Math 185 - Homework 5 Practice

Problem 1:

(a) Recall that $F(z) := \int_\gamma \frac{dw}{w}$. To show that Ω is simply connected, we construct a homotopy by the following. $\gamma[0,1] \to \Omega$ is any curve with $\gamma(0) = 1$ and $\gamma(1) = z$. Then, let γ be defined as in the previous sentence. Then construct the homotopy $H_s(t) = (1-s)\gamma(t) + s(1)$, where $s, t \in [0,1)$. With this homotopy, we observe that we begin with any curve in Ω, and as s goes from 0 to 1, $H_s(t)$ continuously deforms the curve into the point $1 \in \Omega$.

(b) To show that F is well-defined, we aim to prove that F is completely dependent on z, or in other words, F is path-independent. So, let γ_0, γ_1 be two such paths in Ω with $\gamma_0(0) = \gamma_1(0) = 1$ and $\gamma_0(1) = \gamma_1(1) = z$. Then, show that $\int_{\gamma_0} \frac{dw}{w} = \int_{\gamma_1} \frac{dw}{w}$, which implies $\int_{\gamma_0 - \gamma_1} \frac{dw}{w}$. We have that the path $\gamma_0 - \gamma_1$ is a closed curve (loop) in Ω. Thus, by a corollary to the Cauchy-Goursat theorem, every loop Γ in a simply connected has $\int_\Gamma f dz = 0$. Thus, we have that $\int_{\gamma_0 - \gamma_1} \frac{dw}{w} = 0$, since $\frac{1}{w}$ is holomorphic on the simply connected Ω. Thus, we have that $\int_{\gamma_0} \frac{dw}{w} = \int_{\gamma_1} \frac{dw}{w}$, and so F is well-defined.

(c) Let $\tilde{\Omega} = \mathbb{C} \setminus \{0\}$. Then, the function F on $\tilde{\Omega}$ is not a well-defined function, since we can consider the following two path integrals. Define $\gamma_0(t) = e^{it}$ and $\gamma_1(t) = e^{-it}$ for $t \in [0,1)$. Then, the integrals result in $i\pi$ and $-i\pi$, respectively. Since $i\pi = -i\pi$, F is not path-independent, so therefore, it follows that F is not well-defined on $\tilde{\Omega}$.

(d) $\tilde{\Omega}$ is not simply connected, since we can consider the unit circle about the origin, which is homotopic to the point 0, the origin. However, since $0 \notin \tilde{\Omega}$, thus, we have that $\tilde{\Omega}$ is not simply connected.

Problem 2: