Physics 1200
Lecture 23
Spring 2024

Relativistic Momentum and Energy, Energy and Mass Units, Nuclear Energy

Momentum in Relativity

- Einstein argued that the principle of conservation of momentum, as originally introduced in classical (Newtonian) mechanics, must also hold true in the special theory of relativity.
  - ➢ He showed that conservation of momentum occurs in all inertial frames, so long as the momentum of an object is defined as

$$\boxed{\vec{\boldsymbol{p}} = \gamma m \vec{\boldsymbol{v}}} \ ,$$

$\vec{\boldsymbol{p}} =$ <u>the relativistic momentum</u> of an object in a reference frame where it has velocity $\vec{\boldsymbol{v}}$.

$m =$ the mass of the object,

$\boxed{\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}}$ is the Lorentz factor of the particle moving at speed $v = |\vec{\boldsymbol{v}}|$.

Momentum in Relativity (2)

- The mass m is sometimes referred to as the “rest mass” of the object.
  - This is an obsolete terminology used in older textbooks. We will just identify it by what it is – the mass of the object. Sometimes it may be called the “invariant mass.”
- Derivation of relativistic momentum expression can be carried out in several equally valid ways:

  1. Study collisions of particles in linked inertial reference frames, using Lorentz transformation for the velocities measured in the different frames. Einstein found that total system momentum was conserved during the collision, so long as the momenta of the particles were expressed by the form given on the preceding slide.

  2. Introduce quantities known as “four-vectors” (a four-component vector with 3 space-like dimensions + 1 time-like dimension). This is the easiest and most natural way to find the form of the relativistic momentum. This concept is slightly beyond what is normally studied in Physics 1200.

Momentum in Relativity (3)

- The relativistic momentum ($\vec{\boldsymbol{p}} = \gamma m \vec{\boldsymbol{v}}$) also preserves Newton's 2nd law in its general form:

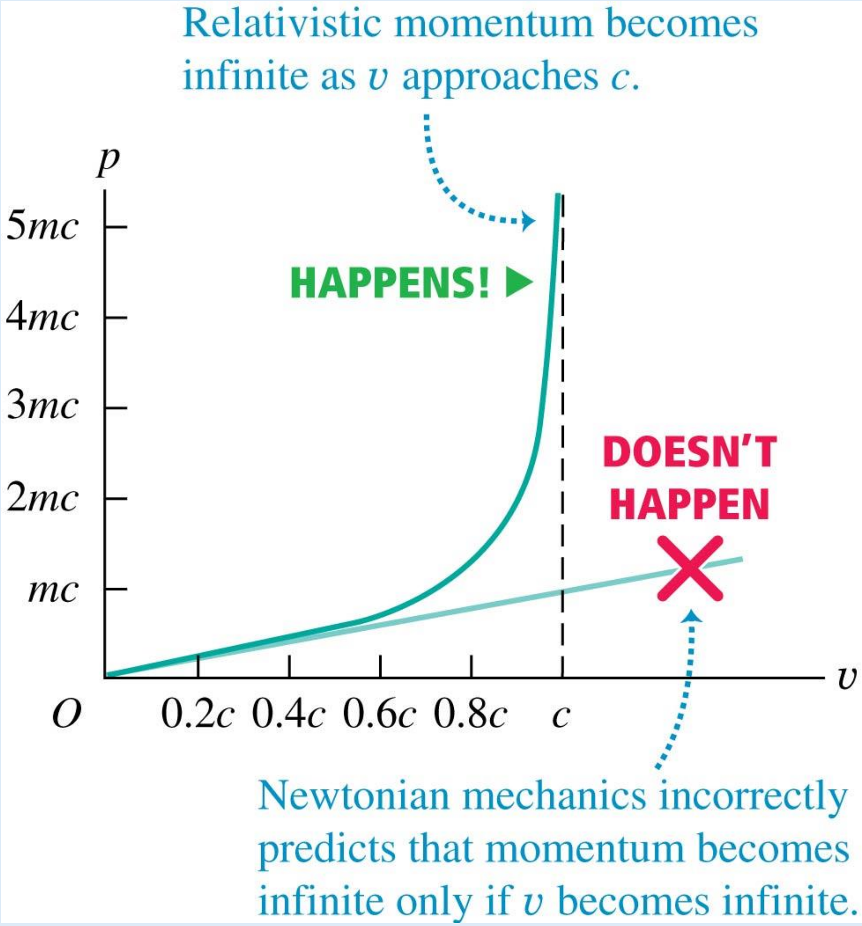
$$\boxed{\frac{d\vec{\boldsymbol{p}}}{dt} = \vec{\boldsymbol{F}}}\,.$$

- For $\frac{v}{c} \ll 1, \gamma \approx 1,$ and $\vec{\boldsymbol{p}} \approx m\vec{\boldsymbol{v}}$. ∴ In the low-speed limit we recover classical Newtonian mechanics.

- When proving conservation of momentum in relativity theory, Einstein showed that conservation of energy also resulted.
  - It turns out that there is a relativistic Lorentz transformation for energy and momentum, completely analogous to the relativistic Lorentz transformation for spacetime coordinates. The momentum and energy conservation principles together are more appropriately seen as "conservation of momentum-energy."

Relativistic vs. Classical Momentum

- Shown is a graph of the magnitude of the relativistic momentum of a particle of mass m as a function of speed v.
- Also shown is the magnitude of the Newtonian momentum ($p = mv$), which gives correct results only at speeds $v \ll c$.



