Math 113 Definitions.

1. **Set.** A set is an unordered collection of elements.
2. **Map.** A map from X to Y is $f : X \to Y$ (a rule that assigns elements to Y to elements in X). So, for any $x \in X$ there exists a unique $y \in Y$ such that $f(x) = y$.
3. **Cartesian Product.** The Cartesian product of X and Y is the set $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$.
4. **Equivalence Relation.** An equivalence relation $R, \sim$ on X is a subset $R \subseteq X \times X$ such that
   (a) Reflexive. ($(x, x) \in R$ for all $x \in X$).
   (b) Symmetric. (if $(x, y) \in R$, then $(y, x) \in R$).
   (c) Transitive. (if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$).
5. **Equivalence Class.** Let X be a set and R be an equivalence relation on X. Then, an equivalence class of $x \in X$ is the set $[x] = [x]_R = [x]_\sim = \{a \in X \mid x \sim a\}$.
6. $\mathbb{Z}/m\mathbb{Z}$. The set of distinct equivalence classes of $\equiv \pmod{n}$ is $\mathbb{Z}/m\mathbb{Z}$.
7. **Group.** A group G (denote: $(G, \star)$) is a set G with a closed binary operation $\star : G \times G \to G$ such that:
   (a) Associativity: $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in G$.
   (b) Identity: There exists an $e \in G$ such that for any $a \in G$, we have $a \star e = e \star a = a$.
   (c) Inverse: For any $a \in G$, there exists an $a^{-1} \in G$ such that $a \star a^{-1} = a^{-1} \star a = e$.
8. **Symmetric Group.** The symmetric group on n letters is S_n.
9. **Disjoint Cycles.** Two cycles $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_l)$ are disjoint if $a_i \neq b_j$ for all i, j.
10. **Transpositions.** The simplest permutation is a cycle of length 2, which is called a transposition.
11. **Even, Odd Permuatations.** A permutation is even if it can be expressed as an even number of transpositions. A permutation is odd if it can be expressed as an odd number of transpositions.
12. **Subgroup.** A subgroup H of a group G is a subset H of G such that when the group operation of G is restricted to H, then H is a group.
13. **Trivial/Proper Subgroup.** The trivial subgroup of a group G is $\{e\}$ and a proper subgroup is a subgroup H of G where H is a proper subset of G.

14. **General/Special Linear Group.** $GL_2(\mathbb{R})$ is the set of 2x2 invertible matrices with real entries. $SL_2(\mathbb{R})$ is the set of 2x2 invertible matrices with real entries and with determinant 1.

15. **Cyclic Group.** A cyclic group is a group generated by one element.

16. **Isomorphism.** An isomorphism is a homomorphism which is bijective.

17. **Kernel of homomorphism.** If $\phi : G \to H$ is a homomorphism, then $\ker\phi$ is the pre-image of $e_H \in H$, that is, $\ker\phi\{g \in G \mid \phi(g) = e_H\}$.

18. **Coset.** Let $(G, \star) \geq (H, \star)$ and $g \in G$. Then, an $H-$ coset of g is a (sub)set of G where $gH = g \star H = \{g \star h \mid h \in H\}$ (left coset) and $Hg = \{h \star g \mid h \in H\}$ (right coset).

19. **Index.** A set of distinct equivalence classes with respect to $_H$ is G/H, a quotient of G by H. Then, $|G/H| = [G : H]$ is the index of H in G.

20. The following are definitions listed in the homeworks:

    (a) Group of units in $\mathbb{Z}/n\mathbb{Z}$ is the set $(\mathbb{Z}/n\mathbb{Z})^\times = \mathbb{Z}/n\mathbb{Z}^\times := \{[a] \in \mathbb{Z} \mid \exists [b] \in \mathbb{Z}/n\mathbb{Z}$ with $[a] \times [b] = [1]\}$.

    (b) If G is a group, then the center of G is the set $Z(G) := \{a \in G \mid ga = ag \forall g \in G\}$.

    (c) $\mathbb{C}^\times$ is the set of nonzero complex numbers.

    (d) $\mathbb{R}^\times$ is the set of nonzero real numbers.

    (e) $GL(n, K)$ is the set of n x n invertible matrices with entries in K.

    (f) If G is a group, then the torsion subgroup of G is called G_T, which is the set of all elements of G with finite order.

    (g) The Klein four-group is V is a subgroup of S_4 and consists of $V = \{\mathrm{id}, (12), (34), (12)(34)\}$.

