# Math 115 - Midterm 1+2 Theorems

1. **Corollary.** Let $a = \prod_p p^{\alpha(p)}$, $b = \prod_p p^{\beta(p)}$, $c = \prod_p p^{\gamma(p)}$.
   (a) $ab = c \iff \alpha(p) + \beta(p) = \gamma(p) \;\; \forall p$.
   (b) $a \mid c \iff \alpha(p) \leq \gamma(p) \;\; \forall p$.
   (c) $c$ is a common divisor of $a$ and $b$ iff $\gamma(p) \leq \min(\alpha(p), \beta(p)) \;\; \forall p$.
   (d) $\gcd(a,b) = \prod_p p^{\min(\alpha(p),\beta(p))}$.
   (e) $\operatorname{lcm}(a,b) = \prod_p p^{\max(\alpha(p),\beta(p))}$.
   (f) $c$ is the square of an integer iff $\gamma(p)$ is even for all $p$.
2. **Pascal's Identity.** $\binom{\alpha+1}{k+1} = \binom{\alpha}{k+1} + \binom{\alpha}{k}$.
3. **Binomial Theorem.** $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$.
4. **Theorem.** If $a \equiv b \pmod{m}$, then $\gcd(a,m) = \gcd(b,m)$.
5. **Euler's Theorem.** If $\gcd(a,m) = 1$, then $a^{\phi(m)} \equiv 1 \pmod{m}$.
6. **Fermat's Little Theorem.** Let $p$ be a prime. Then:
   (a) $\forall a \in \mathbb{Z}$ and $a$ not a multiple of $p$, then $a^{p-1} \equiv 1 \pmod{p}$.
   (b) $\forall a \in \mathbb{Z}$, $a^p \equiv a \pmod{p}$.
7. **Wilson's Theorem.** If $p$ is prime, then $(p-1)! \equiv -1 \pmod{p}$.
8. **Solvability of $x^2 \equiv -1 \pmod{p}$.** Let $p$ be a prime. Then, $x^2 \equiv -1 \pmod{p}$ hasa solution $x \in \mathbb{Z}$ iff $p = 2$ or $p \equiv 1 \pmod{4}$.
9. **Fermat's Theorem on Sum of Squares.** Let $p$ be a prime such that $p \equiv 1 \pmod{4}$. Then $p$ can be written as $p = a^2 + b^2$ with $a, b \in \mathbb{Z}$.
10. **Solving Degree 1 Congruences.** Let $a, b \in \mathbb{Z}$ and let $g = \gcd(a,m)$. Then:
    (a) The congruence $ax \equiv b \pmod{m}$ has a solution iff $g \mid b$.
    (b) If (a) is true, then $\frac{a}{g} x \equiv \frac{b}{g} \pmod{\frac{m}{g}}$ has a solution modulo $\frac{m}{g}$.
11. **Chinese Remainder Theorem.** If $x \equiv a_1 \pmod{m_1}, \ldots, x \equiv a_k \pmod{m_k}$ (where the $m_i$'s are pairwise relatively prime), then let $M = m_1 m_2 \cdots m_k$ and $y_i = \text{inverse}\left(\frac{M}{m_i} \pmod{m_i}\right)$. Then, a solution to the simultaneous congruence is given by $x \equiv a_1 \frac{M}{m_1} y_1 + \cdots + a_k \frac{M}{m_k} y_k \pmod{M}$.
12. **Theorem.** If $m \in \mathbb{Z}_{>0}$, then $\phi(m) = \left(\prod_{p \text{ prime}, p \mid m} (1 - \frac{1}{p})\right) \cdot m$.
13. **RSA Cryptography Lemma.** Suppose $m \in \mathbb{Z}_{>0}$ and $\gcd(a,m) = 1$. Let $h, h' \in \mathbb{Z}_{>0}$ such that $hh' \equiv 1 \pmod{\phi(m)}$. Then $a^{kk'} \equiv a \pmod{m}$.
14. **Primality Testing.** If there is an integer $a$ such that $0 < a < m$ and $a^{m-1} \not\equiv 1 \pmod{m}$, then $m$ is not prime.

