Math 104 - Tail End Definitions

Definition 6.1. Let $[a, b]$ be the given interval. A partition P on $[a, b]$ is a set of points $x_0, \ldots, x_n$ with $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$. Let $\Delta x_i = x_i - x_{i-1}$ and f be a bounded function on $[a, b]$. Then, let $M_i = \sup f(x)$ on $[x_{i-1}, x_i]$ and $m_i = \inf f(x)$ on $[x_{i-1}, x_i]$. Denote $U(P, f) = \sum_{i=1}^{n} M_i \cdot \Delta x_i$ and $L(P, f) = \sum_{i=1}^{n} m_i \cdot \Delta x_i$. Then, write $\underline{\int_a^b} f dx = \sup L(P, f)$ (which is the lower Riemann integral) and $\overline{\int_a^b} f dx = \inf U(P, f)$ (which is the upper Riemann integral). Note that the inf and sup are taken over all possible partitions of $[a, b]$. Then, if $\underline{\int_a^b} f dx = \overline{\int_a^b} f dx$, then, we say that f is Riemann integrable on $[a, b]$ and write $f \in \mathscr{R}$. Denote the common value as $\int_a^b f dx$ (which is the Riemann integral).

Definition 6.2. Let α be a monotonically increasing function on an interval $[a, b]$. Write $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. Note that $\Delta\alpha_i \geq 0$ since α is monotonically increasing. Let f be a bounded real function on $[a, b]$. Then, denote $U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$ and $L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$. Then, write $\underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha)$ and $\overline{\int_a^b} f d\alpha = \inf U(P, f, \alpha)$. If $\underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha$, denote the common value as $\int_a^b f d\alpha$ (which is the Riemann-Stieltjes Integral of f on $[a, b]$).

Definition 6.3. $P^\star$ is a refinement of a partition P if $P^\star \supset P$. P is a common refinement of P_1 and P_2 if $P = P_1 \cup P_2$.

Definition 7.1. Suppose $\{f_n\}$ is a series of functions defined on a set E. Then, also suppose that $\{f_n(x)\}$ is a sequence of numbers that converges for every $x \in E$. Then, we define a function f:

$$f(x) = \lim_{n \to \infty} f_n(x)$$

$\{f_n\}$ converges on E and f is the limit of $\{f_n\}$. $\{f_n\}$ converges pointwise to f for every $x \in E$. If $\sum f_n(x)$ converges, and if we define $f(x) = \sum_{n=1}^{\infty} f_n(x)$, then f is the sum of the series $\sum f_n$.

Definition of Norm. Let f be a bounded function with $f : E \to \mathbb{R}$. Then, define the norm of f to be

$$||f|| = \sup_{x \in E} |f(x)|.$$

Definition 7.7. $\{f_n\}$ converges pointwise on E if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ $||f - f_n|| < \epsilon$ (for all $x \in E$).