Physics 1200
Lecture 04
Spring 2024

Electric Potential Energy,

Electric Potential,

Equipotential Surfaces

Electric Potential Energy

- Work done by a force $\vec{\boldsymbol{F}}$ on an object moving from point A (at $\vec{\boldsymbol{r}}_A$) to point B (at $\vec{\boldsymbol{r}}_B$):

$$W_{AB} = \int_{\vec{\boldsymbol{r}}_A}^{\vec{\boldsymbol{r}}_B} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{s}} \, ,$$

where

$$\boxed{d\vec{\boldsymbol{s}} = dx\hat{\boldsymbol{\imath}} + dy\hat{\boldsymbol{\jmath}} + dz\hat{\boldsymbol{k}}}$$

is path increment to get from A to B.

- Forces for which W_{AB} is <u>independent</u> of the path between the two endpoints are <u>conservative forces</u>.
  - ➢ Work by a conservative force depends only on locations of endpoints. Not the path to get from one point to the other.
  - ➢ Work done by a conservative force $\vec{\boldsymbol{F}}_{cons}$ for a <u>closed path</u>

$$W = \oint \vec{\boldsymbol{F}}_{cons} \cdot d\vec{\boldsymbol{s}} = 0.$$

Electric Potential Energy (2)

- For conservative forces, potential energy difference defined as

$$\Delta U = U(\vec{\boldsymbol{r}}_B) - U(\vec{\boldsymbol{r}}_A) \equiv -W_{AB} = -\int_{\vec{\boldsymbol{r}}_A}^{\vec{\boldsymbol{r}}_B} \vec{\boldsymbol{F}}_{cons} \cdot d\vec{\boldsymbol{s}}.$$

- Potential energy U is a <u>scalar</u> function of position. It is <u>not</u> a vector function!

- For <u>static</u> electric fields, the electric force is a <u>conservative force</u>. For a charge q in an electric field $\vec{\boldsymbol{E}}$,

$$\vec{\boldsymbol{F}}_E = q\vec{\boldsymbol{E}},$$

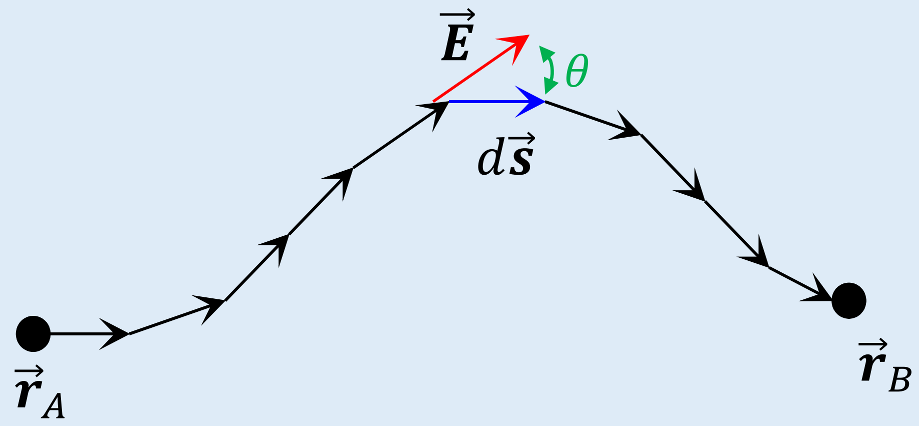
and the change in electrical potential energy

$$\boxed{\Delta U_E = -\int_{\vec{\boldsymbol{r}}_A}^{\vec{\boldsymbol{r}}_B} q\vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}} = -q\int_{\vec{\boldsymbol{r}}_A}^{\vec{\boldsymbol{r}}_B} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}}\ .}$$

SI unit of work and energy is the Joule (J).

Integral Dot (Scalar) Product – Line Integrals

- When integrating along path, use component of $\vec{\boldsymbol{E}}$ in the direction of the local path increment $d\vec{\boldsymbol{s}}$:



Using dot products, can write

$$\int_{\vec{r}_A}^{\vec{r}_B} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}} = \int_{\vec{\boldsymbol{r}}_A}^{\vec{\boldsymbol{r}}_B} E \cos\theta \, ds \quad \text{(angle } \theta \text{ in diagram),}$$

or, $$\int_{\vec{r}_A}^{\vec{r}_B} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}} = \int_{x_A}^{x_B} E_x \, dx + \int_{y_A}^{y_B} E_y \, dy + \int_{z_A}^{z_B} E_z \, dz.$$

These types of expressions are known as '<u>line integrals</u>'.

