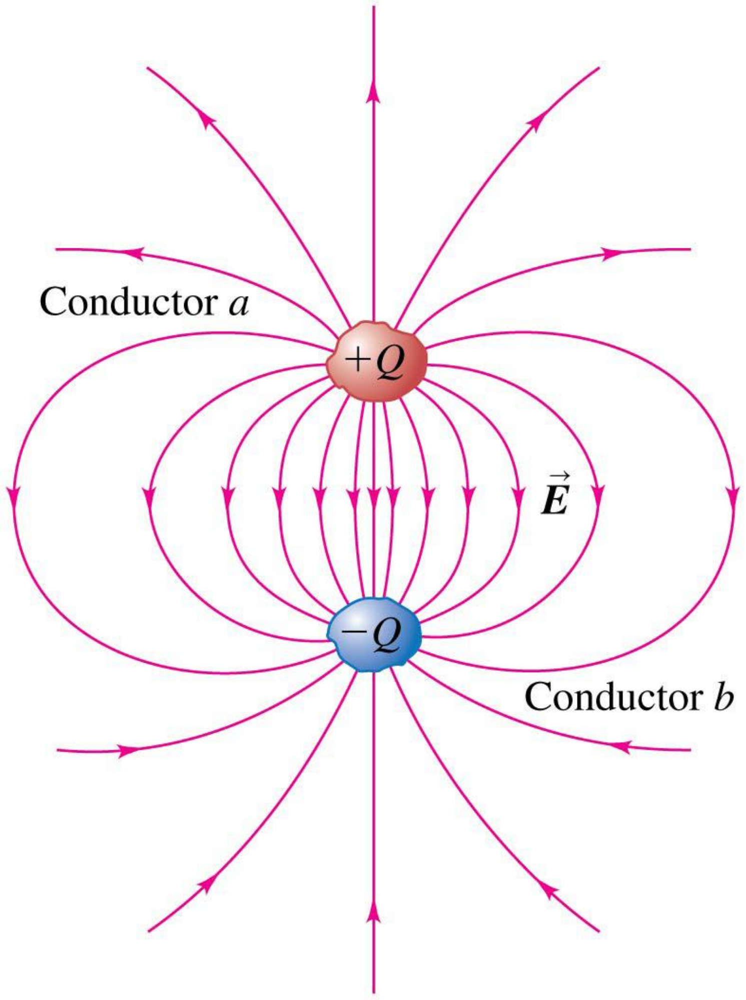
Physics 1200
Lecture 05
Spring 2024

Capacitance, Capacitors in Circuits, Energy Storage, Electric Field Energy, & Dielectrics

Capacitors and Capacitance

- A <u>capacitor</u> is a device that stores energy when an electric potential is applied it. Any two conductors separated by an insulator (or a vacuum) form a capacitor.
- When a capacitor is <u>charged</u>, the two conductors have charges with equal magnitude and opposite sign, and the net charge on the whole capacitor is zero. The charge Q of the capacitor is the charge on the positively-charged conductor



Capacitors and Capacitance (2)

- Can charge a capacitor by connecting the two conductors to opposite terminals of a battery; this creates a potential difference ΔV between them that is equal to the voltage of the battery. (Note: voltage difference across a capacitor is also often written as V.)

- Changing the magnitude of charge on each conductor changes the potential difference between the conductors; however, the *ratio* of charge to potential difference does not change. This ratio is defined to be the capacitance of the capacitor:

