# Note

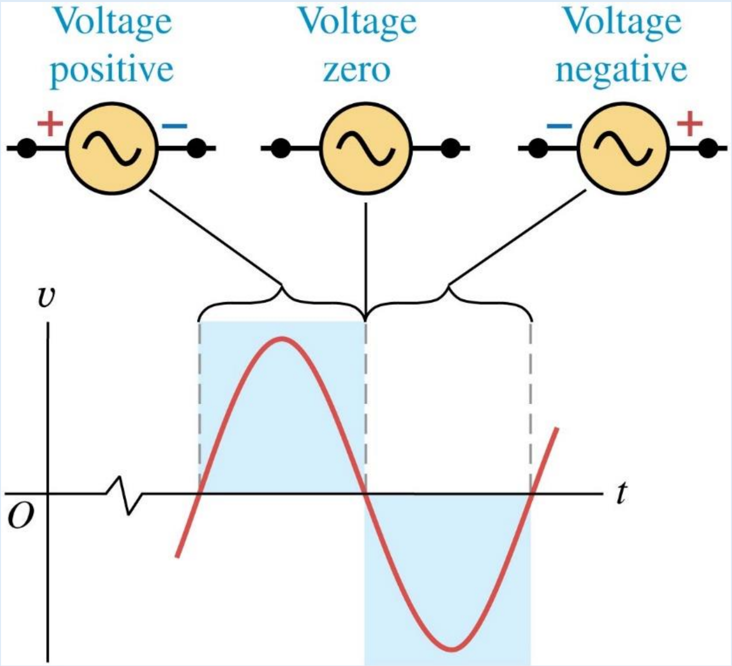
- The topic of today's class (Class 16) will be the last of the new material that will be covered on Exam 2 (Wednesday, March 27).
- The topics of next week's classes (Classes 17 and 18) will appear instead on Exam 3 (Wednesday, April 17).

# Physics 1200
# Lecture 16
# Spring 2024

Alternating Current Circuits, Phasors, Reactance, Impedance, Power, Transformers

# Alternating Current Sources and Uses

- Modern power generation distribution systems operate as alternating current (AC, or ac) systems.
  - ➢ Power generation and transmission.
  - ➢ Step-up and step-down transformers.
- Appliances that you plug into a wall outlet use AC.
- An <u>AC source</u> is a device that supplies a sinusoidally varying voltage.



# Alternating Current Sources and Uses (2)

- Sinusoidal voltage described by a function such as:

$$v(t) = V\cos(\omega t) \ ,$$

$v(t) =$ instantaneous potential difference at time $t$, $V$ is voltage amplitude, and $\omega = 2\pi f$ is angular frequency ($f$ is linear frequency).

- In the U.S. and Canada, commercial power distribution systems use $f = 60 \text{ Hz}$.
- Corresponding sinusoidal alternating current is:

$$\boxed{i(t) = I\cos(\omega t)} \ ,$$

where $I =$ current amplitude.

# Root-Mean-Square Values of Alternating Quantities

- AC circuits oscillate in time. Useful to define a type of average over a regular time interval (such as period of oscillation $T = \frac{1}{f} = \frac{2\pi}{\omega}$ ) for quantities.
- One type of average has trivial result: average function over a period. For instance, average current over 1 period in an AC circuit is

$$i_{\mathrm{av}} = \frac{1}{T} \int_0^T i(t)dt = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} I\cos(\omega t)\, dt$$

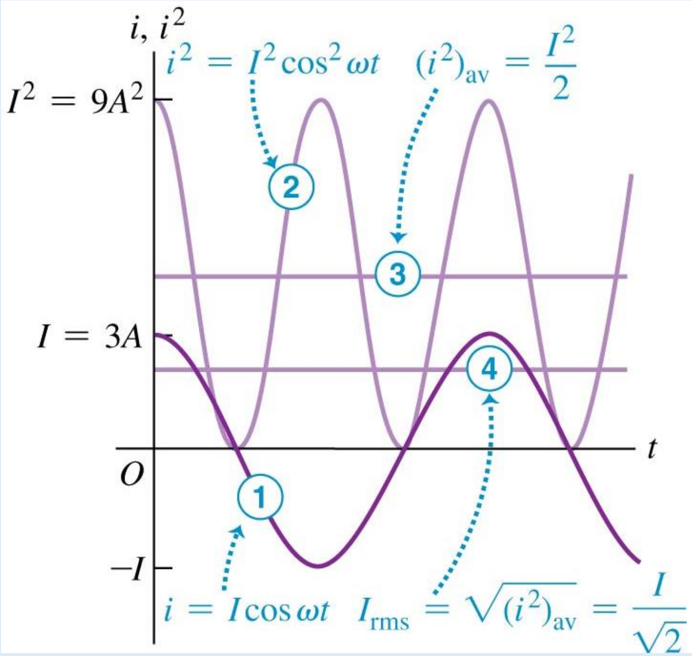
$$= \frac{I}{2\pi} [\sin(2\pi) - \sin(0)] = 0$$

This result is for any sinusoidal function averaged over 1 period.(Has limited use.)

- More useful (non-trivial) type of average is the <u>average of the square of a sinusoidal function, then take the square root of that quantity</u>. This is known as the “root-mean-square” (rms) average.

# Root-Mean-Square Values of Alternating Quantities (2)

- Calculate rms value of sinusoidal current:
  1. Graph current $i(t)$.
  2. Square current $i^2(t)$
  3. Take average (mean) value of $i^2(t)$ over one period.
  4. Take square root of average.




Solution: $(i^2)_{\mathrm{av}} = \frac{1}{T} \int_0^T i^2(t)\, dt = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} I^2 \cos^2(\omega t)\, dt$

$$= \frac{\omega}{2\pi} I^2 \int_0^{2\pi/\omega} \left[\frac{1}{2} + \frac{1}{2}\cos(2\omega t)\right] dt = \frac{1}{2} I^2.$$

$$\Rightarrow (i^2)_{\mathrm{av}} = \frac{1}{2} I^2 \quad \Rightarrow \boxed{I_{\mathrm{rms}} \equiv \sqrt{(i^2)_{\mathrm{av}}} = \frac{1}{\sqrt{2}} I}.$$

# Root-Mean-Square Values of Alternating Quantities (3)

- For sinusoidal ac sources, the rms current and voltage values are:

$$\boxed{I_{\mathrm{rms}} = \frac{1}{\sqrt{2}}\, I}\,, \quad \text{and} \quad \boxed{V_{\mathrm{rms}} = \frac{1}{\sqrt{2}}\, V}\,.$$

- In U.S., wall sockets have voltage amplitude $V$ = 170 V, meaning that voltage alternates between +170 V and −170 V.
  - ➢ The rms voltage is $V_{\mathrm{rms}} = 120\ \mathrm{V}$.
- A listing of rms wall-socket values for various countries around the world can be found at:

https://www.worldstandards.eu/electricity/plug-voltage-by-country/

# Representation of Harmonic Quantities: Phasors

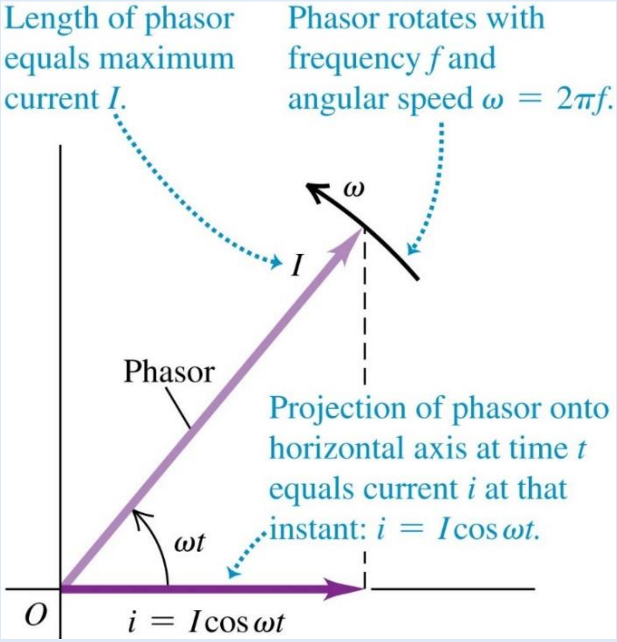
- When dealing with harmonic quantities (sinusoidal functions), operations such as analysis and addition often carried out using "phase vectors", also known as "<u>phasors</u>".
  - Phasors are planar "vectors" emanating from the origin in a Cartesian "<u>phase plane</u>". They rotate in the phase plane about the origin in the counter-clockwise direction at angular velocity $\omega$ that is equal to the angular frequency of the oscillation.
  - Allows use of vector properties to perform operations such as addition.
  - Note: phasors aren't real vectors, such as velocity, momentum, etc. They don't represent spatial direction of any true physical quantity. The horizontal and vertical axes of the phase plane that the phasors reside in <u>do not</u> correspond to any physical axes in the system being considered.

