Physics 1200
Lecture 14
Spring 2024

Mutual & Self Inductance, EMF and Current in Circuits, Magnetic Field Energy

Inductance

- Last class: time-changing magnetic flux induces an emf in a region of space and can induce current in a conductor located where emf is occurring.
- Inductance is the effect an emf exerts on a device by its own doing (self-inductance) or on other devices (mutual inductance).
- Earlier calculations: found magnetic field generated by currents typically had form $B = G_{fac} f(\vec{\boldsymbol{r}}) i$, where,
  - $B =$ the magnetic field strength,
  - $G_{fac} =$ numerical factor dependent only on physical constants and geometry of the conductor,
  - $f(\vec{\boldsymbol{r}}) =$ function of the location of field point position $\vec{\boldsymbol{r}}$ relative to conductor-current region,
  - $i =$ current. To be consistent with your textbook, we now use lower-case symbol for current instead of upper-case symbol, a notation which will indicate possibly time-changing current $[i(t)]$ instead of just steady current (I).

Inductance (2)

- Magnetic flux from a current will generally have form:
  $\Phi_B = \int \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{A}} = G'_{fac} i,$
  where $G'_{fac} =$ physical-geometric constant numerical factor incorporating area/dimension of a device or devices magnetically linked in some way (sometimes referred to as "flux linkage").
- G'_{fac} motivates idea that there is a quantity known as <u>inductance</u>, associating physical and geometric aspects of a conducting device with its magnetic properties.
- <u>Self-inductance L of a device</u> defined by relation

$$\boxed{\Phi_B = Li},$$

  Φ_B = total magnetic flux contained in a region of the device.
- Invert relation to calculate self-inductance if flux and current are known:

$$L = \frac{\Phi_B}{i}.$$

Example: Self-Inductance of a Solenoid

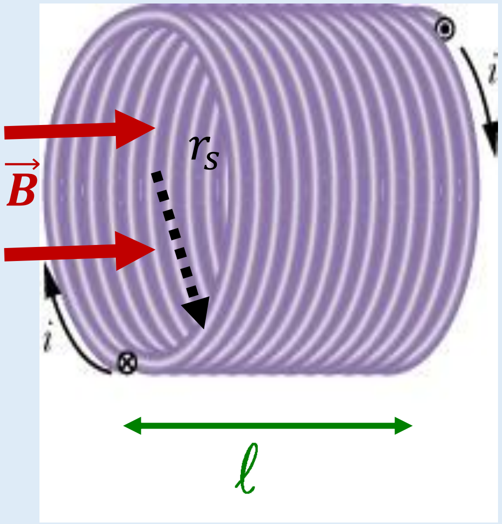
- Calculate self-inductance (often just called the inductance) L of an ideal solenoid of N turns, length l, and radius r_s :

$$L = \frac{\Phi_B}{i} = \frac{N\Phi_{B,loop}}{i} = \frac{NBA_{loop}}{i} = \frac{N(\mu_o ni)A_{loop}}{i}$$

$$\Rightarrow \boxed{L = \mu_0 N\left(\frac{N}{l}\right)A_{loop} = \frac{\mu_0 N^2 \pi r_s^2}{l} = \pi\mu_0 n^2 r_s^2 l},$$

$\Phi_{B,loop}$ = flux of a single loop, and

$A_{loop} = \pi r_s^2$ = area of a loop.



- L is <u>independent</u> of current or magnetic field of the solenoid.

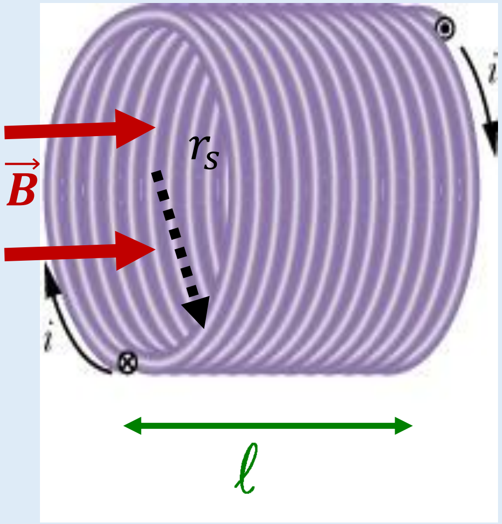
- SI unit of inductance is the henry (H):

  1 H = 1 Wb/A = 1 T m^2/A .

Example: Self-Inductance of a Solenoid (2)

