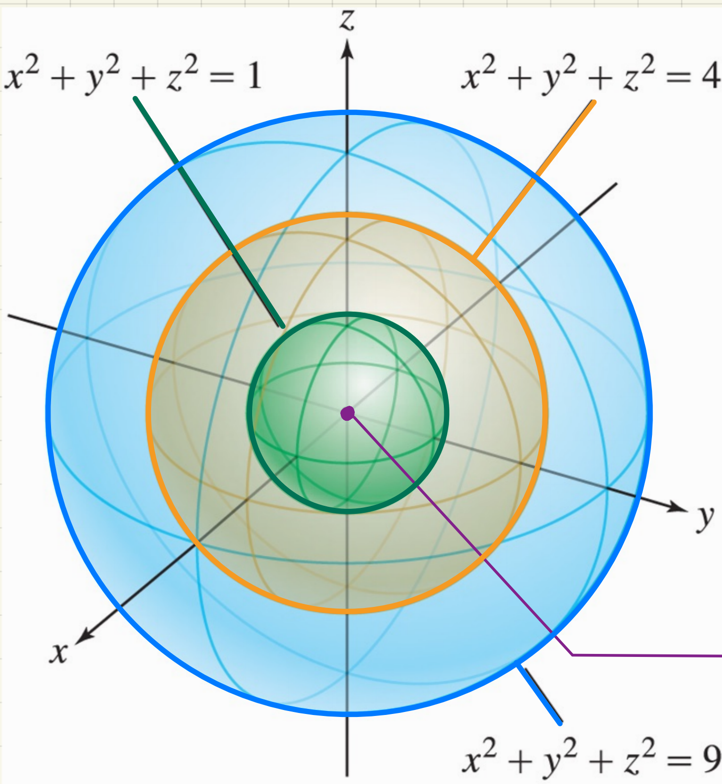
(A) Vertical traces parallel to yz-plane   (B) Vertical traces parallel to xz-plane



$= \{(x, y, f(x,y)); f(x,y) = c\}$

$\longrightarrow$ Projection of the horizontal trace at $z = c$ onto the xy-plane.

- What about functions of more than two variables?
  - no longer able to plot the graph
  - for $f(x,y,z)$ can plot the level surfaces $f(x,y,z) = c$



Level surfaces: $x^2 + y^2 + z^2 = c$

$x^2 + y^2 + z^2 = 0$

For functions of more than 3 variables, we cannot even plot level surfaces.

Comments on continuity...

- Most functions of 2 or more variables that we will consider are compositions of elementary functions, and so are continuous.

- Many exceptions occur at points where the function has a zero in the denominator. In this case, you have

$$f(x_0, y_0) = \frac{\text{Something}}{0}$$

If "Something" $\neq 0$, then the function blows up at (x_0, y_0).

Section 14.3: Partial Derivatives

- For functions of one variable $f(x)$, $f'(x) = \frac{df}{dx}$ is the rate of change of f with respect to x.

equivalent notations

- The **partial derivatives** are the rates of change with respect to each variable seperately.

Given $z = f(x,y)$

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \Bigg\} \; y \text{ is fixed}$$

- Slope of surface in x direction
- rate of change with respect to x

partial derivative of f wrt x

and

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x,y+k) - f(x,y)}{k} \Bigg\} \; x \text{ is fixed}$$

- Slope of surface in y direction.
- rate of change with respect to y.

partial derivative of f wrt y

Notation: Two-variable functions $z = f(x,y)$

Partial derivative with respect to x: f_x, $f_x(x,y)$, z_x, $\frac{\partial f}{\partial x}$, $\frac{\partial z}{\partial x}$