# Representation of Harmonic Quantities: Phasors (2)

- Phasors used to represent harmonic voltages and currents in AC circuits. For oscillations we assume in this class (and your text), that currents have form $i(t) = I\cos(\omega t)$. Instantaneous values of any AC quantity (e.g., $i(t)$, $v(t)$) correspond to projection of the phasor onto the horizontal phase plane axis.
  - ➢ Note: some texts use AC currents of the form $i(t) = I\sin(\omega t)$. For that convention, instantaneous AC values correspond to projection of phasor onto the vertical phase plane axis.
- At right is a phasor diagram for sinusoidal current.
  - ➢ Instantaneous value of current

$$i(t) = I\ \cos(\omega t)$$

is the projection of the phasor on the horizontal axis at the time shown in the diagram.



# Adding Voltages in an AC Circuit = Adding Phasors

- AC circuits: Kirchoff's loop and current rules

$$\boxed{\sum_{j=1}^{N} v_j(t) = 0} \; ,$$

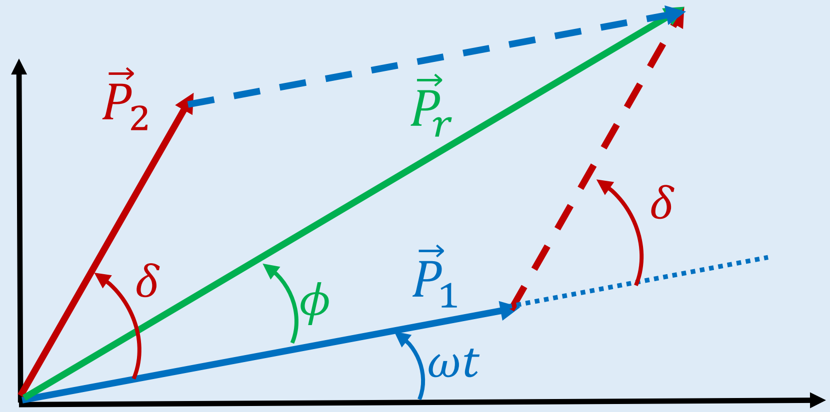
$$\boxed{\sum_{j=1}^{M} i_j(t) = 0} \; ,$$

  are still valid.

- Voltages across different elements in an AC circuit (resistors, capacitors, inductors) will not be in phase.
  - When adding voltages in Kirchoff's loop rule, phases <u>have to be taken into account</u>.
  - Using phasors and the rules of phasor (= vector) addition always gives correct result.

# Phasor / Vector Addition

- AC circuits: addition of harmonics $P_1\cos(\omega t)$ and $P_2\cos(\omega t+\delta)$ ($P_1$ and $P_2$ are constant amplitudes, and $\delta$ is a constant phase angle) can be done using addition relation, $\vec{P}_r = \vec{P}_1 + \vec{P}_2$ , where $\vec{P}_1$ and $\vec{P}_2$ are the two input phasors corresponding to the two input harmonics, and $\vec{P}_r$ is the resultant of the phasor/vector addition.




- From diagram, find:

1. $P_1\cos(\omega t) + P_2\cos(\omega t+\delta) = P_r\cos(\omega t+\phi)$ .
2. $P_r^2 = P_1^2 + P_2^2 + 2P_1P_2\cos\delta$ .
3. $\tan\phi = \dfrac{P_2\sin\delta}{P_1+P_2\cos\delta}$ .

# Phasor / Vector Addition - Example

- Using phasors, find result for

  $\cos(\omega t) + 0.5\cos\left(\omega t + \frac{\pi}{4}\right)$ .

- <u>Solution</u>. Make life easier, find result for $\omega t = 0$ first:

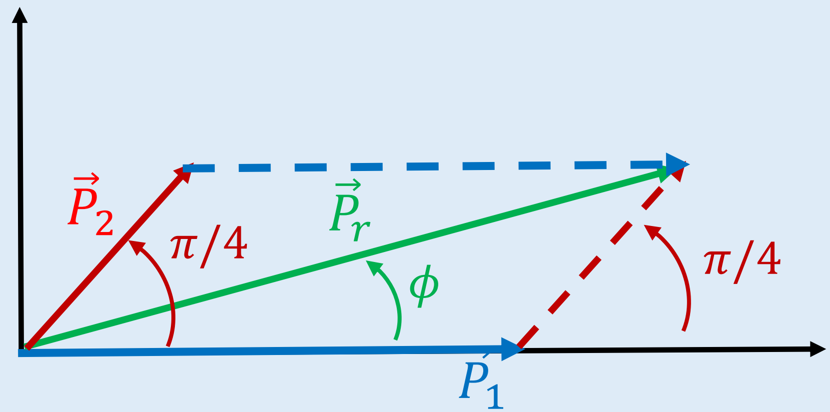
$$P_r = \sqrt{(1)^2 + (0.5)^2 + 2(1)(0.5)\cos(\pi/4)}$$

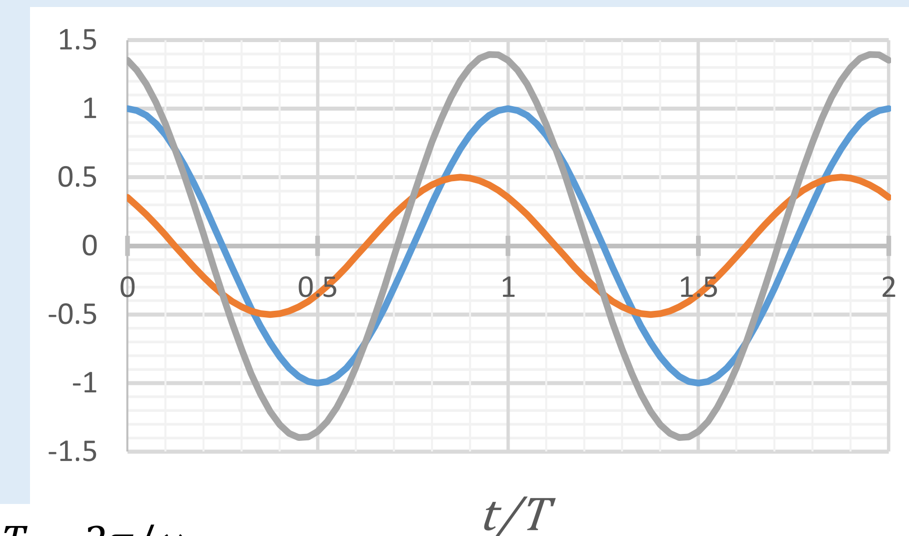
$$= \frac{1}{2}\sqrt{5 + 2\sqrt{2}} = 1.40,$$

$$\phi = \tan^{-1}\left(\frac{0.5\ \sin\left(\frac{\pi}{4}\right)}{1+0.5\cos\left(\frac{\pi}{4}\right)}\right) = 0.261 \text{ rad} .$$

$$\therefore\ \cos(\omega t) + 0.5\cos(\omega t + \pi/4)$$

$$= 1.40\cos(\omega t + 0.261 \text{ rad}) \ .$$




$T = 2\pi/\omega$
$\cos(\omega t)$,
$0.5\cos(\omega t + \pi/4)$ ,
$1.40\ \cos(\omega t + 0.261 \text{ rad})$

# Lecture Question 16.1

- What is the amplitude of the sum of the voltages $v_1(t) = 10\text{ V}\cos(\omega t)$ and $v_2(t) = 10\text{ V}\sin(\omega t)$ ?
  [Recall that $\sin x = \cos\left(x - \frac{\pi}{2}\right)$ .]

