Math 113 Definitions.

1. **Set.** A set is an unordered collection of elements.

2. **Map.** A map from X to Y is $f : X \to Y$ (a rule that assigns elements to Y to elements in X). So, for any $x \in X$ there exists a unique $y \in Y$ such that $f(x) = y$.

3. **Cartesian Product.** The Cartesian product of X and Y is the set $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$.

4. **Equivalence Relation.** An equivalence relation $R, \sim$ on X is a subset $R \subseteq X \times X$ such that

   (a) Reflexive. ($(x, x) \in R$ for all $x \in X$).

   (b) Symmetric. (if $(x, y) \in R$, then $(y, x) \in R$).

   (c) Transitive. (if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$).

5. **Equivalence Class.** Let X be a set and R be an equivalence relation on X. Then, an equivalence class of $x \in X$ is the set $[x] = [x]_R = [x]_\sim = \{a \in X \mid x \sim a\}$.

6. $\mathbb{Z}/m\mathbb{Z}$. The set of distinct equivalence classes of $\equiv \pmod{n}$ is $\mathbb{Z}/m\mathbb{Z}$.

7. **Group.** A group G (denote: $(G, \star)$) is a set G with a closed binary operation $\star : G \times G \to G$ such that:

   (a) Associativity: $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in G$.

   (b) Identity: There exists an $e \in G$ such that for any $a \in G$, we have $a \star e = e \star a = a$.

   (c) Inverse: For any $a \in G$, there exists an $a^{-1} \in G$ such that $a \star a^{-1} = a^{-1} \star a = e$.

8. **Symmetric Group.** The symmetric group on n letters is S_n.

9. **Disjoint Cycles.** Two cycles $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_l)$ are disjoint if $a_i \neq b_j$ for all i, j.

10. **Transpositions.** The simplest permutation is a cycle of length 2, which is called a transposition.

11. **Even, Odd Permuatations.** A permutation is even if it can be expressed as an even number of transpositions. A permutation is odd if it can be expressed as an odd number of transpositions.

12. **Subgroup.** A subgroup H of a group G is a subset H of G such that when the group operation of G is restricted to H, then H is a group.

13. **Trivial/Proper Subgroup.** The trivial subgroup of a group G is $\{e\}$ and a proper subgroup is a subgroup H of G where H is a proper subset of G.

14. **General/Special Linear Group.** $GL_2(\mathbb{R})$ is the set of 2x2 invertible matrices with real entries. $SL_2(\mathbb{R})$ is the set of 2x2 invertible matrices with real entries and with determinant 1.

15. **Cyclic Group.** A cyclic group is a group generated by one element.

16. **Isomorphism.** An isomorphism is a homomorphism which is bijective.

17. **Kernel of homomorphism.** If $\phi : G \to H$ is a homomorphism, then $\ker \phi$ is the pre-image of $e_H \in H$, that is, $\ker \phi \{g \in G \mid \phi(g) = e_H\}$.

18. **Coset.** Let $(G, \star) \geq (H, \star)$ and $g \in G$. Then, an $H-$ coset of g is a (sub)set of G where $gH = g \star H = \{g \star h \mid h \in H\}$ (left coset) and $Hg = \{h \star g \mid h \in H\}$ (right coset).

19. **Index.** A set of distinct equivalence classes with respect to $_H$ is G/H, a quotient of G by H. Then, $|G/H| = [G : H]$ is the index of H in G.

20. The following are definitions listed in the homeworks:

    (a) Group of units in $\mathbb{Z}/n\mathbb{Z}$ is the set $(\mathbb{Z}/n\mathbb{Z})^\times = \mathbb{Z}/n\mathbb{Z}^\times := \{[a] \in \mathbb{Z} \mid \exists [b] \in \mathbb{Z}/n\mathbb{Z} \text{ with } [a] \times [b] = [1]\}$.

    (b) If G is a group, then the center of G is the set $Z(G) := \{a \in G \mid ga = ag \forall g \in G\}$.

    (c) $\mathbb{C}^\times$ is the set of nonzero complex numbers.

    (d) $\mathbb{R}^\times$ is the set of nonzero real numbers.

    (e) $GL(n, K)$ is the set of n x n invertible matrices with entries in K.

    (f) If G is a group, then the torsion subgroup of G is called G_T, which is the set of all elements of G with finite order.

    (g) The Klein four-group is V is a subgroup of S_4 and consists of $V = \{\mathrm{id}, (12), (34), (12)(34)\}$.

