Physics 1200
Lecture 12
Spring 2024

Ampere's Law, Calculating Magnetic Fields from Ampere's Law, Magnetic Moments and Magnetism

Ampere's Law

- Electrostatics: relation between vector integration of electric field over a closed, spatially bounded region (= net electric flux through a closed Gaussian surface) and electric charge (source for electric field) within the bounded region. Idea encapsulated by Gauss's law for electric fields:

$$\underbrace{\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}}}_{\text{Spatially-integrated field}} = \underbrace{\frac{Q_{enc}}{\epsilon_0}}_{\text{Source of field}}$$

- For situations having symmetry, electric field could be constant on Gaussian surface and extracted from flux integral. Allows field solution in terms of system geometry and charge enclosed within spatial region of interest.

Ampere's Law (2)

- There's an analogous concept for magnetic fields and electrical current (source of magnetic field) when current and magnetic field are not changing over time (magnetostatics).
- Ampere's law: for magnetostatic situations, line integral of magnetic field over a closed path is proportional to total electrical current enclosed within the path.
- Mathematically:

$$\boxed{\underbrace{\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{l}}}_{\text{Spatially-integrated field}} = \underbrace{\mu_0 I_{enc}}_{\text{Source of field}}}$$

where I_{enc} is the net current enclosed within the closed ("Amperian") path.

Ampere's Law (3)

- Magnetostatic situations: Ampere's law is true for all chosen paths, regardless of shape.

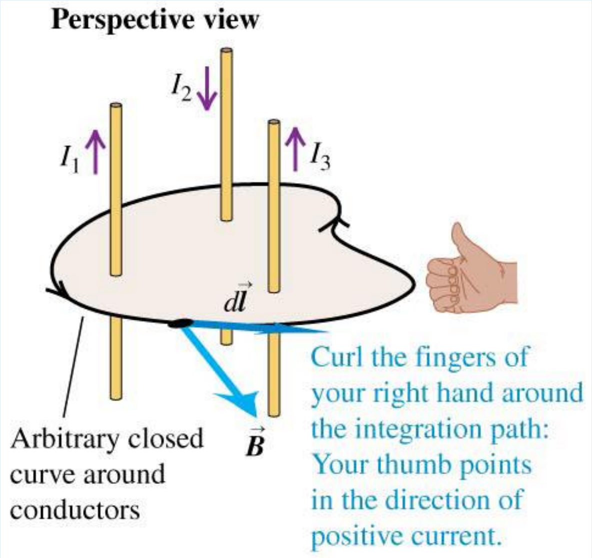
- For case $I_{enc} = 0, \oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{l}} = 0$.

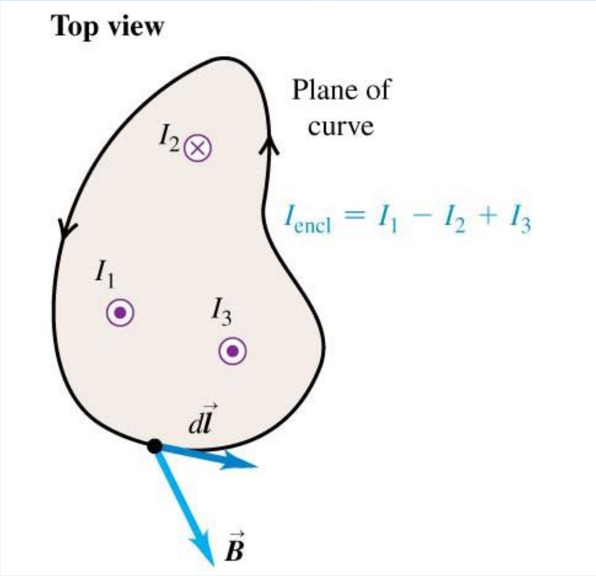
    - Note: this does <u>not</u> necessarily mean that $\vec{\boldsymbol{B}} = 0$ for the region being analyzed!

- I_{enc} = total net current passing through surface area bound by the Amperian path. It is the <u>algebraic sum</u> of the currents passing through the path.

Currents in Ampere's Law

- Positive and negative currents are defined using a right-hand rule:
  1. Curl fingers of your right hand in direction taken by the closed Amperian path.
  2. Direction your right thumb points is taken as positive direction for current. ∴ Currents going in direction of your thumb are positive, and currents going in opposite direction are negative.





