Section 1.3 : Consistent Systems of Linear Equations (cont.)

Recall:

Given a $(m \times n)$ system of equations,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned} \Rightarrow \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] = [A \mid \vec{b}]$$

Recall: Given any system of linear equations, there are at most 3 possibilities:

1.) The system has infinitely many solutions.

2.) The system has no solution.

3.) The system has a unique solution.

A system of linear equations is consistent if it has either one solution or infinitely many solutions. (1 and 3)

A system is inconsistent if it has no solution. (2)

Goal: How do we determine which one?

- If consistent, look at reduced system
  - Free columns $\Rightarrow$ infinitely many
  - All leading 1 columns $\Rightarrow$ unique

Def: A $(m \times n)$ system of linear equations is homogeneous if $b_1 = b_2 = \dots = b_m = 0$, i.e., the system is:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned} \Rightarrow \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & 0 \end{array}\right] = [A \mid \underbrace{\vec{0}}_{\vec{0} \in \mathbb{R}^m}]$$

OR: $A\vec{x} = \vec{0}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

- Every homogeneous system is consistent since $x_1 = x_2 = \dots = x_n = 0$ is always a solution. This is called the trivial solution. Any other solution is called a nontrivial solution.

So, every homogeneous system has either 1 unique solution

No free variables/ all columns leading 1 columns only trivial solution.

OR $\underbrace{\text{infinitely many solutions}}_{\text{free columns / free variables in reduced system.}}$.

- If $m < n$, then the system has nontrivial solutions, i.e., it has infinitely many solutions. (less equations than unknowns or there will be free columns)

Example: $2x_1 + 3x_2 - x_3 = 0$ (2x3) homogeneous system

$x_1 - 5x_2 - 2x_3 = 0$

$2 < 3 \Rightarrow$ infinitely many solutions,

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 1 & -5 & -2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -5 & -2 & 0 \\ 2 & 3 & -1 & 0 \end{array}\right]$$

$$R_2 \to R_2 - 2R_1 \quad \left[\begin{array}{ccc|c} 1 & -5 & -2 & 0 \\ 0 & 13 & 3 & 0 \end{array}\right] \quad R_2 \to \frac{1}{13}R_2 \quad \left[\begin{array}{ccc|c} 1 & -5 & -2 & 0 \\ 0 & 1 & 3/13 & 0 \end{array}\right]$$

free column: x_3 is free variable

$-\frac{26}{13} + \frac{15}{13}$

$$R_1 \to R_1 + 5R_2 \quad \left[\begin{array}{ccc|c} 1 & 0 & -11/13 & 0 \\ 0 & 1 & 3/13 & 0 \end{array}\right]$$

column never changed!

$$\Rightarrow \begin{aligned} x_1 \quad - 11/13\, x_3 &= 0 \\ x_2 + 3/13\, x_3 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= 11/13\, x_3 \\ x_2 &= -3/13\, x_3 \\ x_3 &= x_3 \end{aligned}$$

x_3 free variable

$x_3 \in \mathbb{R}$

In vector form: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 11/13 \\ -3/13 \\ 1 \end{bmatrix}$

Example:
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ x_1 + 3x_3 &= 0 \end{aligned}$$

(3×3) homogeneous system

$3 = 3 \Rightarrow$ either 1 unique solution or infinitely many

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 3 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_1$
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right]$$

$R_3 \to R_3 + 2R_2$
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

$R_3 \to \frac{1}{4} R_3$
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Echelon: all columns are leading 1 columns.
$\Rightarrow$ unique solution: trivial solution

$R_1 \to R_1 - 3R_3$
$R_2 \to R_2 - 2R_3$
$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$R_1 \to R_1 - 2R_2$
$$\underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]}_{\text{rref}} \Rightarrow \begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned} \Rightarrow \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Theorem: Let C be an $(m \times n)$ matrix in reduced row echelon form and let r be the number of nonzero rows of C. Then, for the system represented by $[C \mid \vec{d}\,]$ there are three possibilities:

1.) The system is inconsistent.

