

1. **Complex Numbers,  $\mathbb{C}$ .** The set of complex numbers  $\mathbb{C}$  is the real vector space  $\mathbb{R}^2$  with the properties  $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$  and  $a(x_1, y_1) = (ax_1, ay_1)$  and we write  $z = x + iy = (x, y)$  for any  $z \in \mathbb{C}$ .
2. **Polar Form.** Let  $z = a + bi \in \mathbb{C}$ . Then norm (a.k.a. modulus, absolute value) of  $z \in \mathbb{C}$  is written  $|z| \in \mathbb{R}$  and is defined by  $|z| = \sqrt{a^2 + b^2}$ . Then, define the argument of  $z$  as  $\arg(z) = \theta \in [0, 2\pi)$  as the angle  $z$  makes with the real axis. Then, the polar form of  $z$  is written as  $z = |z|(\cos \theta + i \sin \theta)$ .
3. **Rectangular form.** If  $z \in \mathbb{C}$  is written as  $z = x + iy$ , with  $x, y \in \mathbb{R}$ , then  $z$  is written in rectangular form.
4. **Complex Conjugate.** If  $z = a + ib$ , then its complex conjugate is  $\bar{z} = a - ib$ .
5. **Open Sets.** A set  $\Omega \subseteq \mathbb{C}$  is called open if for each  $z_0 \in \Omega$ , there is an  $\varepsilon > 0$  such that  $D_\varepsilon(z_0) \subseteq \Omega$ .
6. **Neighborhoods.** An  $\varepsilon$ -neighborhood of a point  $z_0$  is a set  $N$  which contains some open disk  $D_\varepsilon(z_0)$ .
7.  **$\varepsilon$ -deleted neighborhoods.** An  $\varepsilon$ -deleted neighborhood of a point  $z_0$  is a set  $N$  which contains a "punctured" open disk  $D_\varepsilon(z_0) \setminus \{z_0\}$ .
8. **Homeomorphism.** A function is a homeomorphism if it is continuous with a continuous inverse.
9. **Periodic.**  $f(z)$  is  $w$ -periodic (with  $w \in \mathbb{C}$ ) if  $f(z + nw) = f(z)$  for all  $z \in \mathbb{C}, n \in \mathbb{Z}$ .
10. **Limits.** Let  $f : \Omega \rightarrow \mathbb{C}$  where  $\Omega$  is an  $r$ -deleted neighborhood of a point  $z_0$ . Then  $f$  has a limit as  $z \rightarrow z_0$ , and write  $\lim_{z \rightarrow z_0} f(z) = a$ . This means that for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that if  $z \in \Omega$  has  $|z - z_0| < \delta$ , then  $|f(z) - a| < \varepsilon$ .
11. **Continuity.** Let  $\Omega \subseteq \mathbb{C}$  be an open set. Then  $f : \Omega \rightarrow \mathbb{C}$  is continuous at a point  $z_0 \in \Omega$  if  $\lim_{z \rightarrow z_0} f(z) = f(z_0)$ .
12. **Closed Sets.** A subset  $F \subseteq \mathbb{C}$  is called closed if its complement  $\mathbb{C} \setminus F$  is open.
13. **Compact.** A subset  $K \subseteq \mathbb{C}$  is called compact if every open cover of  $K$  has a finite subcover.
14. **Uniform Convergence.** A sequence of functions  $f_n : \Omega \rightarrow \mathbb{C}$  converges uniformly to a function  $f : \Omega \rightarrow \mathbb{C}$  if for all  $\varepsilon > 0$  there exists an  $N \in \mathbb{N}$  such that if  $n \geq N$ , then  $|f_n(z) - f(z)| < \varepsilon$  for all  $z \in \Omega$ .
15. **Real Differentiable.**  $f$  is real differentiable at  $z_0$  if the following limit exists:  $\lim_{h \rightarrow 0} \frac{f(z+h) - f(z) - D(f'(z))h}{h}$ , where  $D(f'(z))$  is the Jacobian matrix.
16. **Derivative.** Let  $f : \Omega \rightarrow \mathbb{C}$ , where  $\Omega$  is a neighborhood of  $z_0$ . The derivative of  $f$  at  $z_0$  is the limit  $f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$ . We call  $f$  **complex differentiable** at  $z_0$  if this limit exists. If  $f'(z_0)$  exists for all  $z_0 \in \Omega$ , we call  $f$  **holomorphic** on  $\Omega$ .
17. **Branch of the Argument.** This is a choice of interval (here,  $[-\pi, \pi)$  or  $[a, a + 2\pi)$ ).
18. **Principal Branch of the Logarithm.** Pick a branch  $[a, a + 2\pi)$ . Then,  $\log : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{R} \times i[a, a + 2\pi)$  is defined by  $\log z = \log |z| + i \arg z$ , where  $\arg z \in [a, a + 2\pi)$ . We call the branch  $[-\pi, \pi)$  the principal branch.
19. **Exponentiation of complex numbers  $a, b$ .** Choose a branch of log, with  $\log : \Omega \rightarrow \mathbb{C}$  and  $a, b \in \mathbb{C}$ . Then, define  $a^b := e^{b \log a}$ .
20. **Contour Integral.** Suppose  $f$  is continuous on an open set  $\Omega$  and  $\gamma : [a, b] \rightarrow \Omega$  is a smooth curve. Then the contour integral of  $f$  along  $\gamma$  is defined to be  $\int_\gamma f := \int_\gamma f(z) dz := \int_a^b f(\gamma(t)) \gamma'(t) dt$ .
21. **Re-parametrization of  $\gamma$ .** A piecewise smooth  $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \rightarrow \mathbb{C}$  is called a re-parametrization of  $\gamma$  if there exists a continuously differentiable  $\alpha : [a, b] \rightarrow [\tilde{a}, \tilde{b}]$  such that  $\alpha(a) = \tilde{a}$  and  $\alpha(b) = \tilde{b}$ , and  $\alpha'(t) > 0$  with  $\gamma(t) = \tilde{\gamma}(\alpha(t))$ .
22. **Primitive.** We say a function  $f : \Omega \rightarrow \mathbb{C}$  has a primitive on  $\Omega$  if there exists a holomorphic function  $F : \Omega \rightarrow \mathbb{C}$  such that  $F'(z) = f(z)$  for all  $z \in \Omega$ .
23. **Path-connected.** We say an open set  $\Omega \subseteq \mathbb{C}$  is path-connected if for any pair of points  $z_0, z_1 \in \Omega$  there exists a continuous path  $\gamma : [0, 1] \rightarrow \Omega$  such that  $z_0 = \gamma(0)$  and  $z_1 = \gamma(1)$ , with  $\gamma([0, 1]) \subseteq \Omega$ .
24. **Path-independence.** If  $z_0, z_1 \in \Omega$ , then any paths  $\gamma, \tilde{\gamma}$  (with shared endpoints  $\gamma(0) = \tilde{\gamma}(0)$ ,  $\gamma(1) = \tilde{\gamma}(1)$ ) have  $\int_\gamma f(z) dz = \int_{\tilde{\gamma}} f(z) dz$ .
25. **Homotopy.** Let  $\gamma_{0,1} : [a, b] \rightarrow \mathbb{C}$  be curves with shared endpoints  $z_a, z_b$ . A homotopy is a continuous function  $H : [a, b] \times [0, 1] \rightarrow \mathbb{C}$  with  $t \times s \rightarrow H_s(t)$  such that  $H_0(t) = \gamma_0(t)$  and  $H_1(t) = \gamma_1(t)$ .
26. **Simply-connected.** A set  $A$  is called simply-connected if every closed curve (loop) is homotopic to a point in  $A$ , with  $H_s(t) \in A$  for all  $s, t$ . (Note: a point in  $A$  is a constant loop).
27. **Winding Number.** Let  $\gamma$  be a loop in  $\mathbb{C}$  and  $z_0 \in \mathbb{C}$  but not on  $\gamma$ . Then the winding number of  $\gamma$  (with respect to  $z_0$ ) is  $I(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} dz$ .
28. **Entire.** We call a function  $f : \mathbb{C} \rightarrow \mathbb{C}$  entire if it is holomorphic on  $\mathbb{C}$ .
29. **Closure.** The closure of a set  $A$ , written  $\bar{A}$ , is  $\bar{A} = \{\text{limit points of } A\}$ .
