

Math 113 Definitions.

1. **Set.** A set is an unordered collection of elements.
2. **Map.** A map from X to Y is $f : X \rightarrow Y$ (a rule that assigns elements to Y to elements in X). So, for any $x \in X$ there exists a unique $y \in Y$ such that $f(x) = y$.
3. **Cartesian Product.** The Cartesian product of X and Y is the set $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$.
4. **Equivalence Relation.** An equivalence relation R, \sim on X is a subset $R \subseteq X \times X$ such that
 - (a) Reflexive. $((x, x) \in R \text{ for all } x \in X)$.
 - (b) Symmetric. $(\text{if } (x, y) \in R, \text{ then } (y, x) \in R)$.
 - (c) Transitive. $(\text{if } (x, y) \in R \text{ and } (y, z) \in R, \text{ then } (x, z) \in R)$.
5. **Equivalence Class.** Let X be a set and R be an equivalence relation on X . Then, an equivalence class of $x \in X$ is the set $[x] = [x]_R = [x]_\sim = \{a \in X \mid x \sim a\}$.
6. $\mathbb{Z}/m\mathbb{Z}$. The set of distinct equivalence classes of $\equiv \pmod{n}$ is $\mathbb{Z}/m\mathbb{Z}$.
7. **Group.** A group G (denote: (G, \star)) is a set G with a closed binary operation $\star : G \times G \rightarrow G$ such that:
 - (a) Associativity: $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in G$.
 - (b) Identity: There exists an $e \in G$ such that for any $a \in G$, we have $a \star e = e \star a = a$.
 - (c) Inverse: For any $a \in G$, there exists an $a^{-1} \in G$ such that $a \star a^{-1} = a^{-1} \star a = e$.
8. **Symmetric Group.** The symmetric group on n letters is S_n .
9. **Disjoint Cycles.** Two cycles (a_1, \dots, a_k) and (b_1, \dots, b_l) are disjoint if $a_i \neq b_j$ for all i, j .
10. **Transpositions.** The simplest permutation is a cycle of length 2, which is called a transposition.
11. **Even, Odd Permutations.** A permutation is even if it can be expressed as an even number of transpositions. A permutation is odd if it can be expressed as an odd number of transpositions.
12. **Subgroup.** A subgroup H of a group G is a subset H of G such that when the group operation of G is restricted to H , then H is a group.
13. **Trivial/Proper Subgroup.** The trivial subgroup of a group G is $\{e\}$ and a proper subgroup is a subgroup H of G where H is a proper subset of G .

14. **General/Special Linear Group.** $GL_2(\mathbb{R})$ is the set of 2x2 invertible matrices with real entries. $SL_2(\mathbb{R})$ is the set of 2x2 invertible matrices with real entries and with determinant 1.
 15. **Cyclic Group.** A cyclic group is a group generated by one element.
 16. **Isomorphism.** An isomorphism is a homomorphism which is bijective.
 17. **Kernel of homomorphism.** If $\phi : G \rightarrow H$ is a homomorphism, then $\ker \phi$ is the pre-image of $e_H \in H$, that is, $\ker \phi = \{g \in G \mid \phi(g) = e_H\}$.
 18. **Coset.** Let $(G, \star) \geq (H, \star)$ and $g \in G$. Then, an H -coset of g is a (sub)set of G where $gH = g \star H = \{g \star h \mid h \in H\}$ (left coset) and $Hg = \{h \star g \mid h \in H\}$ (right coset).
 19. **Index.** A set of distinct equivalence classes with respect to \sim_H is G/H , a quotient of G by H . Then, $|G/H| = [G : H]$ is the index of H in G .
 20. The following are definitions listed in the homeworks:
 - (a) Group of units in $\mathbb{Z}/n\mathbb{Z}$ is the set $(\mathbb{Z}/n\mathbb{Z})^\times = \mathbb{Z}/n\mathbb{Z}^\times := \{[a] \in \mathbb{Z}/n\mathbb{Z} \mid \exists [b] \in \mathbb{Z}/n\mathbb{Z} \text{ with } [a] \times [b] = [1]\}$.
 - (b) If G is a group, then the center of G is the set $Z(G) := \{a \in G \mid ga = ag \forall g \in G\}$.
 - (c) \mathbb{C}^\times is the set of nonzero complex numbers.
 - (d) \mathbb{R}^\times is the set of nonzero real numbers.
 - (e) $GL(n, K)$ is the set of $n \times n$ invertible matrices with entries in K .
 - (f) If G is a group, then the torsion subgroup of G is called G_T , which is the set of all elements of G with finite order.
 - (g) The Klein four-group is V is a subgroup of S_4 and consists of $V = \{\text{id}, (12), (34), (12)(34)\}$.
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21. **Dihedral group.** This is the group of symmetries on a regular n -gon with r being rotation s flip. We have $r^n = \text{id}$, $s^2 = \text{id}$, and $sr = rs^{-1}$.
 22. **(External) Direct Product** Let $G = (G, \star)$ and $H = (H, \circ)$ be groups. Then, $G \times H = \{G \times H, (\star, \circ)\}$.
 23. **Normal Subgroup.** Let G be a group and H a subgroup of G . Then H is a normal subgroup (write $H \trianglelefteq G$) iff for all $g \in G$, $gH = Hg$, or equivalently, for all $h \in H$, $ghg^{-1} \in H$ for all $g \in G$.
 24. **Quotient (factor) group.** The quotient group of a group G and a normal subgroup N of G is the group G/N (where G/N is the group of cosets of N in G) under the operation $(aN)(bN) = abN$.

