Math 104 - Midterm 2 Definitions (Attempt 2)

Definition 3.1. A sequence $\{p_n\}$ in a metric space (X, d) is said to converge if there exists a point $p \in X$ with the following property: for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then $d(p_n, p) < \epsilon$. We denote the convergence of $\{p_n\}$ to p in the following three ways:

1. $\{p_n\}$ converges to p.
2. $p_n \to p$.
3. $\lim_{n\to\infty} p_n = p$.

We say that if $\{p_n\}$ does not converge, it diverges.

Definition 3.5. Let $\{p_n\}$ be a sequence in a metric space (X, d). Let $\{n_i\}$ be a sequence of natural numbers with $n_1 < n_2 < n_3 < \dots$. We denote $\{p_{n_i}\}$ to be a sequence of $\{p_n\}$. If $\{p_{n_i}\}$ converges to some $p \in X$, we say that p is a subsequential limit of $\{p_n\}$.

Definition 3.8. A sequence $\{p_n\}$ in a metric space (X, d) is said to be a Cauchy sequence if for every $\epsilon > 0$ there exists an $N \in \mathbb{R}$ such that if $n, m \geq N$, then $d(p_n, p_m) < \epsilon$.

Definition 3.12. A metric space in which every Cauchy sequence converges is said to be complete.

Definition 3.13. Let $\{s_n\}$ be a sequence of real numbers. We define the following:

1. if $s_n \leq s_{n+1} (n = 1, 2, 3, \dots)$, we say that $\{s_n\}$ is monotonically increasing.

2. if $s_n \geq s_{n+1} (n = 1, 2, 3, \dots)$, we say that $\{s_n\}$ is monotonically decreasing.

Definition 3.16. Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of subsequential limits of $\{s_n\}$. Denote $s^\star = \sup E$ and $s_\star = \inf E$. Then, $s^\star$ is the upper limit of $\{s_n\}$ and $s_\star$ is the lower limit of $\{s_n\}$. We say that $\lim_{n\to\infty} \sup s_n = s^\star$ and $\lim_{n\to\infty} \inf s_n = s_\star$.

Definition 3.21. Given $\{a_i\}\Big|_{i=1}^{\infty}$, let $\sum_{i=k}^{q} a_i = a_k + a_{k+1} + \cdots + a_q$. We denote $s_n = \sum_{k=1}^{n} a_k$ to be the n^{th} partial sum.

Definition 4.1. Let (X, d_x) and (Y, d_y) be metric spaces. Let $E \subset X$, $f : E \to Y$, and p is a limit point of E. We say that $\lim_{x\to p} f(x) = q$ if there is a a point $q \in Y$ such that for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in E$, if $0 < d_x(x, p) < \delta$, then $d_y(f(x), q) < \epsilon$.

Definition 4.5. Let (X, d_x) and (Y, d_y) be metric spaces. Let $E \subset X$, $f : E \to Y$, and $p \in E$. We say that f is continuous at p, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in E$ if $d_x(x, p) < \delta$, then $d_y(f(x), f(p)) < \epsilon$.

Definition 4.13. A mapping $f : E \to \mathbb{R}^k$ is bounded if there exists a real number M such that $|f(x)| \leq M$ for all $x \in E$.

Definition 4.18. Let $f : X \to Y$, where X and Y are metric spaces. We say that f is uniformly continuous on X if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $p, q \in X$, if $d_x(p, q) < \delta$, then $d_y(f(p), f(q)) < \epsilon$.

Definition 5.1. A function $f : [a, b] \to \mathbb{R}$ is differentiable if the following limit exists:

$$\lim_{t\to x} \frac{f(t) - f(x)}{t - x} = f'(x)$$

where f' is the first derivative of f. Note that $t \in [a, b]$ and $t \neq x$.

Definition of Sequentially Compact. Let S be a set. We say that S is sequentially compact if every sequence in S has a subsequence whose limit is in S.

Definition 5.7. Let $f : X \to \mathbb{R}$. f has a local minimum at a point $p \in X$ if there exists an $\epsilon > 0$ such that $f(q) \geq f(p)$ for every $q \in X$ such that q is in the $\epsilon-$ ball of p. f has a local maximum at a point $p \in X$ if there exists an $\epsilon > 0$ such that $f(q) \leq f(p)$ for every $q \in X$ such that q is in the $\epsilon-$ ball of p.