30. **Boundary of  $A$ .** The boundary  $\partial A$  of a set  $A \subseteq \mathbb{C}$  is  $\partial A = \bar{A} \cap \overline{(\mathbb{C} \setminus A)}$ .
31. **Reflection:**  $\tilde{z} = \frac{R^2}{\bar{z}}$ .  $\tilde{z} = \frac{R^2}{\bar{z}}$  is the reflection over the line the circle  $|\xi| = R$ .
32. **Analytic.** A function  $f : \Omega \rightarrow \mathbb{C}$  is analytic at  $z_0 \in \Omega$  if there is a neighborhood  $\mathcal{U}$  of  $z_0$  on which  $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$  (for all  $z \in \mathcal{U}$ ) where the RHS is a convergent power series.
33. **Pole of Order  $m$ .** If  $f$  is holomorphic on a deleted neighborhood  $\mathcal{U} \setminus z_0$ , we say  $f$  has a pole of order  $m$  if  $\frac{1}{f}$  has a zero of order  $m$ .
34. **Simple Pole.** If  $f$  has a pole of order 1 at  $z = z_0$ , then  $f$  has a simple pole at  $z_0$ .
35. **Residue of  $f$  at  $z = z_0$ .** Consider the principal part of the Laurent expansion of  $f$  at  $z = z_0$ . Then, the coefficient  $a_{-1}$  is the residue of  $f$  at  $z_0$  and we write it as  $\text{Res}_{z_0}(f) = a_{-1}$ .
36. **Meromorphic.** A function  $f : \Omega \rightarrow \mathbb{C}$  is meromorphic if it is holomorphic on all of  $\Omega$  except at a discrete set of poles.
37. **Essential Singularity.** Let  $f$  be holomorphic on  $\Omega$  except at a point  $z_0$ . We call  $z_0$  an essential singularity if  $z_0$  is neither a pole nor a removable singularity.
38. **Alternative Definition of Essential Singularity.** Let  $f$  be holomorphic except possibly at a point  $z_0$ . Let  $C_1 = \{z_0\}$  and  $C_2 = \partial D_r(z_0)$ . Then,  $z_0$  is an essential singularity if there are infinitely many  $a_{-n}$  in the Laurent series of  $f$ , where still  $\text{Res}_{z_0}(f) = a_{-1}$ .
39. **Holomorphic at Infinity.** Let  $\mathcal{U} \subseteq \mathbb{C}$  be an open set containing  $\mathbb{C} \setminus \overline{D_R(0)}$ . A function  $f : \mathcal{U} \rightarrow \mathbb{C}$  is holomorphic at infinity if  $g(z) = f(1/z)$  has a removable singularity at 0, which in that case, we define  $f(\infty) = g(0)$ .
40. **Zero (respectively, pole) of order at Infinity.** Let  $\mathcal{U} \subseteq \mathbb{C}$  be an open set containing  $\mathbb{C} \setminus \overline{D_R(0)}$ . We say that  $f : \mathcal{U} \rightarrow \mathbb{C}$  has a zero (respectively, pole) of order at  $\infty$  if  $g(z) = f(1/z)$  has a zero (respectively, pole) at  $z = 0$  (of order  $m$ ).
41. **Logarithmic Derivative.** Let  $f : \Omega \rightarrow \mathbb{C}$  be meromorphic. Then, the logarithmic derivative of  $f$  is  $f'/f$ .
42. **Locally injective.** We call a function  $f : \Omega \rightarrow \mathbb{C}$  locally injective near  $z_0$  if there exists a neighborhood  $\mathcal{U}$  of  $z_0$  such that  $f : \mathcal{U} \rightarrow \mathbb{C}$  is injective.
43. **Infinite Product.** Suppose the sequence of finite products  $P_N := \prod_{n=1}^N c_n = c_1 c_2 \dots c_N$  converges to a finite number (where  $c_i \in \mathbb{C}$  for all  $i$ ). We define  $\prod_{n=1}^{\infty} c_n = \lim_{N \rightarrow \infty} \prod_{n=1}^N c_n$  to be the infinite product, and say that this infinite product converges.
44. **Riemann-Zeta Function.** We define the Riemann-Zeta function to be  $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ .
45. **Conformal Maps.** Smooth, invertible maps  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  whose Jacobian at a point can be factored as (scaling)  $\cdot$  (rotation) are called conformal maps. (note: by Cauchy-Riemann equations, we have that  $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = |a + bi|^2 \cdot \frac{1}{|a + bi|^2} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ , where the  $|a + bi|^2$  factor represents scaling and the other factors together represent the orthogonal scaling matrix).
46. **Conformal Maps (2nd definition).** Let  $\Omega$  and  $\Omega'$  be open connected regions in  $\mathbb{C}$ . We say that a map  $g : \Omega \rightarrow \Omega'$  is conformal if it is holomorphic and invertible with  $g^{-1}$  holomorphic.
47. **Conformally Equivalent.** We call  $\Omega, \Omega'$  conformally equivalent (write:  $\Omega \sim \Omega'$ ) if there exists a conformal map  $g : \Omega \rightarrow \Omega'$ .
48. **Set of holomorphic functions / meromorphic functions.** Let  $\mathcal{H}(\Omega) = \{\text{holomorphic functions } f : \Omega \rightarrow \mathbb{C}\}$  and  $\mathcal{M}(\Omega) = \{\text{meromorphic functions } f : \Omega \rightarrow \mathbb{C}\}$ .
49. **Conformal Automorphisms.** We call a conformal map  $g : \Omega \rightarrow \Omega$  a conformal automorphism and write  $\text{Aut}(\Omega)$  to denote the collection of automorphisms (strictly conformal) on  $\Omega$ .
50. **Prop.** Let  $z_1, z_2 \in \mathbb{C}$ . Then  $|z_1 z_2| = |z_1| |z_2|$  and  $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \pmod{2\pi}$ .
51. **Theorem.**  $\mathbb{C}$  is a field.
52. **De Moivre's Formula.** If  $z = r(\cos \theta + i \sin \theta)$  and  $n \in \mathbb{Z}_{>0}$ , then  $z^n = r^n(\cos n\theta + i \sin n\theta)$ .
53. **Prop.** Let  $z, w \in \mathbb{C}$ . Then:
  - (a)  $\overline{z + w} = \bar{z} + \bar{w}$ .
  - (b)  $\overline{zw} = \bar{z} \cdot \bar{w}$ .
  - (c)  $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$  (with  $w \neq 0$ ).
  - (d)  $z\bar{z} = |z|^2$ . If  $z \neq 0$ , then  $z^{-1} = \frac{\bar{z}}{|z|^2}$ .
  - (e) If  $z = \bar{z}$ , then  $z \in \mathbb{R}$  and so  $z = \text{Re}(z)$ .
  - (f)  $\text{Re}(z) = \frac{z + \bar{z}}{2}$  and  $\text{Im}(z) = \frac{z - \bar{z}}{2i}$ .
  - (g)  $\bar{\bar{z}} = z$ .
54. **Prop.** Let  $z, w \in \mathbb{C}$ . Then:
  - (a)  $|z| \geq 0$  and if  $|z| = 0$ , then  $z = 0$ .
  - (b)  $|zw| = |z| |w|$ .
  - (c) If  $w \neq 0$ , then  $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ .
  - (d)  $|\text{Re}(z)| \leq |z|$  and  $|\text{Im}(z)| \leq |z|$ .
  - (e)  $|\bar{z}| = |z|$ .
  - (f)  $|z + w| \leq |z| + |w|$ .
  - (g)  $||z| - |w|| \leq |z - w|$ .
  - (h)  $|z_1 w_1 + \dots + z_n w_n| \leq \sqrt{|z_1|^2 + \dots + |z_n|^2} \sqrt{|w_1|^2 + \dots + |w_n|^2}$ .
55. **Prop.** Fix  $r > 0$  and  $z \in \mathbb{C}$ . The open disk  $D_\varepsilon(z_0)$  is an open set.
56. **Prop.** The following are true:
  - (a)  $\mathbb{C}$  is open.
  - (b) The empty set  $\emptyset$  is open.
  - (c) The union of open sets is open.
  - (d) The intersection of finitely many open sets is open.

