Announcements:

- Exam 2 is on Wednesday, 2/26 during Test Block.
  - Exam 2 information and additional exam review problems are posted on LMS.

- The UTA's are offering a review session on Tuesday 2/25 from 4-6 PM in Amos Eaton 214. This is optional.
  - There will be no UTA office hours on Wednesday 2/26.

Sections 1.1: Introduction to Matrices & Systems of Equations

(Matrix Algebra)

Motivation: Linear (Matrix) Algebra is one of the most widely used tools in all applied math, with applications in differential equations, optimization, linear regression, Fourier Analysis, computer graphics, etc.

We will start with the underlying theory of solving systems of linear equations.

A linear equation in n unknowns has the general form:

$$a_1 x_1' + a_2 x_2' + \dots + a_n x_n' = b$$

where $a_1, a_2, \dots a_n$, b are $\underbrace{\text{constants}}_{\#}$ and $x_1, x_2, \dots x_n$ are $\underbrace{\text{unknowns}}_{\text{variables}}$.

Examples of linear equations:

$x = 5$

$y = -3$

$-5x_1 + 2x_2 - 3x_3 + 9x_5 = 13$

$z = \underset{\#}{a}(x - \underset{\#}{a_1}) + \underset{\#}{b}(y - \underset{\#}{b_1}) + \underbrace{f(a_1, b_1)}_{\#}$

Non-Examples:

$3\sqrt{x_1} + 2x_2 = 7$

$-2x_1^{-2} + 3x_2^2 = 9$

$4x_1x_2 - 3x_3 = 5$

When we have multiple linear equations we want to solve simultaneously, we have a system of linear equations, (or a linear system).

A general $(m \times n)$ system of linear equations has m equations and n unknowns:

$$\left.\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 && (E_1)\\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 && (E_2)\\
\vdots \qquad\qquad\qquad\quad \vdots \quad &\quad \vdots && \ \vdots\\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m && (E_m)
\end{aligned}\right\} \text{m equations}$$

Where a_{ij} is the coefficient in equation i multiplying variable j.

A Solution to a system of equations can be written as a sequence $s_1, s_2, \dots, s_n$ of numbers that simultaneously makes each equation in the system a true statement when $s_1, s_2, \dots, s_n$ are substituted for $x_1, x_2, \dots, x_n$, respectively.

The set of all possible solutions is called the solution set of the linear system.

Two linear systems are called equivalent if they have the same solution set.

Solving Systems of Linear Equations:

Example: $\begin{cases} 2x - 4y = 14 & (E_1) \\ -x + 3y = -9 & (E_2) \end{cases}$ is a linear system with
2 linear equations
2 unknowns
so it is a (2×2) linear system.

Let's start by considering two methods we know to solve a (2×2) linear system.

Method 1: Substitution

Solve for one variable in equation, then substitute into the other equation.

From E_2: $x = 3y + 9$

Substitute into E_1: $2(3y+9) - 4y = 14$

$$6y + 18 - 4y = 14$$
$$2y = -4$$
$$y = -2$$

Plug $y = -2$ into $x = 3y + 9$: $x = 3(-2) + 9 = 3$

$\therefore$ Solution: $(x, y) = (3, -2)$

Method 2: Elimination

$$\begin{cases} 2x - 4y = 14 & (E_1) \\ -x + 3y = -9 & (E_2) \end{cases}$$

Multiply E_2 by 2, then add to E_1:

