Math 185 Theorems

1. **Prop.** Let $z_1, z_2 \in \mathbb{C}$. Then $|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \pmod{2\pi}$.
2. **Theorem.** $\mathbb{C}$ is a field.
3. **De Moivre's Formula.** If $z = r(\cos\theta + i\sin\theta)$ and $n \in \mathbb{Z}_{>0}$, then $z^n = r^n(\cos n\theta + i\sin n\theta)$.
4. **Prop.** Let $z, w \in \mathbb{C}$. Then:
   (a) $\overline{z+w} = \overline{z} + \overline{w}$.
   (b) $\overline{zw} = \overline{z} \cdot \overline{w}$.
   (c) $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$ (with $w \neq 0$).
   (d) $z\overline{z} = |z|^2$. If $z \neq 0$, then $z^{-1} = \frac{\overline{z}}{|z|^2}$.
   (e) If $z = \overline{z}$, then $z \in \mathbb{R}$ and so $z = \mathrm{Re}(z)$.
   (f) $\mathrm{Re}(z) = \frac{z+\overline{z}}{2}$ and $\mathrm{Im}(z) = \frac{z-\overline{z}}{2i}$.
   (g) $\overline{\overline{z}} = z$.
5. **Prop.** Let $z, w \in \mathbb{C}$. Then:
   (a) $|z| \geq 0$ and if $|z| = 0$, then $z = 0$.
   (b) $|zw| = |z||w|$.
   (c) If $w \neq 0$, then $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$.
   (d) $|\mathrm{Re}(z)| \leq |z|$ and $|\mathrm{Im}(z)| \leq |z|$.
   (e) $|\overline{z}| = |z|$.
   (f) $|z+w| \leq |z| + |w|$.
   (g) $||z| - |w|| \leq |z - w|$.
   (h) $|z_1 w_1 + \cdots + z_n w_n| \leq \sqrt{|z_1|^2 + \cdots + |z_n|^2}\sqrt{|w_1|^2 + \cdots + |w_n|^2}$.
6. **Prop.** Fix $r > 0$ and $z \in \mathbb{C}$. The open disk $D_\varepsilon(z_0)$ is an open set.
7. **Prop.** The following are true:
   (a) $\mathbb{C}$ is open.
   (b) The empty set ϕ is open.
   (c) The union of open sets is open.
   (d) The intersection of finitely many open sets is open.
8. **Prop.** Limits are unique (if they exist).
9. **Prop.** If $\lim_{z\to z_0} f(z) = a$ and $\lim_{z\to z_0} g(z) = b$, then:
   (a) $\lim_{z\to z_0}(f(z) + g(z)) = a + b$.
   (b) $\lim_{z\to z_0}((f(z)g(z))) = ab$.
   (c) $\lim_{z\to z_0}\left(\frac{f(z)}{g(z)}\right) = \frac{a}{b}$ (with $b \neq 0$).
10. **Prop.** The following are true:
    (a) If $\lim_{z\to z_0} f(z) = a$ and h is continuous at a, then $\lim_{z\to z_0} h(f(z)) = h(a)$.

(b) If f is continuous on an open set $\Omega \subseteq \mathbb{C}$, and h is continuous on $f(\Omega)$, then $h \circ f$ is continuous on Ω, with $(h \circ f)(z) = h(f(z))$.

11. **Prop.** The following are true:

    (a) The empty set ϕ is closed.

    (b) $\mathbb{C}$ is closed.

    (c) The intersection of a collection of closed sets is closed.

    (d) The union of finitely many closed sets is closed.

12. **Prop.** A set F is closed iff whenever $z_1, z_2, z_3, \ldots$ is a sequence of points in F converging to $\lim_{k\to\infty} z_k = w$, then $w \in F$.

13. **Prop.** If $f : \mathbb{C} \to \mathbb{C}$, TFAE:

    (a) f is continuous.