Kinetic Energy in Relativity

- The expression for kinetic energy can be derived from the Work-Kinetic Energy Theorem, which is still valid for all inertial reference frames.
- Work-Kinetic Energy Theorem:

$$\Delta K = W_{net} = \int \vec{\boldsymbol{F}}_{net} \cdot d\vec{\boldsymbol{r}} \,.$$

For an object initially at rest, $K_0 = 0$, and $\Delta K = K$.

Using Newton's 2nd law, $\vec{\boldsymbol{F}}_{net} = \frac{d\vec{\boldsymbol{p}}}{dt} = \frac{d(\gamma m\vec{\boldsymbol{v}})}{dt}$,

$$K = \int \frac{d(\gamma m\vec{\boldsymbol{v}})}{dt} \cdot d\vec{\boldsymbol{r}} = \int \frac{d(\gamma m\vec{\boldsymbol{v}})}{dt} \cdot \frac{d\vec{\boldsymbol{r}}}{dt} dt = \int \vec{\boldsymbol{v}} \cdot \frac{d(m\gamma\vec{\boldsymbol{v}})}{dt} dt = m \int \vec{\boldsymbol{v}} \cdot \left[\gamma \frac{d\vec{\boldsymbol{v}}}{dt} + \vec{\boldsymbol{v}} \frac{d\gamma}{dt}\right] dt$$

$$= m \int \gamma\, \vec{\boldsymbol{v}} \cdot d\vec{\boldsymbol{v}} + m \int v^2\, d\gamma \;= mc^2 \int \gamma \left(\frac{v}{c}\right) d\left(\frac{v}{c}\right) + mc^2 \int \left(\frac{v}{c}\right)^2 d\gamma$$

$$= mc^2 \int_0^{v/c} \frac{\left(\frac{v'}{c}\right)}{\sqrt{1-\left(\frac{v'}{c}\right)^2}} d\left(\frac{v'}{c}\right) + mc^2 \int_1^{\gamma} \left(1 - \frac{1}{\gamma'^2}\right) d\gamma'$$

$$\Rightarrow \boxed{K = (\gamma - 1)mc^2} \,.$$ Relativistic kinetic energy.

Kinetic Energy in Relativity (2)

- In the derivation we used the relations

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \Rightarrow \left(\frac{v}{c}\right)^2 = 1 - \frac{1}{\gamma^2}$$, with $\gamma = 1$ for $v = 0$,

and $d(\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}) = d(v^2) \Rightarrow 2\vec{\boldsymbol{v}} \cdot d\vec{\boldsymbol{v}} = 2v dv$.

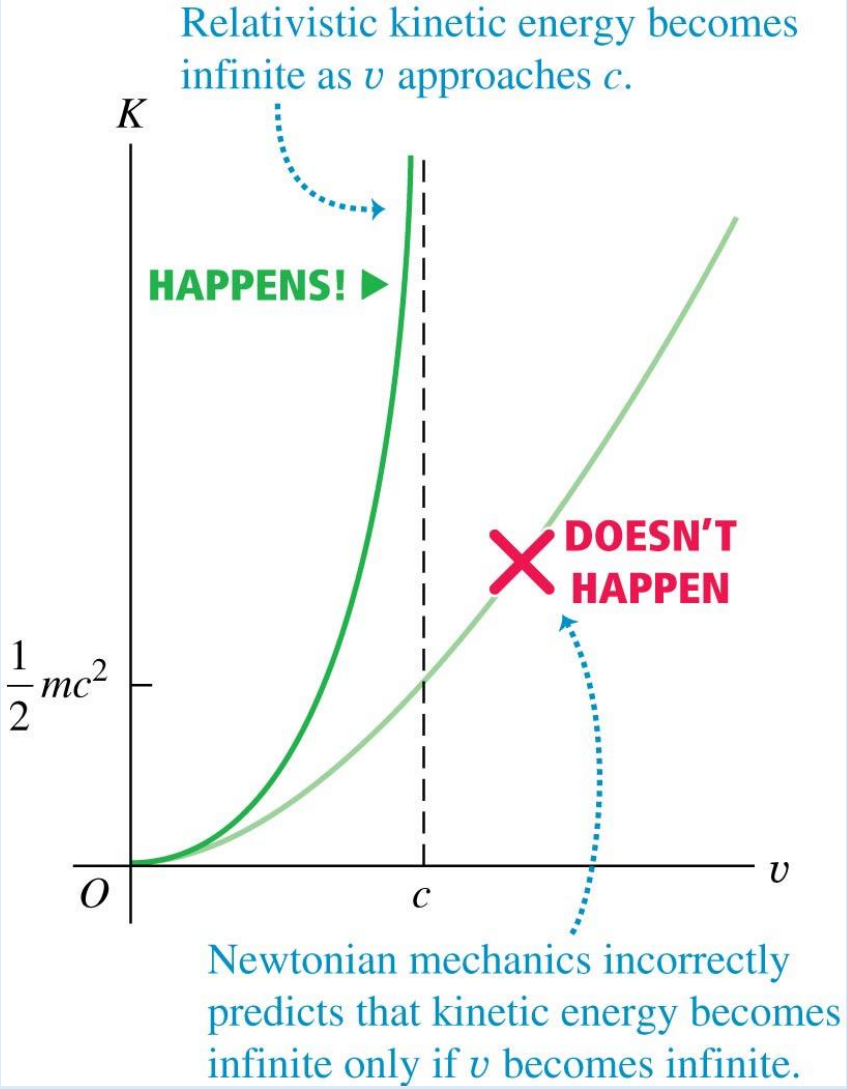
- For $\left(\frac{v}{c}\right)^2 \ll 1$, $\quad \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \approx 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2$.

