

# Math 185 Definitions

1. **Complex Numbers,  $\mathbb{C}$ .** The set of complex numbers  $\mathbb{C}$  is the real vector space  $\mathbb{R}^2$  with the properties  $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$  and  $a(x_1, y_1) = (ax_1, ay_1)$  and we write  $z = x + iy = (x, y)$  for any  $z \in \mathbb{C}$ .
2. **Polar Form.** Let  $z = a + bi \in \mathbb{C}$ . Then norm (a.k.a. modulus, absolute value) of  $z \in \mathbb{C}$  is written  $|z| \in \mathbb{R}$  and is defined by  $|z| = \sqrt{a^2 + b^2}$ . Then, define the argument of  $z$  as  $\arg(z) = \theta \in [0, 2\pi)$  as the angle  $z$  makes with the real axis. Then, the polar form of  $z$  is written as  $z = |z|(\cos \theta + i \sin \theta)$ .
3. **Rectangular form.** If  $z \in \mathbb{C}$  is written as  $z = x + iy$ , with  $x, y \in \mathbb{R}$ , then  $z$  is written in rectangular form.
4. **Complex Conjugate.** If  $z = a + ib$ , then its complex conjugate is  $\bar{z} = a - ib$ .
5. **Open Sets.** A set  $\Omega \subseteq \mathbb{C}$  is called open if for each  $z_0 \in \Omega$ , there is an  $\varepsilon > 0$  such that  $D_\varepsilon(z_0) \subseteq \Omega$ .
6. **Neighborhoods.** An  $\varepsilon$ -neighborhood of a point  $z_0$  is a set  $N$  which contains some open disk  $D_\varepsilon(z_0)$ .
7.  **$\varepsilon$ -deleted neighborhoods.** An  $\varepsilon$ -deleted neighborhood of a point  $z_0$  is a set  $N$  which contains a "punctured" open disk  $D_\varepsilon(z_0) \setminus \{z_0\}$ .
8. **Homeomorphism.** A function is a homeomorphism if it is continuous with a continuous inverse.
9. **Periodic.**  $f(z)$  is  $w$ -periodic (with  $w \in \mathbb{C}$ ) if  $f(z + nw) = f(z)$  for all  $z \in \mathbb{C}, n \in \mathbb{Z}$ .
10. **Limits.** Let  $f : \Omega \rightarrow \mathbb{C}$  where  $\Omega$  is an  $r$ -deleted neighborhood of a point  $z_0$ . Then  $f$  has a limit as  $z \rightarrow z_0$ , and write  $\lim_{z \rightarrow z_0} f(z) = a$ . This means that for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that if  $z \in \Omega$  has  $|z - z_0| < \delta$ , then  $|f(z) - a| < \varepsilon$ .
11. **Continuity.** Let  $\Omega \subseteq \mathbb{C}$  be an open set. Then  $f : \Omega \rightarrow \mathbb{C}$  is continuous at a point  $z_0 \in \Omega$  if  $\lim_{z \rightarrow z_0} f(z) = f(z_0)$ .
12. **Closed Sets.** A subset  $F \subseteq \mathbb{C}$  is called closed if its complement  $\mathbb{C} \setminus F$  is open.
13. **Compact.** A subset  $K \subseteq \mathbb{C}$  is called compact if every open cover of  $K$  has a finite subcover.
14. **Uniform Convergence.** A sequence of functions  $f_n : \Omega \rightarrow \mathbb{C}$  converges uniformly to a function  $f : \Omega \rightarrow \mathbb{C}$  if for all  $\varepsilon > 0$  there exists an  $N \in \mathbb{N}$  such that if  $n \geq N$ , then  $|f_n(z) - f(z)| < \varepsilon$  for all  $z \in \Omega$ .
15. **Real Differentiable.**  $f$  is real differentiable at  $z_0$  if the following limit exists:  $\lim_{h \rightarrow 0} \frac{f(z+h) - f(z) - D(f'(z))h}{h}$ , where  $D(f'(z))$  is the Jacobian matrix.
16. **Derivative.** Let  $f : \Omega \rightarrow \mathbb{C}$ , where  $\Omega$  is a neighborhood of  $z_0$ . The derivative of  $f$  at  $z_0$  is the limit  $f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$ . We call  $f$  **complex differentiable** at  $z_0$  if this limit exists. If  $f'(z_0)$  exists for all  $z_0 \in \Omega$ , we call  $f$  **holomorphic** on  $\Omega$ .