    (b) If $F \subseteq \mathbb{C}$ is closed, then $f^{-1}(F)$ is closed.

    (c) If Ω is open, then $f^{-1}(\Omega)$ is also open.

14. **Prop. (Heine-Borel + Sequential Compactness).** For $K \subseteq \mathbb{C}$, TFAE:

    (a) K is compact.

    (b) K is closed and bounded.

    (c) Every sequence of points in K has a convergent subsequence converging in K (sequentially compact).

15. **Prop.** If K is compact and $f : K \to \mathbb{C}$ is continuous, then the image $f(K)$ is compact.

16. **Theorem (Extreme Value Theorem).** If K is compact and $f : K \to \mathbb{R}$ is continuous, then f attains its minimum and maximum.

17. **Stereographic Projection / Riemann Sphere.** Identify the plane $\overline{\mathbb{C}} = S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$. If $f : \mathbb{C} \to S^2 \setminus \{N\}$, then we have $(u,v) \mapsto \frac{1}{1+u^2+v^2}(2u, 2v, -1+u^2+v^2)$ is a homeomorphism (is continuous with continuous inverse) $f^{-1} : S^2 \setminus \{N\} \to \mathbb{C}$ with $(x,y,z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$.

18. **Prop. (Uniform Convergence).** If $f_n \to f$ uniformly and each f_n is continuous, then f is continuous.

19. **Euler's Formula.** $e^{iz} = \cos z + i \sin z$ for all $z \in \mathbb{C}$.

20. **Theorem.** $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$.

21. **Properties of e.** Let $x, y \in \mathbb{R}$ and $z, w \in \mathbb{C}$. Then:

    (a) $e^{z+w} = e^z e^w$.

    (b) $|e^{x+iy}| = |e^x e^{iy}| = |e^x||e^{iy}| = e^x$.

    (c) $\arg(e^{x+iy}) = y \pmod{2\pi}$.

    (d) $e^z \neq 0$ for all $z \in \mathbb{C}$.

    (e) $e^z = 1$ iff $z = 2\pi i n$ for some $n \in \mathbb{Z}$.

    (f) $e^z = e^{z+2\pi n i}$.

22. **Prop. (Chain Rule).** Let $\Omega, A \subseteq \mathbb{C}$ be open sets, and let $f : \Omega \to A$, $g : A \to \mathbb{C}$ be holomorphic functions. Then, $g \circ f : \Omega \to \mathbb{C}$ is holomorphic and $\frac{d}{dz}(f \circ g)(z) = \frac{dg}{df}(f(z)) \cdot \frac{df}{dz}(z)$.

23. **Prop.** Let $f : \Omega \to \mathbb{C}$ be holomorphic. Then $f : \Omega \to \mathbb{R}^2$ is real differentiable at all $(x,y) \in \Omega$.

24. **Cauchy-Riemann Equations.** Let Ω be an open set in $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be given by $f(x,y) = u(x,y) + iv(x,y)$. Then:

    (a) $f'(z)$ exists at $z \in \Omega$ iff f is real differentiable and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (these are the Cauchy-Riemann equations).

    (b) $f(z)$ is holomorphic on Ω iff partials are continuous and satisfy the CR equations.

    (c) If $f'(z_0)$ exists, then $f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y} = \frac{1}{i}\frac{\partial f}{\partial y}$.

25. **Inverse Function Theorem for $\mathbb{R}^2$.** If $f:\Omega\to\mathbb{R}^2$ is continuously differentiable and the Jacobian $Df(z_0)$ has $\det(Df(z_0))\neq 0$, then there are neighborhoods $U\ni z_0$ and $V\ni f(z_0)$ such that $f:U\to V$is bijective with continuously differentiable $f^{-1}:V\to U$ such that $Df^{-1}(z_0)=[Df(z_0)]^{-1}$, which is the inverse matrix of $Df(z_0)$.

