Quantum Physics 1 2025

Class 2

Complex numbers – Exponential representation

Wave Superposition

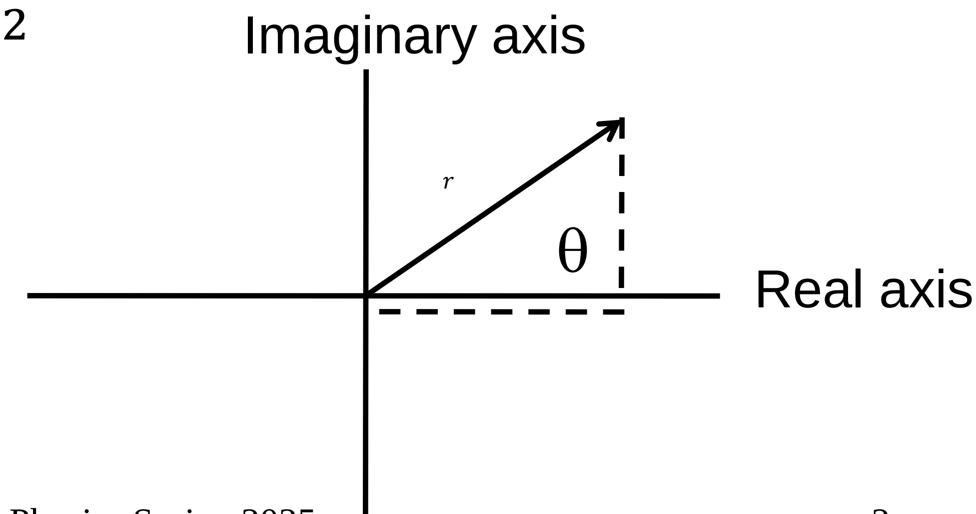
Complex numbers represented by vectors

A complex number includes both real and imaginary parts: $\tilde{z} = a + jb$

A complex number can also be written as an exponential using Euler's relation.

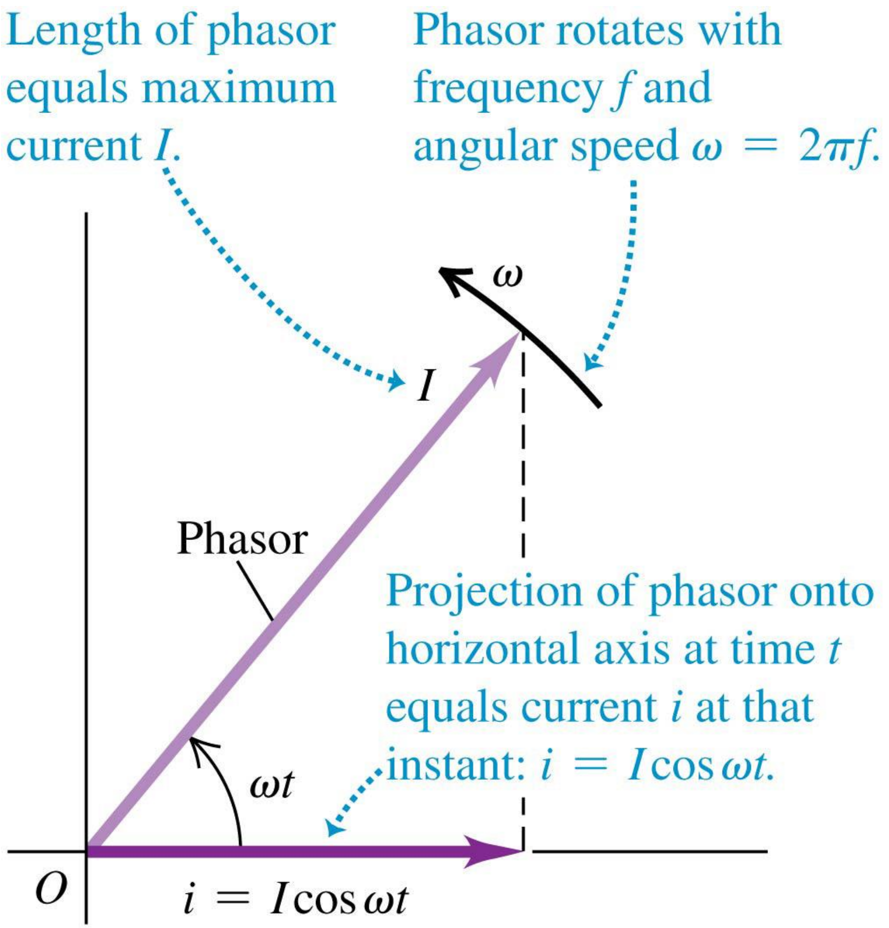
$$\tilde{z} = x + jy$$
$$= r(\cos\theta + j\sin\theta) = re^{j\theta}$$
$$\text{with } r = \sqrt{\vec{r} \bullet \vec{r}} = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}(y/x)$$

$$j \equiv \sqrt{-1} = e^{\frac{j\pi}{2}}$$



Note that we use j, not i, for the $\sqrt{-1}$ *because sometimes we use i for current.*

Phasor representation of a harmonic quantity



A harmonic quantity $v = V_0 \cos(\omega \text{t}+\varphi)$ can be represented by a rotating vector known as a phasor using the following conventions:

1. Phasors rotate in the counterclockwise direction with angular speed ω
2. The length of each phasor is proportional to the ac quantity amplitude
3. The projection of the phasor on the horizontal axis gives the real part of the quantity

The phase effect of multiplying a harmonic function by $j = \sqrt{-1}$

$$\tilde{A}(t) = jae^{j\omega t} = ae^{j\frac{\pi}{2}}e^{j\omega t} = ae^{j\left(\omega t+\frac{\pi}{2}\right)}$$

Remember:

$$\cos(\omega t) = \sin\left(\omega t + \frac{\pi}{2}\right) \quad \sin(\omega t) = \cos\left(\omega t - \frac{\pi}{2}\right)$$

j increases the phase by $\pi/2$
(rotates the phasor 90º ccw)

Basic complex arithmetic

$$If\ z_1 = A_1 e^{j\varphi}; \quad z_2 = A_2 e^{j(\varphi+\delta)}$$

$$z_1 + z_2 = e^{i\varphi}\left(A_1 + A_2 e^{j\delta}\right)$$

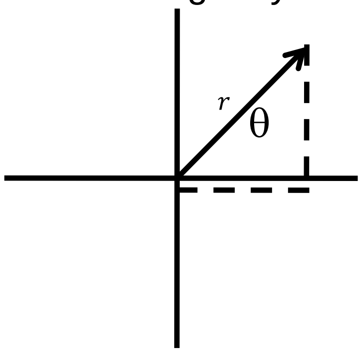
$$= e^{j\varphi}[(A_1 + A_2 \cos\delta) + iA_2 \sin\delta]$$

$$z_1 + z_2 = e^{j\varphi} M e^{j\beta} = M e^{j(\beta+\varphi)}$$

where:

$$M = \sqrt{(A_1 + A_2 \cos\delta)^2 + (A_2 \sin\delta)^2};$$

$$\beta = \tan^{-1}\left(\frac{(A_2 \sin\delta)}{(A_1 + A_2 \cos\delta)}\right)$$



$$\mathrm{If}\ \mathrm{z}_1 = r_1 \mathrm{e}^{\mathrm{j}\theta_1}\ \mathrm{and}\ z_2 = r_2 \mathrm{e}^{\mathrm{j}\theta_2}$$

$$\mathrm{z}_1 \mathrm{z}_2 = r_1 \mathrm{e}^{\mathrm{j}\theta_1} r_2 \mathrm{e}^{\mathrm{j}\theta_2} = r_1 r_2 \mathrm{e}^{\mathrm{j}(\theta_1+\theta_2)}$$

$$\frac{\mathrm{z}_1}{\mathrm{z}_2} = r_1 \mathrm{e}^{\mathrm{j}\theta_1}\left(\frac{1}{r_2}\right)\mathrm{e}^{-\mathrm{j}\theta_2} = \frac{r_1}{r_2}\mathrm{e}^{\mathrm{j}(\theta_1-\theta_2)}$$

