Math 104 - Tail End Definitions

Definition 6.1. Let S be a set. An order on S is a relation, denoted by $<$, with the following properties:

1. if $x, y \in S$, then only one of the following is true: $x < y$, $x = y$, $y < x$.
2. if $x, y, z \in S$ and $x < y$ and $y < z$ then $x < y < z$ and so $x < z$.

Definition 6.1. An ordered set is a set in which an order is defined.

Definition 6.1. Let S be an ordered set and $E \subset S$. Then if there exists a $\beta \in S$ with $\beta \leq x$ for all $x \in E$, then β is a lower bound of E and E is bounded below. If there is a $\beta in S$ with $\beta \geq x$ for all $x \in E$, then β is an upper bound of E and E is bounded above.

Definition 6.1. Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there is an $\alpha \in S$ with the following properties:

1. $\alpha \geq x$ for all $x \in E$.
2. if $\gamma < \alpha$ then γ is not an upper bound of E.

Then $\alpha = \sup E$ is the least upper bound of E. The definition of $\inf E$ being the greatest lower bound of E is defined similarly.

Definition 6.1. An ordered set S is said to have the least upper-bound property if the following statement is true: there is an $E \subset S$, E is nonempty, E is bounded above, then $\sup E \in S$.

Definition 6.1. An ordered field F is a field which is also an ordered set with:

1. if $x, y, z \in F$ and $y < z$, then $x + y < x + z$.

2. if $x, y \in F$ with $x, y > 0$, then $xy > 0$.

Definition 6.1. Consider two sets A and B. If there is a manner in which elements in A are mapped to elements in B, then call this manner a function f. f is also called a mapping of A onto B. Then, A is the domain of f and each $f(x) \in B$ is a value of f. Then the set of all $f(x) \in B$ is the range of f.

Definition 6.1. Let $f : A \to B$, and $E \subset A$. Then $f(E) = \{f(x) \mid x \in E\}$ is the image of E under f. Instead, if $E \subset B$, then $f^{-1}(E) = \{x \in A \mid f(x) \in B\}$ is the inverse image of E under f. f is onto if every element of B appears in the image of f. f is 1-1 if $(f(x) = f(y)) \implies (x = y)$. If f is both onto and 1-1 then f is bijective.

Definition 6.1. If there exists a 1-1 mapping of A onto B, then we say that A and B can be put into 1-1 correspondence. Then, we say that A and B have the same cardinal number, or that they are equivalent, denoted by $A \sim B$. If the relation $A \sim B$ satisfies:

1. Reflexive: $A \sim A$.
2. Symmetric: $A \sim B \implies B \sim A$.
3. Transitive: $(A \sim B \wedge B \sim C) \implies A \sim C$.

Then $A \sim B$ is called an equivalence relation.

Definition 6.1. Let $n \in \mathbb{N}$, $J = \mathbb{N}$, and J_n be the set of the first n positive integers. Then, define the following for a set A:

1. A is finite if $A \sim J_n$ for some n.
2. A is infinite if it is not finite.
3. A is countable if $A \sim J$.
4. A is uncountable if it is neither countable nor finite.
5. A is at most countable if it is either countable or finite.

Definition 6.1. A sequence is a function from the natural numbers to a set X.

Definition 6.1. A set X, whose elements we call elements, is a metric space if for each $p, q \in X$ we can find a real number that represents the distance between p and q, denoted by d, with the following:

1. $d(p,q) > 0$ if $p \neq q$. $d(p,p) = 0$.
2. $d(p,q) = d(q,p)$.
3. $d(p,q) \leq d(p,r) + d(r,q)$ for any $r \in X$.

Any d with these properties is called a distance function or metric.

Definition 6.1. For any set X, in the discrete metric space, the metric is defined to be d_D with $d_D = 0$ if $x = y$ and $d_D = 1$ if $x \neq y$ (for any $x, y \in X$).

Definition 6.1. A sequence space is a space of all bounded sequences of real numbers.

Definition 6.1. l^p is the set of all sequences where each element is a sequence of the form $\{x_i\}$ with $\sum_{i=j}^{\infty} |x_j|^p < \infty$.

Definition 6.1. The L^p metric is defined to be: $d(x,y) = \left(\sum_{j=1}^{n} |x_j - y_j|^p\right)^{1/p}$. For sequences, $d(\{x_i\}, \{y_i\}) = \left(\sum_{j=1}^{\infty} |x_j - y_j|^p\right)^{1/p}$

Definition 6.1. For $x \in \mathbb{R}^n$ the open ball of radius $r > 0$ about x is the set $\{y \in \mathbb{R}^n \mid d(x,y) < r\}$. The closed ball is $\{y \in \mathbb{R}^n \mid d(x,y) \leq r\}$.

