Announcements:

- Quiz 5 is the week we return from Spring Break. The Quiz Schedule in the Homework & Quizzes tab on LMS will be updated by the end of the day with the sections to be covered on Quiz 5.

Echelon Form and Gauss-Jordan Elimination (section 1.2) (cont.)

Recall:

The idea of elimination is to eliminate variables in a systematic way using operations called elementary operations, that yield an equivalent system of equations (same solution set).

We can use any of the following elementary (Row) operations: (ERO's)

1.) Interchange any two equations ($E_i \leftrightarrow E_j$)

Interchange any two Rows ($R_i \leftrightarrow R_j$)

2.) Multiply both sides of an equation by a nonzero number (kE_i, $k \neq 0$)

Multiply a row by a nonzero number (kR_i, $k \neq 0$)

3.) Add a constant multiple of one equation to another ($E_i + kE_j$)

Add a multiple of one row to another row ($R_i + kR_j$)

Note: (3) is an elimination step. We only want to use it to eliminate one variable from one equation/row.

The goal is to reduce the augmented matrix until the coefficient matrix is in $\underbrace{\text{Echelon}}_{\text{easier}}$ or $\underbrace{\text{Reduced Row Echelon}}_{\text{easiest system to solve}}$ form.

- A leading entry of a row refers to the leftmost nonzero entry (in a nonzero row).

Def: A $(m \times n)$ matrix is in (Row) Echelon Form if it satisfies all three of the following conditions:

1.) All nonzero rows are above any rows of all zeros.

2.) The leading entry of each nonzero row is a 1 (leading 1) and the leading 1 is in a column to the right of the leading 1 in the row above it.

3.) All entries in a column below a leading 1 are zero.

If a matrix satisfies the following additional condition, then it is in Reduced (Row) Echelon Form:

4.) Each leading 1 is the only nonzero entry in its column.

A leading 1 position (or pivot position) in a matrix A is a location that corresponds to a leading 1 in the reduced row echelon form of A. A leading 1 column (or pivot column) is a column of A that contains a leading 1 position.

The Row Reduction Algorithm:

Example: Solve the linear system by transforming the augmented matrix first to echelon form, then to reduced row echelon form.

$$\begin{aligned} x_1 - 2x_2 - x_3 &= 3 \\ 3x_1 - 6x_2 - 5x_3 &= 3 \\ 2x_1 - x_2 + x_3 &= 0 \end{aligned}$$

unknowns

(3x3) system

equations

Step 0: Write the augmented matrix

already a leading 1

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 3 & -6 & -5 & 3 \\ 2 & -1 & 1 & 0 \end{array}\right]$$

rows of augmented matrix represent eqns

columns of the coefficient matrix correspond to variables.

Step 1: Begin with the leftmost nonzero column. This is a leading 1 column. The leading 1 position is at the top. If necessary, interchange rows, so there is a nonzero entry a in the leading 1 position. If the leading entry a is not 1, multiply by $\frac{1}{a}$.

(You can also wait until the end to make all the entries in leading 1 positions equal to 1. Can avoid working with fractions.)

<u>Step 2</u>: Use elimination to create zeros in all positions below the leading 1 position by adding appropriate multiples of Row 1 to each of the other rows.

already a leading 1

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ \boxed{3} & -6 & -5 & 3 \\ \boxed{2} & -1 & 1 & 0 \end{array}\right] \qquad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & 0 & -2 & -6 \\ 0 & 3 & 3 & -6 \end{array}\right]$$

<u>Step 3</u>: Repeat steps 1 and 2 (ignoring Row 1).

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & 0 & -2 & -6 \\ 0 & 3 & 3 & -6 \end{array}\right] \qquad R_2 \leftrightarrow R_3 \qquad \left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & \boxed{3} & 3 & -6 \\ 0 & 0 & -2 & -6 \end{array}\right]$$

$$R_2 \to \tfrac{1}{3} R_2 \qquad \left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & \boxed{-2} & -6 \end{array}\right]$$

<u>Step 4</u>: Repeat until matrix is in echelon form.

$$R_3 \to -\tfrac{1}{2} R_3 \qquad \left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Echelon Form

(*) It is ok to do more than one step at a time, if the steps do not involve one another.