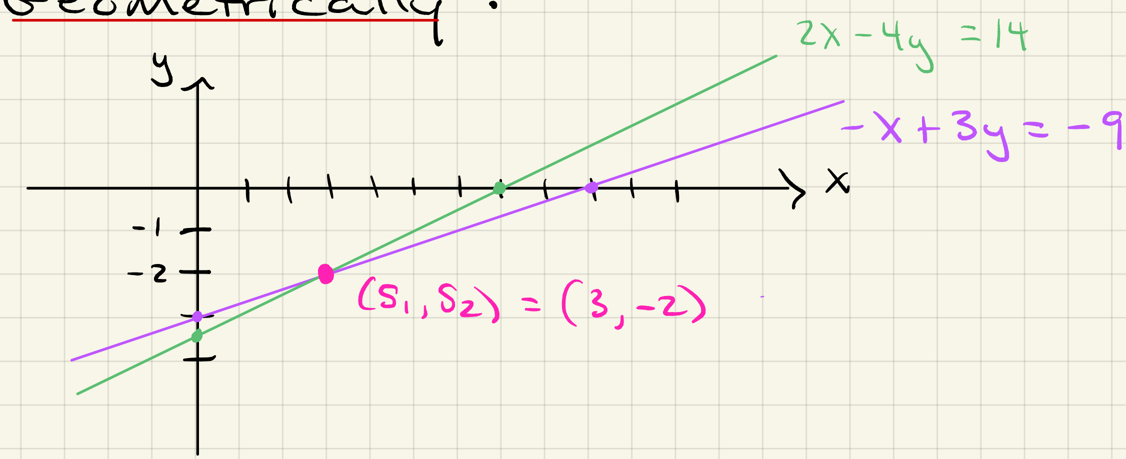
$$\begin{array}{rrcl} & 2\cancel{x} - 4y & = & 14 \\ + & -2\cancel{x} + 6y & = & -18 \\ \hline & 2y & = & -4 \\ & y & = & -2 \end{array}$$

Plug into either equation:

E_2: $-x + 3(-2) = -9 \Rightarrow x = 9 - 6 = 3$

$\therefore$ Solution: $(x, y) = (3, -2)$

Geometrically:



This explains why there is only one solution. The lines are not parallel or coincident.

Given any system of linear equations, there are at most 3 possibilities:

1.) The system has infinitely many solutions.

2.) The system has no solution.

3.) The system has a unique solution. (exactly one solution!)

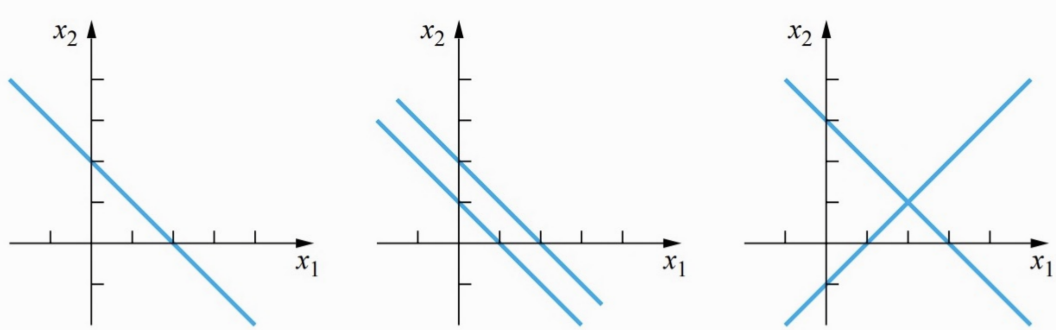
A system of linear equations is consistent if it has either one solution or infinitely many solutions.

A system is inconsistent if it has no solution.

- Linear equations of two variables are represented as lines in the plane. $\mathbb{R}^2$

3 possibilities for solutions:

1) The two lines are coincident (same line) => infinitely many solutions
2.) The two lines are parallel => no solution
3) The lines intersect at a single point => unique solution



Coincident lines
Infinitely many solutions

Parallel lines
No solution

Intersecting lines
Unique solution

• The graph of a linear equation in three variables is a plane in space ($\mathbb{R}^3$)

(2x3) System: $\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \end{array}\Bigg\}$ represents 2 planes

eqns ↑   # unknowns ↑

2 possibilities for solutions:

1.) The two planes are coincident or intersect in a line ⇒ infinitely many solutions

2.) The two planes are parallel ⇒ no solution

(3x3) System: $\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{array}\Bigg\}$ represents 3 planes

3 possibilities for solutions:

1.) Infinitely many solutions (planes coincident or intersect along line)

2.) no solution (parallel planes or intersect pairwise along 3 lines but have no point in common for all 3 planes).