Electric Potential

- Electric potential V is a <u>scalar</u> field created by electric charges. Useful for electrostatic situations.

- Change in potential V between the points A and B is defined by the relation

$$\boxed{\Delta V \equiv \frac{\Delta U_E}{q} = -\int_{\vec{\boldsymbol{r}}_A}^{\vec{\boldsymbol{r}}_B} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}} \ .}$$

Electric potential is the electrostatic potential energy per Coulomb (i.e., per unit charge).

- SI unit of electric potential is the Volt, 1V = J/C.

Be careful: don't confuse the potential (V) with its unit (V).

Sometimes the term 'voltage' is used for potential.

Electric Potential (2)

- From the defining relation, change in electric potential energy U_E of a charge q in a potential V is

$$\boxed{\Delta U_E = q\,\Delta V}\,.$$

- For a system of charges of finite extent, reference point where $V = 0$ is usually taken at infinity (i.e., very far away from the system). Follows that

$$\boxed{V(r) - V(\infty) = V(r) - 0 = -\int_{\infty}^{r} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}}}\,.$$

Example: Potential of a Point Charge

- For a point charge q located at the origin,

$$\vec{\boldsymbol{E}} = \frac{kq}{r^2}\,\hat{\boldsymbol{r}},$$

where r is the distance from the charge.

- Since solution of the line integral for a conservative field is independent of path taken, choose the easiest path: $d\vec{\boldsymbol{s}} = dr\,\hat{\boldsymbol{r}}$.

- Follows that

$$V(r) = -\int_{\infty}^{r} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}} = -\int_{\infty}^{r} \frac{kq}{r'^2}\hat{\boldsymbol{r}} \cdot dr'\,\hat{\boldsymbol{r}} = -\int_{\infty}^{r} \frac{kq}{r'^2}\,dr'$$

$$\Rightarrow \quad \boxed{V(r) = \frac{kq}{r}}.$$

Potential due to a point charge, relative to $V(\infty) = 0$.

Potential of a Point Charge (2)

- Notes on point-charge potential:
  - ➢ It is a scalar function. Not a vector.
  - ➢ $V(r) \propto \frac{1}{r}$, not $\frac{1}{r^2}$!

    Don't get potential and electric fields for a point charge mixed up.
  - ➢ The sign of the charge q must be included when calculating potential. V can be positive, negative, or zero.

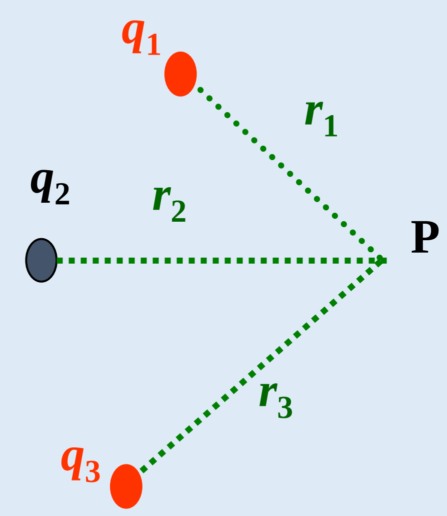
Potential for a System of Charges

- For N discrete charges, the net electric potential is due to the net electric field. The potential at some position $\vec{\boldsymbol{r}}$ will be

$$V(r) = -\int_{\infty}^{\vec{\boldsymbol{r}}} \vec{\boldsymbol{E}}_{\text{net}} \cdot d\vec{\boldsymbol{s}} = -\int_{\infty}^{\vec{\boldsymbol{r}}} \sum_{i=1}^{N} \vec{\boldsymbol{E}}_i \cdot d\vec{\boldsymbol{s}} = -\sum_{i=1}^{N} \int_{\infty}^{\vec{\boldsymbol{r}}} \vec{\boldsymbol{E}}_i \cdot d\vec{\boldsymbol{r}}$$

$$= \sum_{i=1}^{N} V_i = \sum_{i=1}^{N} \frac{kq_i}{|\vec{r}-\vec{R}_i|}$$

$$\Rightarrow \boxed{V(r) = \sum_{i=1}^{N} \frac{kq_i}{r_i}}.$$



q_i = i-th charge, $V_i(r_i)$ = potential due to q_i,

$\vec{R}_i$ = position of q_i, and

$r_i = |\vec{r} - \vec{R}_i| =$ distance from q_i.

