TEST

DEFS

1. a set is an unordered collection of elements.
2. a map from a set X to a set Y (write: $f : X \to Y$) is a rule that assigns elements of Y to elements of X, that is, for each $x \in X$ there exists a unique $y \in Y$ such that $f(x) = y$.
3. Let X and Y be sets. then their cartesian product is $X \times Y = \{(x,y) \mid x \in X, y \in Y\}$.
4. let X be a set and consider a relation $R \subseteq X \times X$. Then R is an equivalence relation if:
   (a) $x \sim x$ for all $x \in R$.
   (b) $x \sim y$ means $y \sim x$ for all $x, y \in R$.
   (c) $x \sim y$ and $y \sim z$ means $x \sim z$ for all $x, y, z \in R$.
5. let R be an equivalence relation on X. then the equivalence class of $x \in R$ is $[x] = \{a \in X \mid x \sim a\}$.
6. $\mathbb{Z}/n\mathbb{Z}$ is the set of equivalence classes of integers mod n (the relation being $\equiv_n$).
7. a group G is a set G with a binary operation $\circ$ such that the following hold:
   (a) if $a, binG$, then $a \circ b \in G$.
   (b) $(a \circ b) \circ = a \circ (b \circ c)$ for all $a, b, c \in G$.
   (c) there exists $e \in G$ such that $a \circ e = e \circ a = a$ for all $a \in G$.
   (d) for each $a \in G$ there exists $a' \in G$ such that $a \circ a' = a' \circ a = e \in G$.
8. the symmetric group S_n is the group of all permutations of n letters, and the binary operation on S_n is the operation of composition of permutations on the n letters.
9. two cycles $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_l)$ are disjoint if $a_i \neq b_j$ for all i, j.
10. a cycle of length 2 is called a transposition (the simplest type of permutation).
11. a permutation is even if can be expressed as a product of an even number of transpositions, and similarly for odd permutations.
12. Let G be a group. H is a subgroup of G if H is a subset of G and H is a group under the same binary operation as defined on G.
13. let G be a group. The trivial subgroup of G is just $\{e\}$. a proper subgroup of G is a subgroup of G that is also a proper subset of G.
14. the general linear group is $GL_2(\mathbb{R})$, which is the set of 2x2 invertible matrices of real entries. the special linear group is $SL_2(\mathbb{R})$, which is the set of 2x2 matrices of real entries and determinant 1.
15. a cyclic group is a group that is generated by one of its elements.
16. an isomorphism is a homomorphism that is bijective.
17. let $\phi : G \to H$ be a homomorphism. then the kernel of ϕ is the set $\ker\phi = \{g \in G \mid \phi(g) = e_H\}$.
18. let G be a group and H a subgroup. then the left H-coset of $g \in G$ is the set $gH = \{gh \mid h \in H\}$. the right H-coset of $g \in G$ is $Hg = \{hg \mid h \in H\}$. if the left and right H-cosets of $g \in G$ are indistinguishable, then we just call them both cosets.
19. let G be a group and H a subgroup. then we define G/H to be the set of equivalence classes with respect to H in G. then we say $[G : H] = |G/H|$ is the index of H in G.

THMS

1. the relation $\equiv_n$ is an equivalence relation on $\mathbb{Z}$.
2. $\mathbb{Z}/n\mathbb{Z}$ has exactly n elements.
   (a) if $i \in [j]$ then $j \in [i]$ (in $\mathbb{Z}/n\mathbb{Z}$).

(b) if $[i] \cap [j] \neq \emptyset$, then $[i] = [j]$.
(c) if $i \neq j$ and $0 \leq i < j \leq n-1$ then $[i] \cap [j] = \emptyset$.
(d) each $x \in \mathbb{Z}$ lies in exactly one of $[0], \ldots, [n-1]$.

3. addition is correctly & and well-defined on $\mathbb{Z}/n\mathbb{Z}$ to be $[a]+[b] = [a+b]$.
4. the identity element in a group G is unique.
5. if G is a group, the inverse of $g \in G$ is unique.
6. for all a, b in a group G, $(ab)' = b'a'$.
7. for all $ginG$ where G is a group, then $g'' = g$.
8. let σ and τ be disjoint cycles on S_X then $\sigma\tau = \tau\sigma$.
9. every permutation in S_n can be written as the product of disjoint cycles.
10. any permutation of a finite set consisting of at least 2 elements can be written as the product of transpositions.
11. if the identity id is written as the product of r transpositions, then r is even.
12. if a permutation σ can be written as the product of an even number of transpositions, then any product of transpositions equaling σ must contain an even number of transpositions. Similarly for odd.
13. a subset H is a subgroup of a group G iff:
    (a) $e_G \in G$ is also the identity element in H.
    (b) if a, b in H, then $ab \in H$.
    (c) for each $a \in H$, $a' \in H$.
14. let H be a subset of a group G. then H is a subgroup of G iff $H \neq \emptyset$ and $gh' \in H$ for all $g, h \in H$.
15. take a group G and an element $a \in G$. then the cyclic subgroup $\langle a \rangle = \{a^k \mid k \in \mathbb{Z}\}$ is a minimal subgroup of G containing a. minimality is that if H is a subgroup of G and $a \in H$, then $\langle a \rangle$ is a subgroup of H.
16. every cyclic group is abelian.
17. let $\phi : G \to H$ be a homomorphism. Then:
    (a) $\phi(e_G) = e_H$.
    (b) $\phi(g') = \phi(g)'$ for all $g \in G$.
    (c) if K is a subgroup of G, then $\phi(K) = \{\phi(g) \mid g \in K\}$ is a subgroup of H.
    (d) $\phi(G) = \{\phi(g) \mid g \in G\}$ is a subgroup of H.
    (e) if M is a subgroup of H, then its pre-image $\phi'(M) = \{g \in G \mid \phi(g) \in M\}$ is a subgroup of G.
18. let G be a group and $H \leq G$. also let $g_1, g_2 \in G$. then, TFAE:
    (a) $g_1H = g_2H$.
    (b) $Hg_1' = Hg_2'$.
    (c) $g_2 \in g_1H$.
    (d) $g_1H \subseteq g_2H$.
    (e) $g_2'g_1 \in H$.
19. left H-cosets partition G.
20. lagrange's theorem. let G be a finite group and H a subgroup of G. then $[G:H] = \frac{|G|}{|H|}$, or $|G| = [G:H] \cdot |H|$.
21. cor. let G be a finite group and H a subgroup. then $|H|$ and $[G:H]$ divide $|G|$.
22. cor. let G be a finite group and H, K subgroups with $K \leq H \leq G$. then $[G:K] = [G:H] \cdot [H:K]$.