$$\boxed{C \equiv \frac{Q}{\Delta V} \; .}$$

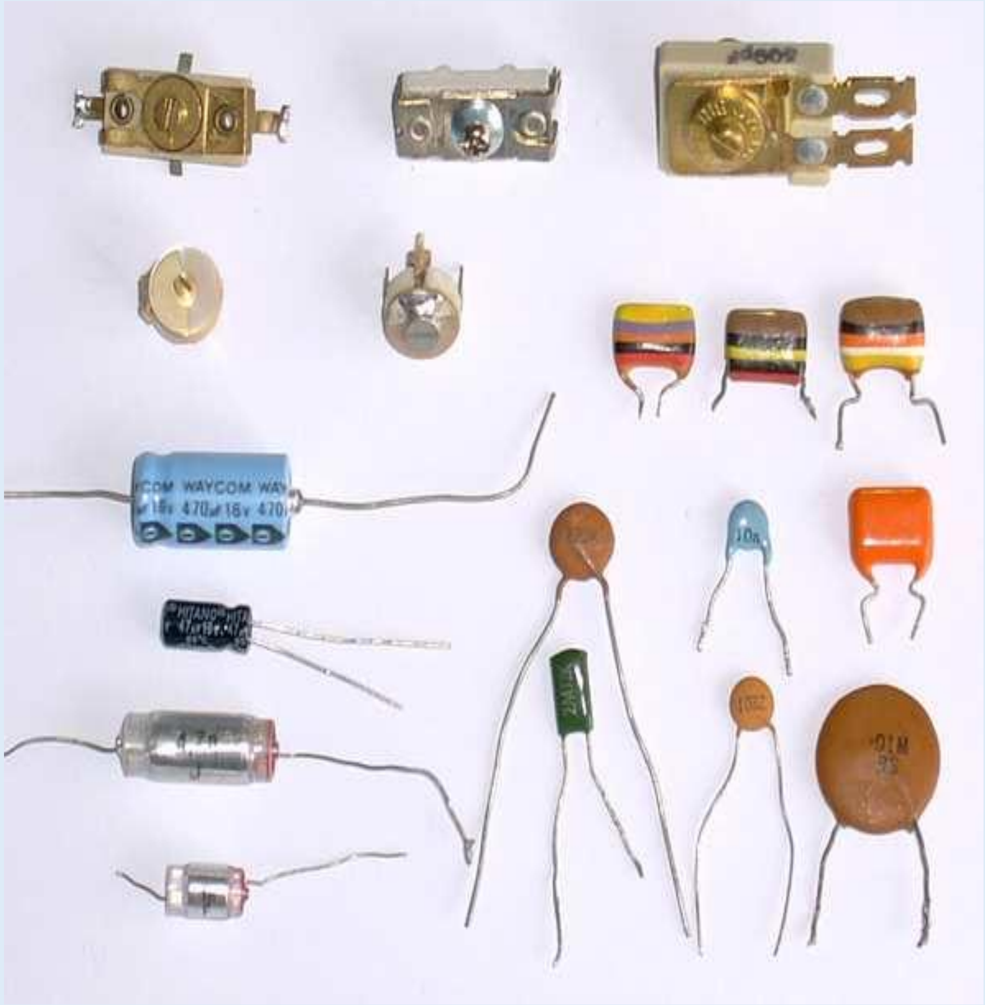
(Note: Q and ΔV in this expression are always positive. $\therefore$ C is also always positive.)

- SI unit of capacitance is the Farad: 1 F = 1 C/V = 1 C^2/J .

- 1 F is a large unit, because 1 C is a large unit.
  - Capacitors with capacitances of $1\ \mu\mathrm{F}\ (=10^{-6}\ \mathrm{F})$, $1\ \mathrm{nF}\ (=10^{-9}\ \mathrm{F})$, and $1\ \mathrm{pF}\ (=10^{-12}\ \mathrm{F})$ are common.

Capacitors and Capacitance (3)

- Some common electrical circuit capacitors.



Calculating Capacitance

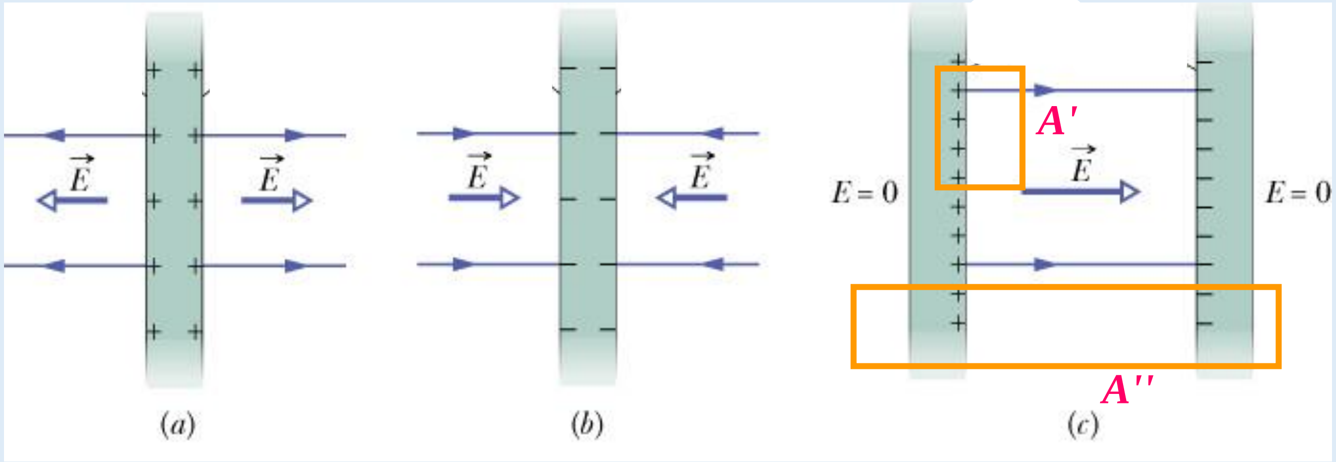
- Basic strategy: given capacitor charge Q, determine electric field $\vec{\boldsymbol{E}}$ between the conductors making up the capacitor.
- Use electric field to calculate the potential ΔV between the conductors, from the relation (last class):

$$\Delta V = V_{+} - V_{-} = -\int_{-}^{+} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}} \,,$$

“$+$” referring to the conductor at higher potential and “$-$" the conductor at lower potential.