$\therefore$ Net potential at a particular location is the arithmetic (not vector!) sum of the potentials from each individual point charge in the system.

Potential for a Continuous Charge Distribution

- For this case, summation becomes integration, charge increments dq' create the total system electrical potential at position (x, y, z):

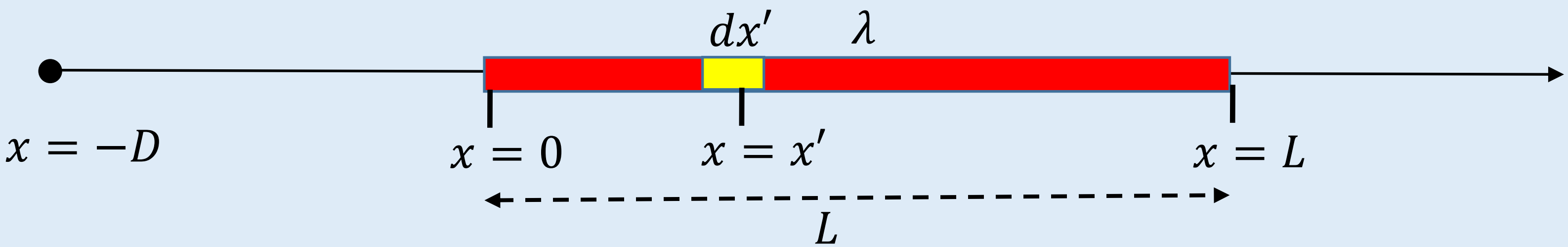
$$\boxed{V(x,y,z) = k \int \frac{dq'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}}.$$

Charge dq' is located at (x', y', z'), and integration is performed over the spatial extent of the charge distribution.

- Object with linear charge density λ: length element ds' has $dq' = \lambda\, ds'$.
- Surface with surface charge density σ: area element dA' has charge $dq' = \sigma\, dA'$.
- Volume with charge density ρ: volume element dv' has charge $dq' = \rho\, dv'$. (Note: using "v"here for volume, so as not to confuse it with potential "V".)

Example: Potential at a point a distance from the end of a uniform line charge

- Choose axis along the rod.
- Calculate potential from small segment of the rod.
- Sum up contributions from all segments. That is, integrate. Include limits.



Amount of charge in dx': $dq' = \lambda\, dx'$. Potential at $x = -D$ just from charge dq' at x' is $dV = \frac{k dq'}{r'} = k \frac{\lambda dx'}{D+x'}$.

Net potential at $x = -D$ (for constant λ) is:

$$V(x = -D) = \int dV = k \int_0^L \frac{\lambda\, dx'}{D + x'} = k\, \lambda \int_0^L \frac{dx'}{D + x'} = k\lambda \int_D^{D+L} \frac{du}{u}$$

$$\boxed{V(x = -D) = k\lambda \ln\left(\frac{D+L}{D}\right).}$$

Finding Field from the Potential

- For small changes in position, the variation in the electric potential $V(x, y, z)$ is:

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} \, dy + \frac{\partial V}{\partial z} \, dz \,.$$

- Additionally, we know for small changes,

$$dV = -\vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}} = -\left(E_x \, \hat{\boldsymbol{\imath}} + E_y \, \hat{\boldsymbol{\jmath}} + E_z \, \widehat{\boldsymbol{k}}\right) \cdot \left(dx \, \hat{\boldsymbol{\imath}} + dy \hat{\boldsymbol{\jmath}} + dz \, \widehat{\boldsymbol{k}}\right)$$

$$\Rightarrow dV = \; -E_x \, dx \; - E_y \, dy \; - E_z \, dz \,.$$

Equating expressions gives:

$$\boxed{E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z} \;.}$$

$\partial V / \partial x$ is the "partial derivative of the potential with respect to x",

$\Rightarrow$ differentiation with respect to x carried out while keeping the y and z values constant. Similar operations for $\partial V / \partial y$ and $\partial V / \partial z$.

Finding Field from the Potential (2)

- Expression for the electric field in terms of the potential can be written more compactly as

$$\boxed{\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}} V\ ,}$$

where

$$\boxed{\vec{\boldsymbol{\nabla}} \equiv \hat{\boldsymbol{\imath}} \frac{\partial}{\partial x} + \hat{\boldsymbol{\jmath}} \frac{\partial}{\partial y} + \hat{\boldsymbol{k}} \frac{\partial}{\partial z}}$$

is known as the "gradient" (or, "gradient operator").

Question 4.1

- The electric potential in a region of space is given by the function

$$V(x, y, z) = 3x + 2y^2 + 5.$$

The resulting electric field for this potential is given by

A. $\frac{3}{2}x^2\hat{\boldsymbol{\imath}} + \frac{2}{3}y^3\hat{\boldsymbol{\jmath}} + 5z\widehat{\boldsymbol{k}}$.

B. $3\hat{\boldsymbol{\imath}} + 4y\hat{\boldsymbol{\jmath}} + 0\widehat{\boldsymbol{k}}$.

C. $-\frac{3}{2}x^2\hat{\boldsymbol{\imath}} - \frac{2}{3}y^3\hat{\boldsymbol{\jmath}} - 5z\widehat{\boldsymbol{k}}$.

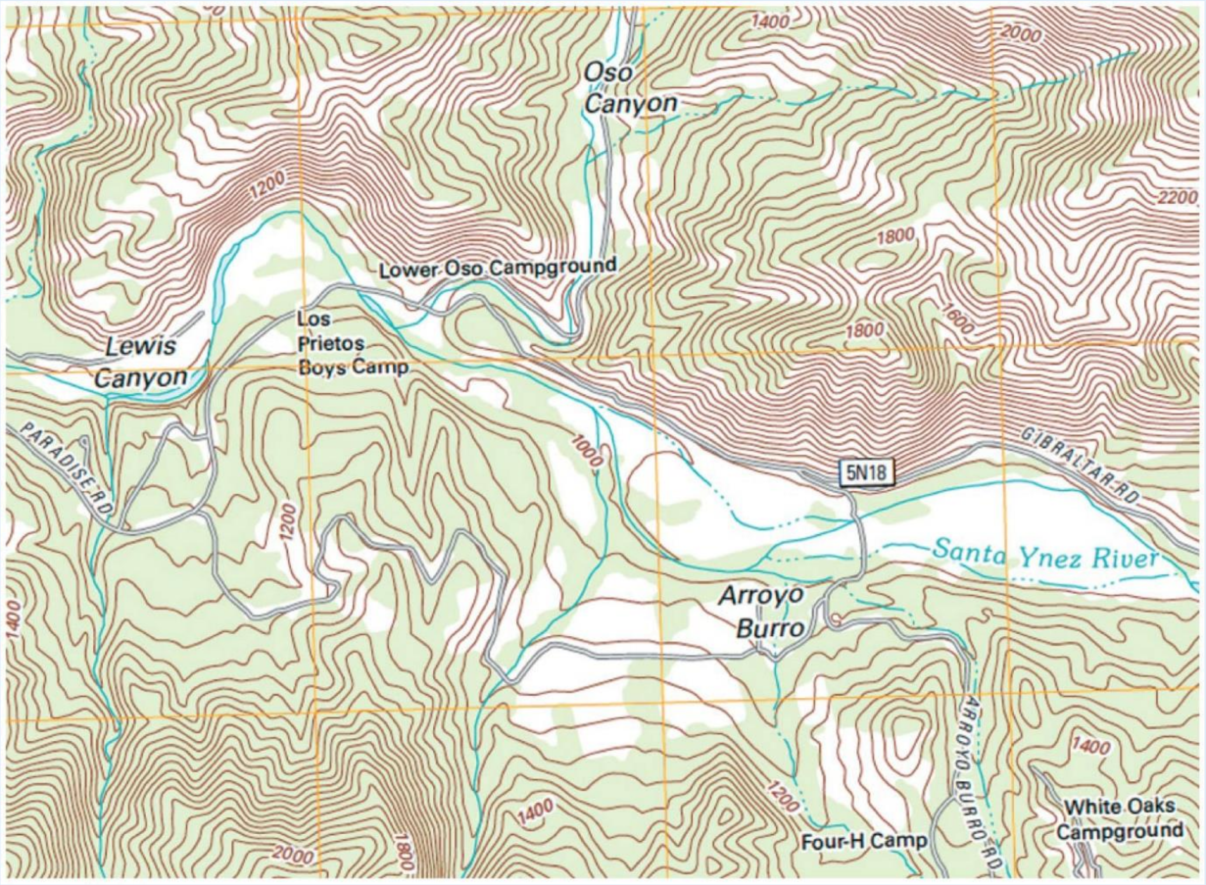
D. $3x\hat{\boldsymbol{\imath}} + 2y^2\hat{\boldsymbol{\jmath}} + 5\widehat{\boldsymbol{k}}$.