26. **Inverse Function Theorem for $\mathbb{C}$.** Let $f:\Omega\to\mathbb{C}$ be holomorphic (with continuous $f'(z_0)$), and $f'(z)\neq 0$ for some $z_0\in\Omega$. Then there exists a neighborhood $U\ni z_0$ and $V\ni f(z_0)$ such that $f:U\to V$ is bijective with holomorphic inverse $f^{-1}:V\to U$ such that for all $z_0\in U$, $\frac{d}{dw}f^{-1}(w)=\frac{1}{f'(w)}$ with $w=f(z)$.

27. **Prop.** Pick a branch $[a,a+2\pi)$. Then $\log z:\mathbb{C}\setminus\{0\}\to\mathbb{R}\times i[a,a+2\pi)$ is the inverse of $\exp:\mathbb{R}\times i[a,a+2\pi)\to\mathbb{C}$.

28. **Prop.** $\log:\mathbb{C}\setminus\mathbb{R}_{\leq 0}\to\mathbb{R}\times i(-\pi,\pi)$ is holomorphic with $\frac{d}{dz}\log z=\frac{1}{z}$.

29. **Prop.** If $z_1,z_2\in\mathbb{C}\setminus\mathbb{R}_{\leq 0}$, then $\log(z_1z_2)=\log z_1+\log z_2 \pmod{2\pi i}$.

30. **Prop.** By choosing different branches of log, we have the following:

    (a) a^b is independent of the branch iff $b\in\mathbb{Z}$.

    (b) a^b takes on exactly q different values iff $b\in\mathbb{Q}$, so $b=\frac{p}{q}$ (with p,q coprime).

    (c) a^b takes on infinitely many values iff b is irrational or $\operatorname{Im}(b)\neq 0$.

31. **Cor.** Choose a branch of log. Then the nth root function is given by $z^{1/n}=e^{\log(z/n)}$, where the nth root function has n branches.

32. **Prop.** Let $a,b\in\mathbb{C}$. Then:

    (a) For any choice of branch of log, the function $\mapsto a^z$ is holomorphic on $\mathbb{C}$, and $z\mapsto(\log a)a^z$.

    (b) Choose a branch of log. Then the function $z\mapsto z^b$ is holomorphic on the domain of log with derivative $z\mapsto bz^{b-1}$.

33. **Prop. (Re-parametrization).** If $\tilde{\gamma}$ is a re-parametrization of γ, then $\int_\gamma f=\int_{\tilde{\gamma}} f$ for any continuous f on Ω.

34. **Fundamental Theorem of Line Integrals.** Let $F:\Omega\to\mathbb{C}$ be holomorphic on an open Ω and let $\gamma:[0,1]\to\Omega$ be piecewise smooth. Then, $\int_\gamma F'(z)dz=F(\gamma(1))-F(\gamma(0))$.

35. **Path-independence and Primitives Theorem.** Let $f:\Omega\to\mathbb{C}$ be continuous and Ω is open and connected. Then, TFAE:

    (a) (path-independence) if $z_0,z_1\in\Omega$, then any paths $\gamma,\tilde{\gamma}$ with shared endpoints $\gamma(0)=\tilde{\gamma}(0)$ and $\gamma(1)=\tilde{\gamma}(1)$ have $\int_\gamma f(z)dz=\int_{\tilde{\gamma}} f(z)dz$.

    (b) (integral along loops is 0) if Γ is a loop, with $\Gamma(1)=\Gamma(0)$, then $\int_\Gamma f(z)dz=0$.

    (c) (f has a primitive) There is a primitive F for f on Ω.

36. **Cauchy-Goursat Theorem.** Let $f:\Omega\to\mathbb{C}$ be holomorphic on Ω, simply connected, and open. Then for any loop $\Gamma\subseteq\Omega$, $\int_\Gamma f(z)dz=0$.

