

HelioCampus/AEFIS Course Survey Participation Incentive Program

- The HelioCampus/AEFIS end-of-course survey is now open, and will remain so until Thursday, April 25.
- We value your participation in the course survey, and any comments or suggestions that you may make to help us improve this class.
- To encourage your participation in the survey we are offering the following incentive program:
 - ✓ If the overall class response to the survey reaches the 70% participation mark by 6 pm, Thursday April 25, the entire class will start the Final Exam with a score of 10 exam points.
 - ✓ If the overall class response to the survey reaches the 80% participation mark by 6 pm, Thursday April 25, the entire class will start the Final Exam with a score of 20 exam points.
 - ✓ If the overall class response to the survey reaches the 90% participation mark by 6 pm, Thursday April 25, the entire class will start the Final Exam with a score of 30 exam points.

Physics 1200

Lecture 27

Spring 2024

Final Exam and Grade Information
Matter Waves & Wave Functions,
Wave Functions and Probability,
Heisenberg Uncertainty Principle

Final Exam

- The PHYS-1200 Final Exam is at 8 am (ET), Tuesday April 30.
 - Final Exam runs from 8 am to 11 am.
- Exam rooms are:

Section(s)	Instructor	Exam Room
1, 2	Robles-Sanchez	DCC 324
3, 4	Zheng	DCC 318
5	Persans	DCC 318
6, 7	Schroeder/Ciolek	DCC 308
8,9	Martin	DCC 308
Accommodations	All	2C14 J-Rowl SC

Final Exam (2)

- Exam will be worth 200 points.
 - Will cover new material from Classes 26 & 27 (last class, and this lecture and its associated lab).
 - Will be comprehensive exam for the other material covered throughout the semester, including
 - ✓ All assigned homework (paper and online), reading, lab and class activity assignments, quizzes, lecture quizzes, and Exams 1, 2, & 3, are eligible to be materials covered on the Final Exam.
- Exam Structure
 - Multiple choice and numerical short answer questions for ~ 70% to 80% of exam pts.
 - ✓ ~ ½ conceptual and scaling questions
 - ✓ ~ ½ numerical solution
 - Free response questions for the remaining % of exam pts.

Final Exam (3)

- The Final Exam total score will be divided by 2; resulting in two 100-point equivalent exams.
 - Means that you will have a total of five 100-point equivalent exams for the semester.
 - To calculate your semester exam average score, we will drop the lowest of the five equivalent scores that you will have received during the semester.
 - Your final course average exam score is determined by the average of your four remaining exam scores.

Final Exam – Additional Information

- Allowed resources –
 - You may use you a calculator with capabilities and functions up to the TI-Nspire. Devices with communication/internet capability are not allowed.
 - **For the final exam you are allowed the use of two pages of crib sheets (= four sides, counting front and back of each of the two pages). Paper pages can't be bigger than 8.5" × 11". Must be two pieces of paper, maximum – no stapling, pasting, or taping of single pages together.**
 - Writing instruments (pencils, pens). For pens, darker inks are preferred. Absolutely no red ink!
- A short table of physical constants (same as done for Exams 1, 2, & 3) will be included on the first page of the Final Exam test booklet.
- Monitor your Rensselaer email and the LMS announcements at regular intervals in case there are exam updates.

Final Course Score & Grades

- Final course score determined by following weighting, as described in syllabus:
 - Exams : 60.0 %
 - Class lab activities : 22.5 %
 - Lecture quizzes : 5.0 %
 - Online homework assignments : 12.5 %
 - Any questions/concerns you may have about recorded grades, etc. should be addressed to your section's professor or graduation TA. Get those questions answered fast! Will be difficult to make changes after April 24.
- The course numerical grade range for each letter grade is:
 - A 92 to 100; A- 89 to 91.999
 - B+ 86 to 88.999; B 82 to 85.999; B- 79 to 81.999
 - C+ 76 to 78.999; C 72 to 75.999; C- 67 to 71.999
 - D 55 to 66.999.
- Rounding of numbers will be performed by the LMS grade book (or an excel spreadsheet). Whatever your numerical course value is as it is represented by your LMS score will be the value that will determine your letter final grade.
- You can calculate your current grade and estimate the effect of the final exam on your grade by downloading the "Course Grade Estimator" spreadsheet that can be found in our LMS "General Information" folder.

