Physics 1200
Lecture 15
Spring 2024

Circuit Transients: RC, RL, LC, & RLC

Transient Behavior in Electrical Circuits

- So far, we've considered only situations in which current is steady (i.e., constant).
- Now allow time-dependent behavior. Includes:
    - Transients. Transition from one steady state to another steady state. Can occur by closing or opening a switch or connecting or disconnecting from a steady power supply or source of emf. Topic of this class.
    - Alternating current/voltage in a circuit. This is the study of AC circuits, discussed next class.

Analysis of Non-Steady Circuits

- Kirchoff's loop and junction rules still valid, even for circuits with voltages and currents changing over time, because they are rooted in fundamental conservation laws (energy and charge) and are true even for non-steady situations.

- Kirchoff's loop rule for time-dependent circuits:

$$\boxed{\sum_{j=1}^{N} V_j\,(t) = 0},$$

$V_j(t) =$ change in electrical potential across circuit element j at time t.

- Kirchoff's junction rule for time-dependent circuits:

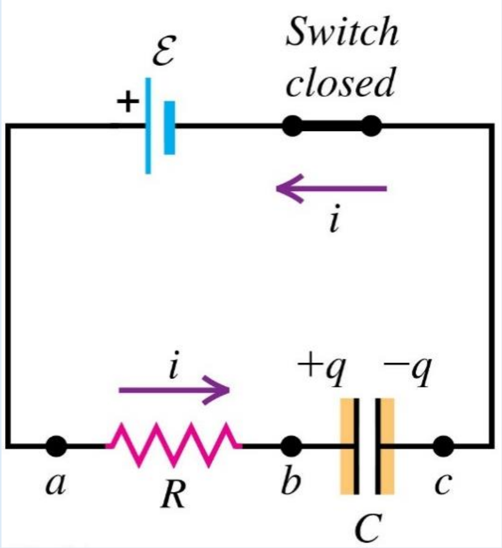
$$\boxed{\sum_{j=1}^{N} i_j(t) = 0},$$

$i_j(t) = j$-th current entering a junction at time t.

Analysis of Non-Steady Circuits (2)

- Application of Kirchoff's rules for non-steady currents and voltages give fundamental equations for circuits in those situations.
- Equations will contain first- or second-derivatives of the current or charge (e.g., $i = dq/dt$, where $q(t)$ is the charge at a time t).
- Solution of differential equations determines how circuit properties (e.g., voltage, current, charge) behave as functions of time.

RC Circuit: Charging of a Capacitor



- Circuit composed of an initially uncharged capacitor C connected to resistor R and emf $\mathcal{E}$ in an open circuit.
- Switch closed at time $t = 0$.
    - Current $i(t)$ from $\mathcal{E}$ flows in circuit for $t > 0$.
    - Positive charge $q(t)$ accumulates on capacitor's high-voltage plate.
- Kirchoff's loop rule for closed circuit yields: $\boxed{\mathcal{E} - iR - \frac{q}{C} = 0}$.
- Inserting $i = \frac{dq}{dt}$, and rearranging, gives

$$\frac{dq}{dt} = -\frac{(q - C\mathcal{E})}{RC}, \quad \text{and has solution} \quad \boxed{q(t) = C\mathcal{E}\left(1 - e^{-t/RC}\right)}$$

for the charge on a charging capacitor with initial value $q(0) = 0$.

- Using $i = \frac{dq}{dt}$, find: $\boxed{i(t) = \frac{\mathcal{E}}{R}\, e^{-t/RC}}$.

RC Circuit: Charging of a Capacitor (2)

- Note:
  - For $t \simeq 0$, $q \simeq 0$, and $i \simeq \mathcal{E}/R$. $\therefore$ At very early times after circuit is closed, potential change across the capacitor is very small $\Rightarrow$ capacitor is 'invisible' to the circuit.
  - For $t \to \infty$, solution gives $i \to 0$, and $q \to Q_{\mathrm{f}} = C\mathcal{E}$. $\therefore$ Capacitor gets 'filled up' to its maximum charge when $V_C \to \mathcal{E}$, $\Rightarrow$ current in circuit stops flowing. Also, in this limit, $V_R = iR \to 0$ (resistor is 'invisible' to the circuit).