See the math appendix in Y&F or Complex Numbers in Schaum's

Complex Conjugate

- Complex Conjugate – wherever a *j* appears in a complex number, negate it.
  $(a + jb)^* = a - jb$ and $\left(Re^{j\theta}\right)^* = Re^{-j\theta}$
- Multiplying by the complex conjugate always results in a real positive number:
  $$(a + jb)(a - jb) = a^2 + b^2$$
- Multiplying by the complex conjugate and taking the square root yields the magnitude of a complex number.

Manipulating complex ratios (1)

It is frequently useful to take a complex ratio, like $\frac{a+jb}{c+jd}$ and separate it into real and imaginary components.

This allows us to rapidly determine how much of the complex number is "in-phase" and "out-of-phase" with the driving signal.

To do this, multiply top and bottom by the complex conjugate of the bottom.

$$\frac{a+jb}{c+jd}\left(\frac{c-jd}{c-jd}\right) = \frac{(ac+bd)+j(-ad+bc)}{c^2+d^2}$$

$$= \left(\frac{ac+bd}{c^2+d^2}\right) + j\left(\frac{-ad+bc}{c^2+d^2}\right)$$

Manipulating complex ratios (2)

It is also useful to find the magnitude of a complex ratio.
To do this, multiply by the complex conjugate and take the square root.

$$A = \frac{a + jb}{c + jd}$$

$$|A| = \sqrt{AA^*} = \sqrt{\left(\frac{a + jb}{c + jd}\right)\left(\frac{a - jb}{c - jd}\right)} = \sqrt{\frac{a^2 + b^2}{c^2 + d^2}}$$

Do Class 2 Activity on Complex Numbers

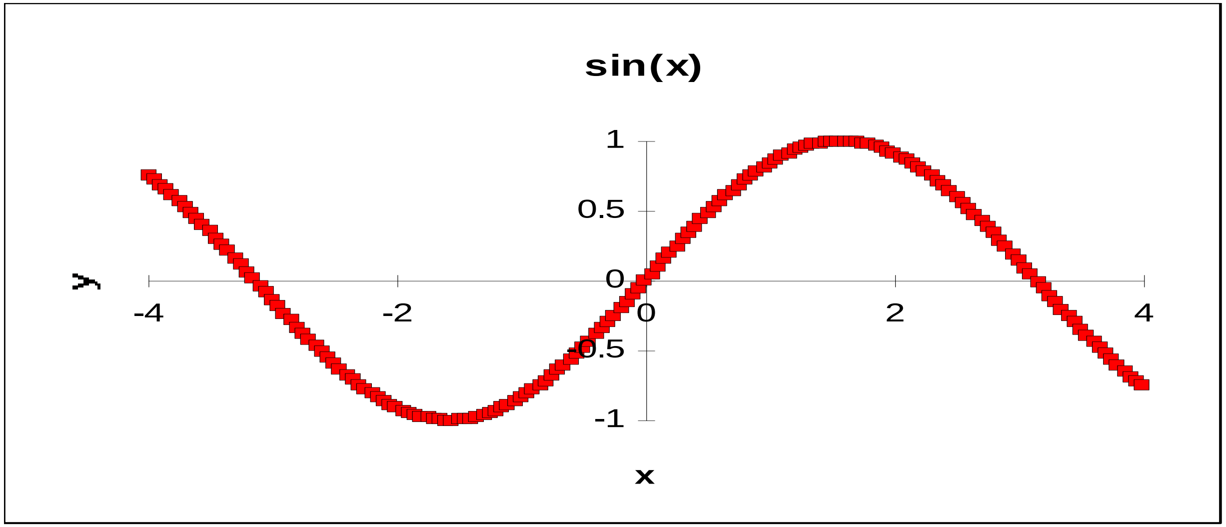
A review of wave superposition and interference

Many of the neat observations of quantum physics can be understood in terms of the addition (superposition) of harmonic waves of different frequency and phase.

