Math 113 Definitions.

1. **Set.** A set is an unordered collection of elements.

2. **Map.** A map from X to Y is $f : X \to Y$ (a rule that assigns elements to Y to elements in X). So, for any $x \in X$ there exists a unique $y \in Y$ such that $f(x) = y$.

3. **Cartesian Product.** The Cartesian product of X and Y is the set $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$.

4. **Equivalence Relation.** An equivalence relation $R, \sim$ on X is a subset $R \subseteq X \times X$ such that

    (a) Reflexive. ($(x, x) \in R$ for all $x \in X$).

    (b) Symmetric. (if $(x, y) \in R$, then $(y, x) \in R$).

    (c) Transitive. (if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$).

5. **Equivalence Class.** Let X be a set and R be an equivalence relation on X. Then, an equivalence class of $x \in X$ is the set $[x] = [x]_R = [x]_\sim = \{a \in X \mid x \sim a\}$.

6. $\mathbb{Z}/m\mathbb{Z}$. The set of distinct equivalence classes of $\equiv \pmod{n}$ is $\mathbb{Z}/m\mathbb{Z}$.

7. **Group.** A group G (denote: $(G, \star)$) is a set G with a closed binary operation $\star : G \times G \to G$ such that:

    (a) Associativity: $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in G$.

    (b) Identity: There exists an $e \in G$ such that for any $a \in G$, we have $a \star e = e \star a = a$.

    (c) Inverse: For any $a \in G$, there exists an $a^{-1} \in G$ such that $a \star a^{-1} = a^{-1} \star a = e$.

8. **Symmetric Group.** The symmetric group on n letters is S_n.

9. **Disjoint Cycles.** Two cycles $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_l)$ are disjoint if $a_i \neq b_j$ for all i, j.

10. **Transpositions.** The simplest permutation is a cycle of length 2, which is called a transposition.

11. **Even, Odd Permuatations.** A permutation is even if it can be expressed as an even number of transpositions. A permutation is odd if it can be expressed as an odd number of transpositions.

12. **Subgroup.** A subgroup H of a group G is a subset H of G such that when the group operation of G is restricted to H, then H is a group.

13. **Trivial/Proper Subgroup.** The trivial subgroup of a group G is $\{e\}$ and a proper subgroup is a subgroup H of G where H is a proper subset of G.

14. **General/Special Linear Group.** $GL_2(\mathbb{R})$ is the set of 2x2 invertible matrices with real entries. $SL_2(\mathbb{R})$ is the set of 2x2 invertible matrices with real entries and with determinant 1.

15. **Cyclic Group.** A cyclic group is a group generated by one element.

16. **Isomorphism.** An isomorphism is a homomorphism which is bijective.

17. **Kernel of homomorphism.** If $\phi : G \to H$ is a homomorphism, then $\ker \phi$ is the pre-image of $e_H \in H$, that is, $\ker \phi\{g \in G \mid \phi(g) = e_H\}$.

18. **Coset.** Let $(G, \star) \geq (H, \star)$ and $g \in G$. Then, an $H-$ coset of g is a (sub)set of G where $gH = g \star H = \{g \star h \mid h \in H\}$ (left coset) and $Hg = \{h \star g \mid h \in H\}$ (right coset).

19. **Index.** A set of distinct equivalence classes with respect to $_H$ is G/H, a quotient of G by H. Then, $|G/H| = [G : H]$ is the index of H in G.

20. The following are definitions listed in the homeworks:

    (a) Group of units in $\mathbb{Z}/n\mathbb{Z}$ is the set $(\mathbb{Z}/n\mathbb{Z})^\times = \mathbb{Z}/n\mathbb{Z}^\times := \{[a] \in \mathbb{Z} \mid \exists [b] \in \mathbb{Z}/n\mathbb{Z}$ with $[a] \times [b] = [1]\}$.

    (b) If G is a group, then the center of G is the set $Z(G) := \{a \in G \mid ga = ag \forall g \in G\}$.

    (c) $\mathbb{C}^\times$ is the set of nonzero complex numbers.

    (d) $\mathbb{R}^\times$ is the set of nonzero real numbers.

    (e) $GL(n, K)$ is the set of n x n invertible matrices with entries in K.

    (f) If G is a group, then the torsion subgroup of G is called G_T, which is the set of all elements of G with finite order.

    (g) The Klein four-group is V is a subgroup of S_4 and consists of $V = \{\text{id}, (12), (34), (12)(34)\}$.

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21. **(External) Direct Product** Let $G = (G, \star)$ and $H = (H, \circ)$ be groups. Then, $G \times H = \{G \times H, (\star, \circ)\}$.

22. **Normal Subgroup.** Let G be a group and H a subgroup of G. Then H is a normal subgroup (write $H \trianglelefteq G$) iff for all $g \in G$, $gH = Hg$, or equivalently, for all $hinH$, $ghg^{-1} \in H$ for all $g \in G$.

23. **Internal Direct Product.** Let G be a group and $H, K \leq G$. G is an internal direct product of H and K iff:

    (a) $G = H \cdot K := \{h \cdot k \mid h \in H, k \in K\}$.

    (b) $H \cap K = \{e_G\}$ ("as small as possible").

    (c) $h \cdot k = k \cdot h$ for all $h \in H, k \in K$.