Matter Waves

- de Broglie (1924) hypothesized that all objects with momentum – not just light – have a wavelength associated with the object.

➤ He proposed that the wavelength of an object is,

$$\lambda = h/p .$$

(Anything that has momentum.)

- For light (photons), the waves are waves in the electromagnetic field. That is, space-time oscillations in the functions representing the electric and magnetic fields of the electromagnetic (EM) waves/photons.
 - As discussed several classes ago, the EM wave functions and their propagation in vacuum are well-described by the vacuum EM wave equations derived from Maxwell's equations.

Matter Waves (2)

- Question: what is a matter wave? What functions describe them, and do they represent solutions to a particular wave equation?
- Noted last class: individual photon interpretation of the intensity pattern for interference and diffraction of light led to relating the pattern to the probability of a photon impacting at a particular point on a viewing screen.
 - Intensity pattern understood to measure the likelihood of where a photon strikes the screen.
 - Interference and diffraction effects manifest themselves by creating regions where a photon strikes the screen with greater frequency (bright regions of high intensity) and regions which have zero probability of any photon strikes at all (zero-intensity, or dark regions).
- Last class: learned that matter waves produce interference/diffraction patterns identical to that of light.
 - Reasonable to expect that the probability-picture of the intensity picture for photons also carries over to the understanding of matter-wave intensity patterns (e.g., electron diffraction/interference experiments).

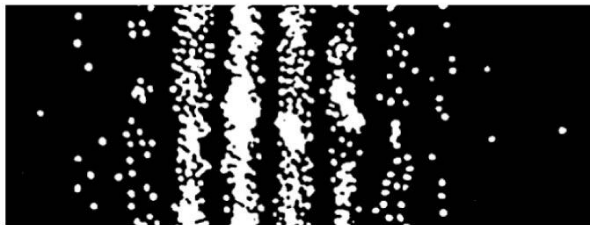
Intensity Patterns

- Comparison of two-slit interference/diffraction patterns for photons (left) and electrons (right). Clearly the same behavior!

After 21 photons reach the screen



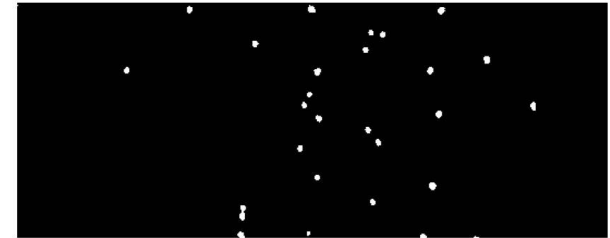
After 1000 photons reach the screen



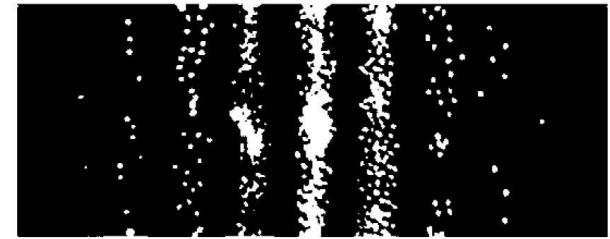
After 10,000 photons reach the screen



After 28
electrons



After 1000
electrons



After 10,000
electrons



Wave Functions and Matter Waves

- Intensity pattern for EM wave interference and diffraction was derived by calculating the net EM wave at a point by superposition of individual waves (this introduced interference into the problem), and then calculating the square of the net electric field $E(x, y, z, t)$ (i.e., the wave function for the EM field) to get the total intensity at the point.
- Erwin Schrödinger (1926) surmised that a similar situation occurs for matter waves, which are described by a wave function $\Psi(x, y, z, t)$. By a postulated extension of non-relativistic classical (“Hamiltonian”) mechanics (an advanced form of Newtonian mechanics) he proposed a wave equation governing the time evolution and spatial behavior of the wave function, now universally referred to as the Schrödinger Wave Equation.
 - This was a pivotal/crowning moment in the development of quantum mechanics (QM). Using his equation – and the solution to the wave equation – he was able to solve for the energy spectrum of the hydrogen atom.
 - Schrödinger won the Nobel prize in Physics in 1933, along with P. A. M. Dirac, another founder of QM; Dirac was able to derive a wave equation valid for relativistic particles (“the Dirac equation”), which predicted the existence of positrons (i.e., anti-electrons) before they were experimentally verified a few years later.