In that limit, $K = (\gamma - 1)mc^2 \approx \left(\left[1 + \frac{1}{2}\left(\frac{v}{c}\right)^2\right] - 1\right)mc^2 = \frac{1}{2}mv^2$

∴ The low-speed limit of relativistic kinetic energy is classical Newtonian kinetic energy!

Kinetic Energy in Relativity (3)

- Graph of kinetic energy of a particle of mass m as a function of speed v.
- Also shown is the Newtonian prediction, which gives correct results only at speeds $v \ll c$.



Rest Energy and Total Relativistic Energy

- The rest energy of an object is defined as

$$\boxed{E_0 = mc^2}.$$

- The total relativistic energy of an object is

$$\boxed{E = K + E_0 = K + mc^2 = (\gamma - 1)mc^2 + mc^2 = \gamma mc^2}\ .$$

(Note: this relation doesn't include any potential energy the object might have.)

- In the frame that an object is at rest, $\gamma = 1$, and $E = E_0$.
- Rest energies for some fundamental particles:
  - Electron: $m_e = 9.11 \times 10^{-31}\ \text{kg} \quad \Rightarrow \quad m_e c^2 = 8.19 \times 10^{-14}\ \text{J}$.
  - Proton: $m_p = 1.67 \times 10^{-27}\ \text{kg} \quad \Rightarrow \quad m_p c^2 = 1.50 \times 10^{-10}\ \text{J}$.

Energy-Momentum Relation

- Using relations $|\vec{\boldsymbol{p}}| = p = \gamma m|\vec{\boldsymbol{v}}| = \gamma m v$, $E = \gamma mc^2$, and $\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$, find

$$\frac{p}{E} = \frac{\gamma m v}{\gamma m c^2} \quad \Rightarrow \quad \frac{pc}{E} = \frac{v}{c} \quad \Rightarrow \left(\frac{pc}{E}\right)^2 = \left(\frac{v}{c}\right)^2 = 1 - \frac{1}{\gamma^2}$$

$$\Rightarrow (pc)^2 = E^2 - \left(\frac{E}{\gamma}\right)^2 = E^2 - \left(\frac{\gamma mc^2}{\gamma}\right)^2 = E^2 - (mc^2)^2 \ .$$

$$\Rightarrow \boxed{E^2 = (pc)^2 + (mc^2)^2} \ .$$

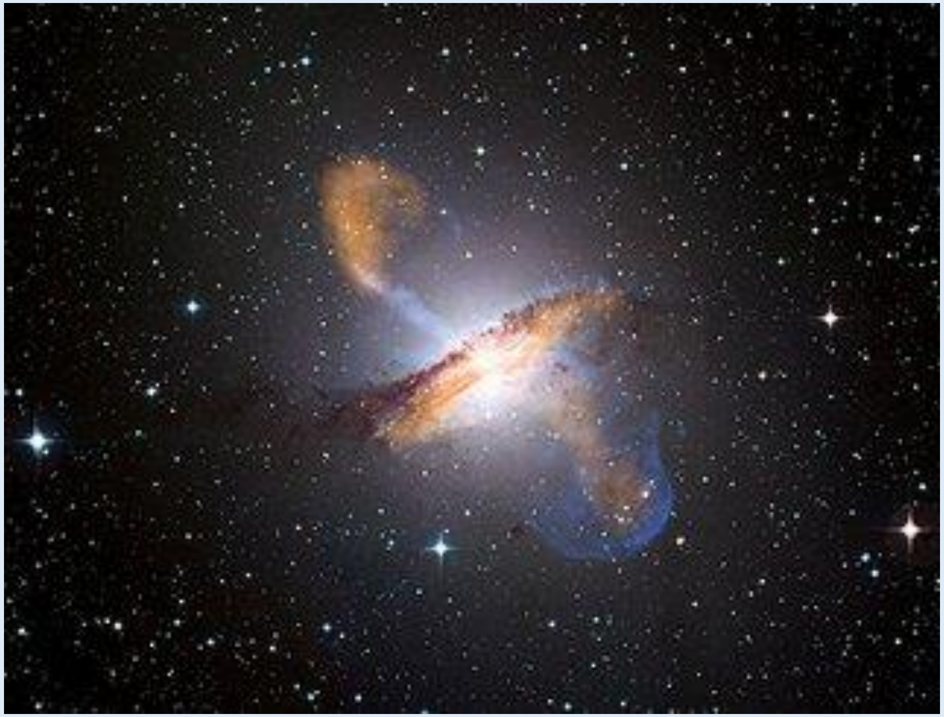
(<u>Relativistic energy-momentum relation</u>.)

- ➢ For objects with no mass, $m = 0$, find $E = pc$. This result will be applied to electromagnetic light quanta (photons), which were proposed by Einstein (1905) to explain the phenomenon known as the photoelectric effect. (Discussed in next class.)

- ➢ For extremely relativistic objects, $\frac{p}{mc} \gg 1$, and $E^2 \approx (pc)^2$, which is equivalent to saying $E \approx pc$ for those objects as well.