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21. **Dihedral group.** This is the group of symmetries on a regular n-gon with r being rotation s flip. We have $r^n = \mathrm{id}$, $s^2 = \mathrm{id}$, and $srs = r^{-1}$.

22. **(External) Direct Product** Let $G = (G, \star)$ and $H = (H, \circ)$ be groups. Then, $G \times H = \{G \times H, (\star, \circ)\}$.

23. **Normal Subgroup.** Let G be a group and H a subgroup of G. Then H is a normal subgroup (write $H \trianglelefteq G$) iff for all $g \in G$, $gH = Hg$, or equivalently, for all $h \in H$, $ghg^{-1} \in H$ for all $g \in G$.

24. **Quotient (factor) group.** The quotient group of a group G and a normal subgroup N of G is the group G/N (where G/N is the group of cosets of N in G) under the operation $(aN)(bN) = abN$.

25. **Internal Direct Product.** Let G be a group and $H, K \leq G$. G is an internal direct product of H and K iff:

   (a) $G = H \cdot K := \{h \cdot k \mid h \in H, k \in K\}$.

   (b) $H \cap K = \{e_G\}$ ("as small as possible").

   (c) $h \cdot k = k \cdot h$ for all $h \in H, k \in K$.

26. **Simple group.** A group G is simple if the only normal subgroups are $\{e_G\}$ are G.

27. **Symmetry.** A symmetry of X is a bijective map $\sigma : X \to X$ preserving the structure where X is some set with some additional structure.

28. **Group of permutations on a set X.** G is a group of permutations on a set X if $\phi : G \to \mathrm{Sym}(X) = S_X = S_{|X|}$ is a homomorphism that is 1-1.

29. **G acts on a set X.** G acts on a set X is a homomorphism $\phi : G \to \mathrm{Sym}(X)$.

30. **Stabilizer of $x \in X$.** Let G be a group and X a set. Then, the stabilizer of $x \in X$ is $\mathrm{Stab}_G(x) = \{g \in G \mid g(x) = x\}$, which are elements of g that preserve $x \in X$.

31. **Orbit of $x \in X$.** Take $x \in X$. Then the orbit of X is $\mathrm{orb}_G(x) = \mathcal{O}_G(x) = \mathcal{O}(x) = \{g(x) \mid g \in G\} \subseteq X$.

32. **G acts on a set X (equivalent) def.** A group G acts on a set X iff $\Phi : G \times X \to X$ with $(g, x) \mapsto \Phi(g, x) = g \circ x$ such that:

   (a) for all $x \in X$, $\Phi(e_G, x) = x$.

   (b) for all $x \in X$, $g, h \in G$, $\Phi(gh, x) = \Phi(g, \Phi(h, x))$.

33. **Left regular action of G on G.** Define $\Lambda : G \times G \to G$ be a group action, where G is a group (and a set) such that $(g, h) \mapsto g \circ h$. Equivalently, the left regular action of G on G is defined as the homomorphism $\lambda : G \to \mathrm{Sym}(G)$ such that $g \mapsto (\lambda_g : h \mapsto \lambda_g(h) = g \circ h)$, where λ_g is a permutation on the set G for $g \in G$.

34. **Ring.** A ring $R = (R, +, \times)$ is a set with two closed binary operations ($+$ and $\times$) such that:

   (a) $(R, +)$ is an abelian group.

   (b) $(R, \times)$ is associative.

   (c) Both distrbutive properties hold, i.e. $a \times (b + c) = (a \times b) + (a \times c)$ and $(a + b) \times c = (a \times c) + (b \times c)$ for all $a, b, c \in R$.

35. **$R^\times$.** Let R be a ring. Then, we define $R^\times = \{a \in R \mid \exists b \in R : ab = 1_R\}$, where $(R^\times, \times)$ is a group, possibly abelian.

36. **Field.** Let $R = (R, +, \times)$ be a commutative ring with 1_R being the multiplicative identity. Then if $R^\times = R \setminus \{0\}$, then R is a field.

37. **Ring Homomorphism.** Let $R = (R, +, \times)$ and $S = (S, +, \times)$ be rings. Then a map $\phi : R \to S$ is a homomorphism of rings iff $\phi(a +_R b) = \phi(a) +_S \phi(b)$ and $\phi(a \times_R b) = \phi(a) \times_S \phi(b)$. Additionally, axiomatically, $\phi(1_R) = 1_S$ and a property is $\phi(0_R) = 0_S$. Also, define $\ker\phi := \{r \in R \mid \phi(r) = 0_S\}$.