- To get a sense of a typical inductance value, an inductor in an electrical circuit might have 1000 turns, a radius of 0.50 cm, and a length of 1.0 cm:

$$L = \frac{\pi\left(4\pi\times10^{-7}\frac{\mathrm{T\ m}}{\mathrm{A}}\right)(1000)^2}{(0.01\ \mathrm{m})}\,(0.0050\ \mathrm{m})^2 = 9.9\times10^{-3}\ \mathrm{H}.$$



Inductance and EMF

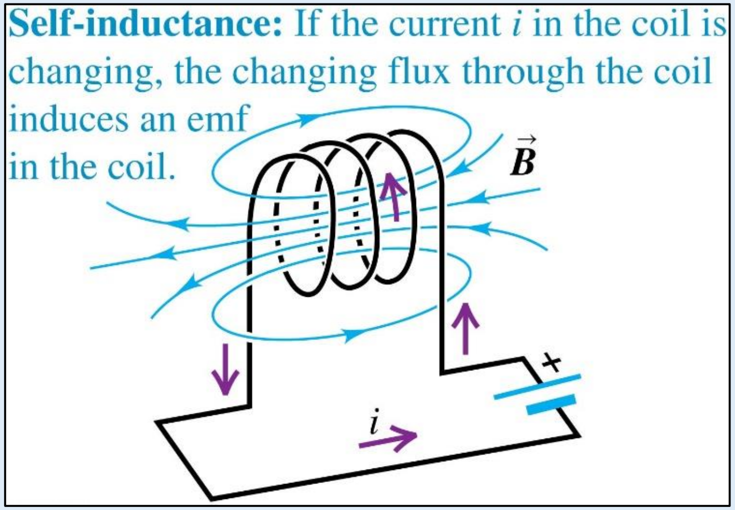
- Differentiating inductance – current relation:

$$\frac{d\Phi_B}{dt} = \frac{d}{dt}\,(Li) = L\,\frac{di}{dt}\,.$$

Combined with Faraday's law, yields:

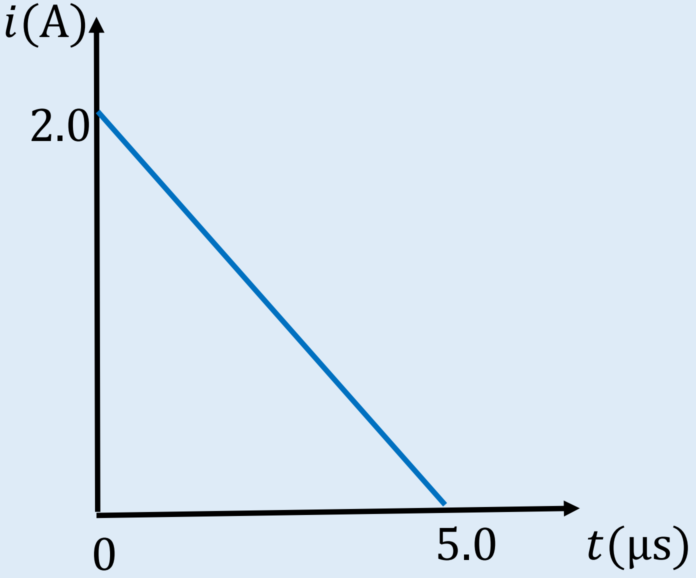
$$\boxed{\mathcal{E}_L = -L\,\frac{di}{dt}}\,.$$

  - ➢ When current changes in an inductor, an emf is induced that opposes the imposed change (Lenz's law).
  - ➢ Physical consequence: when current is turned on or off in a circuit (e.g., closing a switch), a "<u>back emf</u>" $\mathcal{E}_L$ is generated in the inductor that opposes that change. If the change in current is sudden, the back emf can be quite large in magnitude!



Lecture Question 14.1

- An ideal solenoid has length $l = 1.5\ \mathrm{cm}$, cross-sectional area $A_{loop} = 0.78\ \mathrm{cm}^2$, and number of turns $N = 2500$. At time $t = 0.0\ s$ a steady current $i_0 = 2.0\ \mathrm{A}$ flows through the solenoid. Suppose the current were to be cutoff as shown in the graph at right. The magnitude of the emf induced in the inductor is closest to