E. $-3\hat{\boldsymbol{\imath}} - 4y\hat{\boldsymbol{\jmath}} - 0\widehat{\boldsymbol{k}}$.

Equipotential Surfaces

- An equipotential surface is a surface that has a constant value of potential.

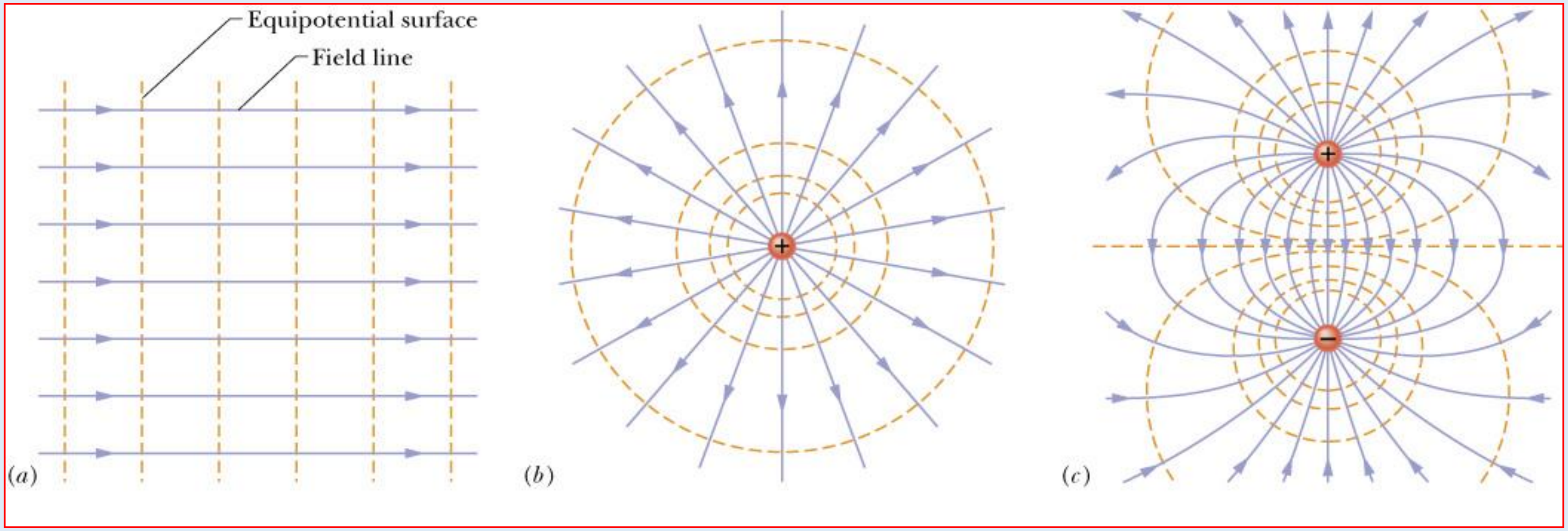
- Example: contour lines on a topographic map are curves of constant elevation = curves of constant gravitational potential energy.