Definition 6.1. The segment (a,b) is the set $\{x \in \mathbb{R} \mid a < x < b\}$ and the interval $[a,b]$ is the set $\{x \in \mathbb{R} \mid a \leq x \leq b\}$.

Definition 6.1. Let (X,d) be a metric space. Then, define the following:

1. A neighborhood about a point $p \in X$ is the set $N_r(p) = \{q \in X \mid d(p,r) < r\}$ for a radius $r > 0$.
2. A point p is a limit point of a set $E \subseteq X$ if every neighborhood of p contains a point $q \neq p$ with $q \in E$.
3. p is an isolated point of E if $p \in E$ and p is not a limit point of E.
4. E is closed if every limit point of E is a point of E.

5. A point p is an interior point of E if there is a neighborhood N of p (with radius $r > 0$) with $N \subset E$.

6. E is open if every point of E is an interior point of E.

7. The complement of E is the set $E^c\{x \in X \mid x \notin E\}$.

8. E is perfect if E is closed and every point of E is a limit point of E.

Definition 6.1. If X is a metric space, and if $E \subset X$, let E' be the set of all limit points of E. Then, $\bar{E} = E \cup E'$ is called the closure of E.

Definition 6.1. AN open cover of E is a collection of open set $\{G_\alpha\}$ such that the union of all G_α contains E. A subcover is a subcollection of $\{G_\alpha\}$ that still covers E.

Definition 6.1. A subset K of a metric space (X, d) is said to be compact if every open cover of K contains a finite subcover.

Definition 6.1. If $a_i < b_i$ for all $i = 1, \dots, k$, then the set of all x with $a_i \leq x_i \leq b_i$ (in $\mathbb{R}^k$) is called a k-cell.

Definition 6.1. A sequence $\{p_n\}$ in a metric space (X, d) is said to converge if there is a point $p \in X$ with the property: for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then $d(p_n, p) < \epsilon$. We have the following phrasings:

1. $\{p_n\}$ converges to p.

2. $p_n \to p$.

3. $\lim_{n\to\infty} p_n = p$.

If $\{p_n\}$ does not converge, it diverges.

Definition 6.1. Given a sequence $\{p_n\}$, let $\{n_i\}$ be a sequence of solely natural numbers with $n_1 < n_2 < \dots$. Then $\{p_{n_i}\}$ is a subsequence of $\{p_n\}$. If $\{p_{n_i}\}$ converges, its limit is called a subsequential limit of $\{p_n\}$.

Definition 6.1. A sequence $\{p_n\}$ in a metric space (X, d) is said to be a Cauchy sequence if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n, m \geq N$, then $d(p_n, p_m) < \epsilon$.

Definition 6.1. A metric space in which every Cauchy sequence converges is said to be complete.

Definition 6.1. A sequence $\{p_n\}$ of real numbers is said to be

1. monotonically increasing if $p_n \leq p_{n+1}$ for all n.

2. monotonically decreasing if $p_n \geq p_{n+1}$ for all n.

Definition 6.1. Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of all subsequential limits of $\{s_n\}$. Then, E possibly includes $\infty, -\infty$. Define $s^\star = \sup E$ to be the upper limit of $\{s_n\}$ and $s_\star = \inf E$ to be the lower limit of $\{s_n\}$. Then, we have that $s^\star = \lim_{n\to\infty} \sup s_n$ and $s_\star = \lim_{n\to\infty} \inf s_n$.

Definition 6.1. Given a sequence $\{a_i\}$, let $\sum_{i=p}^{q} a_i = a_p + a_{p+1} + \cdots + a_q$. Then, let $s_n = \sum_{i=1}^{n} s_n$ be the n^{th} partial sum.

Definition 6.1. Let (X, d_x) and (Y, d_y) be metric spaces, $E \subset X$, $f : E \to Y$, and p be a limit point of E. Then, $\lim_{x\to p} f(x) = q$ if there is a $q \in Y$ with: for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in E$ if $0 < d_x(x, p) < \delta$, then $d_y(f(x), q) < \epsilon$.

Definition 6.1. Retain the same preliminaries notations and conditions as the previous definition, except now instead, $p \in E$. Then, f is continuous at p if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in E$ if $d_x(x, p) < \delta$, then $d_y(f(x), f(q)) < \epsilon$.

Definition 6.1. A mapping $f : E \to \mathbb{R}^k$ is said to be bounded if there is a real number M such that $|f(x)| \leq M$ for all $x \in E$.

Definition 6.1. Let $f : X \to Y$. f is uniformly continuous on X if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $p, q \in X$, if $d_x(p, q) < \delta$, then $d_y(f(p), f(q)) < \epsilon$.

