Algebra Definitions

On Rings, Polynomials, and Fields

1 Section 16.1 - Rings

1. A ring R is a set that is closed under two binary operations, $+$ and $\times$. The following conditions must also be satisfied:
   (a) Additive commutativity.
   (b) Additive associativity.
   (c) Additive identity.
   (d) Additive inverse.
   (e) Multiplicative associativity.
   (f) Multiplicative distributivity 1 & 2.
2. A ring with unity (or with identity) is a ring R that has multiplicative identity.
3. A commutative ring is a ring R that has multiplicative commutativity.
4. An integral domain is a commutative ring R with identity such that for all $a, b \in R$ $ab = 0$ implies $a = 0$ or $b = 0$.
5. A division ring is a ring R that has multiplicative inverse for all nonzero $a \in R$.
6. A zero divisor of a commutative ring R is an $a \in R$ $(a \neq 0)$ such that there exists a nonzero $b \in R$ such that $ab = 0$.
7. The ring of quaternions is the set $\mathbb{H} = \{a + b\hat{i} + c\hat{j} + d\hat{k} \mid a, b, c, d \in \mathbb{R}\}$, where $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \hat{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \hat{j} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \hat{k} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

2 Section 16.2 - Integral Domains and Fields

1. A field is a commutative division ring.

2. The characteristic of a ring R is the least positive integer n such that $nr = 0$ for all $r \in R$. If no such n exists, the characteristic of R is defined to be 0. (denote the characteristic of R by char$R$).