Equipotential Surfaces (2)

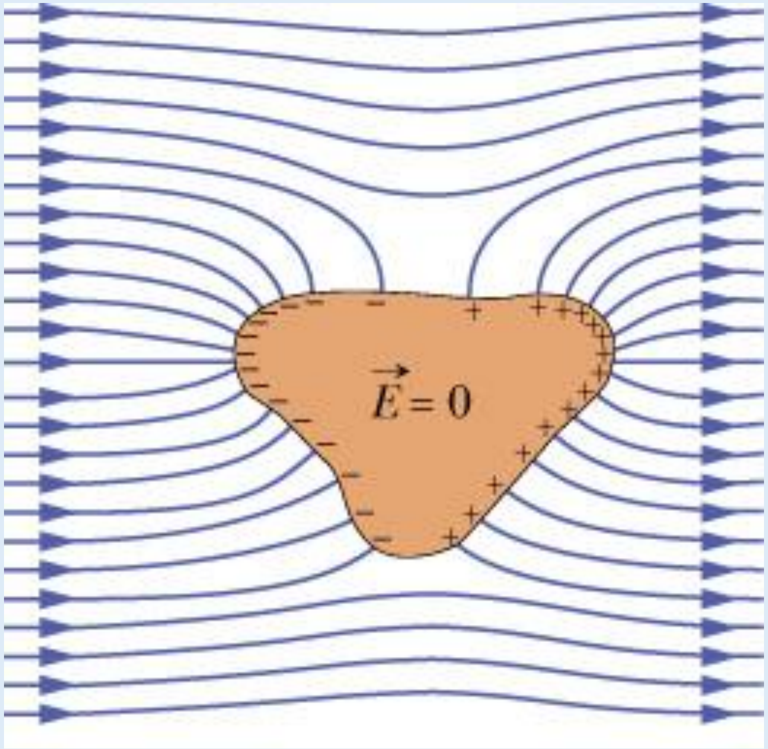
- From the relation between change in electric potential and the path integral of electric field, find that there is no electric field along an equipotential surface.

  $\therefore$ <u>The electric field is always perpendicular to an equipotential surface.</u>



Conductors and Equipotentials

- When all charges are at rest (electrostatic situation):
  - ➢All points of a conductor are at the same potential.
  - ➢The electric field just outside the surface of a conductor is always perpendicular to the surface.



Electrostatic Potential Energy of a System

- Electrostatic potential energy of system of two charges is the 'assembly energy' of the system = work to bring one charge in the potential field of the other in from 'infinity' to a final distance r between them.

- For two charges q_1 and q_2 separated by a distance r_{12}, the electrostatic potential energy of the system is:

$$U_E = q_1 V_2 = q_1 \left(k \frac{q_2}{r_{12}} \right) = k \frac{q_1 q_2}{r_{12}} \ ,$$

or, equivalently, because distance $r_{21} = r_{12}$,

$$U_E = q_2 V_1 = q_2 \left(k \frac{q_1}{r_{21}} \right) = k \ \frac{q_1 q_2}{r_{21}} = k \ \frac{q_1 q_2}{r_{12}} \ .$$

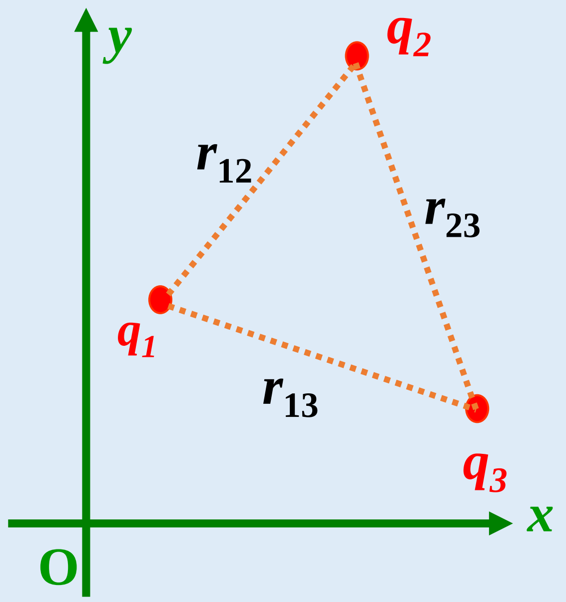
  - Electrostatic energy for two charge-system can be thought of being the result of bringing $q_1$in from infinity to r_{12} in the field of q_2, or q_2 in from infinity to r_{21} in the field of q_1. Result is the same for both cases.

- Must include signs of the charges! System energy can be positive, negative or zero, depending on charge signs.