15. **Hensel's Lemma.** To solve the congruence $f(x) \equiv 0 \pmod{p^k}$, first find the solutions to $f(x) \equiv 0 \pmod{p}$. Then, for each solution $a_1$ to $f(x) \equiv 0 \pmod{p}$, "lift" its solution by the recurrence relation $a_2 = a_1 - f(a_1)\overline{f'(a_1)}$, where $\overline{f'(a_1)}$ is found by solving $f'(a_1)\overline{f'(a_1)} \equiv 1 \pmod{p}$ for $\overline{f'(a_1)}$. To higher powers, we generalize this recurrence relation to $a_{j+1} = a_j - f(a_j)\overline{f'(a_1)}$.

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16. **Hensel's Lemma (General Case).** Let $f \in \mathbb{Z}[x]$, $a \in \mathbb{Z}$, $j \in \mathbb{Z}_{>0}$, and $\tau \in \mathbb{N}$. Assume that $f(a) \equiv 0 \pmod{p^j}$, $p^\tau \,||\, f'(a)$ and $j \geq 2\tau + 1$. Then:
    (a) There is a $\tau \in \mathbb{Z}$, unique modulo $p$, such that $f(a + tp^{j-\tau}) \equiv 0 \pmod{p^{j+1}}$.
    (b) If $b \equiv a \pmod{p^{j-\tau}}$, then $f(b) \equiv f(a) \pmod{p^j}$ and $p^j \,||\, f'(b)$.
17. **Corollary to Hensel's Lemma.** Let $f \in \mathbb{Z}[x]$, $p$ be prime, $a \in \mathbb{Z}$, $\tau \in \mathbb{N}$, and let $l \in \mathbb{Z}$. Assume that $p^\tau \,||\, f'(a)$, $f(a) \equiv 0 \pmod{p^l}$, and $l \geq 2\tau + 1$. Then, for any $\alpha \geq l$, there exists a $b \in \mathbb{Z}$, unique modulo $p^{\alpha-\tau}$, such that $b \equiv a \pmod{p^{l-\tau}}$ and $f(b) \equiv 0 \pmod{p^\alpha}$.
18. **Lemma.** Let $f \in \mathbb{Z}[x]$ and $p$ prime. Assume that $a_1, \ldots, a_r$ are roots of $f \pmod{p}$, with $r > 0$ and $a_i \equiv a_j \pmod{p}$ for all $i \neq j$. Then there is a polynomial $g \in \mathbb{Z}[x]$ such that $f(x) \equiv (x - a_1)g(x) \pmod{p}$. Also, for any such $g$, $a_1, \ldots, a_r$ are roots of $g$ modulo $p$.
19. **Theorem.** If $f(x) \equiv 0 \pmod{p}$ has (at least) $r$ solutions $x \equiv a_1, \ldots, a_r \pmod{p}$, with $a_i \not\equiv a_j \pmod{p}$ (for all $i \neq j$), then there is a polynomial $q \in \mathbb{Z}[x]$ such that $f(x) \equiv (x - a_1)\cdots(x - a_r)q(x) \pmod{p}$.
20. **Theorem 2.26.** The congruence $f(x) \equiv 0 \pmod{p}$ of degree $n \geq 0$ has at most $n$ solutions.
21. **Corollary 2.27.** If $f \in \mathbb{Z}[x]$ has degree $n \geq 0$ (thus, $f \neq 0$), and the congruence $f(x) \equiv 0 \pmod{p}$ has more than $n$ distinct solutions, then $f \equiv 0 \pmod{p}$ (as polynomials).
22. **Lemma.** Let $f \in \mathbb{Z}[x]$ be a monic polynomial of degree $n$. If the congruence $f(x) \equiv 0 \pmod{p}$ has $n$ solutions, $x \equiv a_1, \ldots, a_n \pmod{p}$, distinct modulo $p$, then $f(x) \equiv (x - a_1)\cdots(x - a_n) \pmod{p}$.
23. **Proposition.** Let $f \in \mathbb{Z}[x]$. Then there is a well-defined function $\tilde{f}$ with $\tilde{f} : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ given by $\tilde{f}(\tilde{a}) = \widetilde{f(a)}$ for all $\tilde{a} \in \mathbb{Z}/m\mathbb{Z}$.
24. **Proposition.** Let $f, g \in \mathbb{Z}[x]$. If $f \equiv g \pmod{m}$ (as polynomials), then $\tilde{f} = \tilde{g}$ (as functions $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$).
25. **Corollary.** Let $\psi : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ be any function. If $\psi$ can be given by a polynomial (i.e. $\psi = f$ for some $f \in \mathbb{Z}[x]$), then it can be given by a polynomial of degree less than $p$.
26. **Theorem 2.28.** If $F : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$, then there is a polynomial $f \in \mathbb{Z}[x]$ with degree at most $p - 1$ such that $F(x) \equiv f(x) \pmod{p}$ for all residue classes $x$ modulo $p$.
27. **Theorem.** The polynomials in Theorem 2.28 are unique modulo $p$.
28. **Corollary 2.30.** Suppose $d > 0$ and $d \mid (p - 1)$, then the congruence $x^d \equiv 1 \pmod{p}$ has exactly $d$ solutions.
29. **Proposition.** Let $a$ have order $h$ modulo $m$, and let $n \in \mathbb{N}$. Then, $a^n \equiv 1 \pmod{m}$ iff $n$ is a multiple of $h$.
30. **Corollary.** If $a$ has order $h$ (modulo $m$), then $h \mid \phi(m)$.
31. **Corollary.** Let $m, m' \in \mathbb{Z}_{>0}$ and $a \in \mathbb{Z}$. Assume that $a$ has orders $h$ and $h'$ modulo $m$ and $m'$, respectively (i.e. $\gcd(a, m) = \gcd(a, m') = 1$). Then, if $m \mid m'$, then $h \mid h'$.
32. **Proposition.** Suppose $g$ is a primitive root modulo $m$. Then:
    (a) $1, g, \ldots, g^{\phi(m)-1}$ are distinct modulo $m$.
    (b) The above numbers are a reduced residue system modulo $m$.