A. 1.2×10^5 V.

B. 1.3×10^3 V.

C. 21 V.

D. 1.6×10^4 V.

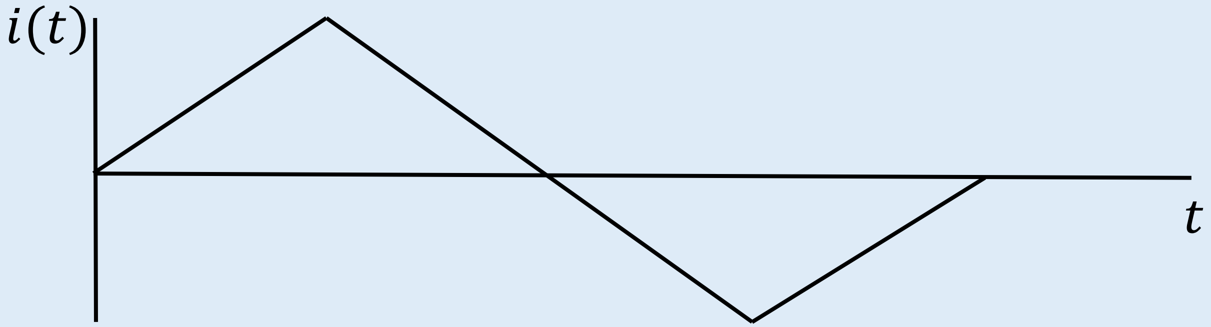
E. 55 V.

Inductors as Surge Protectors

- Because they oppose sudden changes in current, inductors can be used as surge protectors in electrical circuits and systems.
- Large inductors are used as surge protectors against lightning strikes for outdoor power transmission systems.

Calculus with an Inductor

- Because emf in an inductor is proportional to the time derivative of the current, can use this to perform basic calculus with inductors in circuits.
- Example: if the current in a circuit is a triangle wave, how does emf across the inductor behave?



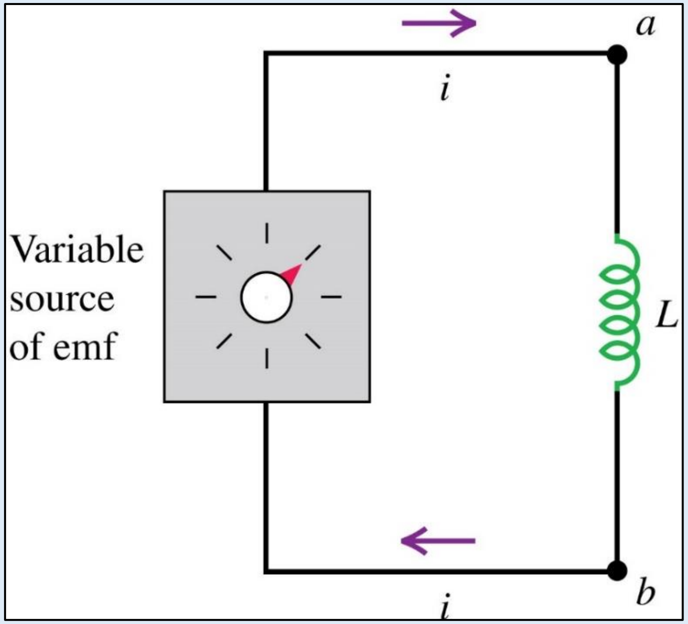
- You will be doing this in your lab today.

Change in Potential Across an Inductor

- Last class: Faraday's law relates change in magnetic flux to the line-integral of a non-conservative electric field $\vec{\boldsymbol{E}}_{nc}$:

$$\oint \vec{\boldsymbol{E}}_{nc} \cdot d\vec{\boldsymbol{l}} = -\frac{d\Phi_B}{dt} \,.$$

- Because magnetic flux change in a circuit is confined within inductor region, $\vec{\boldsymbol{E}}_{nc}$ is also confined between the terminals of the inductor (typically labeled points a and b in your text).



- Because $\vec{\boldsymbol{E}}_{nc} \neq 0$ only between a and b, integrating the line integral in Faraday's law clockwise around the loop gives

$$\int_a^b \vec{\boldsymbol{E}}_{nc} \cdot d\vec{\boldsymbol{l}} = -\frac{d\Phi_B}{dt} = -L\,\frac{di}{dt} \;,$$

