Physics 1200
Lecture 13
Spring 2024

Magnetic Flux, Faraday's Law, Lenz's Law, Electromagnetic Induction, Displacement Current, Maxwell's Equations

Magnetic Flux

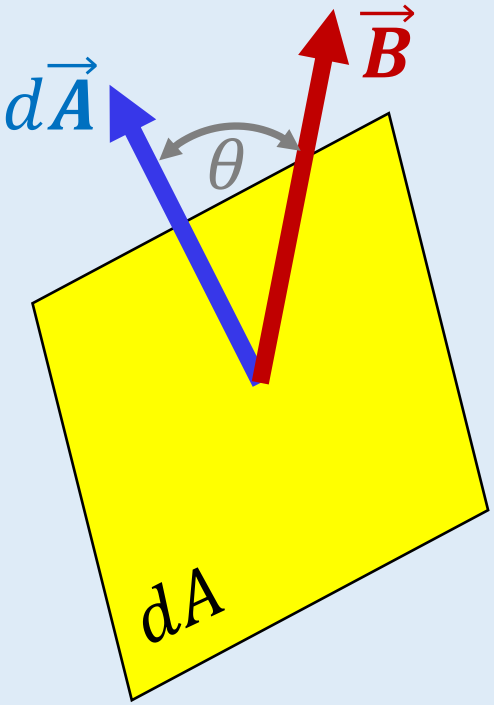
- Magnetic flux through a surface:

$$\boxed{\Phi_B = \int \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{A}} = \int B\, dA \cos\theta} ,$$

  where $\vec{\boldsymbol{B}}$ is magnetic field at location of element of area dA, having normal vector $d\vec{\boldsymbol{A}}$ at angle θ with respect to the field.

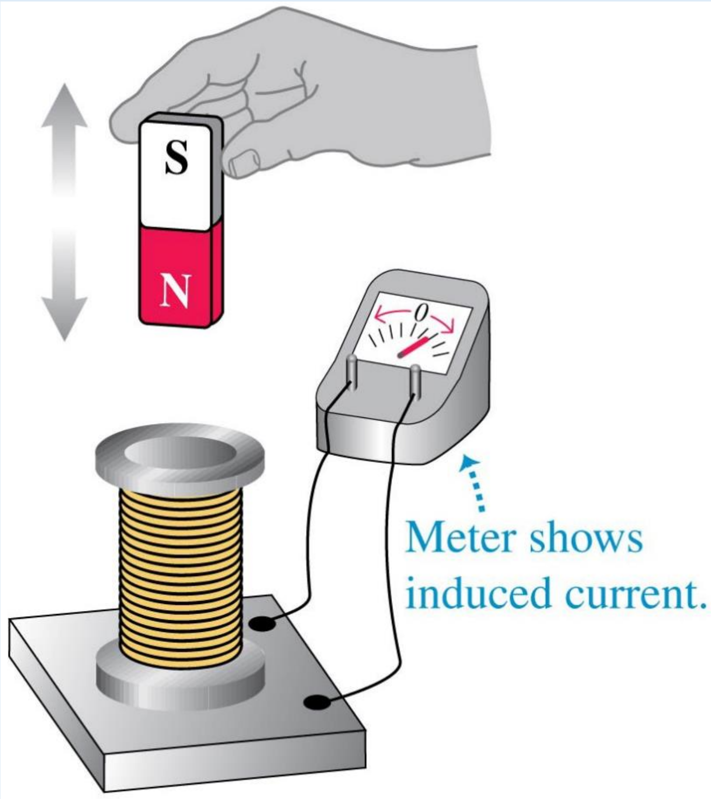
- Magnetic flux through a surface is a measure of number of magnetic field vectors/field lines passing through area of the surface.

- SI unit for magnetic flux is the Weber (Wb). 1 Wb = 1 T m^2.



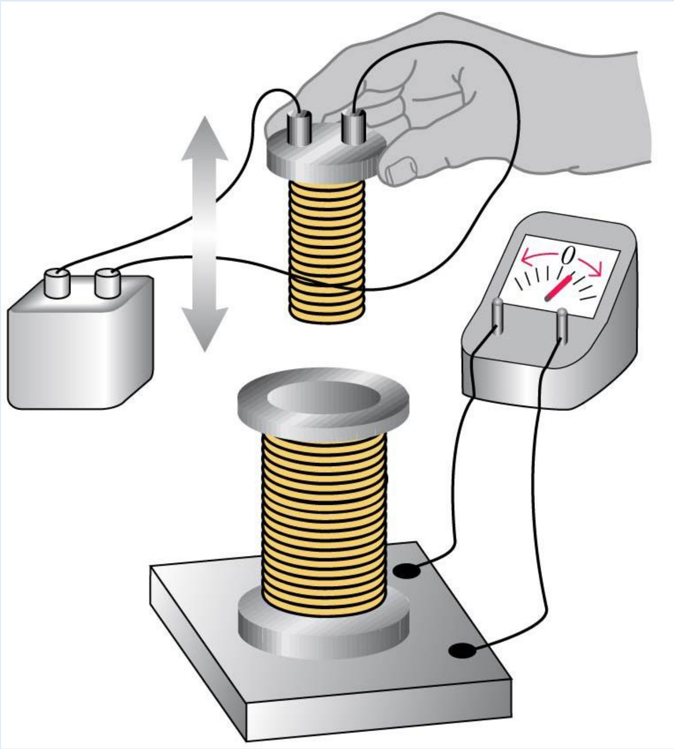
Time-Changing Magnetic Fields and Fluxes: Induced EMFs and Currents

- Experiments in 1830s by Michael Faraday and others found: time-changing magnetic field near a conducting loop, emf and current are generated (induced) inside the conductor.
- Experiment 1: changing magnetic field by moving a magnet relative to a stationary solenoid.
  - Holding magnet fixed, no current measured in solenoid.
  - Moving magnet up or down induces a current in the solenoid.
  - Direction of induced current depends upon whether magnet is moving upward or downward.



