Math 104 - Midterm 1 Definitions

Definition 1.5. Let S be a set. An **order** on S is a relation that satisfies both of the following properties:

1. Let $x, y \in S$. Only one of the following statements is true. $x < y, x = y, x > y$.

2. If $x, y, z \in S$ and $x < y$ and $y < z$, then $x < z$ and it follows that $x < y < z$.

Definition 1.6. An **ordered set** S is a set in which an order is defined.

Definition 1.7. Suppose S is an ordered set and $E \subset S$. E is **bounded above** if there exists a $\beta \in S$ such that $\beta \geq x$ for all $x \in E$. E is **bounded below** if there exists a $\beta \in S$ such that $\beta \leq x$ for all $x \in E$.

Definition 1.8. Suppose S is an ordered set, $E \subset S$, E is bounded above. Suppose there exists an $\alpha \in S$ with the following two properties:

1. α is an upper bound of E.

2. If $\gamma < \alpha$, then γ is not an upper bound.

then α is the **least upper-bound** of E, denoted by $\alpha = \sup E$.
Suppose there is a $\beta \in S$ with the following two properties:

1. β is a lower bound of E.

2. If $\gamma > \beta$, then γ is not a lower bound of E.

then β is the **greatest lower-bound** of E, denoted by $\beta = \inf E$.

Definition 1.10. An ordered set S is said to have the **least upper-bound property** if the following statement is true: $E \subset S$, E is nonempty, E is bounded above, and $\sup E \in S$.

Definition 1.17. An **ordered field** is a field F which is also an **ordered set** which satisfies the following two properties:

1. $x + y < x + z$ if $x, y, z \in F$ and $y < z$.

2. $xy > 0$ if $x, y \in F$, $x > 0$, and $y > 0$.

Definition 2.1. Consider two sets A and B which can contain any objects whatsoever. Suppose that with each element $x \in A$, we associate an element in B through some manner. Let this assignment be denoted by $f(x)$ where f is a **function**. We can also say that there is a mapping of A into B. We use the following notation: $f : A \to B$. A is called the **domain** of f (we also say that f is defined on A). The elements $f(x)$ in B are called the **values** of f. The set of all $f(x)$ is called the **range** of f.

Definition 2.2. Let $f : A \to B$. If $E \subset A$, then the **image** of E under f is the set $\{f(x) \mid x \in E\}$. If $E \subset B$, then the **inverse image** of E under f is the set $\{x \in A \mid f(x) \in E\}$. If $y \in B$, $f^{-1}(y) = \{x \in A \mid f(x) = y\}$. If f is **onto**, then every in element in B appears in the image of E under f. If $(f(x) = f(y)) \implies (x = y)$, then f is 1-1. If f is both onto and 1-1, then f is **bijective**.

Definition 2.3. If there is a 1-1 mapping of A onto B, then we say that A and B can be put into **1-1 correspondence**. If this is true, A and B have the same **cardinal number**, or that A and B are **equivalent**, denoted by $A \sim B$. If this is true, then the relation $A \sim B$ has the following 3 properties:

1. **Reflexive**: $A \sim A$.

2. **Symmetric**: $(A \sim B) \implies (B \sim A)$.

3. **Transitive**: $(A \sim B \wedge B \sim C) \implies (A \sim C)$.

Any relation with these 3 properties is called an **equivalence relation**.

Definition 2.4. Let A be a set, $n \in \mathbb{N}$, J_n denotes the set of the first n positive integers, and $J = \mathbb{N}$. We have some terms to define:

1. A is **finite** if the relation $A \sim J_n$ exists for some n.
2. A is **infinite** if A is not finite.
3. A is **countable** if the relation $A \sim J$ exists.
4. A is **uncountable** if it is not finite and not countable.
5. A is **at most countable** if it is finite or countable.

Definition 2.7. A sequence is a function $f(n)$ that is defined on $\mathbb{N}$. If $f(n) = x_n$ for all n, then we denote $\{x_n\}$ to be the entire sequence $f(n)$ applied to all $n \in \mathbb{N}$.