25. **Internal Direct Product.** Let G be a group and $H, K \leq G$. G is an internal direct product of H and K iff:
- (a) $G = H \cdot K := \{h \cdot k \mid h \in H, k \in K\}$.
 - (b) $H \cap K = \{e_G\}$ (“as small as possible”).
 - (c) $h \cdot k = k \cdot h$ for all $h \in H, k \in K$.
26. **Simple group.** A group G is simple if the only normal subgroups are $\{e_G\}$ and G .
27. **Symmetry.** A symmetry of X is a bijective map $\sigma : X \rightarrow X$ preserving the structure where X is some set with some additional structure.
28. **Group of permutations on a set X .** G is a group of permutations on a set X if $\phi : G \rightarrow \text{Sym}(X) = S_X = S_{|X|}$ is a homomorphism that is 1-1.
29. **G acts on a set X .** G acts on a set X is a homomorphism $\phi : G \rightarrow \text{Sym}(X)$.
30. **Stabilizer of $x \in X$.** Let G be a group and X a set. Then, the stabilizer of $x \in X$ is $\text{Stab}_G(x) = \{g \in G \mid g(x) = x\}$, which are elements of g that preserve $x \in X$.
31. **Orbit of $x \in X$.** Take $x \in X$. Then the orbit of x is $\text{orb}_G(x) = \mathcal{O}_G(x) = \mathcal{O}(x) = \{g(x) \mid g \in G\} \subseteq X$.
32. **G acts on a set X (equivalent) def.** A group G acts on a set X iff $\Phi : G \times X \rightarrow X$ with $(g, x) \mapsto \Phi(g, x) = g \circ x$ such that:
- (a) for all $x \in X$, $\Phi(e_G, x) = x$.
 - (b) for all $x \in X$, $g, h \in G$, $\Phi(gh, x) = \Phi(g, \Phi(h, x))$.
33. **Left regular action of G on G .** Define $\Lambda : G \times G \rightarrow G$ be a group action, where G is a group (and a set) such that $(g, h) \mapsto g \circ h$. Equivalently, the left regular action of G on G is defined as the homomorphism $\lambda : G \rightarrow \text{Sym}(G)$ such that $g \mapsto (\lambda_g : h \mapsto \lambda_g(h) = g \circ h)$, where λ_g is a permutation on the set G for $g \in G$.
34. **Ring.** A ring $R = (R, +, \times)$ is a set with two closed binary operations $(+)$ and (\times) such that:
- (a) $(R, +)$ is an abelian group.
 - (b) (R, \times) is associative.
 - (c) Both distributive properties hold, i.e. $a \times (b + c) = (a \times b) + (a \times c)$ and $(a + b) \times c = (a \times c) + (b \times c)$ for all $a, b, c \in R$.
35. **R^\times .** Let R be a ring. Then, we define $R^\times = \{a \in R \mid \exists b \in R : ab = 1_R\}$, where (R^\times, \times) is a group, possibly abelian.
36. **Field.** Let $R = (R, +, \times)$ be a commutative ring with 1_R being the multiplicative identity. Then if $R^\times = R \setminus \{0\}$, then R is a field.