(c) Let $a \in \mathbb{Z}$, with $\gcd(a,m) = 1$. Then there exists an $i \in \mathbb{Z}$ such that for all $j \in \mathbb{N}$, $g^j \equiv a \pmod{m}$ iff $j \equiv i \pmod{\phi(m)}$.

33. If there exists a primitive root $g$ modulo $m$, then you have a theory of <u>discrete logarithms</u> modulo $m$.

34. **Generalization of Corollary 2.30.** Assume that there is a primitive root modulo $m$ and let $d$ be a positive divisor of $\phi(m)$. Then, the congruence $x^d \equiv 1 \pmod{m}$ has exactly $d$ solutions.

35. **Generalization of Theorem 2.37.** Assume that there is a primitive root modulo $m$ and let $n \in \mathbb{Z}_{>0}$, and let $a \in \mathbb{Z}$ coprime to $m$. Then the congruence $x^n \equiv a \pmod{m}$ has $\gcd(n, \phi(m))$ solutions if $a^{\phi(m)/\gcd(n,\phi(m))} \equiv 1 \pmod{m}$ or has no solutions otherwise.

36. **Euler's Criterion.** Let $p$ be an odd prime and let $a \in \mathbb{Z}$ with $p \nmid a$. Assume that there is a primitive root modulo $p$. Then, $x^2 \equiv a \pmod{p}$ has two solutions if $a^{\frac{p-1}{2}} \equiv 1 \pmod{p}$ or has no solutions otherwise.

37. **Lemma.** Suppose $a \in \mathbb{Z}$ has order $h$ modulo $m$. Then:

    (a) If $d > 0$, and $d \mid h$, then $a^d$ has order $\frac{h}{d}$ modulo $m$.

    (b) For all $k \in \mathbb{N}$, $a^k$ has order $\frac{h}{\gcd h,k} \pmod{m}$.

38. **Corollary.** If there is a primitive root modulo $m$, then there are $\phi(\phi(m))$ of them (as residue classes modulo $m$).

39. **Lemma.** Suppose that $a, b \in \mathbb{Z}$ have order $h$ and $k$, respectively modulo $m$, and that $h$ and $k$ are coprime. Then, $ab$ has order $hk$ modulo $m$.

40. If $p$ is prime, then there exists a primitive root modulo $p$.

41. **Lemma.** Let $m, m' \in \mathbb{Z}_{>0}$ with $m \mid m'$. Let $a \in \mathbb{Z}$, with $\gcd(a, m') = 1$. Then:

    (a) $\gcd(a,m) = 1$.

    (b) If $h, h'$ are the orders of $a$ and $m'$ modulo $m$, respectively, then $h \mid h'$.

42. **Theorem.** Let $p$ be an odd prime and let $\alpha \in \mathbb{Z}_{>0}$. THen there exists a primitive root modulo $p^\alpha$.

43. **Theorem 2.41** There exists a primitive root modulo $m$ iff $m = 1, 2, 4, p^\alpha, 2p^\alpha$, where $\alpha \in \mathbb{Z}_{>0}$ and $p$ an odd prime.