Electrostatic Potential Energy of a System (2)

- For systems with more than two charges, idea of 'assembly energy' still holds.
- Calculate work done by bringing in the first charge from infinity in the presence of zero (i.e., no) charges. Then calculate work of bringing a second charge in from infinity in the field of the first. Then the work done by bringing in a third charge from infinity in the fields of the first and second. And so on…
- For a three-charge system this yields total system energy:

$$U_E = k\frac{q_1 q_2}{r_{12}} + k\frac{q_1 q_3}{r_{13}} + k\frac{q_2 q_3}{r_{23}}\,.$$



<u>Note: total electrostatic potential energy of a system of point charges is the sum of the potential energy of all of the unique, individual pairs of charges.</u>

Electrostatic Potential Energy of a System (3)

- For a system of N charges, the total electrostatic potential energy of the system can be calculated in two different (but equally valid) ways:

$$\text{(A)} \quad U_E = \sum_{i=1}^{N} \sum_{j<i} k \, \frac{q_i q_j}{r_{ij}} \; .$$

$$\text{(B)} \quad U_E = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} k \, \frac{q_i q_j}{r_{ij}} \;, \; j \neq i.$$

- Factor of ½ in (B) corrects for double-counting in summation of electrostatic energy pairs (e.g., $\frac{q_1 q_2}{r_{12}} = \frac{q_2 q_1}{r_{21}}$).
- Exclusion of $j = i$ in summation prevents occurrence of terms $r_{11} = 0$, $r_{22} = 0$, etc. , which would lead to divergences (i.e., infinities) in the electrostatic potential energy.
- For either (A) or (B), still find that total energy is equal to the total sum of each unique, individual pair of charges.

The Electron Volt

- From defining relation, change in potential energy ΔU_E for a charge q having a change in potential ΔV is

$$\Delta U_E = q\Delta V\,.$$

- If a charge $q = e$ has a potential change of 1 V,

$$\Delta U_E = (1\;e)(1\;\mathrm{V}) = (1.602 \times 10^{-19}\mathrm{C})(1\;\mathrm{V}) = 1.602 \times 10^{-19}\;\mathrm{J}.$$

- The unit of energy known as the electron volt (eV) is defined to be

$$\boxed{1\;\mathrm{eV} = 1.602 \times 10^{-19}\;\mathrm{J}\,.}$$

  - This unit appears frequently in many fields. It's the natural energy scale for atomic and molecular physics, chemistry, and biochemistry.

- The keV (kilo-electron volt $= 10^3\;\mathrm{eV}$) and the MeV (mega-electron volt $= 10^6\;\mathrm{eV}$) are often used in x-ray and medical physics, nuclear physics and engineering, and particle physics.

Question 4.2

- The 'classical' model of the hydrogen atom has an electron ($q_{\mathrm{e}} = -e = -1.60 \times 10^{-19}\mathrm{C}$) in a circular orbit of radius $r = 5.30 \times 10^{-11}\mathrm{m}$ about a fixed central proton ($q_{\mathrm{p}} = +e$). Calculate the electrostatic potential energy (in eV) of the 'classical' hydrogen atom. Use $k = 9.0 \times 10^{9}\mathrm{N}\frac{\mathrm{m}^2}{\mathrm{C}^2}$.

A. 13.6 eV.

B. –27.2 eV.

C. –13.6 eV.

D. –3.34 eV.

E. –1.51 eV.

F. 27.2 eV.

Example Problem 4.1

An object with charge $q = -4.00\ \mathrm{nC}$ is placed in a region of uniform electric field $\vec{\boldsymbol{E}} = -E\hat{\boldsymbol{\imath}}$ and is released from rest at point *A* on the *x*-axis. After the charge has moved to point *B* ($+0.500\ \mathrm{m}$ to the right of point *A*) it has a kinetic energy $5.00 \times 10^{-7}\ \mathrm{J}$.

1. If the electric potential at point *A* is $+30.0\ \mathrm{V}$, what is the electric potential at point *B*?

2. What is the magnitude of the electric field $\left|\vec{\boldsymbol{E}}\right|$?

Example Problem 4.2

A total electric charge of +2.00 nC is distributed uniformly over the surface of a metal sphere with a radius of 30.0 cm . The potential is zero at a point at infinity.

1. Find the potential 50.0 cm from the center of the sphere.

2. Find the potential 25.0 cm from the center of the sphere.