Math 104 - Tail End Definitions

Definition 6.1.

Definition 6.1. A sequence $\{p_n\}$ in a metric space (X, d) is said to be a Cauchy sequence if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n, m \geq N$, then $d(p_n, p_m) < \epsilon$.

Definition 6.1. A metric space in which every Cauchy sequence converges is called complete.

Definition 6.1. A sequence $\{p_n\}$ of real numbers is said to

1. be monotonically increasing if $p_n \leq p_{n+1}$ for all n.

2. be monotonically decreasing if $p_n \geq p_{n+1}$ for all n.

Definition 6.1. Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of all subsequential limits of $\{s_n\}$. E possibly includes $\infty, -\infty$. Then, let $s_\star = \inf E$ to be the lower limit of $\{s_n\}$ and $s^\star = \sup E$ be the upper limit of $\{s_n\}$. Then, it follows that $s^\star = \lim sups_n$ and $s_\star = \lim\inf s_n$.

Definition 6.1. Given a sequence $\{a_i\}$, let $\sum_{i=p}^{q} a_i = a_p + a_{p+1} + \cdots + a_q$. Then $s_n = \sum_{i=1}^{n} a_i$ is the n^{th} partial sum.

Definition 6.1. Let (X, d_x) and (Y, d_y) be metric spaces, $E \subset X$, $f : E \to Y$, and $p \in E'$. Then, $\lim_{x \to p} f(x) = q$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in E$, if $0 < d_x(x, p) < \delta$, then $d_y(f(x), q) < \epsilon$.

Definition 6.1. Use the same preliminaries as before except $p \in E$. Then f is continuous at p if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in E$, if $d_x(x, p) < \delta$ then $d_y(f(x), f(p)) < \epsilon$.

Definition 6.1. A mapping $f : X \to \mathbb{R}^k$ is bounded if there exists an $M\mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in E$.

Definition 6.1. Let $f : X \to Y$. f is uniformly continuous on X if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $p, q \in X$ if $d_x(p, q) < \delta$ then $d_y(f(p), f(q)) < \epsilon$.

Definition 6.1. Let $f : [a, b] \to \mathbb{R}$. f is differentiable at $x \in [a, b]$ if the following limit exists:

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

where f' is the first derivative of f and $t \in [a, b]$ with $t \neq x$.

Definition 6.1. Let X be a set. X is sequentially compact if every sequence in X has a subsequence that converges to a point in X.

Definition 6.1. Let $f : X \to \mathbb{R}$. f has a local minimum at p if there exists a $\delta > 0$ such that $f(q) \geq f(p)$ for every $q \in X$ with $d(p, q) < \delta$. f has a local maximum at p if there exists a $\delta > 0$ such that $f(q) \leq f(p)$ for every $q \in X$ with $d(p, q) < \delta$.

Definition 6.1. Let $[a, b]$ be the given interval. A partition P is a set of points $x_0, \ldots, x_n$ with $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$. Then, $\Delta x_i = x_i - x_{i-1}$ and let f be a bounded function on $[a, b]$. Then, let $M_i := \sup f(x)$ on $[x_{i-1}, x_i]$ and $m_i := \inf f(x)$ on $[x_{i-1}, x_i]$. Then, let $L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$ and $U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$. Then, let $\int_{\underline{a}}^{b} f dx = \sup L(P, f)$ be the lower Riemann integral of f on $[a, b]$ and let $\int_{a}^{\overline{b}} f dx = \inf U(P, f)$ be the upper Riemann integral of f on $[a, b]$. Then, if $\int_{\underline{a}}^{b} f dx = \int_{a}^{\overline{b}} f dx$, then f is Riemann integrable on $[a, b]$ and denote the common value as $\int_a^b f dx$ (called the Riemann integral of f on $[a, b]$).

Definition 6.1. Let α be a monotonically increasing function with $\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. Note that $\Delta \alpha_i \geq 0$ since α is monotonically increasing. Let f be a real bounded function on $[a, b]$. Then, let $L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta \alpha_i$ and $U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta \alpha_i$. Then, let $\int_{\underline{a}}^{b} f d\alpha = \sup L(P, f, \alpha)$ and $\int_{a}^{\overline{b}} f d\alpha = \inf U(P, f, \alpha)$. If $\int_{\underline{a}}^{b} f d\alpha = \int_{a}^{\overline{b}} f d\alpha$ then f is Riemann-Stieltjes integrable on $[a, b]$ and denote the common value as $\int_a^b f d\alpha$.

Definition 6.1. The partition P' is a refinement of the partition P if $P' \supset P$. P is the common refinement of partitions P_1 and P_2 if $P = P_1 \cup P_2$.

Definition 6.1. Suppose $\{f_n\}$ is a sequence of functions defined on E and $\{f_n(x)\}$ is a sequence of numbers that converges for every $x \in E$. Then, define the following function:

$$f(x) = \lim_{n \to \infty} f_n(x)$$

for all $x \in E$. Then, $\{f_n\}$ converges on E if f is the limit of $\{f_n\}$. $\{f_n\}$ converges pointwise to f on E. If $\sum f_n(x)$ converges for every $x \in E$, and if we define

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

then f is the sum of the series $\sum f_n$.

Definition 6.1. Let f be a bounded function with $f : E \to \mathbb{R}$. Then, let the following represent the norm of f.

$$||f|| = \sup_{x \in E} |f(x)|$$

Definition 6.1. $\{f_n\}$ converges uniformly if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then, $||f_n - f|| < \epsilon$ for every $x \in E$.

Definition 6.1. If (X, d), let $\mathscr{C}(x) = \{f : X \to \mathbb{C} : f \text{ is bounded and continous}\}$. Then, for $f \in \mathscr{C}(x)$, let its supremum norm be defined by $||f|| = \sup_{x \in X} |f(x)|$. Also define $d_{\mathscr{C}(x)}(f, g) = ||f - g||$, where $\mathscr{C}(x)$ is a metric space.

Definition 6.1. Let $\{f_n\}$ be a sequence of bounded functions on E. $\{f_n\}$ is pointwise bounded if $\{f_n(x)\}$ is bounded for every $x \in E$, that is, there is a real-valued function ϕ, also defined on E, such that $|f_n(x)| < \phi$ for every $x \in E$. $\{f_n\}$ is uniformly bounded on E if there exists an $M \in \mathbb{R}$ such that $|f_n(x)| \leq M$ for every $x \in E$ and natural number n.