37. **Green's Theorem.** Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be a vector field and let γ be a loop, and A a region in the loop γ. Let $f(x,y)=(P(x,y),Q(x,y))$. Then, $\int_\gamma P(x,y)dx+Q(x,y)dy=\iint \operatorname{curl} F\,dA=\iint\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$.

38. **Prop.** If $f(x+iy)=u(x,y)+iv(x,y)$, then $\int_\gamma f=\int_\gamma udx-vdy+i\int_\gamma udx+vdy$.

39. **Cauchy-Goursat Theorem (Weaker Version).** Let $f:\Omega\to\mathbb{C}$ be holomorphic with $f'(z)$ continuous and $\gamma:[0,1]\to\mathbb{C}$ a simple closed curve and Ω an open & simply connected set. Then, $\int_\gamma f=0$.

40. **Cauchy-Goursat Theorem (for rectangles).** Let R be a rectangle with R and its interior are contained in an open set Ω. Let $f:\Omega\to\mathbb{C}$ be holomorphic. Then, $\int_R f=0$.

41. **Cauchy-Goursat Theorem (for disks).** Suppose $f:D\to\mathbb{C}$ is holomorphic on an open disk $D:=D_\rho(z_0)$. Then:

    (a) f has a primitive F on D.

    (b) if Γ is any loop in D, then $\int_\Gamma=0$.

42. **Deformation Theorem.** Suppose f is holomorphic on an open set Ω and γ_0,γ_1 are piecewise continuously differentiable. Then there are continuously differentiable curves in Ω. Then:

    (a) If γ_0,γ_1 are paths from z_0 to z_1, which are homotopic in Ω, then $\int_{\gamma_0}F=\int_{\gamma_1}F$.

(b) If γ_0, γ_1 are loops homotpic in Ω, then $\int_{\gamma_0} F = \int_{\gamma_1} F$.

43. **Cauchy-Goursat Theorem (restated).** Let $f : \Omega \to \mathbb{C}$ be holomorphic with Ω open and a loop (let γ) be homotopic to a point in Ω. Then, $\int_\gamma f dz = 0$.

44. **Cor.** If Ω is simply connected, then every loop γ has $\int_\gamma f dz = 0$.

45. **Cor.** Let $f : \Omega \to \mathbb{C}$ be holomorphic on a simply connected oen set Ω. Then, f has a primitive F on Ω (unique up to constants).

46. **Winding number (as an index).** Let $\gamma : [a,b] \to \mathbb{C}$ (a piecewise continuous) loop and $z \notin \gamma([a,b])$. Then, the winding number of γ around z_0 is an integer.

47. **Cauchy's Integral Formula.** Let f be holomorphic on Ω and γ a loop in Ω hommotopic to a point. Let $z_0 \in \Omega$ but $z_0 \notin \gamma$. Then,
$$f(z_0) \cdot I(\gamma, z_0) = \frac{1}{2\pi i} \cdot \int_\gamma \frac{f(z)}{z - z_0} dz.$$

48. **Cauchy's Integral Formula for Derivatives.** Let f be holomorphic on Ω. Then f is infinitely differentiable (complex) and if γ is a loop homotopic to a point (simple loop) $I(\gamma, z_0) = 1$, then:
$$f^{(n)}(z_0) = \frac{n}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi.$$

49. **Cor. Cauchy-Type Integrals.** Let γ be a loop $\gamma : [a,b] \to \mathbb{C}$ and g a continuous function on γ. Set $\tilde{g}(z) := \int_\gamma \frac{g(\xi)}{\xi - z} d\xi$. Then, $\tilde{g}(z)$ is holomorphic inside γ and so $\tilde{g}(z)$ is infinitely differentiable.

50. **Prop. (Cauchy Inequalities).** Let f be holomorphic on Ω and let $\overline{D_R(z_0)} \subseteq \Omega$ with boundary γ. Suppose $f(z)$ is bounded above $|f(z)| \leq M$ for all $z \in \gamma$. Then for all $k = 1, 2, \ldots$, the kth derivative is also upper bounded with $|f^{(k)}(z_0)| \leq \frac{k_i}{R^k} M$.

