# Section 15.1 : Integration in Two variables

**Calc I:** $\int_a^b f(x)\,dx$ : Signed area between the curve $y=f(x)$ and the $x$-axis from $x=a$ to $x=b$.

For functions of two variables:

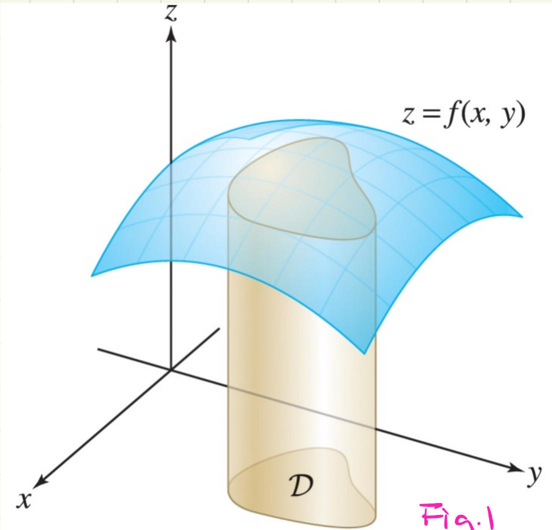



Fig. 1

$\iint_D f(x,y)\,dA$ : represents the signed volume of the solid region between the graph of $z=f(x,y)$ and the region $D$ in the $xy$-plane.

We will start by solving double integrals when $D$ is a rectangle ($R$):

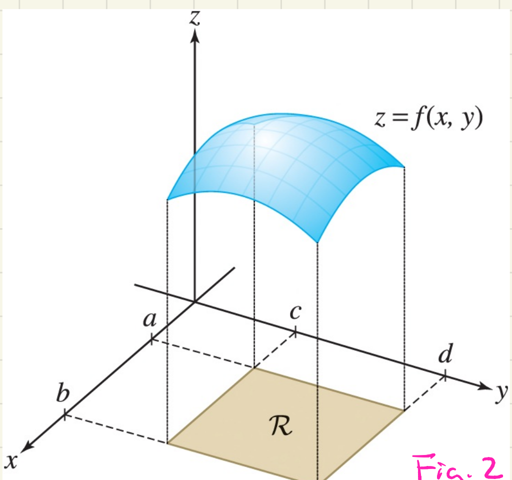



Fig. 2

$$R = \underbrace{[a, b]}_{x\text{-values}} \times \underbrace{[c, d]}_{y\text{-values}}$$

