1. **Uniform Convergence.** A sequence of functions $f_n : \Omega \to \mathbb{C}$ converges uniformly to a function $f : \Omega \to \mathbb{C}$ if for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(z) - f(z)| < \varepsilon$ for all $z \in \Omega$.

2. **Real Differentiable.** f is real differentiable at z_0 if the following limit exists: $\lim_{h\to 0} \frac{f(z+h)-f(z)-D(f'(z))h}{h}$, where $D(f'(z))$ is the Jacobian matrix.

3. **Derivative.** Let $f : \Omega \to \mathbb{C}$, where Ω is a neighborhood of z_0. The derivative of f at z_0 is the limit $f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$. We call f **complex differentiable** at z_0 if this limit exists. If $f'(z_0)$ exists for all $z_0 \in \Omega$, we call f **holomorphic** on Ω.

4. **Principal Branch of the Logarithm.** Pick a branch $[a, a+2\pi)$. Then, $\log : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times i[a, a+2\pi)$ is defined by $\log z = \log|z| + i \arg z$, where $\arg z \in [a, a+2\pi)$. We call the branch $[-\pi, \pi)$ the principal branch.

5. **Exponentiation of complex numbers a, b.** Choose a branch of log, with $\log : \Omega \to \mathbb{C}$ and $a, b \in \mathbb{C}$. Then, define $a^b := e^{b \log a}$.

6. **Contour Integral.** Suppose f is continuous on an open set Ω and $\gamma : [a,b] \to \Omega$ is a smooth curve. Then the contour integral of f along γ is defined to be $\int_\gamma f := \int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt$.

7. **Re-parametrization of γ.** A piecewise smooth $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \mathbb{C}$ is called a re-parametrization of γ if there exists a continuously differentiable $\alpha : [a,b] \to [\tilde{a}, \tilde{b}]$ such that $\alpha(a) = \tilde{a}$ and $\alpha(b) = \tilde{b}$, and $\alpha'(t) > 0$ with $\gamma(t) = \tilde{\gamma}(\alpha(t))$.

8. **Path-connected.** We say an open set $\Omega \subseteq \mathbb{C}$ is path-connected if for any pair of points $z_0, z_1 \in \Omega$ there exists a continuous path $\gamma : [0,1) \to \mathbb{C}$ such that $z_0 = \gamma(0)$ and $z_1 = \gamma(1)$, with $\gamma([0,1)) \subseteq \Omega$.

9. **Path-independence.** If $z_0, z_1 \in \Omega$, then any paths $\gamma, \tilde{\gamma}$ (with shared endpoints $\gamma(0) = \tilde{\gamma}(0)$, $\gamma(1) = \tilde{\gamma}(1)$) have $\int_\gamma f(z)dz = \int_{\tilde{\gamma}} f(z)dz$.

10. **Simply-connected.** A set A is called simply-connected if every closed curve (loop) is homotopic to a point in A, with $H_s(t) \in A$ for all s, t. (Note: a point in A is a constant loop).

11. **Winding Number.** Let γ be a loop in $\mathbb{C}$ and $z_0 \in \mathbb{C}$ but not on γ. Then the winding number of γ (with respect to z_0) is $I(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z-z_0} dz$.

12. **Reflection:** $\tilde{z} = \frac{R^2}{\bar{z}}$. $\tilde{z} = \frac{R^2}{\bar{z}}$ is the reflection over the line the circle $|\xi| = R$.

13. **Analytic.** A function $f : \Omega \to \mathbb{C}$ is analytic at $z_0 \in \Omega$ if there is a neighborhood $\mathscr{U}$ of z_0 on which $f(z) = \sum_{k=0}^\infty a_k (z - z_0)^k$ (for all $z \in \mathscr{U}$) where the RHS is a convergent power series.

14. **Pole of Order m.** If f is holomorphic on a deleted neighborhood $\mathscr{U} \setminus z_0$, we say f has a pole of order m if $\frac{1}{f}$ has a zero of order m.

15. **Meromorphic.** A function $f : \Omega \to \mathbb{C}$ is meromorphic if it is holomorphic on all of Ω except at a discrete set of poles.

16. **Alternative Definition of Essential Singularity.** Let f be holomorphic except possibly at a point z_0. Let $C_1 = \{z_0\}$ and $C_2 = \partial D_r(z_0)$. Then, z_0 is an essential singularity if there are infinitely many a_{-n} in the Laurent series of f, where still $\text{Res}_{z_0}(f) = a_{-1}$.