A. $20.0\text{ V}$ .
B. $14.1\text{ V}$ .
C. $10.0\text{ V}$ .
D. $0.00\text{ V}$.
E. None of the above.

# Solution

$v_1(t) = 10\text{ V}\cos(\omega t)$ ,

$v_2(t) = 10\text{ V } \sin(\omega t) = 10\text{ V}\cos(\omega t - \frac{\pi}{2})$

From phasor addition amplitude rule:

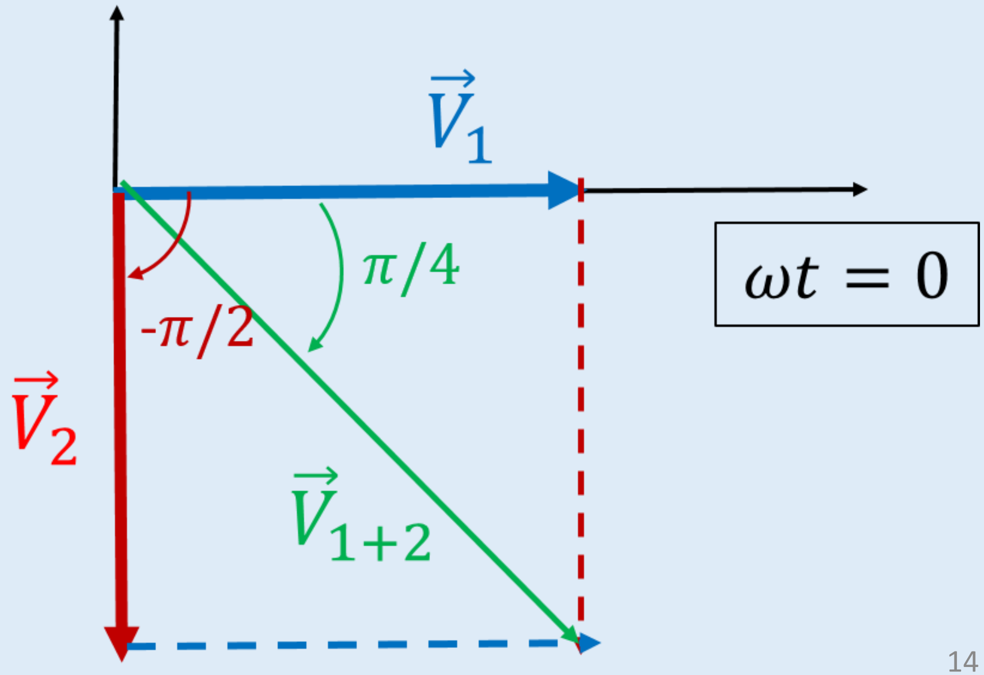
$P_r^2 = P_1^2 + P_2^2 + 2P_1P_2\cos(\delta)$

$V_{1+2}^2 = (10\text{ V})^2 + (10\text{ V})^2 + 2(10\text{ V})(10\text{ V})\cos\left(-\frac{\pi}{2}\right) = 200\text{ V}^2$

$\Rightarrow V_{1+2} = 14.1\text{ V}$

Answer B.

Phasor diagram:

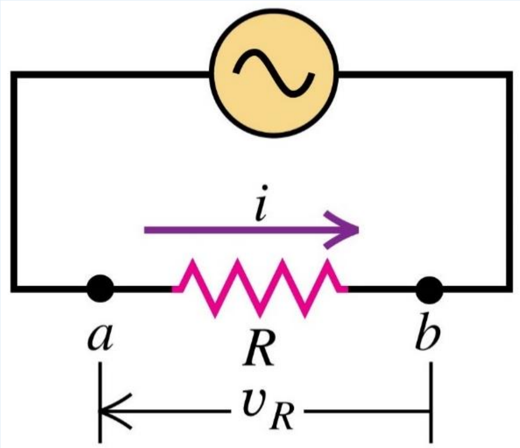


# Resistor in an AC Circuit

- Resistances of resistors typically frequency-independent.
- Consider circuit with sinusoidal emf /current source and resistor.
- AC current in the circuit is

$$i(t) = I\cos(\omega t)\ .$$

- Kirchoff's loop rule (= voltage-current relation for a resistor) for circuit gives instantaneous voltage across resistor:




$$\boxed{v_R(t) = i(t)R = IR\cos(\omega t) = V_R\cos(\omega t)}\ .$$

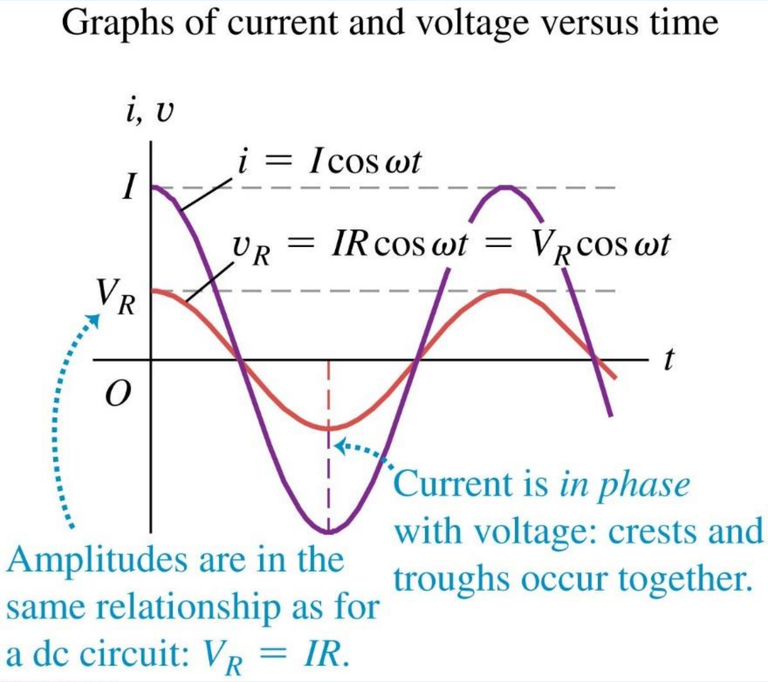
$\therefore$ Voltage phasor across resistor and current phasor always in-phase.

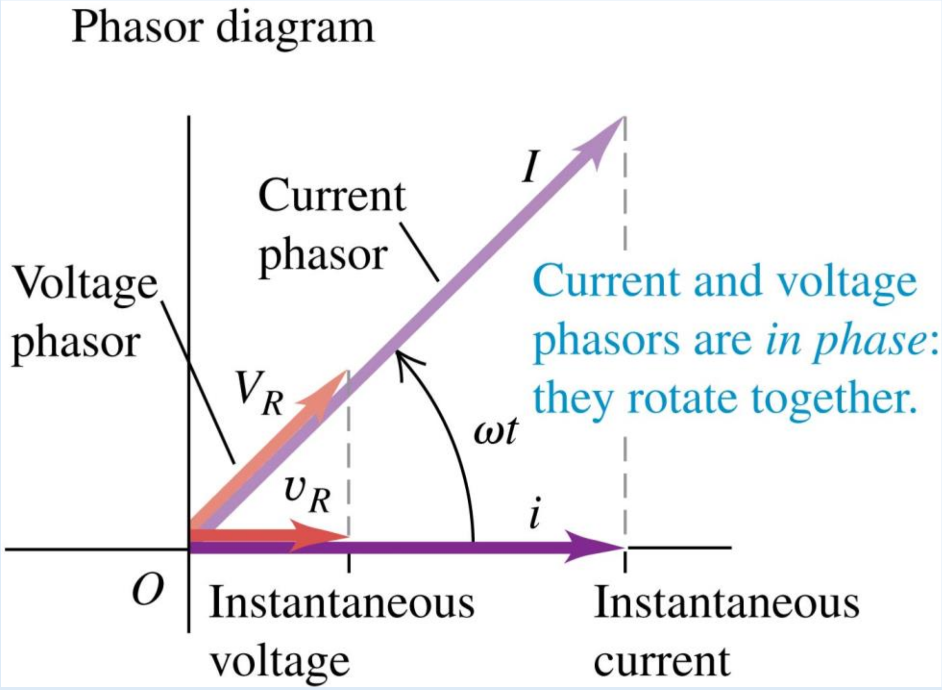
Relation between voltage and current amplitudes is

$$\boxed{V_R = IR}\ .$$

<u>Independent of frequency</u>.