Applications of Relativity Theory

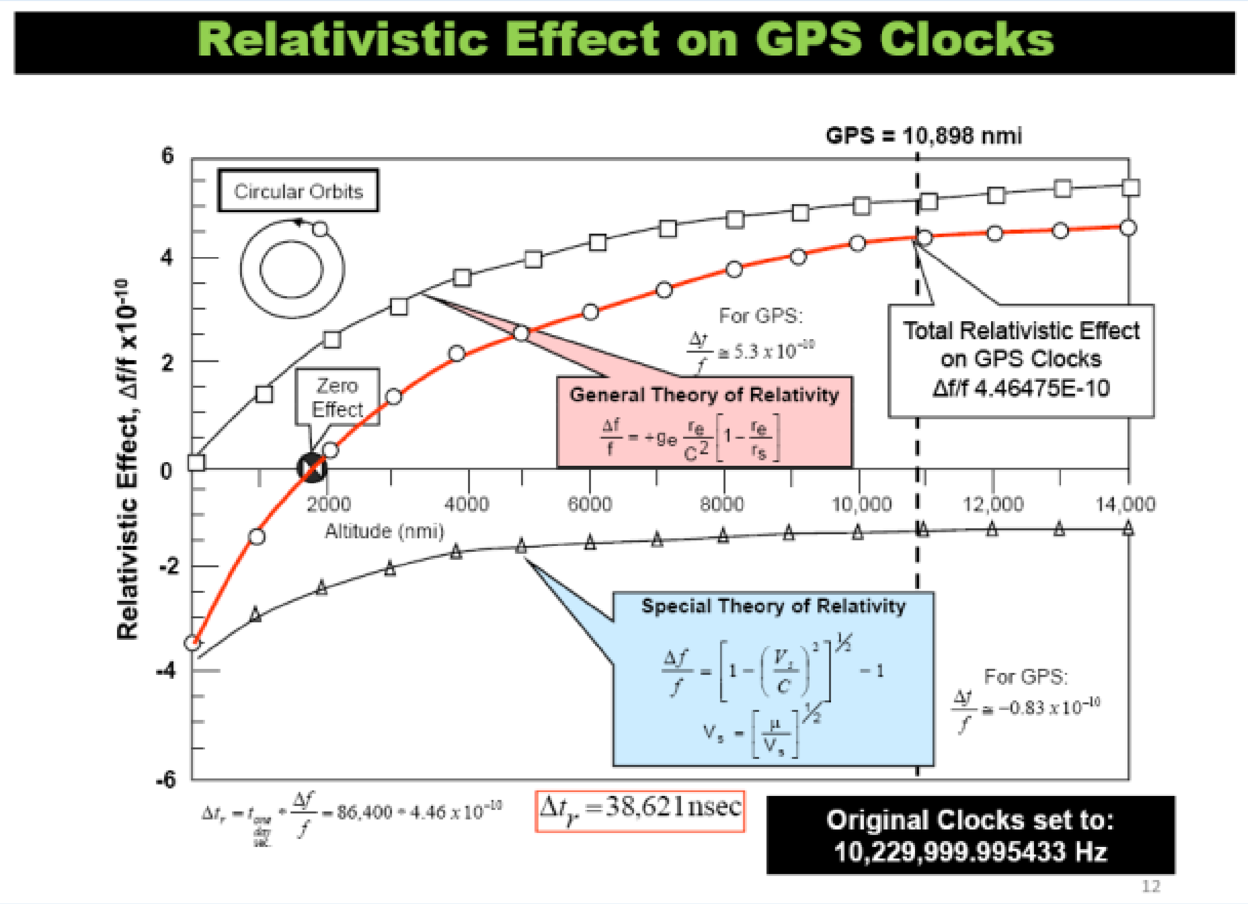
- The theory of relativity is the fundamental and correct formulation of the laws of physics. However, for many terrestrial, day-to-day situations, the low-speed Newtonian limit of relativistic theory is sufficient in dealing with those situations.
- There are many situations, though, where Newtonian physics is not adequate. For those cases, relativistic theory must be used.
  - Many astrophysical/space environments have particle speeds or energies such that Newtonian physics is inapplicable.



  - Shown is Centaurus A, which is a distant galaxy emitting a pair of relativistic jets. Gaseous streams of particles are being ejected by its active galactic nucleus (AGN – basically a supermassive black hole surrounded by an accretion disk) at speeds comparable to c. Due to relativistic motions, free electrons in the jets emit radio waves, visible light, x-rays, and gamma rays.

Applications of Relativity Theory (2)

- Terrestrial technologies such as GPS. Due to high orbital speeds of GPS satellites with respect to the surface of the Earth, must use special relativistic time-dilation effect in their position calculations. General relativistic (gravitational) time-dilation also needs to be considered.



Applications of Relativity Theory (3)

- Nuclear and particle physics deal with particles at speeds and energies where Newtonian physics will give incorrect results when studying those phenomena.
- We will consider some nuclear phenomena as a specific application of relativistic mechanics.

Energy Units

- Atomic and Nuclear Physicists and Engineers often use the energy unit: electron-volt or eV.
- It is the energy that an electron gains when it is accelerated through a potential difference of one volt.
- The electron volt is a good unit for atomic physics – binding energies are of the order of 1 to 10 eV. For instance, the ground-state binding energy of the hydrogen atom is 13.6 eV.
- Recall that, in SI units, $1\ \mathrm{eV} = 1.6 \times 10^{-19}\ \mathrm{J}$.
- Nuclear reactions tend to occur on the mega-electron-volt or MeV scale. $1\ \mathrm{MeV} = 10^{6}\ \mathrm{eV}$.