57. **Prop.** Limits are unique (if they exist).

58. **Prop.** If  $\lim_{z \rightarrow z_0} f(z) = a$  and  $\lim_{z \rightarrow z_0} g(z) = b$ , then:

- (a)  $\lim_{z \rightarrow z_0} (f(z) + g(z)) = a + b$ .
- (b)  $\lim_{z \rightarrow z_0} ((f(z)g(z))) = ab$ .
- (c)  $\lim_{z \rightarrow z_0} \left( \frac{f(z)}{g(z)} \right) = \frac{a}{b}$  (with  $b \neq 0$ ).

59. **Prop.** The following are true:

- (a) If  $\lim_{z \rightarrow z_0} f(z) = a$  and  $h$  is continuous at  $a$ , then  $\lim_{z \rightarrow z_0} h(f(z)) = h(a)$ .
- (b) If  $f$  is continuous on an open set  $\Omega \subseteq \mathbb{C}$ , and  $h$  is continuous on  $f(\Omega)$ , then  $h \circ f$  is continuous on  $\Omega$ , with  $(h \circ f)(z) = h(f(z))$ .

60. **Prop.** The following are true:

- (a) The empty set  $\emptyset$  is closed.
- (b)  $\mathbb{C}$  is closed.
- (c) The intersection of a collection of closed sets is closed.
- (d) The union of finitely many closed sets is closed.

61. **Prop.** A set  $F$  is closed iff whenever  $z_1, z_2, z_3, \dots$  is a sequence of points in  $F$  converging to  $\lim_{k \rightarrow \infty} z_k = w$ , then  $w \in F$ .

62. **Prop.** If  $f: \mathbb{C} \rightarrow \mathbb{C}$ , TFAE:

- (a)  $f$  is continuous.
- (b) If  $F \subseteq \mathbb{C}$  is closed, then  $f^{-1}(F)$  is closed.
- (c) If  $\Omega$  is open, then  $f^{-1}(\Omega)$  is also open.

63. **Prop. (Heine-Borel + Sequential Compactness).** For  $K \subseteq \mathbb{C}$ , TFAE:

- (a)  $K$  is compact.
- (b)  $K$  is closed and bounded.
- (c) Every sequence of points in  $K$  has a convergent subsequence converging in  $K$  (sequentially compact).

64. **Prop.** If  $K$  is compact and  $f: K \rightarrow \mathbb{C}$  is continuous, then the image  $f(K)$  is compact.

65. **Theorem (Extreme Value Theorem).** If  $K$  is compact and  $f: K \rightarrow \mathbb{R}$  is continuous, then  $f$  attains its minimum and maximum.

66. **Stereographic Projection / Riemann Sphere.** Identify the plane  $\overline{\mathbb{C}} = S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ . If  $f: \mathbb{C} \rightarrow S^2 \setminus \{N\}$ , then we have  $(u, v) \mapsto \frac{1}{1+u^2+v^2} (2u, 2v, -1+u^2+v^2)$  is a homeomorphism (is continuous with continuous inverse)  $f^{-1}: S^2 \setminus \{N\} \rightarrow \mathbb{C}$  with  $(x, y, z) \mapsto \left( \frac{x}{1-z}, \frac{y}{1-z} \right)$ .

67. **Prop. (Uniform Convergence).** If  $f_n \rightarrow f$  uniformly and each  $f_n$  is continuous, then  $f$  is continuous.

68. **Euler's Formula.**  $e^{iz} = \cos z + i \sin z$  for all  $z \in \mathbb{C}$ .

69. **Theorem.**  $\cos z = \frac{e^{iz} + e^{-iz}}{2}$  and  $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ .

70. **Properties of  $e$ .** Let  $x, y \in \mathbb{R}$  and  $z, w \in \mathbb{C}$ . Then:

- (a)  $e^{z+w} = e^z e^w$ .
- (b)  $|e^{x+iy}| = |e^x e^{iy}| = |e^x| |e^{iy}| = e^x$ .
- (c)  $\arg(e^{x+iy}) = y \pmod{2\pi}$ .
- (d)  $e^z \neq 0$  for all  $z \in \mathbb{C}$ .
- (e)  $e^z = 1$  iff  $z = 2\pi in$  for some  $n \in \mathbb{Z}$ .
- (f)  $e^z = e^{z+2\pi ni}$ .

71. **Prop. (Chain Rule).** Let  $\Omega, A \subseteq \mathbb{C}$  be open sets, and let  $f: \Omega \rightarrow A, g: A \rightarrow \mathbb{C}$  be holomorphic functions. Then,  $g \circ f: \Omega \rightarrow \mathbb{C}$  is holomorphic and  $\frac{dg}{dz}(f \circ g)(z) = \frac{dg}{dw}(f(z)) \cdot \frac{df}{dz}(z)$ .

72. **Prop.** Let  $f: \Omega \rightarrow \mathbb{C}$  be holomorphic. Then  $f: \Omega \rightarrow \mathbb{R}^2$  is real differentiable at all  $(x, y) \in \Omega$ .

73. **Cauchy-Riemann Equations.** Let  $\Omega$  be an open set in  $\mathbb{C}$  and let  $f: \Omega \rightarrow \mathbb{C}$  be given by  $f(x, y) = u(x, y) + iv(x, y)$ . Then:

- (a)  $f'(z)$  exists at  $z \in \Omega$  iff  $f$  is real differentiable and  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$  and  $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$  (these are the Cauchy-Riemann equations).
- (b)  $f(z)$  is holomorphic on  $\Omega$  iff partials are continuous and satisfy the CR equations.
- (c) If  $f'(z_0)$  exists, then  $f'(z_0) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial y} - i \frac{\partial v}{\partial y} = \frac{1}{i} \frac{\partial f}{\partial y}$ .

74. **Inverse Function Theorem for  $\mathbb{R}^2$ .** If  $f: \Omega \rightarrow \mathbb{R}^2$  is continuously differentiable and the Jacobian  $Df(z_0)$  has  $\det(Df(z_0)) \neq 0$ , then there are neighborhoods  $U \ni z_0$  and  $V \ni f(z_0)$  such that  $f: U \rightarrow V$  is bijective with continuously differentiable  $f^{-1}: V \rightarrow U$  such that  $Df^{-1}(z_0) = [Df(z_0)]^{-1}$ , which is the inverse matrix of  $Df(z_0)$ .

75. **Inverse Function Theorem for  $\mathbb{C}$ .** Let  $f: \Omega \rightarrow \mathbb{C}$  be holomorphic (with continuous  $f'(z_0)$ ), and  $f'(z_0) \neq 0$  for some  $z_0 \in \Omega$ . Then there exists a neighborhood  $U \ni z_0$  and  $V \ni f(z_0)$  such that  $f: U \rightarrow V$  is bijective with holomorphic inverse  $f^{-1}: V \rightarrow U$  such that for all  $z_0 \in U$ ,  $\frac{df}{dw} f^{-1}(w) = \frac{1}{f'(w)}$  with  $w = f(z)$ .

76. **Prop.** Pick a branch  $[a, a + 2\pi)$ . Then  $\log z: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{R} \times i[a, a + 2\pi)$  is the inverse of  $\exp: \mathbb{R} \times i[a, a + 2\pi) \rightarrow \mathbb{C}$ .

77. **Prop.**  $\log: \mathbb{C} \setminus \mathbb{R}_{\leq 0} \rightarrow \mathbb{R} \times i(-\pi, \pi)$  is holomorphic with  $\frac{d}{dz} \log z = \frac{1}{z}$ .

78. **Prop.** If  $z_1, z_2 \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ , then  $\log(z_1 z_2) = \log z_1 + \log z_2 \pmod{2\pi i}$ .

79. **Prop.** By choosing different branches of log, we have the following:

- (a)  $a^b$  is independent of the branch iff  $b \in \mathbb{Z}$ .
- (b)  $a^b$  takes on exactly  $q$  different values iff  $b \in \mathbb{Q}$ , so  $b = \frac{p}{q}$  (with  $p, q$  coprime).
- (c)  $a^b$  takes on infinitely many values iff  $b$  is irrational or  $\text{Im}(b) \neq 0$ .

80. **Cor.** Choose a branch of log. Then the  $n$ th root function is given by  $z^{1/n} = e^{\log(z)/n}$ , where the  $n$ th root function has  $n$  branches.

81. **Prop.** Let  $a, b \in \mathbb{C}$ . Then:

- (a) For any choice of branch of log, the function  $z \mapsto a^z$  is holomorphic on  $\mathbb{C}$ , and  $z \mapsto (\log a) a^z$ .
- (b) Choose a branch of log. Then the function  $z \mapsto z^b$  is holomorphic on the domain of log with derivative  $z \mapsto b z^{b-1}$ .

82. **Prop. (Re-parametrization).** If  $\tilde{\gamma}$  is a re-parametrization of  $\gamma$ , then  $\int_{\tilde{\gamma}} f = \int_{\gamma} f$  for any continuous  $f$  on  $\Omega$ .

83. **Fundamental Theorem of Line Integrals.** Let  $F: \Omega \rightarrow \mathbb{C}$  be holomorphic on an open  $\Omega$  and let  $\gamma: [0, 1] \rightarrow \Omega$  be piecewise smooth. Then,  $\int_{\gamma} F'(z) dz = F(\gamma(1)) - F(\gamma(0))$ .

84. **Path-independence and Primitives Theorem.** Let  $f: \Omega \rightarrow \mathbb{C}$  be continuous and  $\Omega$  is open and connected. Then, TFAE:

- (a) (path-independence) if  $z_0, z_1 \in \Omega$ , then any paths  $\gamma, \tilde{\gamma}$  with shared endpoints  $\gamma(0) = \tilde{\gamma}(0)$  and  $\gamma(1) = \tilde{\gamma}(1)$  have  $\int_{\gamma} f(z) dz = \int_{\tilde{\gamma}} f(z) dz$ .
- (b) (integral along loops is 0) if  $\Gamma$  is a loop, with  $\Gamma(1) = \Gamma(0)$ , then  $\int_{\Gamma} f(z) dz = 0$ .
- (c) (f has a primitive) There is a primitive  $F$  for  $f$  on  $\Omega$ .

85. **Cauchy-Goursat Theorem.** Let  $f: \Omega \rightarrow \mathbb{C}$  be holomorphic on  $\Omega$ , simply connected, and open. Then for any loop  $\Gamma \subseteq \Omega$ ,  $\int_{\Gamma} f(z) dz = 0$ .

86. **Green's Theorem.** Let  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a vector field and let  $\gamma$  be a loop, and  $A$  a region in the loop  $\gamma$ . Let  $f(x, y) = (P(x, y), Q(x, y))$ . Then,  $\int_{\gamma} P(x, y) dx + Q(x, y) dy = \iint \text{curl } F dA = \iint \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$ .

87. **Prop.** If  $f(x + iy) = u(x, y) + iv(x, y)$ , then  $\int_{\gamma} f = \int_{\gamma} u dx - v dy + i \int_{\gamma} u dy + v dx$ .

88. **Cauchy-Goursat Theorem (Weaker Version).** Let  $f: \Omega \rightarrow \mathbb{C}$  be holomorphic with  $f'(z)$  continuous and  $\gamma: [0, 1] \rightarrow \mathbb{C}$  a simple closed curve and  $\Omega$  an open & simply connected set. Then,  $\int_{\gamma} f = 0$ .

89. **Cauchy-Goursat Theorem (for rectangles).** Let  $R$  be a rectangle with  $R$  and its interior are contained in an open set  $\Omega$ . Let  $f: \Omega \rightarrow \mathbb{C}$  be holomorphic. Then,  $\int_R f = 0$ .

90. **Cauchy-Goursat Theorem (for disks).** Suppose  $f: D \rightarrow \mathbb{C}$  is holomorphic on an open disk  $D := D_{\rho}(z_0)$ . Then:

- (a)  $f$  has a primitive  $F$  on  $D$ .
- (b) if  $\Gamma$  is any loop in  $D$ , then  $\int_{\Gamma} = 0$ .

91. **Deformation Theorem.** Suppose  $f$  is holomorphic on an open set  $\Omega$  and  $\gamma_0, \gamma_1$  are piecewise continuously differentiable. Then there are continuously differentiable curves in  $\Omega$ . Then:

- (a) If  $\gamma_0, \gamma_1$  are paths from  $z_0$  to  $z_1$ , which are homotopic in  $\Omega$ , then  $\int_{\gamma_0} F = \int_{\gamma_1} F$ .
- (b) If  $\gamma_0, \gamma_1$  are loops homotpic in  $\Omega$ , then  $\int_{\gamma_0} F = \int_{\gamma_1} F$ .

92. **Cauchy-Goursat Theorem (restated).** Let  $f: \Omega \rightarrow \mathbb{C}$  be holomorphic with  $\Omega$  open and a loop (let  $\gamma$ ) be homotopic to a point in  $\Omega$ . Then,  $\int_{\gamma} f dz = 0$ .

93. **Cor.** If  $\Omega$  is simply connected, then every loop  $\gamma$  has  $\int_{\gamma} f dz = 0$ .

94. **Cor.** Let  $f: \Omega \rightarrow \mathbb{C}$  be holomorphic on a simply connected oen set  $\Omega$ . Then,  $f$  has a primitive  $F$  on  $\Omega$  (unique up to constants).

95. **Winding number (as an index).** Let  $\gamma: [a, b] \rightarrow \mathbb{C}$  (a piecewise continuous) loop and  $z \notin \gamma([a, b])$ . Then, the winding number of  $\gamma$  around  $z_0$  is an integer.

96. **Cauchy's Integral Formula.** Let  $f$  be holomorphic on  $\Omega$  and  $\gamma$  a loop in  $\Omega$  hommotopic to a point. Let  $z_0 \in \Omega$  but  $z_0 \notin \gamma$ . Then,

$$f(z_0) \cdot I(\gamma, z_0) = \frac{1}{2\pi i} \cdot \int_{\gamma} \frac{f(z)}{z - z_0} dz.$$

97. **Cauchy's Integral Formula for Derivatives.** Let  $f$  be holomorphic on  $\Omega$ . Then  $f$  is infinitely differentiable (complex) and if  $\gamma$  is a loop homotopic to a point (simple loop)  $I(\gamma, z_0) = 1$ , then:

$$f^{(n)}(z_0) = \frac{n}{2\pi i} \int_{\gamma} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi.$$

98. **Cor. Cauchy-Type Integrals.** Let  $\gamma$  be a loop  $\gamma: [a, b] \rightarrow \mathbb{C}$  and  $g$  a continuous function on  $\gamma$ . Set  $\tilde{g}(z) := \int_{\gamma} \frac{g(\xi)}{\xi - z} d\xi$ . Then,  $\tilde{g}(z)$  is holomorphic inside  $\gamma$  and so  $\tilde{g}(z)$  is infinitely differentiable.

99. **Prop. (Cauchy Inequalities).** Let  $f$  be holomorphic on  $\Omega$  and let  $\overline{D_R(z_0)} \subseteq \Omega$  with boundary  $\gamma$ . Suppose  $f(z)$  is bounded above  $|f(z)| \leq M$  for all  $z \in \gamma$ . Then for all  $k = 1, 2, \dots$ , the  $k$ th derivative is also upper bounded with  $|f^{(k)}(z_0)| \leq \frac{k!}{R^k} M$ .

100. **Louisville's Theorem.** If  $f$  is entire and bounded (i.e. there exists an  $M \in \mathbb{R}_{>0}$  with  $|f(z)| \leq M$  for all  $z \in \mathbb{C}$ ), then  $f$  is constant.

101. **Fundamental Theorem of Algebra.** Let  $a_0, \dots, a_n \in \mathbb{C}$  with  $a_i \neq 0$  for  $n \geq 1$ . Then the polynomial  $p(z) = a_n z^n + \dots + a_0$  has a zero (root) where  $z_0 \in \mathbb{C}$  with  $p(z_0) = 0$ .

102. **Cor.** A degree  $n$  complex polynomial has exactly  $n$  roots, counting multiplicity.