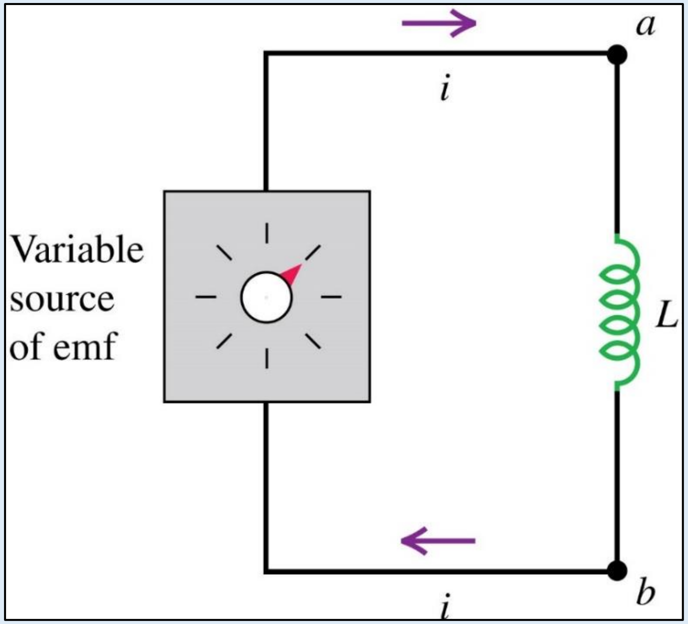
(used the inductance – flux definition).

Change in Potential Across an Inductor (2)

- Ideal inductors are made of conductors with effectively zero resistance. To have finite current density in this type of inductor, the total electric field that acts on charges inside the conductor must be essentially zero:

$$\vec{\boldsymbol{E}}_{tot} = \vec{\boldsymbol{E}}_c + \vec{\boldsymbol{E}}_{nc} = 0 \quad \Rightarrow \vec{\boldsymbol{E}}_c = -\vec{\boldsymbol{E}}_{nc} .$$

  - Conservative electric field $\vec{\boldsymbol{E}}_c$ in the inductor is opposite in direction and has same magnitude as induced non-conservative field, $\vec{\boldsymbol{E}}_{nc}$.



- Electric potential is defined in terms of conservative electric fields; follows that potential difference between inductor leads is

$$V(b) - V(a) = -\int_a^b \vec{\boldsymbol{E}}_c \cdot d\vec{\boldsymbol{l}} = \int_a^b \vec{\boldsymbol{E}}_{nc} \cdot d\vec{\boldsymbol{l}} = -L\frac{di}{dt}$$

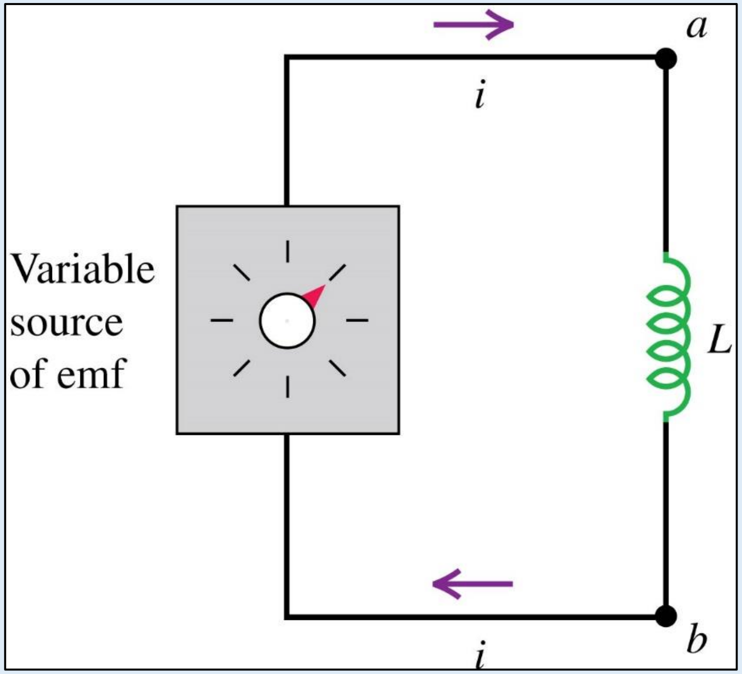
$$\Rightarrow V(a) - V(b) = L\frac{di}{dt} .$$

