TEST

DEFS

1. A set is an unordered collection of elements.
2. A map from a set X to a set Y is a rule (writ: $f : X \to Y$) such that for each $x \in X$ there exists a unique $y \in Y$ such that $f(x) = y$.
3. Let X and Y be sets. then their cartesian prdouct is $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$.
4. Let X be a set. then an equivalence relation is $R \subseteq X \times X$ with:
   (a) reflexive: $x \sim x$ for all $x \in R$.
   (b) symmetric: $x \sim y$ means $y \sim x$ for all $x, y \in R$.
   (c) transitive: $x \sim y$ and $y \sim z$ means $x \sim z$ for all $x, y, z \in R$.
5. Let R be an equivalence relation with $R \subseteq X \times X$. Then, the equivalence class of $x \in R$ is $[x] = \{a \in R \mid x \sim a\}$.
6. $\mathbb{Z}/m\mathbb{Z}$ is the set of equivalence classes of the integers mod m.
7. A group G is a set G equipped with a binary operation $\circ$ such that:
   (a) if $a, b \in G$, then $a \circ b \in G$.
   (b) $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a, b, c \in G$.
   (c) there exists $e \in G$ such that $a = a \circ e = e \circ a$ for all $a \in G$.
   (d) for each $a \in G$, there exists $a' \in G$ such that $a \circ a' = a' \circ a = e \in G$.
8. Symmetric group is the group of permutations on n letters, write S_n.
9. let $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_m)$ be cycles. Then, they are disjoint cycles if $a_i \neq b_j$ for all i, j.
10. a transposition is a cycle of length 2, the simplest possible permutation.
11. a permutation is called odd if it can be written as the product of an odd number of transpositions, similarly for even permuattions.
12. if G is a group, then we call H a subgroup of G if H is a subset of G and H is a group under the same binary operation as on G.
13. let G be a group. Then the trivial subgroup is $\{e\}$ and H is a proper subgroup of G if H is a proper subset of H and H is a subgroup of G.
14. The general linear group $GL_2(\mathbb{R})$ is the set of 2x2 invertible matrices of real entries and the special linear group $SL_2(\mathbb{R})$ is the set of 2x2 matrices of real entries and determinant 1. (both are groups under the binary operation of multiplication of their respective elements).
15. A cyclic group is a group such that the entire group is genereated by a single element.
16. an isomorphism is a homomorphism that is bijective (that is, 1-1 and onto).
17. Let $f : G \to H$ be homomorphism. then $\ker f$ is the set $\{g \in G \mid f(g) = e_H\}$.
18. Let G be a group and H, a subgroup. then the left H-coset of $g \in G$ is the set $gH = \{gh \mid h \in H\}$ and the right H-coset of $g \in G$ is the set $Hg = \{hg \mid h \in H\}$. If the right and left H-cosets are indistinguishable, then we call them both just cosets.
19. Let G be a group and H, a subgroup. we define then G/H to be the quotient group, that is the group of all equivalence classes with respect to H in G. then, we say the index of H in G is $|G/H| = [G : H]$.

THMS

1. the relation $\equiv \pmod{n}$ is an equivalence relation on $\mathbb{Z}$.
2. $\mathbb{Z}/n\mathbb{Z}$ has exactly n elements.
   (a) if $i \in [j]$, then $j \in [i]$ (in $\mathbb{Z}/n\mathbb{Z}$).

   (b) if $[i] \cap [j] \neq \emptyset$, then $[i] = [j]$.
   (c) if $i \neq j$ and $0 \leq i < j \leq n-1$, then $[i] \cap [j] = \emptyset$.
   (d) every $x \in Z$ belongs to one of $[0], \ldots, [n-1]$.

3. Addition is correctly and well-defined on $\mathbb{Z}/n\mathbb{Z}$ as $[a] + [b] = [a+b]$.
4. the identity element in a group is unique.
5. the inverse for an element in a group is unique.
6. for any $a, b \in G$, $(ab)' = b'a'$
7. for any $g \in G$, $g'' = g$.
8. S_n is a group with $n!$ elements with the binary operation being composition of permutations.
9. Let σ, τ be disjoint cycles in S_X. then $\sigma\tau = \tau\sigma$.
10. every permutation in S_n can be written as the product of disjoint cycles.
11. any permutation of a finite set of at least 2 elements can be written as the product of transpositions.
12. if the identity id is written as the product of r transpositions, then r is even.
13. if a permutation σ can be expressed as the product of an even number of transpositions, then any product of transpositions equaling σ must contain an even number of cycles. similarly, for odd.
14. H is a subgroup of a group G iff:
    (a) if $a \in H$, then $a' \in H$.
    (b) if $a, b \in H$, then $ab \in H$.
    (c) the identity of G exists in H, and is H's identity element.
15. let H be a subset of a group G. then H is a subgroup iff $H \neq \emptyset$ and $gh' \in H$ for all $g, h \in H$.
16. take a group G and $a \in G$. consider the cyclic subgroup $\langle a \rangle$. then $\langle a \rangle$ is a minimal subgroup of G that contains a, where minimality means that if H is a subgroup of G and $a \in H$, then $\langle a \rangle$ is a subgroup of H.
17. every cyclic group is abelian.
18. let $\phi : G \to H$ be a homomorphism. Then:
    (a) $\phi(e_G) = e_H$.
    (b) $\phi(g)' = \phi(g')$ for all $g \in G$.
    (c) let K be a subgroup of G. then $\phi(K) = \{phi(g) \mid g \in K\}$ is a subgroup of H.
    (d) $\phi(G) \leq H$, where $\phi(G) = \{\phi(g) \mid g \in G\}$.
    (e) let M be a subgroup of H. then $\phi'(M) = \{g \in G \mid \phi(g) \in M\}$ is a subgroup of G.
19. TFAE: (let G be a group, and H a subgroup of G) and $g_1, g_2 \in G$.
    (a) $g_1 H = g_2 H$.
    (b) $Hg_1' = Hg_2'$.
    (c) $g_1 \in g_2 H$.
    (d) $g_1 H \subseteq g_2 H$.
    (e) $g_1' g_2 \in H$.
20. left H-cosets partition G.
21. lagrange's theorem. let G be a finite group and H a subgroup of G. then $[G : H] = \frac{|G|}{|H|}$, or $|G| = |H| \cdot [G : H]$.
22. cor. let G be a finite group and H a subgroup of G. then $|H|$ divides $|G|$ and $[G : H]$ divides $|G|$.
23. let G be a finite group and $G \geq H \geq K$. Then $[G : K] = [G : H] \cdot [H : K]$.