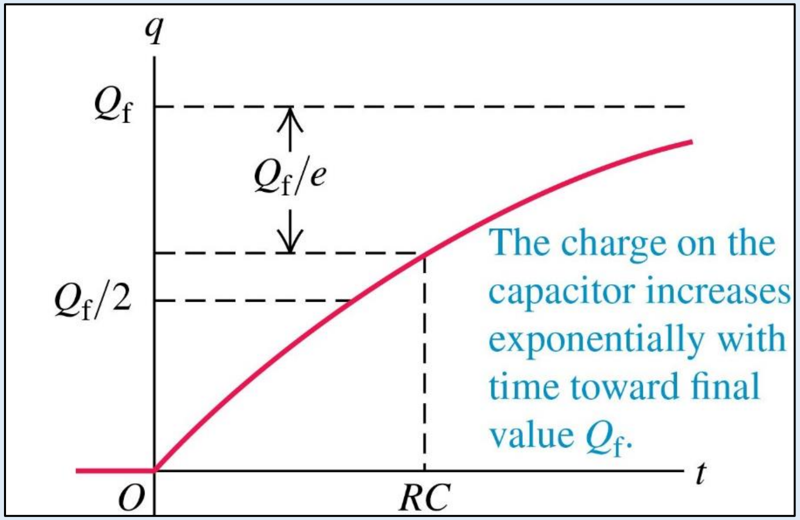
- Quantity

$$\boxed{\tau_{RC} = RC}$$

is the 'RC time constant' of the circuit. It is a characteristic time scale for the capacitor charging process: at $t = \tau_{RC}$,

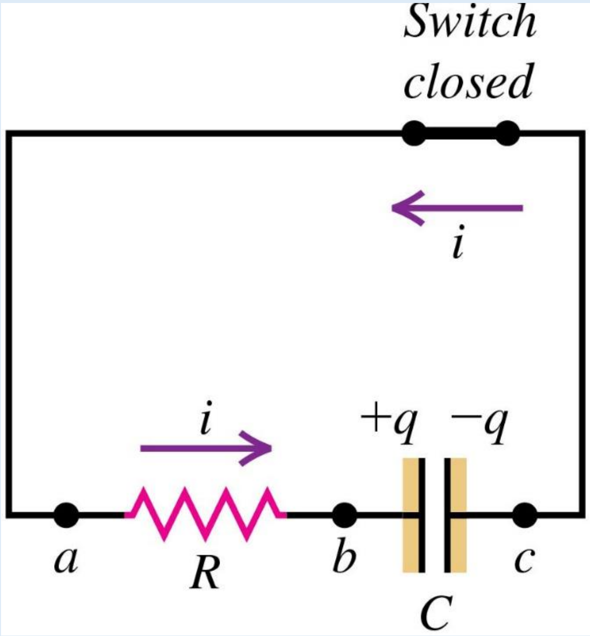
$$q(t = \tau_{RC}) = C\mathcal{E}(1 - e^{-1})$$

$$= Q_{\mathrm{f}}(1 - e^{-1}) \simeq \frac{2}{3}\, Q_{\mathrm{f}}.$$



RC Circuit: Discharge of a Capacitor

- Discharge of capacitor through resistor after closing a switch in an open RC series. Or closing a switch that disconnects an emf from a resistor-capacitor network.
- Switch closed at time $t = 0$, makes a closed circuit.
  - Positive charge $q(t)$ on high potential-plate of capacitor has path to negative charge on lower-potential plate. $\therefore$Capacitor is discharging.



- Kirchoff's loop rule: $-iR - \frac{q}{C} = 0$.
- Rearranging: $iR = \frac{dq}{dt}R = -\frac{q}{C} \quad \Rightarrow \frac{dq}{q} = -\frac{dt}{RC} \quad \Rightarrow \boxed{q(t) = Q_0 e^{-t/RC}}$,

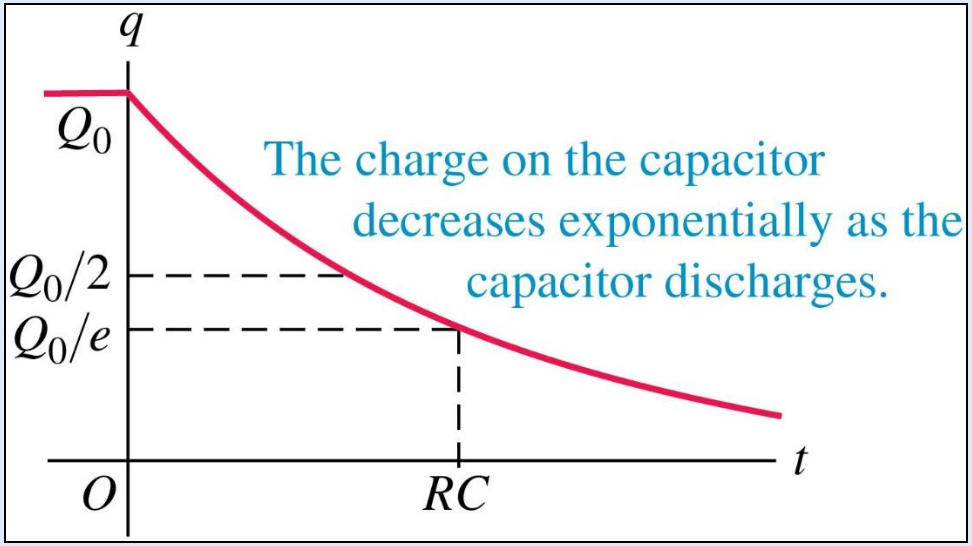
  and $\boxed{i(t) = \frac{dq}{dt} = -\frac{Q_0}{RC}\, e^{-t/RC}}$,

Q_0 = charge on positive capacitor plate at start ($t = 0$) of discharge.