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21. **(External) Direct Product** Let $G = (G, \star)$ and $H = (H, \circ)$ be groups. Then, $G \times H = \{G \times H, (\star, \circ)\}$.

22. **Normal Subgroup.** Let G be a group and H a subgroup of G. Then H is a normal subgroup (write $H \trianglelefteq G$) iff for all $g \in G$, $gH = Hg$, or equivalently, for all $h \in H$, $ghg^{-1} \in H$ for all $g \in G$.

23. **Quotient (factor) group.** The quotient group of a group G and a normal subgroup N of G is the group G/N (where G/N is the group of cosets of N in G) under the operation $(aN)(bN) = abN$.

24. **Internal Direct Product.** Let G be a group and $H, K \leq G$. G is an internal direct product of H and K iff:

    (a) $G = H \cdot K := \{h \cdot k \mid h \in H, k \in K\}$.

(b) $H \cap K = \{e_G\}$ ("as small as possible").
(c) $h \cdot k = k \cdot h$ for all $h \in H, k \in K$.

25. **Simple group.** A group G is simple if the only normal subgroups are $\{e_G\}$ are G.

26. **Symmetry.** A symmetry of X is a bijective map $\sigma : X \to X$ preserving the structure where X is some set with some additional structure.

27. **Group of permutations on a set X.** G is a group of permutations on a set X if $\phi : G \to \mathrm{Sym}(X) = S_X = S_{|X|}$ is a homomorphism that is 1-1.

28. G **acts on a set X.** G acts on a set X is a homomorphism $\phi : G \to \mathrm{Sym}(X)$.

29. **Stabilizer of $x \in X$.** Let G be a group and X a set. Then, the stabilizer of $x \in X$ is $\mathrm{Stab}_G(x) = \{g \in G \mid g(x) = x\}$, which are elements of g that preserve $x \in X$.

30. **Orbit of $x \in X$.** Take $x \in X$. Then the orbit of X is $\mathrm{orb}_G(x) = \mathcal{O}_G(x) = \mathcal{O}(x) = \{g(x) \mid g \in G\} \subseteq X$.

31. G **acts on a set X (equivalent) def.** A group G acts on a set X iff $\Phi : G \times X \to X$ with $(g, x) \mapsto \Phi(g, x) = g \circ x$ such that:

    (a) for all $x \in X$, $\Phi(e_G, x) = x$.
    (b) for all $x \in X$, $g, h \in G$, $\Phi(gh, x) = \Phi(g, \Phi(h, x))$.

32. **Left regular action of G on G.** Define $\Lambda : G \times G \to G$ be a group action, where G is a group (and a set) such that $(g, h) \mapsto g \circ h$. Equivalently, the left regular action of G on G is defined as the homomorphism $\lambda : G \to \mathrm{Sym}(G)$ such that $g \mapsto (\lambda_g : h \mapsto \lambda_g(h) = g \circ h)$, where λ_g is a permutation on the set G for $g \in G$.

33. **Ring.** A ring $R = (R, +, \times)$ is a set with two closed binary operations ($+$ and $\times$) such that:

    (a) $(R, +)$ is an abelian group.
    (b) $(R, \times)$ is associative.
    (c) Both distrbutive properties hold, i.e. $a \times (b + c) = (a \times b) + (a \times c)$ and $(a + b) \times c = (a \times c) + (b \times c)$ for all $a, b, c \in R$.

34. $R^\times$**.** Let R be a ring. Then, we define $R^\times = \{a \in R \mid \exists b \in R : ab = 1_R\}$, where $(R^\times, \times)$ is a group, possibly abelian.

35. **Field.** Let $R = (R, +, \times)$ be a commutative ring with 1_R being the multiplicative identity. Then if $R^\times = R \setminus \{0\}$, then R is a field.

36. **Ring Homomorphism.** Let $R = (R, +, \times)$ and $S = (S, +, \times)$ be rings. Then a map $\phi : R \to S$ is a homomorphism of rings iff $\phi(a +_R b) = \phi(a) +_S \phi(b)$ and $\phi(a \times_R b) = \phi(a) \times_S \phi(b)$. Additionally, axiomatically, $\phi(1_R) = 1_S$ and a property is $\phi(0_R) = 0_S$. Also, define $\ker \phi := \{r \in R \mid \phi(r) = 0_S\}$.