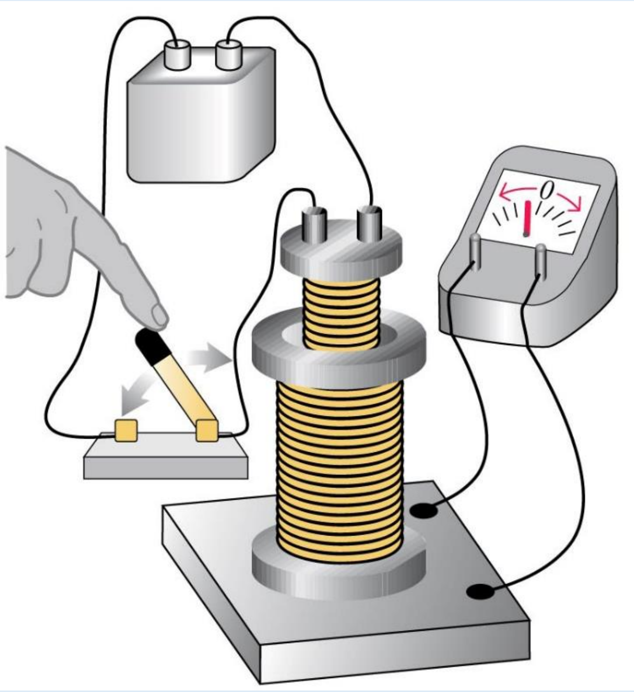
Time-Changing Magnetic Fields and Fluxes: Induced EMFs and Currents (2)

- Experiment 2: changing magnetic field by moving current-carrying solenoid relative to stationary solenoid.
  - Solenoid with battery current flowing through it creates an internal magnetic field, as discussed last class. (Created magnetic field in battery-connected solenoid is like a dipole magnet.)
  - Holding battery-connected solenoid at rest with respect to stationary solenoid: no current induced in stationary solenoid.
  - Moving battery-connected solenoid up or down: induces a current in the stationary solenoid.
  - Direction of induced current in stationary solenoid depends upon whether battery-connected solenoid is moving upward or downward.



Time-Changing Magnetic Fields and Fluxes: Induced EMFs and Currents (3)

- Example experiment 3: changing magnetic field by changing current in a battery-connected solenoid within an outer, unconnected solenoid.
  - Before closing switch: inner solenoid has no current flowing through it, and no magnetic field generated inside. No current measured in unconnected outer solenoid.
  - Instant switch is closed: an induced current flows in outer, unconnected solenoid.
  - Shortly afterward: induced current in outer solenoid goes to zero, even though a current still flows in the battery-connected, close-switched solenoid.
  - If switch is opened: the instant switch is opened a current is induced in outer, unconnected solenoid. Direction of induced current is <u>opposite</u> to what it was earlier.
  - Shortly afterward: induced current in outer solenoid goes to zero.



Repeating experiment with polarity of battery connections reversed results in same behavior, except that observed induced currents are in <u>opposite</u> direction!

Induced EMFs and Currents: Faraday's Law

- The results of the experiments are codified in Faraday's law:

$$\boxed{\mathcal{E} = \oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}} = -\frac{d\Phi_B}{dt}},$$

where $\mathcal{E}$ is induced emf, and the line integral involves an induced non-conservative electric field.

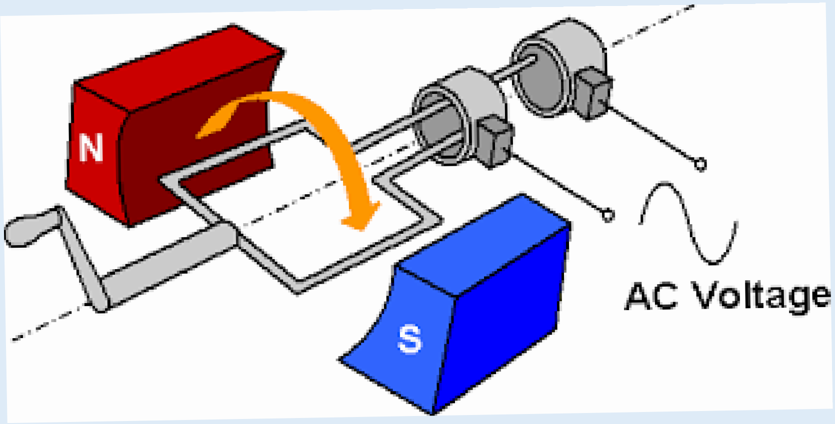
- Faraday didn't derive this law. (He wasn't too good at math.) However, others deduced the form of the law from his experiments.