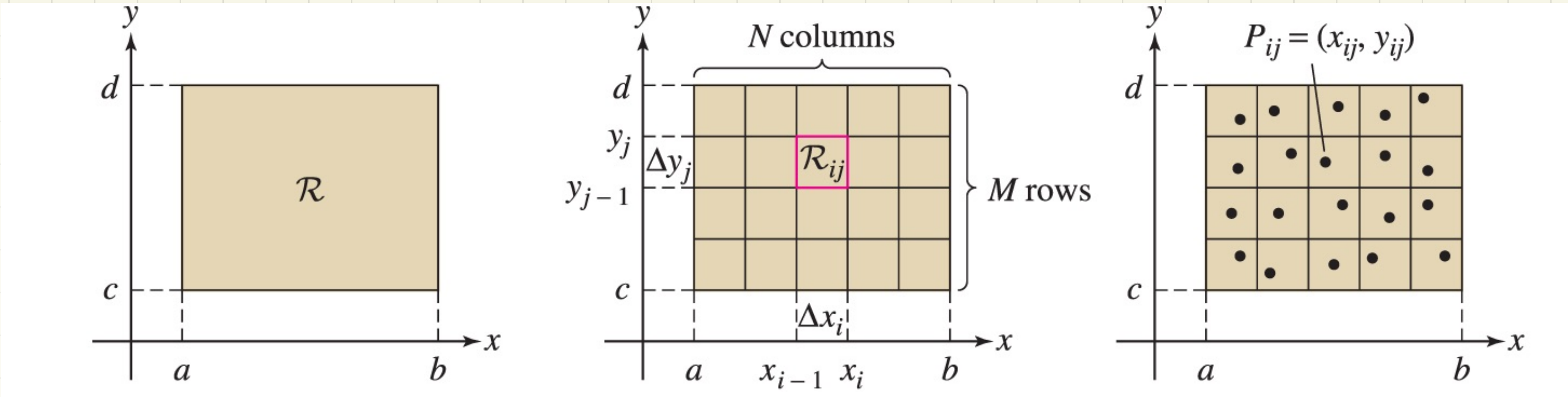
OR

$$R: \begin{array}{l} a \le x \le b \\ c \le y \le d \end{array}$$

The limits of integration for <u>both</u> variables are <u>both</u> constants

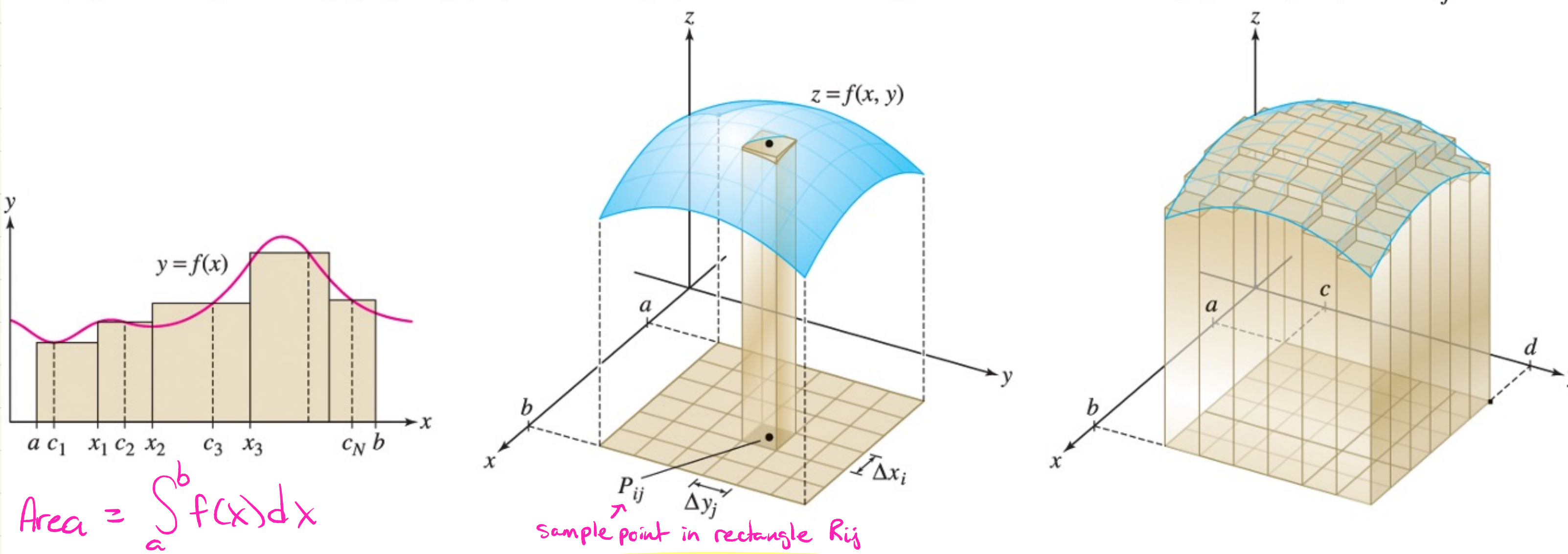
# Integration on a Rectangle

Similiarly to what we do for functions of a single variable in Calc I, we define double integrals in terms of Reimann sums.