103. **Morera's Theorem. (partial converse to Cauchy-Goursat)** Let  $f$  continuous on an open  $\Omega$  and suppose that  $\int_\gamma f = 0$  for every loop in  $\Omega$ . Then,  $f$  is holomorphic on  $\Omega$  and  $f$  has a primitive  $F$  on  $\Omega$ .

104. **Cor. to Morera's Theorem (Removable Singularities Theorem).** Let  $f$  be continuous on an open  $\Omega$  in  $\mathbb{C}$  and holomorphic on  $\Omega \setminus \{z_0\}$ , with  $z_0 \in \mathbb{C}$ . Then,  $f$  is holomorphic on  $\Omega$ .

105. **Another Cor. to Morera's Theorem.** If  $f$  is holomorphic on  $\Omega \setminus \{z_0\}$  and bounded on a neighborhood of  $z_0$ , there is unique holomorphic extension  $\tilde{f}$  of  $f$  to  $\gamma$  defined by  $\tilde{f}(z) = f(z)$  if  $z \neq z_0$  and  $\tilde{f}(z) = \lim_{z \rightarrow z_0} f(z)$  if  $z = z_0$ .

106. **Mean Value Property.** Let  $f$  be holomorphic on  $\overline{D_R(z_0)}$ . Then  $f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$ .

107. **Maximum Principle, local version.** Let  $f$  be holomorphic on a neighborhood  $\Omega$  of  $z_0$ , and suppose that  $|f|$  has a relative max at  $z_0$ . Then,  $f$  is constant on some neighborhood  $U$  of  $z_0$ .

108. **Prop.** The following are true:

- (a)  $A \subseteq \bar{A}$ .
- (b)  $\bar{A}$  is closed.
- (c)  $A$  is closed iff  $A = \bar{A}$ .
- (d) If  $A \subseteq C$  and  $C$  closed, then  $\bar{A} \subseteq C$ .

109. **Maximum Modulus Principle.** Let  $A$  be an open, connected, bounded set in  $\mathbb{C}$  and suppose  $f: \bar{A} \rightarrow \mathbb{C}$  is holomorphic on  $A$  and continuous on  $\bar{A}$ . Then  $|f|$  has a finite maximum value on  $\bar{A}$  which is achieved on  $\partial A$ . If  $|f|$  is attained in  $A$ , then  $f$  is constant.

110. **Prop.** Let  $u: \Omega \rightarrow \mathbb{R}$  be an twice-continuous harmonic function on an open set  $\Omega \subseteq \mathbb{C}$ . Then  $u$  is infinitely differentiable, so  $u$  is  $C^\infty$ , and in the neighborhood  $U$  of  $z_0 \in \Omega$ , there exists a holomorphic function  $f: U \rightarrow \mathbb{C}$  such that  $u = \text{Re}(f)$ .

111. **Dirichlet Problem.**  $\Delta u = 0$ ,  $u|_{\partial\Omega}(\theta) = g(\theta)$ .

112. **Prop.** Let  $u, \tilde{u}$  solve the Dirichlet Problem. Then,  $u = \tilde{u}$ , so the solution to the Dirichlet Problem is unique.

113. **Solution to the Dirichlet Problem.** This is given by:

$$u(re^{i\phi}) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \cdot \frac{R^2 - r^2}{|Re^{i\theta} - re^{i\phi}|} d\theta.$$

114. **Analytic Convergence Theorem.** Let  $f_n: \Omega \rightarrow \mathbb{C}$  be a sequence of holomorphic functions. If  $f_n \rightarrow f$  uniformly on every closed disk in  $\Omega$ , then:

- (a)  $f$  is holomorphic on  $\Omega$ .
- (b)  $f'_n$  converges to  $f'$  uniformly on every closed disk, and pointwise on  $\Omega$ .

115. **Prop.** Let  $\gamma: [a, b] \rightarrow \Omega$  be a contour and  $f_n: \gamma([a, b]) \rightarrow \mathbb{C}$  be a sequence of continuous functions. If  $f_n \rightarrow f$  uniformly on  $\gamma([a, b])$ , then  $\int_\gamma f_n \rightarrow \int_\gamma f$ .

116. **Power Series Convergence Theorem.** Let  $\sum_{n=0}^\infty a_n(z - z_0)^n$  be a power series. Then there is a unique  $R \geq 0$ , possibly  $R = \infty$ , such that:

- (a) If  $|z - z_0| < R$ , the series converges pointwise.
- (b) If  $|z - z_0| \leq R - \epsilon$ , the series converges uniformly.
- (c) If  $|z - z_0| > R$ , the series diverges.
- (d) If  $|z - z_0| = R$ , need to check.

117. **Cor.** A power series is analytic on its disk of convergence and so, holomorphic by the analytic convergence theorem.

118. **Cor 2.** We can apply term-by-term differentiation to the power series of an analytic function.

119. **Cor 3.** Power series expansions around some center  $z_0$  are unique.

120. **Taylor Series Theorem.** Let  $f$  be holomorphic on a region  $\Omega$ , and let  $D_r(z_0) \subseteq \Omega$  with  $r > 0$ . Then for every  $z \in D_r(z_0)$ , the power series  $f(z) = \sum_{n=0}^\infty \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$  converges on  $D_r(z_0)$  and equal to  $f(z)$ . We call this the Taylor series of  $f$  centered at  $z_0$ .

121. **Theorem.**  $f$  is analytic iff  $f$  is holomorphic.

122. **Prop.** If  $f$  is holomorphic on an open, connected set  $\Omega$  and the zero set  $\{z \in \Omega \mid f(z) = 0\}$  contains a limit point, then  $f = 0$  on  $\Omega$ .

123. **Cor. (Identity Theorem).** Let  $f, g$  be holomorphic on an open, connected  $\Omega$  and  $f(z) = g(z)$  for a set of  $z$  with a limit point in  $\Omega$ . Then,  $f = g$  on  $\Omega$ .

124. **Cor. (Zeros are Isolated).** If  $f$  is holomorphic on  $\Omega$ , and not identically zero on  $\Omega$ , then for any zero  $z_0$  of  $f$ , there is a deleted neighborhood  $U \setminus \{z_0\}$  on which  $f(z) \neq 0$  for all  $z \in U \setminus \{z_0\}$ .

125. **Cor. (Analytic Continuation).** If  $f$  is holomorphic on an open, connected set  $\Omega$  and  $f_+$  is holomorphic on an open connected  $\Omega_+ \supseteq \Omega$  with  $f_+ = f$  on  $\Omega$ , then  $f_+$  is the unique such extension, i.e. if there exists another such extension,  $\tilde{f}_+$ , then  $f_+ = \tilde{f}_+$ .