- Finally, use definition $C = \frac{Q}{\Delta V}$.

Example: Parallel Plate Capacitor (2)

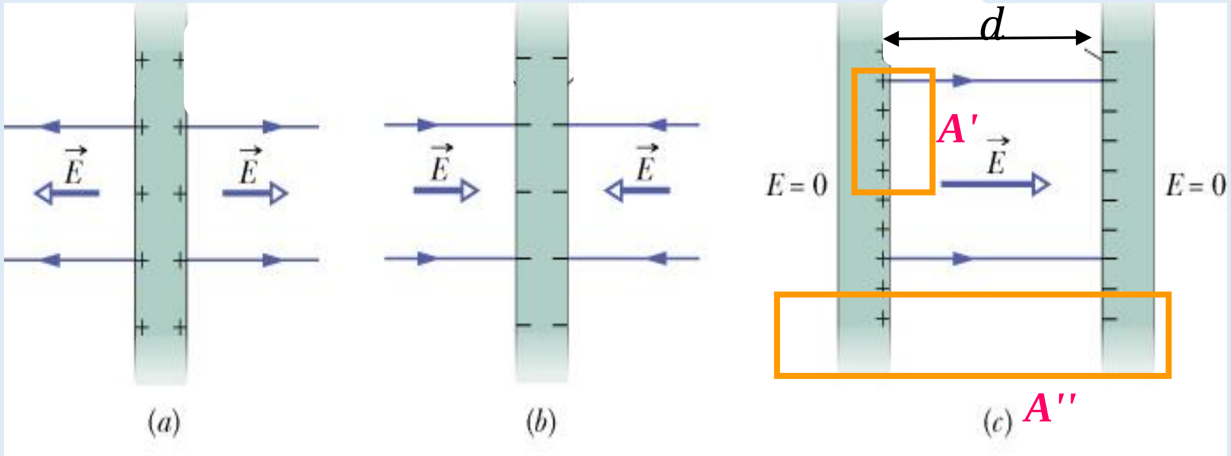


- Charge on one plate is drawn to the other.
- Inside metal: $\vec{\boldsymbol{E}} = 0$.
- Outside plates: $\vec{\boldsymbol{E}} = 0$.
- Between plates:

$$\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \frac{Q_{\text{enc}}}{\epsilon_0} \Rightarrow EA' = \frac{Q'}{\epsilon_0} \Rightarrow E = \frac{Q'}{\epsilon_0 A'} = \frac{\sigma A'}{\epsilon_0 A'} = \frac{\sigma}{\epsilon_0}$$

(Surface charge density σ is uniform when charge is evenly distributed over the inner-facing surfaces.)

Example: Parallel Plate Capacitor (3)



$$\Delta V = V_+ - V_- = -\int_d^0 \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}}$$

$$= -\int_d^0 (E\hat{\boldsymbol{\imath}}) \cdot (dx\hat{\boldsymbol{\imath}}) = -\int_d^0 \left(\frac{\sigma}{\varepsilon_0}\hat{\boldsymbol{\imath}}\right) \cdot (dx\hat{\boldsymbol{\imath}}) = \frac{\sigma}{\varepsilon_0}\int_0^d dx = \frac{\sigma d}{\varepsilon_0} = \frac{Qd}{\varepsilon_0 A}$$

$$C = \frac{Q}{\Delta V} = \frac{Q}{\frac{Qd}{\varepsilon_0 A}} = \frac{\epsilon_0 A}{d} \quad \Rightarrow \boxed{C = \frac{\varepsilon_0 A}{d}.}$$

C is <u>independent</u> of Q and ΔV.

<u>It depends only on the dimensions (i.e., the geometry) of the capacitor.</u>

Question 5.1

The charge on the square plates of a parallel-plate capacitor is 2.5 μC. The potential across the plates is kept at a constant voltage of 10 V by a battery as they are pulled apart to three times their original separation, which is small compared to the dimensions of the plates. The amount of charge on the plates is now equal to:

A. 0.83 μC.

B. 0.25 μC.

C. 7.5 μC.

D. 0.0 μC.

E. 2.5 μC.

F. Not enough information given.

Example: Spherical Capacitor

- Spherically symmetric system of metal shell having a charge $-q$ on its inner surface, surrounded by smaller interior conducting sphere of charge $+q$.
- Gauss's law:

$$\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = EA = E(4\,\pi r^2) = \frac{q}{\varepsilon_0}$$