(A) Rectangle $\mathcal{R} = [a, b] \times [c, d]$ (B) Create $N \times M$ grid. (C) Sample point $P_{ij}$

(Fig. 3)




Area $= \int_a^b f(x)\,dx$

sample point in rectangle $R_{ij}$

(A) In one variable, a Riemann sum approximates the area under the curve by a sum of areas of rectangles.

(B) The volume of the box is $f(P_{ij})\Delta A_{ij}$, where $\Delta A_{ij} = \Delta x_i \Delta y_j$.

(C) The Riemann sum $S_{N,M}$ is the sum of the volumes of the boxes.

(Fig. 4)

$$S_{N,M} = \sum_{i=1}^{N} \sum_{j=1}^{M} \underbrace{f(P_{ij})}_{\text{Height}} \underbrace{\Delta A_{ij}}_{\text{Area of the Base}}$$

**Reimann Sum**

height $\times$ Area $=$ Volume of box

For any given partition $\mathcal{P}$ and sampling, this sum gives an approximation of the integral.

Def: The double integral of $f(x,y)$ over a rectangle $R$ is

$$\iint_R f(x,y)\,dA = \lim_{\underbrace{\|P\|\to 0}_{M,N\to\infty \text{ for regular partition}}} \sum_{i=1}^{N}\sum_{j=1}^{M} f(P_{ij})\Delta A_{ij}$$

If this limit exists, we say $f$ is integrable over $R$.

Special case: For a Regular partition,

$$\Delta x = \frac{b-a}{N}, \quad \Delta y = \frac{d-c}{M}, \quad \Delta A = \Delta x \Delta y$$

Note: If $f(x,y) \geq 0$, then the volume $V$ of the solid region between the graph of $f$ and the rectangle $R$ is

$$V = \iint_R f(x,y)\,dA$$

Note: Some double integrals over a rectangle R can be found using geometry. See Example 2 in section 15.1 of the textbook.

**Theorem:** If a function $f$ of two variables is continuous on a rectangle $R$, then $f(x,y)$ is integrable over $R$.

**Theorem 2 (Linearity of the double integral):** (can be for $R$ or more general region $D$)

Assume that $f(x,y)$ and $g(x,y)$ are integrable. Then,

(i) $\iint_R \left(f(x,y) + g(x,y)\right) dA = \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA$

(ii) For any constant $c$, $\iint_R c\, f(x,y)\, dA = c \iint_R f(x,y)\, dA$

**Note:** If $f(x,y) = C$ is a constant function, then

$$\iint_R C\, dA = C \cdot \text{Area}(R)$$

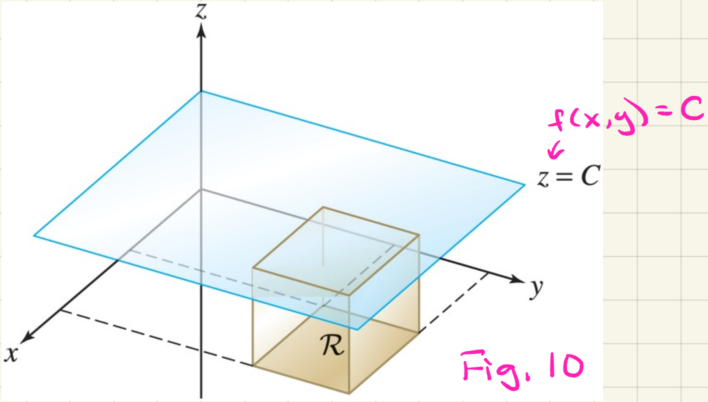



Fig. 10

**Iterated Integrals:** Let $f$ be a function of 2 variables that is integrable on $R=[a,b] \times [c,d]$. To compute $\int_{y=c}^{d} f(x,y)\,dy$, fix $x$ and integrate with respect to $y$ on $[c,d]$. This is called partial anti-differentiation with respect to $y$. This gives a function of $x$, $S(x)=\int_c^d f(x,y)\,dy$, which can be then be integrated with respect to $x$.

$$\int_a^b S(x)\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx$$

parenthesis are not needed and typically are not included. Work from the "inside" out.