17. **Holomorphic at Infinity.** Let $\mathscr{U} \subseteq \mathbb{C}$ be an open set containing $\mathbb{C} \setminus \overline{D_R(0)}$. A function $f : \mathscr{U} \to \mathbb{C}$ is holomorphic at infinity if $g(z) = f(1/z)$ has a removable singularity at 0, which in that case, we define $f(\infty) = g(0)$.

18. **Zero (respectively, pole) of order at Infinity.** Let $\mathscr{U} \subseteq \mathbb{C}$ be an open set containing $\mathbb{C} \setminus \overline{D_R(0)}$. We say that $f : \mathscr{U} \to \mathbb{C}$ has a zero (respectively, pole) of order at ∞ if $g(z) = f(1/z)$ has a zero (respectively, pole) at $z = 0$ (of order m).

19. **Logarithmic Derivative.** Let $f : \Omega \to \mathbb{C}$ be meromorphic. Then, the logarithmic derivative of f is f'/f.

20. **Locally injective.** We call a function $f : \Omega \to \mathbb{C}$ locally injective near z_0 if there exists a neighborhood $\mathscr{U}$ of z_0 such that $f : \mathscr{U} \to \mathbb{C}$ is injective.

21. **Conformal Maps.** Smooth, invertible maps $f : \mathbb{R}^2 \to \mathbb{R}^2$ whose Jacobian at a point can be factored as (scaling) $\cdot$ (rotation) are called conformal maps. (note: by Cauchy-Riemann equations, we have that $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = |a+bi|^2 \cdot \frac{1}{|a+bi|^2} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, where the $|a+bi|^2$ factor represents scaling and the other factors together represent the orthogonal scaling matrix).

22. **Conformal Maps (2nd definition).** Let Ω and Ω' be open connected regions in $\mathbb{C}$. We say that a map $g : \Omega \to \Omega'$ is conformal if it is holomorphic and invertible with g^{-1} holomorphic.

23. **Conformal Automorphisms.** We call a conformal map $g : \Omega \to \Omega$ a conformal automorphism and write $\text{Aut}(\Omega)$ to denote the collection of automorphisms (strictly conformal) on Ω.

24. **De Moivre's Formula.** If $z = r(\cos\theta + i \sin\theta)$ and $n \in \mathbb{Z}_{>0}$, then $z^n = r^n(\cos n\theta + i \sin n\theta)$.

25. **Theorem (Extreme Value Theorem).** If K is compact and $f : K \to \mathbb{R}$ is continuous, then f attains its minimum and maximum.

26. **Stereographic Projection / Riemann Sphere.** Identify the plane $\overline{\mathbb{C}} = S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. If $f : \mathbb{C} \to S^2 \setminus \{N\}$, then we have $(u,v) \mapsto \frac{1}{1+u^2+v^2}(2u, 2v, -1+u^2+v^2)$ is a homeomorphism (is continuous with continuous inverse) $f^{-1} : S^2 \setminus \{N\} \to \mathbb{C}$ with $(x,y,z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$.

27. **Prop. (Uniform Convergence).** If $f_n \to f$ uniformly and each f_n is continuous, then f is continuous.

28. **Theorem.** $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$.

29. **Properties of e.** Let $x, y \in \mathbb{R}$ and $z, w \in \mathbb{C}$. Then:
    (a) $e^{z+w} = e^z e^w$.
    (b) $|e^{x+iy}| = |e^x e^{iy}| = |e^x||e^{iy}| = e^x$.
    (c) $\arg(e^{x+iy}) = y \pmod{2\pi}$.
    (d) $e^z \neq 0$ for all $z \in \mathbb{C}$.
    (e) $e^z = 1$ iff $z = 2\pi i n$ for some $n \in \mathbb{Z}$.
    (f) $e^z = e^{z+2\pi n i}$.

30. **Prop.** Let $f : \Omega \to \mathbb{C}$ be holomorphic. Then $f : \Omega \to \mathbb{R}^2$ is real differentiable at all $(x,y) \in \Omega$.

31. **Cauchy-Riemann Equations.** Let Ω be an open set in $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be given by $f(x,y) = u(x,y) + iv(x,y)$. Then:
    (a) $f'(z)$ exists at $z \in \Omega$ iff f is real differentiable and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (these are the Cauchy-Riemann equations).
    (b) $f(z)$ is holomorphic on Ω iff partials are continuous and satisfy the CR equations.
    (c) If $f'(z_0)$ exists, then $f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y} = \frac{1}{i}\frac{\partial f}{\partial y}$.