Wave Functions and Matter Waves (2)

- For one-dimensional motion of a non-relativistic particle of mass m , the Schrödinger wave equation is

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + U(x)\Psi(x,t) = i\hbar \frac{\partial \Psi(x,t)}{\partial t},$$

$\Psi(x, t)$ = wave function of the particle,

$U(x)$ = potential energy of the particle,

$\hbar = \frac{h}{2\pi}$ = reduced Planck's constant ("h-bar"),

$i = \sqrt{-1}$ = imaginary unit number.

- Imaginary unit number i ($= \sqrt{-1}$) in the Schrödinger equation tells us that, in general, the solution wave function $\Psi(x, t)$ to the wave equation is a complex quantity. (Has both real and imaginary parts.)
- Because it is complex (unlike the case of an EM wave), the wave function Ψ of a matter wave cannot be physically measured, because only real quantities can be measured.

Wave Function and Matter Waves (3)

- Special case of no potential: $U(x) = 0$.

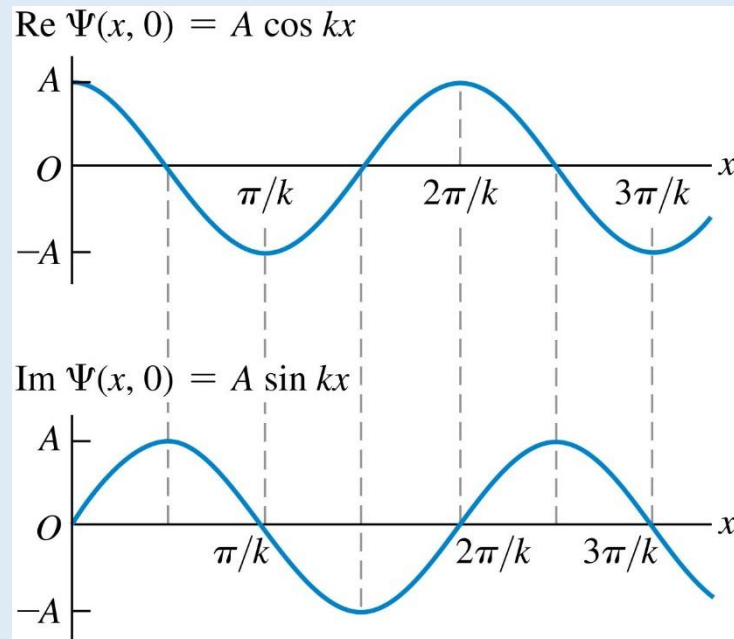
- Schrödinger equation has free-particle solution:

$$\Psi(x, t) = Ae^{i(kx - \omega t)} = A \cos(kx - \omega t) + iA \sin(kx - \omega t),$$

wave number $k = 2\pi/\lambda \Rightarrow$ momentum $p = \frac{h}{\lambda} = \hbar k$,

angular frequency $\omega = 2\pi f \Rightarrow$ energy $E = hf = \hbar\omega$.

Plots of real and imaginary parts
of $\Psi(x, t)$ at time $t = 0$.



Where Does the Schrodinger Equation Come From?

(Following is for “cultural enlightenment”. This slide and next won’t be on any test.)

- Fundamentally, the Schrodinger equation can be motivated/postulated from the principle of conservation of mechanical energy.

- For conservative systems, the total mechanical energy E of a particle with mass m , potential energy U , kinetic energy K , and speed v is

$$K + U = \frac{1}{2}mv^2 + U = \frac{p^2}{2m} + U = E$$

where $p = mv$ is the particle’s momentum. (Non-relativistic motion assumed here.)