- Current is negative because capacitor is discharging: positive charge on high-voltage capacitor plate decreases with time.

RC Circuit: Discharge of a Capacitor (2)

- At very early times ($t \simeq 0$): $q \simeq Q_0$, $V_C \simeq \frac{Q_0}{C}$, and $i \simeq -\frac{Q_0}{RC}$.
- At later times ($t \to \infty$): $q \to 0$, $V_C \to 0$, and $i \to 0$.
- RC time constant $\tau_{RC} = RC$ is the natural decay time of the charge and current in a discharging RC-circuit.
  - Decay time is e-folding time: $q(t = \tau_{RC}) = Q_0 e^{-1} \simeq \frac{Q_0}{3}$.
  - Decay behavior of circuit similar to decay of radioactive species, with characteristic decay time τ_{rad}.



Lecture Question 15.1

A $2.00 \times 10^{-7} \mathrm{F}$ capacitor is charged to $100 \mathrm{~V}$ and then discharged through a $30.0 \times 10^{4} \Omega$ resistor. How long does it take for the voltage of the capacitor to drop from its initial value to $3.00 \mathrm{~V}$?

A. 0.0335 s.

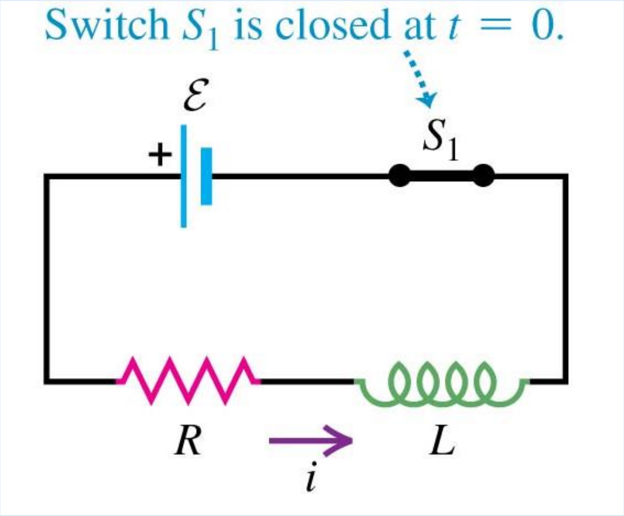
B. 0.210 s.

C. 2.00 s.

D. 17.1 s.

E. 58.4 s.

RL Circuit: Build-Up of Current

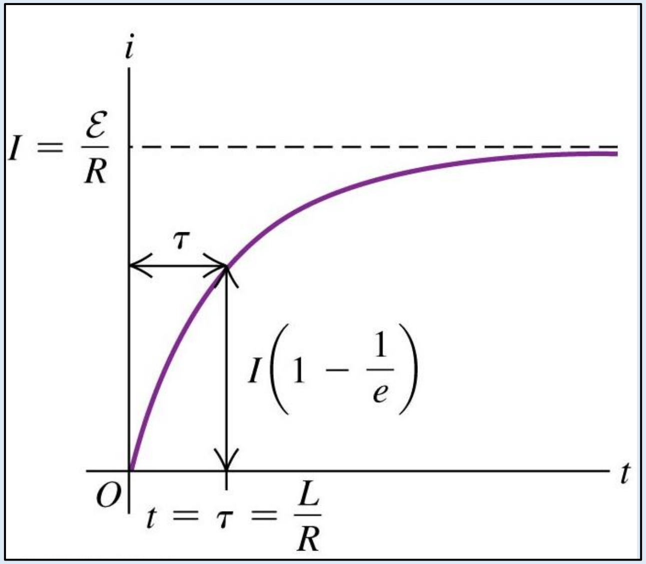


- Like RC circuit: first, close switch.
- Kirchoff's loop rule for closed circuit gives: $\boxed{\mathcal{E} - iR - L\frac{di}{dt} = 0}$.
- Rearranging: $\frac{di}{\left(i - \frac{\mathcal{E}}{R}\right)} = -\frac{dt}{(L/R)}$.
- Equation has solution: $\boxed{i(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-t/(L/R)}\right)}$.
- Result is identical in form to charging of RC circuit.

