Math 185 Definitions

1. **Complex Numbers, $\mathbb{C}$.** The set of complex numbers $\mathbb{C}$ is the real vector space $\mathbb{R}^2$ with the properties $(x_1,y_1)+(x_2,y_2)+(x_1+x_2,y_1+y_2)$ and $a(x_1,y_1)=(ax_1,ay_1)$ and we write $z=x+iy=(x,y)$ for any $z\in\mathbb{C}$.
2. **Polar Form.** Let $z=a+bi\in\mathbb{C}$. Then norm (a.k.a. modulus, absolute value) of $z\in\mathbb{C}$ is written $|z|\in\mathbb{R}$ and is defined by $|z|=\sqrt{a^2+b^2}$. Then, define the argument of z as $\arg(z)=\theta\in[0,2\pi)$ as the angle z makes with the real axis. Then, the polar form of z is written as $z=|z|(\cos\theta+i\sin\theta)$.
3. **Rectangular form.** If $z\in\mathbb{C}$ is written as $z=x+iy$, with $x,y\in\mathbb{R}$, then z is written in rectangular form.
4. **Complex Conjugate.** If $z=a+ib$, then its complex conjugate is $\bar{z}=a-ib$.
5. **Open Sets.** A set $\Omega\subseteq\mathbb{C}$ is called open if for each $z_0\in\mathbb{C}$, there is an $\varepsilon>0$ such that $D_\varepsilon(z_0)\subseteq\Omega$.
6. **Neighborhoods.** An ε-neighborhood of a point z_0 is a set N which contains some open disk $D_\varepsilon(z_0)$.
7. **ε-deleted neighborhoods.** An ε-deleted neighborhood of a point z_0 is a set N which contains a "punctured" open disk $D_\varepsilon(z_0)\setminus\{z_0\}$.
8. **Homeomorphism.** A function is a homeomorphism if it is continuous with a continuous inverse.
9. **Periodic.** $f(z)$ is w-periodic (with $w\in\mathbb{C}$) if $f(z+nw)=f(z)$ for all $z\in\mathbb{C},n\in\mathbb{Z}$.
10. **Limits.** Let $f:\Omega\to\mathbb{C}$ where Ω is an r-deleted neighborhood of a point z_0. Then f has a limit as $z\to z_0$, and write $\lim_{z\to z_0}f(z)=a$. This means that for every $\varepsilon>0$ there exists a $\delta>0$ such that if $z\in\Omega$ has $|z-z_0|<\delta$, then $|f(z)-a|<\varepsilon$.
11. **Continuity.** Let $\Omega\subseteq\mathbb{C}$ be an open set. Then $f:\Omega\to\mathbb{C}$ is continuous at a point $z_0\in\Omega$ if $\lim_{z\to z_0}f(z)=f(z_0)$.
12. **Closed Sets.** A subset $F\subseteq\mathbb{C}$ is called closed if its complement $\mathbb{C}\setminus F$ is open.
13. **Compact.** A subset $K\subseteq\mathbb{C}$ is called compact if every open cover of K has a finite subcover.
14. **Uniform Convergence.** A sequence of functions $f_n:\Omega\to\mathbb{C}$ converges uniformly to a function $f:\Omega\to\mathbb{C}$ if for all $\varepsilon>0$ there exists an $N\in\mathbb{N}$ such that if $n\geq N$, then $|f_n(z)-f(z)|<\varepsilon$ for all $z\in\Omega$.
15. **Real Differentiable.** f is real differentiable at z_0 if the following limit exists: $\lim_{h\to 0}\frac{f(z+h)-f(z)-D(f'(z))h}{h}$, where $D(f'(z))$ is the Jacobian matrix.
16. **Derivative.** Let $f:\Omega\to\mathbb{C}$, where Ω is a neighborhood of z_0. The derivative of f at z_0 is the limit $f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$. We call f **complex differentiable** at z_0 if this limit exists. If $f'(z_0)$ exists for all $z_0\in\Omega$, we call f **holomorphic** on Ω.
17. **Branch of the Argument.** This is a choice of interval (here, $[-\pi,\pi)$ or $[a,a+2\pi)$).
18. **Principal Branch of the Logarithm.** Pick a branch $[a,a+2\pi)$. Then, $\log:\mathbb{C}\setminus\{0\}\to\mathbb{R}\times i[a,a+2\pi)$ is defined by $\log z=\log|z|+i\arg z$, where $\arg z\in[a,a+2\pi)$. We call the branch $[-\pi,\pi)$ the principal branch.