- Multiplying both sides of above relation by any non-zero number preserves the equality; this includes non-zero wave functions Ψ :

$$\left[\frac{p^2}{2m} + U \right] \Psi = E\Psi$$

- Recall that deBroglie waves have momentum and energy relations (last class) same as for photons:

$$p = \frac{h}{\lambda} = \hbar k, E = hf = \hbar\omega.$$

Inserting into energy-conservation statement above gives:

$$\left[\frac{\hbar^2 k^2}{2m} + U \right] \Psi = \hbar\omega\Psi$$

Where Does the Schrodinger Equation Come From? (2)

- For 1-dimensional motion (say, along x -axis) of a free particle (no force, set potential energy $U(x) = 0$), energy conservation relation simplifies:

$$\frac{\hbar^2 k^2}{2m} \Psi = \hbar \omega \Psi$$

- As mentioned above, free particle moving in one dimension has complex wave solution:

$$\Psi = A e^{i(kx - \omega t)}.$$

- Note that (using $i^2 = -1$):

$$\frac{\partial \Psi}{\partial x} = i k A e^{i(kx - \omega t)}, \quad \frac{\partial^2 \Psi}{\partial x^2} = i^2 k^2 A e^{i(kx - \omega t)} = -k^2 A e^{i(kx - \omega t)} = -k^2 \Psi,$$

$$\frac{\partial \Psi}{\partial t} = -i \omega A e^{i(kx - \omega t)} = -i \omega \Psi,$$

$$\Rightarrow \text{can rewrite } k^2 \Psi = -\frac{\partial^2 \Psi}{\partial x^2}, \text{ and } \omega \Psi = -\frac{1}{i} \frac{\partial \Psi}{\partial t} = i \frac{\partial \Psi}{\partial t}.$$

- \therefore Energy conservation equation for the 1-D free particle is equivalent to:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} = i \hbar \frac{\partial \Psi}{\partial t}$$

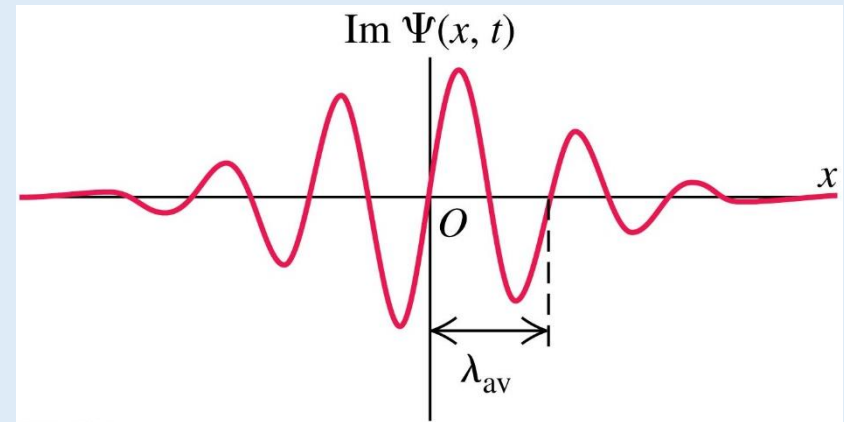
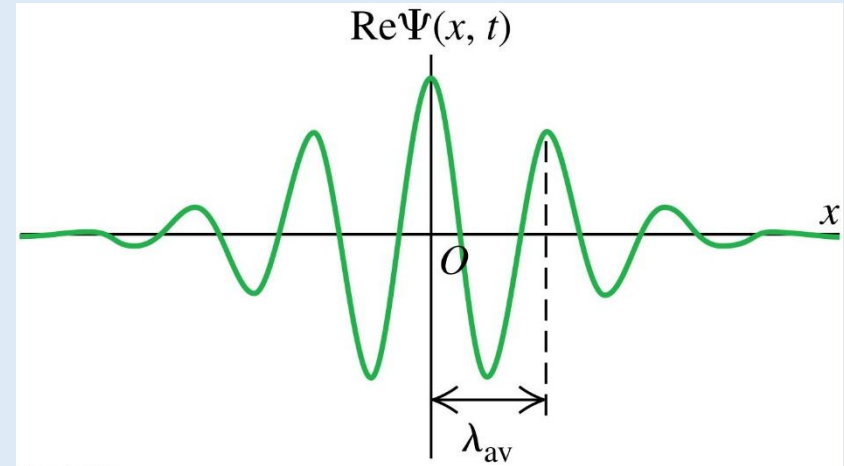
this is the 1-dimensional Schrodinger wave equation for a free particle!

