Section 14.7: Optimization in Several Variables (Cont.)

Theorem (Existence and Location of Global Extrema):
Let $f(x,y)$ be a continuous function on a closed, bounded domain D in $\mathbb{R}^2$. Then

1.) $f(x,y)$ takes on both a global maximum and a global minimum value on D.

2.) The extreme values occur either at critical points in the interior of D or at points on the boundary of D.

This is also called the extreme value theorem for functions of 2 variables.

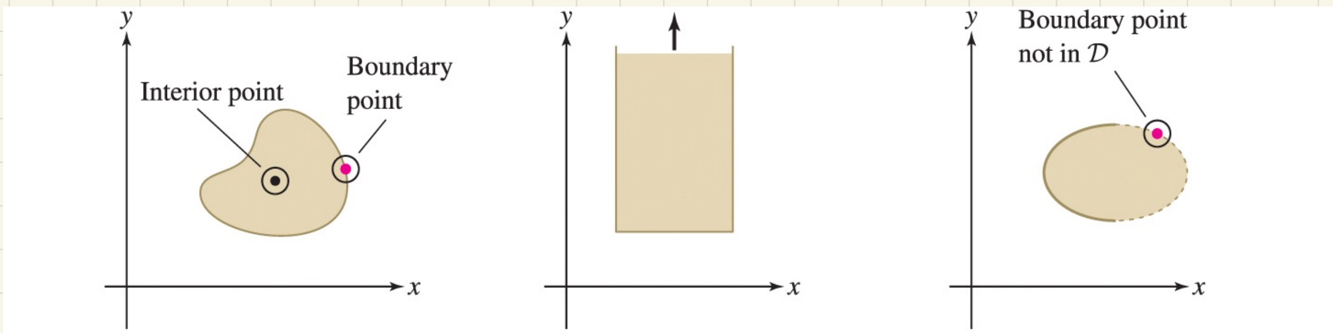


Fig. 11 (A) This domain is bounded and closed (contains all boundary points). (B) An unbounded domain (contains points arbitrarily far from the origin). (C) A nonclosed domain (contains some but not all boundary points).

To find the Absolute Max and Min:

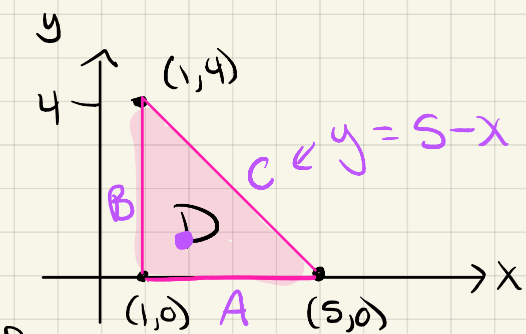
1.) Find the values of f at the critical points of f in D.

2.) Find the extreme values of f on the boundary of D.

3.) The largest from 1 & 2 is the $\underbrace{\text{absolute}}_{\text{global}}$ max & the smallest is the absolute min.

Example: Find the absolute max & min values of $f(x,y) = 3 + xy - x - 2y$ on the domain D which is the closed region with vertices $(1,0)$, $(5,0)$, $(1,4)$.

1.) $\begin{cases} f_x = y - 1 = 0 \Rightarrow y = 1 \\ f_y = x - 2 = 0 \Rightarrow x = 2 \end{cases}$ $\therefore$ critical point: $(2,1)$



$f(2,1) = 3 + \cancel{2} - \cancel{2} - 2 = 1$

2.) A: $\underbrace{y=0,\ 1 \le x \le 5}_{\text{defines line segment for "A"}}$: $f(x,0) = \overbrace{3 - x}^{\text{This is the function on the line segment "A"}}$ is a decreasing function

max is $f(1,0) = 3 - 1 = 2$ & min is $f(5,0) = 3 - 5 = -2$

B: $x = 1$, $0 \le y \le 4$: $f(1,y) = 3 + y - 1 - 2y = 2 - y$ is a decreasing function

max is $\underbrace{f(1,0) = 2}_{\text{already found}}$ & min is $f(1,4) = 2 - 4 = -2$

C: $\underbrace{y = 5 - x,\ 1 \le x \le 5}_{\text{defines line segment for "C"}}$: $f(x, 5-x) = 3 + x(5-x) - x - 2(5-x) = -x^2 + 6x - 7$

* Try to find the max & min values of $f(x, 5-x) = -x^2 + 6x - 7$ on the closed interval $1 \le x \le 5$ for next class. *

Hint: Let $g(x) = -x^2 + 6x - 7$ and use calc 1.

Section 14.8: Lagrange Multipliers – Optimizing with a Constraint

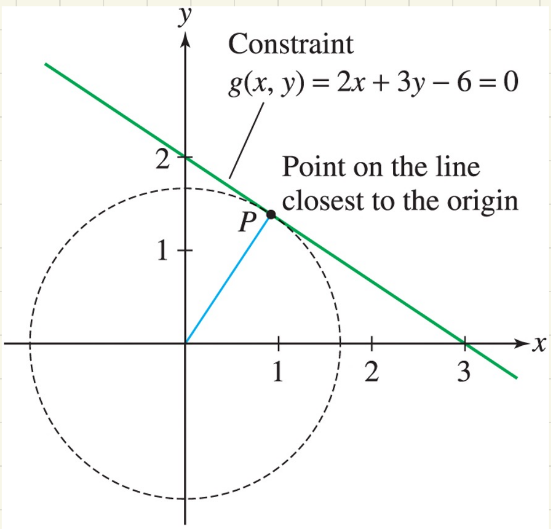
Problem: Find the min (or max) value of $f(x,y)$ subject to the constraint $g(x,y)=0$.