Mass Units

- In addition to the SI unit of mass (kg), when dealing with nuclear reactions scientists and engineers may also use the unit MeV/c^2.
- Convenient to use for energy calculations, especially for rest energies, since rest energy

$$\boxed{E_0 = mc^2}.$$

- To convert from kg to MeV/c^2, use

$$\boxed{m = m\left(\frac{c^2}{c^2}\right) = \frac{mc^2}{c^2} = \frac{E_0}{c^2}}.$$

Example Problem 23.1

- The neutral pion (π^0) is a subatomic particle that has a mass $m_{\pi^0} = 2.406 \times 10^{-28}$ kg. Calculate its mass in MeV/c^2.

$$m_{\pi^0} = (2.406 \times 10^{-28}\ \mathrm{kg})\left(\frac{c^2}{c^2}\right)$$

$$= \frac{\left(2.406 \times 10^{-28}\ \mathrm{kg}\right)\left(3 \times 10^8 \frac{\mathrm{m}}{\mathrm{s}}\right)^2 \left(\frac{1\mathrm{eV}}{1.6 \times 10^{-19}\ \mathrm{J}}\right)\left(\frac{1\mathrm{MeV}}{10^6\ \mathrm{eV}}\right)}{c^2}$$

$$\rightarrow m_{\pi^0} = 135\ \frac{\mathrm{MeV}}{c^2}$$

Example Problem 23.2

- The charged B meson is a subatomic particle that has a mass $m_{B_c} = 6276\ \mathrm{MeV}/c^2$. Calculate (a) its rest energy, and (b) its mass in kg.

(a) $E_{0,B_c} = m_{B_c}c^2 = \left(6276\,\frac{\mathrm{MeV}}{c^2}\right)c^2 = 6276\ \mathrm{MeV}$

$$= (6276 \times 10^6\ \mathrm{eV})\left(\frac{1.6\times10^{-19}\mathrm{J}}{1\ \mathrm{eV}}\right) = 1.004 \times 10^{-9}\ \mathrm{J}.$$

(b) $m_{B_c} = 6276\,\frac{\mathrm{MeV}}{c^2} = \frac{6276\ \mathrm{MeV}}{c^2} = \frac{1.004\times10^{-9}\ \mathrm{J}}{\left(3\times10^8\frac{m}{s}\right)^2} = 1.12 \times 10^{-26}\ \mathrm{kg}.$

Atomic and Nuclear Constituents

A partial list of constituents:

- Electron e^- (and positron e^+):
  - Mass $m_e = 9.11 \times 10^{-31}$ kg $= 0.511\ \frac{\text{MeV}}{c^2} = 0.0005486$ u .
  - Charge $q_e = -1.6 \times 10^{-19}$ C $= -e$ ($= +e$ for positron).
- Proton p:
  - Mass $m_p = 1.673 \times 10^{-27}$ kg $= 938.3\ \frac{\text{MeV}}{c^2} = 1.007$ u .
  - Charge $q_p = +1.6 \times 10^{-19} C = +e$.
- Neutron n :
  - Mass $m_n = 1.675 \times 10^{-27}$ kg $= 939.6\ \frac{\text{MeV}}{c^2} = 1.009$ u .
  - Charge $q_n = 0$ C.
- Muon μ:
  - Mass $m_\mu = 1.88 \times 10^{-28}$ kg $= 105\ \frac{\text{MeV}}{c^2} = 0.1134$ u .
  - Charge $q_{\mu_\mp} = \mp 1.6 \times 10^{-19} C = \mp e$ (depends on type).
- Note: 1u = unified atomic mass unit $= 1.660539 \times 10^{-27}$ kg $= 931.5\ \frac{\text{MeV}}{c^2}$.

Nuclides and Isotopes

- The general format for an element El is ${}^{A}_{Z}\mathrm{El}$
- For example, the most common isotope of chlorine has $A = 35$, $Z = 17$, and is written ${}^{35}_{17}\mathrm{Cl}$ and usually pronounced “chlorine-35.”
- This name of the element determines the atomic number Z, so the subscript Z is sometimes omitted, as in ${}^{35}\mathrm{Cl}$.

The Strong Nuclear Force

- The <u>nuclear force</u> binds protons and neutrons together. It is an example of the <u>strong interaction</u>.
- Important characteristics of the strong nuclear force:
  - It does not depend on charge. Protons <u>and</u> neutrons are bound. It has a short range, of the order of nuclear dimensions.
  - Because of its short range, a nucleon (i.e., a proton or neutron) only interacts with those in its immediate vicinity.
  - It favors binding of <u>pairs</u> of protons or neutrons with opposite spins (= an intrinsic angular momentum of elementary particles) and with <u>pairs of pairs</u> (a pair of protons and a pair of neutrons, each pair having opposite spins).