19. **Exponentiation of complex numbers a, b.** Choose a branch of log, with $\log : \Omega \to \mathbb{C}$ and $a, b \in \mathbb{C}$. Then, define $a^b := e^{b \log a}$.

20. **Contour Integral.** Suppose f is continuous on an open set Ω and $\gamma : [a,b] \to \Omega$ is a smooth curve. Then the contour integral of f along γ is defined to be $\int_\gamma f := \int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt$.

21. **Re-parametrization of γ.** A piecewise smooth $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \mathbb{C}$ is called a re-parametrization of γ if there exists a continuously differentiable $\alpha : [a,b] \to [\tilde{a}, \tilde{b}]$ such that $\alpha(a) = \tilde{a}$ and $\alpha(b) = \tilde{b}$, and $\alpha'(t) > 0$ with $\gamma(t) = \tilde{\gamma}(\alpha(t))$.

22. **Primitive.** We say a function $f : \Omega \to \mathbb{C}$ has a primitive on Ω if there exists a holomorphic function $F : \Omega \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in \Omega$.

23. **Path-connected.** We say an open set $\Omega \subseteq \mathbb{C}$ is path-connected if for any pair of points $z_0, z_1 \in \Omega$ there exists a continuous path $\gamma : [0,1) \to \mathbb{C}$ such that $z_0 = \gamma(0)$ and $z_1 = \gamma(1)$, with $\gamma([0,1)) \subseteq \Omega$.

24. **Path-independence.** If $z_0, z_1 \in \Omega$, then any paths $\gamma, \tilde{\gamma}$ (with shared endpoints $\gamma(0) = \tilde{\gamma}(0)$, $\gamma(1) = \tilde{\gamma}(1)$) have $\int_\gamma f(z)dz = \int_{\tilde{\gamma}} f(z)dz$.

25. **Homotopy.** Let $\gamma_{0,1} : [a,b] \to \mathbb{C}$ be curves with shared endpoints z_a, z_b. A homotopy is a continuous function $H : [a,b] \times [0,1] \to \mathbb{C}$ with $t \times s \to H_s(t)$ such that $H_0(t) = \gamma_0(t)$ and $H_1(t) = \gamma_1(t)$.

26. **Simply-connected.** A set A is called simply-connected if every closed curve (loop) is homotopic to a point in A, with $H_s(t) \in A$ for all s, t. (Note: a point in A is a constant loop).

27. **Winding Number.** Let γ be a loop in $\mathbb{C}$ and $z_0 \in \mathbb{C}$ but not on γ. Then the winding number of γ (with respect to z_0) is $I(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} dz$.

28. **Entire.** We call a function $f : \mathbb{C} \to \mathbb{C}$ entire if it is holomorphic on $\mathbb{C}$.

29. **Closure.** The closure of a set A, written $\overline{A}$, is $\overline{A} = \{\text{limit points of } A\}$.

30. **Boundary of A.** The boundary ∂A of a set $A \subseteq \mathbb{C}$ is $\partial A = \overline{A} \cap \overline{(\mathbb{C} \setminus A)}$.

31. **Reflection: $\tilde{z} = \frac{R^2}{\bar{z}}$.** $\tilde{z} = \frac{R^2}{\bar{z}}$ is the reflection over the line the circle $|\xi| = R$.

32. **Analytic.** A function $f : \Omega \to \mathbb{C}$ is analytic at $z_0 \in \Omega$ if there is a neighborhood $\mathscr{U}$ of z_0 on which $f(z) = \sum_{k=0}^\infty a_k (z - z_0)^k$ (for all $z \in \mathscr{U}$) where the RHS is a convergent power series.

33. **Pole of Order m.** If f is holomorphic on a deleted neighborhood $\mathscr{U} \setminus z_0$, we say f has a pole of order m if $\frac{1}{f}$ has a zero of order m.

34. **Simple Pole.** If f has a pole of order 1 at $z = z_0$, then f has a simple pole at z_0.