51. **Louisville's Theorem.** If f is entire and bounded (i.e. there exists an $M \in \mathbb{R}_{>0}$ with $|f(z)| \leq M$ for all $z \in \mathbb{C}$), then f is constant.

52. **Fundamental Theorem of Algebra.** Let $a_0, \ldots, a_n \in \mathbb{C}$ with $a_i \neq 0$ for $n \geq 1$. Then the polynomial $p(z) = a_n z^n + \cdots + a_0$ has a zero (root) where $z_0 \in \mathbb{C}$ with $p(z_0) = 0$.

53. **Cor.** A degree n complex polynomial has exactly n roots, counting multiplicity.

54. **Morera's Theorem. (partial converse to Cauchy-Goursat)** Let f continuous on an open Ω and suppose that $\int_\gamma f = 0$ for every loop in Ω. Then, f is holomorphic on Ω and f has a primitive F on Ω.

55. **Cor. to Morera's Theorem (Removable Singularities Theorem).** Let f be continuous on an open Ω in $\mathbb{C}$ and holomorphic on $\Omega \setminus \{z_0\}$, with $z_0 \in \mathbb{C}$. Then, f is holomorphic on Ω.

56. **Another Cor. to Morera's Theorem.** If f is holomorphic on $\Omega \setminus \{z_0\}$ and bounded on a neighborhood of z_0, there is unique holomorphic extension $\tilde{f}$ of f to γ defined by $\tilde{f}(z) = f(z)$ if $z \neq z_0$ and $\tilde{f}(z) = \lim_{z \to z_0} f(z)$ if $z = z_0$.

57. **Mean Value Property.** Let f be holomorphic on $\overline{D_R(z_0)}$. Then $f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$.

58. **Maximum Principle, local version.** Let f be holomorphic on a neighborhood Ω of z_0, and suppose that $|f|$ has a relative max at z_0. Then, f is constant on some neighborhood U of z_0.

59. **Prop.** The following are true:

    (a) $A \subseteq \overline{A}$.

    (b) $\overline{A}$ is closed.

    (c) A is closed iff $A = \overline{A}$.

    (d) If $A \subseteq C$ and C closed, then $\overline{A} \subseteq C$.

60. **Maximum Modulus Principle.** Let A be an open, connected, bounded set in $\mathbb{C}$ and suppose $f : \overline{A} \to \mathbb{C}$ is holomorphic on A and continuous on $\overline{A}$. Then $|f|$ has a finite maximum value on $\overline{A}$ which is achieved on ∂A. IF $|f|$ is attained in A, then f is constant.

61. **Prop.** Let $u : \Omega \to \mathbb{R}$ be an twice-continuous harmonic function on an open set $\Omega \subseteq \mathbb{C}$. Then u is infinitely differentiable, so u is C^∞, and in the neighborhood U of $z_0 \in \Omega$, there exists a holomorphic function $f : U \to \mathbb{C}$ such that $u = \mathrm{Re}(f)$.

62. **Dirichlet Problem.** $\Delta u = 0$, $u\,|_{\partial\Omega}\,(\theta) = g(\theta)$.

63. **Prop.** Let $u, \tilde{u}$ solve the Dirichlet Problem. Then, $u = \tilde{u}$, so the solution to the Dirichlet Problem is unique.

64. **Solution to the Dirichlet Problem.** This is given by:

$$u(re^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta}) \cdot \frac{R^2 - r^2}{|Re^{i\theta} - re^{i\theta}|} d\theta.$$

65. **Analytic Convergence Theorem.** Let $f_n : \Omega \to \mathbb{C}$ be a sequence of holomorphic functions. If $f_n \to f$ uniformly on every closed disk in Ω, then:

    (a) f is holomorphic on Ω.

    (b) f_n' converges to f' uniformly on every closed disk, and pointwise on Ω.