In the next several pages I review some of the basic relations and phenomena that are useful in understanding wave phenomena.

Harmonic waves

$y = A \sin(k(x \pm vt) + \varphi)$ or $y = A\sin(kx \pm \omega t + \varphi)$



$$\lambda = wavelength; \quad k = \frac{2\pi}{\lambda}$$

$$T = period; \quad \omega = \frac{2\pi}{T} = 2\pi\nu$$

Phase and phase velocity

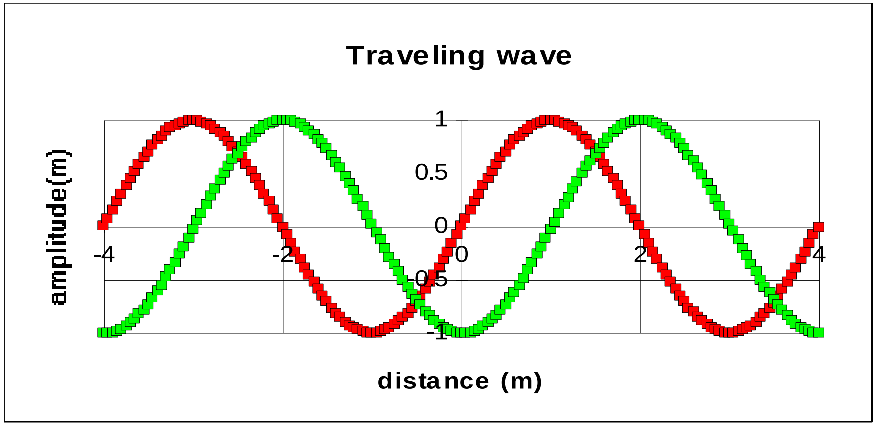
$y = A\sin(kx \pm \omega t)$

$(kx \pm \omega t)$= phase

when change in phase=2π = repeat

<u>Phase velocity</u> = speed with which point of constant phase moves in space.

$v_{phase} = \omega/k = \lambda\nu$



Complex Representation of Travelling Waves

Doing arithmetic for waves is frequently easier using the complex representation using:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

From which it follows that:

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \, and \, \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

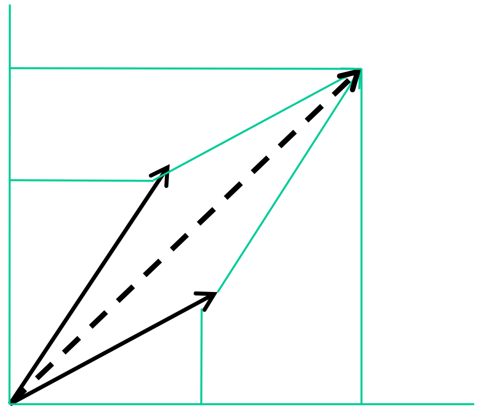
So that a harmonic travelling cosine wave is represented as:

$$\psi(x,t) = A\cos(\,kx - \omega t + \varepsilon)$$
$$\psi(x,t) = \mathrm{Re}\left[Ae^{i(kx-\omega t+\varepsilon)}\right]$$

General addition of two harmonic waves
– Phasors are useful!

$\psi_1 = \psi_{01} \sin(\omega t + \varphi_1)$
$\psi_2 = \psi_{02} \sin(\omega t + \varphi_2)$
Writing the result in the form:
$\psi_R = \psi_1 + \psi_2 = \psi_0 \sin(\omega t + \alpha)$
we have:
$$\left|\psi_0^2\right| = \psi_{01}^2 + \psi_{02}^2 + 2\psi_{01}\psi_{02}\cos(\varphi_1 - \varphi_2)$$
and:
$$\tan\alpha = \frac{\psi_{01}\sin\varphi_1 + \psi_{02}\sin\varphi_2}{\psi_{01}\cos\varphi_1 + \psi_{02}\cos\varphi_2}$$

This is easy to see if we use the phasor approach.