Step 5: Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.

suppose $\left[\begin{array}{ccc|c} & \vdots & & \vdots \\ 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow 0x_1 + 0x_2 + 0x_3 = 1$

↑ or any nonzero #

$\Rightarrow 0 = 1$ (Not possible!)

Inconsistent

∴ No solution

Step 6: Beginning with the rightmost leading 1 position and working upward to the left, create zeros above each leading 1 until matrix is in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & -2 & \boxed{-1} & 3 \\ 0 & 1 & \boxed{1} & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 + R_3 \\ R_2 \to R_2 - R_3 \end{array} \quad \left[\begin{array}{ccc|c} 1 & \boxed{-2} & 0 & 6 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$R_1 \to R_1 + 2R_2 \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

The row reduction algorithm is complete.

To solve the system, write the system of equations corresponding to the system at the end of step 6.

The solution is: $x_1 = -4$

$x_2 = -5$

$x_3 = 3$

in vector form $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \\ 3 \end{bmatrix}$

Example: Consider the following augmented matrix.

rref

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

leading 1 columns

(4×5) consistent system

This reduced matrix corresponds to the reduced system

$$\begin{array}{l} x_1 + 2x_2 + 3x_5 = 2 \\ x_3 - x_5 = 4 \\ x_4 - 2x_5 = 3 \end{array} \Rightarrow \begin{array}{l} x_1 = 2 - 2x_2 - 3x_5 \\ x_2 = x_2 \\ x_3 = 4 + x_5 \\ x_4 = 3 + 2x_5 \\ x_5 = x_5 \end{array}$$

$x_2, x_5 \in \mathbb{R}$ (any real #)

The variables x_1, x_3, and x_4 are called leading variables (or basic variables, or dependent variables, or constrained variables.) These are the variables in the leading 1 positions of the reduced matrix.

The variables x_2 and x_5 are free variables (or independent or unconstrained variables)

Whenever a system is consistent, the solution set can be described explicitly by solving the reduced system of equations for the leading variables in terms of the free variables.

Using Row Reduction to Solve a Linear System (Gauss-Jordan Elimination)

1.) Write the augmented matrix of the system.

2.) Use the row reduction algorithm to obtain a row equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to next step.

3.) Continue the row reduction algorithm to obtain the reduced row echelon form.

4.) Write the system of equations corresponding to the system in Step 3.

5.) Rewrite each nonzero equation from step 4 so that each leading variable is expressed in terms of any free variables appearing in the reduced equations.

Example: Solve the system.

$$x_1 - x_2 - 2x_3 = 1$$
$$2x_1 + x_2 + x_3 = 5$$

(2x3) system

$$\begin{bmatrix} 1 & -1 & -2 & | & 1 \\ 2 & 1 & 1 & | & 5 \end{bmatrix} \quad R_2 \to R_2 - 2R_1 \quad \begin{bmatrix} 1 & -1 & -2 & | & 1 \\ 0 & 3 & 5 & | & 3 \end{bmatrix}$$

$$R_2 \to \tfrac{1}{3} R_2 \quad \begin{bmatrix} 1 & -1 & -2 & | & 1 \\ 0 & 1 & 5/3 & | & 1 \end{bmatrix}$$

Echelon form: consistent, know infinitely many solutions.

$$R_1 \to R_1 + R_2 \quad \begin{bmatrix} 1 & 0 & -1/3 & | & 2 \\ 0 & 1 & 5/3 & | & 1 \end{bmatrix}$$

free column: x_3 is free variable.

2 leading 1 columns

rref

$$x_1 \quad - \tfrac{1}{3} x_3 = 2$$
$$x_2 + \tfrac{5}{3} x_3 = 1$$

$$\Rightarrow \quad \begin{aligned} x_1 &= 2 + \tfrac{1}{3} x_3 \\ x_2 &= 1 - \tfrac{5}{3} x_3 \\ x_3 &= \quad\quad x_3 \end{aligned}$$

Infinitely many solutions. (x_3 free variable) ($x_3 \in \mathbb{R}$)

in vector form
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 + \tfrac{1}{3} x_3 \\ 1 - \tfrac{5}{3} x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1/3 \\ -5/3 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R}$$

Example: Solve the system.

$$x_1 - x_2 - 2x_3 = 2$$
$$2x_1 + x_2 + x_3 = 5$$
$$x_1 + 2x_2 + 3x_3 = 4$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 2 & 1 & 1 & 5 \\ 1 & 2 & 3 & 4 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & 3 & 5 & 1 \\ 0 & 3 & 5 & 2 \end{array}\right]$$

$$R_3 \to R_3 - R_2 \quad \left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & 3 & 5 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

inconsistent $0 \neq 1$!

No solution!

Example: Each of the following matrices are the augmented matrix for a system of linear equations in reduced row echelon form. Give the system of equations and describe the solution.

a.) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \underbrace{0}_{0=0\checkmark} & 0 \end{array}\right]$ consistent

1 unique solution

$x_1 = 3$
$x_2 = -2$
$x_3 = 0$

b.) $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & \underbrace{0}_{0\neq 1} & 1 \end{array}\right]$

Inconsistent ; no solutions

c.) Free columns: x_2 & x_4 free variables

$$\begin{array}{c} \begin{array}{cccc|c} & \downarrow & & \downarrow & \end{array} \\ \left[\begin{array}{cccc|c} 1 & -3 & 0 & 4 & 2 \\ 0 & 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$\underbrace{\qquad\qquad}_{0=0}$

$\Rightarrow$ $x_1 - 3x_2 + 4x_4 = 2$
$x_3 - 5x_4 = 1$

$\Rightarrow$ $x_1 = 2 + 3x_2 - 4x_4$
$x_2 = x_2$
$x_3 = 1 + 5x_4$
$x_4 = x_4$

$$\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

Section 1.3 : Consistent Systems of Linear Equations

How to Determine How Many Solutions

Given a $(m \times n)$ system of equations,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \Rightarrow \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] = [A \mid \vec{b}]$$

Recall: Given any system of linear equations, there are at most 3 possibilities:

1.) The system has infinitely many solutions.

2.) The system has no solution.

3.) The system has a unique solution.

A system of linear equations is consistent if it has either one solution or infinitely many solutions. (1 and 3)

A system is inconsistent if it has no solution. (2)

Goal: How do we determine which one?

Example: Find all values of a for which the given system is consistent and find the solution set for these a values.

$$\begin{aligned} x_1 - x_2 + 7x_3 &= 1 \\ x_1 - 2x_2 + 9x_3 &= 0 \\ 2x_1 - 3x_2 + 16x_3 &= a \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 7 & 1 \\ 1 & -2 & 9 & 0 \\ 2 & -3 & 16 & a \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & -1 & 7 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & 2 & a-2 \end{array}\right]$$

need same RHS $> -1 = a-2$

$\Rightarrow a = 1$

$$R_2 \to -R_2 \quad \left[\begin{array}{ccc|c} 1 & -1 & 7 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 2 & a-2 \end{array}\right]$$

$$\begin{array}{l} R_1 \to R_1 + R_2 \\ R_3 \to R_3 + R_2 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & a-1 \end{array}\right]$$

From 3rd row/eqn system only consistent if $0 = a-1 \Rightarrow a = 1$

free column! x_3 is free variable!

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{aligned} x_1 + 5x_3 &= 2 \\ x_2 - 2x_3 &= 1 \end{aligned} \Rightarrow \begin{aligned} x_1 &= 2 - 5x_3 \\ x_2 &= 1 + 2x_3 \\ x_3 &= x_3 \end{aligned}$$

infinitely many solutions!

Def: A $(m \times n)$ system of linear equations is **homogeneous** if $b_1 = b_2 = \dots = b_m = 0$, i.e., the system is:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned} \Rightarrow \underbrace{\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right]}_{} = [A \mid \underbrace{\vec{0}}_{\vec{0} \in \mathbb{R}^m}]$$

OR: $A\vec{x} = \vec{0}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

- Every homogeneous system is consistent since $x_1 = x_2 = \dots = x_n = 0$ is always a solution. This is called the **trivial solution**. Any other solution is called a **nontrivial solution**.