# Resistor in an AC Circuit (2)



# Inductor in an AC circuit

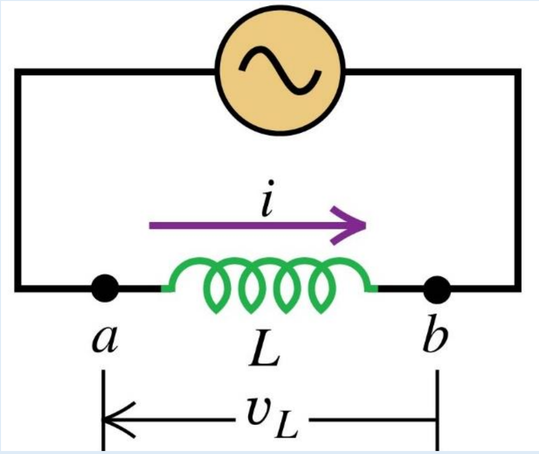
- Consider circuit with AC source and an inductor.
- The instantaneous voltage across the inductor is

$$v_L(t) = L\frac{di}{dt} = L\frac{d}{dt}\left[I\cos(\omega t)\right] = -\omega L I \sin(\omega t)$$

Recalling trigonometric relation

$$-\sin x = \cos\left(x + \frac{\pi}{2}\right) \text{ yields:}$$

$$\boxed{v_L(t) = \omega L I \cos\left(\omega t + \frac{\pi}{2}\right) = V_L \cos\left(\omega t + \frac{\pi}{2}\right)}.$$




- Voltage phasor across inductor <u>leads</u> current phasor by $\pi/2$ radians ($= 90°$).
- Defining <u>AC reactance of an inductor</u> as $\boxed{X_L = \omega L}$,

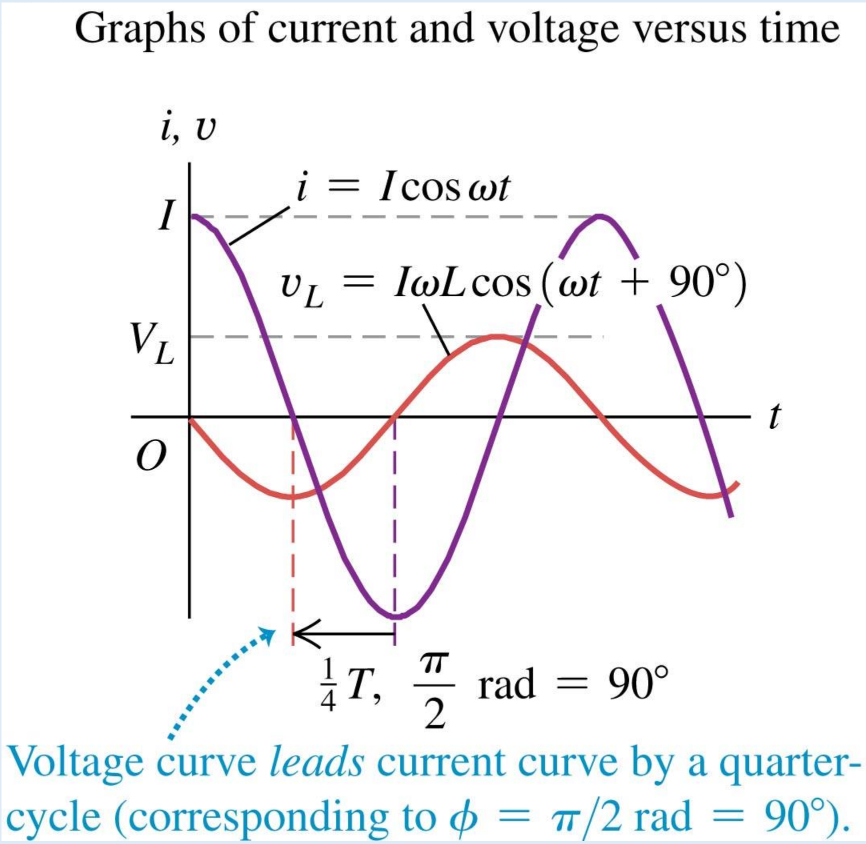
relationship between AC current and inductance amplitudes can be written as

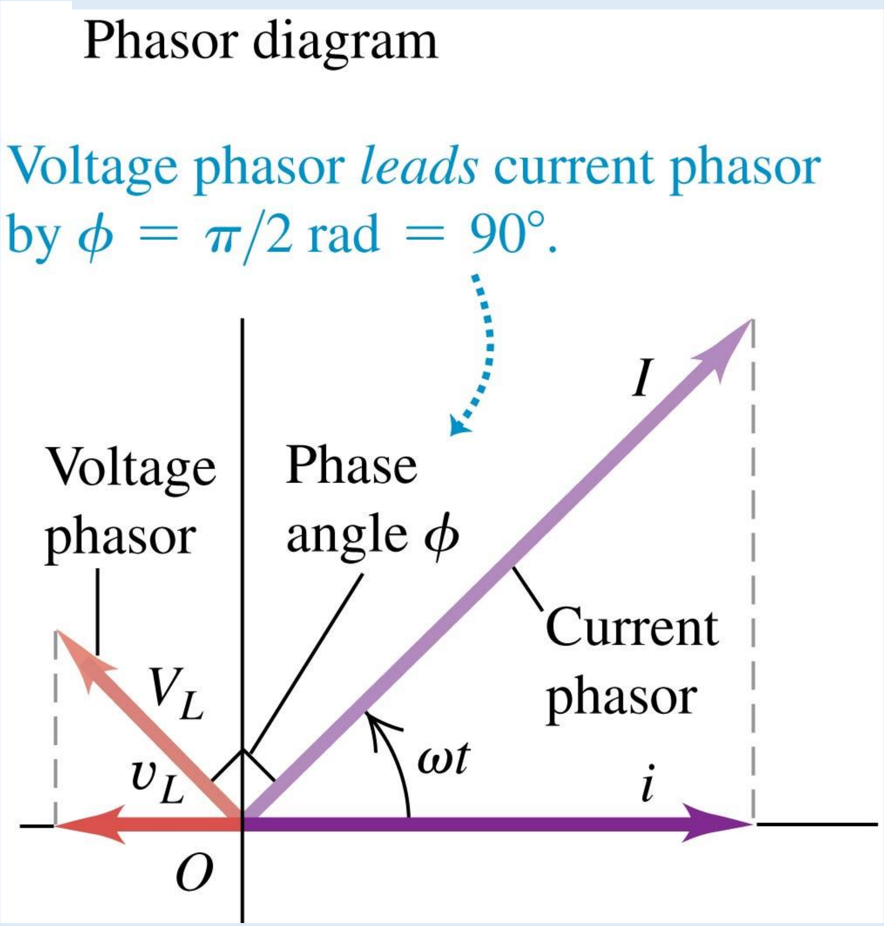
$$\boxed{V_L = I X_L}.$$

Looks like an "Ohm's law" for inductor voltage amplitudes.

> ➢ SI unit of reactance is the ohm (Ω), same as for DC resistance.
>
> ➢ Reactance of an inductor is $\propto \omega$. Increases linearly with increasing frequency.

# Inductor in an AC circuit (2)



# Lecture Question 16.2

- For which of the following frequencies is the reactance of an inductor the largest?

A. 1 Hz.

B. 100 Hz.

C. 1000 Hz.

D. There is insufficient information to answer this question.

# Solution

$X_L = \omega L.$

$X_L \propto \omega.$

$\therefore X_L$ is largest when $\omega$ is largest.

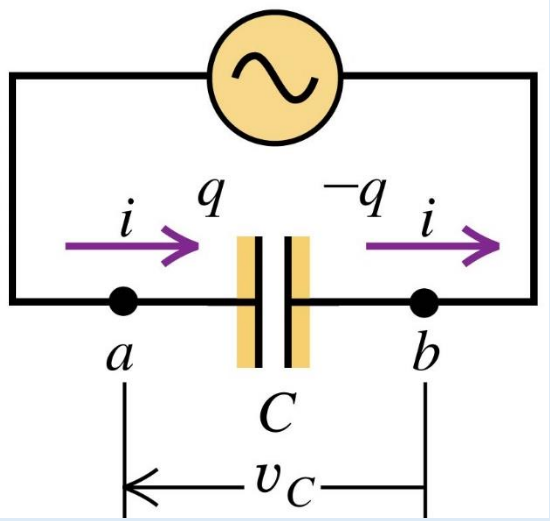
Recall that $\omega = 2\pi f.$

$\Rightarrow X_L$ is largest when $f$ is largest.

Largest $f$ in list is choice 1000 Hz (C).

Answer C.

# Capacitor in an AC circuit



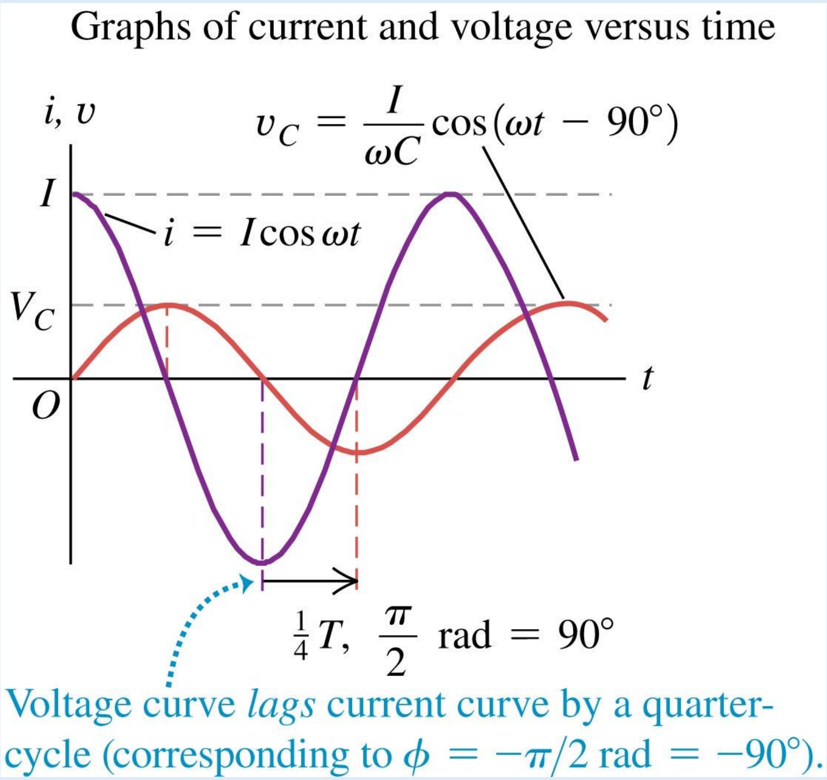

- Consider circuit consisting with AC source and a capacitor.
- Instantaneous voltage across inductor is $v_C(t) = \frac{q(t)}{C}$, $q(t)$ is charge on high-potential capacitor plate.
- Since $i = dq/dt$