2.) The system is consistent and $\underbrace{r<n}_{\text{free columns}}$. Thus there are infinitely many solutions.

3.) The system is consistent and $\underbrace{r=n}_{\text{all columns are leading 1 columns.}}$. Thus, there is a unique solution.

$$[A \mid \vec{b}\,] \rightarrow \text{EROs} \rightarrow [C \mid \vec{d}\,]$$

Corollary: Consider an $(m \times n)$ system of linear equations. If $m<n$, then either the system is inconsistent or it has infinitely many solutions.

In other words, consider

$$\underset{(m\times n)}{[A\,|\,\vec{b}\,]} \xrightarrow{\text{EROs}} [\,C\,|\,\vec{d}\,] = \left[\begin{array}{c|c} \text{non zero} & d_1 \\ \text{rows} & \vdots \\ & d_r \\ \hline \text{zero} & d_{r+1} \\ \text{rows} & \vdots \\ & d_m \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}a\\b\\c\end{array}}\right\} r \\ \left.\vphantom{\begin{array}{c}a\\b\\c\end{array}}\right\} m-r \end{array}$$

(the left block of the matrix is labelled rref)

Cases:

I.) No zero rows ($m-r=0 \iff r=m$)

a) $m=n \Rightarrow$ exactly 1 solution (every column is leading 1 column)

b) $m\neq n \Rightarrow$ infinitely many solutions (free column / free variable)

II.) one or more zero rows ($r\neq m$)

a) If even one of $d_{r+1}, d_{r+2}, \dots, d_m$ is non-zero $\Rightarrow$ no solution

b) If $d_{r+1}=d_{r+2}=\dots=d_m=0$:

- $r=n \Rightarrow$ exactly one solution (every column is leading 1 column)
- $r\neq n$ $(r<n) \Rightarrow$ infinitely many solutions : $n-r$ $\underbrace{\text{free variables.}}_{n-r \text{ free columns}}$
  (can be assigned arbitrary values)

Examples:

1.) $[A \mid \vec{b}] \rightarrow$ EROs $\rightarrow$ $[\underbrace{C}_{\text{rref}} \mid \vec{d}] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $\begin{array}{l} \} \; r = 2 \text{ nonzero rows} \\ \} \; 2 \text{ zero rows} \end{array}$

2 free columns: x_2 and x_4 are free variables.

2 leading 1 columns

A is (4×4): $m = 4$, $n = 4$

$a \neq 0$: No solution (inconsistent)

$a = 0$: consistent and infinitely many solutions.

when $a = 0$:

$$\begin{aligned} x_1 \quad\quad + 3x_4 &= 1 \\ x_3 - 2x_4 &= 4 \end{aligned} \Rightarrow \begin{aligned} x_1 &= 1 - 3x_4 \\ x_2 &= x_2 \quad \text{(free var)} \\ x_3 &= 4 + 2x_4 \\ x_4 &= x_4 \quad \text{(free)} \end{aligned}$$

Vector form: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ $\quad x_2, x_4 \in \mathbb{R}$

2.) $[A \mid \vec{b}] \rightarrow$ EROs $\rightarrow$ $[C \mid \vec{d}] = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\begin{array}{l} \} \; r = 2 \text{ nonzero rows} \\ \} \; \underbrace{3}_{} \text{ zero rows} \end{array}$

A is (5×2): $m = 5$, $n = 2$

both columns leading 1 columns,

$m - r = 5 - 2 = 3$

Consistent: $0 = 0$

unique solution $\quad x_1 = 2, \; x_2 = 0 \quad$ or $\quad \vec{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Example: Determine conditions on b_1, b_2, b_3 for the system to be consistent.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= b_1 \\ 2x_1 - 3x_2 + 2x_3 &= b_2 \\ -x_1 \qquad + 5x_3 &= b_3 \end{aligned}$$

free column — x_3 is free variable

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 2 & -3 & 2 & b_2 \\ -1 & 0 & 5 & b_3 \end{array}\right] \longrightarrow \text{EROs} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -5 & -3b_1 + 2b_2 \\ 0 & 1 & -4 & -2b_1 + b_2 \\ 0 & 0 & 0 & -3b_1 + 2b_2 + b_3 \end{array}\right]$$

(verify this!)

rref

3rd row: $0x_1 + 0x_2 + 0x_3 = -3b_1 + 2b_2 + b_3 = 0$

Conclusion:

- if $-3b_1 + 2b_2 + b_3 \neq 0$, then no solution (inconsistent)
- if $-3b_1 + 2b_2 + b_3 = 0$, then consistent infinitely many solution

Consider $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $(b_1 = b_2 = b_3 = 1)$

Check condition: $-3(1) + 2(1) + 1 = 0$ $\therefore$ consistent

Consider $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Check condition: $-3(1) + 2(1) + 0 = -1 \neq 0$ $\therefore$ inconsistent

Sections 1.5 and 1.6: Matrix Operations and Algebraic Properties

Recall matrix notation: For a $(m \times n)$ matrix A,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

m = # of rows

n = # of columns

a_{ij} = entry in i^{th} row, j^{th} column

$A = (a_{ij})_{\underbrace{m \times n}}$

can be omitted if size given

Def: Two matrices are equal if they have the same size and their corresponding entries are equal. That is, if $A = (a_{ij})$ and $B = (B_{ij})$ are $(m \times n)$ matrices, $A = B$ if and only if $a_{ij} = b_{ij}$ for all i, j, $1 \le i \le m$, $1 \le j \le n$.

Def: If A and B are $(m \times n)$ matrices, then their sum $A+B$ is the $(m \times n)$ matrix defined by $(A+B)_{ij} = a_{ij} + b_{ij}$ (add corresponding entries)

Def: If A is an $(m \times n)$ matrix and c is a scalar, then the scalar multiple cA is the $(m \times n)$ matrix defined by $(cA)_{ij} = ca_{ij}$ (multiply every entry by the scalar c)

Note: If $c = -1$, then $cA = -1A = -A$ and $(-A)_{ij} = -a_{ij}$. Then, $A - B = A + (-B)$.

Def: If $A = (a_{ij})$ is an $(m \times n)$ matrix, then the transpose of A, denoted A^T, is the $(n \times m)$ matrix with $(A^T)_{ij} = a_{ji}$, i.e., rows of A are columns of A^T.

(and vice versa)

Examples: Consider $A = \begin{bmatrix} 4 & -1 \\ 0 & 3 \\ 5 & 2 \end{bmatrix}_{(3\times 2)}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}_{(2\times 3)}$

Find the following, if defined. If undefined state why.

a.) $2B = 2\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix}_{(2\times 3}$

b.) $\underset{(3\times 2)}{A} + \underset{(2\times 3)}{2B}$ undefined: not same size!

c.) $\underset{(2\times 3)}{A^T} + \underset{(2\times 3)}{2B} = \underbrace{\begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}}_{A^T} + \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 7 \\ 5 & 13 & 16 \end{bmatrix}$

d.) $\underset{(3\times 2)}{A} - \underbrace{2B^T}_{\substack{(2B)^T \\ (3\times 2)}} = \begin{bmatrix} 4 & -1 \\ 0 & 3 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ 2 & 10 \\ 2 & 14 \end{bmatrix} = \begin{bmatrix} 2 & -7 \\ -2 & -7 \\ 3 & -12 \end{bmatrix}$

Recall:

An n-dimensional vector in $\mathbb{R}^n$ can be written as an $(n\times 1)$ matrix called a column vector.

$$\vec{x} = \langle x_1, x_2, \dots, x_n \rangle \iff \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Vector Form of the General Solution: (or Vector form of solution)

Example: Given the Matrix $B = \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right]$ is the augmented matrix for a homogeneous system of linear equations. Give the vector form of the general solution.

