Weekend Problem

Define the floor function, $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \lfloor x \rfloor$.

(a) Let $a \notin \mathbb{Z}$. Use the $\delta - \epsilon$ definition to show that f is continuous at a.
Proof: Let $\epsilon > 0$ be given. Then, we have the ϵ-ball $N_\epsilon(f(a)) \subset \mathbb{R}$. Since $a \notin \mathbb{Z}$, $a \in \mathbb{R} \setminus \mathbb{Z}$. Let $g = a - \lfloor a \rfloor$ represent the non-integer component of a; innately, $0 < g < 1$ as $a \notin \mathbb{Z}$. If $g < \frac{1}{2}$, then choose $\delta = \frac{g}{2}$ and so $f\left(N_\delta(a)\right) \subset N_\epsilon(f(a))$, which is obvious since every $x \in (N_\delta(a))$ has the function value $f(a)$. If $g = \frac{1}{2}$, then choose $\delta = \frac{1}{4}$, and so every $x \in \left(a - \frac{1}{4}, a + \frac{1}{4}\right)$ has a function value $f(a) \in (f(a) - \epsilon, f(a) + \epsilon)$. If $g > \frac{1}{2}$, choose $\delta = \frac{1-g}{2}$ and so every $x \in (a - \delta, a + \delta)$ has a function value $f(a) \in (f(a) - \epsilon, f(a) + \epsilon)$. We have shown, for each case of g, a $\delta > 0$ exists, so thus, f is continuous at all $a \notin \mathbb{Z}$. $\square$

(b) Let $a \in \mathbb{Z}$. Use the $\delta - \epsilon$ definition to show that f is not continuous at a.
Proof: Suppose for contradiction that f is continuous for an integer a. Then, by definition, for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in E$ (where $E \subset \mathbb{R}$) if $d_x(x, a) < \delta$, then $d_y(f(x), f(a)) < \epsilon$. Let an $\epsilon > 0$ that is sufficiently small be given. Then, we must show that there is a $\delta > 0$ such that for all $x \in E = N_\epsilon(a)$, $f(E) \subset N_\epsilon(f(a))$. By definition, $N_\delta(a) = (a - \delta, a + \delta)$ and $N_\epsilon(f(a)) = (a - \epsilon, a + \epsilon)$; the latter is due to definition that $a = f(a)$ for integers a. So, all $x \in (a - \delta, a + \delta)$ have their function values in $(a - \delta, a + \delta)$. Since f returns exclusively integers, any element $x \in (a - \delta, a + \delta)$ has a function value of a, since we choose ϵ to be sufficiently small. Let $x_0 = \frac{(a-\delta)+a}{2} \in (a - \delta, a + \delta)$. Since $x_0 < a$, thus by definition of f, $f\left(x_0\right) \leq a - 1$ as any $x \geq a$ has a function value at least a. Therefore, we have the contradiction that $f\left(x_0\right) = a$ while also $f\left(x_0\right) < a$, so f is not continuous at any integer a. $\square$