44. **Diffie-Hellman Key Exchange.** This is a process used in order to initialize the secure line before message transfers occur. Suppose Alice and Bob are the participants. Then:

    (a) They (publicly) agree on a large prime $p$ (600 digits...) and a primitive root $g$ modulo $p$.

    (b) Alice thinks up a number $a$, $1 < a < p-1$, and sends $g^a \underline{\text{mod}} p$ to Bob.

    (c) Bob thinks up a number $b$, $1 < b < p-1$, and sends $g^b \underline{\text{mod}} p$ to Alice.

    (d) Alice computes $(g^b)^a \underline{\text{mod}} p$ and Bob computes $(g^a)^b \underline{\text{mod}} p$, which becomes their shared key.

45. **Solving Quadratic Congruences Modulo $p \neq 2$.** Let $ax^2 + bx + c \equiv 0 \pmod{p}$ be a quadratic congruence with $a \not\equiv 0 \pmod{p}$. First, multiply it by $\overline{a} \underline{\text{mod}} p$ to get $x^2 + \overline{a}bx + \overline{a}c \equiv 0 \pmod{p}$. Then, complete the square to get $(x + \overline{2}\overline{a}b)^2 + \overline{a}c - (\overline{2}\overline{a}b)^2$. Then, solve the resulting congruence.

46. **Theorem 3.1.** Let $a, b \in \mathbb{Z}$. Then:

    (a) $\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod{p}$.

    (b) $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$.

    (c) If $a \equiv b \pmod{p}$, then $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$.

(d) If $p \nmid a$, then $\left(\frac{a^2}{p}\right) = 1$ and $\left(\frac{a^2 b}{p}\right) = \left(\frac{b}{p}\right)$.

(e) $\left(\frac{1}{p}\right) = 1$ and $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$.

47. **Theorem 3.2 (Lemma of Gauss).** Let $a \in \mathbb{Z}$ be relatively prime to $p$ and let $n$ be the number of elements of the set $\{j \in \{1,2,\ldots,p-1\} \mid ja\underline{\mathrm{mod}}p > \frac{p}{2}\}$. Then, $\left(\frac{a}{p}\right) = (-1)^n$.

48. **Part of Theorem 3.3.** Let $a \in \mathbb{Z}$ be relatively prime to $p$. Assume also that $a$ is odd. Let $t = \sum_{j=1}^{(p-1)/2} \left[\frac{ja}{p}\right]$. Then, $\left(\frac{a}{p}\right) = (-1)^t$.

49. **Quadratic Reciprocity.** We have:

    (a) $\left(\frac{-1}{p}\right) = 1$ if $p \equiv 1 \pmod 4$ and $-1$ if $p \equiv -1 \pmod 4$.

    (b) $\left(\frac{2}{p}\right) = 1$ if $p \equiv \pm 1 \pmod 8$ and $-1$ if $p \equiv \pm 3 \pmod 8$.

    (c) For all odd primes $p, q$ with $p \neq q$, we have that $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}$.

50. **Variation of Theorem 3.4.** If $p, q$ are odd primes, then $\left(\frac{q}{p}\right) =$

    (a) $\left(\frac{p}{q}\right)$ if $p \equiv 1 \pmod 4$ or $q \equiv 1 \pmod 4$.

    (b) $-\left(\frac{p}{q}\right)$ if $p \equiv q \equiv -1 \pmod 4$.

51. **Quadratic Reciprocity for Jacobi Symbols** Let $Q$ be an odd positive integer. Then:

    (a) $\left(\frac{-1}{Q}\right) = (-1)^{\frac{Q-1}{2}}$.

    (b) $\left(\frac{2}{Q}\right) = (-1)^{\frac{Q^2-1}{8}}$.

    (c) If $P$ is an odd positive integer, then $\left(\frac{P}{Q}\right) = (-1)^{\frac{P-1}{2}\cdot\frac{Q-1}{2}}\left(\frac{Q}{P}\right)$

52. **Lemma.** For all odd positive integers $a$ and $b$, we have $\frac{ab-1}{2} \equiv \frac{a-1}{2} + \frac{b-1}{2} \pmod 2$, so therefore, $(-1)^{\frac{ab-1}{2}} = (-1)^{\frac{a-1}{2}}(-1)^{\frac{b-1}{2}}$.