- Note that $\frac{d\Phi_B}{dt} = \frac{d}{dt}\int \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{A}} = \frac{d}{dt}\int B\, dA\, \cos\theta$. This means that an emf can be induced by:

  - a time-changing magnetic field.
  - a time-changing area.
  - a time-changing angle between the field and the area.
  - combinations of the above.
  - Key is that at least one of these quantities must change over some time interval.

Induced EMFs and Currents: Faraday's Law (2)

- Line-integral bounding curve in Faraday's law is a path that encloses region of time-changing magnetic flux. It is independent of whether there is anything material in the path or not. It could be along a conducting loop, or in vacuum. Faraday's law says that there will be an emf along the path in either case.
- If bounding curve consists of a conducting loop, the emf induces a current in it.



➢This is how electrical current is generated in power stations: A conducting loop is rotated in a magnetic field (by mechanical action of some other agent, such as a steam driving a piston, or falling water driving a water wheel), causing the flux to change because the angle between the plane of the loop and the magnetic field changes with time. This creates an emf (a non-conservative electric field) in the loop that drives a current connected to transmission cables passing to the outside world.

Faraday’s Law for Multiple-Loop Coil

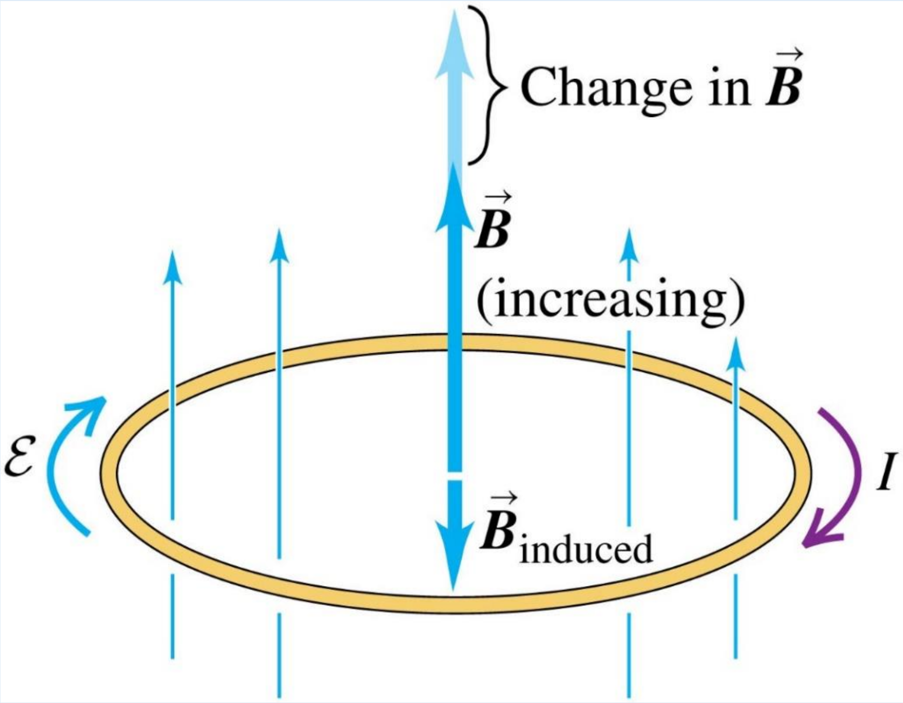
- If a coil has N identical turns, and if the flux varies at the same rate in each turn, the total emf induced across the entire coil is

$$\boxed{\mathcal{E}_{\text{tot}} = -N\frac{d\Phi_B}{dt}}\ ,$$

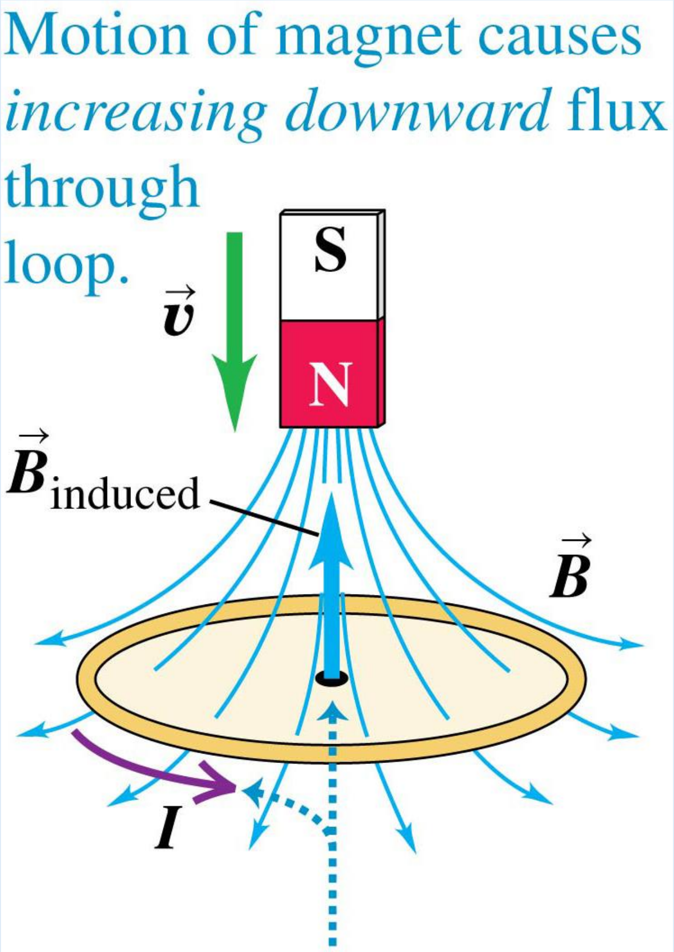
where Φ_B is the flux through a single loop.