Free columns : x_3 & x_4 free variables

(2x4) system

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right] \Rightarrow \begin{aligned} x_1 \quad - x_3 - 2x_4 &= 0 \\ x_2 + 2x_3 + 3x_4 &= 0 \end{aligned} \Rightarrow \begin{aligned} x_1 &= x_3 + 2x_4 \\ x_2 &= -2x_3 - 3x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

leading 1 columns

Vector form of general solution is what we have been calling "Vector form"

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}}$$

must write as a "linear combination" of vectors multiplied by free variables. In non-homogeneous systems may also have constant vector.

The general solution of an $(m\times n)$ system has the form: $\vec{x} = \vec{d} + a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_p\vec{v}_p$ where $\vec{d}, \vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ are fixed vectors in $\mathbb{R}^n$. The scalars $a_1, a_2, \dots, a_p \in \mathbb{R}$ correspond to the p free variables in the reduced system.

Matrix Multiplication:

If A is an $(m \times n)$ matrix and B is an $(n \times p)$ matrix, then their product AB is defined and is an $(m \times p)$ matrix. The entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{ij}$ denotes the ij^{th}-entry in AB, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

The ijth entry of AB is the dot product of the i^{th} row of A and jth column of B.

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & & b_{nj} & & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{bmatrix}$$

$(m \times n)$ $(n \times p)$ $(m \times p)$

match

size of AB

Matrix-Vector Multiplication:

Let $A = \begin{bmatrix} | & | & & | \\ \vec{A}_1 & \vec{A}_2 & \cdots & \vec{A}_n \\ | & | & & | \end{bmatrix}_{m \times n}$ where $\vec{A}_j$ is the j^{th} column of A, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}_{n \times 1}$

$$A\vec{v} = v_1\vec{A}_1 + v_2\vec{A}_2 + \dots + v_n\vec{A}_n = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \dots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \dots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \dots + a_{mn}v_n \end{bmatrix}$$

Example: $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

For each of the expressions below, determine whether the product is defined. If it is, calculate it.

a) AC

b) CA

c) $A^T C$

d) $C A^T$

e) BA

f) $B A^T$

g) $A\vec{v}$

h) $B\vec{v}$

Note: Matrix multiplication is not commutative, even when AB & BA are both defined.

Note: $AB = AC$ does not imply $B = C$

Example: $A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} -3 & -5 \\ 2 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -21 & -21 \\ 7 & 7 \end{bmatrix}$$

$$AC = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -3 & -5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -21 & -21 \\ 7 & 7 \end{bmatrix}$$

Note: An $(m \times n)$ system of linear equations may be expressed as a matrix equation:

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{matrix} \longleftrightarrow \begin{bmatrix} a_{11} & a_{12} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & a_{m2} \dots & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

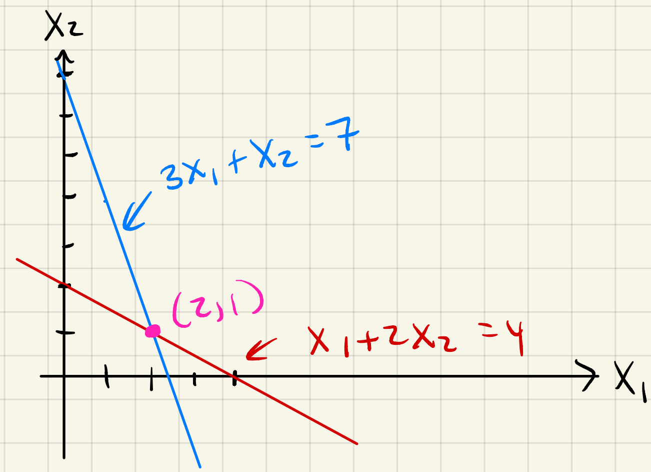
Therefore, we want to solve the system is $A\vec{x} = \vec{b}$

We can interpret solving $A\vec{x} = x_1\vec{A}_1 + x_2\vec{A}_2 + \dots + x_n\vec{A}_n = \vec{b}$ as determining the scalars $x_1, x_2, \dots, x_n$ such that $\vec{b}$ is a linear combination of the column vectors $\vec{A}_1, \vec{A}_2, \dots, \vec{A}_n$ of the coefficient matrix A.