38. BELOW IS ADDITIONAL DEFS FOR MT2

39. A **ring with unity (or with identity)** is a ring R that has multiplicative identity.

40. A **commutative ring** is a ring R that has multiplicative commutativity.

41. An **integral domain** is a commutative ring R with identity such that for all $a, b \in R$ $ab = 0$ implies $a = 0$ or $b = 0$.

42. A **division ring** is a ring R that has multiplicative inverse for all nonzero $a \in R$.

43. A **zero divisor** of a commutative ring R is an $a \in R$ ($a \neq 0$) such that there exists a nonzero $b \in R$ such that $ab = 0$.

44. The **ring of quaternions** is the set $\mathbb{H} = \{a + b\hat{i} + c\hat{j} + d\hat{k} \mid a, b, c, d \in \mathbb{R}\}$, where $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \hat{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \hat{j} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \hat{k} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

45. A **field** is a commutative division ring.

46. The **characteristic** of a ring R is the least positive integer n such that $nr = 0$ for all $r \in R$. If no such n exists, the characteristic of R is defined to be 0. (denote the characteristic of R by $\mathrm{char}R$).

47. A **ring homomorphism** is a map $\phi : R \to S$ (where R, S are rings) such that $\phi(a + b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in R$.

48. A **ring isomorphism** is a bijective map $\phi : R \to S$ where R, S are rings.

49. The **kernel** of a ring homomorphism $\phi : R \to S$ is the set $\ker\phi := \{r \in R \mid \phi(r) = 0\}$.

50. An **evaluation homomorphism** is a ring homomorphism of the form $\phi_\alpha : C[a, b] \to \mathbb{R}$ or other such related homomorphisms.

51. An **ideal** of a ring R is a subring I such that
    (a) $(I, +)$ is a subgroup of $(R, +)$.
    (b) if $a \in I$ and $r \in R$, then $ar, ra \in I$.

52. The **trivial ideals** of a ring R are the subrings $\{0\}$ and R.

53. A **principal ideal** of a commutative ring R (with identity) is an ideal of the form $\langle a \rangle = \{ar \mid r \in R\}$.

54. A **two-sided ideal** I is a subring of a ring R such that $rI \subset I$ and $Ir \subset I$ for all $r \in R$.

55. A **one-sided ideal** I is a subring of a ring R is one such that $rI \subset I$ for all $r \in R$ (a **left ideal**) or $Ir \subset I$ for all $r \in R$ (a **right ideal**).

56. **Quotient ring.** Let R be a ring and I a two-sided ideal of R. Then the quotient ring R/I is defined to be the set of all cosets of I with respect to $+$ and $\times$.

57. **Natural/canonical homomorphism.** The map $\phi : R \to R/I$ is called the natural/canonical homomorphism.

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58. **proper ideal.** $I \trianglelefteq R$ is a proper ideal of R iff $I \neq \{0_R\}$ and $I \neq R$.

59. **Integral domain.** A commutative ring R with 1_R is an integral domain if there are no (nonzero) zero-divisors.

60. **Prime ideal.** An ideal I of a ring R is a prime ideal if $ab \in I$ means $a \in I$ or $b \in I$.

61. **Prime.** Let $p \in D$, where D is an integral domain and p a non-unit. p is prime iff if $p \mid ab$, then $p \mid a$ or $p \mid b$.

62. **Irreducible.** Let $x \in D$, where D is an integral domain and x a non-unit. x is irreducible iff if $x = ab$ means a is a unit or b is a unit.

63. **Principal ideal domain (PID).** A principal ideal is an integral domain in which every ideal is a principal ideal.

64. **Unique factorization domain (UFD).** An integral domain D is a unique factorization doman (UFD) if:

    (a) Let $a \in D$ such that $a \neq 0$ and a is a non-unit. Then a can be written as the product of irreducible elements of D.

    (b) Let $a = p_1 \cdots p_r = q_1 \cdots q_s$, where p_i, q_k are irreducbile. Then $r = s$ and there is a $\pi \in S_r$ such that p_i and $q_{\pi(j)}$ are associates for $j = 1, \ldots, r$.

65. **Euclidean domain.** Let D be an integral domain such that there is a function $v : D \setminus \{0\} \to \mathbb{N}$ such that:

    (a) If a, b are nonzero elements of D, then $v(a) \leq v(ab)$.