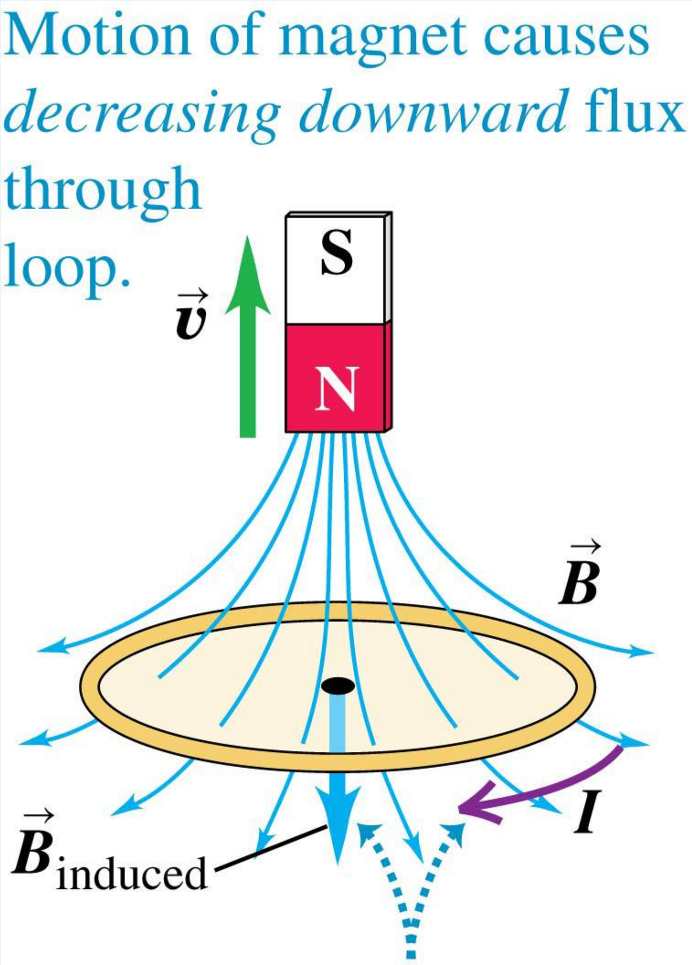
Lenz’s Law

- Lenz’s law is a corollary to Faraday’s law. States that <u>the emf induced by a changing magnetic flux induces a response that opposes the magnetic flux change</u>.
- If the Faraday loop is a conducting wire, induced current in the wire generates a magnetic field that opposes the externally-directed magnetic flux change in the wire:

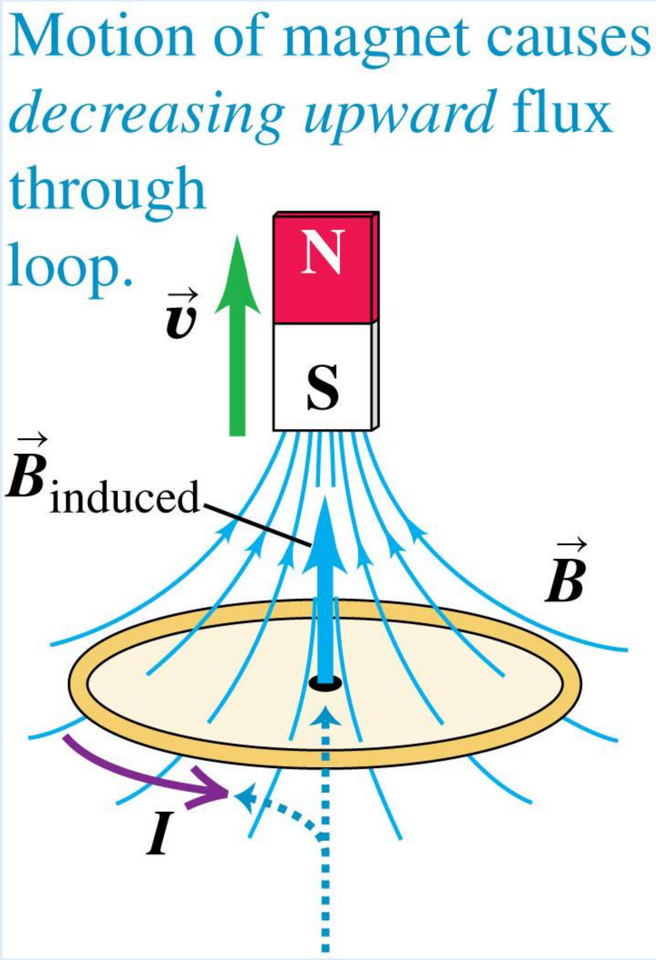


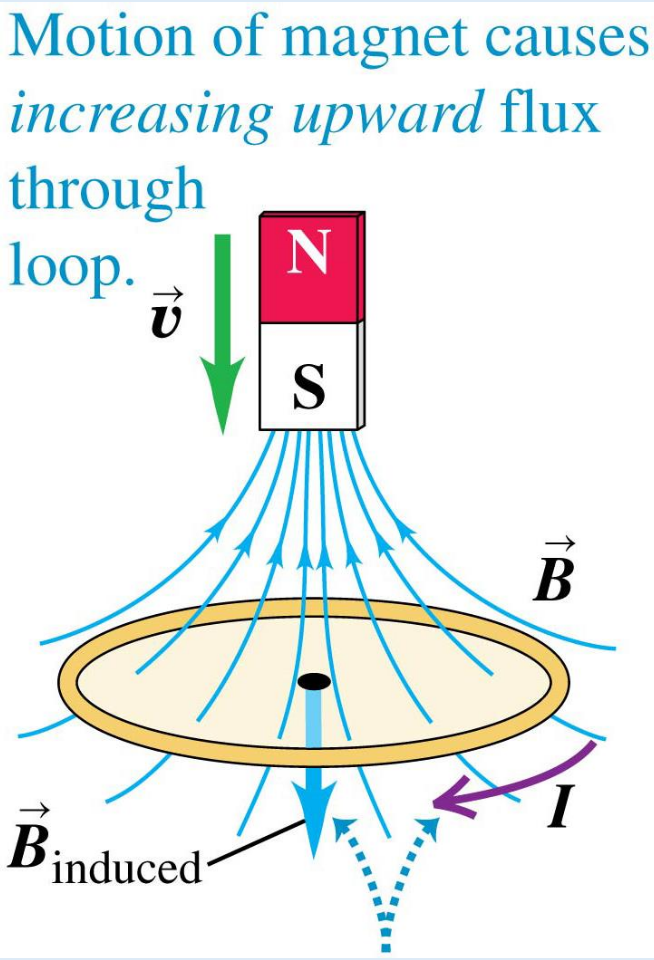
Lenz’s Law Examples





Lenz’s Law Examples (2)





Lecture Question 13.1

- A conducting loop is pulled out of a region of magnetic field at a <u>constant</u> velocity. For the instant of time shown in the figure, what is the direction of the induced current in the loop?

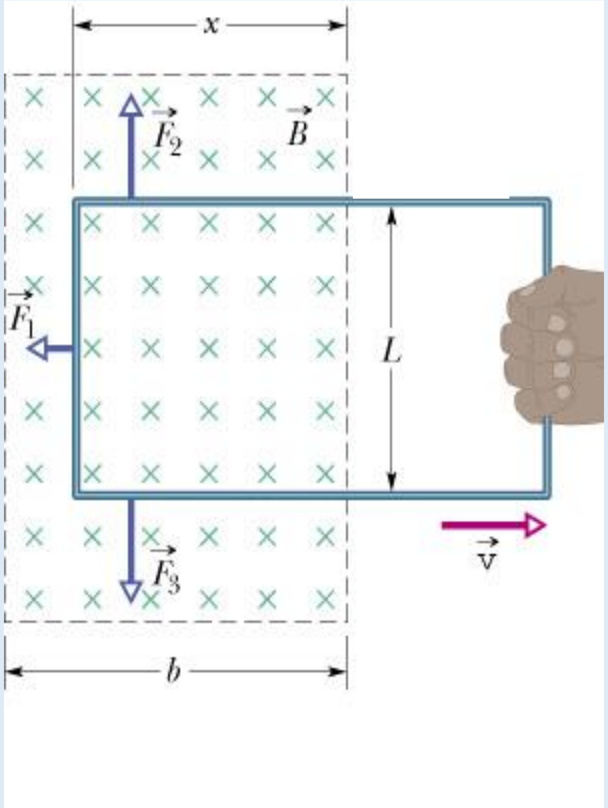
A. Clockwise.

B. Counterclockwise.

C. Out of the page.

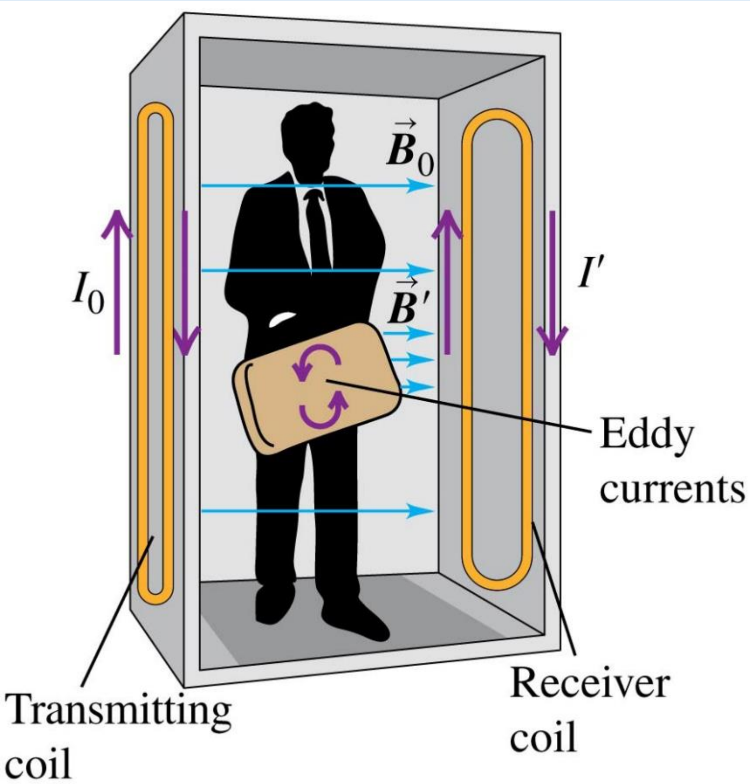
D. Into the page.