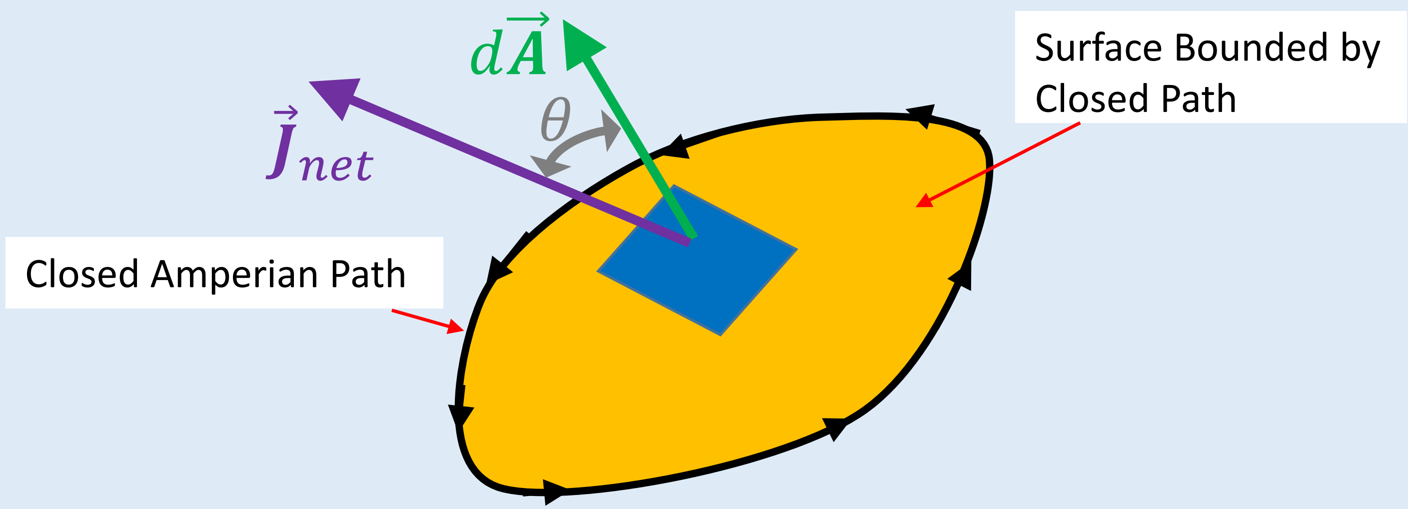
Currents in Ampere's Law (2)

- If current is spread throughout a conductor with net current density $\vec{J}_{net}$, enclosed current in Ampere's law is

$$I_{enc} = \int \vec{J}_{net} \cdot d\vec{A} \,,$$

where the total area being integrated over <u>is the area of the surface enclosed by the Amperian path</u>.

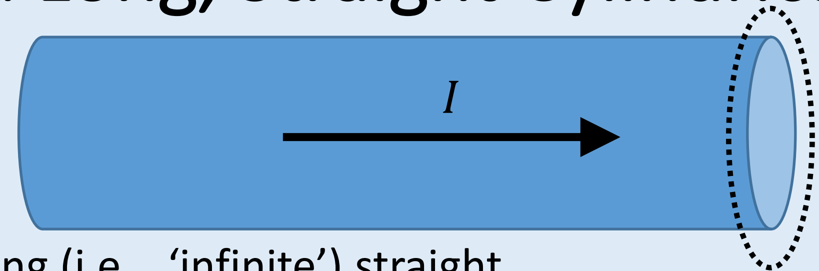
  - Direction of surface area vector found by using same right-hand rule used to determine whether a current is positive or negative (preceding slide): curl fingers on right hand in direction path is traversed $\Rightarrow$ direction thumb points is direction of perpendicular area vector in integral above.



Using Ampere’s Law to Calculate Magnetic Field

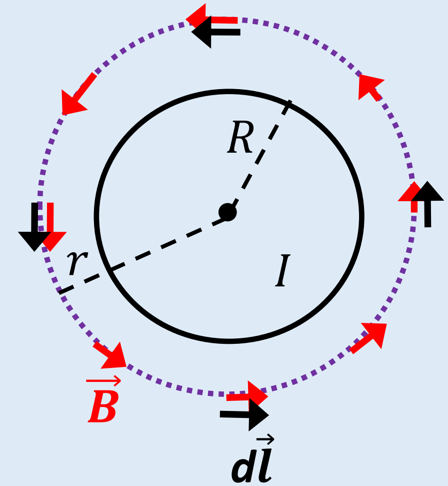
- Strategy is like that for using Gauss’s law to find electric field.
  - Take Amperian path consistent with any symmetry (if any) in situation being considered.
  - To be able to extract magnetic field from line integral, need to have portions of a closed path for which magnetic field is constant and/or zero.
  - If field is successfully extracted from line integral, divide each side of Ampere’s law by total path length. This gives an expression for the magnetic field in terms of the total current and the geometry of the problem being considered.

Example: Magnetic Field Due to a Uniform Current in a Long, Straight Cylindrical Conductor



- Consider a long (i.e., 'infinite') straight, cylindrical conductor of radius R, carrying steady current I. Consider case where current is uniformly distributed across the cross-sectional area of the conductor.

Current I is coming <u>out</u> of page.



- First: find field for distances $r > R$ from axis of symmetry.
- Field for a long wire (last class), we know that magnetic field lines for this kind of geometry are circular. $\therefore$ Take circular Amperian path of constant radius r, as shown in figure.
- Because $B(r)$ is constant on indicated path, Ampere's law yields:

$$\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{l}} = \oint B\,dl \cos 0° = B \oint dl = B(2\pi r) = \mu_0 I_{enc} = \mu_0 I,$$

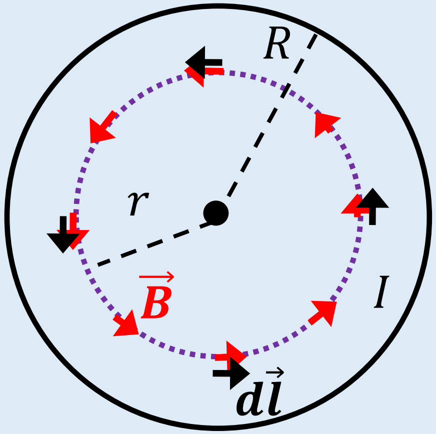
since all of conductor current I is enclosed within path.