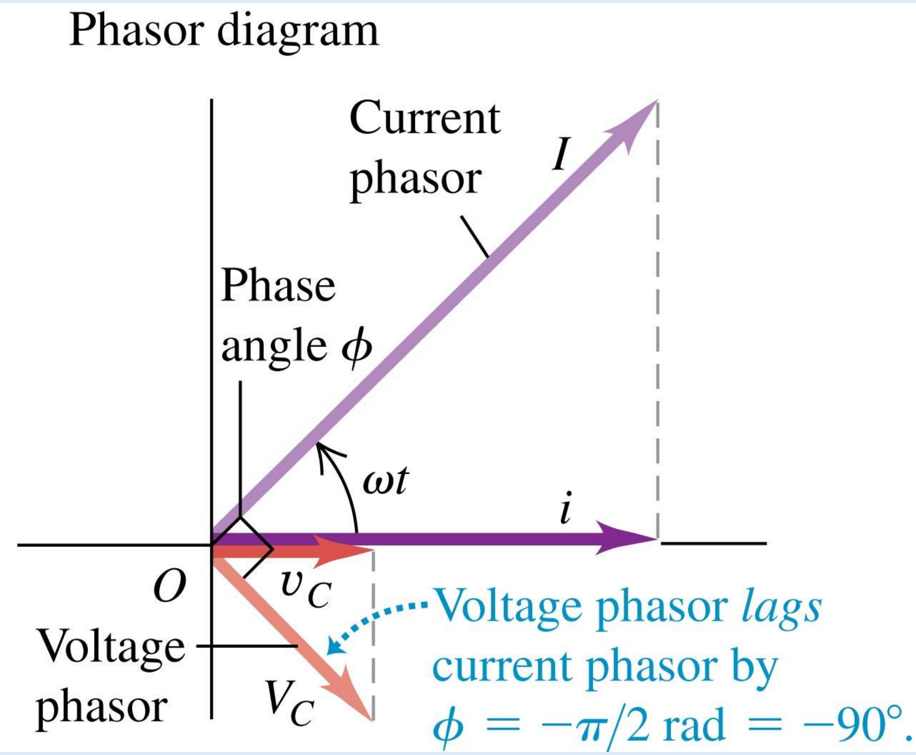
  $\Rightarrow \; q(t) = \int i\, dt = \int I \cos(\omega t)\, dt = \frac{I}{\omega} \sin(\omega t)$ .
- Capacitor voltage: $v_C(t) = \frac{q(t)}{C} = \frac{I}{\omega C} \sin(\omega t)$ .
- Using trig relation $\sin x = \cos(x - \pi/2)$ , voltage across

  capacitor can be written: $\boxed{v_C(t) = \frac{I}{\omega C} \cos\left(\omega t - \frac{\pi}{2}\right) = V_C \cos\left(\omega t - \frac{\pi}{2}\right)}$ .
  - ➢ Voltage phasor for a capacitor <u>lags behind</u> current phasor in AC circuit by $\pi/2$ radians ($= 90°$).
- Defining <u>capacitive reactance:</u> $\boxed{X_C = 1/\omega C}$ , relation between AC current and voltage amplitudes for a capacitor is

  $\boxed{V_C = I X_C}$ . (Looks like an “Ohm’s law” for amplitudes.)
  - ➢ SI unit for $X_C$ is the ohm (Ω).
  - ➢ $X_C \propto 1/\omega$. ∴ Capacitive reactance greater at smaller frequencies.

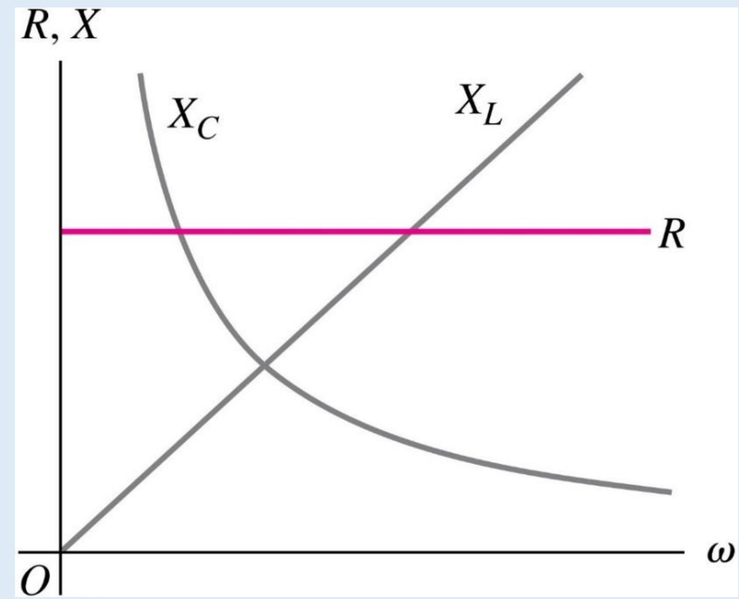
# Capacitor in an AC circuit (2)



# Frequency Response of Circuit Elements

- Frequency behavior of the resistance or reactance of a circuit element varied for each type of device discussed. Sketch at right shows frequency dependencies.
- Key results:
  - ➢ Resistance $R$ is independent of frequency.
  - ➢ If $\omega = 0$, corresponding to a DC circuit (no alternation of current), there is no current through a capacitor because $X_C = \frac{1}{\omega C} \to \infty$. Capacitors react more strongly against low-frequency phenomena.
  - ➢ Limit $\omega \to \infty$: current through an inductor becomes vanishingly small. Inductors react more strongly against high-frequency phenomena.

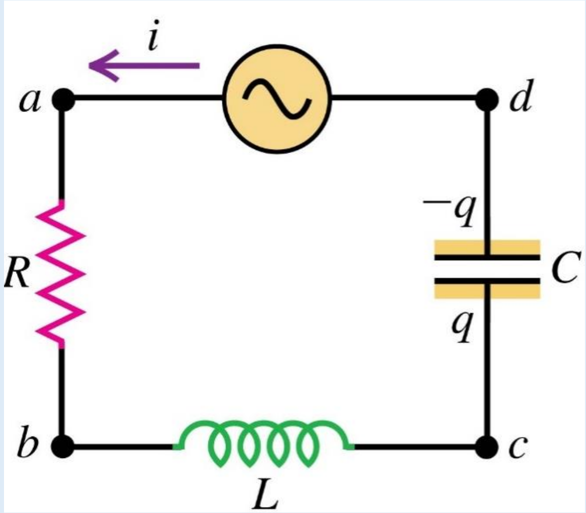


# The Series LRC AC Circuit

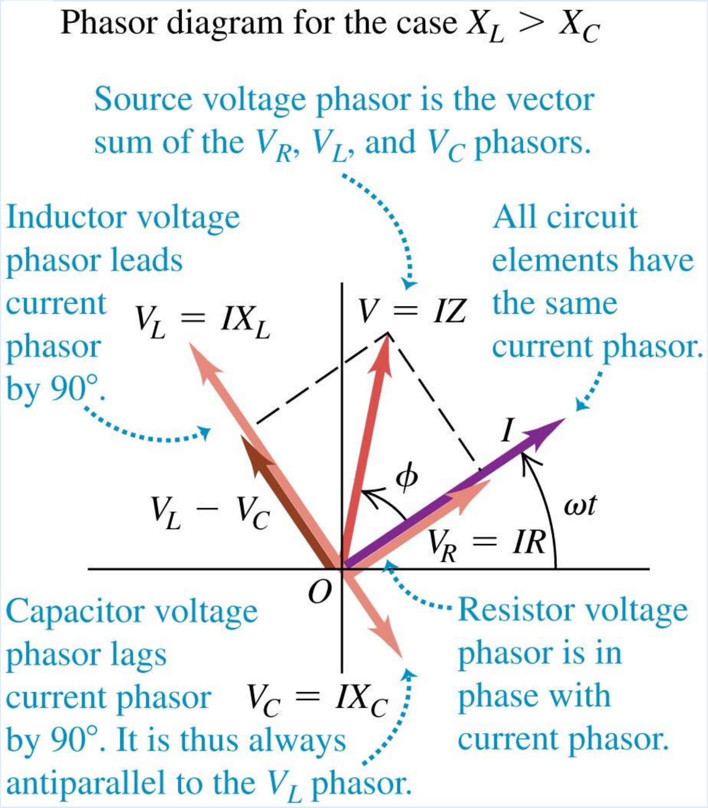
- AC circuit with inductor, capacitor, and resistor in series. Each circuit element has same current $i(t)$ flowing through it.
- Kirchoff's loop rule:

$$v(t) = v_R(t) + v_L(t) + v_C(t)\ .$$

- Analyze circuit using phasors. Kirchoff's loop equation corresponds to projection of summed phasors of the series resistor, inductor, and capacitor when they have the same current flowing through each circuit element.
  - ➢ Use earlier results for amplitudes of the phasors for each element in the phasor diagram.