E. There is no induced current.



Eddy Currents

- When a piece of metal moves through a magnetic field or is within a changing magnetic field, eddy currents of electric current are induced in the metal.
- Metal detectors used at airport security checkpoints operate by detecting eddy currents induced in metallic objects.



Lecture Question 2

- A loop of area $5.0\ \mathrm{m}^2$ lies in a plane, as shown in the figure. In the region of space where the loop is located there is a magnetic field
$$\vec{\boldsymbol{B}} = [(2.0\hat{\boldsymbol{\imath}} + 3.0\hat{\boldsymbol{\jmath}})t^2]\frac{\mathrm{T}}{\mathrm{s}^2}.$$
- The magnitude of the induced emf in the loop at $t = 5.0\ s$ is

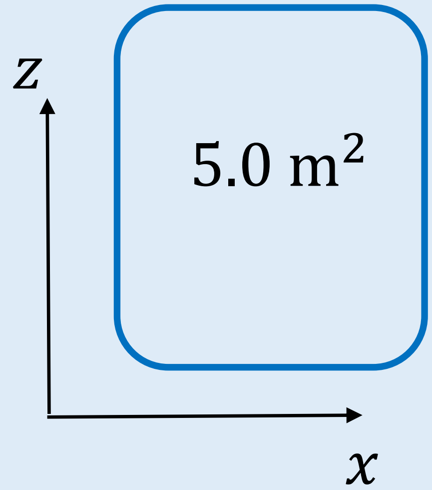
A. 0 V.

B. 100 V.

C. 125 V.

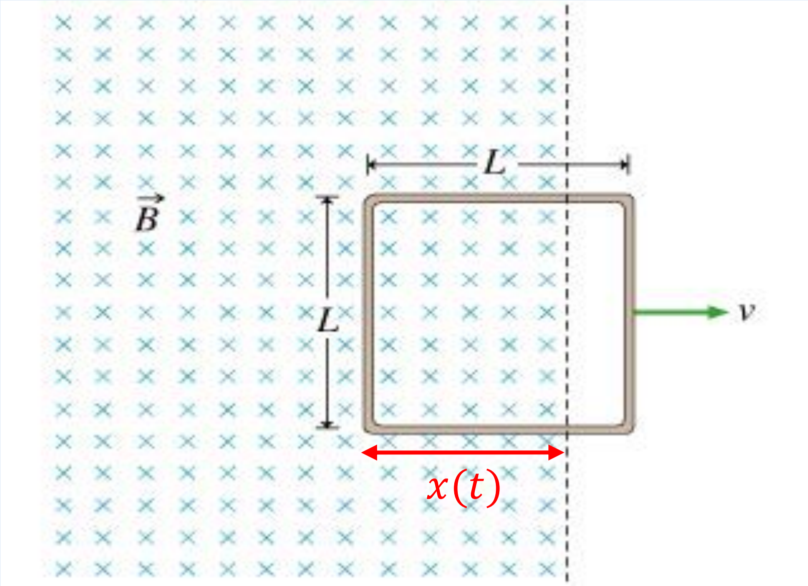
D. 150 V.

E. 375 V.



Example Problem

- A single square loop of side $L = 3.0\ \mathrm{cm}$ is pulled out of a region of magnetic field $B = 1.0\ \mathrm{T}$ at a speed $v = 2.0\ \mathrm{m/s}$. Loop has an internal resistance $R = 2.0\ \Omega$. Find the magnitude of the induced emf and current, and the direction of the induced current in the loop.



- At the instant of time shown, area of loop that has magnetic field through it is $A_{\mathrm{flux}} = Lx(t)$, where $x(t)$ is measured from the trailing left edge of the loop.
- At time shown, $\Phi_B = BA_{\mathrm{flux}} = BLx(t)$.
- Magnitude of emf: $|\mathcal{E}| = \left|\frac{d\Phi_B}{dt}\right| = \left|\frac{d}{dt}\ (BLx)\right| = BL\left|\frac{dx}{dt}\right|$

$$= BLv = (1.0\ \mathrm{T})(0.03\ \mathrm{m})\left(2.0\frac{\mathrm{m}}{\mathrm{s}}\right) = 0.060\ \mathrm{V}.$$

- For current, use $\mathcal{E} = IR \Rightarrow I = \frac{\mathcal{E}}{R} = \frac{0.060\ \mathrm{V}}{2.0\ \Omega} = 0.030\ \mathrm{A}$. Lenz's law: induced current will add downward magnetic field (to recover flux being lost in the coil). To create that field, right-hand 'thumb rule' yields the induced current must be clockwise in the coil.