Inductors in Circuits

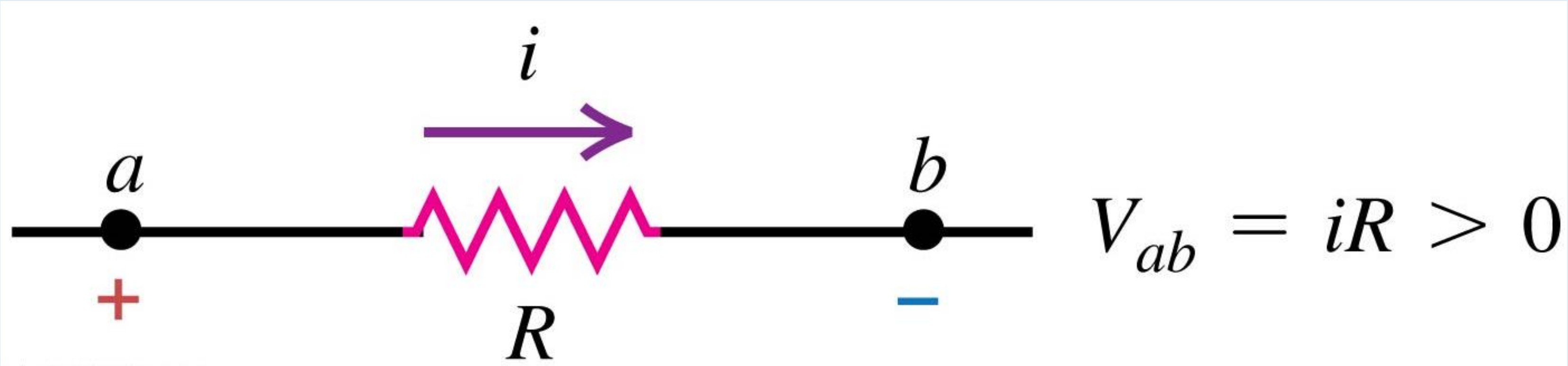
- Circuit shown: box enables us to control the current i in the circuit.
- Potential difference between the terminals of the inductor L is:

$$\boxed{V_{ab} = V(a) - V(b) = L\frac{di}{dt}}.$$

- Consider inductor response for different situations.

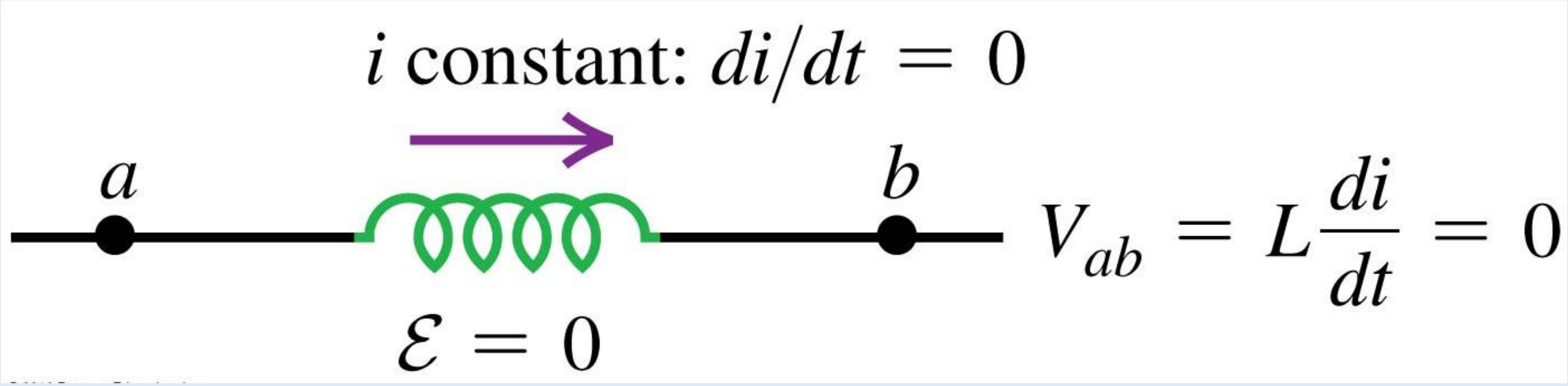


Inductors in Circuits (2)