Similiarly, $\int_c^d \left(\int_a^b f(x,y)\,dx\right) dy$

**Fubini's Theorem:** The double integral of a continuous function $f(x,y)$ over a rectangle $R=[a,b]\times[c,d]$ is equal to the iterated integral (in either order)

$$\iint_R f(x,y)\,dA = \int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dy\,dx \qquad \int_{y=c}^{d}\int_{x=a}^{b} f(x,y)\,dx\,dy$$

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**Notes:**

- Integration rules/techniques are the same as from Calc I and II where one variable is fixed (treated as a constant) in the first integral.
- Each time you integrate with respect to one variable, you evaluate using the Fundamental Theorem of Calculus (FTC), Part 1.

Calc I: $\int_a^b f(x)\,dx = F(b)-F(a)$ where $F(x)$ is the antiderivative of $f(x)$ $(F'(x)=f(x))$

## Examples: Evaluate

a) $\int_0^3 \int_1^2 x^2 y \, dy \, dx = \int_0^3 \underbrace{\frac{1}{2} x^2 y^2 \Big|_{y=1}^{y=2}}_{\text{Apply FTC}} dx = \int_0^3 \underbrace{\left(\cancel{\frac{1}{2}} x^2 (2)^{\cancel{2}} - \frac{1}{2} x^2 (1)^2\right)}_{\text{simplify}} dx$

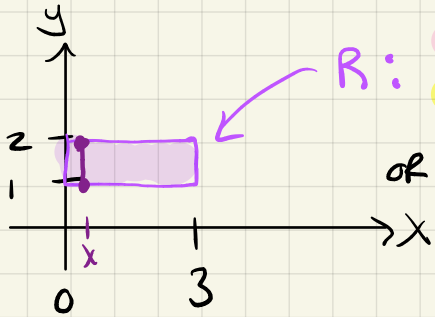
① Fix $x$ & integrate with respect to (wrt) $y$ first.

$$= \int_0^3 \frac{3}{2} x^2 \, dx = \underbrace{\cancel{\frac{3}{2}} \cdot \frac{1}{\cancel{3}} x^3 \Big|_{x=0}^{x=3}}_{\text{Apply FTC}}$$

$$= \left(\frac{1}{2} \cdot 3^3 - \frac{1}{2} \cdot 0^3\right)$$

$$= \frac{27}{2}$$

What is $R$?




$R$: $0 \le x \le 3$, $1 \le y \le 2$

or $R = [0,3] \times [1,2]$

Check that reversing the order of integration will give the same result:

$$\int_1^2 \int_0^3 x^2 y \, dx \, dy = \int_1^2 \underbrace{\frac{1}{3} x^3 y \Big|_{x=0}^{x=3}}_{\text{FTC}} dy = \int_1^2 9y \, dy = \underbrace{\frac{9}{2} y^2 \Big|_{y=1}^{y=2}}_{\text{Apply FTC}} = \frac{9}{2}(2)^2 - \frac{9}{2}(1)^2$$

$$= \frac{9}{2}(4-1)$$

$$= \frac{27}{2}$$

Fix $y$ and integrate wrt $x$ first.

$f(x,y) = x^2 y$ : check $\int_1^2 \int_0^3 x^2 y \, dx dy = \left(\int_0^3 x^2 \, dx\right)\left(\int_1^2 y \, dy\right)$