37. **Ring Homomorphism.** Let $R = (R, +, \times)$ and $S = (S, +, \times)$ be rings. Then a map $\phi : R \rightarrow S$ is a homomorphism of rings iff $\phi(a +_R b) = \phi(a) +_S \phi(b)$ and $\phi(a \times_R b) = \phi(a) \times_S \phi(b)$. Additionally, axiomatically, $\phi(1_R) = 1_S$ and a property is $\phi(0_R) = 0_S$. Also, define $\ker \phi := \{r \in R \mid \phi(r) = 0_S\}$.
38. BELOW IS ADDITIONAL DEFS FOR MT2
39. A **ring with unity (or with identity)** is a ring R that has multiplicative identity.
40. A **commutative ring** is a ring R that has multiplicative commutativity.
41. An **integral domain** is a commutative ring R with identity such that for all $a, b \in R$ $ab = 0$ implies $a = 0$ or $b = 0$.
42. A **division ring** is a ring R that has multiplicative inverse for all nonzero $a \in R$.
43. A **zero divisor** of a commutative ring R is an $a \in R$ ($a \neq 0$) such that there exists a nonzero $b \in R$ such that $ab = 0$.
44. The **ring of quaternions** is the set $\mathbb{H} = \{a + b\hat{i} + c\hat{j} + d\hat{k} \mid a, b, c, d \in \mathbb{R}\}$, where $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \hat{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \hat{j} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \hat{k} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.
45. A **field** is a commutative division ring.
46. The **characteristic** of a ring R is the least positive integer n such that $nr = 0$ for all $r \in R$. If no such n exists, the characteristic of R is defined to be 0. (denote the characteristic of R by $\text{char}R$).
47. A **ring homomorphism** is a map $\phi : R \rightarrow S$ (where R, S are rings) such that $\phi(a + b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in R$.
48. A **ring isomorphism** is a bijective map $\phi : R \rightarrow S$ where R, S are rings.
49. The **kernel** of a ring homomorphism $\phi : R \rightarrow S$ is the set $\ker \phi := \{r \in R \mid \phi(r) = 0\}$.
50. An **evaluation homomorphism** is a ring homomorphism of the form $\phi_\alpha : C[a, b] \rightarrow \mathbb{R}$ or other such related homomorphisms.
51. An **ideal** of a ring R is a subring I such that
- $(I, +)$ is a subgroup of $(R, +)$.
 - if $a \in I$ and $r \in R$, then $ar, ra \in I$.
52. The **trivial ideals** of a ring R are the subrings $\{0\}$ and R .
53. A **principal ideal** of a commutative ring R (with identity) is an ideal of the form $\langle a \rangle = \{ar \mid r \in R\}$.

