Algebra Theorems

On Rings, Polynomials, and Fields

1 Section 16.2 - Integral Domains and Fields

1. **Prop. 16.15. Cancellation Law.** Let D be a commutative ring with identity. Then D is an integral domain iff for all nonzero elements $a \in D$ with $ab = ac$, we have $b = c$.

2. **Theorem 16.16.** Every finite integral domain is a field.

3. **Lemma 16.18.** Let R be a ring with identity. If 1 has order n, then the characteristic of R is n.

4. **Theorem 16.19.** The characteristic of an integral domain is either prime or zero.

2 Section 16.3 - Ring Homomorphisms and Ideals

1. **Prop. 16.22.** Let $\phi : R \to S$ be a ring homomorphism. Then:
   (a) If R is a commutative ring, then $\phi(R)$ is a commutative ring.
   (b) $\phi(0) = 0$.
   (c) Let 1_R and 1_S be the identities for R and S, respectively. If ϕ is onto, then $\phi(1_R) = 1_S$.
   (d) If R is a field and $\phi(R) \neq \{0\}$, then $\phi(R)$ is a field.

2. **Theorem 16.25.** Every ideal in the ring of integers $\mathbb{Z}$ is a principal ideal.

3. **Prop. 16.27.** The kernel of any ring homomorphism $\phi : R \to S$ is an ideal in R.

3 Section 17.1 - Polynomial Rings

1. **Theorem 17.3.** Let R be a commutative ring with identity. Then $R[x]$ is a commutative ring with identity.

2. **Prop. 17.4.** Let $p(x), q(x) \in R[x]$, where R is an integral domain. Then $\deg p(x) + \deg q(x) = \deg(pq(x))$. Furthermore, $R[x]$ is an integral domain.

3. **Theorem 17.5.** Let R be a commutative ring with identity and $\alpha \in R$. Then we have a ring homomorphism $\phi_\alpha : R[x] \to R$ defined by $\phi_\alpha(p(x)) = p(\alpha) = a_n\alpha^n + \cdots + a_0$, where $p(x) = a_n x^n + \cdots + a_0$.