32. **Inverse Function Theorem for $\mathbb{R}^2$.** If $f : \Omega \to \mathbb{R}^2$ is continuously differentiable and the Jacobian $Df(z_0)$ has $\det(Df(z_0)) \neq 0$, then there are neighborhoods $U \ni z_0$ and $V \ni f(z_0)$ such that $f : U \to V$ is bijective with continuously differentiable $f^{-1} : V \to U$ such that $Df^{-1}(z_0) = [Df(z_0)]^{-1}$, which is the inverse matrix of $Df(z_0)$.

33. **Inverse Function Theorem for $\mathbb{C}$.** Let $f : \Omega \to \mathbb{C}$ be holomorphic (with continuous $f'(z_0)$), and $f'(z) \neq 0$ for some $z_0 \in \Omega$. Then there exists a neighborhood $U \ni z_0$ and $V \ni f(z_0)$ such that $f : U \to V$ is bijective with holomorphic inverse $f^{-1} : V \to U$ such that for all $z_0 \in U$, $\frac{d}{dw} f^{-1}(w) = \frac{1}{f'(w)}$ with $w = f(z)$.

34. **Prop.** Pick a branch $[a, a+2\pi)$. Then $\log z : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times i[a, a+2\pi)$ is the inverse of $\exp : \mathbb{R} \times i[a, a+2\pi) \to \mathbb{C}$.

35. **Prop.** If $z_1, z_2 \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, then $\log(z_1 z_2) = \log z_1 + \log z_2 \pmod{2\pi i}$.

36. **Prop.** By choosing different branches of log, we have the following:
    (a) a^b is independent of the branch iff $b \in \mathbb{Z}$.
    (b) a^b takes on exactly q different values iff $b \in \mathbb{Q}$, so $b = \frac{p}{q}$ (with p, q coprime).
    (c) a^b takes on infinitely many values iff b is irrational or $\text{Im}(b) \neq 0$.

37. **Cor.** Choose a branch of log. Then the nth root function is given by $z^{1/n} = e^{\log(z/n)}$, where the nth root function has n branches.

38. **Prop.** Let $a, b \in \mathbb{C}$. Then:
    (a) For any choice of branch of log, the function $\mapsto a^z$ is holomorphic on $\mathbb{C}$, and $z \mapsto (\log a)a^z$.
    (b) Choose a branch of log. Then the function $z \mapsto z^b$ is holomorphic on the domain of log with derivative $z \mapsto bz^{b-1}$.

39. **Prop. (Re-parametrization).** If $\tilde{\gamma}$ is a re-parametrization of γ, then $\int_\gamma f = \int_{\tilde{\gamma}} f$ for any continuous f on Ω.

40. **Fundamental Theorem of Line Integrals.** Let $F : \Omega \to \mathbb{C}$ be holomorphic on an open Ω and let $\gamma : [0,1] \to \Omega$ be piecewise smooth. Then, $\int_\gamma F'(z)dz = F(\gamma(1)) - F(\gamma(0))$.

41. **Path-independence and Primitives Theorem.** Let $f : \Omega \to \mathbb{C}$ be continuous and Ω is open and connected. Then, TFAE:
    (a) (path-independence) if $z_0, z_1 \in \Omega$, then any paths $\gamma, \tilde{\gamma}$ with shared endpoints $\gamma(0) = \tilde{\gamma}(0)$ and $\gamma(1) = \tilde{\gamma}(1)$ have $\int_\gamma f(z)dz = \int_{\tilde{\gamma}} f(z)dz$.
    (b) (integral along loops is 0) if Γ is a loop, with $\Gamma(1) = \Gamma(0)$, then $\int_\Gamma f(z)dz = 0$.
    (c) (f has a primitive) There is a primitive F for f on Ω.

42. **Cauchy-Goursat Theorem.** Let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω, simply connected, and open. Then for any loop $\Gamma \subseteq \Omega$, $\int_\Gamma f(z)dz = 0$.

43. **Prop.** If $f(x+iy) = u(x,y) + iv(x,y)$, then $\int_\gamma f = \int_\gamma u dx - v dy + i \int_\gamma u dx + v dy$.