54. A **two-sided ideal** I is a subring of a ring R such that $rI \subset I$ and $Ir \subset I$ for all $r \in R$.
55. A **one-sided ideal** I is a subring of a ring R is one such that $rI \subset I$ for all $r \in R$ (a **left ideal**) or $Ir \subset I$ for all $r \in R$ (a **right ideal**).
56. **Quotient ring.** Let R be a ring and I a two-sided ideal of R . Then the quotient ring R/I is defined to be the set of all cosets of I with respect to $+$ and \times .
57. **Natural/canonical homomorphism.** The map $\phi : R \rightarrow R/I$ is called the natural/canonical homomorphism.
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58. **proper ideal.** $I \trianglelefteq R$ is a proper ideal of R iff $I \neq \{0_R\}$ and $I \neq R$.
59. **Integral domain.** A commutative ring R with 1_R is an integral domain if there are no (nonzero) zero-divisors.
60. **Prime ideal.** An ideal I of a ring R is a prime ideal if $ab \in I$ means $a \in I$ or $b \in I$.
61. **Prime.** Let $p \in D$, where D is an integral domain and p a non-unit. p is prime iff if $p \mid ab$, then $p \mid a$ or $p \mid b$.
62. **Irreducible.** Let $x \in D$, where D is an integral domain and x a non-unit. x is irreducible iff if $x = ab$ means a is a unit or b is a unit.
63. **Principal ideal domain (PID).** A principal ideal is an integral domain in which every ideal is a principal ideal.
64. **Unique factorization domain (UFD).** An integral domain D is a unique factorization domain (UFD) if:
- (a) Let $a \in D$ such that $a \neq 0$ and a is a non-unit. Then a can be written as the product of irreducible elements of D .
 - (b) Let $a = p_1 \cdots p_r = q_1 \cdots q_s$, where p_i, q_k are irreducible. Then $r = s$ and there is a $\pi \in S_r$ such that p_i and $q_{\pi(j)}$ are associates for $j = 1, \dots, r$.
65. **Euclidean domain.** Let D be an integral domain such that there is a function $v : D \setminus \{0\} \rightarrow \mathbb{N}$ such that:
- (a) If a, b are nonzero elements of D , then $v(a) \leq v(ab)$.
 - (b) Let $a, b \in D$ and suppose $b \neq 0$. Then There exist elements $q, r \in D$ such that $a = bq + r$ and either $r = 0$ or $v(r) < v(b)$.

Then D is a Euclidean domain.

66. **Gaussian Integers.** The set of Gaussian integers is the set $\{a + bi \mid a, b \in \mathbb{Z}, i^2 = -1\} =: \mathbb{Z}[i]$.

67. **Norm.** Let $z \in \mathbb{Z}[i]$. Then we define the norm of z to be $N(z) = z \cdot \bar{z}$, or if $z = a + bi \in \mathbb{Z}[i]$, then $N(z) = a^2 + b^2$.
68. **Norm (again).** Norm of $z = a + b\sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]$ is $a^2 + 5b^2 = z\bar{z} \in \mathbb{Z}$.
69. **Prime (again).** Let $p \in R$, non-unit. p is prime iff if $p \mid ab$ then $p \mid a$ or $p \mid b$ for all $a, b \in R$.
70. **Irreducible (again).** $q \in R$ is irreducible if for any $a, b \in R$ such that $q = a \cdot b$, then a or b is a unit.
71. **Product of ideals.** Let I, J be ideals in R . Then define the product of ideals as $I \cdot J = \{\sum_{i=1}^n a_i b_i \mid a_i \in I, b_i \in J, n \in \mathbb{Z}_{>0}\}$.
72. **Finitely generated ideal.** Let R be a ring and $x_1, \dots, x_n \in R$. Then a finitely generated ideal of R is $\langle x_1, \dots, x_n \rangle := \{a_1 x_1 + \dots + a_n x_n \mid a_1, \dots, a_n \in R\}$.
73. **Associates.** Let R be a commutative ring with identity. Then nonunits elements $x, y \in R$ are associates if there exists a unit $u \in R$ such that $x = uy$.
74. **Finite field notation.** We write a finite field with p^n elements as $GF(p^n)$ or \mathbb{F}_{p^n} .
75. **Vector spcae.** A vector space V over a field F is
- (a) an abelian group (addition of vectors) with $V \times V \rightarrow V$ by $(v, w) \mapsto v + w$.
 - (b) operation of multiplication by elements of F with $F \times v \mapsto V$ by $(\lambda, v) \mapsto \lambda \cdot v$.
 - (c) $\alpha(\beta v) = (\alpha\beta)v$.
 - (d) $(\alpha + \beta)v = \alpha v + \beta v$.
 - (e) $\alpha(u + v) = \alpha u + \alpha v$.
 - (f) $1 \cdot v = v$.

where $u, v \in V$, and $\alpha, \beta \in F$.

76. **Linear map.** A linear map is $\phi : V \rightarrow W$, where V, W are F -vector spaces, where $\phi(v + w) = \phi(v) + \phi(w)$ and $\phi(\lambda \cdot v) = \lambda\phi(v)$.