Special case: If $f(x,y) = g(x) h(y)$, then the double integral over $R = [a,b] \times [c,d]$ is

$$\underbrace{\int_{x=a}^{b} \int_{y=c}^{d} f(x,y) \, dy \, dx}_{\text{either order}} = \left(\int_{x=a}^{b} g(x) \, dx\right)\left(\int_{y=c}^{d} h(y) \, dy\right)$$

b.) $\iint_R y\sin(xy)\,dA$ , $R = [1,2] \times [0, \pi/2]$

$1 \le x \le 2$ $\quad$ $0 \le y \le \pi/2$

$$\int_0^{\pi/2}\int_1^2 y\sin(xy)\,dx\,dy$$

Recall for constant $k \neq 0$:

$$\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + C$$

$$= \int_0^{\pi/2} \underbrace{\cancel{y}\cdot \frac{-1}{\cancel{y}}\cos(xy)\Big|_{x=1}^{x=2}}_{\text{Apply FTC}} \,dy$$

$$= \int_0^{\pi/2} \left(-\cos(2y) + \cos(y)\right) dy$$

$$= \left(-\frac{1}{2}\sin(2y) + \sin(y)\right)\Big|_{y=0}^{y=\pi/2}$$

$$= \left(-\frac{1}{2}\underbrace{\sin(\pi)}_{0} + \underbrace{\sin(\pi/2)}_{1}\right) - \left(-\frac{1}{2}\underbrace{\sin(0)}_{0} + \underbrace{\sin(0)}_{0}\right)$$

$$= 1$$

c.) $\displaystyle\int_0^8\int_1^2 \frac{x}{\sqrt{x^2+y}}\,dx\,dy$ Fix $y$ and integrate wrt $x$ first

Find the antiderivative (indefinite integral) "side work"

$u = x^2+y$

$du = 2x\,dx$ : $\displaystyle\int \frac{x}{\sqrt{x^2+y}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\int u^{-1/2}\,du = \cancel{\tfrac{1}{2}}\cdot\cancel{2}\,u^{1/2}+C$

$\frac{1}{2}du = x\,dx$

$= (x^2+y)^{1/2}+C$

$$\int_0^8\int_1^2 \frac{x}{\sqrt{x^2+y}}\,dx\,dy = \int_0^8 \underbrace{(x^2+y)^{1/2}\Big|_{x=1}^{x=2}}_{\text{FTC}}\,dy$$

$$= \int_0^8 \left[(4+y)^{1/2} - (1+y)^{1/2}\right] dy$$

$$= \left[\frac{2}{3}(4+y)^{3/2} - \frac{2}{3}(1+y)^{3/2}\right]\Big|_{y=0}^{y=8}$$

$$= \frac{2}{3}\left[\left(\underbrace{(12)^{3/2}}_{(\sqrt{4\cdot 3})^3 = (2\sqrt{3})^3} - \underbrace{9^{3/2}}_{27}\right) - \left(\underbrace{4^{3/2}}_{8} - \underbrace{1^{3/2}}_{1}\right)\right]$$

$$= 16\sqrt{3} - \frac{68}{3}$$

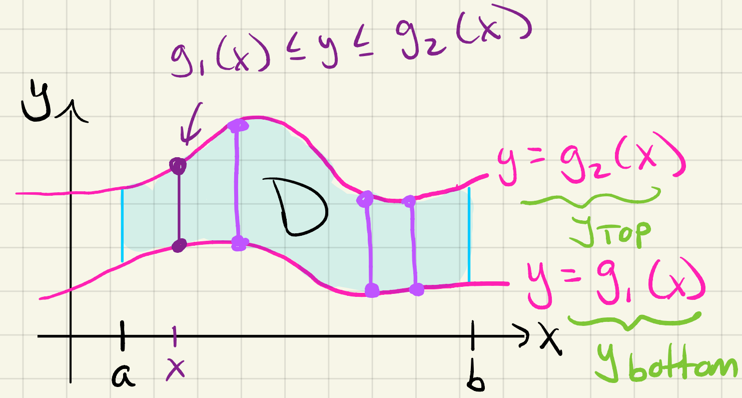
On you own: reverse the order of integration : $\displaystyle\int_1^2\int_0^8 \frac{x}{\sqrt{x^2+y}}\,dy\,dx$
to verify you get same answer

# Section 15.2: Double Integrals Over More General Regions

Now we consider $\iint_D f(x,y)\,dA$ where $D$ is not a rectangle.