Nuclear Binding Energy

- Energy must be added to a nucleus to separate it into its individual protons and neutrons, the total rest energy E_0 of the separated nucleons (i.e., protons and neutrons) is greater than the rest energy of the nucleus.
- $\therefore$ Rest energy of a nucleus is $= E_0 - E_B$.
- Added energy needed to separate the nucleons is called the binding energy E_B:

$$\boxed{E_B = (ZM_H + NM_N - {}^A_Z M)c^2} \ ,$$

$M_H =$ mass of atomic hydrogen,

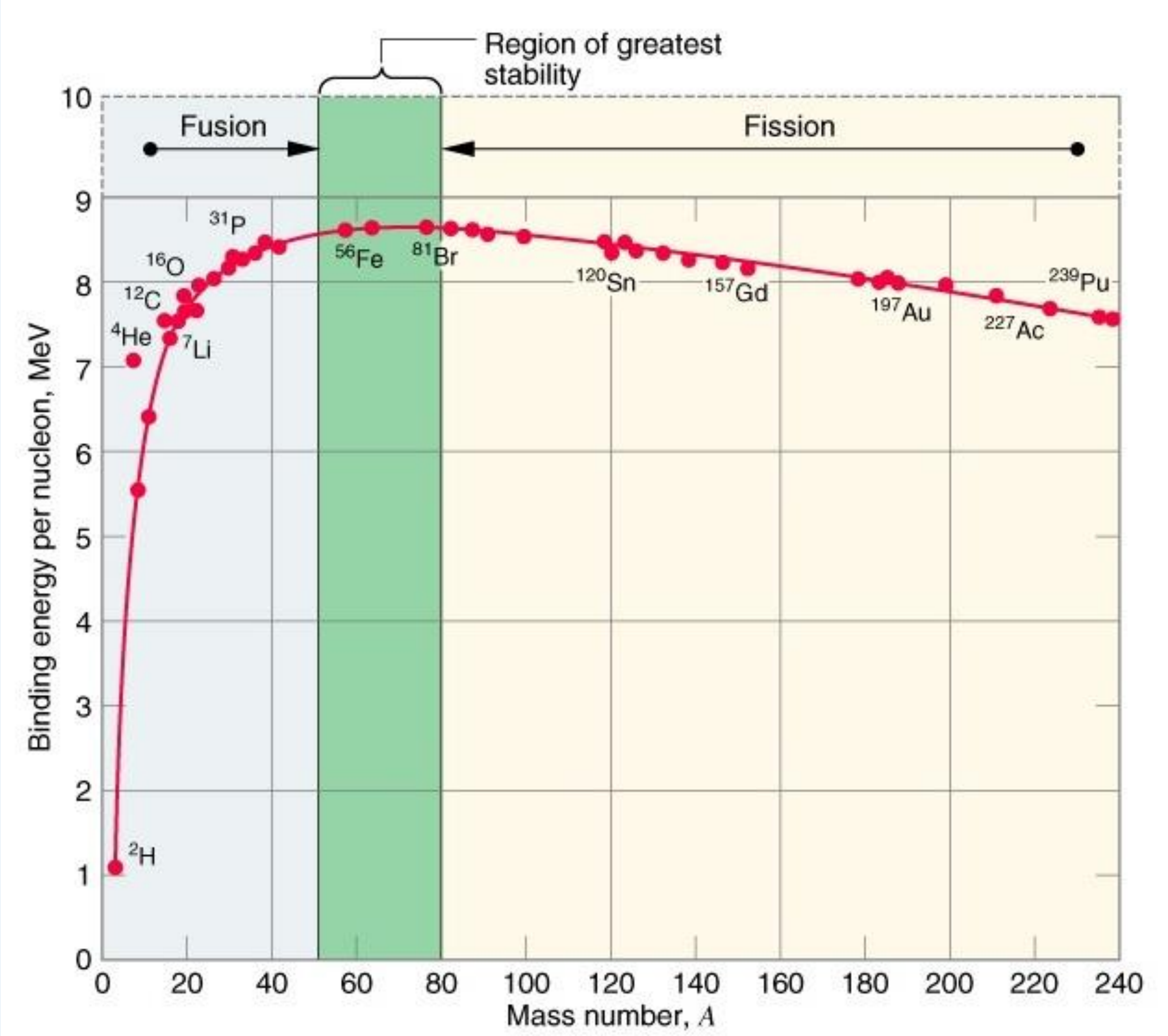
$M_N =$ mass of a neutron,

$N = A - Z =$ number of neutrons in the nucleus, and

${}^A_Z M =$ mass of the neutral atom containing the nucleus of atomic number Z and isotope mass number A.