Definition 6.1. A function $f : [a, b] \to \mathbb{R}$ is differentiable at $x \in [a, b]$ if the following limit exists:

$$f'(x) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x}$$

where f' is the first derivative of f. Note that $t \in [a, b]$ and $t \neq x$.

Definition 6.1. A set X is sequentially compact if every sequence in X has a subsequence that converges to a point in X.

Definition 6.1. Let $f : X \to \mathbb{R}$. f has a local minimum at $p \in X$ if there exists a $\delta > 0$ such that $f(q) \geq f(p)$ for all $q \in X$ with $d(p, q) < \delta$. f has a local maximum at $p \in X$ if there exists a $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \delta$.

Definition 6.1. Let $[a, b]$ be the given interval. A partition P of $[a, b]$ is a set of points $x_0, \ldots, x_n$ such that $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$. Let $\Delta x_i = x_i - x_{i-1}$ (for $i = 1, \ldots, n$) and f be a bounded function on $[a, b]$. Then, let $M_i = \sup f(x)$ on $[x_{i-1}, x_i]$ and $m_i = \inf f(x)$ on $[x_{i-1}, x_i]$. Then, let $U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$ and $L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$. Then, let $\int_{\underline{a}}^{b} f dx = \sup L(P, f)$ be the lower Riemann integral of f on $[a, b]$ and $\int_{a}^{\overline{b}} f dx = \inf U(P, f)$ be the upper Riemann integral of f on $[a, b]$. Then, if $\int_{\underline{a}}^{b} f dx = \int_{a}^{\overline{b}} f dx$, then f is Riemann integrable on $[a, b]$ (write: $f \in \mathscr{R}$) and denote the common value as $\int_a^b f dx$ (called the Riemann integral of f on $[a, b]$).

Definition 6.1. Let α be a monotonically increasing function. Then, let $\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. Then, $\Delta \alpha_i \geq 0$ since α is monotonically increasing. Let f be a bounded function on $[a, b]$ and P be a partition of $[a, b]$. Let $U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta \alpha_i$ and $L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta \alpha_i$. Then, let $\int_{\underline{a}}^{f} f d\alpha = \sup L(P, f, \alpha)$ and $\int_{a}^{\overline{b}} f d\alpha = \inf U(P, f, \alpha)$. Then, if $\int_{\underline{a}}^{b} f d\alpha = \int_{a}^{\overline{b}} f d\alpha$, then f is Riemann-Stieltjes integrable on $[a, b]$ (write $f \in \mathscr{R}(\alpha)$) and let the common value be $\int_a^b f d\alpha$ be the Riemann-Stieltjes integral of f on $[a, b]$.

Definition 6.1. A partition $P^\star$ is a refinement of a partition P if $P^\star \supset P$. P is the common refinement of partitions P_1 and P_2 if $P = P_1 \cup P_2$.

Definition 6.1. Suppose $\{f_n\}$ is a sequence of functions on a set E. Then, suppose $\{f_n(x)\}$ is a sequence of numbers that converges for every $x \in E$. Then, define the following function (with $x \in E$):

$$f(x) = \lim_{n \to \infty} f_n(x)$$

$\{f_n\}$ converges on E if f is the limit of $\{f_n\}$. $\{f_n\}$ converges pointwise to f on E. If $\sum f_n(x)$ converges for all $x \in E$ and if we define $f(x) = \sum_{n=1}^{\infty} f_n(x)$ (for $x \in E$), then f is the sum of the series $\sum f_n$.

Definition 6.1. For a bounded function $f : E \to \mathbb{R}$, let the norm of f be $||f|| = \sup_{x \in E} |f(x)|$.

Definition 6.1. $\{f_n\}$ converges uniformly on E to a function f if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then $||f_n - f|| < \epsilon$ for every $x \in E$.

Definition 6.1. If (X, d), then, let $\mathscr{C}(X) = \{f : X \to \mathbb{C} : f$ is bounded and continuous$\}$. For $f \in \mathscr{C}(X)$, define the supremum norm of f to be $||f|| = \sup_{x \in X} |f(x)|$ and define $d_{\mathscr{C}(X)}(f, g) = ||f - g||$, where $\mathscr{C}(X)$ is a metric space.

Definition 6.1. Let $\{f_n\}$ be a sequence of bounded functions on E. Then, $\{f_n\}$ is pointwise bounded if $\{f_n(x)\}$ is bounded (for each $x \in E$), that is, there is a real-valued function ϕ on E with $|f_n(x)| \leq \phi(x)$ for each $x \in E$. $\{f_n\}$ is uniformly bounded on E if there exists an $M \in \mathbb{R}$ such that $|f_n(x)| \leq M$ for all $x \in E$, $n \in \mathbb{N}$.