# Math 104 Lecture Notes

Chapter 4 - Continuity

July 10, 2024

## Notes

In previous lectures, we discussed the $\epsilon - \mathbb{N}$ definition for convergence of a sequence $\{x_n\}$, namely $\lim_{x\to\infty} x_n = a$ for some $a \in \mathbb{R}$. We would like the general approach for finding limits of the form $\lim_{x\to p} f(x) = q$ $(= f(p))$ for some finite values $p$ and $q$. First, we define a function $f : X \to Y$, where $(X, d_x)$ and $(Y, d_y)$ are metric spaces. We consider a mapping of $X$ into $Y$; in particular, we observe the behavior of $f$ as $x$ tends to $p$.

**Definition 4.1.** Let $(X, d_x)$ and $(Y, d_y)$ be metric spaces, $E \subset X$, $f : X \to Y$, and $p$ is a limit point of $E$. The statement "$f(x) \to q$ as $x \to p$" is written as $\lim_{x\to p} f(x) = q$. We say that $\lim_{x\to p} f(x) = q$ if there is a point $q \in Y$ such that for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in E$, if $0 < d_x(x, p) < \delta$, then $d_y(f(x), q) < \epsilon$.

Note that a function may not be defined at a certain point but the limit of the function may exist at that point. Consider the following example:

$$f(x) = \frac{x(x+1)}{x+1}, \quad f(-1) \text{ DNE}, \quad \lim_{x\to -1} f(x) = -1$$

If we take the inequality $0 < d_x(x, p) < \delta$, we don't look at what $f$ does at $p$, just what $f$ does near $p$.

**Theorem 4.2.** *Let $X, Y, E, f, p$ be as from Definition 4.1. $\lim_{x\to p} f(x)$ if and only if for every sequence $\{p_n\}$ in $E$ such that $p_n \neq p$ and $\lim_{n\to\infty} p_n = p$, we have that $\lim_{n\to\infty} f(p_n) = q$.*

*Proof.* ($\rightarrow$): Suppose $\lim_{x\to p} f(x) = q$. Let $\{p_n\}$ in $E$ satisfy $p_n \neq p$ and $p_n \to p$. Let $\epsilon > 0$ be given. We know there exists a $\delta > 0$ such that if $x \in E$ and $0 < d_x(x,p) < \delta$, then $d_y(f(x), q) < \epsilon$. Since $p_n \to p$, we know there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then $d(p_n, p) < \delta$. So for this $N$, if $n \geq N$, then $d(f(p_n), q) < \epsilon$.
($\leftarrow$): Consider the contrapositive of ($\rightarrow$). Suppose $\lim_{x\to p} f(x) \neq q$, then there exists an $\epsilon > 0$ such that for all $\delta > 0$, there exists an $x \in E$ for which $d_y(f(x), q) \geq \epsilon$ but $0 < d_x(x,p) < \delta$. Let $\delta_n = \frac{1}{n}$ $(n = 1, 2, 3, \dots)$ and choose $p_n$ such that $0 < d(p_n, p) < \frac{1}{n} = \delta_n$. However, $d(f(p_n), p) \geq \epsilon$. Then, $p_n \to p$ but $f(p_n) \nrightarrow q$, since all images of $p_n$ are at least $\epsilon$ away from $q$. $\square$

Corollary: if $f$ has a limit at $p$, then the limit is unique.

**Theorem 4.4.** *Suppose $(X, d)$ is a metric space, $E \subset X$, $p$ is a limit point of $E$, and both $f$ and $g$ send $E \to \mathbb{R}$. Additionally, $\lim_{x\to p} f(x) = A$ and $\lim_{x\to p} g(x) = B$. Then,*

1. $\lim_{x\to p}(f+g)(x) = A + B$
2. $\lim_{x\to p}(fg)(x) = AB$
3. $\lim_{x\to p}\left(\frac{f}{g}\right)(x) = \frac{A}{B}$ *if* $B \neq 0$

*Proof.* (Comes from Theorem 3.3 and Theorem 4.2) $\square$

# Practice Problems

1. Suppose $\Sigma a_n$ converges and $a_n \geq 0$ for all $n$. Show that $\Sigma a_n^2$, $\Sigma\sqrt{a_{n+1}\cdot a_n}$, and $\Sigma\frac{\sqrt{a_n}}{n}$ all converge. (Hint: $(a-b)^2 \geq 0$ for all $a, b \in \mathbb{R}$)
2. Suppose we have the following function:

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

   Let $a \in \mathbb{R}$. Show that $\lim_{x\to a} f(x)$ does not exist.