44. **Deformation Theorem.** Suppose f is holomorphic on an open set Ω and γ_0, γ_1 are piecewise continuously differentiable. Then there are continuously differentiable curves in Ω. Then:
    (a) If γ_0, γ_1 are paths from z_0 to z_1, which are homotopic in Ω, then $\int_{\gamma_0} F = \int_{\gamma_1} F$.
    (b) If γ_0, γ_1 are loops homotpic in Ω, then $\int_{\gamma_0} F = \int_{\gamma_1} F$.

45. **Cor.** If Ω is simply connected, then every loop γ has $\int_\gamma f dz = 0$.

46. **Cor.** Let $f : \Omega \to \mathbb{C}$ be holomorphic on a simply connected oen set Ω. Then, f has a primitive F on Ω (unique up to constants).

47. **Winding number (as an index).** Let $\gamma : [a,b] \to \mathbb{C}$ (a piecewise continuous) loop and $z \notin \gamma([a,b])$. Then, the winding number of γ around z_0 is an integer.

48. **Cauchy's Integral Formula.** Let f be holomorphic on Ω and γ a loop in Ω hommotopic to a point. Let $z_0 \in \Omega$ but $z_0 \notin \gamma$. Then,
$$f(z_0) \cdot I(\gamma, z_0) = \frac{1}{2\pi i} \cdot \int_\gamma \frac{f(z)}{z - z_0} dz.$$

49. **Cauchy's Integral Formula for Derivatives.** Let f be holomorphic on Ω. Then f is infinitely differentiable (complex) and if γ is a loop homotopic to a point (simple loop) $I(\gamma, z_0) = 1$, then:
$$f^{(n)}(z_0) = \frac{n}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi.$$

50. **Cor. Cauchy-Type Integrals.** Let γ be a loop $\gamma : [a,b] \to \mathbb{C}$ and g a continuous function on γ. Set $\tilde{g}(z) := \int_\gamma \frac{g(\xi)}{\xi - z} d\xi$. Then, $\tilde{g}(z)$ is holomorphic inside γ and so $\tilde{g}(z)$ is infinitely differentiable.

51. **Prop. (Cauchy Inequalities).** Let f be holomorphic on Ω and let $\overline{D_R(z_0)} \subseteq \Omega$ with boundary γ. Suppose $f(z)$ is bounded above $|f(z)| \leq M$ for all $z \in \gamma$. Then for all $k = 1, 2, \ldots$, the kth derivative is also upper bounded with $|f^{(k)}(z_0)| \leq \frac{k!}{R^k} M$.

52. **Louisville's Theorem.** If f is entire and bounded (i.e. there exists an $M \in \mathbb{R}_{>0}$ with $|f(z)| \leq M$ for all $z \in \mathbb{C}$), then f is constant.

53. **Morera's Theorem. (partial converse to Cauchy-Goursat)** Let f continuous on an open Ω and suppose that $\int_\gamma f = 0$ for every loop in Ω. Then, f is holomorphic on Ω and f has a primitive F on Ω.

54. **Cor. to Morera's Theorem (Removable Singularities Theorem).** Let f be continuous on an open Ω in $\mathbb{C}$ and holomorphic on $\Omega \setminus \{z_0\}$, with $z_0 \in \mathbb{C}$. Then, f is holomorphic on Ω.

55. **Another Cor. to Morera's Theorem.** If f is holomorphic on $\Omega \setminus \{z_0\}$ and bounded on a neighborhood of z_0, there is unique holomorphic extension $\tilde{f}$ of f to γ defined by $\tilde{f}(z) = f(z)$ if $z \neq z_0$ and $\tilde{f}(z) = \lim_{z \to z_0} f(z)$ if $z = z_0$.

56. **Mean Value Property.** Let f be holomorphic on $\overline{D_R(z_0)}$. Then $f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$.

57. **Maximum Modulus Principle.** Let A be an open, connected, bounded set in $\mathbb{C}$ and suppose $f : \overline{A} \to \mathbb{C}$ is holomorphic on A and continuous on $\overline{A}$. Then $|f|$ has a finite maximum value on $\overline{A}$ which is achieved on ∂A. IF $|f|$ is attained in A, then f is constant.

58. **Prop.** Let $u : \Omega \to \mathbb{R}$ be an twice-continuous harmonic function on an open set $\Omega \subseteq \mathbb{C}$. Then u is infinitely differentiable, so u is C^∞, and in the neighborhood U of $z_0 \in \Omega$, there exists a holomorphic function $f : U \to \mathbb{C}$ such that $u = \mathrm{Re}(f)$.