$\Rightarrow \boxed{B = \frac{\mu_0 I}{2\pi r}}$, for $r > R$. Same result as infinitely long wire!

Example: Magnetic Field Due to a Uniform Current in a Long, Straight Cylindrical Conductor (2)

- Second: find $B(r)$ for $r \leq R$.
- Situation still has cylindrical symmetry, expect field lines are still circular, even inside conductor. $\therefore$ Take another circular Amperian path, as shown in figure.
- Inside conductor, uniform current density

$\vec{\boldsymbol{J}} = \frac{I}{\pi R^2}\widehat{\boldsymbol{k}}$, and differential area vector $d\vec{\boldsymbol{A}} = dA\widehat{\boldsymbol{k}}$.



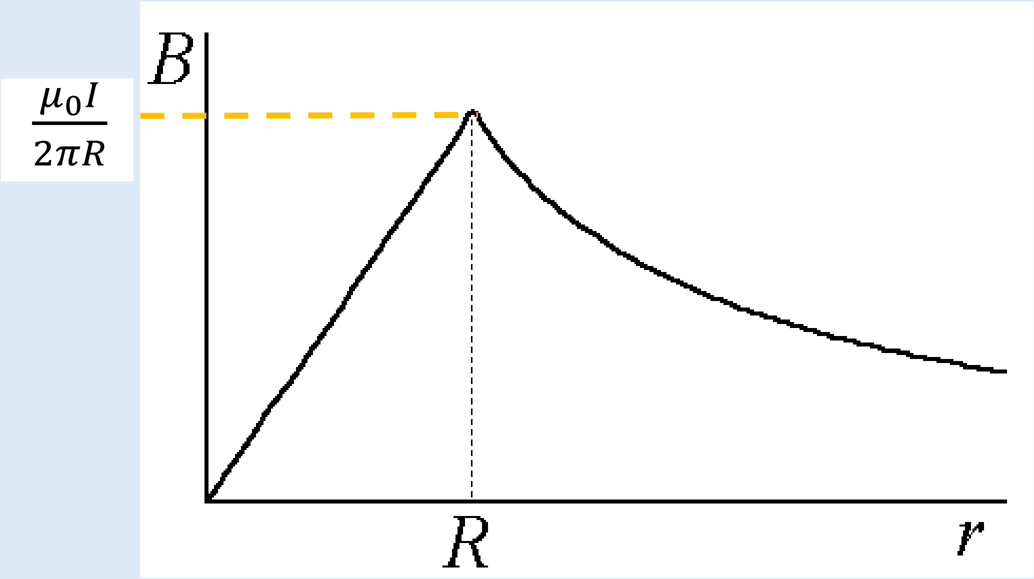
Current I is coming <u>out</u> of page.

- Ampere's law:

$$\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{l}} = B(2\pi r) = \mu_0 I_{enc} = \mu_0 \int \vec{\boldsymbol{J}} \cdot d\vec{\boldsymbol{A}} = \mu_0 \int J\, dA = \mu_0 J A = \left(\frac{\mu_0 I}{\pi R^2}\right)(\pi r^2).$$

$$\Rightarrow \boxed{B = \left(\frac{\mu_0 I}{2\pi R^2}\right) r}\text{, for } r \leq R.$$

Example: Magnetic Field Due to a Uniform Current in a Long, Straight Cylindrical Conductor (3)



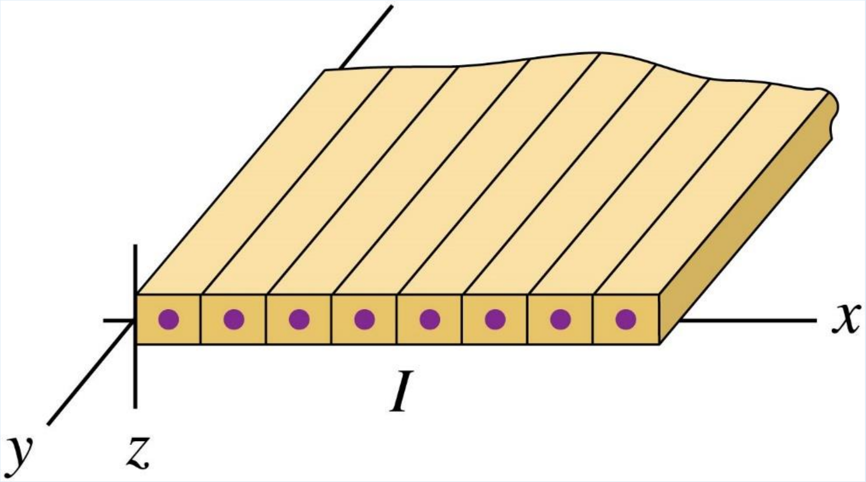
For $r \leq R$:

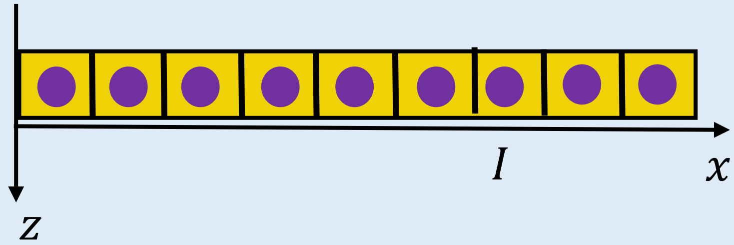
$$B = \left(\frac{\mu_0 I}{2\pi R^2}\right) r$$