17. **Branch of the Argument.** This is a choice of interval (here,  $[-\pi, \pi)$  or  $[a, a + 2\pi)$ ).
18. **Principal Branch of the Logarithm.** Pick a branch  $[a, a + 2\pi)$ . Then,  $\log : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{R} \times i[a, a + 2\pi)$  is defined by  $\log z = \log |z| + i \arg z$ , where  $\arg z \in [a, a + 2\pi)$ . We call the branch  $[-\pi, \pi)$  the principal branch.

19. **Exponentiation of complex numbers  $a, b$ .** Choose a branch of  $\log$ , with  $\log : \Omega \rightarrow \mathbb{C}$  and  $a, b \in \mathbb{C}$ . Then, define  $a^b := e^{b \log a}$ .
20. **Contour Integral.** Suppose  $f$  is continuous on an open set  $\Omega$  and  $\gamma : [a, b] \rightarrow \Omega$  is a smooth curve. Then the contour integral of  $f$  along  $\gamma$  is defined to be  $\int_\gamma f := \int_\gamma f(z) dz := \int_a^b f(\gamma(t)) \gamma'(t) dt$ .
21. **Re-parametrization of  $\gamma$ .** A piecewise smooth  $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \rightarrow \mathbb{C}$  is called a re-parametrization of  $\gamma$  if there exists a continuously differentiable  $\alpha : [a, b] \rightarrow [\tilde{a}, \tilde{b}]$  such that  $\alpha(a) = \tilde{a}$  and  $\alpha(b) = \tilde{b}$ , and  $\alpha'(t) > 0$  with  $\gamma(t) = \tilde{\gamma}(\alpha(t))$ .
22. **Primitive.** We say a function  $f : \Omega \rightarrow \mathbb{C}$  has a primitive on  $\Omega$  if there exists a holomorphic function  $F : \Omega \rightarrow \mathbb{C}$  such that  $F'(z) = f(z)$  for all  $z \in \Omega$ .
23. **Path-connected.** We say an open set  $\Omega \subseteq \mathbb{C}$  is path-connected if for any pair of points  $z_0, z_1 \in \Omega$  there exists a continuous path  $\gamma : [0, 1] \rightarrow \Omega$  such that  $z_0 = \gamma(0)$  and  $z_1 = \gamma(1)$ , with  $\gamma([0, 1]) \subseteq \Omega$ .
24. **Path-independence.** If  $z_0, z_1 \in \Omega$ , then any paths  $\gamma, \tilde{\gamma}$  (with shared endpoints  $\gamma(0) = \tilde{\gamma}(0)$ ,  $\gamma(1) = \tilde{\gamma}(1)$ ) have  $\int_\gamma f(z) dz = \int_{\tilde{\gamma}} f(z) dz$ .
25. **Homotopy.** Let  $\gamma_{0,1} : [a, b] \rightarrow \mathbb{C}$  be curves with shared endpoints  $z_a, z_b$ . A homotopy is a continuous function  $H : [a, b] \times [0, 1] \rightarrow \mathbb{C}$  with  $t \times s \rightarrow H_s(t)$  such that  $H_0(t) = \gamma_0(t)$  and  $H_1(t) = \gamma_1(t)$ .
26. **Simply-connected.** A set  $A$  is called simply-connected if every closed curve (loop) is homotopic to a point in  $A$ , with  $H_s(t) \in A$  for all  $s, t$ . (Note: a point in  $A$  is a constant loop).
27. **Winding Number.** Let  $\gamma$  be a loop in  $\mathbb{C}$  and  $z_0 \in \mathbb{C}$  but not on  $\gamma$ . Then the winding number of  $\gamma$  (with respect to  $z_0$ ) is  $I(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} dz$ .
28. **Entire.** We call a function  $f : \mathbb{C} \rightarrow \mathbb{C}$  entire if it is holomorphic on  $\mathbb{C}$ .
29. **Closure.** The closure of a set  $A$ , written  $\bar{A}$ , is  $\bar{A} = \{\text{limit points of } A\}$ .