59. **Dirichlet Problem.** $\Delta u = 0$, $u\mid_{\partial\Omega}(\theta) = g(\theta)$. Let $u, \tilde{u}$ solve the Dirichlet Problem. Then, $u = \tilde{u}$, so the solution to the Dirichlet Problem is unique. Solution given by:

$$u(re^{i\phi}) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \cdot \frac{R^2 - r^2}{|Re^{i\theta} - re^{i\theta}|} d\theta.$$

60. **Analytic Convergence Theorem.** Let $f_n : \Omega \to \mathbb{C}$ be a sequence of holomorphic functions. If $f_n \to f$ uniformly on every closed disk in Ω, then:

    (a) f is holomorphic on Ω.

    (b) f_n' converges to f' uniformly on every closed disk, and pointwise on Ω.

61. **Prop.** Let $\gamma : [a,b] \to \Omega$ be a contour and $f_n : \gamma([a,b]) \to \mathbb{C}$ be a sequence of continuous functions. If $f_n \to f$ uniformly on $\gamma([a,b])$, then $\int_\gamma f_n \to \int_\gamma f$.

62. **Prop.** If f is holomorphic on an open, connected set Ω and the zero set $\{z \in \Omega \mid f(z) = 0\}$ contains a limit point, then $f = 0$ on Ω.

63. **Cor. (Identity Theorem).** Let f, g be holomorphic on an open, connected Ω and $f(z) = g(z)$ for a set of z with a limit point in Ω. Then, $f = g$ on Ω.

64. **Cor. (Zeros are Isolated).** If f is holomorphic on Ω, and not identically zero on Ω, then for any zero z_0 of f, there is a deleted neighborhood $U \setminus \{z_0\}$ on which $f(z) \neq 0$ for all $z \in U \setminus \{z_0\}$.

65. **Cor. (Analytic Continuation).** If f is holomorphic on an open, connected set Ω and f_+ is holomorphic on an open connected $\Omega_+ \supseteq \Omega$ with $f_+ = f$ on Ω, then f_+ is the unique such extension, i.e. if there exists another such extension, $\tilde{f}_+$, then $\tilde{f}_+ = f_+$.

66. **Lemma.** Let $f : \Omega \to \mathbb{C}$ be holomorphic, not identically 0, with a zero z_0. Then in a neighborhood U of z_0, we may write $f(z) = (z - z_0)^m g(z)$ for all $z \in U$, where $g(z) \neq 0$ and m is unique.

67. **Lemma.** A function f has a pole of order m at z_0 iff there is a neighborhood U of z_0 on which $f(z) = (z - z_0)^{-m} g(z)$ for all $z \in U \setminus \{z_0\}$ with a nonzero $g(z)$ and g holomorphic on U.

68. **Theorem.** If f has a pole of order m at z_0, then it can be represented uniquely as:

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z - z_0} + G(z)$$

where $G(z)$ is holomorphic on a neighborhood U of z_0 and $a_{-m}, \ldots, a_{-1} \in \mathbb{C}$ with $a_{-m} \neq 0$.

69. **Residue Theorem, Simple Closed Loops.** Let Ω be open, connected and γ a simple loop homotopic to a point in Ω. Let f be a function $f : \Omega \to \mathbb{C}$ be holomorphic except at a finite set of points $z_1, \ldots, z_N$ inside γ. Then, $\int_\gamma f(z) dz = 2\pi i \sum_{k=1}^N \mathrm{Res}_{z_k}(f)$.

70. **Laurent Series Theorem.** Let C_1, C_2 be two circles centered at z_0 (it is fine if $C_1 = \{z_0\}$ and C_2 "encloses" $\mathbb{C}$). Call R the region the annulus between C_1 and C_2. Let f be holomorphic on R. Then f can be expanded uniquely as a (absolutely) convergent power series in R by:

$$f(z) = \sum_{n=1}^\infty \frac{a_{-n}}{(z - z_0)^n} + \sum_{n=0}^\infty a_n (z - z_0)^n.$$

where the first infinite series is called the principal part and second one is called the Taylor series / holomorphic part.

71. **Casorati-Weierstrauss Theorem.** If f is holomorphic in a deleted $D_r(z_0) \setminus \{z_0\}$ and has an essential singularity at z_0, then the image of $f(D_r(z_0) \setminus \{z_0\})$ is dense in $\mathbb{C}$.

72. **Prop. (Fourier Transform).** Let $f : \mathbb{R} \to \mathbb{R}$ be a real function. The Fourier Transform of f is the function $\hat{f} : \mathbb{R} \to \mathbb{R}$ given by $\hat{f}(k) = \int_{-\infty}^\infty f(x) \cdot e^{-ikx} dx$.

73. **Jordan's Lemma.** $\int_0^\pi e^{-R\sin\theta} d\theta \leq \frac{\pi}{R}$.

74. **Argument Principle.** Let $f : \Omega \to \mathbb{C}$ be meromorphic and γ a simple loop in Ω bounding a simply connected region R_γ, with $\overline{R_\gamma} \subseteq \Omega$. Let f have no zeros or poles on γ. Then:

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = N - P$$

where N is the number of zeros of f inside R_γ (counting multiplicity) and P is the number of poles of f inside R_γ (counting multiplicity).

75. **Rouche's Theorem.** Let $f, g : \Omega \to \mathbb{C}$ be holomorphic and let γ be a simple loop bounding a simply connected open U, with $\overline{U} \subseteq \Omega$. If $|f(z)| > |g(z)|$ on γ, then f and $f + g$ have the same number of zeros inside U.

76. **Open-Mapping Theorem.** Any nonconstant holomorphic function is an open map, meaning it maps open sets to open sets.

77. **Lemma. (Local Injectivity).** Let $f : \Omega \to \mathbb{C}$ be holomorphic and $z_0 \in \Omega$. If $f'(z_0) \neq 0$, then f is locally injective near z.

78. **Theorem.** If $g : \Omega \to \mathbb{C}$ is holomorphic and Ω is simply connected, and $g \neq 0$, there exists a holomorphic function $F : \Omega \to \mathbb{C}$ satisfying $e^{F(z)} = g(z)$, where $F(z)$ is unique up to $2\pi i k$, with $k \in \mathbb{Z}$.

79. **Theorem (Local description of holomorphic).** Let $f : \Omega \to \mathbb{C}$ be holomorphic, Ω open. Let $z_0 \in \Omega$ and let $k \geq 1$ denote the order of the zero $f(z) - f(z_0)$ at z_0. Then, there exists an open neighborhood U of z_0 (and $r > 0$) and a function $\phi : U \to D_r(z_0)$ such that:

    (a) ϕ is holomorphic with a holomorphic inverse.

    (b) $\phi(z_0) = 0$.

    (c) We have $f(z) = f(z_0) + (\phi(z))^k$ with $z \in U$.

80. **Prop.** Let $\{a_n\}_{n=1}^\infty$ be a sequence of complex numbers. If $\sum_{n=1}^\infty |a_n| < \infty$, then $\prod_{n=1}^\infty (1 + a_n)$ converges and its value is 0 iff one of the $1 + a_n$ factors is zero.

81. **Prop.** Let $\{f_n\}_{n=1}^\infty$ be a sequence of holomorphic functions on Ω. If $\sum_{n=1}^\infty |f_n|$ converges uniformly on compact subsets of Ω, then so does $\prod_{n=1}^\infty (1 + f_n(z))$. Moreover, the limiting function is holomorphic and nonzero everywhere except at points z such that $1 + f_n(z) = 0$ (for some n).

82. **Prop. (Partial fractions expansion for log derivatives).** Same assumptions as the above proposition. Then, the log derivative of product = sum of log derivatives. i.e.

$$\frac{\left(\prod_{n=1}^\infty (1 + f_n)\right)'}{\prod_{n=1}^\infty (1 + f_n)} = \sum_{n=1}^\infty \frac{f_n'}{1 + f_n}.$$

83. **Cor.** $\sin(\pi z) = \pi z \cdot \prod_{n=1}^\infty (1 - \frac{z}{n})(1 + \frac{z}{n})$ for $z \in \mathbb{C}$, $\cos(\pi z) = \prod_{n=1}^\infty \left(1 - \frac{z^2}{\left(n - \frac{1}{2}\right)^2}\right)$, $e^z - 1 = ze^{z/2} \cdot \prod_{n=1}^\infty \left(1 + \frac{z^2}{4\pi^2 n^2}\right)$, and $\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^\infty \frac{2z}{z^2 - n^2}$ (for the contangent expression, $z \in \mathbb{C} \setminus \mathbb{Z}$).