RL Circuit: Build-Up of Current (2)

- At very early times: $t \simeq 0$, $i \simeq 0$, $V_R = iR \simeq 0$, and $V_L = L\frac{di}{dt} \simeq \mathcal{E}$. Resistor is 'invisible' to circuit, and the voltage across the inductor balances the emf of the power supply.
- As $t \to \infty$: $i \to \mathcal{E}/R$, $V_R \to \mathcal{E}$, and $V_L \to 0$. Inductor becomes 'invisible' to circuit, and voltage across the resistor becomes equal to emf of power supply in the circuit.
- Time scale $\tau_{RL} = L/R$ is the 'RL time-constant'; a characteristic time scale for current build-up:

$$i(t = \tau_{RL}) = \frac{\mathcal{E}}{R}(1 - e^{-1}) \simeq \frac{2}{3}\frac{\mathcal{E}}{R}.$$



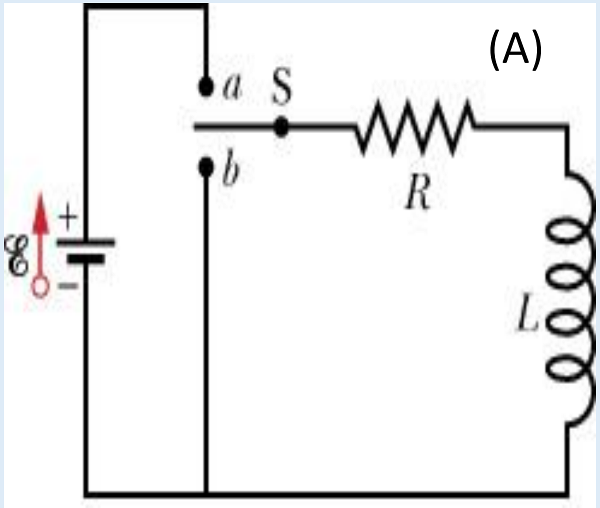
RL Circuit: Decay of Current

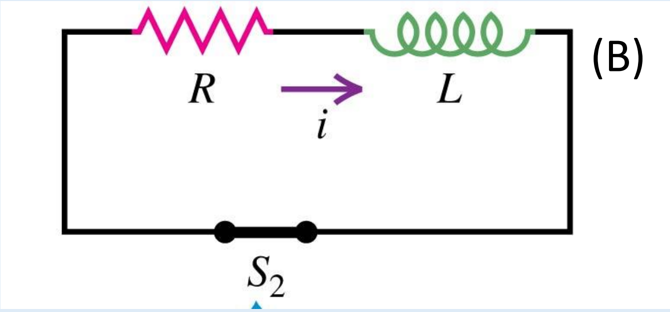
- Discharge circuit: first, have switch closed at position a, and allow emf $\mathcal{E}$ to bring current in RL circuit up to steady value i_0 [Figure (A)].
- Then, move switch to position b: cuts $\mathcal{E}$ out of circuit (same as setting $\mathcal{E} = 0$). Allows decay of current in circuit. Resulting circuit is same as the simplified circuit in Figure (B).
- Kirchoff's loop rule gives:

$$-iR - L\frac{di}{dt} = 0$$

$$\Rightarrow \frac{di}{i} = -\frac{dt}{(L/R)}.$$

- Solution: $\boxed{i(t) = i_0 e^{-t/(L/R)}}$.
- Form of solution is identical to that of a discharging capacitor.

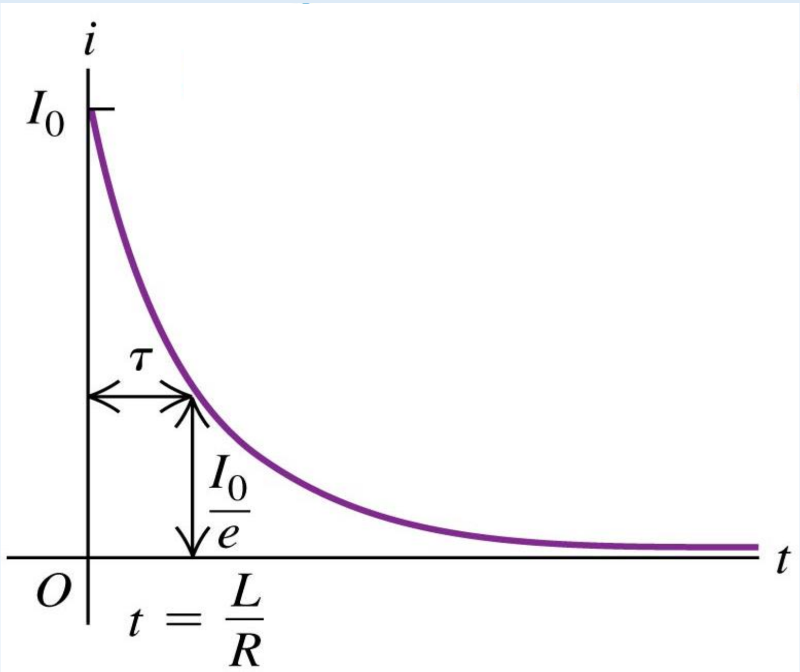




RL Circuit: Decay of Current (2)

