Math 104 - Midterm 2 Definitions

Definition 3.1. A sequence $\{p_n\}$ in a metric space (X, d) is said to **converge** if there is a point $p \in X$ with the following property: for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then $d(p_n, p) < \epsilon$. We have the following three ways of phrasing the convergence of $\{p_n\}$ to p;

1. $\{p_n\}$ converges to p.
2. $p_n \to p$.
3. $\lim_{n\to\infty} p_n = p$.

If $\{p_n\}$ does not converge, we say that it diverges.

Definition 3.5. Let $\{p_n\}$ be a sequence. Consider the sequence of positive integers $\{n_i\}$ where $n_1 < n_2 < n_3 < \dots$. Then, $\{p_{n_i}\}$ is a **subsequence** of $\{p_n\}$. If $\{p_{n_i}\}$ converges to p, then p is a **subsequential limit** of $\{p_n\}$.

Definition 3.8. Let $\{p_n\}$ be a sequence in some metric space (X, d). We say that $\{p_n\}$ is a **Cauchy sequence** if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n, m \geq N$, then $d(p_n, p_m) < \epsilon$.

Definition 3.12. A metric space in which every Cauchy sequence converges is said to be **complete**.

Definition 3.13. Let $\{p_n\}$ be a sequence of real numbers. Then, we define the following:

1. $\{p_n\}$ is **monotonically increasing** if $p_n \leq p_{n+1} (p = 1, 2, 3, \dots)$.
2. $\{p_n\}$ is **monotonically decreasing** if $p_n \geq p_{n+1} (p = 1, 2, 3, \dots)$.

Definition 3.16. Let $\{p_n\}$ be a sequence of real numbers. Let E be the set of all subsequential limits of $\{p_n\}$. E possibly includes ∞ or $-\infty$. Let $s^\star = \sup E$ and $s_\star = \inf E$. Then, we define $s^\star$ to be the **upper limit** of $\{p_n\}$ and $s_\star$ to be the **lower limit** of $\{p_n\}$. We also have that $\lim_{n\to\infty} \sup p_n = s^\star$ and $\lim_{n\to\infty} \inf p_n = s_\star$.

Definition 3.21. Let $\{a_i\}_{i=1}^{\infty}$ be a sequence. We have that $\sum_{i=k}^{q} a_i = a_k + a_{k+1} + \cdots + a_q$. Let $s_n = \sum_{n}^{i=1} a_i$ to be the n^{th} **partial sum**.

Definition 4.1. Let (X, d_x) and (Y, d_y) be metric spaces, $E \subset X$, $f : E \to Y$, and p is a limit point of E. Then, $\lim_{x\to p} f(x) = q$ if there is a $q \in Y$ such that for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in E$, if $0 < d_x(x, p) < \delta$, then $d_y(f(x), q) < \epsilon$.

Definition 4.5. Let (X, d_x) and (Y, d_y) be metric spaces, $E \subset X$, $f : E \to X$, and $p \in E$. We say that f is **continuous** at p if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in E$ if $d_x(x, p) < \delta$ then $d_y(f(x), f(p)) < \epsilon$.

Definition 4.18. Let $f : X \to Y$, where X and Y are metric spaces. We say that f is **uniformly continuous** on X if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $p, q \in X$ if $d_x(p, q) < \delta$, then $d_y(f(p), f(q)) < \epsilon$.

Definition 5.1. Let $f : [a, b] \to \mathbb{R}$. f is **differentiable** at $x \in [a, b]$ if the following limit exists:

$$f'(x) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x}$$

where f' is the first derivative of f. Also, $t \in [a, b]$ and $t \neq x$.

Definition of Sequentially Compact. Let X be a set. We say that X is **sequentially compact** if any sequence in X has a subsequence whose limit is in X.

Definition 5.7. Let $f : X \to \mathbb{R}$. f has a **local maximum** at a point $p \in X$ if there exists a $\delta > 0$ such that $f(q) \leq f(p)$ for any $q \in X$ with $d(q, p) < \delta$. Likewise, f has a **local minimum** at a point $p \in X$ if there exists a $\delta > 0$ such that $f(q) \geq f(p)$ for any $q \in X$ with $d(q, p) < \delta$.