3.) A unique solution

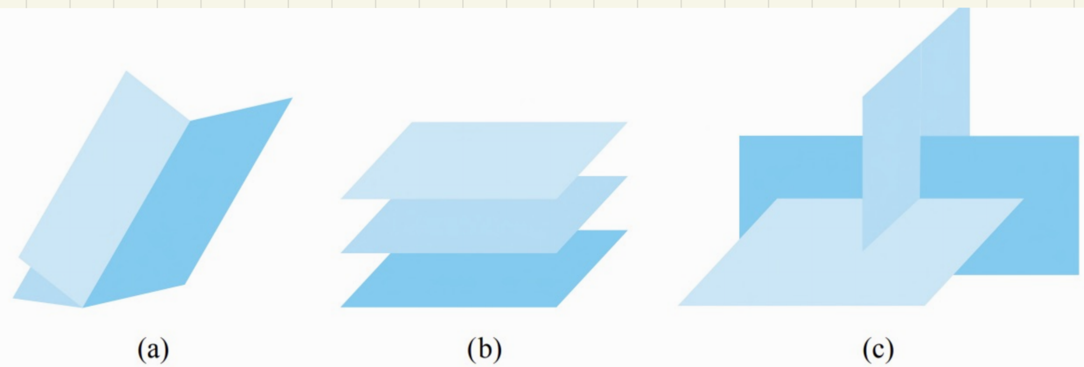


Figure 1.2 The general (3 × 3) system may have (a) infinitely many solutions, (b) no solution, or (c) a unique solution.

Which method should we use: Elimination or substitution?

Athough substitution is fairly straight forward, it becomes "messy" with more variables. We will have a systematic way to use elimination to solve $(m \times n)$ systems.

First, let's use matrices to represent linear systems.

- A matrix is a rectangular array of numbers.
- An $(m \times n)$ matrix A has m rows and n columns. If $m=n$, we say A is a square matrix.

\# of rows always listed 1st!

We often use the notation $A = (a_{ij})_{m \times n}$. The number a_{ij} is the ij-entry of the $(m \times n)$ matrix A, that is, the number in row i and column j.

Example:

$$A = \begin{bmatrix} 2 & 1 & -1 & 4 \\ -3 & -2 & 4 & 0 \\ 8 & -3 & -5 & 10 \end{bmatrix}$$

3 rows
4 columns

What is the size of A? (3×4)

Identify the following entries (if they exist):

a_{23}: 4 ; a_{32}: -3 ; a_{14}: 4 ; a_{41}: Not possible! A has only 3 rows!

- A matrix with one column is called a vector or column vector, and a matrix with one row is called a row vector.

A general $(m \times n)$ system of linear equations has m equations and n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

The augmented matrix of the system is

not always included, represents "=" in the original system

$$[\underbrace{A}_{\text{matrix}} \mid \underbrace{\vec{b}}_{\text{column vector}}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right], \quad \text{where}$$

rows of $[A \mid \vec{b}]$ represent equations

columns of A correspond to variables.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

A is the coefficient matrix.

We want to find the values for $x_1, x_2, \dots, x_n$ that solve the system (if the system is consistent).

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$ is the solution vector.

The idea of elimination is to eliminate variables in a systematic way using operations called elementary operations, that yield an equivalent system of equations (same solution set).