Adding waves using trig identities
same wavelength and direction, different phase

$\psi_1 = \psi_{01} \sin(kx + \omega t + \varphi_1)$
$\psi_2 = \psi_{02} \sin(kx + \omega t + \varphi_2)$

taking the form:

$\psi_T = \psi_1 + \psi_2 = \psi_{T0} \sin(kx + \omega t + \alpha)$

we find:

$|\psi_{T0}^2| = \psi_{01}^2 + \psi_{02}^2 + 2\psi_{01}\psi_{02} \cos(\varphi_1 - \varphi_2)$

and:

$$\tan\alpha = \frac{\psi_{01} \sin\varphi_1 + \psi_{02} \sin\varphi_2}{\psi_{01} \cos\varphi_1 + \psi_{02} \cos\varphi_2}$$

The sum of the waves displays interference.

Adding waves using the complex exponential representation

$$\psi_1 = \psi_{01} e^{j(kx+\omega t+\phi_1)}; \quad \psi_2 = \psi_{02} e^{j(kx+\omega t+\phi_2)}$$

$$\psi_T = e^{j\omega t}\left(\psi_{01} e^{j\phi_1} + \psi_{02} e^{j\phi_2}\right)$$

$$\psi_T = e^{j\left(\omega t+\frac{\phi_1+\phi_2}{2}\right)}\left[\psi_{01} e^{-\frac{j(\phi_2-\phi_1)}{2}} + \psi_{02} e^{\frac{j(\phi_2-\phi_1)}{2}}\right]$$

$$\psi_T = e^{j\left(\omega t+\frac{\phi_1+\phi_2}{2}\right)}\left[\psi_{01} e^{-j\frac{\delta}{2}} + \psi_{02} e^{j\frac{\delta}{2}}\right]$$

with $\delta = (\phi_2 - \phi_1)$

$$\psi_T \psi_T^* = \psi_{01}^2 + \psi_{02}^2 + \psi_{01}\psi_{02} e^{-2j\frac{\delta}{2}} + \psi_{01}\psi_{02} e^{2j\frac{\delta}{2}}$$

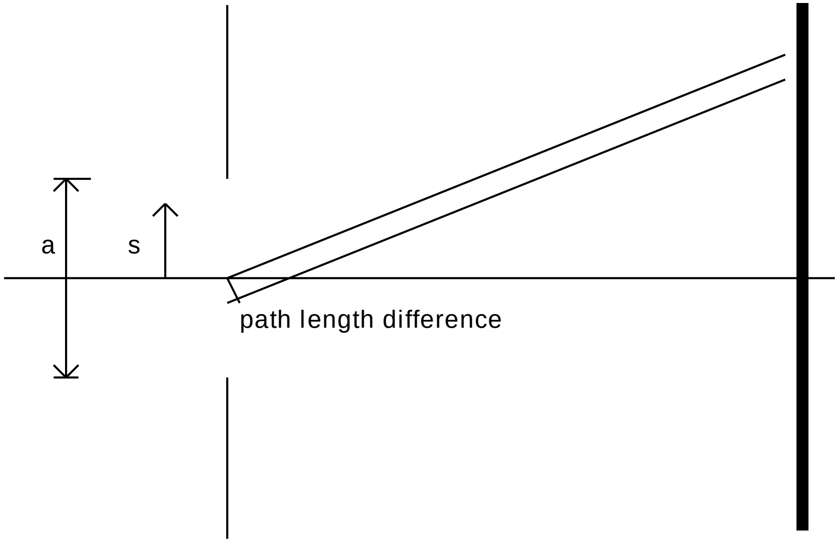
and using: $\cos\delta = (e^{j\delta} + e^{-j\delta})/2$,

$$\psi_T \psi_T^* = \psi_{01}^2 + \psi_{02}^2 + 2\psi_{01}\psi_{02}\cos(\delta)\,\psi$$

Adding like-waves, in words

- Amplitude
  - When two waves are in phase, the resultant amplitude is just the sum of the amplitudes.
  - When two waves are 180^0 out of phase, the resultant is the difference between the two.
- Phase
  - The resultant phase is always between the two component phases. (Halfway when they are equal; closer to the larger wave when they are not.)