126. **Lemma.** Let  $f: \Omega \rightarrow \mathbb{C}$  be holomorphic, not identically 0, with a zero  $z_0$ . Then in a neighborhood  $U$  of  $z_0$ , we may write  $f(z) = (z - z_0)^m g(z)$  for all  $z \in U$ , where  $g(z) \neq 0$  and  $m$  is unique.

127. **Lemma.** A function  $f$  has a pole of order  $m$  at  $z_0$  iff there is a neighborhood  $U$  of  $z_0$  on which  $f(z) = (z - z_0)^{-m} g(z)$  for all  $z \in U \setminus \{z_0\}$  with a nonzero  $g(z)$  and  $g$  holomorphic on  $U$ .

128. **Theorem.** If  $f$  has a pole of order  $m$  at  $z_0$ , then it can be represented uniquely as:

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \dots + \frac{a_{-1}}{z - z_0} + G(z)$$

where  $G(z)$  is holomorphic on a neighborhood  $U$  of  $z_0$  and  $a_{-m}, \dots, a_{-1} \in \mathbb{C}$  with  $a_{-m} \neq 0$ .

129. **Residue Theorem, Simple Version.** Let  $f$  be holomorphic on a set  $\Omega \supseteq \overline{D_R(z_0)}$ ,  $\gamma = \partial D_R(z_0)$  except at  $z_0$ . Then  $\int_\gamma f(z) dz = 2\pi i \text{Res}_{z_0}(f)$ .

130. **Residue Theorem, Simple Closed Loops.** Let  $\Omega$  be open, connected and  $\gamma$  a simple loop homotopic to a point in  $\Omega$ . Let  $f$  be a function  $f: \Omega \rightarrow \mathbb{C}$  be holomorphic except at a finite set of points  $z_1, \dots, z_N$  inside  $\gamma$ . Then,  $\int_\gamma f(z) dz = 2\pi i \sum_{k=1}^N \text{Res}_{z_k}(f)$ .

131. **Laurent Series Theorem.** Let  $C_1, C_2$  be two circles centered at  $z_0$  (it is fine if  $C_1 = \{z_0\}$  and  $C_2$  "encloses"  $\mathbb{C}$ ). Call  $R$  the region the annulus between  $C_1$  and  $C_2$ . Let  $f$  be holomorphic on  $R$ . Then  $f$  can be expanded uniquely as a (absolutely) convergent power series in  $R$  by:

$$f(z) = \sum_{n=1}^\infty \frac{a_{-n}}{(z - z_0)^n} + \sum_{n=0}^\infty a_n (z - z_0)^n.$$

where the first infinite series is called the principal part and second one is called the Taylor series / holomorphic part.

132. **Casorati-Weierstrauss Theorem.** If  $f$  is holomorphic in a deleted  $D_r(z_0) \setminus \{z_0\}$  and has an essential singularity at  $z_0$ , then the image of  $f(D_r(z_0) \setminus \{z_0\})$  is dense in  $\mathbb{C}$ .

133. **Prop. (Fourier Transform).** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a real function. The Fourier Transform of  $f$  is the function  $\hat{f}: \mathbb{R} \rightarrow \mathbb{R}$  given by  $\hat{f}(k) = \int_{-\infty}^\infty f(x) \cdot e^{-ikx} dx$ .

134. **Jordan's Lemma.**  $\int_0^\pi e^{-R \sin \theta} d\theta \leq \frac{\pi}{R}$ .

135. **Cauchy Principal Value.** Take the real integral symmetrically, so we can find an indefinite integral (with discontinuity in the interval) by approaching "the same way" from both sides of the discontinuity.

136. **Argument Principle.** Let  $f: \Omega \rightarrow \mathbb{C}$  be meromorphic and  $\gamma$  a simple loop in  $\Omega$  bounding a simply connected region  $R_\gamma$ , with  $\bar{R}_\gamma \subseteq \Omega$ . Let  $f$  have no zeros or poles on  $\gamma$ . Then:

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = N - P$$

where  $N$  is the number of zeros of  $f$  inside  $R_\gamma$  (counting multiplicity) and  $P$  is the number of poles of  $f$  inside  $R_\gamma$  (counting multiplicity).

137. **Rouche's Theorem.** Let  $f, g: \Omega \rightarrow \mathbb{C}$  be holomorphic and let  $\gamma$  be a simple loop bounding a simply connected open  $U$ , with  $\bar{U} \subseteq \Omega$ . If  $|f(z)| > |g(z)|$  on  $\gamma$ , then  $f$  and  $f + g$  have the same number of zeros inside  $U$ .

138. **Open-Mapping Theorem.** Any nonconstant holomorphic function is an open map, meaning it maps open sets to open sets.

139. **Lemma. (Local Injectivity).** Let  $f: \Omega \rightarrow \mathbb{C}$  be holomorphic and  $z_0 \in \Omega$ . If  $f'(z_0) \neq 0$ , then  $f$  is locally injective near  $z$ .

140. **Theorem.** If  $g: \Omega \rightarrow \mathbb{C}$  is holomorphic and  $\Omega$  is simply connected, and  $g \neq 0$ , there exists a holomorphic function  $F: \Omega \rightarrow \mathbb{C}$  satisfying  $e^{F(z)} = g(z)$ , where  $F(z)$  is unique up to  $2\pi i k$ , with  $k \in \mathbb{Z}$ .

141. **Theorem (Local description of holomorphic).** Let  $f: \Omega \rightarrow \mathbb{C}$  be holomorphic,  $\Omega$  open. Let  $z_0 \in \Omega$  and let  $k \geq 1$  denote the order of the zero  $f(z) - f(z_0)$  at  $z_0$ . Then, there exists an open neighborhood  $U$  of  $z_0$  (and  $r > 0$ ) and a function  $\phi: U \rightarrow D_r(z_0)$  such that:

- (a)  $\phi$  is holomorphic with a holomorphic inverse.
- (b)  $\phi(z_0) = 0$ .
- (c) We have  $f(z) = f(z_0) + (\phi(z))^k$  with  $z \in U$ .

142. **Prop.** Let  $\{a_n\}_{n=1}^\infty$  be a sequence of complex numbers. If  $\sum_{n=1}^\infty |a_n| < \infty$ , then  $\prod_{n=1}^\infty (1 + a_n)$  converges and its value is 0 iff one of the  $1 + a_n$  factors is zero.