53. For all odd positive integers $a$ and $b$, we have $\frac{(ab)^2-1}{8} \equiv \frac{a^2-1}{8} + \frac{b^2-1}{8} \pmod 2$, so therefore, $(-1)^{\frac{(ab)^2-1}{8}} = (-1)^{\frac{a^2-1}{8}}(-1)^{\frac{b^2-1}{8}}$.

54. **Theorem 3.10.** Let $f(x,y) = ax^2 + bxy + cy^2$ be a nonzero binary quadratic form with integer coefficients, and let $d = b^2 - 4ac$ be its discriminant. Then:

    (a) If $d$ is a perfect square (including 0), then $f$ can be factored into two linear factors with integer coefficients.

    (b) If $d$ is not a perfect square, then $f$ cannot be factored into linear factors with rational coefficients.

55. **Theorem.** Let $d \in \mathbb{Z}$. Then there exists a binary quadratic form of discriminant $d$ iff $d \equiv 0 \pmod 4$ or $d \equiv 1 \pmod 4$.

56. **Theorem 3.13.** Let $n, d \in \mathbb{Z}$ with $n \neq 0$. Then there exists a form of discriminant $d$ that properly represents $n$ iff the congruence $x^2 \equiv d \pmod{4|n|}$ has a solution.

57. **Corollary.** Let $p$ be an odd prime and $d \in \mathbb{Z}$. Then there is a form of discriminant $d$ that (properly) represents $p$ iff $\left(\frac{d}{p}\right) = 0$ or 1.

58. **Corollary.** Let $p$ be a prime. Then there exists a binary quadratic form of discriminant $-4$ that represents $p$ iff p is represented by $x^2+y^2$.

59. **Theorem.** Let $f$ be a positive definite quadratic form of discriminant -4. Then an integer $n$ is represented by $f$ iff it is represented by $x^2+y^2$.

60. **Theorem.** Let $d \in \mathbb{Z}$ and assume $d \equiv 0 \pmod 4$ or $d \equiv 1 \pmod 4$. Then there is a finite list $f_1, \ldots, f_n$ of forms of discriminant $d$ such that for all $n \in \mathbb{Z}$, $n$ is represented by some form of discriminant $d$ iff $f$ is represented by one of $f_1, \ldots, f_n$.

61. **Theorem.** For any $d \equiv 0$ (mod 4) or $d \equiv 1$ (mod 4), there are infinitely many forms of discriminant $d$.

62. **Theorem.** Let $a,b,c,d \in \mathbb{R}$ and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then:

    (a) $T_M(\mathbb{Z}^2) \subseteq \mathbb{Z}^2$ (i.e. $T_M(x,y) \in \mathbb{Z}^2$ for all $(x,y) \in \mathbb{Z}^2$) iff $a,b,c,d \in \mathbb{Z}$.

    (b) $T_M$ maps $\mathbb{Z}^2$ bijectively to $\mathbb{Z}^2$ iff $a,b,c,d \in \mathbb{Z}$ and $\det M = \pm 1$.

63. **Theorem.** Let $\sim$ be the relation that determines whether two binary quadratic forms are equivalent. Then, $\sim$ is an equivalence relation.

64. **Theorem.** Let $f,g$ be equivalent forms. Then:

    (a) $f$ and $g$ represent the same numbers.

    (b) $f$ and $g$ properly represent the same numbers.

65. **Reducing Quadratic forms** Begin with $f(x,y) = ax^2+bxy+cy^2$. Carry out the following procedure:

    (a) *Step 1:* If $|c| < |a|$, then $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \Gamma$ takes $f$ to $f(y,-x) = cx^2 - bxy + ay^2$. So, after doing this if necessary, we may assume that $|a| \leq |c|$.

    (b) *Step 2:* Notice that, for any $m \in \mathbb{Z}$, the matrix $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in \Gamma$ takes $f(x,y)$ to $f(x+my,y) = ax^2 + (2am+b)xy + (am^2+bm+c)y^2$. Choose $m$ (unique) such that $-|a| < 2am+b \leq |a|$.

    (c) *Step 3:* If $|c| < |a|$, go back to Step 1. Otherwise, continue.

    (d) *Step 4:*

    - If $|c| > |a|$, done; we have a reduced form.
    - If $|c| = |a|$ and $b \geq 0$, done; we have a reduced form.
    - If $|c| = |a|$ and $b < 0$, then use $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ again. Your form is reduced, because $0 < b < |a| = |c|$.

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