For $r > R$:

$$B = \frac{\mu_0 I}{2\pi r}$$

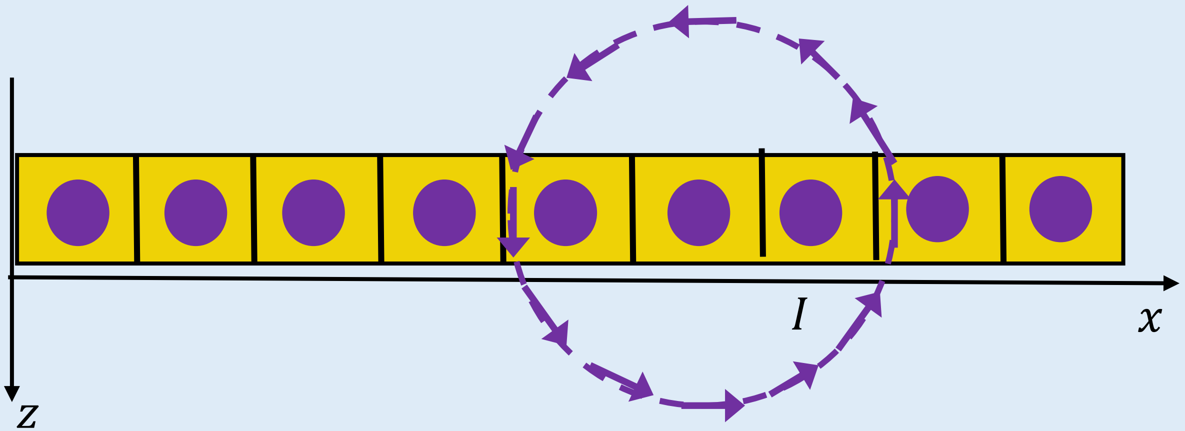
Magnetic Field Near a Flat Current Sheet



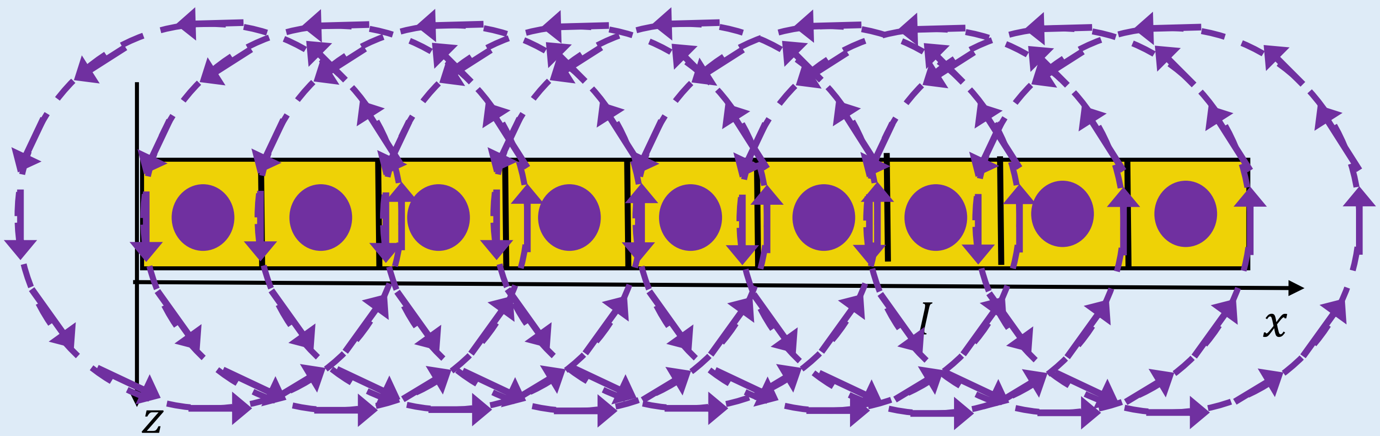


- Problem in lab activity for today.
- First: find *B*-field direction in different regions for this system.
  - ➢ Hint: RHR "thumb rule" for finding field direction from current is helpful.
- Take an Amperian path that exploits expected $\vec{\boldsymbol{B}}$-field orientation and allows removal of $\vec{\boldsymbol{B}}$ from line integral in Ampere's law.
- Result of problem is starting point for calculating magnetic field of an infinitely long solenoid using Ampere's law.

