Midterm 2 Attempt.

1. Problem 0.

   (a) A zero divisor in a commutative ring T with 1_T is a nonzero element $a \in T$ such that there exists a nonzero $b \in T$ such that $ab = 0$. A unit in a commutative ring T with 1_T is an element $a \in T$ such that there exists a $b \in T$ such that $ab = 1_T$.

   (b) Consider the ring $R = \mathbb{Z}$ and the ideal $5\mathbb{Z}$. The ideal $5\mathbb{Z}$ is a proper ideal of $\mathbb{Z}$, and $4 \notin 5\mathbb{Z}$ with $4 \in \mathbb{Z}$, and $5 \in 5\mathbb{Z}$ with $5 \notin \{0\}$.

2. Problem 1.

   (a) We first find $\text{orb}(A) = \{\sigma(A) \mid \sigma \in S_4\}$. Our set is $X = \{A, B, C\}$. By definition of how S_4 acts on X, we have that $(12) \circ A = \{\{2,1\},\{3,4\}\} = \{\{1,2\},\{3,4\}\} = A$, so $A \in \text{orb}(A)$. We have $(23) \circ A = \{\{1,3\},\{2,4\}\} = B \in \text{orb}(A)$ and $(123) \circ A = \{\{2,3\},\{1,4\}\} = C$, and so $\text{orb}(A) = X$. Now, find $\text{Stab}(A) = \{\sigma \in S_4 \mid \sigma(A) = A\}$. Clearly, id fixes A, and also (12), (34), and $(12)(34)$ fix A, as A is a set of sets. Additionally, we may also flip the order of the subsets while preserving the elements in each subset. We see that the permutations $(13)(24), (1423), (1324), (14)(23)$ do this and only these permutations do this. Thus, $\text{Stab}(A)$ is the union of these permutations with the Klein 4-group.

   (b) We see that $(12) \circ A = A$, and (12) sends B to C and C to B; thus, $\phi((12)) = (A)(BC)$. $(12)(34) \circ A = A$, $(12)(34) \circ B = B$, and since $(12)(34)$ is a permutation, we must have $(12)(34) \circ C = C$, and so $\phi((12)(34)) = (A)(B)(C) = \text{id}$. $(123) \circ A = C$, $(123) \circ B = A$, $(123) \circ C = B$, so thus $\phi((123)) = (ACB)$. $(1234) \circ A = C$, $(1234) \circ B = B$, $(1234) \circ C = A$, so $\phi((1234)) = (AC)(B)$.

   (c) $\ker \phi = \{\sigma \in S_4 \mid \phi(\sigma) = \text{id}\}$. We see that, trivially, id sends each $x \in X$ to itself, so $\text{id} \in \ker \phi$. Also, $(12)(34), (13)(24), (14)(23)$, send each $x \in X$ to itself, so they are also included in $\ker \phi$. It can also be verified that any for any $\sigma \in S_4$ other than these, there exists an $x \in X$ such that $\sigma(x) \neq x$, and so these σ cannot be in $\ker \phi$.

   (d) We now use the first isomorphism theorem of groups and Lagrange's theorem to solve this, since $G = S_4$ is a finite group. Since the first isomorphism theorem gives us that $S_4/\ker \phi \cong \text{Im}\phi$, thus, by Lagrange's theorem, we get that $|\text{Im}\phi| = \frac{|S_4|}{|\ker \phi|} = \frac{24}{4} = 6$.

3. Problem 2.

   (a) (check solutions).

   (b) Distributivity follows from high school algebra (as showed on my exam copy). We now find the multiplicative identity in $(\mathbb{Z}, \star, \circ)$. Fix $a \in (\mathbb{Z}, \star, \circ)$. Find x such that $a \circ x = a$, which is equivalent to finding x such that $ax + 4a + 4x + 12 = a$. We have $ax + 4a + 4x + 12 = a$ gives $ax + 3a = -4x - 12$, giving $a(x+3) = -4(x+3)$, giving $x = -3$ to not contradict our required condition. Thus, $x = -3$ is the multiplicative identity.

4. Problem 3.

   (a) $\ker(ev)$ consists precisely of those polynomials of the form $p(x) = a^2+bx+c$ (with $a, b, c \in \mathbb{R}$) such that $p(5) = 0$, namely 5 is a root of p, which is to say we may write $p(x) = (x-5)g(x)$, where $g(x) \in \mathbb{R}[x]$, with $\deg g = 1$.

   (b) We now show that $I[x]$ is an ideal of $R[x]$. First, we show $I[x]$ is a abelian subgroup of $R[x]$. We see $0 \in I \subseteq R$, as I is ideal of R. Then, we must have that for any $a_nx^n + \cdots + a_0 \in I[x]$, $a_nx^n + \cdots + a_0 = (a_nx^n + \ldots a_0) + 0 = (a_n+0)x^n + \cdots + (a_0+0) = (0+a_n)x^n + \cdots + (0+a_0) = 0 + (a_nx^n + \cdots + a_0)$, and so 0 is the identity in $I[x]$. Associativity of $I[x]$ holds as $I[x] \subseteq R[x]$. We also that if $a_nx^n + \ldots a_0 \in I[x]$, then $(-a_n)x^n + \cdots + (-a_0) \in I[x]$ (as $a_i, -a_i \in I$, as I is an ideal of R) such that it commutes with $a_nx^n + \cdots + a_0$ to give 0 in both cases. Take now $(a_n)x^n + \cdots + a_0, b_mx^m + \cdots + b_0 \in I[x]$. Then, their product is a polynomial of degree $n+m$ with coeffients in I, by definition of multiplying polynomials, and thus their product is in $I[x]$. Also, $I[x]$ is abelian, since polynomial addition is commutative, as seen in class. Now take $a(x) = a_nx^n + \ldots a_0 \in I[x]$ and $b(x) = b_mx^m + \ldots b_0 \in R[x]$. The product $(ab)(x)$ has degree $n+m$. Put $c(x) = (ab)(x) = c_{m+n}x^{m+n} + \cdots + c_0$. By definition of polynomial multiplication, we have that any coefficient c_i of c is a sum of products, where each product consists of an element from I and an element from R. Since I is an ideal in R, thus each of the products (which are summands) lies in I, and since I is an abelian subgroup of R under addition, thus the sum of the products lies in I, and so $c_i \in I$ for all $i \in \{0, \ldots, m+n\}$. Thus, $c(x) \in I[x]$, and so $I[x]$ is an ideal of $R[x]$.