- Very short times: $t \simeq 0$, $i \simeq i_0$, and $|V_R| = |V_L| \simeq i_0 R$.
- For $t \to \infty$: $i \to 0$, $V_R \to 0$, and $V_L \to 0$ (Loop rule gives $|V_R(t)| = |V_L(t)|$). Current dissipates away at large t.
- The characteristic decay time for the loss of current is τ_{RL}:

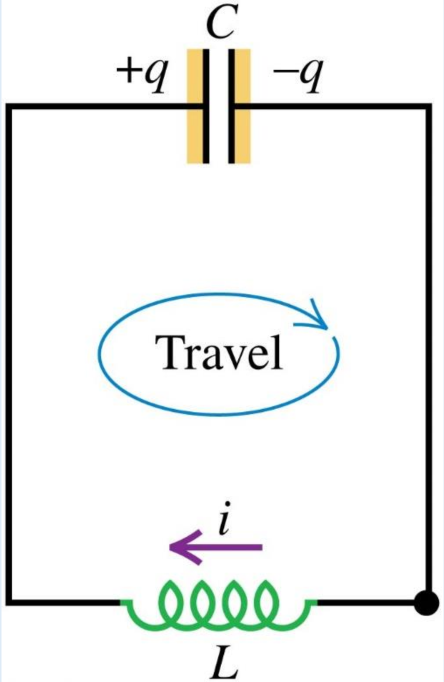
$$i(t = \tau_{RL}) = i_0 e^{-1} \simeq \frac{i_0}{3} \ .$$



LC Circuit: LC Oscillations

- Consider initially open circuit consisting of an ideal charged capacitor C and an ideal inductor L. How does the charge $q(t)$ on the capacitor change after closing the switch?
  - Current flows in circuit, because opposite charges on capacitor plates are drawn to each other.
  - Kirchoff's loop rule:

$$-L\frac{di}{dt} - \frac{q}{C} = 0 \Rightarrow \boxed{L\frac{di}{dt} = -\frac{q}{C}}\ .$$



- Using $i = dq/dt$, and rearranging, gives the governing equation for the charge on the capacitor as a function of time:

$$\boxed{\frac{d^2q}{dt^2} = -\frac{1}{LC}\ q}\ .$$

This is an equation you would have encountered in PHYS 1100, and you know the solution to it: it is identical to the equation for a simple harmonic oscillator!

LC Circuit: LC Oscillations (2)

- PHYS 1100: studied 1-dimensional motion of mass connected to an ideal spring.

  For that case: $F = -kx \quad \Rightarrow \quad ma = m\dfrac{d^2x}{dt^2} = -kx$

  $\Rightarrow \boxed{\dfrac{d^2x}{dt^2} = -\dfrac{k}{m}\,x}$ = equation of motion for simple harmonic oscillator (SHO).
- Equation has the general solution: $\boxed{x(t) = A\cos(\omega t + \phi)}$,

  where $A =$ amplitude of the oscillation, $\phi =$ phase constant, and

  $\boxed{\omega = \sqrt{k/m}}$ is the angular frequency of the oscillation.
- Equation for charge $q(t)$ on capacitor in LC circuit, $\boxed{\dfrac{d^2q}{dt^2} = -\dfrac{1}{LC}\,q}$, <u>identical in form to equation of motion for a SHO</u>.
- Direct analogy: solution is $\boxed{q(t) = Q\cos(\omega t + \phi)}$,

  where $Q =$ maximum (i.e., amplitude) of charge on capacitor plate,

  $\phi =$ phase constant of oscillation, and $\boxed{\omega = \sqrt{1/LC}}$ is the angular frequency of the oscillation.

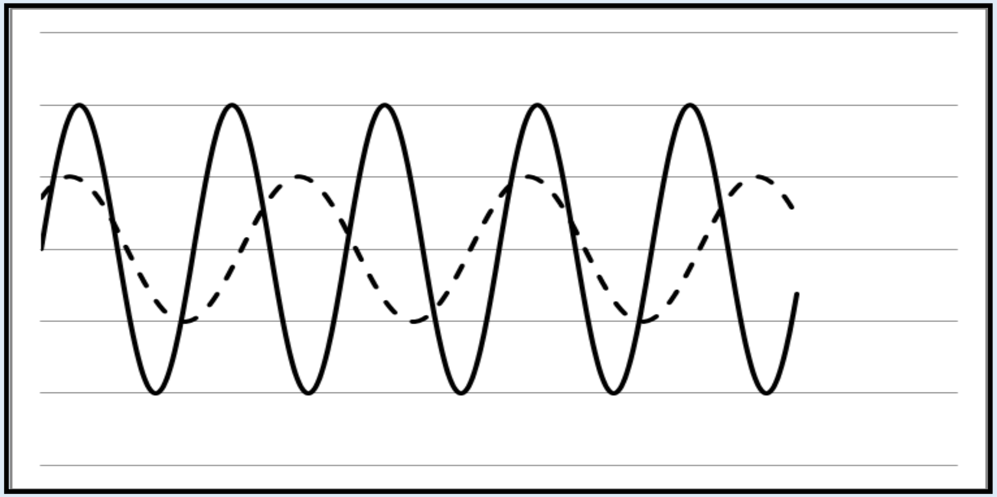
LC Circuit: LC Oscillations (3)

