

## Math 113 Definitions.

1. **Set.** A set is an unordered collection of elements.
2. **Map.** A map from  $X$  to  $Y$  is  $f : X \rightarrow Y$  (a rule that assigns elements to  $Y$  to elements in  $X$ ). So, for any  $x \in X$  there exists a unique  $y \in Y$  such that  $f(x) = y$ .
3. **Cartesian Product.** The Cartesian product of  $X$  and  $Y$  is the set  $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ .
4. **Equivalence Relation.** An equivalence relation  $R, \sim$  on  $X$  is a subset  $R \subseteq X \times X$  such that
  - (a) Reflexive.  $((x, x) \in R \text{ for all } x \in X)$ .
  - (b) Symmetric.  $(\text{if } (x, y) \in R, \text{ then } (y, x) \in R)$ .
  - (c) Transitive.  $(\text{if } (x, y) \in R \text{ and } (y, z) \in R, \text{ then } (x, z) \in R)$ .
5. **Equivalence Class.** Let  $X$  be a set and  $R$  be an equivalence relation on  $X$ . Then, an equivalence class of  $x \in X$  is the set  $[x] = [x]_R = [x]_{\sim} = \{a \in X \mid x \sim a\}$ .
6.  $\mathbb{Z}/m\mathbb{Z}$ . The set of distinct equivalence classes of  $\equiv \pmod{n}$  is  $\mathbb{Z}/m\mathbb{Z}$ .
7. **Group.** A group  $G$  (denote:  $(G, \star)$ ) is a set  $G$  with a closed binary operation  $\star : G \times G \rightarrow G$  such that:
  - (a) Associativity:  $(a \star b) \star c = a \star (b \star c)$  for all  $a, b, c \in G$ .
  - (b) Identity: There exists an  $e \in G$  such that for any  $a \in G$ , we have  $a \star e = e \star a = a$ .
  - (c) Inverse: For any  $a \in G$ , there exists an  $a^{-1} \in G$  such that  $a \star a^{-1} = a^{-1} \star a = e$ .
8. **Symmetric Group.** The symmetric group on  $n$  letters is  $S_n$ .
9. **Disjoint Cycles.** Two cycles  $(a_1, \dots, a_k)$  and  $(b_1, \dots, b_l)$  are disjoint if  $a_i \neq b_j$  for all  $i, j$ .
10. **Transpositions.** The simplest permutation is a cycle of length 2, which is called a transposition.

11. **Even, Odd Permutations.** A permutation is even if it can be expressed as an even number of transpositions. A permutation is odd if it can be expressed as an odd number of transpositions.
  12. **Subgroup.** A subgroup  $H$  of a group  $G$  is a subset  $H$  of  $G$  such that when the group operation of  $G$  is restricted to  $H$ , then  $H$  is a group.
  13. **Trivial/Proper Subgroup.** The trivial subgroup of a group  $G$  is  $\{e\}$  and a proper subgroup is a subgroup  $H$  of  $G$  where  $H$  is a proper subset of  $G$ .
  14. **General/Special Linear Group.**  $GL_2(\mathbb{R})$  is the set of  $2 \times 2$  invertible matrices with real entries.  $SL_2(\mathbb{R})$  is the set of  $2 \times 2$  invertible matrices with real entries and with determinant 1.
  15. **Cyclic Group.** A cyclic group is a group generated by one element.
  16. **Isomorphism.** An isomorphism is a homomorphism which is bijective.
  17. **Kernel of homomorphism.** If  $\phi : G \rightarrow H$  is a homomorphism, then  $\ker \phi$  is the pre-image of  $e_H \in H$ , that is,  $\ker \phi = \{g \in G \mid \phi(g) = e_H\}$ .
  18. **Coset.** Let  $(G, \star) \geq (H, \star)$  and  $g \in G$ . Then, an  $H$ -coset of  $g$  is a (sub)set of  $G$  where  $gH = g \star H = \{g \star h \mid h \in H\}$  (left coset) and  $Hg = \{h \star g \mid h \in H\}$  (right coset).
  19. **Index.** A set of distinct equivalence classes with respect to  $\sim_H$  is  $G/H$ , a quotient of  $G$  by  $H$ . Then,  $|G/H| = [G : H]$  is the index of  $H$  in  $G$ .
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20. **(External) Direct Product** Let  $G = (G, \star)$  and  $H = (H, \circ)$  be groups. Then,  $G \times H = \{G \times H, (\star, \circ)\}$ .