Survey of Fundamental Electric and Magnetic Laws

- Up to this point, we have encountered the following fundamental relations that govern the behavior of electric and magnetic fields.
- Gauss's law for electric fields:

$$\boxed{\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \frac{Q_{enc}}{\epsilon_0}} \ .$$

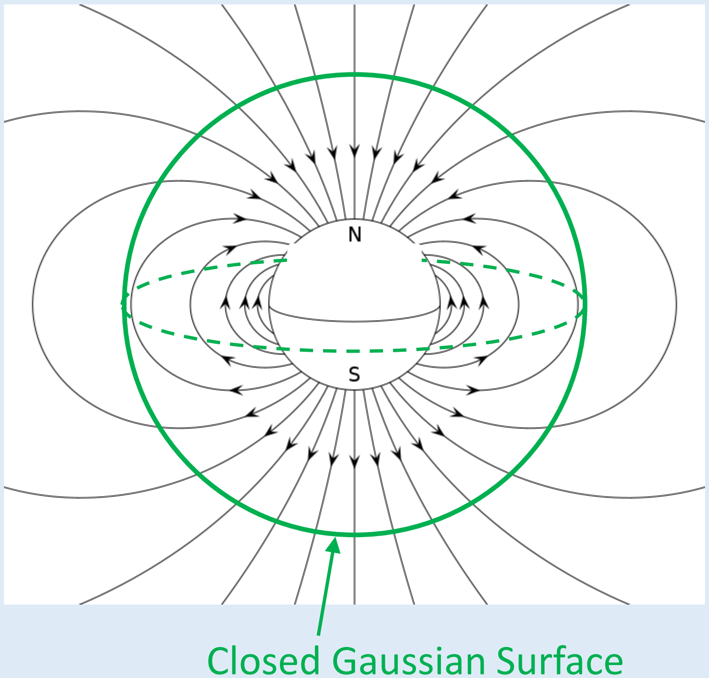
- We also came across an implied Gauss's law for magnetic fields. Fundamental relation expressed by Gauss's law for magnetic fields is simply a statement of the fact that there are no real monopole (i.e., "magnetic charge") sources for the magnetic field.

Gauss's Law for Magnetic Fields

- Like electric field case, Gauss's law can be derived for magnetic fields.
- Because there are no magnetic monopoles, magnetic field lines are always closed loops.
  - ➢ Has consequence that, for any closed Gaussian surface, just as many magnetic field lines go into the surface as come out.
  - ➢ ∴ Gauss's law for magnetic fields has the form

$$\boxed{\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{A}} = \oint B\, dA \cos\theta = 0\,.}$$

In words: The net magnetic flux through any closed surface is zero.



Faraday's Law

- Introduced this class, Faraday's law relates time change of magnetic flux to the line integral of electric field around a closed path:

$$\boxed{\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}} = -\frac{d\Phi_B}{dt}} \quad .$$

Ampere's Law

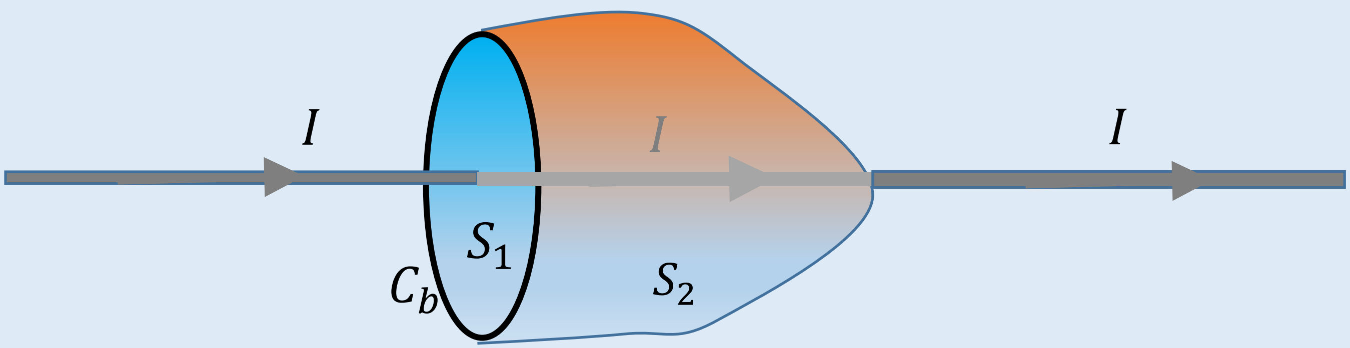
- Ampere's law was considered for cases of steady current flow. Valid for cases in which magnetic fields and current are not changing in time (i.e., steady):

$$\boxed{\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{l}} = \mu_0 I_{enc}}.$$

- In 1861-1862, while examining the electric-magnetic relations we've highlighted, James Clerk Maxwell (1831 – 1879) concluded that Ampere's law was <u>incomplete</u>. The law needed to be changed to account for situations when current and fields were not constant in time.