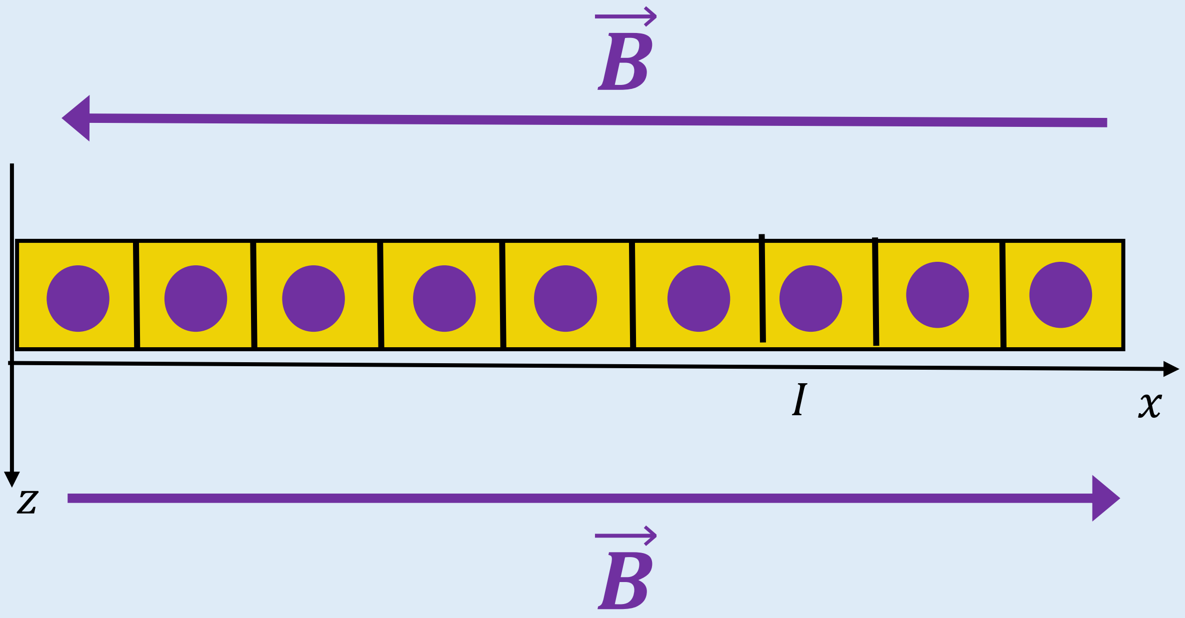
Magnetic Field Near a Flat Current Sheet



Magnetic Field Near a Flat Current Sheet

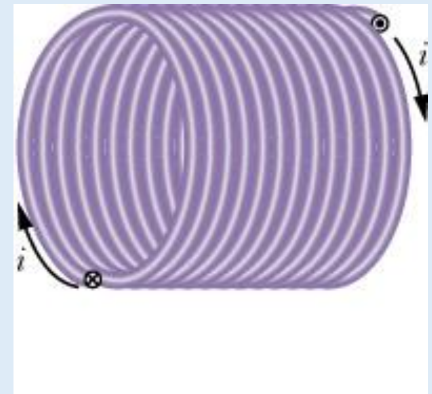


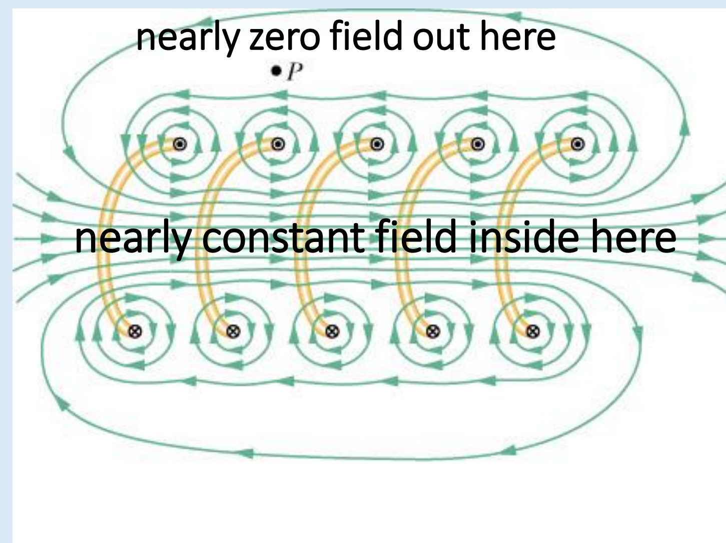
Magnetic Field Near a Flat Current Sheet



Solenoids

- A <u>solenoid</u> is a helical winding of wire on a cylinder.
- Approximately uniform field created within a very long solenoid having very tight winding.
- Field between wires cancels.
- Field outside coil cancels.
- Field inside coil adds.
  - Very long solenoid like a magnetic field 'pipe', condensing a nearly uniform magnetic field within it.
- Finite-sized solenoids have a dipole-field behavior. One end of solenoid acts as a north pole, and other end as a south pole.





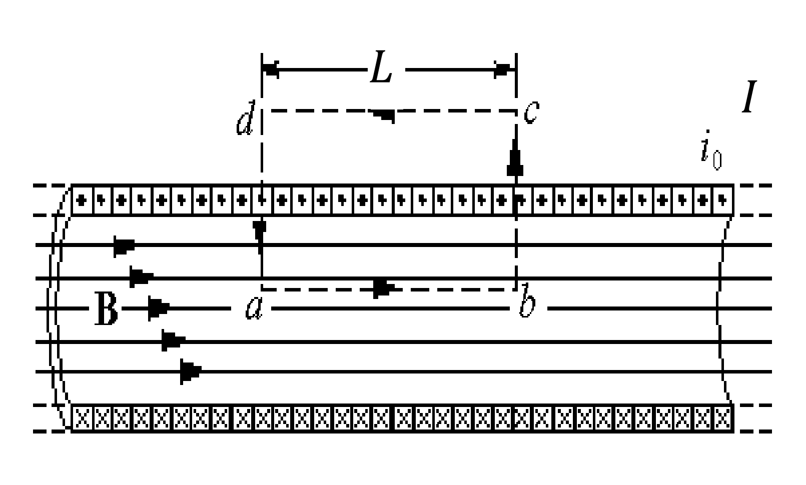
Demonstration: Magnetic Field of a Solenoid

- Generation of a finite solenoid field, driven by an external power supply.

https://www.youtube.com/watch?v=FzPO92Jxxt8

Using Ampere's Law to Calculate the Magnetic Field Inside a Long Solenoid

- Assumptions:
  - Solenoid is very long, and magnetic field is essentially constant inside, directed along axis of solenoid.
  - Field outside the solenoid is negligible.