Another useful example of added waves: diffraction from a slit



Let's take the field $d\psi_{screen}$ at the view screen from an element of length on the aperture *ds* as $A\psi_{aperture}ds$ where A is a constant to make the magnitude and dimensions work out

Ignore effects of distance except in the path length.

Single slit diffraction

$$d\psi_{screen} = \mathrm{Re}(A\, e^{j(kz(s)-\omega t)} ds)$$

$$z(s) = z_0 + s \sin\theta$$

$$\psi(\theta) = Ae^{j(kz_0-\omega t)} \int_{-a/2}^{a/2} e^{jks\sin\theta}\, ds$$

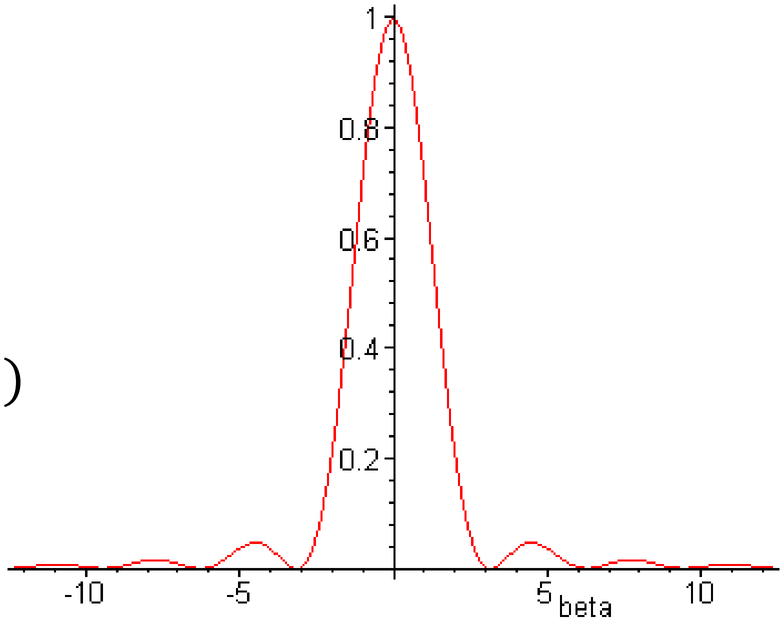
$$= \frac{Ae^{i(kz_0-\omega t)}}{jk\sin\theta}\left(e^{\frac{jka}{2}\sin\theta} - e^{-\frac{jka}{2}\sin\theta}\right)$$

$$= Ae^{i(kz_0-\omega t)} \frac{a\sin\beta}{\beta}$$

where

$$\beta = \frac{ak}{2}\sin\theta$$

$$\psi^2 = (Aa)^2 \frac{\sin^2\beta}{\beta^2}$$



First zero at $\beta = \pi$, so $\sin\theta = \frac{2\pi}{ak} = \frac{\lambda}{a}$

Adding waves: traveling in opposite directions

Assuming equal amplitudes$*$:

$$\psi_R = \psi_0(\sin(kx - \omega t + \varepsilon_1) + \sin(kx + \omega t + \varepsilon_2))$$