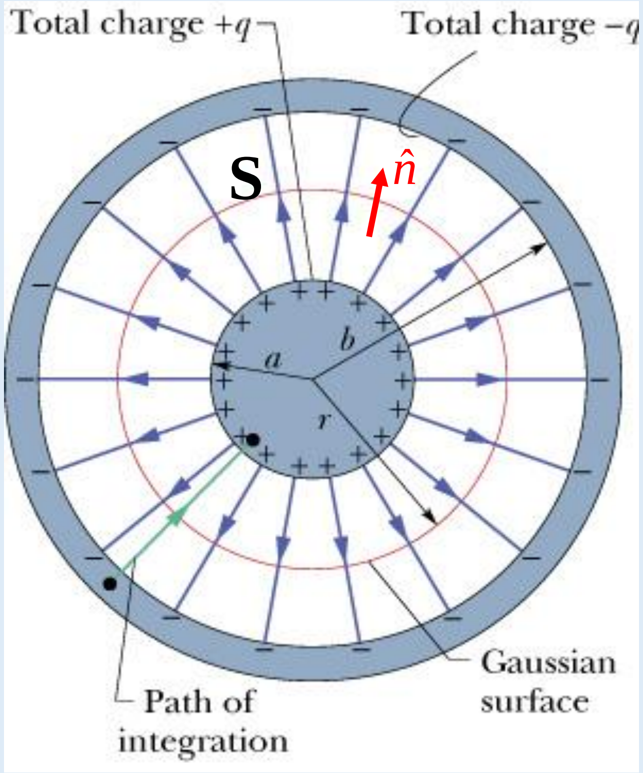
$$\Rightarrow E = \frac{q}{4\pi\varepsilon_0 r^2}$$ for field between conductors.

- Potential difference between outer shell and inner sphere is

$$\Delta V = V_a - V_b = -\int_b^a \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}} = \int_a^b \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}}$$

$$= \int_a^b E\hat{\boldsymbol{r}} \cdot dr\hat{\boldsymbol{r}} = \int_a^b \frac{q}{4\pi\varepsilon_0 r^2} dr$$

$$\Delta V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$



$$\Rightarrow \boxed{C = \frac{Q}{\Delta V} = \frac{q}{\left[\frac{q}{4\pi\varepsilon_0\left(\frac{1}{a} - \frac{1}{b}\right)}\right]} = \frac{4\pi\varepsilon_0}{\left(\frac{1}{a} - \frac{1}{b}\right)}}$$

C again depends only on dimensions of the capacitor!

Example: Capacitance of a Van de Graaff Generator Sphere

- The top sphere of a Van de Graaff generator can be treated as the inner sphere of a spherical capacitor having outer shell at $b \to \infty$.
- For that case, spherical capacitance relation gives

$$C = 4\pi\varepsilon_0 a.$$

- Physics 1200 owns a Van de Graaf sphere that has

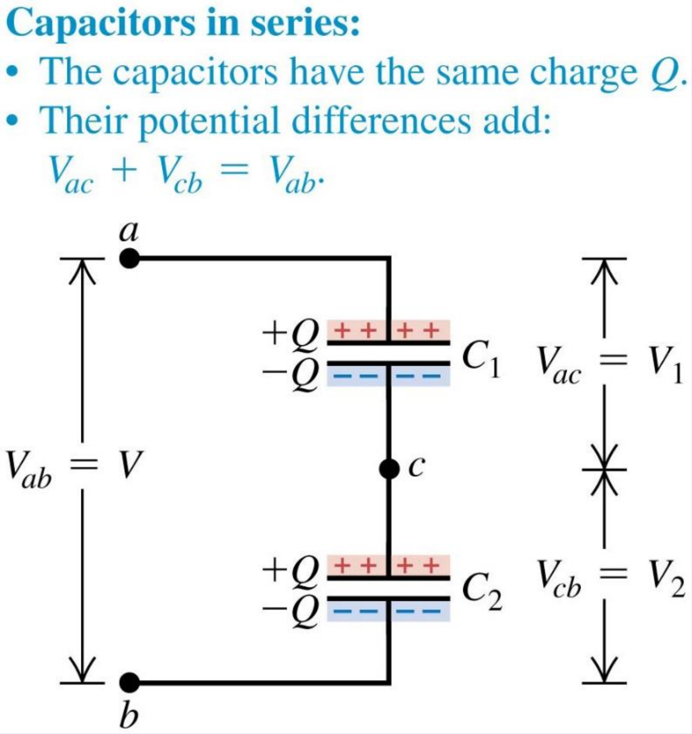
$a \approx 0.15\ m$, and

$$C = 4\pi\left(8.85 \times 10^{-12}\,\frac{\mathrm{F}}{\mathrm{m}}\right)(0.15\ \mathrm{m})$$
$$= 1.67 \times 10^{-11}\mathrm{F} = 16.7\ \mathrm{pF}.$$

- If generator sphere is raised to potential $\Delta V = 10^5\mathrm{V}$, charge on the sphere is $Q = C\Delta V = 1.67 \times 10^{-6}\ \mathrm{C}$.