- Path a-b-c-d shown in sketch, traveling counter-clockwise (indicated by the arrows), Ampere's law gives:

$$\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{l}} = \int_a^b B dl \cos 0° + \int_b^c B dl \cos 90° + \int_c^d 0 dl + \int_d^a B dl \cos 90°$$

$$= B(b-a) = BL = \mu_0 I_{enc} = \mu_0 N I,$$

where N = number of turns (loops) in the length L of the solenoid.

$\therefore B = \mu_0 \left(\frac{N}{L}\right) I$, or, defining number of turns per unit length, $n \equiv \frac{N}{L}$,

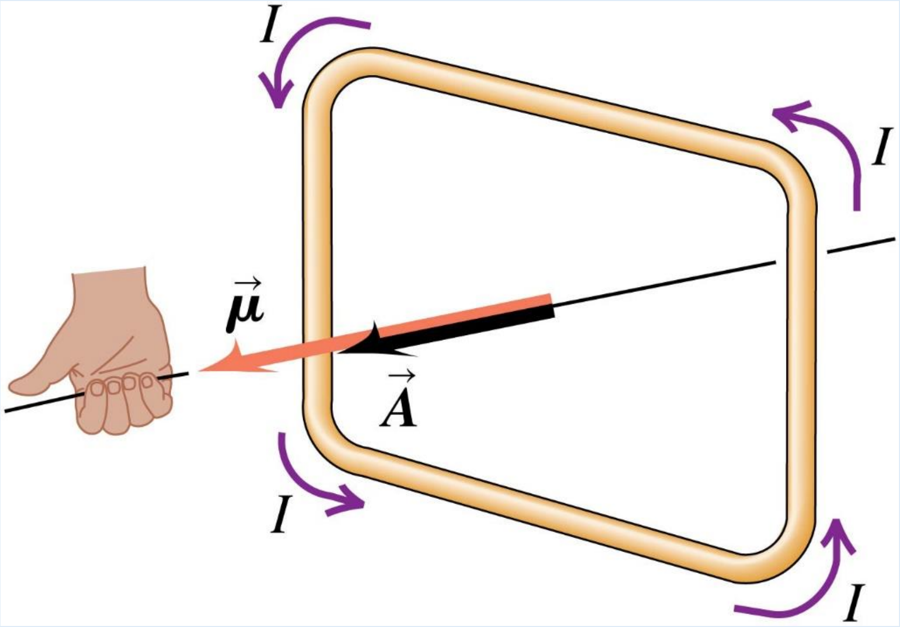
$\Rightarrow \boxed{B = \mu_0 n I}$. (Field inside an ideal solenoid.)

Magnetic Materials, Atomic Magnetism

- Learned current is source of magnetic fields.
- Many naturally occurring magnetic materials. Magnetism of these materials can be thought of as result of <u>internal current loops on the atomic scale</u>, e.g., due to orbital motion of electrons around the nuclear core in an atom. Orbital motion of electrons ⇒ atomic currents. Can lead to non-zero atomic-level <u>magnetic moments</u>.
  - ➢ Means that magnetic moment of a particular material is related to orbital angular momentum of atomic electrons making up material.
  - ➢ Magnetic moments can also arise due to another property of electrons known as <u>spin</u>. This is a type of angular momentum but should not be thought of as being result of classical orbital or rotational behavior. It has no classical (that is, non-quantum mechanical) analog.

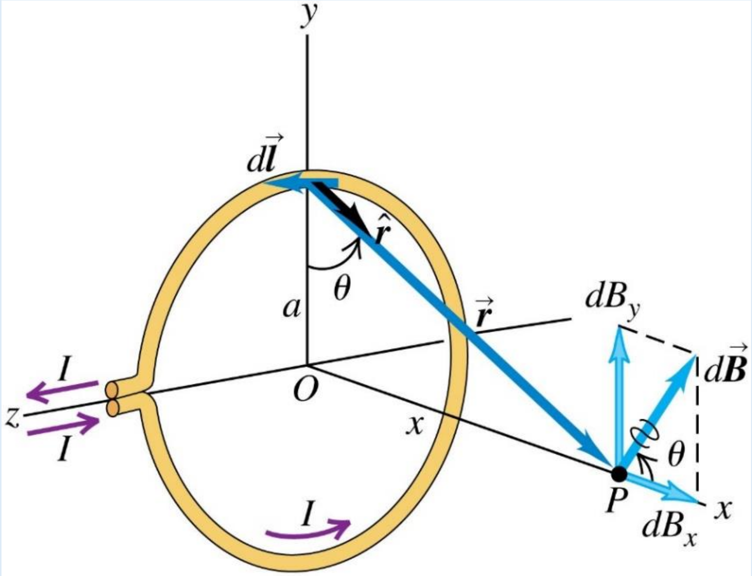
Magnetic Moment of a Current Loop

- The magnetic moment of a current loop is defined as $\vec{\boldsymbol{\mu}} \equiv I\,\vec{\boldsymbol{A}}$, where $\vec{\boldsymbol{A}}$ is the area vector of the current loop.
  - ➢ Direction of $\vec{\boldsymbol{A}}$ found by using the same right-hand thumb rule we introduced earlier:



Magnetic Moment and Magnetic Field of a Current Loop

- Magnetic moment appears naturally in calculated magnetic field of a current loop. Consider field a distance along the symmetry axis of a circular loop of radius a:



- Find field at distance x along symmetry axis, as shown in diagram.
- Shown current element $Id\vec{\boldsymbol{l}} = Idl\widehat{\boldsymbol{k}}$, located at $a\hat{\boldsymbol{j}}$.
- Position vector from current element to field point P is $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} - a\hat{\boldsymbol{j}}$.
- Biot-Savart law:

$$d\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi}\frac{Id\vec{\boldsymbol{l}}\times\vec{\boldsymbol{r}}}{r^3} = \frac{\mu_0}{4\pi}\frac{Idl\widehat{\boldsymbol{k}}\times(x\hat{\boldsymbol{\imath}}-a\hat{\boldsymbol{j}})}{(x^2+a^2)^{3/2}}.$$

Magnetic Moment and Magnetic Field of a Current Loop (2)

- Carrying expression further, get

$$d\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi}\frac{(Iadl\hat{\boldsymbol{\imath}}+Ixdl\hat{\boldsymbol{\jmath}})}{(x^2+a^2)^{3/2}}$$

- Other current elements also create $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ field components. When summed up to get total magnetic field, they cancel out, leaving only the x-component of field along symmetry axis:

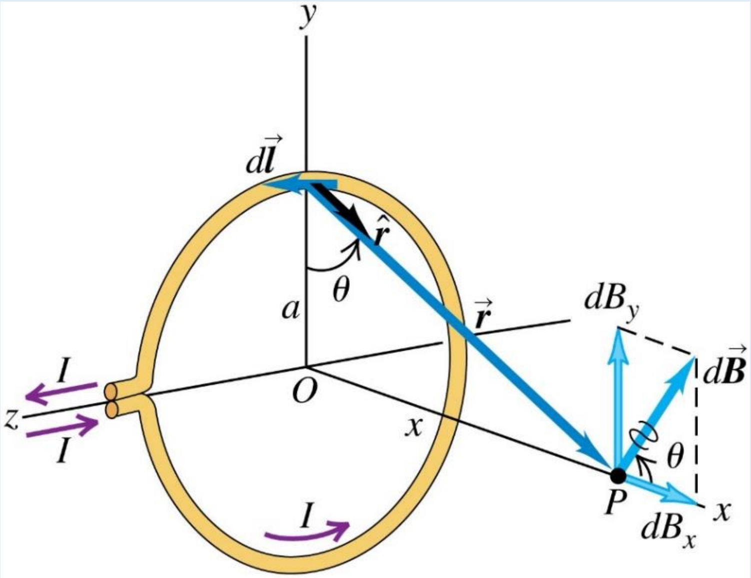
$$\vec{\boldsymbol{B}} = B_x\hat{\boldsymbol{\imath}} = \int dB_x\hat{\boldsymbol{\imath}} = \int \frac{\mu_0}{4\pi}\frac{Ia}{(x^2+a^2)^{\frac{3}{2}}}dl\hat{\boldsymbol{\imath}}.$$

Integrating around circumference of loop ($l = 2\pi a$) yields

$$\boxed{B_x = \frac{\mu_0 I a^2}{2(x^2+a^2)^{3/2}} = \frac{\mu_0\mu}{2\pi(x^2+a^2)^{3/2}}},$$

used the magnetic moment relation:

$\mu = IA = I\pi a^2$.



∴ Field from a current loop can be written in terms of its magnetic moment!

Note: very far away ($x \gg a$),

$B_x \propto \mu/x^3$ (typical dipole result).

Magnetic Torque on a Uniform Current Loop

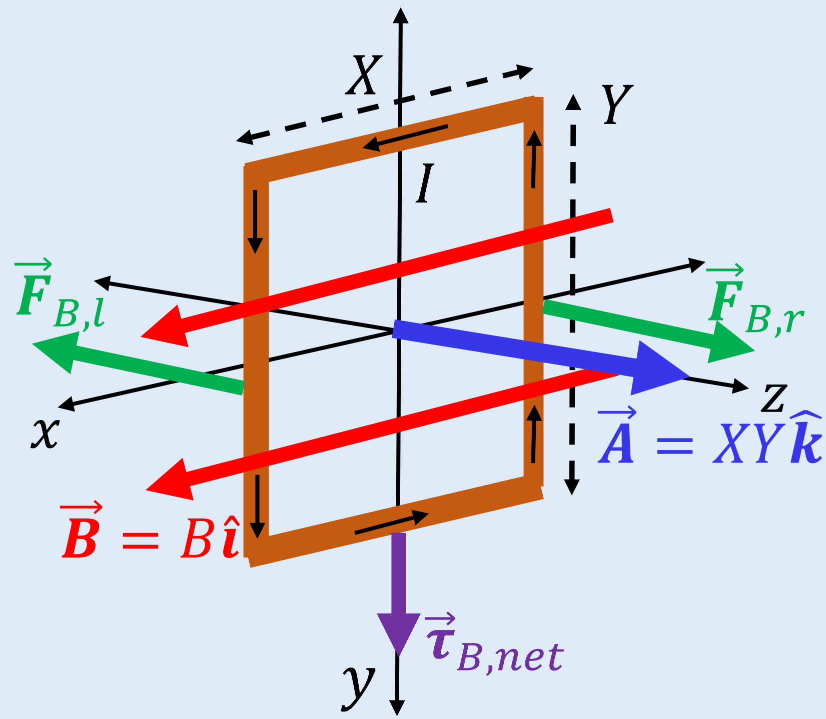
- Consider uniform current loop in uniform magnetic field.
- Net magnetic force on the loop is (recall that $\vec{\boldsymbol{F}}_B = I\vec{\boldsymbol{l}} \times \vec{\boldsymbol{B}}$):

