Compute $\left(\frac{-42}{61}\right)$.

For this, we utilize the following results.

Theorem 2.37*. Let $n = 2$ and $\gcd(a, p) = 1$ with $a \in \mathbb{Z}$. Then a is a quadratic residue modulo p iff $a^{(p-1)/2} \equiv 1 \pmod{p}$.

Theorem 3.1(2). The second part of Theorem 3.1 states that if $a, b \in \mathbb{Z}$ and p an odd prime, then

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$$

Theorem 3.3(2). The second part of Theorem 3.3 states that if p is an odd prime,

$$\left(\frac{2}{p}\right) = (-1)^{\left(p^2-1\right)/8}.$$

Theorem 3.4. The Gaussian reciprocity law. If p and q are distinct odd primes, then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}.$$

Solution: Set $p = 61$ and note $-42 = -1 \cdot 2 \cdot 3 \cdot 7$. By Theorem 3.1(2), we have

$$\left(\frac{-42}{61}\right) = \left(\frac{-1}{61}\right)\left(\frac{2}{61}\right)\left(\frac{3}{61}\right)\left(\frac{7}{61}\right)$$

.

If $a = -1$, $a^{(p-1)/2} = (-1)^{30} = 1 \equiv 1 \pmod{61}$. By Theorem 2.37*, -1 is a quadratic residue modulo $p = 61$, so $\left(\frac{-1}{61}\right) = 1$. If $a = 2$, then by Theorem 3.3(2), $\left(\frac{2}{61}\right) = (-1)^{(61^2-1)/8} = -1$. If $a = 3$, by Theorem 3.4, $\left(\frac{3}{61}\right)\left(\frac{61}{3}\right) = (-1)^{\frac{2}{2}\cdot\frac{60}{2}} = 1$. Since both 3 and 61 are prime, $\left(\frac{61}{3}\right) = \pm 1$, so our equation becomes $\left(\frac{3}{61}\right) = \left(\frac{61}{3}\right) \cdot 1 = \left(\frac{61}{3}\right)$. Theorem 2.37* yields $61^{2/2} = 61 \equiv 1 \pmod{3}$, so by definition, $\left(\frac{61}{3}\right) = 1 = \left(\frac{3}{61}\right)$. If $a = 7$, then Theorem 3.4 gives $\left(\frac{7}{61}\right)\left(\frac{61}{7}\right) = (-1)^{\frac{6}{2}\cdot\frac{60}{2}} = (-1)^{90} = 1$. Using a similar argument as the previous sentence gives $\left(\frac{7}{61}\right) = \left(\frac{61}{7}\right)$. Then, $61^{6/2} = 61^3 = 226981 \equiv 6 \not\equiv 1 \pmod{7}$, so $\left(\frac{61}{7}\right) = -1 = \left(\frac{7}{61}\right)$. By multiplication, we have $\left(\frac{-42}{61}\right) = 1$. $\square$