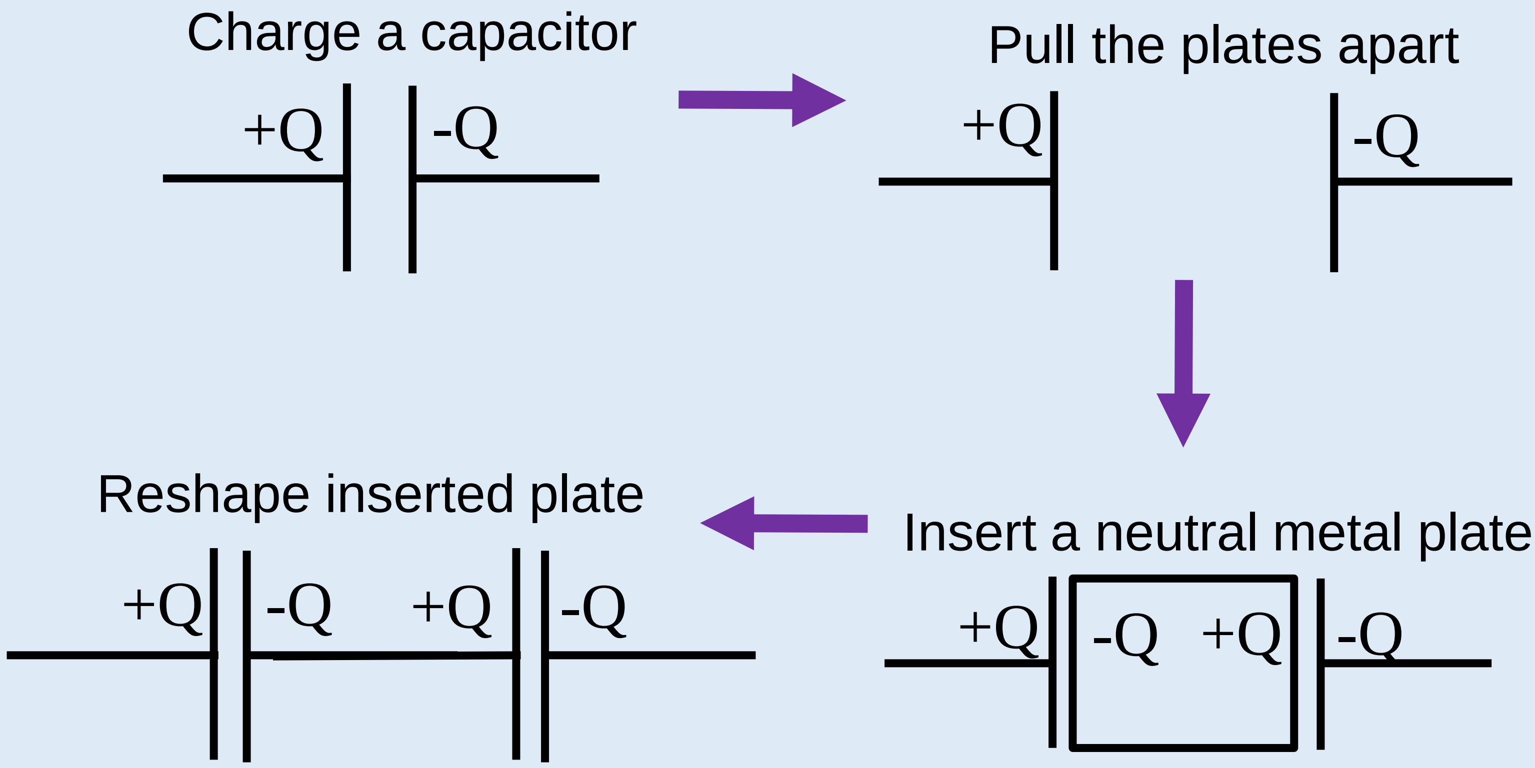
Capacitors in Series

- Capacitors are in series if they are connected one after the other. For that case, the same current that flows through one will flow through all the others.



Capacitors in Series (2)

- Capacitors in series have the same charge:



Capacitors in Series (3)

- The total potential across a series of capacitors is given by

$$V = V_1 + V_2 + V_3 + \cdots$$

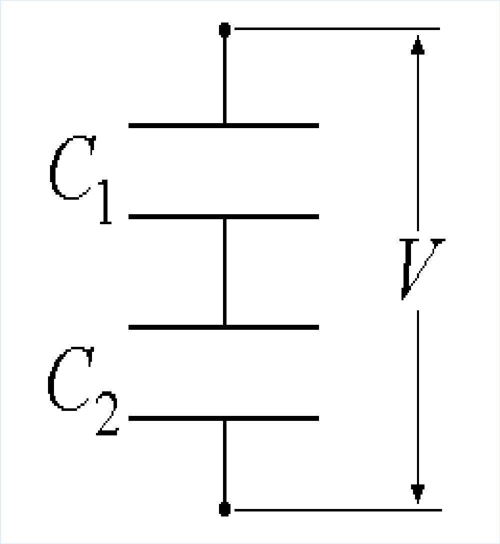
$$V = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} + \cdots$$

(Charge on all capacitors is the same when they're in series.)