Example: $3x_1 + x_2 = 7$
$x_1 + 2x_2 = 4$

Augmented matrix: $\left[\begin{array}{cc|c} 3 & 1 & 7 \\ 1 & 2 & 4 \end{array}\right] \xrightarrow{\text{ERO's}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$

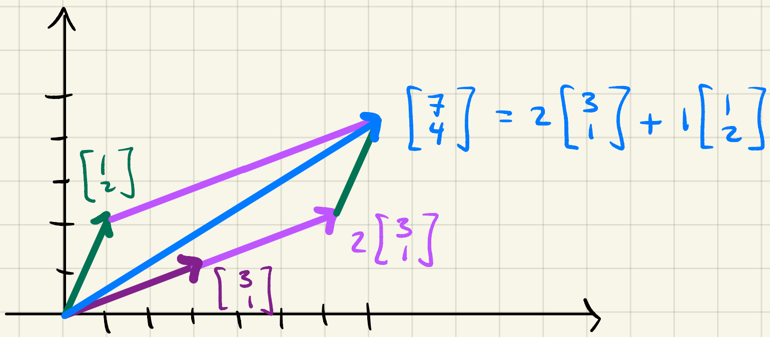
We interpret the solution of this system as the intersection of two lines in the x_1x_2-plane:



Alternatively, we write the system as a linear combination of the columns of A:

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \Rightarrow A\vec{x} = x_1\begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

So the problem amounts to writing the vector $\begin{bmatrix} 7 \\ 4 \end{bmatrix}$ as a linear combination of vectors $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$



Algebraic Properties of Matrix Operations:

Theorem 7 (p. 61): Let A, B, and C be $(m \times n)$ matrices. Then,

1.) $A+B = B+A$ addition is commutative

2.) $(A+B)+C = A+(B+C)$ addition is associative

3.) There exists a unique $(m \times n)$ matrix $\mathcal{O}$ (Zero Matrix) such that $A + \mathcal{O} = A$ for every $(m \times n)$ matrix A. Additive identity

4.) For every $(m \times n)$ matrix A, there exists a matrix $-A$ such that $A+(-A)=\mathcal{O}$.

Additive inverse

Theorem 8 (p. 62): Let A be an $(m \times n)$ matrix, B be an $(n \times p)$ matrix and C be a $(p \times q)$ matrix. Let r, s be scalars.

1.) $A(BC) = (AB)C$ matrix multiplication is associative.

2.) $r(sA) = (rs)A$

3.) $s(AB) = (sA)B = A(sB)$

Theorem 9 (p. 63): Distributivity

1.) If A, B are $(m \times n)$ matrices and C is an $(n \times p)$ matrix, then $(A+B)C = AC+BC$.

2.) If A is an $(m \times n)$ matrix and B, C are $(n \times p)$ matrices, then $A(B+C) = AB+AC$.

3.) If r, s are scalars and A is an $(m \times n)$ matrix, then $(r+s)A = rA+sA$.

4.) If r is a scalar and A, B are $(m \times n)$ matrices, then $r(A+B) = rA+rB$.

Def: A matrix is **symmetric** if $A^T = A$.

- A symmetric matrix must be **square**: $m = n$

Theorem 10 (p. 64): Let A, B be $(m \times n)$ matrices and C be an $(n \times p)$ matrix.

1.) $(A+B)^T = A^T + B^T$

2.) $(AC)^T = C^T A^T$

3.) $(A^T)^T = A$

Def: The $(n \times n)$ **identity matrix**, denoted I_n, is the matrix with ones on the main diagonal and zeros everywhere else.

$$I_1 = [1], \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ldots$$

- If A is an $(n \times n)$ matrix, then $AI_n = I_n A$, i.e., I_n is the **multiplicative identity** of A.

- If B is an $(p \times q)$ matrix, then $I_p B = B$ and $B I_q = B$.

Example: Let $A = \begin{bmatrix} -1 & -3 \\ 4 & 6 \end{bmatrix}$. Show that $AI_2 = I_2 A$