Incompleteness of Ampere's Law

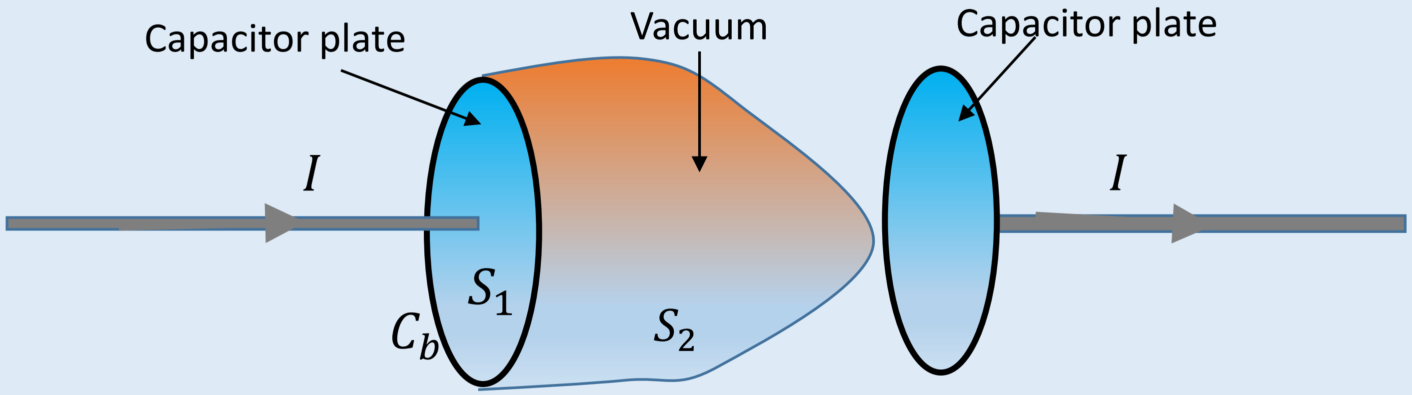
- Ampere's law as formulated should be valid for <u>any</u> surface bounded by an Amperian path. For example, a current-carrying conductor (a wire) could be represented by:



Both surfaces S_1 and S_2 have the same bounding curve C_b. Both also have the same current I passing through them. $\therefore\ I_{enc} = I$ for both S_1 and S_2. <u>Continuity of current is preserved here</u>.

Incompleteness of Ampere's Law (2)

- However, continuity of charge is not the case for all surfaces having the same bounding curve in a circuit with a capacitor in it:



Surfaces S_1 and S_2 still have same bounding curve C_b. But they do not have the same current I passing through them. In fact, there is no "convection" current (physically moving charges) in the vacuum region between the capacitor plates. $\therefore\ I_{enc} \neq I$ for S_1 and S_2. <u>Continuity of charge is not preserved in this case</u>.

Maxwell’s “Fix” to Ampere’s Law: Addition of the Displacement Current

- Maxwell deduced that there needs to be an additional term added to Ampere’s law to preserve charge continuity. He realized that, although physical charge didn’t cross between the capacitor plates, electric field did. For an air-filled parallel plate capacitor,

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A},$$

where $Q =$ charge on the positive capacitor plate, and $A =$ area of the plate. Rewriting as $Q = \epsilon_0 EA$, and differentiating with respect to time gives the rate at which a capacitor plate charges:

$$\frac{dQ}{dt} = \frac{d}{dt}(\epsilon_0 EA) = \epsilon_0 \frac{d}{dt}\int \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \epsilon_0 \frac{d\Phi_E}{dt}.$$

- Maxwell identified the quantity

$$\boxed{I_d = \epsilon_0 \frac{d\Phi_E}{dt}}$$

as an effective current between the plates of the capacitor, which was dubbed the “<u>displacement current</u>”.

Maxwell's "Fix" to Ampere's Law: Addition of the Displacement Current (2)

- Maxwell proposed that the true form of Ampere's law should be

$$\boxed{\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{l}} = \mu_o (I_c + I_d)_{enc} = \mu_0 \; I_{c,enc} + \mu_0 \epsilon_0 \frac{d\Phi_{E,enc}}{dt}},$$

where I_c = "convection current" = motion of physical charges, and I_d is displacement current, which can occur even in regions of vacuum (i.e., no physical charges present). As in the "regular Ampere's law", <u>only the enclosed currents (convection and displacement) by the Amperian path are used in the calculation</u>.

- The "fixed" equation is referred to as the Ampere-Maxwell equation.
- Note that in cases of no time-changing electric fields, $\frac{d\Phi_E}{dt} = 0$, and the equation reverts to the original form we first encountered.

Maxwell's Equations

- In honor of his achievement, which led to a unification of electricity and magnetism into a single, unified system known as electromagnetism, the set of governing equations for the electric and magnetic fields are now commonly known as "Maxwell's Equations".
- The wave equation for light is derived from these equations. (Maxwell was the first to do that.) Maxwell's equations are a milestone in physical science, and a testament to his genius.
- The combined equations (in integral form) are:

Gauss's law for $\vec{E}$:

Flux of electric field through a closed surface

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\mathrm{encl}}}{\epsilon_0}$$

Charge enclosed by surface

Electric constant

Gauss's law for $\vec{B}$:

Flux of magnetic field through any closed surface ...

$$\oint \vec{B} \cdot d\vec{A} = 0$$

... equals zero.

Faraday's law for a stationary integration path:

Line integral of electric field around path

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

Negative of the time rate of change of magnetic flux through path

Ampere's law for a stationary integration path:

Line integral of magnetic field around path

Electric constant

Time rate of change of electric flux through path

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \left(i_C + \epsilon_0 \frac{d\Phi_E}{dt} \right)_{\mathrm{encl}}$$

Magnetic constant

Conduction current through path

Displacement current through path