- Dividing by Q on both sides of equation gives:

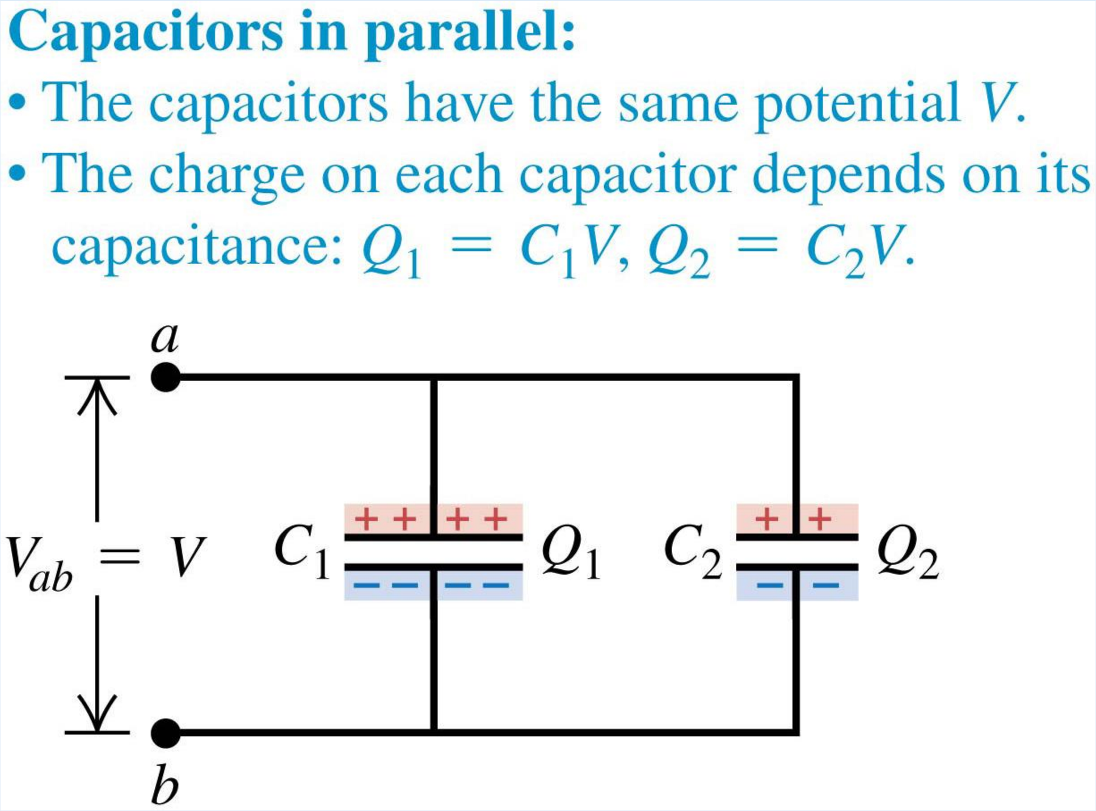
$$\boxed{\frac{V}{Q} = \frac{1}{C_{eff,s}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots}$$

for series capacitors. <u>A network of series capacitors can be replaced by a single effective capacitor with value</u> $C_{eff,s}$.



Capacitors in Parallel

- Capacitors are connected in <u>parallel</u> between points *a* and *b* if the potential difference V_{ab} is the same for all the capacitors.



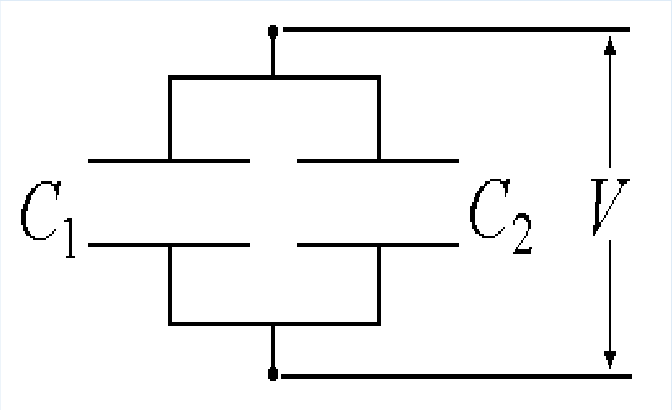
Capacitors in Parallel (2)

- Total charge on plates of parallel capacitors:

$$Q = Q_1 + Q_2 + Q_3 + \cdots$$

$$Q = C_1 V + C_2 V + C_3 V + \cdots$$

(Potential is the same for capacitors in parallel.)



- Dividing both sides of the relation by the common factor of the potential V gives

$$\frac{Q}{V} = \boxed{C_{eff,p} = C_1 + C_2 + C_3 + \cdots}$$

- Capacitors in parallel can be replaced by a single capacitor of effective capacitance $C_{eff,p}$.

Energy Storage by Capacitors

- Moving a small amount of charge dq from one plate of a capacitor to the other adds increment of potential energy to the system:

$$dU_E = dq\,V = dq\,\left(\frac{q}{C}\right) = \frac{q}{C}\,dq\,,$$

where q is the charge on the other plate as it is being charged up (to a final charge Q). V is the potential difference between the two plates.