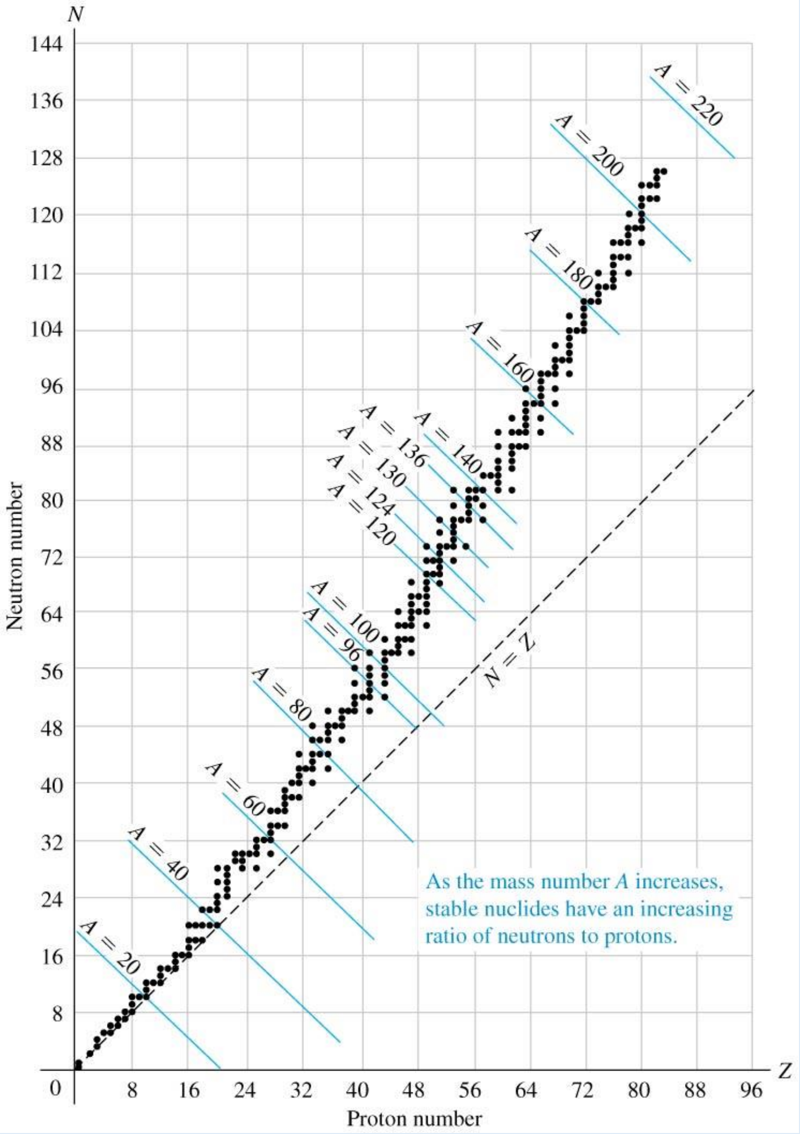
Nuclear Binding Energy Per Nucleon As A Function of Atomic Mass Number A

- The larger the binding energy per nucleon, the more stable the nucleus. (Stronger "glue" holding the nucleus together.)



Nuclear Stability

- Among about 2500 known nuclides, fewer than 300 are stable.
- The others are unstable structures that decay to form other nuclides by emitting particles and electromagnetic radiation, a process called radioactivity.
- The stable nuclides are shown by dots on a Segrè chart (shown), where the neutron number N and proton number (or atomic number) Z for each nuclide are plotted.



Nuclear Reactions

- A nuclear reaction is a rearrangement of nuclear components due to bombardment by a particle rather than a spontaneous natural process.
- The difference in masses before and after the reaction corresponds to the reaction energy.
- If initial particles A and B interact to produce final particles C and D,

$$\mathrm{A} + \mathrm{B} \rightarrow \mathrm{C} + \mathrm{D}\,,$$

the reaction energy Q (also known as the Q-value) is defined as:

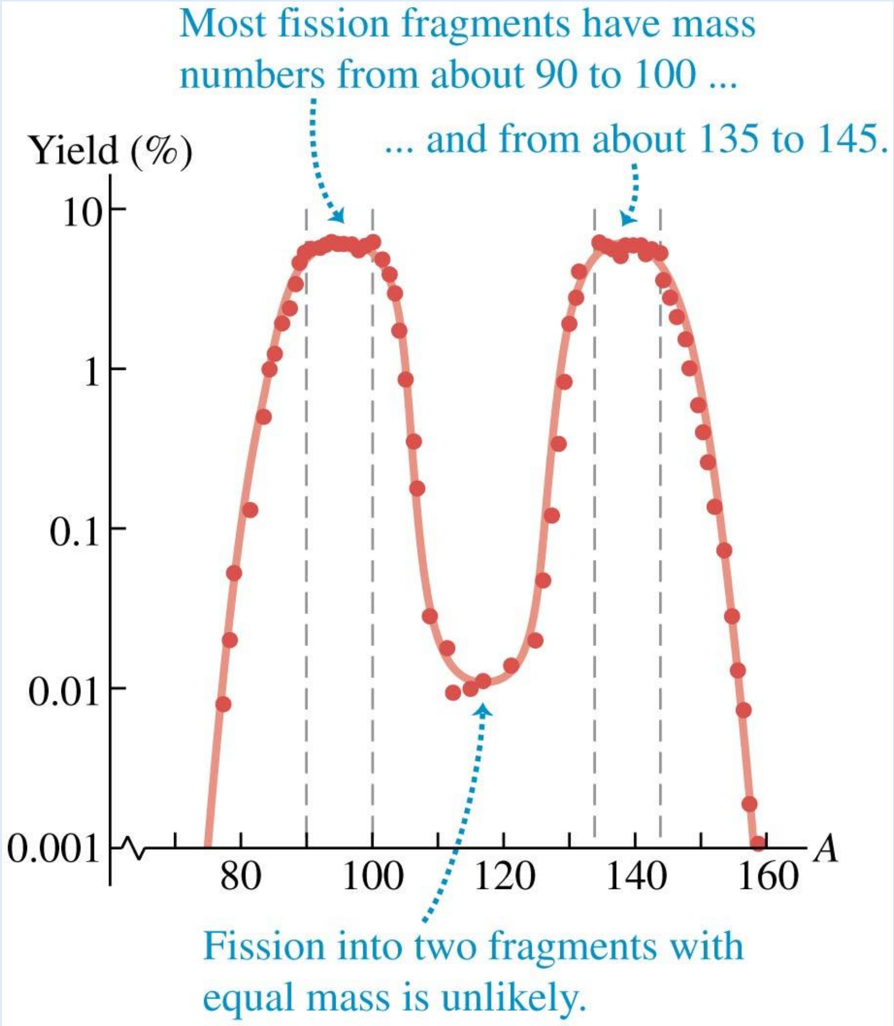
$$M_{\mathrm{A}}c^2 + M_{\mathrm{B}}c^2 = M_{\mathrm{C}}c^2 + M_{\mathrm{D}}c^2 + Q$$

$$\Rightarrow \boxed{Q = (M_{\mathrm{A}} + M_{\mathrm{B}} - M_{\mathrm{C}} - M_{\mathrm{D}})c^2} \quad \text{(Reaction energy).}$$

