

Math 113 Definitions.

1. **Set.** A set is an unordered collection of elements.
2. **Map.** A map from  $X$  to  $Y$  is  $f : X \rightarrow Y$  (a rule that assigns elements to  $Y$  to elements in  $X$ ). So, for any  $x \in X$  there exists a unique  $y \in Y$  such that  $f(x) = y$ .
3. **Cartesian Product.** The Cartesian product of  $X$  and  $Y$  is the set  $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ .
4. **Equivalence Relation.** An equivalence relation  $R, \sim$  on  $X$  is a subset  $R \subseteq X \times X$  such that
  - (a) Reflexive.  $((x, x) \in R \text{ for all } x \in X)$ .
  - (b) Symmetric.  $(\text{if } (x, y) \in R, \text{ then } (y, x) \in R)$ .
  - (c) Transitive.  $(\text{if } (x, y) \in R \text{ and } (y, z) \in R, \text{ then } (x, z) \in R)$ .
5. **Equivalence Class.** Let  $X$  be a set and  $R$  be an equivalence relation on  $X$ . Then, an equivalence class of  $x \in X$  is the set  $[x] = [x]_R = [x]_\sim = \{a \in X \mid x \sim a\}$ .
6.  $\mathbb{Z}/m\mathbb{Z}$ . The set of distinct equivalence classes of  $\equiv \pmod{n}$  is  $\mathbb{Z}/m\mathbb{Z}$ .
7. **Group.** A group  $G$  (denote:  $(G, \star)$ ) is a set  $G$  with a closed binary operation  $\star : G \times G \rightarrow G$  such that:
  - (a) Associativity:  $(a \star b) \star c = a \star (b \star c)$  for all  $a, b, c \in G$ .
  - (b) Identity: There exists an  $e \in G$  such that for any  $a \in G$ , we have  $a \star e = e \star a = a$ .
  - (c) Inverse: For any  $a \in G$ , there exists an  $a^{-1} \in G$  such that  $a \star a^{-1} = a^{-1} \star a = e$ .
8. **Symmetric Group.** The symmetric group on  $n$  letters is  $S_n$ .
9. **Disjoint Cycles.** Two cycles  $(a_1, \dots, a_k)$  and  $(b_1, \dots, b_l)$  are disjoint if  $a_i \neq b_j$  for all  $i, j$ .
10. **Transpositions.** The simplest permutation is a cycle of length 2, which is called a transposition.
11. **Even, Odd Permutations.** A permutation is even if it can be expressed as an even number of transpositions. A permutation is odd if it can be expressed as an odd number of transpositions.
12. **Subgroup.** A subgroup  $H$  of a group  $G$  is a subset  $H$  of  $G$  such that when the group operation of  $G$  is restricted to  $H$ , then  $H$  is a group.
13. **Trivial/Proper Subgroup.** The trivial subgroup of a group  $G$  is  $\{e\}$  and a proper subgroup is a subgroup  $H$  of  $G$  where  $H$  is a proper subset of  $G$ .