84. **Theorem (Schwarz Reflection Principle).** Let A be a region in the upper-half plane with $\partial A \cap \mathbb{R}$ nonempty and containing $[a,b] \subseteq \mathbb{R}$. Let f be holomorphic on A and continuous on $\partial A \cap [a,b]$ and real on $[a,b]$. Then f can be uniquely extended to a holomorphic function on $A \cup (a,b) \cup A_{\mathrm{ref}}$, where $A_{\mathrm{ref}} = \{\overline{z} \mid z \in A\}$ with $f(z) = \overline{f(\overline{z})}$ for all $z \in A_{\mathrm{ref}}$.

85. **Theorem (Gamma Function).** There is a unique $\Gamma(s)$ with the following:

    (a) $\Gamma(s)$ is meromorphic.

    (b) (Factorial). $\Gamma(n+1) = n!$ for $n = 0, 1, 2, \ldots$.

    (c) (Special Value). $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

    (d) (Integral Representation). $\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx$ (for $\mathrm{Re}(s) > 0$).

    (e) (Infinite Product Representation). $\Gamma(s) = s^{-1} e^{\gamma s} \prod_{n=1}^\infty \left(1 + \frac{s}{s}\right)^{-1} e^{s/n}$ where $\gamma = \lim_{n \to \infty} \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right) \approx 0.58...$ (Euler-Mascheroni constant) (for $s \in \mathbb{C}$, except poles).

    (f) (Limit of finite products). $\Gamma(s) = \lim_{n \to \infty} \frac{n! n^s}{s(s+1)\ldots(s+n)}$ (for $s \in \mathbb{C}$, except poles).

    (g) (Zeros). $\Gamma(s)$ has no zeros.

    (h) (Poles). $\Gamma(s)$ has poles at nonpositive integers $s = 0, -1, -2, \ldots$ and is holomorphic everywhere else. At $s = -n$, the pole is simple and $\mathrm{Res}_{-n}(\Gamma) = -\frac{1}{n!}$.

    (i) (Functional Equation). $\Gamma(s+1) = s\Gamma(s)$ (for $s \in \mathbb{C}$, except at poles).

    (j) (Reflection Formula). $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$ (for $s \in \mathbb{C}$, except at poles).

86. **Cor.** Holomorphic f is locally injective iff $f'(z_0) \neq 0$.

87. **Conformal Equivalence Classes.** These are the following:

    (a) Complex plane, $\mathbb{C}$.

    (b) Punctured plane, $\mathbb{C} \setminus \{0\}$.

    (c) Unit disk, $D_1(0)$.

    (d) Upper-half plane, $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.

    (e) Riemann sphere*, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (not a subset of $\mathbb{C}$ but we can still talk about holomorphic/meromorphic functions on it).

    (f) The slit plane, $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

    (g) Strip (similar to critical region from Riemann hypothesis).

(h) Rectangle.

(i) Annulus.

(j) Blob

88. **Prop.** If $g : \Omega \to \Omega'$ is holomorphic and invertible, then g^{-1} is holomorphic (i.e. g is conformal).

89. **Lemma.** $\mathrm{Aut}(\Omega)$ is a group, with function composition. In other words, let $f, g, h \in \mathrm{Aut}(\Omega)$. Then:

    (a) $(g \circ f) \circ h = g \circ (f \circ h)$.

    (b) If $g \in \mathrm{Aut}(\Omega)$, then $g^{-1} \in \mathrm{Aut}(\Omega)$.

    (c) There is an identity map $\mathrm{id} \in \mathrm{Aut}(\Omega)$ such that $\mathrm{id} \circ g(z) = g(z) = g \circ \mathrm{id}(z) = g(z)$.

90. **Theorem.** Let $g : \mathbb{C} \to \Omega$ be a conformal map between $\mathbb{C}$ and a region Ω. Then, $\Omega = \mathbb{C}$ and $g(z)$ is a conformal automorphism of the form $g(z) = az + b$, with $a \neq 0$ and $b \in \mathbb{C}$.

91. **Theorem (Riemann Sphere).** Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. If $g : \hat{\mathbb{C}} \to \Omega$ is a conformal map, then $\Omega = \hat{\mathbb{C}}$ and g is a conformal automorphism, with $g(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{C}$ (i.e. g is a Möbius transformation).