Hw 3

1. Show that, the set of matrices, $S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$ is a subring of $M_2(\mathbb{R})$.

Closed under Addition:

Let A and $B \in S$ such that $A = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}$ and $B = \begin{pmatrix} g & h \\ 0 & j \end{pmatrix}$ where d, e, f, g, h, and j are all real numbers. Consider

$$A+B = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} + \begin{pmatrix} g & h \\ 0 & j \end{pmatrix} = \begin{pmatrix} d+g & e+h \\ 0+0 & f+j \end{pmatrix} = \begin{pmatrix} d+g & e+h \\ 0 & f+j \end{pmatrix}.$$

Since $A+B = \begin{pmatrix} d+g & e+h \\ 0 & f+j \end{pmatrix}$ where $d+g$, $e+h$, and $f+j$ are all real numbers, $A+B$ takes on the form of S. Therefore $A+B \in S$ and S is closed under addition.

Closed under multiplication:

Let A and $B \in S$ such that $A = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}$ and $B = \begin{pmatrix} g & h \\ 0 & j \end{pmatrix}$ where d, e, f, g, h, and j are all real numbers. Consider

$$AB = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} \begin{pmatrix} g & h \\ 0 & j \end{pmatrix} = \begin{pmatrix} dg & eh \\ 0 & fj \end{pmatrix}.$$

Since $AB = \begin{pmatrix} dg & eh \\ 0 & fj \end{pmatrix}$ where dg, eh, and fj are all real numbers. AB takes on the form of S. Therefore $AB \in S$ and S is closed under multiplication.

Hw 3

Additive Identity :

So, $0_R = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. 0_R is in the form of S because the top row is a row of 0s and the bottom right is a $0 \in R$. Therefore $0_R \in S$ and is closed under the additive identity.

Additive inverse :

Let $A \in S$ such that $A = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}$ where d, e and f are all real numbers. From $M_2(R)$, there is an additive inverse denoted $-A = \begin{pmatrix} -d & -e \\ 0 & -f \end{pmatrix}$. Since $-A = \begin{pmatrix} -d & -e \\ 0 & -f \end{pmatrix}$ where $-d$, $-e$, and $-f$ are all real numbers, $-A$ takes on the form of S, therefore $-A \in S$ and S is closed under additive inverse.

Since all 4 conditions have been proven true, S is a subring of $M_2(R)$.

Hw 3

2. Let R be a rins. Define the $Z(R)$ by $Z(R) = \{a \in R \mid ar = ra \text{ for all } r \in R\}$. Prove that $Z(R)$ is a subrins of R.

Closed under Addition:

Let $a, b \in Z(R)$ such that $ar = ra$ and $br = rb$ for all $r \in R$. Consider

$$\begin{aligned}(a+b)r &= ar + br \\ &= ra + rb \\ &= r(a+b)\end{aligned}$$

Since $(a+b)r = r(a+b)$ for all $r \in R$, $a+b \in Z(R)$. Therefore $Z(R)$ is closed under addition.

Closed under multiplication:

Let $a, b \in Z(R)$ such that $ar = ra$ and $br = rb$ for all $r \in R$. Consider

$$\begin{aligned}(ab)r &= a(br) = a(rb) = (ar)b = r(ab) \\ &= a(rb) \\ &= (ar)b \\ &= (ra)b \\ &= r(ab)\end{aligned}$$

Since $(ab)r = r(ab)$ for all $r \in R$, $ab \in Z(R)$. Therefore $Z(R)$ is closed under multiplication.

Closed under Additive Identity.

Consider 0_R. So, $0_R r = r 0_R$ for all $r \in R$ because 0_R is commutative in the rins of R. Therefore $0_R \in Z(R)$ and $Z(R)$ is closed under the additive identity.

Hw 3

2. Closed under additive Inverse

Let $a \in Z(R)$ such that $ar = ra$ for all $r \in R$. From R, we know there is additive inverse denoted as $-a$. Consider $(-a)r = -(ar)$
$$= -(ra)$$
$$= (-r)a$$
$$= r(-a)$$
Since $(-a)r = r(-a)$ for all $r \in R$, $-a \in Z(R)$. Therefore $Z(R)$ is closed under additive Inverse. Since all 4 conditions have been proven true, $Z(R)$ is a subring of R.

3. Let $S = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$. Show that S is a subring of the real numbers, $\mathbb{R}$.

Closed under addition

Let A and $B \in S$ such that $A = c + d\sqrt{2}$ and $B = e + f\sqrt{2}$ where c, d, e, and f are integers. Consider $A + B = c + d\sqrt{2} + e + f\sqrt{2}$
$$= c + e + d\sqrt{2} + f\sqrt{2}$$
$$= (c+e) + (d+f)\sqrt{2}$$
Since $A + B = (c+e) + (d+f)\sqrt{2}$ where $c+e$ and $d+f$ are integers, $A + B \in S$. Therefore S is closed under addition.

HW3

Closed under multiplication

Let A and $B \in S$ such that $A = c + d\sqrt{2}$ and $B = e + f\sqrt{2}$ where $c, d, e,$ and f are integers. Consider

$$AB = (c + d\sqrt{2})(e + f\sqrt{2})$$
$$= ce + cf\sqrt{2} + ed\sqrt{2} + 2df$$
$$= ce + 2df + cf\sqrt{2} + ed\sqrt{2}$$
$$= (ce + 2df) + (cf + ed)\sqrt{2}$$

Since $AB = (ce + 2df) + (cf + ed)\sqrt{2}$ where $ce + 2df$ and $cf + ed$ are integers $AB \in S$. Therefore S is close under multiplication.

Closed under Additive Identity

Consider $0_R = 0 + 0\sqrt{2}$. Since 0 is an integer $0_R \in S$. therefore S is closed under additive inverse.

Closed under Additive Inverse

Let $A \in S$ such that $A = c + d\sqrt{2}$ where c and d are both integers. Then by $\mathbb{R}$, we know that there is and additive inverse denoted as $-A = -c + -d\sqrt{2}$. Since $-c$ and $-d$ are both integers, $-A \in S$. Therefore A is closed under Additive Inverse.

Since all 4 conditions have been proven true, S is a subring of the real numbers, $\mathbb{R}$.

Note: functions multiplicative identity is 1

Hw 3 4a: Give an example of a function $f \in T$ that has a multiplicative inverse

a. Let $f \in t$ such that $f(x) = 2$ for $R \to R$.

(a) Consider $g(x) = 1/2$ for $R \to R$. So, $f(x) \cdot g(x) = 2 \cdot 1/2 = 1$. Now we know that $g(x)$ is an inverse of $f(x)$.

Since $f(x) \cdot g(x) = 1$, for $R \to R$, $f(x) = 2$ has a mutiplicative inverse. Therefore $f \in t$ with a multiplicative inverse.

(b) Give an example of a function that does not have a multiplicative inverse.

Let $f \in t$ s.t. $f(x) = x$ for $R \to R$. Consider $f(x) \cdot g(x) = 1$.

By substitution, $x \cdot g(x) = 1$ can be shown as $g(x) = \frac{1}{x}$.

However $g(x) = 1/x$ has a denomitor x that can't be 0, otherwise it is undefined. Therefore every real does not map to a real. Therefore $f \in t$, but does not have a multiplicative inverse.

Give an example of two nonzero functions of $f, g \in T$ s.t. $fg = 0$

(c) Let $f \in T$ and $g \in T$ such that $f(x) = \begin{cases} 3 & x \geq 0 \\ 0 & x < 0 \end{cases}$

and $g(x) = \begin{cases} 0 & x \geq 0 \\ 3 & x < 0 \end{cases}$. Consider $f(x) \cdot g(x)$. When

$x \geq 0$, $f(x) \cdot g(x) = 3 \cdot 0 = 0$. Also when $x < 0$,

$f(x) \cdot g(x) = 0 \cdot 3 = 0$. Since $f(x) \neq 0 \in T$ and $g(x) \neq 0 \in T$ such that $fg = 0$, examples have been provided.