Math 104 - Tail End Definitions

Definition 6.1. Let $[a, b]$ be a given interval. A partition P is a set of points $x_0, \ldots, x_n$ with $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$. Let $\Delta x_i = x_i - x_{i-1}$ and f be a bounded function on $[a, b]$. Then, we define $M_i = \sup f(x)(x_{i-1} \leq x \leq x_i)$ and $m_i = \inf f(x)(x_{i-1} \leq x \leq x_i)$. Then, define $U(P, f) = \sum_{i=1}^{n} M_i \cdot \Delta x_i$ and $L(P, f) = \sum_{i=1}^{n} m_i \cdot \Delta x_i$. Then, define $\int_a^{\bar{b}} f dx = \inf U(P, f)$ to be the upper Riemann integral and $\int_{\underline{a}}^{b} f dx = \sup L(P, f)$ to be the lower Riemann integral. The inf and sup are taken over all partitions of $[a, b]$. If $\int_{\underline{a}}^{a} f dx = \int_a^{\bar{b}} f dx$, then, f is Riemann integrable and we write $f \in \mathscr{R}$ and denote the common value of the lower/upper Riemann integrals as $\int_a^b f dx$ (term: Riemann integral).

Definition 6.1. Let $[a, b]$ be the given interval. A partition P is a set of points $x_0, \ldots, x_n$ with $a = x_0 \leq \cdots \leq x_n = b$. Let $\Delta x_i = x_i - x_{i-1}$ and f be a bounded function on $[a, b]$. We define $M_i = \sup f(x)(x_{i-1} \leq x \leq x_i)$ and $m_i = \inf f(x)(x_{i-1} \leq x \leq x_i)$. Then, define $U(P, f) = \sum_{i=1}^{n} M_i \cdot \Delta x_i$ and $L(P, f) = \sum_{i=1}^{n} m_i \cdot \Delta x_i$. Then, define $\int_{\underline{a}}^{b} f dx = \sup L(P, f)$ as the lower Riemann integral and $\int_a^{\bar{b}} f dx = \inf U(P, f)$ as the upper Riemann integral. The inf and sup are taken over all partitions of $[a, b]$. If $\int_{\underline{a}}^{b} f dx = \int_a^{\bar{b}} f dx$, then f is Riemann integrable and we write $f \in \mathscr{R}$ and we denote the common value to be $\int_a^b f dx$ to be the Riemann integral.

Definition 6.2. Let α be a monotonically increasing function on $[a, b]$. For each partition, write $\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. $\Delta \alpha_i \geq 0$, since α is monotonically increasing. Then, denote $U(P, f, \alpha) = \sum_{i=1}^{n} M_i \cdot \Delta \alpha_i$ and $L(P, f, \alpha) = \sum_{i=1}^{n} m_i \cdot \Delta \alpha_i$. Then, denote $\int_{\underline{a}}^{b} f d\alpha = \sup L(P, f, \alpha)$ and $\int_a^{\bar{b}} f d\alpha = \inf U(P, f, \alpha)$. Then, if $\int_{\underline{a}}^{b} f d\alpha = \int_a^{\bar{b}} f d\alpha$, then denote the common value as $\int_a^b f d\alpha$ (which is the Riemann Stieltjes Integral) of f on $[a, b]$.

Definition 6.3. The partition $P^\star$ is a refinement of P if $P^\star \supset P$. $P^\star$ is the common refinement of P_1 and P_2 if $P^\star = P_1 \cup P_2$.

Definition 7.1. Suppose $\{f_n\}(n = 1, 2, 3, \dots)$ is a sequence of functions defined on a set E, and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$. Define a function f:

$$f(x) = \lim_{n\to\infty} f_n(x)(x \in E)$$

$\{f_n\}_{n=1}^{\infty}$ converges on E and f is the limit of $\{f_n\}$. $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f on E. If $\sum f_n(x)$ converges for every $x \in E$, and if we define $f(x) = \sum_{n=1}^{\infty} f_n(x)(x \in E)$, then f is the sum of the series $\sum f_n$.

Definition 7.1. Suppose $\{f_n\}$ is a sequence of functions on a set E and $\{f_n(x)\}$ is a sequence of numbers that converges for every $x \in E$. Then, define the function f as follows:

$$f(x) = \lim_{n\to\infty} f_n(x)$$

Then, $\{f_n\}_{n=1}^{\infty}$ converges on E and f is the limit of $\{f_n\}$. $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f on E. If $\sum f_n(x)$ converges for every $x \in E$, then we define $f(x) = \sum_{n=1}^{\infty} f_n(x)$ and f is the sum of the series $\sum f_n$.

Definition 7.1. Suppose $\{f_n\}$ is a sequence of functions on a set E and $\{f_n(x)\}$ is a sequence of numbers that converges for every $x \in E$. Then, define a function f as the following:

$$f(x) = \lim_{n\to\infty} f_n(x)$$

Then, $\{f_n\}_{n=1}^{\infty}$ converges on E if f is the limit of $\{f_n\}$. $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f on E. $\sum f_n(x)$ converges for every $x \in E$; if we define $f(x) = \sum_{n=1}^{\infty} f_n(x)$, then f is the sum of the series $\sum f_n$.

Definition 7.1. Suppose $\{f_n\}$ is a sequence of functions on a set E and $\{f_n(x)\}$ is a sequence of numbers that converges for every $x \in E$. Then, define a function f as the following:

$$f(x) = \lim_{n\to\infty} f_n(x)$$

$\{f_n\}_{n=1}^{\infty}$ converges on E if f is the limit of $\{f_n\}$. Then, $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f on E. If $\sum f_n(x)$ converges for every $x \in E$, and if we define $f(x) = \sum_{n=1}^{\infty} f_n(x)$, then f is the sum of the series $\sum f_n$.