For example: Find the minimum value of

$$f(x,y)=\sqrt{x^2+y^2}$$

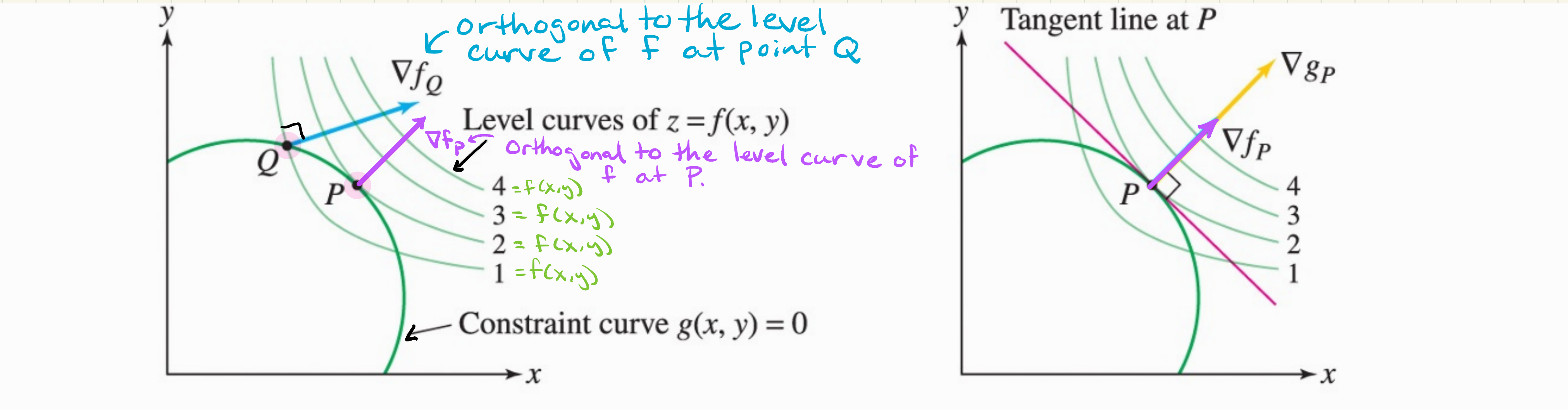
Note: $f(x,y)$ represents the distance of a point (x,y) from the origin.

subject to $\underbrace{2x+3y-6}_{g(x,y)}=0$ (constraint)



Need to find the minimum value of $f(x,y)$ among all points that lie on the line $2x+3y-6=0 \Rightarrow y=2-\frac{2}{3}x$.

The method of Lagrange Multipliers is a procedure for solving optimization problems with constraints.



(A) f increases as we move to the right along the constraint curve.

Fig. 2

(B) The local maximum of f on the constraint curve occurs where ∇f_P and ∇g_P are parallel.

Conclusion: At points that lead to the max/min value of $f(x,y)$, subject to the constraint curve $g(x,y) = 0$, the level curve of f and the constraint curve g are tangent.

∇g_P is orthogonal to the constraint curve at P and thus also orthogonal to the level curve of f at P.

$\therefore$ ∇f_P and ∇g_P are parallel : $\nabla f_P = \underset{\text{Lagrange Multiplier}}{\lambda} \nabla g_P$

Theorem (Lagrange Multipliers):

Assume that $f(x,y)$ and $g(x,y)$ are differentiable functions. If $f(x,y)$ has a local min or local max on the constraint curve $g(x,y)=0$ at $P=(a,b)$, and if $\nabla g_P \neq \vec{0}$, then there is a scalar λ such that

$$\nabla f_P = \lambda \nabla g_P \quad (\text{Lagrange Condition})$$

Written in terms of components:

$$\begin{cases} f_x(a,b) = \lambda g_x(a,b) \\ f_y(a,b) = \lambda g_y(a,b) \end{cases} \quad (\text{Lagrange Equations})$$

Def: A point $P=(a,b)$ satisfying the Lagrange Equations/condition is called a critical point for the optimization problem with constraint and $f(a,b)$ is called a critical value.

General Procedure: To find the extreme values of $f(x,y)$ subject to $g(x,y)=0$:

1.) Solve the system

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 0 \end{cases}$$

to find critical points.

2.) Calculate the critical values by evaluating f at all critical points. The largest critical value is the max and the smallest critical value is the min.

Example: Find the minimum and maximum values of $f(x,y) = x^2 - y^2$ on the circle $x^2+y^2=1$.

Need to solve :

Lagrange Multipliers in 3 variables:

Example: Find the minimum and maximum values of $f(x,y,z) = xy + 2z$ subject to $z^2 = 36 - x^2 - y^2$.

Need to solve:

Critical point	$f(x,y,z) = xy + 2z$

$$\boxed{\begin{array}{l} \text{max} = \\ \text{min} = \end{array}}$$

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