66. **Prop.** Let $\gamma : [a,b] \to \Omega$ be a contour and $f_n : \gamma([a,b]) \to \mathbb{C}$ be a sequence of continuous functions. If $f_n \to f$ uniformly on $\gamma([a,b])$, then $\int_\gamma f_n \to \int_\gamma f$.

67. **Power Series Convergence Theorem.** Let $\sum_{n=0}^\infty a_n(z - z_0)^n$ be a power series. Then there is a unique $R \geq 0$, possibly $R = \infty$, such that:

    (a) If $|z - z_0| < R$, the series converges pointwise.

    (b) If $|z - z_0| \leq R - \varepsilon$, the series converges uniformly.

    (c) If $|z - z_0| > R$, the series diverges.

    (d) If $|z - z_0| = R$, need to check.

68. **Cor.** A power series is analytic on its disk of convergence and so, holomorphic by the analytic convergence theorem.

69. **Cor 2.** We can apply term-by-term differentiation to the power series of an analytic function.

70. **Cor 3.** Power series expansions around some center z_0 are unique.

71. **Taylor Series Theorem.** Let f be holomorphic on a region Ω, and let $D_r(z_0) \subseteq \Omega$ with $r > 0$. Then for every $z \in D_r(z_0)$, the power series $f(z) = \sum_{n=0}^\infty \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$ converges on $D_r(z_0)$ and equal to $f(z)$. We call this the Taylor series of f centered at z_0.

72. **Theorem.** f is analytic iff f is holomorphic.

73. **Prop.** If f is holomorphic on an open, connected set Ω and the zero set $\{z \in \Omega \mid f(z) = 0\}$ contains a limit point, then $f = 0$ on Ω.

74. **Cor. (Identity Theorem).** Let f, g be holomorphic on an open, connected Ω and $f(z) = g(z)$ for a set of z with a limit point in Ω. Then, $f = g$ on Ω.

75. **Cor. (Zeros are Isolated).** If f is holomorphic on Ω, and not identically zero on Ω, then for any zero z_0 of f, there is a deleted neighborhood $U \setminus \{z_0\}$ on which $f(z) \neq 0$ for all $z \in U \setminus \{z_0\}$.

76. **Cor. (Analytic Continuation).** If f is holomorphic on an open, connected set Ω and f_+ is holomorphic on an open connected $\Omega_+ \supseteq \Omega$ with $f_+ = f$ on Ω, then f_+ is the unique such extension, i.e. if there exists another such extension, $\tilde{f}_+$, then $\tilde{f}_+ = f_+$.

77. **Lemma.** Let $f : \Omega \to \mathbb{C}$ be holomorphic, not identically 0, with a zero z_0. Then in a neighborhood U of z_0, we may write $f(z) = (z - z_0)^m g(z)$ for all $z \in U$, where $g(z) \neq 0$ and m is unique.

78. **Lemma.** A function f has a pole of order m at z_0 iff there is a neighborhood U of z_0 on which $f(z) = (z - z_0)^{-m} g(z)$ for all $z \in U \setminus \{z_0\}$ with a nonzero $g(z)$ and g holomorphic on U.

79. **Theorem.** If f has a pole of order m at z_0, then it can be represented uniquely as:

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z - z_0} + G(z)$$

    where $G(z)$ is holomorphic on a neighborhood U of z_0 and $a_{-m}, \ldots, a_{-1} \in \mathbb{C}$ with $a_{-m} \neq 0$.

80. **Residue Theorem, Simple Version.** Let f be holomorphic on a set $\Omega \supseteq \overline{D_R(z_0)}$, $\gamma = \partial\overline{D_R(z_0)}$ except at z_0. Then $\int_\gamma f(z)dz = 2\pi i \operatorname{Res}_{z_0}(f)$.