- Total energy added to system is the integral of the above, from there being no initial charge ($q = 0$) on the plate to a final charge Q:

$$\boxed{U_E = \int dU_E = \int_0^Q \frac{q}{C}\,dq = \frac{Q^2}{2C} = \frac{1}{2}\,QV = \frac{1}{2}CV^2.}$$

Question 5.2

An ideal parallel-plate capacitor has a set of two parallel plates of area A separated by a very small distance d. When the capacitor plates carry charges $+Q$ and $-Q$, the capacitor stores energy U_0. If the separation between the plates is doubled, and the charge kept constant, how much electrical energy is stored in the capacitor?

A) $4U_0$.

B) $2U_0$.

C) U_0.

D) $U_0/2$.

E) $U_0/4$.

Energy Density of the Electric Field

- The electric energy density of energy stored in a capacitor is:

$$u_E = U_E/v,$$

$v =$ volume of capacitor.

- Parallel-plate capacitor: $v = Ad$, $A =$ plate area, $d =$ plate separation distance.
- Parallel-plate capacitor:

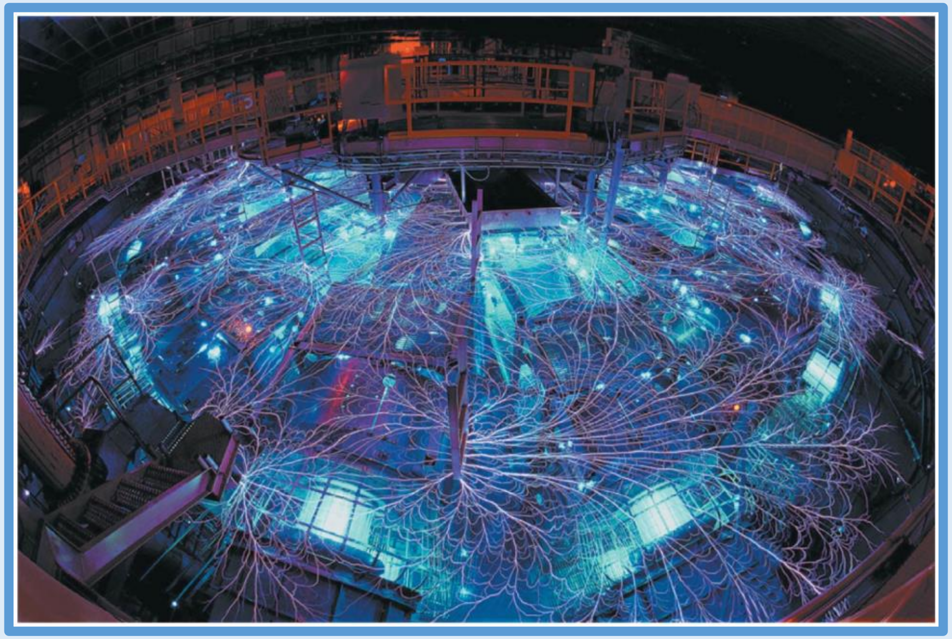
$$u_E = \frac{1}{2}\frac{CV^2}{v} = \frac{1}{2}\frac{\left(\frac{\varepsilon_0 A}{d}\right)(Ed)^2}{Ad} = \frac{1}{2}\varepsilon_0 E^2, \qquad \Rightarrow \boxed{u_E = \frac{1}{2}\varepsilon_0 E^2}.$$

Used potential difference $V = Ed$ for the uniform electric field E between the plates of a parallel-plate capacitor

- Electric energy density is characteristic of all electric fields, even for situations that don't involve capacitors. Can consider it stored field energy per unit volume. In this picture, it "belongs" to the electric field.
- SI units of energy density are: $\frac{\mathrm{J}}{\mathrm{m}^3} = \frac{\mathrm{N}}{\mathrm{m}^2}$. (Last is same as for pressure.)

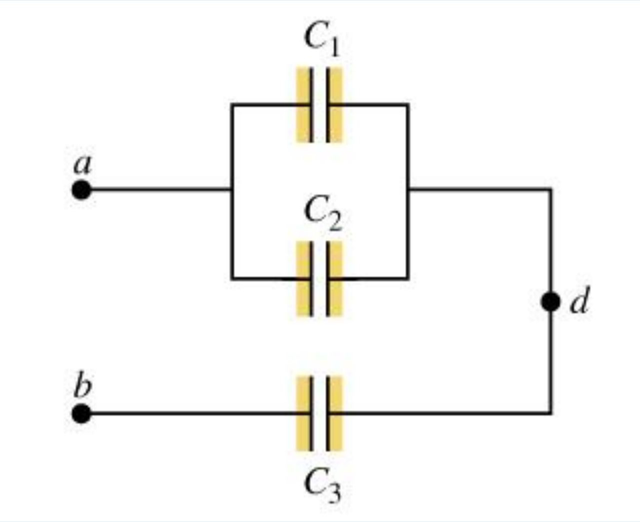
Energy stored in a capacitor

- Capacitors store energy and can release it quickly.
- Extreme example of this is the "Z machine" at Sandia National Laboratories in New Mexico, used in experiments for controlled nuclear fusion.
- Z machine uses a very large number of capacitors in parallel to give a huge equivalent capacitance.
- Arcs (shown) are produced when the capacitors discharge their energy into a target, which is heated to a temperature $T > 2 \times 10^9$ K.