- Generalizing to particles that aren't free (i.e., $U(x) \neq 0$):

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + U(x) \Psi(x,t) = i \hbar \frac{\partial \Psi(x,t)}{\partial t} \quad \checkmark$$

Superposition of Many Wave Functions: Wave Packets

- If matter waves overlap at a single point in space, the net wave function is the superposition (i.e., sum) of the wave functions at the overlap point. Wave interference (constructive and destructive) effects combine to give a net localized wave function known as a “wave packet”.
- Superposing many sinusoidal waves with different wave numbers and appropriate amplitudes can produce a wave pulse that has an average wavelength $\lambda_{av} = 2\pi/k_{av}$ and is localized within a region of space of length Δx .
- Shown are the real and imaginary components of such a wave packet at some time t .



Wave Functions, Matter Waves, and Probability

- Max Born (1926), proposed that the square of the magnitude (i.e., the absolute value) of the wave function $|\Psi|^2 \equiv \Psi^* \Psi$ can be physically measured, since that quantity is always real.

➤ For any complex number $\Psi = \Psi_r + i\Psi_i$, where Ψ_r is a real function representing the real part of the wave function, and Ψ_i is a real function representing the imaginary part of the wave function,

$$\Psi^* \Psi = |\Psi|^2 = (\Psi_r + i\Psi_i)^* (\Psi_r + i\Psi_i) = (\Psi_r - i\Psi_i)(\Psi_r + i\Psi_i)$$

$$\Rightarrow |\Psi|^2 = \Psi_r^2 + \Psi_i^2 \geq 0 \quad . \quad \therefore |\Psi|^2 \text{ is always real.}$$

- Born suggested that, for one-dimensional case, probability of finding a particle between x and $x + dx$ at a particular time t is given by:

$$P(x, x + dx, t) = |\Psi(x, t)|^2 dx \quad ,$$

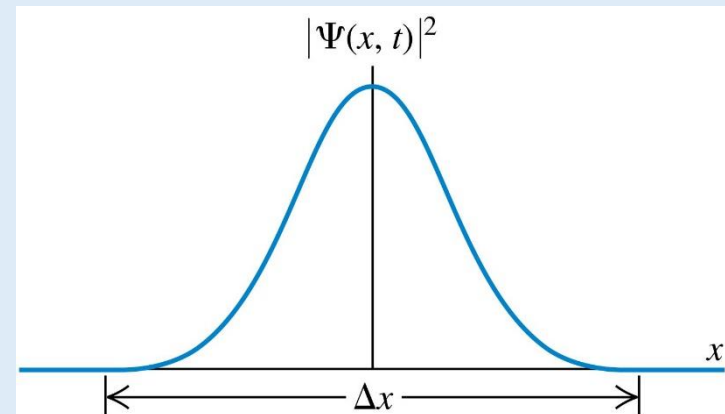
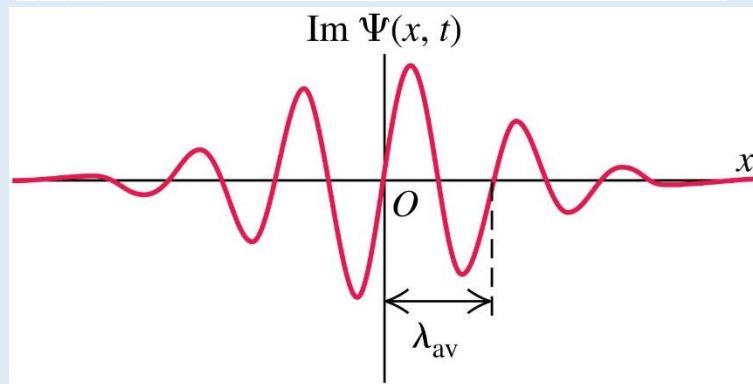
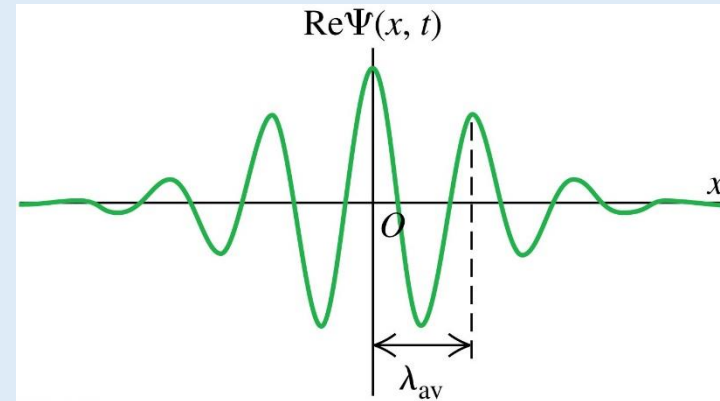
in this picture, $|\Psi(x, t)|^2$ is the one-dimensional probability density.