81. **Residue Theorem, Simple Closed Loops.** Let Ω be open, connected and γ a simple loop homotopic to a point in Ω. Let f be a function $f : \Omega \to \mathbb{C}$ be holomorphic except at a finite set of points $z_1, \dots, z_N$ inside γ. Then, $\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{N} \operatorname{Res}_{z_k}(f)$.

82. **Laurent Series Theorem.** Let C_1, C_2 be two circles centered at z_0 (it is fine if $C_1 = \{z_0\}$ and C_2 "encloses" $\mathbb{C}$). Call R the region the annulus between C_1 and C_2. Let f be holomorphic on R. Then f can be expanded uniquely as a (absolutely) convergent power series in R by:

$$f(z) = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n.$$

where the first infinite series is called the principal part and second one is called the Taylor series / holomorphic part.

83. **Casorati-Weierstrauss Theorem.** If f is holomorphic in a deleted $D_r(z_0) \setminus \{z_0\}$ and has an essential singularity at z_0, then the image of $f(D_r(z_0) \setminus \{z_0\})$ is dense in $\mathbb{C}$.

84. **Prop. (Fourier Transform).** Let $f : \mathbb{R} \to \mathbb{R}$ be a real function. The Fourier Transform of f is the function $\hat{f} : \mathbb{R} \to \mathbb{R}$ given by $\hat{f}(k) = \int_{-\infty}^{\infty} f(x) \cdot e^{-ikx} dx$.

85. **Jordan's Lemma.** $\int_0^\pi e^{-R\sin\theta} d\theta \leq \frac{\pi}{R}$.

86. **Cauchy Principal Value.** Take the real integral symmetrically, so we can find an indefinite integral (with discontinuity in the interval) by approaching "the same way" from both sides of the discontinuity.

87. **Argument Principle.** Let $f : \Omega \to \mathbb{C}$ be meromorphic and γ a simple loop in Ω bounding a simply connected region R_γ, with $\overline{R_\gamma} \subseteq \Omega$. Let f have no zeros or poles on γ. Then:

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = N - P$$

where N is the number of zeros of f inside R_γ (counting multiplicity) and P is the number of poles of f inside R_γ (counting multiplicity).

88. **Rouche's Theorem.** Let $f, g : \Omega \to \mathbb{C}$ be holomorphic and let γ be a simple loop bounding a simply connected open U, with $\overline{U} \subseteq \Omega$. If $|f(z)| > |g(z)|$ on γ, then f and $f + g$ have the same number of zeros inside U.

89. **Open-Mapping Theorem.** Any nonconstant holomorphic function is an open map, meaning it maps open sets to open sets.

90. **Lemma. (Local Injectivity).** Let $f : \Omega \to \mathbb{C}$ be holomorphic and $z_0 \in \Omega$. If $f'(z_0) \neq 0$, then f is locally injective near z.

91. **Theorem.** If $g : \Omega \to \mathbb{C}$ is holomorphic and Ω is simply connected, and $g \neq 0$, there exists a holomorphic function $F : \Omega \to \mathbb{C}$ satisfying $e^{F(z)} = g(z)$, where $F(z)$ is unique up to $2\pi i k$, with $k \in \mathbb{Z}$.

92. **Theorem (Local description of holomorphic).** Let $f : \Omega \to \mathbb{C}$ be holomorphic, Ω open. Let $z_0 \in \Omega$ and let $k \geq 1$ denote the order of the zero $f(z) - f(z_0)$ at z_0. Then, there exists an open neighborhood U of z_0 (and $r > 0$) and a function $\phi : U \to D_r(z_0)$ such that:
    (a) ϕ is holomorphic with a holomorphic inverse.
    (b) $\phi(z_0) = 0$.
    (c) We have $f(z) = f(z_0) + (\phi(z))^k$ with $z \in U$.

93. **Prop.** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. If $\sum_{n=1}^{\infty} |a_n| < \infty$, then $\prod_{n=1}^{\infty}(1 + a_n)$ converges and its value is 0 iff one of the $1 + a_n$ factors is zero.