Definition 2.15. A set X is said to be a **metric space** if for every $p, q \in X$ (elements in X are called **points**) there is associated a real number $d(p,q)$ that satisfies the following 3 properties (a function that has these 3 properties is also called a **distance function** or a **metric**):

1. $d(p,q) > 0$ if $p \neq q$ and $d(p,p) = 0$.
2. $d(p,q) = d(q,p)$.
3. $d(p,q) \leq d(p,r) + d(r,q)$ for any $r \in X$ (triangle inequality).

Definition of Discrete Metric Space. For any set X, we can define $d_D(x,y) = 0$ if $x = y$ and $d_D(x,y) = 1$ if $x \neq y$. Therefore, the pair (X, d_D) denote the **discrete metric space**. Specifically to when $X = \mathbb{R}^n$, we have the notation $(\mathbb{R}^n, d_D)$.

Definition of Sequence Spaces. A **sequence space** is a space of all sequences of real numbers that are bounded.

Definition of l^p. Let l^p denote the set of all sequences $\{x_i\}\Big|_{i=1}^{n}$ such that $\sum_{j=1}^{\infty} |x_j|^p < \infty$.

Definition of L^p-metric. Under the L^p-metric, the standard distance function is defined as $d(x,y) = \left(\sum_{j=1}^{n} |x_j - y_j|^p\right)^{\frac{1}{p}}$ where n represents the dimension (the same n as in $\mathbb{R}^n$). For sequences, $d\left(\{x_i\}\Big|_{i=1}^{\infty}, \{y_i\}\Big|_{i=1}^{\infty}\right) = \left(\sum_{j=1}^{n} |x_j - y_j|^p\right)^{\frac{1}{p}}$

Definition of Open/Closed Ball. Let $x \in \mathbb{R}^n$ and r be a real number with $r > 0$. The **open ball** with center x is defined to be the set $\{y \in \mathbb{R}^n \mid d(x, y) < r\}$ and the **closed ball** with center x is defined to be the set $\{y \in \mathbb{R}^n \mid d(x, y) \leq r\}$.

Definition 2.17. The **segment** (a, b) is defined to be the set $\{x \in \mathbb{R} \mid a < x < b\}$ and the **interval** $[a, b]$ is defined to be the set $\{x \in \mathbb{R} \mid a \leq x \leq b\}$.

Definition 2.18. Let (X, d) be a metric space. We define the following terms:

1. Let p be a point in X with $p \in X$. A **neighborhood** of point p is the set $N_r(p)$ with **radius** $r > 0$ such that $N_r(p) = \{q \in X \mid d(p, q) < r\}$.

2. A point p is a **limit point** of the set $E \subseteq X$ if every neighborhood of p contains a point $q \neq p$ such that $q \in E$.

3. A point p is an **isolated point** in E if p is not a limit point.

4. E is **closed** if every limit point of E is a point of E.

5. A point p is an **interior point** of E if a neighborhood N of p satisfies $N \subset E$.

6. E is **open** if every point of E is an interior point of E.

7. The **complement** of E is the set $E^c = \{x \in X \mid x \notin E\}$.

8. E is **perfect** if E is closed and if every point of E is a limit point of E.

Definition 2.26. If X is a metric space, $E \subset X$, and E' denotes the set of all limit points of E, then $\bar{E} = E \cup E'$ is defined to be the **closure** of E.

Definition 2.31. An **open cover** of a set $E \subset X$ is the collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \cup_\alpha G_\alpha$. A **subcover** of E is a subcollection of that still contains E.

Definition 2.32. A set $E \subset X$ is said to be **compact** if every open cover of E contains a finite subcover.

Definition of k-cell. If $a_i < b_i$ for $i = 1, \ldots, k$, the set of all points $x = \{x_1, \ldots, x_k\}$ in $\mathbb{R}^k$ that satisfy $a_i \leq x_i \leq b_i$ is called a k-**cell**.