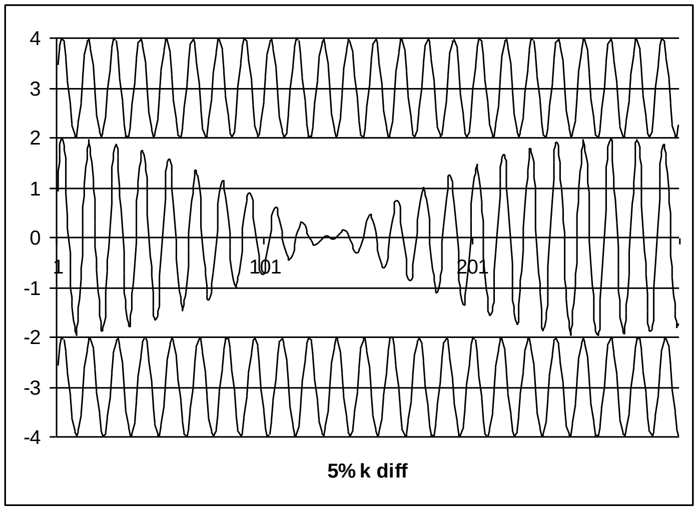
$$\psi_R = 2\psi_0 \sin\left(kx + \frac{\varepsilon_1 + \varepsilon_2}{2}\right)\cos\omega\, t$$

- The resultant wave does not appear to travel – it oscillates in place on a harmonic pattern both in time and space separately

=Standing wave

* If amplitudes are equal, then we can simply use trig identities to find the sum: $\cos A + \cos B = \left(2\cos\frac{A+B}{2}\cos\frac{A-B}{2}\right)$ and $\sin A + \sin B = \left(2\sin\frac{A+B}{2}\cos\frac{A-B}{2}\right)$

Adding waves: different wavelengths



BEATS

$$\psi_1 = \psi_{01}\cos(k_1 x - \omega_1 t)$$
$$\psi_2 = \psi_{01}\cos(k_2 x - \omega_2 t)$$
$$\psi = 2\psi_{01}\cos\frac{1}{2}[(k_1 + k_2)x - (\omega_1 + \omega_2)t]$$
$$\times \cos\frac{1}{2}[(k_1 - k_2)x - (\omega_1 - \omega_2)t]$$
$$= \psi_{01}\cos[\bar{k}x - \bar{\omega}t]\cos\left[\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right]$$

- The resultant wave has a quickly varying part that waves at the average wavelength of the two components.
- It also has an envelope part that varies at the difference between the component wavelengths.

Adding waves: group velocity

- Note that when we add two waves of differing ω and *k* to one another, the envelope travels with a different speed:

$$\text{monochromatic wave 1: } v_{phase_1} = \frac{\omega_1}{k_1}$$

$$\text{monochromatic wave 2: } v_{phase_2} = \frac{\omega_2}{k_2}$$

$$\text{beat envelope: } v_{group} = \frac{\Delta\omega}{\Delta k}$$

Adding many waves to make a pulse

- In order to make a wave pulse of finite width, we have to add many waves of differing wavelengths in different amounts.
- The mathematical approach to finding out how much of each wavelength we need is the Fourier transform:

$$f(x) = \frac{1}{\pi}\left[\int_0^{\infty} A(k)\cos k\,x dk + \int_0^{\infty} B(k)\sin k\,x dk\right]$$

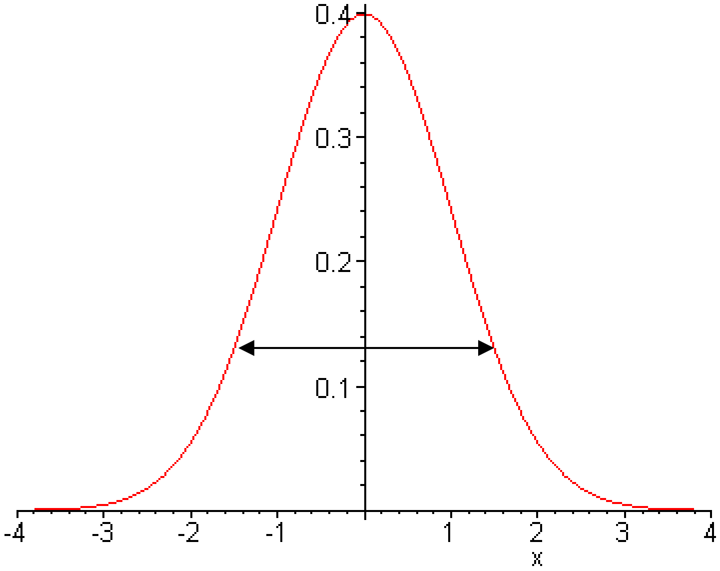
where:

$$A(k) = \int_{-\infty}^{\infty} f(x)\cos k\,x dx$$

$$B(k) = \int_{-\infty}^{\infty} f(x)\sin k\,x dx$$

The Fourier transform of a Gaussian pulse

- We can think of a Gaussian pulse as a localized pulse, whose position we know to a certain accuracy $\Delta x = 2\sigma_x$.



$$f(x) = \frac{1}{\sigma_x\sqrt{2\pi}} e^{-x^2/2\sigma_x^2}$$

Finding the transform

I will drop overall multiplicative constants because I am interested in the shape of A(k)

$$A(k) \propto \int_{-\infty}^{\infty} e^{-x^2/2\sigma_x^2} e^{-ikx} dx = \int_{-\infty}^{\infty} e^{-ax^2} e^{-ikx} dx$$

$where\ a = 1/2\sigma_x^2$

(solving by completing the square:)

$$A(k) \propto \int_{-\infty}^{\infty} e^{-ax^2} e^{-ikx} dx = \int_{-\infty}^{\infty} e^{-\left(x\sqrt{a}-\frac{ik}{2\sqrt{a}}\right)^2 - k^2/4a} dx$$

letting $\beta = x\sqrt{a} - \dfrac{ik}{2\sqrt{a}}$

$$A(k) \propto \frac{1}{\sqrt{a}} e^{-k^2/4a} \int_{-\infty}^{\infty} e^{-\beta^2} d\beta = \sqrt{\frac{\pi}{a}} e^{-k^2/4a} = \sqrt{\frac{\pi}{a}} e^{-k^2\sigma_x^2/2}$$

Transform of a Gaussian pulse: The Heisenberg Uncertainty Principle

We can rewrite this in the standard form of a Gaussian in k:

$$A(k) \propto e^{-k^2/2\sigma_k^2} \quad \text{where} \quad \sigma_k^2 = 1/\sigma_x^2$$

The result then is that $\sigma_x\sigma_k$=1 for a Gaussian pulse. You will find that the product of spatial and wavenumber widths is always equal to or greater than one. Since the deBroglie hypothesis relates wavelength to momentum, p=h/λ we thus conclude that $\sigma_x\sigma_p$>=h/2π. This is a statement of the **Heisenberg Uncertainty Principle**.

The Heisenberg Uncertainty Principle

- This principle states that you cannot know both the position and momentum of a particle simultaneously to arbitrary accuracy.
  - There are many approaches to this idea. Here are two.
    - The act of measuring position requires that the particle intact with a probe, which imparts momentum to the particle.
    - Representing the position of localized wave requires that many wavelengths (momenta) be added together.
    - The act of measuring position by forcing a particle to pass through an aperture causes the particle wave to diffract.

The Heisenberg Uncertainty Principle

- Position and momentum are called conjugate variables and specify the trajectory of a classical particle. We have found that if one wants to specify the position of a Gaussian wave packet, then:

$$\Delta x \Delta p = \hbar$$

- Similarly, angular frequency and time are conjugate variables in wave analysis. (They appear with one another in the phase of a harmonic wave.)

$$\Delta \omega \Delta t = 1$$

- Since energy and frequency are related by the Planck constant we have, for a Gaussian packet:

$$\Delta E \Delta t = \hbar$$

The next stages

- We have seen through experiment that particles behave like waves with wavelength relationship: $p=h/\lambda$.
- The next stage is to figure out the relationship between whatever waves and observable quantities like position, momentum, energy, mass…
- The stage after that is to come up with a differential equation that describes the wavy thing and predicts its behavior.
- There is still a lot more we can do before actually addressing the solution to the wave equation.