- Solution for $q(t)$ can be obtained from LC governing equation using standard techniques for ordinary differential equations (ode's) with constant coefficients.
  - Can verify that solution $q(t)$ is valid by inserting it into governing equation and showing its consistency – that is, resulting right-hand and left-hand sides of the equation are equal.
- From solution $q(t)$, follows that current through the circuit is

$$\boxed{i(t) = \frac{dq}{dt} = \frac{d}{dt}[Q\cos(\omega t + \phi)] = -\omega Q \sin(\omega t + \phi)}.$$

$\therefore$ Charge on the capacitor and the current in the circuit are $90°$ out of phase.

Lecture Question 15.2

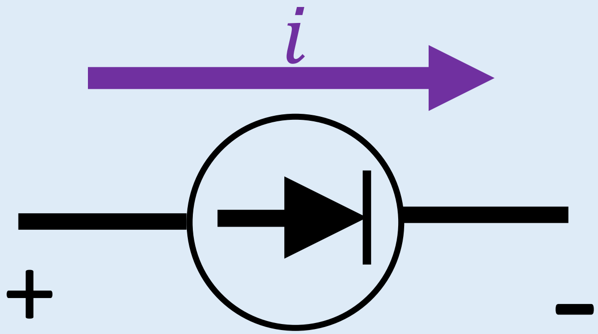


The current through the inductor in each of two different LC circuits is plotted as a function of time. The value of the capacitance is the same for both circuits. For which circuit is the inductance the largest?

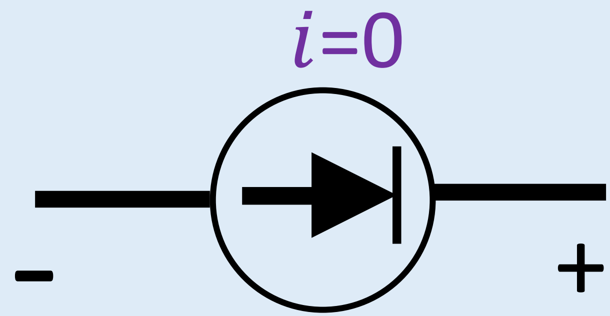
A. The circuit corresponding to the dashed curve.
B. The circuit corresponding to the solid curve.
C. Both circuits have the same inductance.
D. Not enough information to determine.

Lab Activity: Actual Switching

- In your lab today, switching in one experiment will be performed by use of a diode.
  - ➢ Diode acts as a 'one-way' valve: allows current to flow only in one direction, namely, the 'forward' direction of the diode (also known as <u>forward biasing</u>):



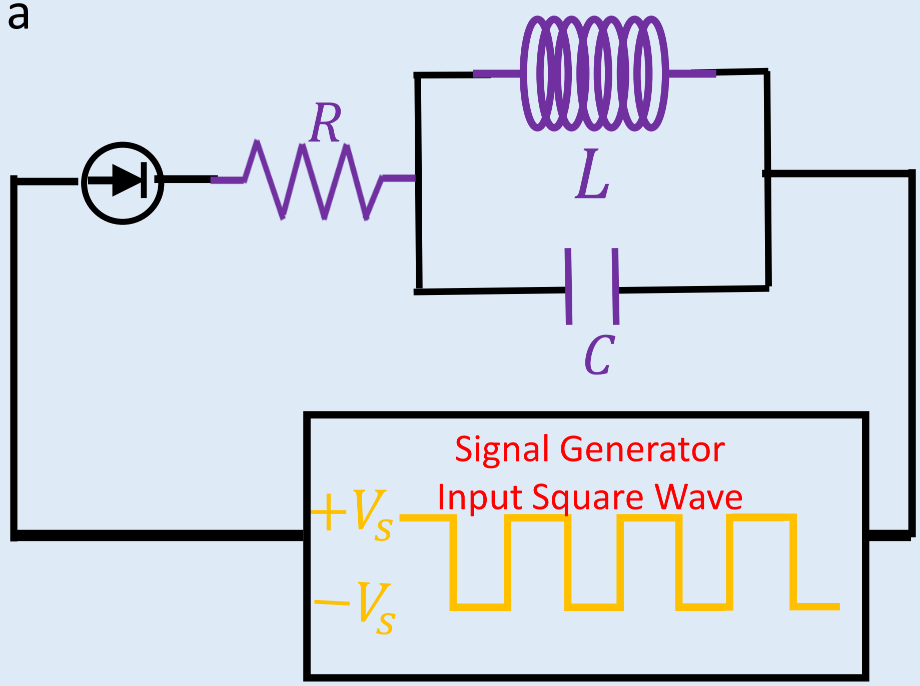
Forward biased, current flows through diode in direction shown.