94. **Prop.** Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions on Ω. If $\sum_{n=1}^{\infty} |f_n|$ converges uniformly on compact subsets of Ω, then so does $\prod_{n-1}^{\infty}(1 + f_n(z))$. Moreover, the limiting function is holomorphic and nonzero everywhere except at points z such that $1 + f_n(z) = 0$ (for some n).

95. **Prop. (Partial fractions expansion for log derivatives).** Same assumptions as the above proposition. Then, the log derivative of product = sum of log derivatives. i.e.

$$\frac{(\prod_{n=1}^{\infty}(1+f_n))'}{\prod_{n=1}^{\infty}(1+f_n)} = \sum_{n=1}^{\infty} \frac{f_n'}{1+f_n}.$$

96. **Infinite products formula for sine.** $\sin(\pi z) = \pi z \cdot \prod_{n=1}^{\infty}(1-\frac{z}{n})(1+\frac{z}{n})$, for $z \in \mathbb{C}$.

97. **Prop.** Fix $z \in \mathbb{C} \setminus \mathbb{Z}$ and a large positive $N \in \mathbb{Z}$. By the residue theorem, the integral $I_N(z) := \int_{\gamma_N} \frac{\pi \cot(\pi z)}{(w+z)^2} dw$.

98. **Cor.** $\cos(\pi z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\left(n-\frac{1}{2}\right)^2}\right)$, $e^z - 1 = ze^{z/2} \cdot \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{4\pi^2 n^2}\right)$, and $\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}$ (for the contangent expression, $z \in \mathbb{C} \setminus \mathbb{Z}$).

99. **Theorem (Schwarz Reflection Principle).** Let A be a region in the upper-half plane with $\partial A \cap \mathbb{R}$ nonempty and containing $[a,b] \subseteq \mathbb{R}$. Let f be holomorphic on A and continuous on $\partial A \cap [a,b]$ and real on $[a,b]$. Then f can be uniquely extended to a holomorphic function on $A \cup (a,b) \cup A_{\mathrm{ref}}$, where $A_{\mathrm{ref}} = \{\bar{z} \mid z \in A\}$ with $f(z) = \overline{f(\bar{z})}$ for all $z \in A_{\mathrm{ref}}$.

100. **Theorem (Gamma Function).** There is a unique $\Gamma(s)$ with the following:

   (a) $\Gamma(s)$ is meromorphic.

   (b) (Factorial). $\Gamma(n+1) = n!$ for $n = 0, 1, 2, \ldots$.

   (c) (Special Value). $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

   (d) (Integral Representation). $\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1} dx$ (for $\mathrm{Re}(s) > 0$).

   (e) (Infinite Product Representation). $\Gamma(s) = s^{-1} e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{s}\right)^{-1} e^{s/n}$ where $\gamma = \lim_{n\to\infty} \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right) \approx 0.58\ldots$ (Euler-Mascheroni constant) (for $s \in \mathbb{C}$, except poles).

   (f) (Limit of finite products). $\Gamma(s) = \lim_{n\to\infty} \frac{n! n^s}{s(s+1)\ldots(s+n)}$ (for $s \in \mathbb{C}$, except poles).

   (g) (Zeros). $\Gamma(s)$ has no zeros.

   (h) (Poles). $\Gamma(s)$ has poles at nonpositive integers $s = 0, -1, -2, \ldots$ and is holomorphic everywhere else. At $s = -n$, the pole is simple and $\mathrm{Res}_{-n}(\Gamma) = -\frac{1}{n!}$.

   (i) (Functional Equation). $\Gamma(s+1) = s\Gamma(s)$ (for $s \in \mathbb{C}$, except at poles).

   (j) (Reflection Formula). $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$ (for $s \in \mathbb{C}$, except at poles).

   (k) ADD THEOREMS FROM NOVEMBER 22 LECTURE