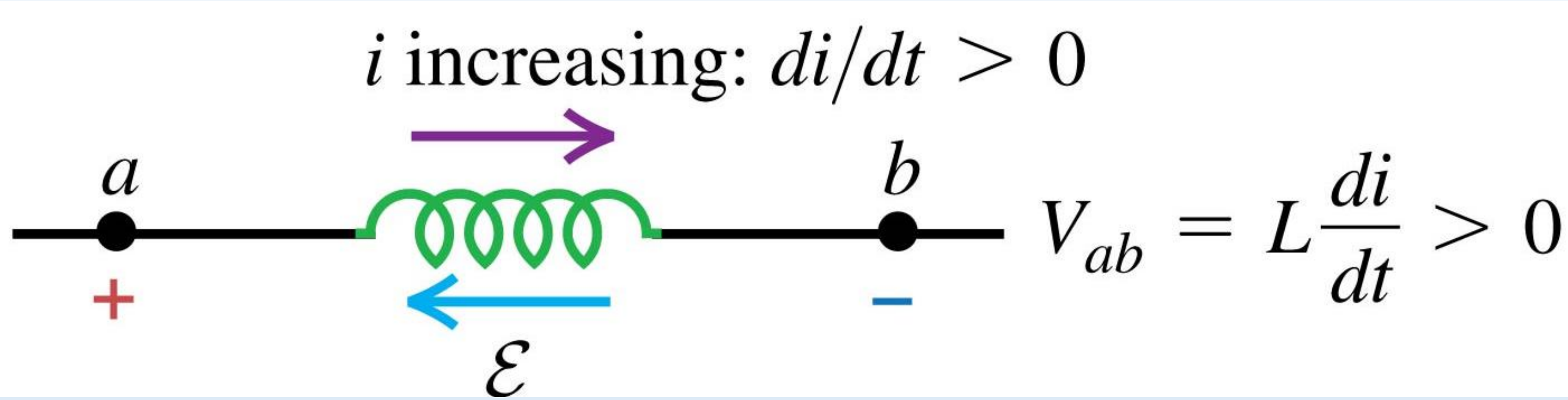
- For resistors in circuits, flow shown above is always a <u>drop</u> in potential across a resistor.
- Result for an inductor in a circuit will depend on how the current is <u>changing</u> with time.

Inductors in Circuits (3)



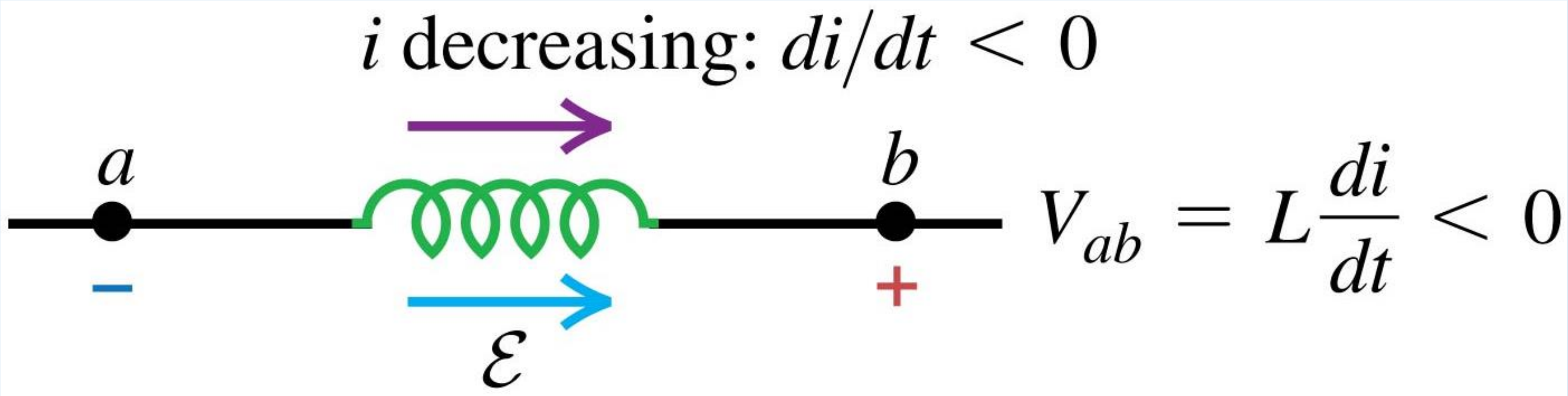
- Steady (i.e., constant) current: no emf and no potential difference across the inductor.

Inductors in Circuits (4)



- Current increasing in direction of the current flow: induced back-emf opposes increase of current in sense shown.
  - For this case, emf due to $\vec{\boldsymbol{E}}_{nc}$ in inductor directed from terminal b to terminal a.
  - Conservative electric field $\vec{\boldsymbol{E}}_{c}$ in inductor directed oppositely, from terminal a to terminal b.
  - Conservative electric field points from high electric potential to low electric potential: follows that $V(a) > V(b) \Rightarrow V_{ab} > 0$.