Reverse biased, current flow blocked by diode.

Lab Activity: Actual Switching (2)

- Lab: construct an LC circuit in series with a resistor and diode, as shown.
- Input voltage signal from a source generator will be a square wave, which alternates between $+V_s$ and $-V_s$ at a specified frequency.
  - When voltage signal $V = +V_s$ diode is forward-biased. Switch (i.e., circuit) is 'closed' and capacitor charges to a value Q_0.
  - When voltage signal $V = -V_s$, diode is reverse-biased. Switch (i.e., circuit) is 'open', and capacitor discharges through inductor.



  - Oscilloscope input leads placed across inductor or capacitor lets you view LC-oscillation curve. Can measure period of oscillation directly from oscilloscope trace and compare to the predicted value for an LC-circuit.

LC Oscillations: Conservation of System Energy

- Like harmonic oscillator system, solution for LC oscillations is consistent with <u>conservation of the total energy of the inductor-capacitor system</u>:
  - Recall that electrostatic energy stored in the capacitor is

$$U_E(t) = \frac{[q(t)]^2}{2C}$$

$$\Rightarrow \quad \boxed{U_E(t) = \frac{1}{2C}\,[Q\cos(\omega t + \phi)]^2 = \frac{Q^2}{2C}\,\cos^2(\omega t + \phi)}.$$

  - Magnetic energy stored in the inductor at any time is

$$U_B(t) = \frac{1}{2}L[i(t)]^2$$

$$\Rightarrow U_B(t) = \frac{1}{2}L\,[-\omega Q\sin(\omega t + \phi)]^2 = \frac{1}{2}\,L\,\omega^2 Q^2\sin^2(\omega t + \phi)$$

Since $\omega = 1/\sqrt{LC}$, follows that $\boxed{U_B(t) = \frac{Q^2}{2C}\sin^2(\omega t + \phi)}$.

  - Total system stored energy is:

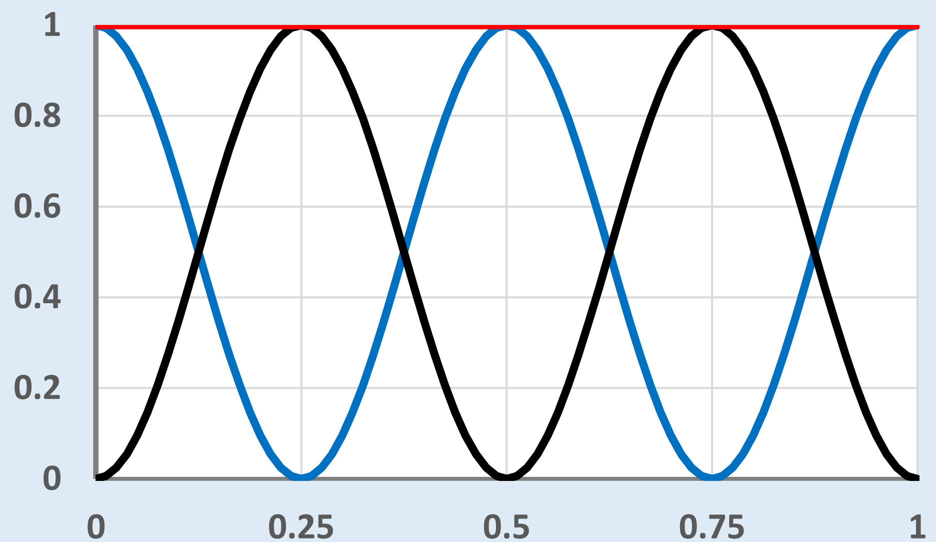
$$U_{tot}(t) = U_E(t) + U_B(t) = \frac{Q^2}{2C}\cos^2(\omega t + \phi) + \frac{Q^2}{2C}\sin^2(\omega t + \phi) = \frac{Q^2}{2C}.$$

$\therefore$ <u>The total stored energy of the LC circuit system is constant and equal to the electrostratic energy stored in the capacitor at its maximum charge Q</u>.

LC Oscillations: Conservation of System Energy (2)

- Plotting the LC-system energies as a function of time, for an example circuit with $\phi = 0$:

$$\frac{U_{tot}}{U_{tot}}, \frac{U_E}{U_{tot}}, \frac{U_B}{U_{tot}}$$



$$\frac{t}{T}$$

$$T = \frac{2\pi}{\omega} = \text{period}$$

$$U_E = \frac{[q(t)]^2}{2C}, \ U_B = \frac{1}{2}L[i(t)]^2, \ U_{tot}(t) = U_E(t) + U_B(t)$$

