Math 104 - Midterm 2 Definitions (Attempt 1)

Definition 3.1. A sequence $\{p_n\}$ in a metric space X converges if the following property is true: there is a point $p \in X$ such that for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that if $n \geq N$, then $d(p_n, p) < \epsilon$. Here are 3 ways of denoting the convergence of $\{p_n\}$ to p:

1. p is the limit of $\{p_n\}$ (also written as $p_n \to p$).
2. $\{p_n\}$ converges to p.
3. $\lim_{n\to\infty} p_n = p$.

We say that if $\{p_n\}$ does not converge, it diverges.

Definition 3.5. Given a sequence $\{p_n\}$ in a metric space X, consider the sequence $\{n_i\}$ with $n_1 < n_2 < n_3 < \dots$. Then, $\{p_{n_i}\}$ is a subsequence of $\{p_n\}$, and if $\{p_{n_i}\}$ converges to a $p \in X$, then, p is a subsequential limit of $\{p_n\}$.

Definition 3.8. A sequence $\{p_n\}$ in a metric space (X, d) is called a Cauchy sequence if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n, m \geq N$, then, $d(p_n, p_m) < \epsilon$. In other words, if $\{p_n\}$ is a Cauchy sequence, then its terms get arbitrarily close to each other.

Definition 3.12. A metric space (X, d) in which every Cauchy sequence converges is called a complete metric space.

Definition 3.13. Let $\{s_n\}$ be a sequence of real numbers. Then, we have the following:

1. If $s_n \leq s_{n+1}$ (for $n = 1, 2, 3, \dots$), then, $\{s_n\}$ is monotonically increasing.

2. If $s_n \geq s_{n+1}$ (for $n = 1, 2, 3, \dots$), then, $\{s_n\}$ is monotonically decreasing.

Definition 3.16. Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of all subsequential limits of $\{s_n\}$, where $E \subseteq \mathbb{R}$. Denote $s^\star = \sup E$ and $s_\star = \inf E$. Then, $s^\star$ is the upper limit of $\{s_n\}$ and $s_\star$ is the lower limit of $\{s_n\}$. Note that $\lim_{n\to\infty} \sup s_n = s^\star$ and $\lim_{n\to\infty} \inf s_n = s_\star$.

Definition 3.21. Let $\{a_i\}\Big|_{i=1}^{\infty}$ be a sequence of real numbers. Realize that $\sum_{n=k}^{q} a_n = a_k + a_{k+1} + \cdots + a_q$. We denote $s_n = \sum_{k=1}^{n} a_k$ to be the n^{th} partial sum.

Definition 4.1. Let (X, d_x) and (Y, d_y) be metric spaces, $E \subset X$, $f : E \to Y$, and p is a limit point of E. Then, we define $\lim_{x\to p} f(x) = q$ if there is a $q \in Y$ such that for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in E$, if $0 < d_x(x, p) < \delta$, then $d_y(f(x), q) < \epsilon$.

Definition 4.5. Retain each notation and its respective denotation from Definition 4.1, except now $p \in E$. We define f to be continuous at $p \in E$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in E$, if $d_x(x, p) < \delta$, then $d_y(f(x), f(p)) < \epsilon$.

Definition 4.13. A mapping $f : E \to \mathbb{R}^k$ is bounded if there exists a real number M such that $|f(x)| \leq M$ for all $x \in E$.

Definition 4.18. Let (X, d_x) and (Y, d_y) be metric spaces. We say that f is uniformly continuous on X if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $p, q \in X$, if $d_x(p, q) < \delta$, then $d_y(f(p), f(q)) < \epsilon$.

Definition 5.1. Let f be a function with $f : [a, b] \to \mathbb{R}$. We say that f is differentiable at a point $x \in [a, b]$ if the following limit exists:

$$\lim_{t\to x} \frac{f(t) - f(x)}{t - x} = f'(x)$$

where f' denotes the first derivative of f. Also, note that $t \in [a, b]$ and $t \neq x$.

Definition of Sequentially Compact. Let X be a set. If every sequence in X has a subsequence that converges to an element in X, then we say that X is sequentially compact.

Definition 5.7. Let $f : X \to \mathbb{R}$. f has a local maximum at a point $p \in X$ if there exists a $\delta > 0$ such that $f(q) \leq f(p)$ for every $q \in X$ such that $d(p, q) < \delta$. f has a local minimum at a point $p \in X$ if there exists a $\delta > 0$ such that $f(q) \geq f(p)$ for every $q \in X$ such that $d(p, q) < \delta$.