I.) $D$ is a vertically simple region:

$D: a \le x \le b,\ g_1(x) \le y \le g_2(x)$



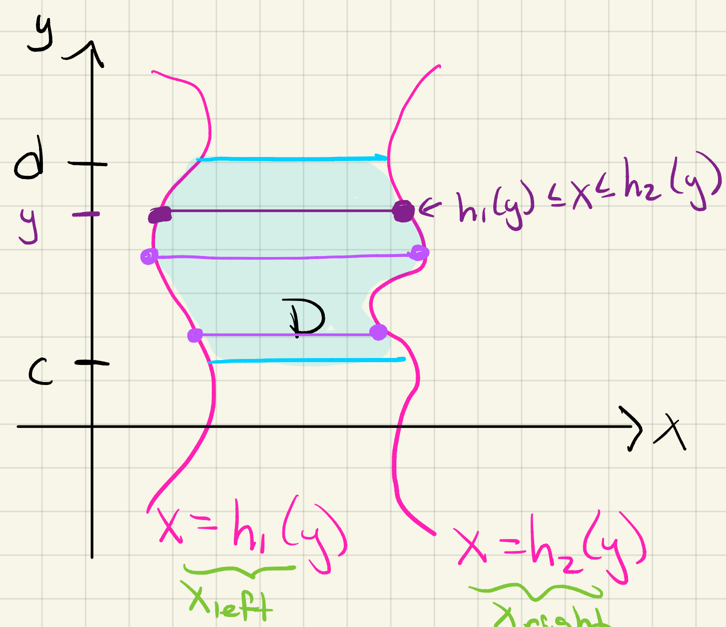

Then,

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Fix $x$ & integrate with respect to $y$ first

II.) $D$ is a horizontally simple region:

$D: c \le y \le d,\ h_1(y) \le x \le h_2(y)$




Then,

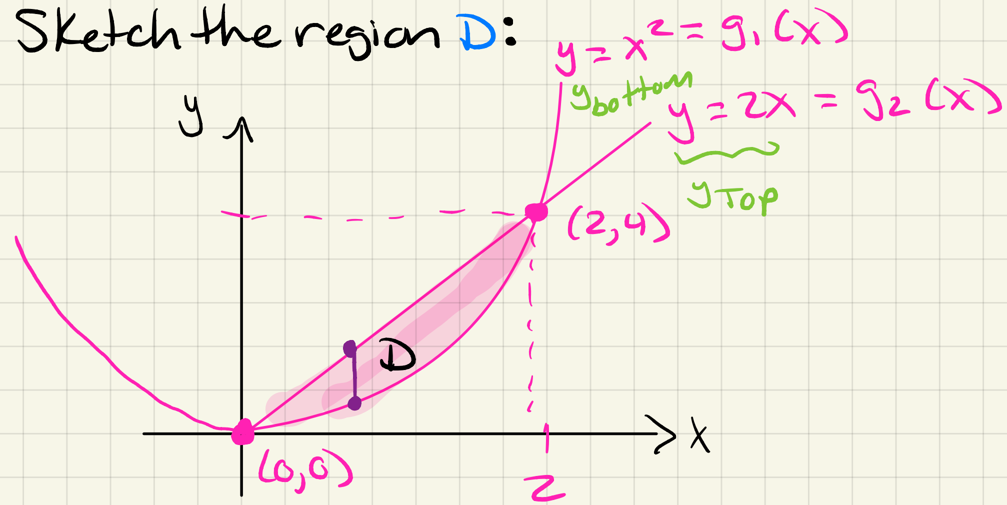
$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Fix $y$ & integrate with respect to $x$ first.