We can use any of the following elementary (Row) operations: (ERO's)

1.) Interchange any two equations ($E_i \leftrightarrow E_j$)

Interchange any two Rows ($R_i \leftrightarrow R_j$)

2.) Multiply both sides of an equation by a nonzero number (kE_i, $k \neq 0$)

Multiply a row by a nonzero number (kR_i, $k \neq 0$)

3.) Add a constant multiple of one equation to another ($E_i + kE_j$)

Add a multiple of one row to another row ($R_i + kR_j$)

Note: (3) is an elimination step. We only want to use it to eliminate one variable from one equation/row.

The goal is to reduce the augmented matrix until the coefficient matrix is in $\underbrace{\text{Echelon}}_{\text{easier}}$ or $\underbrace{\text{Reduced Row Echelon}}_{\text{easiest system to solve}}$ form.

- If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set. Row equivalent matrices come from applying Elementary Row operations (ERO's) on one matrix to get to the other.

Echelon Form and Gauss-Jordan Elimination (Section 1.2)

- A nonzero row or column in a matrix means a row or column contains at least one nonzero entry.
- A leading entry of a row refers to the leftmost nonzero entry (in a nonzero row).

Def: A $(m \times n)$ matrix is in (Row) Echelon Form if it satisfies all three of the following conditions:

1.) All nonzero rows are above any rows of all zeros.

2.) The leading entry of each nonzero row is a 1 (leading 1) and the leading 1 is in a column to the right of the leading 1 in the row above it.

3.) All entries in a column below a leading 1 are zero.

If a matrix satisfies the following additional condition, then it is in Reduced (Row) Echelon Form:

4.) Each leading 1 is the only nonzero entry in its column.

Note: Every matrix can be transformed to echelon form with elementary row operations, but echelon form is not unique. To guarantee uniqueness, need to use reduced row echelon form.

Example: Classify each matrix as

(I) Not in Echelon Form

(II) In Echelon Form but <u>not</u> Reduced Row Echelon Form

(III) In Reduced Row Echelon Form

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \quad (\mathrm{II})$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (\mathrm{I})$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\mathrm{III})$$

$$D = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \mathrm{III})$$

$$E = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (\mathrm{III})$$

Solving a Linear System:

Example: Solve the linear system $\begin{cases} x_1 - 3x_2 = 7 \\ -3x_1 + 4x_2 = -1 \end{cases}$

System:

$$\begin{aligned} x_1 - 3x_2 &= 7 \\ -3x_1 + 4x_2 &= -1 \end{aligned}$$

Augmented Matrix:

leading 1

$$\left[\begin{array}{cc|c} 1 & -3 & 7 \\ -3 & 4 & -1 \end{array}\right]$$

Keep x_1 in the 1st eqn and eliminate it from the 2nd eqn:

new E_2

$E_2 \to E_2 + 3E_1$

$$\begin{aligned} x_1 - 3x_2 &= 7 \\ -5x_2 &= 20 \end{aligned}$$

$R_2 \to R_2 + 3R_1$

$$\left[\begin{array}{cc|c} 1 & -3 & 7 \\ 0 & -5 & 20 \end{array}\right]$$

Make the coefficient for x_2 a 1:

$E_2 \to -\frac{1}{5} E_2$

$$\begin{aligned} x_1 - 3x_2 &= 7 \\ x &= -4 \end{aligned}$$

$R_2 \to -\frac{1}{5} R_2$

$$\left[\begin{array}{cc|c} 1 & -3 & 7 \\ 0 & 1 & -4 \end{array}\right]$$

Echelon form

Use the x_2 in the 2nd eqn to eliminate x_2 in the 1st eqn:

$E_1 \to E_1 + 3E_2$

$$\begin{aligned} x_1 \quad &= -5 \\ x_2 &= -4 \end{aligned}$$

$R_1 \to R_1 + 3R_2$

$$\left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & -4 \end{array}\right]$$

rref

Unique solution: $(-5, -4)$ or in vector form $\vec{x} = \begin{bmatrix} -5 \\ -4 \end{bmatrix}$

Check solution we found satisfies the original system:

$(-5) - 3(-4) = -5 + 12 = 7$ ✓

$-3(-5) + 4(-4) = 15 - 16 = -1$ ✓

A leading 1 position (or pivot position) in a matrix A is a location that corresponds to a leading 1 in the reduced row echelon form of A. A leading 1 column (or pivot column) is a column of A that contains a leading 1 position.