- When Q is positive, the total mass decreases and the total kinetic energy increases. Such a reaction is called an **exoergic** reaction.
- When Q is negative, the mass increases and the kinetic energy decreases, and the reaction is called an **endoergic** reaction.

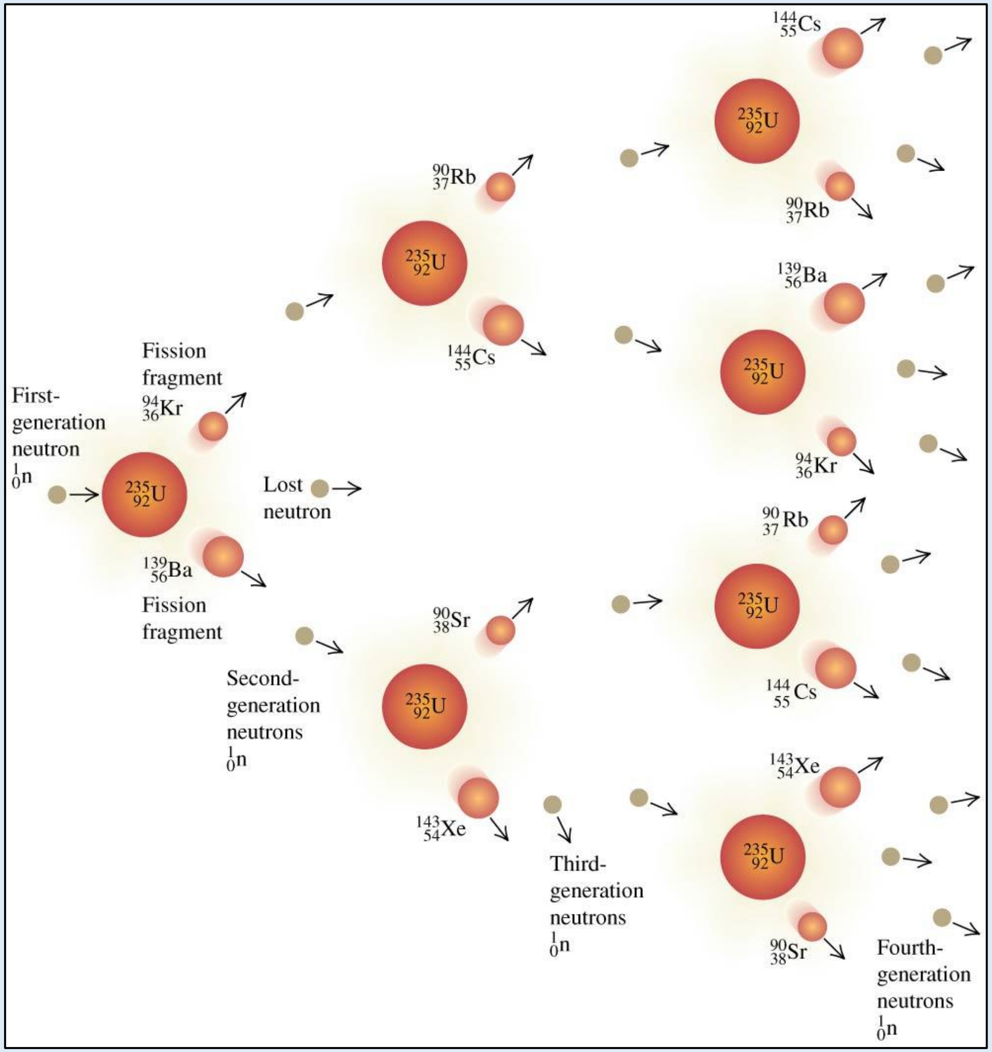
Nuclear Fission

- Nuclear fission is a decay process in which an unstable nucleus splits into two fragments (the fission fragments) of comparable mass.
- Shown is the mass distribution of the fission fragments from the fission of $^{236}U^{*}$.



Nuclear Fission

- Very heavy nuclei ($A > 62$) can break apart, releasing energy.
- Fission of a uranium nucleus, triggered by neutron bombardment, releases other neutrons that can trigger more fissions, suggesting the possibility of a chain reaction (shown).
- The chain reaction may be made to proceed slowly and in a controlled manner in a nuclear reactor, or explosively in a bomb.



Nuclear Fusion

- Light nuclei ($A < 62$) can combine, that is, fuse, releasing energy.
  - Electric (Coulomb) repulsion makes it hard to get them close enough to each other for fusion to be possible. The probability of nuclei fusing is therefore low.
  - Stellar interiors, however, are dense enough that nuclei are on average close enough that even low-probability fusion events can occur over time.
- Fusion reactions release energy for the same reason as fission reactions: the binding energy per nucleon after the reaction is greater than before.
- The figure below illustrates the proton-proton ("p-p") chain, which is the main energy source in our sun.