14. **General/Special Linear Group.**  $GL_2(\mathbb{R})$  is the set of 2x2 invertible matrices with real entries.  $SL_2(\mathbb{R})$  is the set of 2x2 invertible matrices with real entries and with determinant 1.
  15. **Cyclic Group.** A cyclic group is a group generated by one element.
  16. **Isomorphism.** An isomorphism is a homomorphism which is bijective.
  17. **Kernel of homomorphism.** If  $\phi : G \rightarrow H$  is a homomorphism, then  $\ker \phi$  is the pre-image of  $e_H \in H$ , that is,  $\ker \phi = \{g \in G \mid \phi(g) = e_H\}$ .
  18. **Coset.** Let  $(G, \star) \geq (H, \star)$  and  $g \in G$ . Then, an  $H$ -coset of  $g$  is a (sub)set of  $G$  where  $gH = g \star H = \{g \star h \mid h \in H\}$  (left coset) and  $Hg = \{h \star g \mid h \in H\}$  (right coset).
  19. **Index.** A set of distinct equivalence classes with respect to  $\sim_H$  is  $G/H$ , a quotient of  $G$  by  $H$ . Then,  $|G/H| = [G : H]$  is the index of  $H$  in  $G$ .
  20. The following are definitions listed in the homeworks:
    - (a) Group of units in  $\mathbb{Z}/n\mathbb{Z}$  is the set  $(\mathbb{Z}/n\mathbb{Z})^\times = \mathbb{Z}/n\mathbb{Z}^\times := \{[a] \in \mathbb{Z}/n\mathbb{Z} \mid \exists [b] \in \mathbb{Z}/n\mathbb{Z} \text{ with } [a] \times [b] = [1]\}$ .
    - (b) If  $G$  is a group, then the center of  $G$  is the set  $Z(G) := \{a \in G \mid ga = ag \forall g \in G\}$ .
    - (c)  $\mathbb{C}^\times$  is the set of nonzero complex numbers.
    - (d)  $\mathbb{R}^\times$  is the set of nonzero real numbers.
    - (e)  $GL(n, K)$  is the set of  $n \times n$  invertible matrices with entries in  $K$ .
    - (f) If  $G$  is a group, then the torsion subgroup of  $G$  is called  $G_T$ , which is the set of all elements of  $G$  with finite order.
    - (g) The Klein four-group is  $V$  is a subgroup of  $S_4$  and consists of  $V = \{\text{id}, (12), (34), (12)(34)\}$ .
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21. **(External) Direct Product** Let  $G = (G, \star)$  and  $H = (H, \circ)$  be groups. Then,  $G \times H = \{G \times H, (\star, \circ)\}$ .
22. **Normal Subgroup.** Let  $G$  be a group and  $H$  a subgroup of  $G$ . Then  $H$  is a normal subgroup (write  $H \trianglelefteq G$ ) iff for all  $g \in G$ ,  $gH = Hg$ , or equivalently, for all  $h \in H$ ,  $ghg^{-1} \in H$  for all  $g \in G$ .
23. **Quotient (factor) group.** The quotient group of a group  $G$  and a normal subgroup  $N$  of  $G$  is the group  $G/N$  (where  $G/N$  is the group of cosets of  $N$  in  $G$ ) under the operation  $(aN)(bN) = abN$ .
24. **Internal Direct Product.** Let  $G$  be a group and  $H, K \leq G$ .  $G$  is an internal direct product of  $H$  and  $K$  iff:
  - (a)  $G = H \cdot K := \{h \cdot k \mid h \in H, k \in K\}$ .

- (b)  $H \cap K = \{e_G\}$  (“as small as possible”).
  - (c)  $h \cdot k = k \cdot h$  for all  $h \in H, k \in K$ .
25. **Simple group.** A group  $G$  is simple if the only normal subgroups are  $\{e_G\}$  and  $G$ .
26. **Symmetry.** A symmetry of  $X$  is a bijective map  $\sigma : X \rightarrow X$  preserving the structure where  $X$  is some set with some additional structure.
27. **Group of permutations on a set  $X$ .**  $G$  is a group of permutations on a set  $X$  if  $\phi : G \rightarrow \text{Sym}(X) = S_X = S_{|X|}$  is a homomorphism that is 1-1.
28.  **$G$  acts on a set  $X$ .**  $G$  acts on a set  $X$  is a homomorphism  $\phi : G \rightarrow \text{Sym}(X)$ .
29. **Stabilizer of  $x \in X$ .** Let  $G$  be a group and  $X$  a set. Then, the stabilizer of  $x \in X$  is  $\text{Stab}_G(x) = \{g \in G \mid g(x) = x\}$ , which are elements of  $G$  that preserve  $x \in X$ .
30. **Orbit of  $x \in X$ .** Take  $x \in X$ . Then the orbit of  $x$  is  $\text{orb}_G(x) = \mathcal{O}_G(x) = \mathcal{O}(x) = \{g(x) \mid g \in G\} \subseteq X$ .
31.  **$G$  acts on a set  $X$  (equivalent) def.** A group  $G$  acts on a set  $X$  iff  $\Phi : G \times X \rightarrow X$  with  $(g, x) \mapsto \Phi(g, x) = g \circ x$  such that:
- (a) for all  $x \in X$ ,  $\Phi(e_G, x) = x$ .
  - (b) for all  $x \in X$ ,  $g, h \in G$ ,  $\Phi(gh, x) = \Phi(g, \Phi(h, x))$ .
32. **Left regular action of  $G$  on  $G$ .** Define  $\Lambda : G \times G \rightarrow G$  be a group action, where  $G$  is a group (and a set) such that  $(g, h) \mapsto g \circ h$ . Equivalently, the left regular action of  $G$  on  $G$  is defined as the homomorphism  $\lambda : G \rightarrow \text{Sym}(G)$  such that  $g \mapsto (\lambda_g : h \mapsto \lambda_g(h) = g \circ h)$ , where  $\lambda_g$  is a permutation on the set  $G$  for  $g \in G$ .