Analogies Between SHO and LC Systems

Mass-Spring System

Kinetic energy $= \frac{1}{2}mv_x^2$

Potential energy $= \frac{1}{2}kx^2$

$\frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$

$v_x = \pm\sqrt{k/m}\sqrt{A^2 - x^2}$

$v_x = dx/dt$

$\omega = \sqrt{\frac{k}{m}}$

$x = A\cos(\omega t + \phi)$

Inductor-Capacitor Circuit

Magnetic energy $= \frac{1}{2}Li^2$

Electrical energy $= q^2/2C$

$\frac{1}{2}Li^2 + q^2/2C = Q^2/2C$

$i = \pm\sqrt{1/LC}\sqrt{Q^2 - q^2}$

$i = dq/dt$

$\omega = \sqrt{\frac{1}{LC}}$

$q = Q\cos(\omega t + \phi)$

Lecture Question 15.3

In an oscillating LC circuit, the total stored energy is U_{tot} and the maximum charge on the capacitor is Q. When the charge on the capacitor is $\frac{1}{2}Q$, the energy stored in the inductor is:

A. $\frac{1}{4}U_{tot}$.

B. $\frac{1}{2}U_{tot}$.

C. $\frac{3}{4}U_{tot}$.

D. $\frac{4}{3}U_{tot}$.

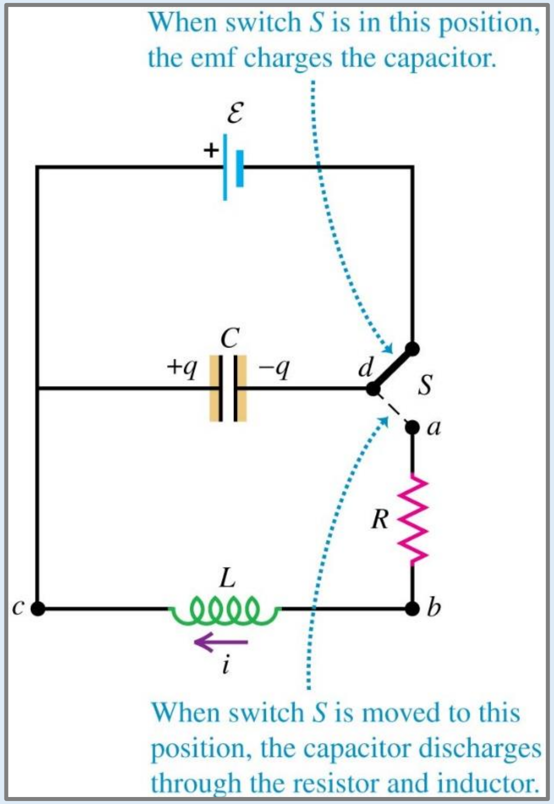
E. $\frac{3}{2}U_{tot}$.

Series LRC Circuit

- Circuit with charged capacitor C and charge q, with an inductor L and a resistor R, such as in the shown figure.
- When switch is moved to cut emf out of circuit, capacitor, inductor, and resistor are all in series $\Rightarrow$ series LRC circuit.
- As in the LC-circuit, when switch is closed at position a, capacitor can charge/discharge, and current i flows in the LRC circuit.
- Kirchoff's loop rule:

$$-iR - L\frac{di}{dt} - \frac{q}{C} = 0 \Rightarrow L\frac{di}{dt} = -\frac{q}{C} - iR$$

$$\Rightarrow \boxed{\frac{d^2q}{dt^2} = -\frac{q}{LC} - \frac{R}{L}\frac{dq}{dt}}\ .$$



This is the governing equation for an LRC circuit. Note: for $R \to 0$, equation reduces to the governing equation for LC oscillations.

Series LRC Circuit (2)

- LRC circuit is more realistic than LC circuit, as LC circuit assumed ideal conditions, i.e., no resistance anywhere in circuit. However, real solenoids and connections have some finite (small, but not $R = 0$) resistance.
- Mechanical analog of governing equation $q(t)$ for LRC circuit is motion $x(t)$ of a spring-mass SHO with friction.
  - Has damping and loss of mechanical energy from spring-mass system.
  - Resistive term in LRC governing equation causes decay and loss of electromagnetic energy in inductor-capacitor system.
- As in LC circuit, standard solution techniques for an ordinary differential equation with constant coefficients gives solution of governing equation for $q(t)$ (when $R^2 < 4L/C$):

$$\boxed{q(t) = Qe^{-t/\tau_d}\cos(\omega' t + \phi)}$$, where

$\boxed{\tau_d \equiv 2L/R}$ is the decay time of the oscillation, and

$\boxed{\omega' \equiv \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}}$ is the angular frequency of oscillation.

Series LRC Circuit (3)

- Behavior of charge $q(t)$, and current $i(t) = dq/dt$ are oscillatory, with an overall exponential decay.

- Note: in limit of weak damping ($R^2 \ll \frac{4L}{C}$) frequency

$$\omega' \simeq \omega = \frac{1}{\sqrt{LC}} .$$

  - Corresponds to LC oscillations with gradual decay of system energy, lost due to ohmic (i.e., resistive) dissipation in circuit.