Example Problem 5.1

- Given C_1 = 6.00 μF, C_2 = 3.00 μF, and C_3 = 5.00 μF. The capacitor network is connected to an applied potential V_{ab}. After the charges on the capacitors have reached their final values, the charge on C_2 is 30.0 μC.
- What is the charge on capacitor C_1?
- What is the charge on capacitor C_3?
- What is the applied voltage V_{ab}?



Example Problem 5.2

- The plates of a parallel-plate capacitor are 3.22 mm apart, and each has an area of 9.72 cm^2 . Each plate carries a charge of magnitude 4.40×10^{-8} C . The plates are in vacuum.
- What is the capacitance?
- What is the voltage across the capacitor?
- What is the magnitude of the electric field between the plates?

Dielectrics

- A dielectric is a material which responds to an applied electric field by polarizing.
- Polarization is the shift in position of charge in a material when an electric field is present.
- Most important dielectrics are insulators.
- The actual shift in position of charge in the dielectric is very small – less than one atom distance. Effect can be large.

Space-Filling Dielectric within a Parallel-Plate Capacitor

Polarization decreases electric field by collecting opposite charge at the edge of the material.

A dielectric medium has a "permittivity"

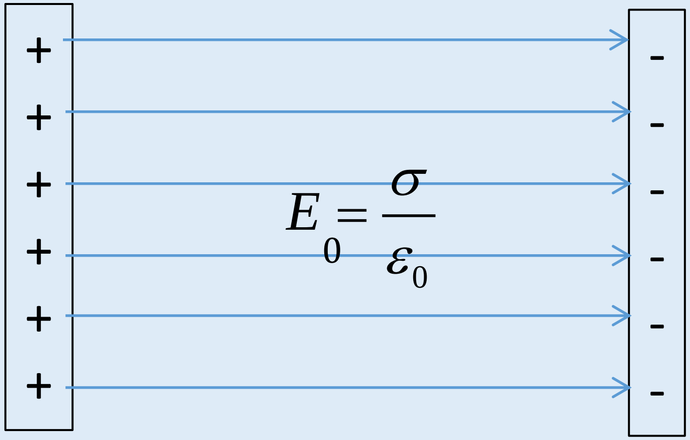
$$\varepsilon = \kappa\varepsilon_0 \; ,$$

where dimensionless constant $\kappa (\geq 1)$ is the dielectric constant of the medium.

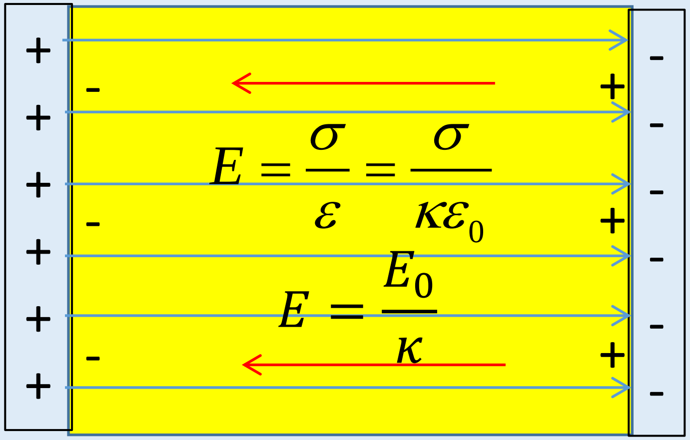
Vacuum (free space) has $\kappa = 1$,

$$\varepsilon = \varepsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F/m}.$$

No dielectric



Dielectric



Space-Filling Dielectric within a Parallel-Plate Capacitor (2)

For a parallel-plate capacitor,

$$Q_{\text{on a capacitor plate}} = CV_{\text{between plates}}$$

$$V = Ed = \left(\frac{\sigma}{\kappa\varepsilon_0}\right)d = \frac{Qd}{A\kappa\varepsilon_0}$$

$$Q = \sigma A$$

$$Q = CV = C\left(\frac{Qd}{A\kappa\varepsilon_0}\right)$$

$$C = \kappa\left(\frac{\varepsilon_0 A}{d}\right) = \kappa C_0$$

($C_0 = \frac{\varepsilon_0 A}{d}$ = capacitance when space between plates is vacuum.)

∴ Capacitance of a capacitor increases when a dielectric fills the space between its plates.