# The Series LRC AC Circuit: Phasor Diagram




- Key points of phasor diagram:
  - ➢ Voltage phasor of resistor in phase with current phasor.
  - ➢ Voltage phasor of inductor leads current (and the resistor) phasor by $90°$. ∴ Inductor voltage phasor is at right angle to resistor voltage phasor.
  - ➢ Voltage phasor of capacitor lags behind current (and resistor voltage) phasor by $90°$. ∴ Capacitor voltage phasor is at right angle to resistor voltage phasor.
  - ➢ Capacitor voltage phasor lags behind inductor voltage phasor by $180°$ . ∴ <u>Voltage phasors of inductor and capacitor are in opposite directions, and subtract from each other</u>. Can be combined into net LC-voltage phasor with magnitude of $|V_L - V_c|$ at a right angle to the resistor voltage phasor.
  - ➢ Net LC-voltage phasor and resistor voltage phasor form a right triangle.

# The Series LRC AC Circuit: Impedance

- From right triangle formed in series LRC phasor diagram, find

$$V = \sqrt{V_R^2 + (V_L - V_C)^2} = \sqrt{(IR)^2 + (IX_L - IX_C)^2} = I\sqrt{R^2 + (X_L - X_C)^2}.$$

- Can be rewritten as $\boxed{V = IZ}$, where $\boxed{Z \equiv \sqrt{R^2 + (X_L - X_C)^2}}$ is the <u>impedance</u> of series LRC AC circuit.
    - Impedance is a generalized, frequency-dependent resistance of the AC circuit. Has same SI unit as resistance and reactance (ohm Ω).
- Impedance expression above also valid when circuit elements are missing from series circuit.
    - No resistor in series circuit: set $R = 0$ in $Z$-equation.
    - No inductor in series circuit: set $L = 0 \;\; \Rightarrow X_L = \omega L = 0$ in $Z$-equation.
    - No capacitor in series circuit: set $C = \infty \;\Rightarrow X_C = \frac{1}{\omega C} = 0$ in $Z$-equation.
- For given AC source voltage amplitude $V$, current amplitude in circuit

$$I = V/Z\ .$$

Because $X_L$ and $X_C$ are functions of frequency, $I$ is also a function of frequency.

# The Series LRC AC Circuit: Phase Angle

- From LRC phasor diagram it follows that phase angle between circuit's net voltage phasor and current (and resistor voltage) phasor can be found from relation:

$$\boxed{\tan\phi = \frac{V_L - V_C}{V_R} = \frac{IX_L - IX_C}{IR} = \frac{X_L - X_C}{R}}.$$

- $X_L > X_C$: $\phi > 0$, net voltage phasor leads current phasor.
- $X_L < X_C$: $\phi < 0$, net voltage phasor lags current phasor.
- $X_L = X_C$: $\phi = 0$, net voltage and current phasors in phase with each other.

# The Series LRC AC Circuit: Resonance

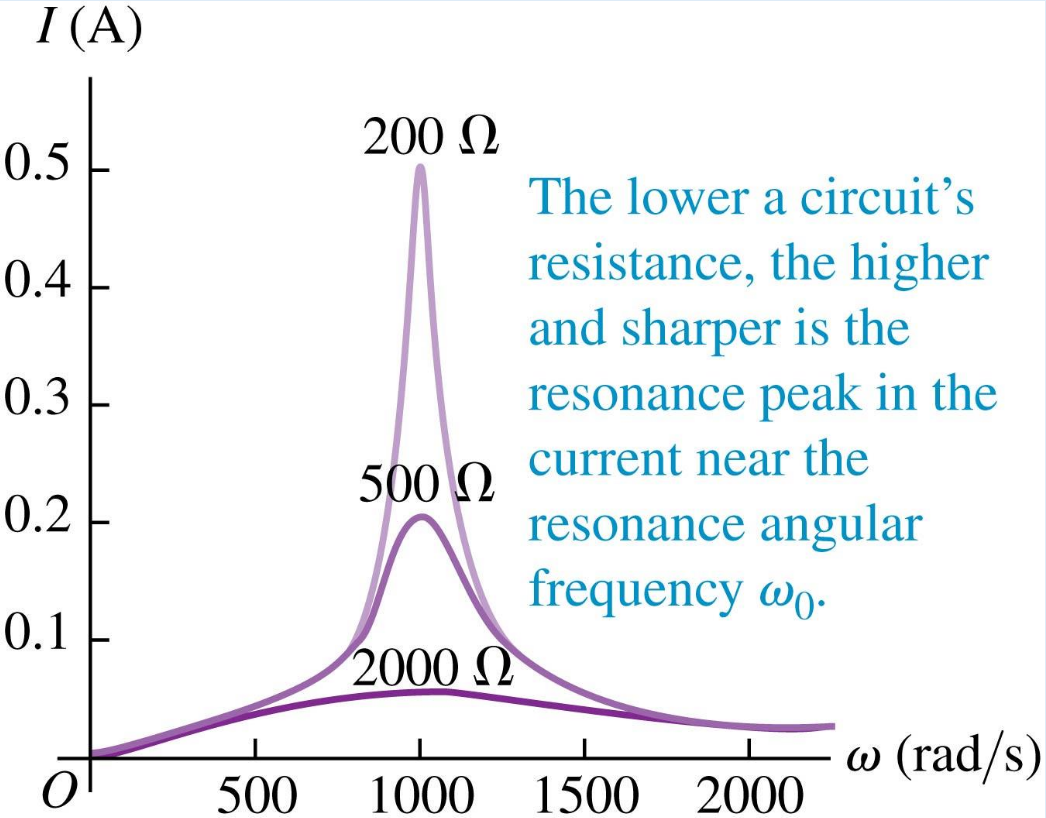
- As angular frequency $\omega$ of source is varied, maximum value of current amplitude $I$ occurs at the frequency for which impedance $Z$ is a minimum.
- Peaking of current amplitude at a certain frequency is called resonance.
- Angular frequency $\omega_0$ at which resonance peak occurs is called the resonance angular frequency.
- At $\omega = \omega_0$, inductive reactance $X_L$ and capacitive reactance $X_C$ are equal:

$$\therefore X_L = X_C \Rightarrow \omega_0 L = \frac{1}{\omega_0 C} \Rightarrow \boxed{\omega_0 = \frac{1}{\sqrt{LC}}}.$$

Resonance angular frequency of a series AC LRC circuit.

- At resonance, $Z = R$.

# The Series LRC AC Circuit: Resonance (2)




- Shown: current amplitude *I* as a function of angular frequency $\omega$ for an LRC series circuit with $V = 100$ V, $L = 2.0$ H, $C = 0.50$ mF, and three different values of resistance $R$.

# AC Circuits in General: Impedance

- Series LRC circuit is an example of an AC circuit.
  - Found that voltage and current amplitudes across circuit can be related by a characteristic impedance of the circuit through an “Ohm’s law” type of relation.
- Other types of AC circuits also can be cast in the form of an “Ohm’s law” for their amplitudes: $V = IZ$ .
- However, the equation for $Z$ will be different in other (i.e., non-series) types of circuits. Impedance for those other kinds of circuits can be also be found using phasor analysis.