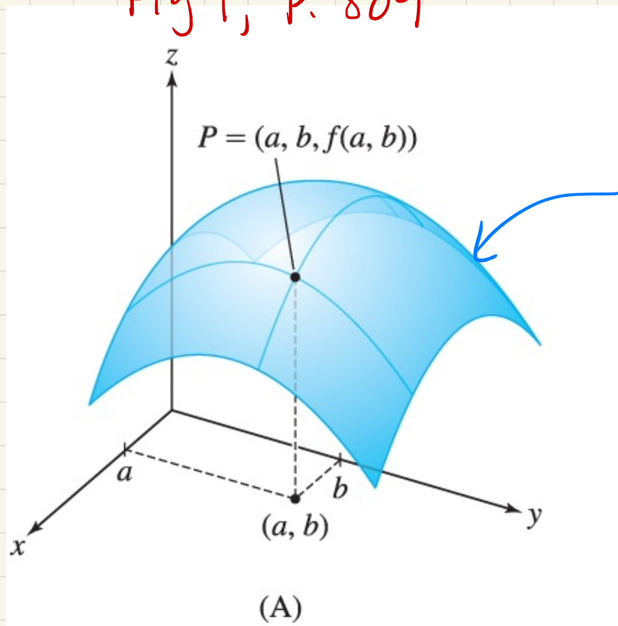
- To find, treat y as a constant and differentiate with respect to x.

Partial derivative with respect to y: f_y, $f_y(x,y)$, z_y, $\frac{\partial f}{\partial y}$, $\frac{\partial z}{\partial y}$

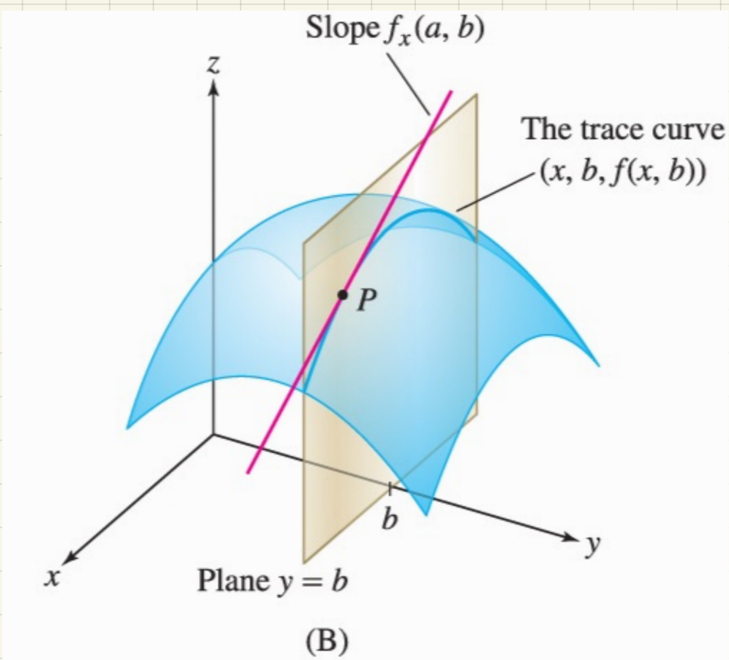
- To find, treat x as a constant and differentiate with respect to y.

Geometric Interpretation of a Partial Derivative :