$$\vec{\boldsymbol{F}}_{B,net} = \vec{\boldsymbol{F}}_{B,l} + \vec{\boldsymbol{F}}_{B,r}$$
$$= -IYB\hat{\boldsymbol{k}} + IYB\hat{\boldsymbol{k}} = 0.$$

- Net magnetic torque on the loop is

$$\vec{\boldsymbol{\tau}}_{B,net} = \vec{\boldsymbol{\tau}}_{B,l} + \vec{\boldsymbol{\tau}}_{B,r}$$
$$= \left|\vec{\boldsymbol{F}}_{B,l}\right|\frac{X}{2}\hat{\boldsymbol{j}} + \left|\vec{\boldsymbol{F}}_{B,r}\right|\frac{X}{2}\hat{\boldsymbol{j}}$$
$$= IXYB\hat{\boldsymbol{j}} = \vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{B}}\,,$$

where $\vec{\boldsymbol{\mu}} = I\vec{\boldsymbol{A}} = IXY\hat{\boldsymbol{k}}$ is the magnetic moment of the loop.



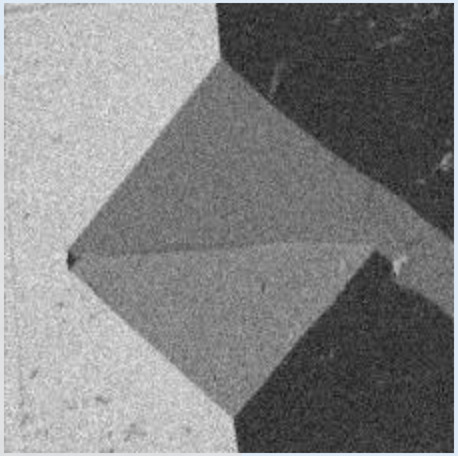
- General result: net magnetic torque on a uniform current loop of arbitrary shape with magnetic moment $\vec{\boldsymbol{\mu}}$ is:

$$\boxed{\vec{\boldsymbol{\tau}}_{B,net} = \vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{B}}}\ .$$

- Torque tries to rotate $\vec{\boldsymbol{\mu}}$ so that it aligns with $\vec{\boldsymbol{B}}$. Effect of magnetic fields on atomic magnetic moments has many tech applications (e.g., magnetic resonance imaging [MRI]).

Ferromagnetic materials

- In **ferromagnetic materials** (such as iron), atomic magnetic moments tend to line up parallel to each other in regions called “magnetic domains.”
- When there is no externally applied field, the domain magnetizations are randomly oriented. No net ordered field results.
- When an external magnetic field is present, the domain boundaries shift; the domains that are magnetized in the field direction grow, and those that are magnetized in other directions shrink.



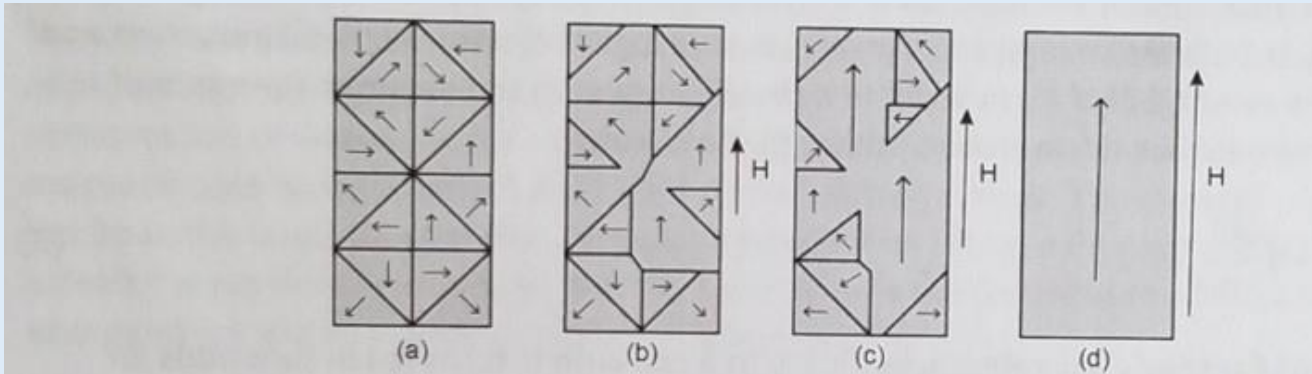


Figure 2: As the magnetic field strength (H) increases from zero (a), the magnetic domains that are aligned parallel to it increase in size at the expense of their neighbouring domains (b), then whole domains start to switch (c) until all domains are aligned (d).