- More generally, probability of finding a particle between x_1 and x_2 at time t is

$$P(x_1, x_2, t) = \int_{x_1}^{x_2} |\Psi(x, t)|^2 dx \quad .$$

Wave Functions, Matter Waves, and Probability (2)

- Plot of the probability density $|\Psi(x, t)|^2$ for the 1-dimensional wave packet described earlier.
- Resulting probability distribution has only one maximum.
- Localized pulse has aspects of both particle and wave.
- It is particle-like in that it is localized in space; if we look from a distance, it may look like a point.
- But it also has a periodic structure that is characteristic of a wave.



Wave Functions, Matter Waves, and Probability (3)

- Extending to 3 dimensions, probability of finding a particle at time t between locations (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$P(x_1, y_1, z_1; x_2, y_2, z_2; t) = \iiint_{x_1, y_1, z_1}^{x_2, y_2, z_2} |\Psi(x, y, z, t)|^2 dx dy dz .$$

- Born awarded 1954 Nobel Prize in Physics for fundamental contributions (including probability interpretation) to the development of QM.
- In probability picture, intensity of a matter-wave interference/diffraction experiment measured at a point on an analyzer screen is proportional to the local value of $|\Psi|^2$ at that location.

Acceptable Wave Functions

- Consideration of the properties of the Schrödinger equation leads to the following conditions that a valid wave function solution Ψ must satisfy:

1. The probability of finding a particle somewhere must be unity (i.e., equal to 1). For the one-dimensional case,

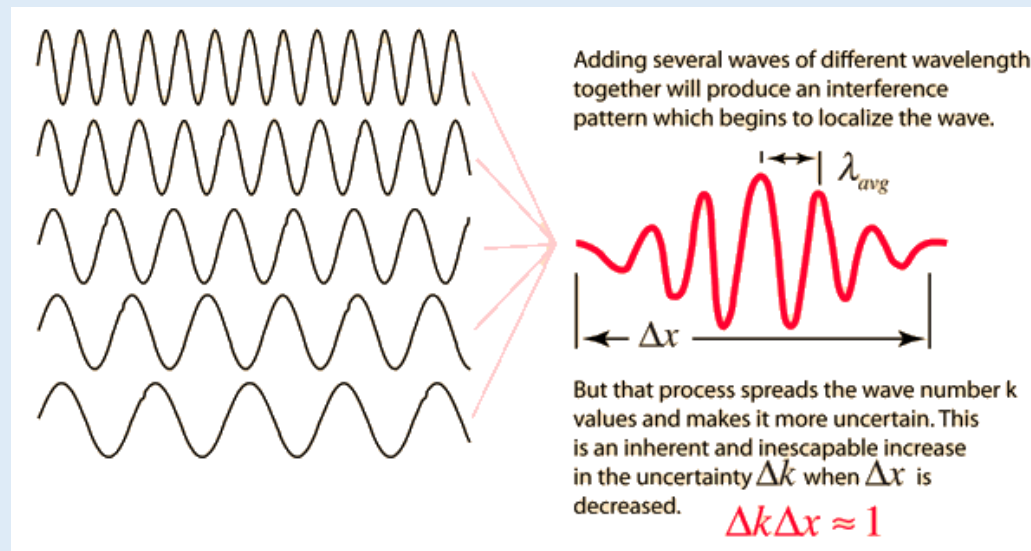
$$P(-\infty, +\infty, t) = \int_{-\infty}^{+\infty} |\Psi(x, t)|^2 dx = 1 .$$

A wave function that satisfies this condition is said to be normalized.

2. The wave function can only have one value at a particular place and time. That is, the function is single-valued.
3. The wave function must be continuous.
4. The first spatial partial derivative ($\partial\Psi/\partial x$) of the wave function must also be continuous.