# Power in AC Circuits: Resistors

- If circuit element is a pure resistor, voltage and current are *in phase*.
- Instantaneous power

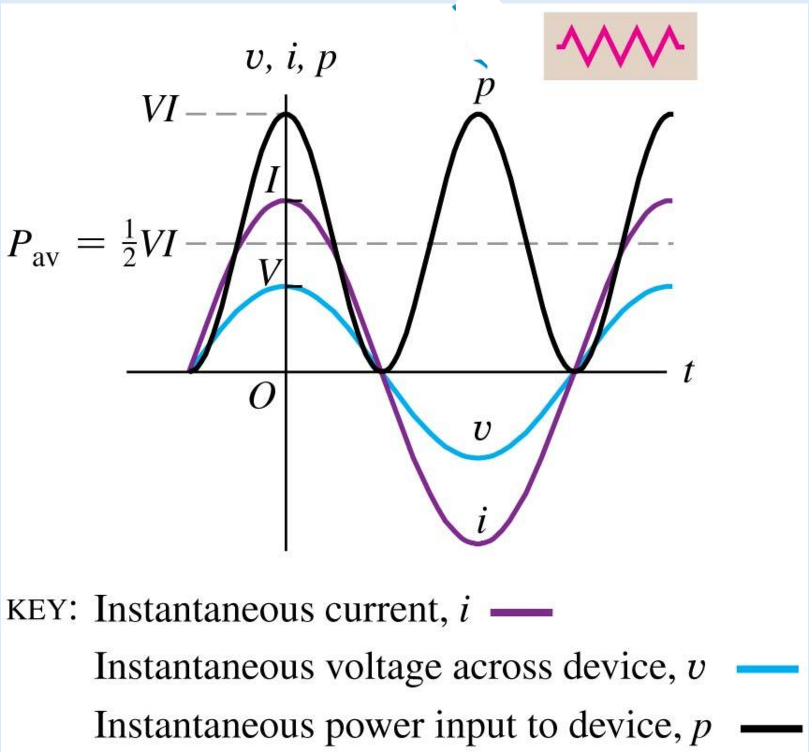
  $$p(t) = i(t)v(t) = [i(t)]^2 R$$

  is always > 0.
- Averaging over 1 period of oscillation, average power dissipated in an AC resistor is

$$P_{\mathrm{av}} = \frac{1}{T}\int_0^{2\pi/\omega} I^2 R[\cos(\omega t)]^2 dt$$

$$\Rightarrow \boxed{P_{\mathrm{av}} = \frac{1}{2}I^2 R = \frac{1}{2}IV = I_{\mathrm{rms}}V_{\mathrm{rms}}}.$$

or, alternatively: $\boxed{P_{\mathrm{av}} = I_{\mathrm{rms}}^2 R}$.



# Power in AC Circuits: Inductors

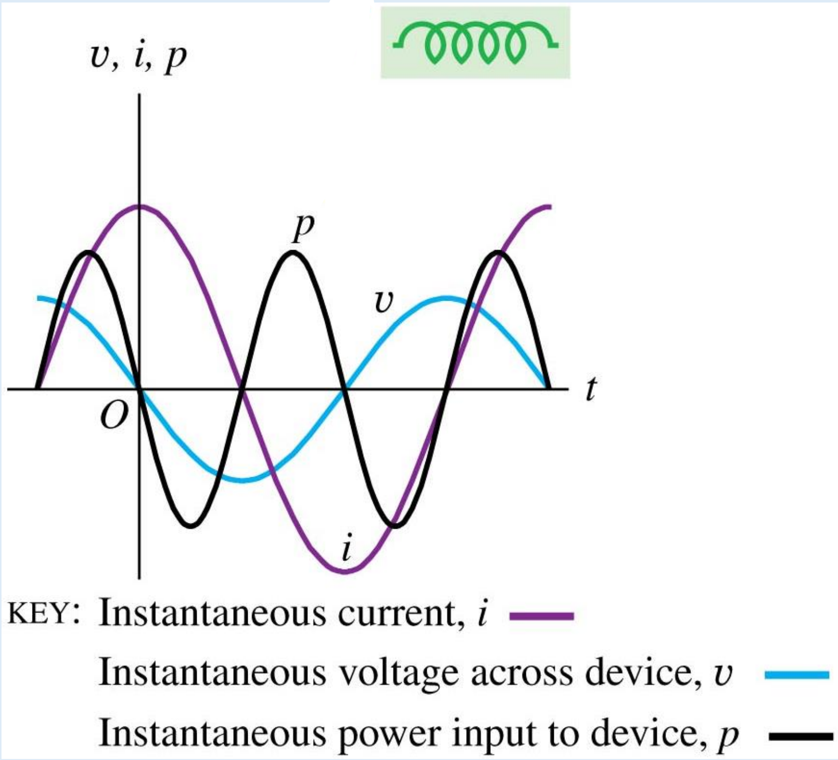
- If circuit element is a pure inductor, voltage *leads* current by 90°.
- Power is < 0 when $v(t)$ and $i(t)$ have opposite signs, and > 0 when they have the same signs.
- Average power over 1 period is zero:

$$P_{av} = \frac{1}{T}\int_0^T i(t)v(t)dt$$

$$= -\frac{\omega}{2\pi}\int_0^{2\pi/\omega} IV\sin(\omega t)\cos(\omega t)\,dt$$

$$= 0.$$

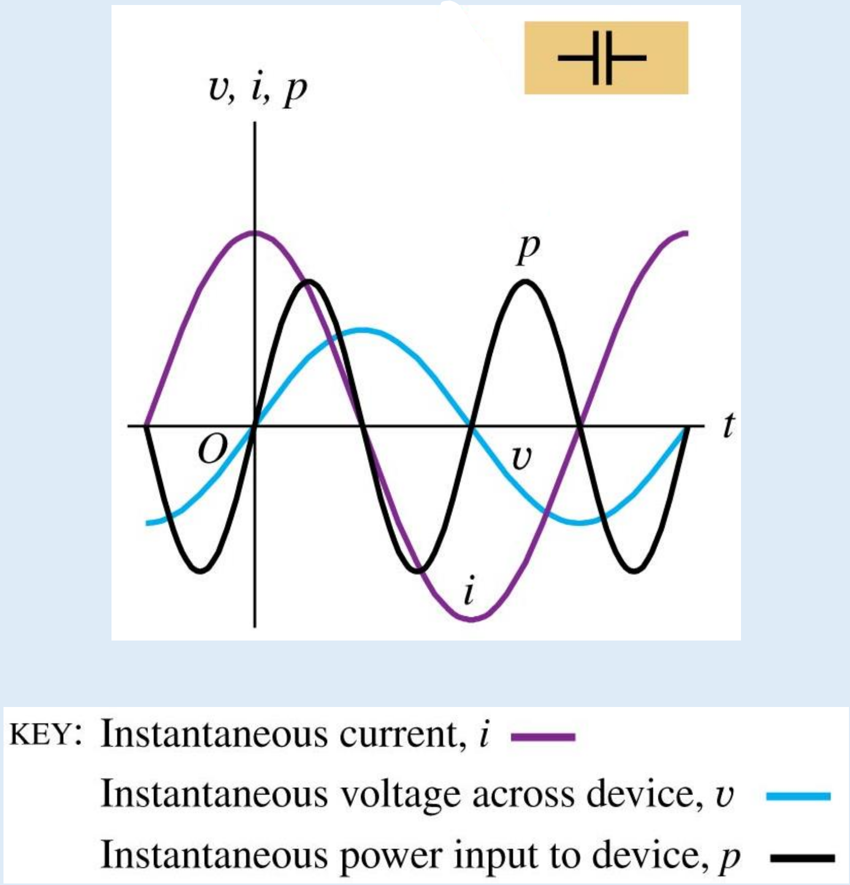
$\therefore \boxed{P_{\text{av}} = 0}$ for an inductor.



# Power in AC Circuits: Capacitors

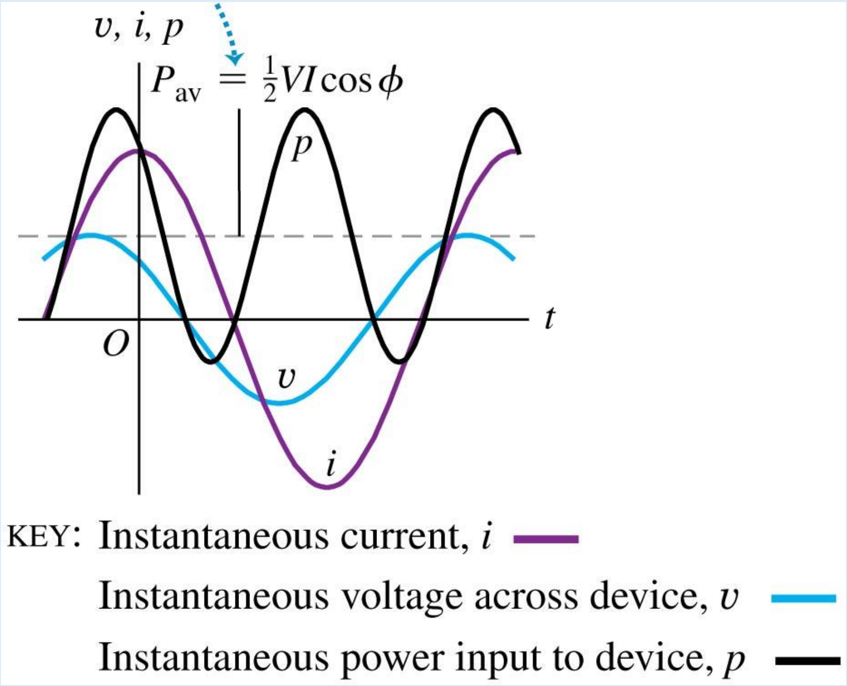
- If circuit element is a pure capacitor, voltage *lags* current by 90°.
- The power is < 0 when $v(t)$ and $i(t)$ have opposite signs, and > 0 when they have the same signs.
- This behavior is the same type as that of the inductor.
- Average power over 1 period of oscillation is also zero for a capacitor:

$$\boxed{P_{\text{av}} = 0}.$$



# Power in AC Circuits: General Circuit

- For an arbitrary combination of resistors, inductors, and capacitors, the average power > 0.