143. **Prop.** Let  $\{f_n\}_{n=1}^\infty$  be a sequence of holomorphic functions on  $\Omega$ . If  $\sum_{n=1}^\infty |f_n|$  converges uniformly on compact subsets of  $\Omega$ , then so does  $\prod_{n=1}^\infty (1 + f_n(z))$ . Moreover, the limiting function is holomorphic and nonzero everywhere except at points  $z$  such that  $1 + f_n(z) = 0$  (for some  $n$ ).

144. **Prop. (Partial fractions expansion for log derivatives).** Same assumptions as the above proposition. Then, the log derivative of product = sum of log derivatives. i.e.

$$\frac{(\prod_{n=1}^\infty (1 + f_n))'}{\prod_{n=1}^\infty (1 + f_n)} = \sum_{n=1}^\infty \frac{f'_n}{1 + f_n}.$$

145. **Infinite products formula for sine.**  $\sin(\pi z) = \pi z \cdot \prod_{n=1}^\infty (1 - \frac{z}{n})(1 + \frac{z}{n})$ , for  $z \in \mathbb{C}$ .

146. **Prop.** Fix  $z \in \mathbb{C} \setminus \mathbb{Z}$  and a large positive  $N \in \mathbb{Z}$ . By the residue theorem, the integral  $I_N(z) := \int_{\gamma_N} \frac{\pi \cot(\pi z)}{(w+z)^2} dw$ .

147. **Cor.**  $\cos(\pi z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\left(n - \frac{1}{2}\right)^2}\right)$ ,  $e^z - 1 = z e^{z/2} \cdot \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right)$ , and  $\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$  (for the contangent expression,  $z \in \mathbb{C} \setminus \mathbb{Z}$ ).

148. **Theorem (Schwarz Reflection Principle).** Let  $A$  be a region in the upper-half plane with  $\partial A \cap \mathbb{R}$  nonempty and containing  $[a, b] \subseteq \mathbb{R}$ . Let  $f$  be holomorphic on  $A$  and continuous on  $\partial A \cap [a, b]$  and real on  $[a, b]$ . Then  $f$  can be uniquely extended to a holomorphic function on  $A \cup (a, b) \cup A_{\text{ref}}$ , where  $A_{\text{ref}} = \{\bar{z} \mid z \in A\}$  with  $f(z) = \overline{f(\bar{z})}$  for all  $z \in A_{\text{ref}}$ .

149. **Theorem (Gamma Function).** There is a unique  $\Gamma(s)$  with the following:

- (a)  $\Gamma(s)$  is meromorphic.
- (b) (Factorial).  $\Gamma(n+1) = n!$  for  $n = 0, 1, 2, \dots$
- (c) (Special Value).  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ .
- (d) (Integral Representation).  $\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1} dx$  (for  $\text{Re}(s) > 0$ ).
- (e) (Infinite Product Representation).  $\Gamma(s) = s^{-1} e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right)^{-1} e^{s/n}$  where  $\gamma = \lim_{n \rightarrow \infty} \left(\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n} - \log n\right) \approx 0.58\dots$  (Euler-Mascheroni constant) (for  $s \in \mathbb{C}$ , except poles).
- (f) (Limit of finite products).  $\Gamma(s) = \lim_{n \rightarrow \infty} \frac{n! n^s}{s(s+1)\dots(s+n)}$  (for  $s \in \mathbb{C}$ , except poles).
- (g) (Zeros).  $\Gamma(s)$  has no zeros.
- (h) (Poles).  $\Gamma(s)$  has poles at nonpositive integers  $s = 0, -1, -2, \dots$  and is holomorphic everywhere else. At  $s = -n$ , the pole is simple and  $\text{Res}_{-n}(\Gamma) = -\frac{1}{n!}$ .
- (i) (Functional Equation).  $\Gamma(s+1) = s\Gamma(s)$  (for  $s \in \mathbb{C}$ , except at poles).
- (j) (Reflection Formula).  $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$  (for  $s \in \mathbb{C}$ , except at poles).

150. **Theorem.** The conformal equivalence, which is a relation, is an equivalence relation.

151. **Cor.** Holomorphic  $f$  is locally injective iff  $f'(z_0) \neq 0$ .

152. **Conformal Equivalence Classes.** These are the following:

- (a) Complex plane,  $\mathbb{C}$ .
- (b) Punctured plane,  $\mathbb{C} \setminus \{0\}$ .
- (c) Unit disk,  $D_1(0)$ .
- (d) Upper-half plane,  $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ .
- (e) Riemann sphere\*,  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  (not a subset of  $\mathbb{C}$  but we can still talk about holomorphic/meromorphic functions on it).
- (f) The slit plane,  $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ .
- (g) Strip (similar to critical region from Riemann hypothesis).
- (h) Rectangle.
- (i) Annulus.
- (j) Blob

153. **Lemma.** If  $\Omega \sim \Omega'$ , then  $\Omega$  and  $\Omega'$  are homeomorphic (a.k.a. there is a continuous map  $g: \Omega \rightarrow \Omega'$  such that  $g^{-1}$  is defined and also continuous).

154. **Lemma.** If  $\Omega \sim \Omega'$ , then they are homeomorphic.

155. **Prop.** If  $g: \Omega \rightarrow \Omega'$  is holomorphic and invertible, then  $g^{-1}$  is holomorphic (i.e.  $g$  is conformal).

156. **Lemma.**  $\text{Aut}(\Omega)$  is a group, with function composition. In other words, let  $f, g, h \in \text{Aut}(\Omega)$ . Then:

- (a)  $(g \circ f) \circ h = g \circ (f \circ h)$ .
- (b) If  $g \in \text{Aut}(\Omega)$ , then  $g^{-1} \in \text{Aut}(\Omega)$ .
- (c) There is an identity map  $\text{id} \in \text{Aut}(\Omega)$  such that  $\text{id} \circ g(z) = g(z) = g \circ \text{id}(z) = g(z)$ .

157. **Theorem.** Let  $g: \mathbb{C} \rightarrow \Omega$  be a conformal map between  $\mathbb{C}$  and a region  $\Omega$ . Then,  $\Omega = \mathbb{C}$  and  $g(z)$  is a conformal automorphism of the form  $g(z) = az + b$ , with  $a \neq 0$  and  $b \in \mathbb{C}$ .

158. **Theorem (Riemann Sphere).** Let  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ . If  $g: \hat{\mathbb{C}} \rightarrow \Omega$  is a conformal map, then  $\Omega = \hat{\mathbb{C}}$  and  $g$  is a conformal automorphism, with  $g(z) = \frac{az+b}{cz+d}$  with  $a, b, c, d \in \mathbb{C}$  (i.e.  $g$  is a Möbius transformation).