30. **Boundary of  $A$ .** The boundary  $\partial A$  of a set  $A \subseteq \mathbb{C}$  is  $\partial A = \bar{A} \cap \overline{(\mathbb{C} \setminus A)}$ .
31. **Reflection:**  $\tilde{z} = \frac{R^2}{\bar{z}}$ .  $\tilde{z} = \frac{R^2}{\bar{z}}$  is the reflection over the line the circle  $|\xi| = R$ .
32. **Analytic.** A function  $f : \Omega \rightarrow \mathbb{C}$  is analytic at  $z_0 \in \Omega$  if there is a neighborhood  $\mathcal{U}$  of  $z_0$  on which  $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$  (for all  $z \in \mathcal{U}$ ) where the RHS is a convergent power series.
33. **Pole of Order  $m$ .** If  $f$  is holomorphic on a deleted neighborhood  $\mathcal{U} \setminus z_0$ , we say  $f$  has a pole of order  $m$  if  $\frac{1}{f}$  has a zero of order  $m$ .
34. **Simple Pole.** If  $f$  has a pole of order 1 at  $z = z_0$ , then  $f$  has a simple pole at  $z_0$ .
35. **Residue of  $f$  at  $z = z_0$ .** Consider the principal part of the Laurent expansion of  $f$  at  $z = z_0$ . Then, the coefficient  $a_{-1}$  is the residue of  $f$  at  $z_0$  and we write it as  $\text{Res}_{z_0}(f) = a_{-1}$ .
36. **Meromorphic.** A function  $f : \Omega \rightarrow \mathbb{C}$  is meromorphic if it is holomorphic on all of  $\Omega$  except at a discrete set of poles.
37. **Essential Singularity.** Let  $f$  be holomorphic on  $\Omega$  except at a point  $z_0$ . We call  $z_0$  an essential singularity if  $z_0$  is neither a pole nor a removable singularity.
38. **Alternative Definition of Essential Singularity.** Let  $f$  be holomorphic except possibly at a point  $z_0$ . Let  $C_1 = \{z_0\}$  and  $C_2 = \partial D_r(z_0)$ . Then,  $z_0$  is an essential singularity if there are infinitely many  $a_{-n}$  in the Laurent series of  $f$ , where still  $\text{Res}_{z_0}(f) = a_{-1}$ .
39. **Holomorphic at Infinity.** Let  $\mathcal{U} \subseteq \mathbb{C}$  be an open set containing  $\mathbb{C} \setminus \overline{D_R(0)}$ . A function  $f : \mathcal{U} \rightarrow \mathbb{C}$  is holomorphic at infinity if  $g(z) = f(1/z)$  has a removable singularity at 0, which in that case, we define  $f(\infty) = g(0)$ .

40. **Zero (respectively, pole) of order at Infinity.** Let  $\mathcal{U} \subseteq \mathbb{C}$  be an open set containing  $\mathbb{C} \setminus \overline{D_R(0)}$ . We say that  $f: \mathcal{U} \rightarrow \mathbb{C}$  has a zero (respectively, pole) of order at  $\infty$  if  $g(z) = f(1/z)$  has a zero (respectively, pole) at  $z = 0$  (of order  $m$ ).
41. **Logarithmic Derivative.** Let  $f: \Omega \rightarrow \mathbb{C}$  be meromorphic. Then, the logarithmic derivative of  $f$  is  $f'/f$ .
42. **Locally injective.** We call a function  $f: \Omega \rightarrow \mathbb{C}$  locally injective near  $z_0$  if there exists a neighborhood  $\mathcal{U}$  of  $z_0$  such that  $f: \mathcal{U} \rightarrow \mathbb{C}$  is injective.
43. **Infinite Product.** Suppose the sequence of finite products  $P_N := \prod_{n=1}^N c_n = c_1 c_2 \dots c_N$  converges to a finite number (where  $c_i \in \mathbb{C}$  for all  $i$ ). We define  $\prod_{n=1}^{\infty} c_n = \lim_{N \rightarrow \infty} \prod_{n=1}^N c_n$  to be the infinite product, and say that this infinite product converges.
44. **Riemann-Zeta Function.** We define the Riemann-Zeta function to be  $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ .
45. RESUME DEFINITIONS FOR NOVEMBER 18 LECTURE ONWARDS