Inductors in Circuits (5)



- Current decreasing with time in direction of current flow.
  - Inductor emf tries to replenish (i.e., add to) diminishing current in sense shown, going from a to b. $\therefore$ $\vec{\boldsymbol{E}}_{nc}$ for this case goes from a to b.
  - Conservative electric field $\vec{\boldsymbol{E}}_c$ is oppositely directed, from b to a.
  - Follows that $V(b) > V(a)$ for this case. That is, $V_{ab} < 0$.

Magnetic Energy Stored in Inductors

- When inductor voltage $V_{ab} \neq 0$ and current $i \neq 0$, power can be supplied to an inductor. Amount of power will be

$$P_L = i\,V_{ab} = iL\frac{di}{dt}\,.$$

- Energy stored in the inductor as magnetic energy, found by integrating supplied power over time:

$$\boxed{U_B = \int P_L\,dt = \int iL\frac{di}{dt}dt = L\int_0^i i'di' = \frac{1}{2}\,Li^2}\,.$$

  - Unlike a resistor, where energy is dissipated (i.e., lost from system), power P_L to the inductor is stored as energy in the magnetic field of the inductor.
  - If current in circuit $i \to 0$, stored magnetic energy is returned to the circuit.

Energy Density of the Magnetic Field

- Like electrostatics, define energy density of a magnetic field as

$$\boxed{u_B \equiv \frac{U_B}{v}}, \; v = \text{volume of a system.}$$

- For a solenoid of length l and cylindrical cross-section area A_{loop}, $v = A_{loop}l = \pi r_s^2 l$, $B = \mu_0 n i$, and $L = \pi\mu_0 n^2 r_s^2 l$. Follows that

$$u_B = \frac{\frac{1}{2}Li^2}{A_{loop}l} = \frac{1}{2}\frac{(\mu_0 \pi r_s^2 n^2 l)i^2}{\pi r_s^2 l} = \frac{1}{2}\mu_0(ni)^2 = \frac{1}{2}\mu_0\left(\frac{B}{\mu_0}\right)^2,$$

$$\Rightarrow \boxed{u_B = \frac{B^2}{2\,\mu_0}}.$$

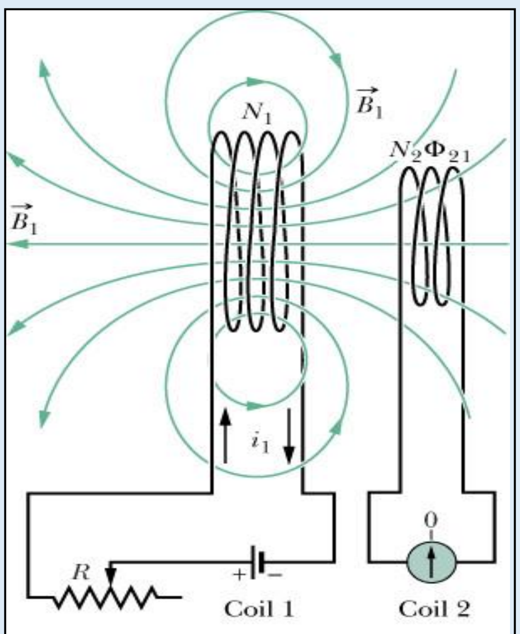
- Although derived for a solenoid, expression found for u_B is independent of device. Above relation is true for all situations.

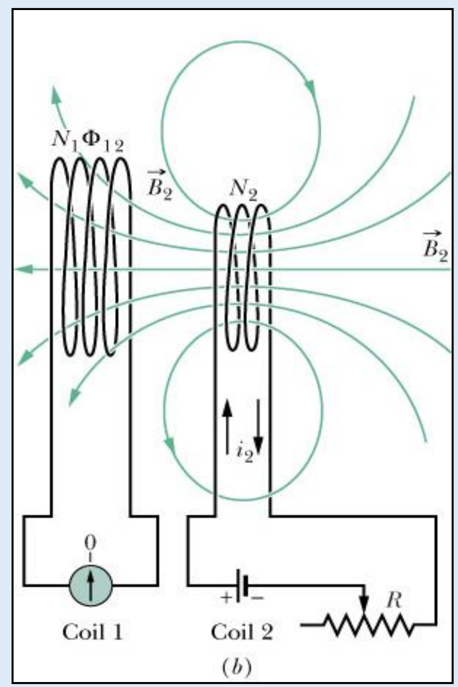
Energy Density of the Magnetic Field (2)

- Energy density u_B can be considered an intrinsic property of a magnetic field. Like that for the energy density of the electric field, $u_E = \frac{1}{2}\, \epsilon_0 E^2$.
- SI units of energy density u_B are $\frac{\mathrm{J}}{\mathrm{m}^3}$. This is equivalent to $\frac{\mathrm{N}}{\mathrm{m}^2}$. These units are the same as for <u>pressure</u>.
  - In plasma physics (physics of ionized gases) and magnetohydrodynamics (MHD), $u_B = \frac{B^2}{2\mu_0}$ explicitly appears in the force equation (= Newton's 2nd law per unit volume) for a plasma as a "magnetic pressure term."
- When we discuss electromagnetic waves and radiation (later in semester), energy densities u_B and u_E naturally appear in discussion of wave energy and momentum (transported by something known as the Poynting vector).