## Examples: Evaluate

1.) $\iint_D (x^2+y)\,dA$, where $D$ is the region bounded by $y=x^2$ & $y=2x$.

Sketch the region $D$:




Vertically simple and it is also horizontally simple.

As vertically simple region:

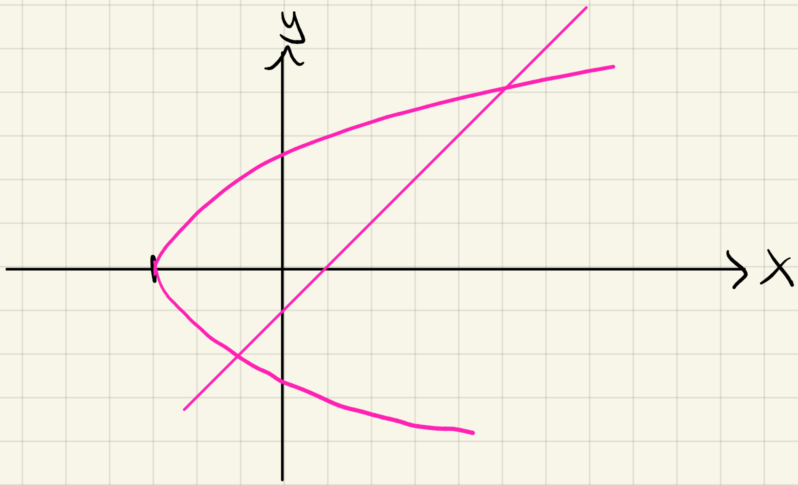
$$D: \quad 0 \le x \le 2$$
$$x^2 \le y \le 2x$$

$$\iint_D (x^2+y)\,dA = \int_0^2 \int_{x^2}^{2x} (x^2+y)\,dy\,dx$$

Find the points of intersection

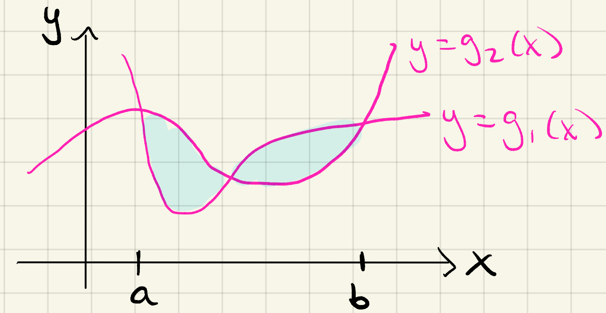
$$x^2 = 2x$$
$$\Rightarrow \quad x^2 - 2x = 0$$
$$\Rightarrow \quad x(x-2) = 0 \qquad \therefore\ x=0,\ x=2$$

2.) $\iint_D xy\,dA$, where $D$ is the region bounded by $y=x-1$ & $y^2=2x+6$ sketched below.



What if $D$ is not a simple region?

- Split into a sum of simple regions!




* not a simple region.

- Double integrals over more general regions $D$ satisfy the same linearity properties (Thm 2, sec. 15.1) as rectangular regions $R$.

Theorem 3: Let $f(x,y)$ and $g(x,y)$ be integrable functions on $D$.

(a) If $f(x,y) \le g(x,y)$ for all $(x,y) \in D$, then

$$\iint_D f(x,y)\,dA \le \iint_D g(x,y)\,dA$$

(b) If $m \le f(x,y) \le M$ for all $(x,y) \in D$, then  $m$ & $M$ constants.

$$m \cdot \text{Area}(D) \le \iint_D f(x,y)\,dA \le M \cdot \text{Area}(D)$$

Decomposing the Domain into Smaller Domains:

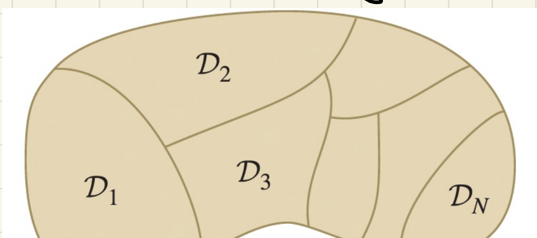
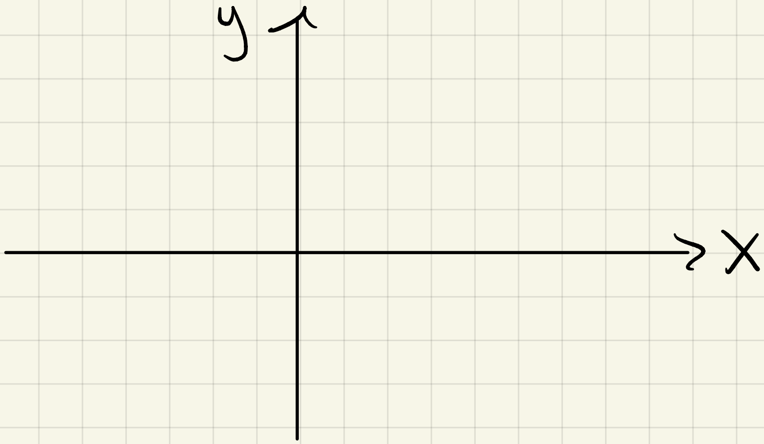


Fig. 19

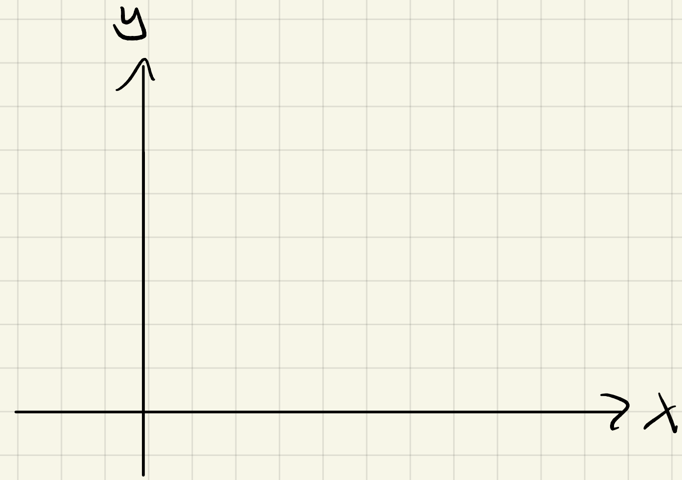
Double integrals are additive with respect to the domain: If $D$ is the union of domains $D_1, D_2, \ldots, D_N$ that do not overlap except possibly on boundary curves (fig. 19), then

$$\boxed{\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \cdots + \iint_{D_N} f(x,y)\,dA}$$

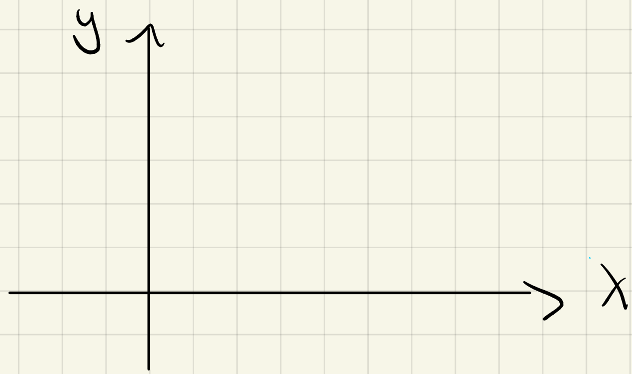
Example: Evaluate $\iint_D xy\,dA$ where $D$: region between $y = x^3$ and $y = x$



Example: Evaluate $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$



Example: $\int_1^e \int_0^{\ln x} f(x,y)\,dy\,dx$. Change the order of integration.




$$\int_1^e \int_0^{\ln x} f(x,y)\,dy\,dx =$$