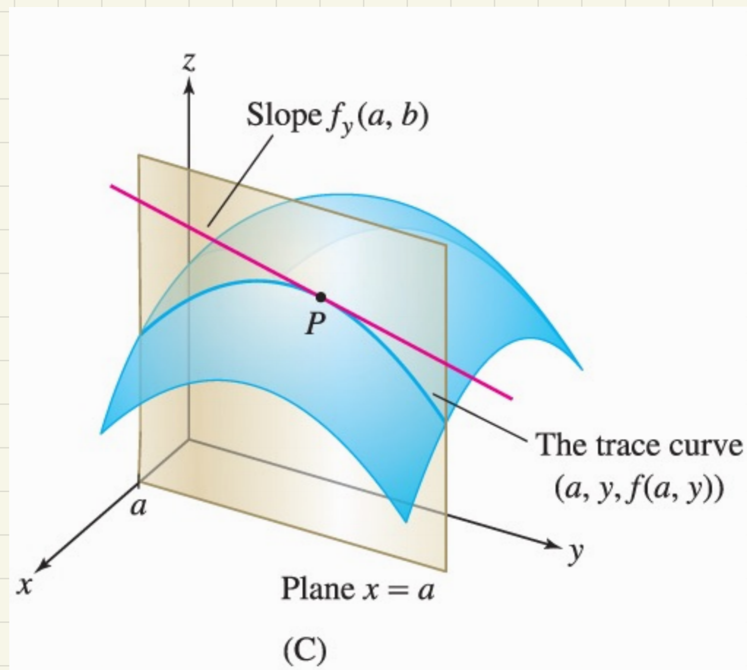
Fig 1, P. 809



$z = f(x, y)$



Fix y: y constant



Fix x: x constant

Examples: Compute f_x and f_y.

1.) $f(x,y) = x^3 y + \ln(x+y)$

$f_x = 3x^2 y + \dfrac{1}{x+y}$ ↳ Treat y as a constant

$f_y = x^3 + \dfrac{1}{x+y}$ ↳ Treat x as a constant

2.) $f(x,y) = x^3 y + \ln\underbrace{(x^2+4y^3)}_{\text{chain rule}}$

$$f_x = 3x^2 y + \frac{1}{x^2+4y^3} \cdot \frac{\partial}{\partial x}\left[x^2+4y^3\right] = 3x^2 y + \frac{2x}{x^2+4y^3}$$

$$f_y = x^3 + \frac{1}{x^2+4y^3}\frac{\partial}{\partial y}\left[x^2+4y^3\right] = x^3 + \frac{12y^2}{x^2+4y^3}$$

3.) $f(x,y) = x^2 e^{xy}$

$$f_x = \frac{\partial}{\partial x}\underbrace{\left(x^2 e^{xy}\right)}_{\text{product rule}} = \frac{\partial}{\partial x}(x^2)e^{xy} + x^2 \underbrace{\frac{\partial}{\partial x}\left(e^{xy}\right)}_{\text{chain rule}}$$

$$= 2x e^{xy} + x^2 e^{xy} y = (2x + x^2 y)e^{xy}$$

$$f_y = \frac{\partial}{\partial y}\left(x^2 e^{xy}\right) = x^2 \frac{\partial}{\partial y}\underbrace{\left(e^{xy}\right)}_{\text{chain rule}} = x^2\left(x e^{xy}\right) = x^3 e^{xy}$$

Higher-Order Partial Derivatives:

$f(x,y)$

- $\frac{\partial}{\partial x}(f)$ → 1st-order partials: f_x
  - $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$ → 2nd-order partials: f_{xx}
    - 3rd-order partials: f_{xxx}, f_{xxy}
  - $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ → $(f_x)_y = f_{xy}$
    - f_{xyx}, f_{xyy}
- $\frac{\partial}{\partial y}(f)$ → f_y
  - $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ → $(f_y)_x = f_{yx}$
    - f_{yxx}, f_{yxy}
  - $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$ → f_{yy}
    - f_{yyx}, f_{yyy}

The order in **mixed partials** does not matter (as long as they are continuous)

$$\boxed{f_{xy} = f_{yx}}$$

* Also true for higher-order partial derivatives: $f_{xyy} = f_{yxy} = f_{yyx}$

Other Notation:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{xy} = \frac{\partial^2 f}{\partial y\, \partial x}, \quad f_{yx} = \frac{\partial^2 f}{\partial x\, \partial y}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

Example: Find all second order partial derivative for $f(x,y) = x^2 e^{xy}$.

Note: Found $f_x = (2x + x^2 y) e^{xy}$ and $f_y = x^3 e^{xy}$ in example 3.

$$f_{xx} = \frac{\partial}{\partial x}(f_x) =$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) =$$

$$f_{yx} = \frac{\partial}{\partial x}(f_y) =$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) =$$

3-Variable Function:

Example: $f(x,y,z) = yz\ln(xy)$. Find the indicated partial derivatives.

$f_x(x,y,z) =$

$f_y(x,y,z) =$

$f_z(x,y,z) =$

Find f_{yzx}.

Find f_{zxy}.