Math 185 Theorems

1. **Prop.** Let $z_1, z_2 \in \mathbb{C}$. Then $|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \pmod{2\pi}$.
2. **Theorem.** $\mathbb{C}$ is a field.
3. **De Moivre's Formula.** If $z = r(\cos\theta + i\sin\theta)$ and $n \in \mathbb{Z}_{>0}$, then $z^n = r^n(\cos n\theta + i\sin n\theta)$.
4. **Prop.** Let $z, w \in \mathbb{C}$. Then:
   (a) $\overline{z+w} = \bar{z} + \overline{w}$.
   (b) $\overline{zw} = \bar{z} \cdot \overline{w}$.
   (c) $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\overline{w}}$ (with $w \neq 0$).
   (d) $z\bar{z} = |z|^2$. If $z \neq 0$, then $z^{-1} = \frac{\bar{z}}{|z|^2}$.
   (e) If $z = \bar{z}$, then $z \in \mathbb{R}$ and so $z = \mathrm{Re}(z)$.
   (f) $\mathrm{Re}(z) = \frac{z+\bar{z}}{2}$ and $\mathrm{Im}(z) = \frac{z-\bar{z}}{2i}$.
   (g) $\bar{\bar{z}} = z$.
5. **Prop.** Let $z, w \in \mathbb{C}$. Then:
   (a) $|z| \geq 0$ and if $|z| = 0$, then $z = 0$.
   (b) $|zw| = |z||w|$.
   (c) If $w \neq 0$, then $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$.
   (d) $|\mathrm{Re}(z)| \leq |z|$ and $|\mathrm{Im}(z)| \leq |z|$.
   (e) $|\bar{z}| = |z|$.
   (f) $|z+w| \leq |z| + |w|$.
   (g) $||z| - |w|| \leq |z - w|$.
   (h) $|z_1 w_1 + \cdots + z_n w_n| \leq \sqrt{|z_1|^2 + \cdots + |z_n|^2}\sqrt{|w_1|^2 + \cdots + |w_n|^2}$.
6. **Prop.** Fix $r > 0$ and $z \in \mathbb{C}$. The open disk $D_\varepsilon(z_0)$ is an open set.
7. **Prop.** The following are true:
   (a) $\mathbb{C}$ is open.
   (b) The empty set ϕ is open.
   (c) The union of open sets is open.
   (d) The intersection of finitely many open sets is open.
8. **Prop.** Limits are unique (if they exist).
9. **Prop.** If $\lim_{z\to z_0} f(z) = a$ and $\lim_{z\to z_0} g(z) = b$, then:
   (a) $\lim_{z\to z_0}(f(z) + g(z)) = a + b$.
   (b) $\lim_{z\to z_0}((f(z)g(z))) = ab$.

(c) $\lim_{z\to z_0}\left(\frac{f(z)}{g(z)}\right)=\frac{a}{b}$ (with $b\neq 0$).

10. **Prop.** The following are true:

    (a) If $\lim_{z\to z_0} f(z)=a$ and h is continuous at a, then $\lim_{z\to z_0} h(f(z))=h(a)$.

    (b) If f is continuous on an open set $\Omega\subseteq\mathbb{C}$, and h is continuous on $f(\Omega)$, then $h\circ f$ is continuous on Ω, with $(h\circ f)(z)=h(f(z))$.

11. **Prop.** The following are true:

    (a) The empty set ϕ is closed.

    (b) $\mathbb{C}$ is closed.

    (c) The intersection of a collection of closed sets is closed.

    (d) The union of finitely many closed sets is closed.

12. **Prop.** A set F is closed iff whenever $z_1,z_2,z_3,\dots$ is a sequence of points in F converging to $\lim_{k\to\infty} z_k=w$, then $w\in F$.

13. **Prop.** If $f:\mathbb{C}\to\mathbb{C}$, TFAE:

    (a) f is continuous.

    (b) If $F\subseteq\mathbb{C}$ is closed, then $f^{-1}(F)$ is closed.

    (c) If Ω is open, then $f^{-1}(\Omega)$ is also open.

14. **Prop. (Heine-Borel + Sequential Compactness).** For $K\subseteq\mathbb{C}$, TFAE:

    (a) K is compact.

    (b) K is closed and bounded.

    (c) Every sequence of points in K has a convergent subsequence converging in K (sequentially compact).

15. **Prop.** If K is compact and $f:K\to\mathbb{C}$ is continuous, then the image $f(K)$ is compact.

16. **Theorem (Extreme Value Theorem).** If K is compact and $f:K\to\mathbb{R}$ is continuous, then f attains its minimum and maximum.

17. **Stereographic Projection / Riemann Sphere.** Identify the plane $\overline{\mathbb{C}}=S^2=\{(x,y,z)\in\mathbb{R}^3\mid x^2+y^2+z^2=1\}$. If $f:\mathbb{C}\to S^2\setminus\{N\}$, then we have $(u,v)\mapsto\frac{1}{1+u^2+v^2}(2u,2v,-1+u^2+v^2)$ is a homeomorphism (is continuous with continuous inverse) $f^{-1}:S^2\setminus\{N\}\to\mathbb{C}$ with $(x,y,z)\mapsto\left(\frac{x}{1-z},\frac{y}{1-z}\right)$.

18. **Prop. (Uniform Convergence).** If $f_n\to f$ uniformly and each f_n is continuous, then f is continuous.

19. **Euler's Formula.** $e^{iz}=\cos z+i\sin z$ for all $z\in\mathbb{C}$.

20. **Theorem.** $\cos z=\frac{e^{iz}+e^{-iz}}{2}$ and $\sin z=\frac{e^{iz}-e^{-iz}}{2i}$.

21. **Properties of e.** Let $x,y\in\mathbb{R}$ and $z,w\in\mathbb{C}$. Then:

    (a) $e^{z+w}=e^ze^w$.

    (b) $|e^{x+iy}|=|e^xe^{iy}|=|e^x||e^{iy}|=e^x$.

    (c) $\arg(e^{x+iy})=y \pmod{2\pi}$.

    (d) $e^z\neq 0$ for all $z\in\mathbb{C}$.

    (e) $e^z=1$ iff $z=2\pi in$ for some $n\in\mathbb{Z}$.

    (f) $e^z=e^{z+2\pi ni}$.

22. **Prop. (Chain Rule).** Let $\Omega, A \subseteq \mathbb{C}$ be open sets, and let $f : \Omega \to A$, $g : A \to \mathbb{C}$ be holomorphic functions. Then, $g \circ f : \Omega \to \mathbb{C}$ is holomorphic and $\frac{d}{dz}(f \circ g)(z) = \frac{dg}{df}(f(z)) \cdot \frac{df}{dz}(z)$.

23. **Prop.** Let $f : \Omega \to \mathbb{C}$ be holomorphic. Then $f : \Omega \to \mathbb{R}^2$ is real differentiable at all $(x,y) \in \Omega$.

24. **Cauchy-Riemann Equations.** Let Ω be an open set in $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be given by $f(x,y) = u(x,y) + iv(x,y)$. Then:

    (a) $f'(z)$ exists at $z \in \Omega$ iff f is real differentiable and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (these are the Cauchy-Riemann equations).

    (b) $f(z)$ is holomorphic on Ω iff partials are continuous and satisfy the CR equations.

    (c) If $f'(z_0)$ exists, then $f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y} = \frac{1}{i}\frac{\partial f}{\partial y}$.

25. **Inverse Function Theorem for $\mathbb{R}^2$.** If $f : \Omega \to \mathbb{R}^2$ is continuously differentiable and the Jacobian $Df(z_0)$ has $\det(Df(z_0)) \neq 0$, then there are neighborhoods $U \ni z_0$ and $V \ni f(z_0)$ such that $f : U \to V$is bijective with continuously differentiable $f^{-1} : V \to U$ such that $Df^{-1}(z_0) = [Df(z_0)]^{-1}$, which is the inverse matrix of $Df(z_0)$.