So, every homogeneous system has either 1 unique solution **OR** infinitely many solutions.

- If $m < n$, then the system has nontrivial solutions, i.e., it has infinitely many solutions.

Example: $2x_1 + 3x_2 - x_3 = 0$

$x_1 - 5x_2 - 2x_3 = 0$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 1 & -5 & -2 & 0 \end{array}\right]$$

Example:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ x_1 \qquad\quad + x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 3 & 0 \end{array}\right]$$

Theorem: Let C be an $(m \times n)$ matrix in reduced row echelon form and let r be the number of nonzero rows of C. Then, for the system represented by $[C \mid \vec{d}\,]$ there are three possibilities:

1.) The system is inconsistent.

2.) The system is consistent and $r < n$. Thus there are infinitely many solutions.

3.) The system is consistent and $r = n$. Thus, there is a unique solution.

Corollary: Consider an $(m \times n)$ system of linear equations. If $m < n$, then either the system is inconsistent or it has infinitely many solutions.

In other words, consider

$$\underset{(m \times n)}{[A \mid \vec{b}]} \xrightarrow{\text{EROs}} [C \mid \vec{d}] = \left[\begin{array}{c|c} \text{non zero} & d_1 \\ \text{rows} & \vdots \\ & d_r \\ \hdashline \text{zero} & d_{r+1} \\ \text{rows} & \vdots \\ & d_m \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c} d_1 \\ \vdots \\ d_r \end{array}}\right\} r \\ \left.\vphantom{\begin{array}{c} d_{r+1} \\ \vdots \\ d_m \end{array}}\right\} m-r \end{array}$$

(the left block, under a brace, is labelled: rref)

Cases:

I.) No zero rows ($m-r=0 \iff r=m$)

a) $m=n \Rightarrow$ exactly 1 solution

b) $m \neq n \Rightarrow$ infinitely many solutions

II.) one or more zero rows ($r \neq m$)

a) If even one of $d_{r+1}, d_{r+2}, \ldots, d_m$ is non-zero $\Rightarrow$ no solution

b) If $d_{r+1} = d_{r+2} = \ldots = d_m = 0$:

- $r=n \Rightarrow$ exactly one solution
- $r \neq n$ $(r<n) \Rightarrow$ infinitely many solutions : $n-r$ free variables. (can be assigned arbitrary values)

Examples:

1.) $[A \mid \vec{b}] \to \text{EROs} \to [\underbrace{C}_{\text{rref}} \mid \vec{d}] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\end{matrix}}\right\} r=2 \text{ nonzero rows} \\ \left.\vphantom{\begin{matrix}1\\1\end{matrix}}\right\} 2 \text{ zero rows} \end{matrix}$

$a \neq 0$:

$a = 0$:

2.) $[A \mid \vec{b}] \to \text{EROs} \to [C \mid \vec{d}] = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\end{matrix}}\right\} r=2 \text{ nonzero rows} \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} 3 \text{ zero rows} \end{matrix}$

Example: Determine conditions on b_1, b_2, b_3 for the system to be consistent.

$$x_1 - 2x_2 + 3x_3 = b_1$$
$$2x_1 - 3x_2 + 2x_3 = b_2$$
$$-x_1 + 5x_3 = b_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 2 & -3 & 2 & b_2 \\ -1 & 0 & 5 & b_3 \end{array}\right] \xrightarrow{\text{EROs}} \left[\begin{array}{ccc|c} 1 & 0 & -5 & -3b_1 + 2b_2 \\ 0 & 1 & -4 & -2b_1 + b_2 \\ 0 & 0 & 0 & -3b_1 + 2b_2 + b_3 \end{array}\right] \text{(verify this!)}$$

$\underbrace{}_{\text{rref}}$

3rd row: $0x_1 + 0x_2 + 0x_3 = -3b_1 + 2b_2 + b_3$

<u>Conclusion</u>:

- if $-3b_1 + 2b_2 + b_3 \neq 0$, then
- if $-3b_1 + 2b_2 + b_3 = 0$, then