    (b) Let $a, b \in D$ and suppose $b \neq 0$. Then There exist elements $q, r \in D$ such that $a = bq + r$ and either $r = 0$ or $v(r) < v(b)$.

    Then D is a Euclidean domain.

66. **Gaussian Integers.** The set of Gaussian integers is the set $\{a + bi \mid a, b \in \mathbb{Z}, i^2 = -1\} =: \mathbb{Z}[i]$.

67. **Norm.** Let $z \in \mathbb{Z}[i]$. Then we define the norm of z to be $N(z) = z \cdot \overline{z}$, or if $z = a + bi \in \mathbb{Z}[i]$, then $N(z) = a^2 + b^2$.

68. **Norm (again).** Norm of $z = a + b\sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]$ is $a^2 + 5b^2 = z\overline{z} \in \mathbb{Z}$.

69. **Prime (again).** Let $p \in R$, non-unit. p is prime iff if $p \mid ab$ then $p \mid a$ or $p \mid b$ for all $a, b \in R$.

70. **Irreducible (again).** $q \in R$ is irreducible if for any $a, b \in R$ such that $q = a \cdot b$, then a or b is a unit.

71. **Product of ideals.** Let I, J be ideals in R. Then define the product of ideals as $I \cdot J = \{\sum_{i=1}^{n} a_i b_i \mid a_i \in I, b_i \in J, n \in \mathbb{Z}_{>0}\}$.

72. **Finitely generated ideal.** Let R be a ring and $x_1, \ldots, x_n \in R$. Then a finitely generated ideal of R is $\langle x_1, \ldots, x_n \rangle := \{a_1 x_1 + \cdots + a_n x_n \mid a_1, \ldots, a_n \in R\}$.

73. **Associates.** Let R be a commutative ring with identity. Then nonunits elements $x, y \in R$ are associates if there exists a unit $u \in R$ such that $x = uy$.

74. **Finite field notation.** We write a finite field with p^n elements as $GF(p^n)$ or $\mathbb{F}_{p^n}$.

75. **Vector spcae.** A vector space V over a field F is

    (a) an abelian group (addition of vectors) with $V \times V \to V$ by $(v, w) \mapsto v + w$.
    (b) operation of multiplication by elements of F with $F \times v \mapsto V$ by $(\lambda, v) \mapsto \lambda \cdot v$.
    (c) $\alpha(\beta v) = (\alpha\beta)v$.
    (d) $(\alpha + \beta)v = \alpha v + \beta v$.
    (e) $\alpha(u + v) = \alpha u + \alpha v$.
    (f) $1 \cdot v = v$.

    where $u, v \in V$, and $\alpha, \beta = \in F$.

76. **Linear map.** A linear map is $\phi : V \to W$, where V, W are F-vector spaces, where $\phi(v + w) = \phi(v) + \phi(w)$ and $\phi(\lambda \cdot v) = \lambda\phi(v)$.

77. **Extension of fields.** Let E, F be fields and $F \leq E$ a subfield. Then we write this extension of fields as

$$\begin{array}{c} E \\ | \\ F \end{array}$$

78. **Simple algebraic extension (def 1).** E over F is a simple algebraic extension iff $E = F[\alpha]$ for some $\alpha \in E$, algebraic element over F.

79. **Simple algebraic extension (def 2).** E over F is a simply algebraic extension iff $E \cong F[x]/\langle p(x) \rangle$, where $p(x)$ is irreducible.

80. $F[p(x)]$**.** We define this to be $F(\alpha_1, \ldots, \alpha_n)$ where $\alpha_1, \ldots, \alpha_n$ are all the roots of $p(x)$.

81. **Splitting field.** Let F be a field and $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a nonconstant polynomial in $F[x]$. An extension E of F is called a splitting field of $p(x)$ if there exist elements $\alpha_1, \ldots, \alpha_n$ in E so that $E = F(\alpha_1, \ldots, \alpha_n)$ and $p(x) = (x - \alpha_1) \cdots (x - \alpha_n)$.

82. **degree of splitting field.** this is the dimension of the vector space 'generated' by splitting field extension.

83. **Automorphism of** E **over** F**.** Let $E \geq F$ be a field extension. Then an automorphicm of E over F is a bijective ring homomorphism $\phi : E \to E$ so that for any $f \in F$, $\phi(f) = f$, i.e. $\phi\,|_F = \mathrm{id}$.