Mutual Inductance

- Inductance effects not limited to emf effect of a device on itself. Inductance can also occur between different devices. In those cases we refer to there being a mutual inductance between the devices.
- Consider two separate, yet nearby coils: magnetic field of one can create magnetic flux in the other (and vice-versa). Change of magnetic field in one (by a change in its current) can induce change in magnetic flux of the other $\Rightarrow$ one coil induces emf in the other.





Mutual Inductance (2)

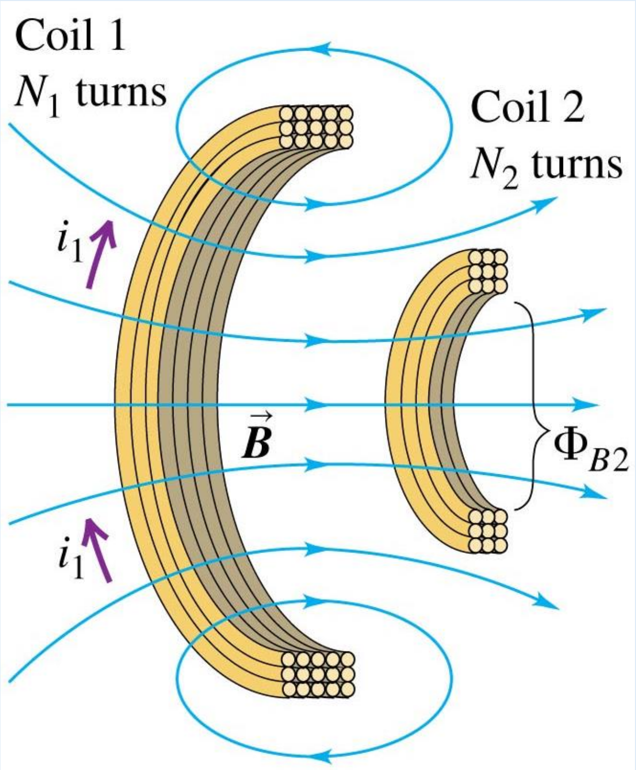
- Consider two neighboring coils of wire. If current in coil 1 changes, it induces an emf in coil 2, and vice versa.
- Proportionality constant for this pair of coils is the mutual inductance, M.
- Define mutual inductance of coil 2 with respect to coil 1 as

$$M_{21} = \frac{N_2 \Phi_{B2}}{i_1},$$

and that of coil 1 with respect to coil 2 as

$$M_{12} = \frac{N_1 \Phi_{B1}}{i_2}.$$

- Being inductances, mutual inductances M_{21} and M_{12} are independent of magnetic field and current in either coil. They depend only on the geometry of the coils and physical constants.



Mutual Inductance (3)

- From their definitions, it follows that coil emfs are

$$\mathcal{E}_1 = -N_1 \frac{d\Phi_{B1}}{dt} = -M_{12} \frac{di_2}{dt} \ ,$$

$$\mathcal{E}_2 = -N_2 \frac{d\Phi_{B2}}{dt} = -M_{21} \frac{di_1}{dt} \ .$$

- With a little more detail and level of math greater than we use in PHYS 1200, one can show from general considerations that there is a <u>reciprocity relation</u>, such that, for general loop circuits 1 and 2,

$$\boxed{M_{12} = M_{21} = M} \ .$$

It follows for the magnetically linked coils,

$$\boxed{\begin{aligned} \mathcal{E}_1 &= -M \frac{di_2}{dt} \\ \mathcal{E}_2 &= -M \frac{di_1}{dt} \end{aligned}}$$

Applications of Inductance

- This electric toothbrush makes use of mutual inductance.
  - Base contains a coil that is supplied with alternating current from a wall socket.
  - Even though there is no direct electrical contact between base and toothbrush, the varying current induces an emf in a coil within the toothbrush itself, recharging the toothbrush battery.