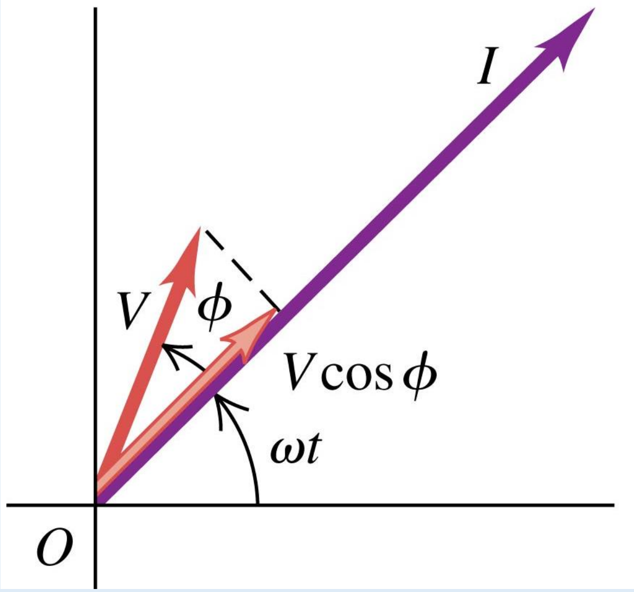
# Power in AC Circuits: General Circuit (2)

- In *any* ac circuit, with any combination of resistors, capacitors, and inductors, voltage $v$ across the entire circuit has some phase angle $\phi$ with respect to the current *i*.
- Comes from relations:

$$i(t) = I\cos(\omega t)\ ,\quad v(t) = V\cos(\omega t + \phi)$$

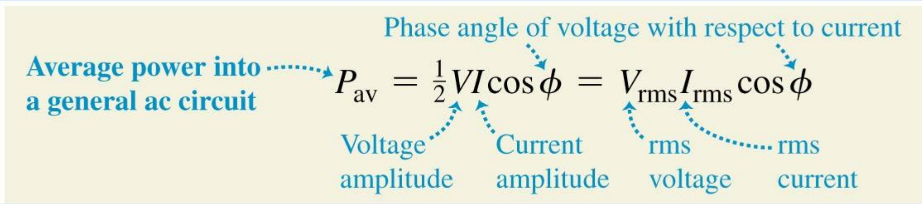
$$\begin{aligned} p(t) = i(t)v(t) &= IV\cos(\omega t)\cos(\omega t + \phi) \\ &= IV\cos(\omega t)\left[\cos(\omega t)\cos\phi - \sin(\omega t)\sin\phi\right] \\ &= IV\cos\phi\left[\cos^2(\omega t) - \cos(\omega t)\sin(\omega t)\tan\phi\right] \end{aligned}$$

$$\rightarrow \boxed{P_{\mathrm{av}} = \frac{1}{2} IV\cos\phi}\ .$$



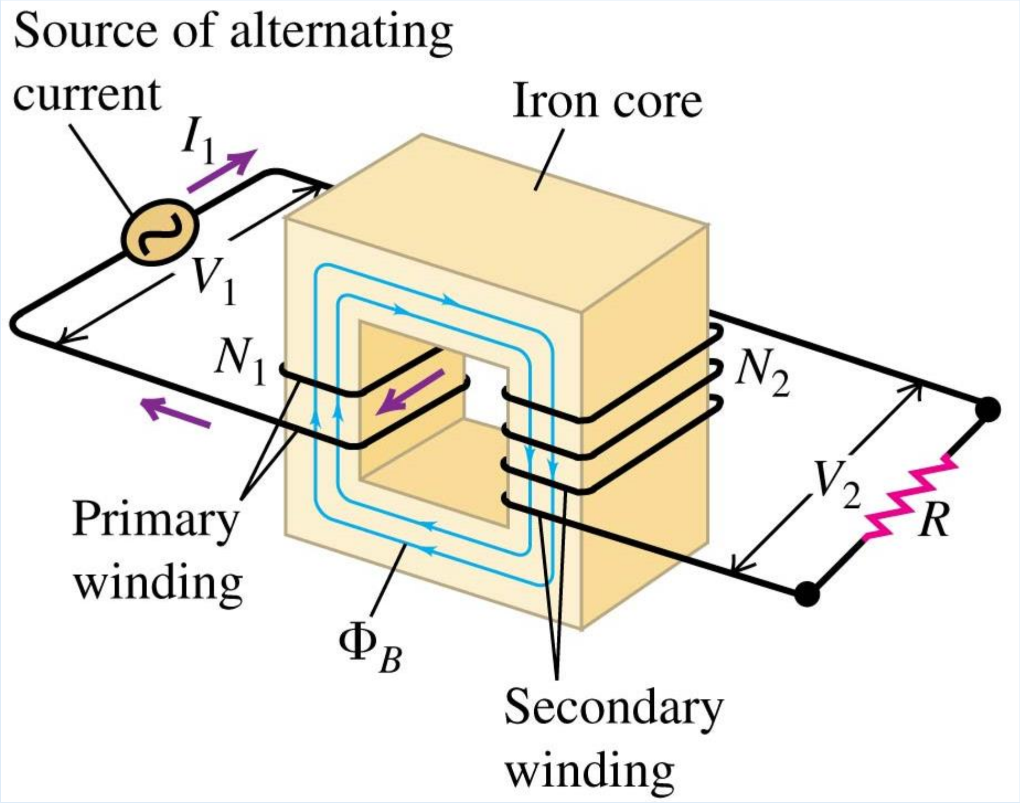

- The factor $\cos\phi$ is called the power factor of the circuit.
- For pure resistor, $\phi = 0$, and power factor is 1.
- For LRC AC series circuit, phasor diagram gives the result:

$$\boxed{\cos\phi = R/Z}\ .$$



# Transformers

- In a <u>transformer</u>, power is supplied to a primary coil, and then a secondary coil delivers power to a resistor.
- Purpose of a <u>step-up transformer</u> (shown), is to increase delivered voltage relative to supplied voltage.



# Transformers (2)

- Voltage transformation occurs due to <u>flux-linkage</u> between the coils 1 and 2: share the same flux $\Phi_B$ passing through the shared iron core.
- Recalling Faraday's law, EMFs created due to changing flux in the coils wrapped around the common iron core are

$$\mathcal{E}_1 = -N_1 \frac{d\Phi_B}{dt} \text{ , } N_1 = \text{number of turns in coil 1,}$$

$$\mathcal{E}_2 = -N_2 \frac{d\Phi_B}{dt} \text{ , } N_2 = \text{number of turns in coil 2.}$$

- Using $|\mathcal{E}_1| = |V_1|$ , and $|\mathcal{E}_2| = |V_2|$, have

$$\frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{V_2}{V_1} = \frac{N_2(\, d\Phi_B/dt)}{N_1(d\Phi_B/dt)} = \frac{N_2}{N_1}$$

# Transformers (3)

- ∴ In an ideal transformer (i.e., no resistive power loss), ratio of the voltages across the primary and secondary coils equals the ratio of the number of turns in the coils:

**Terminal voltages in a transformer:**

Secondary voltage amplitude or rms value

Number of turns in secondary

$$\frac{V_2}{V_1} = \frac{N_2}{N_1}$$

Primary voltage amplitude or rms value

Number of turns in primary

  - If $N_2 > N_1$, then $V_2 > V_1$ $\Rightarrow$ a <u>step-up</u> transformer.
  - If $N_2 < N_1$, then $V_2 < V_1$ $\Rightarrow$ a <u>step-down</u> transformer.