Constructing Wave Packets – Uncertainty

- Addition of several matter waves with varying wave wavelength (or, wave number) results in packets that are localized in space. (Analogous to the phenomena of beats, covered in Physics I. For that case, resultant packet was localized in period \Rightarrow the beat period.)
- To localize packet in space, many waves with many different wave numbers are added together. Since $p = \hbar k$, creating a spread Δk in wave numbers means creating a spread Δp in the momentum of the net wave represented by the packet. \therefore The more certain the location of the particle becomes (essentially the packet spatial size), the momentum of the particle becomes less certain.
- Hence, there is an uncertainty relation between the momentum and location of the particle-wave packet.



Constructing Wave Packets – Uncertainty (2)

- Uncertainty between the momentum and spatial location of the wave packet is due to the packet's wave-particle duality.
- As mentioned last class, wave-particle duality is the basis for uncertainty relations for photon momentum and location, and energy and time, as found by Heisenberg (1926).
- Since wave-particle duality is true for matter waves, the Uncertainty Principle also applies to those objects as well. Hence, for a matter wave, the following uncertainty relations between momentum and position, and energy and time, will be true:

$$\Delta p_x \Delta x \geq \hbar/2, \quad \Delta p_y \Delta y \geq \hbar/2, \quad \Delta p_z \Delta z \geq \hbar/2, \\ \Delta E \Delta t \geq \hbar/2.$$

- \therefore It is impossible to simultaneously measure the x -coordinate of a particle and its x -momentum with absolute precision (or, equivalently, no uncertainty).
- Similar things for the other coordinate directions can be said from the other uncertain relations given above.

Wave Functions: Stationary States

- If a particle has a definite energy E , the wave function $\Psi(x, t)$ is a product of a time-independent wave function and a factor that depends on time t but not position:

The diagram shows the equation $\Psi(x, t) = \psi(x)e^{-iEt/\hbar}$ with annotations. A bracket on the left groups $\Psi(x, t)$ with the text "Time-dependent wave function for a state of definite energy". A bracket on the right groups $\psi(x)$ with the text "Time-independent wave function". A bracket under E is labeled "Energy of state". A bracket under \hbar is labeled "Planck's constant divided by 2π ". Dotted lines connect the text to the corresponding parts of the equation.

$$\Psi(x, t) = \psi(x)e^{-iEt/\hbar}$$

- For such a “stationary state” the probability distribution function

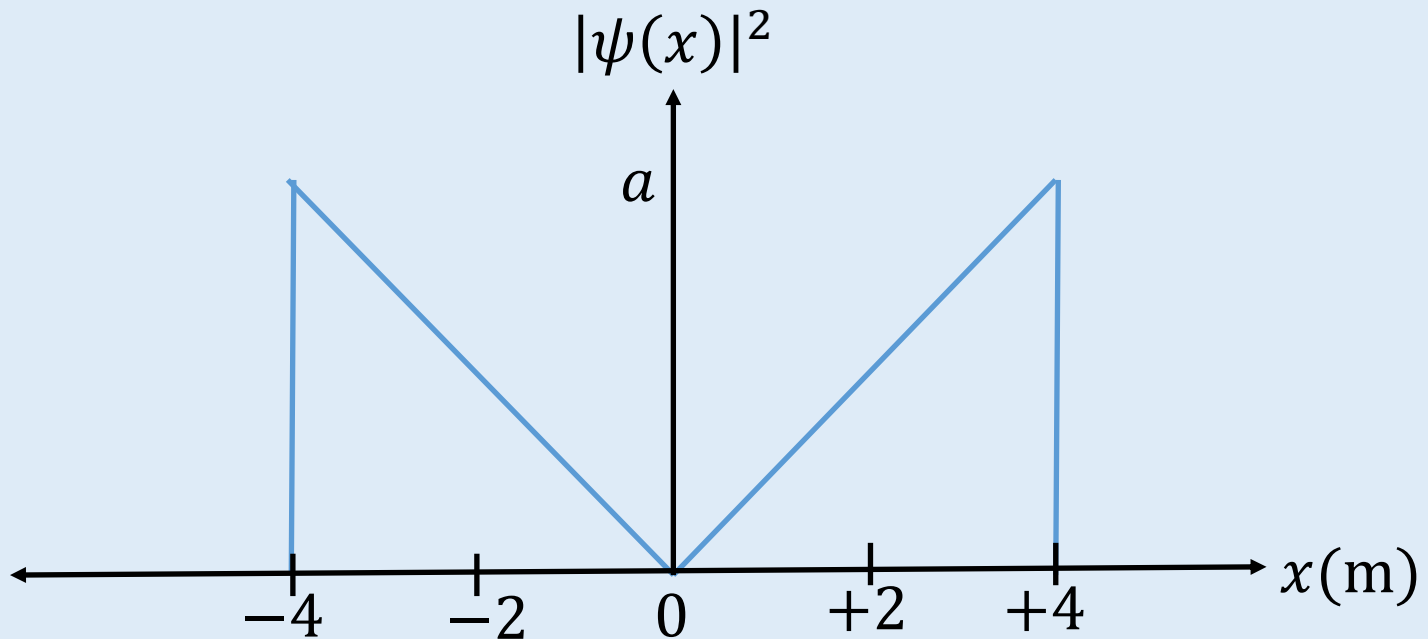
$$|\Psi(x, t)|^2 = \Psi(x, t)^*\Psi(x, t) = |\psi(x)|^2$$