35. **Residue of f at $z = z_0$.** Consider the principal part of the Laurent expansion of f at $z = z_0$. Then, the coefficient a_{-1} is the residue of f at z_0 and we write it as $\text{Res}_{z_0}(f) = a_{-1}$.

36. **Meromorphic.** A function $f : \Omega \to \mathbb{C}$ is meromorphic if it is holomorphic on all of Ω except at a discrete set of poles.

37. **Essential Singularity.** Let f be holomorphic on Ω except at a point z_0. We call z_0 an essential singularity if z_0 is neither a pole nor a removable singularity.

38. **Alternative Definition of Essential Singularity.** Let f be holomorphic except possibly at a point z_0. Let $C_1 = \{z_0\}$ and $C_2 = \partial D_r(z_0)$. Then, z_0 is an essential singularity if there are infinitely many a_{-n} in the Laurent series of f, where still $\text{Res}_{z_0}(f) = a_{-1}$.

39. **Holomorphic at Infinity.** Let $\mathscr{U} \subseteq \mathbb{C}$ be an open set containing $\mathbb{C} \setminus \overline{D_R(0)}$. A function $f : \mathscr{U} \to \mathbb{C}$ is holomorphic at infinity if $g(z) = f(1/z)$ has a removable singularity at 0, which in that case, we define $f(\infty) = g(0)$.

40. **Zero (respectively, pole) of order at Infinity.** Let $\mathscr{U} \subseteq \mathbb{C}$ be an open set containing $\mathbb{C} \setminus \overline{D_R(0)}$. We say that $f : \mathscr{U} \to \mathbb{C}$ has a zero (respectively, pole) of order at ∞ if $g(z) = f(1/z)$ has a zero (respectively, pole) at $z = 0$ (of order m).

41. **Logarithmic Derivative.** Let $f : \Omega \to \mathbb{C}$ be meromorphic. Then, the logarithmic derivative of f is f'/f.

42. **Locally injective.** We call a function $f : \Omega \to \mathbb{C}$ locally injective near z_0 if there exists a neighborhood $\mathscr{U}$ of z_0 such that $f : \mathscr{U} \to \mathbb{C}$ is injective.

43. **Infinite Product.** Suppose the sequence of finite products $P_N := \prod_{n=1}^{N} c_n = c_1 c_2 \ldots c_N$ converges to a finite number (where $c_i \in \mathbb{C}$ for all i). We define $\prod_{n=1}^{\infty} c_n = \lim_{N\to\infty} \prod_{n=1}^{N} c_n$ to be the infinite product, and say that this infinite product converges.

44. **Riemann-Zeta Function.** We define the Riemann-Zeta function to be $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$.

45. **Conformal Maps.** Smooth, invertible maps $f : \mathbb{R}^2 \to \mathbb{R}^2$ whose Jacobian at a point can be factored as (scaling) $\cdot$ (rotation) are called conformal maps. (note: by Cauchy-Riemann equations, we have that $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = |a+bi|^2 \cdot \frac{1}{|a+bi|^2} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, where the $|a+bi|^2$ factor represents scaling and the other factors together represent the orthogonal scaling matrix).

46. **Conformal Maps (2nd definition).** Let Ω and Ω' be open connected regions in $\mathbb{C}$. We say that a map $g : \Omega \to \Omega'$ is conformal if it is holomorphic and invertible with g^{-1} holomorphic.

47. **Conformally Equivalent.** We call Ω, Ω' conformally equivalent (write: $\Omega \sim \Omega'$) if there exists a conformal map $g : \Omega \to \Omega'$.

48. **Set of holomorphic functions / meromorphic functions.** Let $\mathscr{H}(\Omega) = \{\text{holomorphic functions } f : \Omega \to \mathbb{C}\}$ and $\mathscr{M}(\Omega) = \{\text{meromorphic functions } f : \Omega \to \mathbb{C}\}$.

49. **Conformal Automorphisms.** We call a conformal map $g : \Omega \to \Omega$ a conformal automorphism and write $\mathrm{Aut}(\Omega)$ to denote the collection of automorphisms (strictly conformal) on Ω.

50. ADD DEFINITIONS FROM DECEMBER 2 LECTURE