The Row Reduction Algorithm:

Example: Solve the linear system by transforming the augmented matrix first to echelon form, then to reduced row echelon form.

$$\begin{aligned} x_1 - 2x_2 - x_3 &= 3 \\ 3x_1 - 6x_2 - 5x_3 &= 3 \\ 2x_1 - x_2 + x_3 &= 0 \end{aligned}$$

(3x3) system

Step 0: Write the augmented matrix

already a leading 1

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 3 & -6 & -5 & 3 \\ 2 & -1 & 1 & 0 \end{array}\right]$$

Step 1: Begin with the leftmost nonzero column. This is a leading 1 column. The leading 1 position is at the top. If necessary, interchange rows, so there is a nonzero entry a in the leading 1 position. If the leading entry a is not 1, multiply by $\frac{1}{a}$.

(You can also wait until the end to make all the entries in leading 1 positions equal to 1. Can avoid working with fractions.)

Step 2: Use elimination to create zeros in all positions below the leading 1 position by adding appropriate multiples of Row 1 to each of the other rows.

↙ already a leading 1

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ \boxed{3} & -6 & -5 & 3 \\ \boxed{2} & -1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & 0 & -2 & -6 \\ 0 & 3 & 3 & -6 \end{array}\right]$$

Step 3: Repeat steps 1 and 2 (ignoring Row 1).

Step 4: Repeat until matrix is in echelon form.

(*) It is ok to do more than one step at a time, if the steps do not involve one another.

<u>Step 5</u>: Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.

suppose $\left[\begin{array}{ccc|c} & \vdots & & \vdots \\ 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow 0x_1 + 0x_2 + 0x_3 = 1$

$\Rightarrow 0 = 1$

<u>Step 6</u>: Beginning with the rightmost leading 1 position and working upward to the left, create zeros above each leading 1 until matrix is in reduced row echelon form.

The row reduction algorithm is complete.

To solve the system, write the system of equations corresponding to the system at the end of step 6.

The solution is:

Example: Consider the following augmented matrix.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This reduced matrix corresponds to the reduced system

The variables x_1, x_3, and x_4 are called leading variables (or basic variables, or dependent variables, or constrained variables.) These are the variables in the leading 1 positions of the reduced matrix.

The variables x_2 and x_5 are free variables (or independent or unconstrained variables)

Whenever a system is consistent, the solution set can be described explicitly by solving the reduced system of equations for the leading variables in terms of the free variables.

Using Row Reduction to Solve a Linear System (Gauss-Jordan Elimination)

1.) Write the augmented matrix of the system.

2.) Use the row reduction algorithm to obtain a row equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to next step.

3.) Continue the row reduction algorithm to obtain the reduced row echelon form.

4.) Write the system of equations corresponding to the system in Step 3.

5.) Rewrite each nonzero equation from step 4 so that each leading variable is expressed in terms of any free variables appearing in the reduced equations.

Example: Solve the system.

$$\begin{aligned} x_1 - x_2 - 2x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 5 \end{aligned}$$

Example: Solve the system.

$$\begin{aligned} x_1 - x_2 - 2x_3 &= 2 \\ 2x_1 + x_2 + x_3 &= 5 \\ x_1 + 2x_2 + 3x_3 &= 4 \end{aligned}$$

Example: Each of the following matrices are the augmented matrix for a system of linear equations in reduced row echelon form. Give the system of equations and describe the solution.

a.) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

b.) $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

c.) $\left[\begin{array}{cccc|c} 1 & -3 & 0 & 4 & 2 \\ 0 & 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$