

Algebra Definitions

On Rings, Polynomials, and Fields

1 Section 16.1 - Rings

1. A **ring** R is a set that is closed under two binary operations, $+$ and \times . The following conditions must also be satisfied:
 - (a) Additive commutativity.
 - (b) Additive associativity.
 - (c) Additive identity.
 - (d) Additive inverse.
 - (e) Multiplicative associativity.
 - (f) Multiplicative distributivity 1 & 2.
2. A **ring with unity (or with identity)** is a ring R that has multiplicative identity.
3. A **commutative ring** is a ring R that has multiplicative commutativity.
4. An **integral domain** is a commutative ring R with identity such that for all $a, b \in R$ $ab = 0$ implies $a = 0$ or $b = 0$.
5. A **division ring** is a ring R that has multiplicative inverse for all nonzero $a \in R$.
6. A **zero divisor** of a commutative ring R is an $a \in R$ ($a \neq 0$) such that there exists a nonzero $b \in R$ such that $ab = 0$.
7. The **ring of quaternions** is the set $\mathbb{H} = \{a + b\hat{i} + c\hat{j} + d\hat{k} \mid a, b, c, d \in \mathbb{R}\}$, where $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\hat{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\hat{j} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $\hat{k} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

2 Section 16.2 - Integral Domains and Fields

1. A **field** is a commutative division ring.
2. The **characteristic** of a ring R is the least positive integer n such that $nr = 0$ for all $r \in R$. If no such n exists, the characteristic of R is defined to be 0. (denote the characteristic of R by $\text{char}R$).

3 Section 16.3 - Ring Homomorphisms and Ideals

1. A **ring homomorphism** is a map $\phi : R \rightarrow S$ (where R, S are rings) such that $\phi(a + b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in R$.
2. A **ring isomorphism** is a bijective map $\phi : R \rightarrow S$ where R, S are rings.
3. The **kernel** of a ring homomorphism $\phi : R \rightarrow S$ is the set $\ker \phi := \{r \in R \mid \phi(r) = 0\}$.
4. An **evaluation homomorphism** is a ring homomorphism of the form $\phi_a : C[a, b] \rightarrow \mathbb{R}$ or other such related homomorphisms.
5. An **ideal** of a ring R is a subring I such that if $a \in I$ and $r \in R$, then $ar, ra \in I$.
6. The **trivial ideals** of a ring R are the subrings $\{0\}$ and R .
7. A **principal ideal** of a commutative ring R (with identity) is an ideal of the form $\langle a \rangle = \{ar \mid r \in R\}$.
8. A **two-sided ideal** I is a subring of a ring R such that $rI \subset I$ and $Ir \subset I$ for all $r \in R$.
9. A **one-sided ideal** I is a subring of a ring R is one such that $rI \subset I$ for all $r \in R$ (a **left ideal**) or $Ir \subset I$ for all $r \in R$ (a **right ideal**).

4 Section 17.1 - Polynomial Rings

1. A **polynomial over** R is an expression of the form $f(x = \sum_{i=0}^n a_i x^i)$ with **indeterminate** x . Define a_0, \dots, a_n to be the **coefficients** of f and a_n is the **leading coefficient** of f . A polynomial is **monic** if its leading coefficient a_n is 1. The **degree** (write: $\deg f(x) = n$) is the largest nonnegative number for which $a_n \neq 0$. If no such n exists, then $f = 0$, the **zero polynomial** and define the degree of $f = 0$ to be $-\infty$. Denote $R[x]$ to be the set of all polynomials with coefficients in a ring R .
2. $R[x, y]$ is the **ring of polynomials in two indeterminates x, y with coefficients in R** . $R[x_1, \dots, x_n]$ is the **ring of polynomials in n indeterminates with coefficients in R** .