does not depend on time.

- Stationary state wave functions $\psi(x)$ for the hydrogen atom can be found as solutions of the Schrödinger equation. The energy levels for those solutions are in excellent agreement with measured energy levels for hydrogen → quantum physics explains atomic and molecular phenomena!

Question

- Given the probability density for a neutron shown below.
 - What is the value of a ?
 - What is the probability that the particle is located between $x = 0$ and 2 ?



Question - Solution

A) From the graph, have

$$P(x, t) = P(x) = |\psi(x)|^2 = \frac{a}{4m} |x| \text{ for } -4m \leq x \leq +4m, \text{ and } P(x, t) = 0 \text{ for } |x| > 4m.$$

To find constant a , use fact that the wave function is normalized:

$$\begin{aligned} \int_{-\infty}^{+\infty} P(x, t) dx &= 1 = \int_{-4m}^{+4m} \frac{a}{4m} |x| dx = \frac{a}{4m} \int_{-4m}^{+4m} |x| dx = \frac{a}{4m} \int_{-4m}^0 (-x) dx + \frac{a}{4m} \int_0^{+4m} x dx \\ \Rightarrow \frac{a}{4m} \left[\frac{x^2}{2} \right]_0^{-4m} + \frac{a}{4m} \left[\frac{x^2}{2} \right]_0^{+4m} &= 2 \left(\frac{a}{4m} \right) \left[\frac{x^2}{2} \right]_0^{+4m} = 16 \left(\frac{a}{4m} \right) m^2 = 4a m = 1. \quad \therefore a = \frac{1}{4m}. \end{aligned}$$

Could also find by just using:

$$\text{Total probability} = (\text{area under curve in graph}) = 2 \left[\frac{1}{2} (4m)a \right] = 1 \Rightarrow a = \frac{1}{4m} \checkmark$$

B) To find the probability that the neutron is in the range $0 \leq x \leq 2m$:

$$P(0 \text{ m}, 2 \text{ m}) = \text{area under curve} = \frac{1}{2} (2 \text{ m}) \left(\frac{a}{2} \right) = \frac{1}{2} (2 \text{ m}) \left(\frac{1/4m}{2} \right) = \frac{1}{8} = 0.125.$$

$$\begin{aligned} \text{Check: } P(0 \text{ m}, 2 \text{ m}) &= \int_0^{2m} |\psi(x)|^2 dx = \int_0^{2m} \frac{a}{4m} x dx = \frac{a}{4m} \int_0^{2m} x dx = \frac{a}{4m} \left[\frac{x^2}{2} \right]_0^{2m} \\ &= \left(\frac{1/4m}{4m} \right) \left[\frac{(2m)^2}{2} \right] = \left(\frac{1}{16m^2} \right) \left(\frac{4m^2}{2} \right) = \frac{1}{8} = 0.125 \checkmark \end{aligned}$$

Class 27 Labs & Homework

- Regular lab sessions for Class 27 (Monday April 22 / Tuesday April 23).
- Monday labs: assignment due by 11:59 pm, Tuesday April 23.
- Tuesday labs: assignment due by 6:00 pm, Wednesday April 24.
- No homework assignment is due for this class. A set of extra-credit problems on this class topic will be available on MasteringPhysics (due date for extra-credit assignment is 11:59 pm Wednesday, April 24).
- Note, however: material presented in this class will appear on the Final Exam.

