1. **Complex Numbers, $\mathbb{C}$.** The set of complex numbers $\mathbb{C}$ is the real vector space $\mathbb{R}^2$ with the properties $(x_1,y_1)+(x_2,y_2)+(x_1+x_2,y_1+y_2)$ and $a(x_1,y_1)=(ax_1,ay_1)$ and we write $z=x+iy=(x,y)$ for any $z\in\mathbb{C}$.

2. **Polar Form.** Let $z=a+bi\in\mathbb{C}$. Then norm (a.k.a. modulus, absolute value) of $z\in\mathbb{C}$ is written $|z|\in\mathbb{R}$ and is defined by $|z|=\sqrt{a^2+b^2}$. Then, define the argument of z as $\arg(z)=\theta\in[0,2\pi)$ as the angle z makes with the real axis. Then, the polar form of z is written as $z=|z|(\cos\theta+i\sin\theta)$.

3. **Rectangular form.** If $z\in\mathbb{C}$ is written as $z=x+iy$, with $x,y\in\mathbb{R}$, then z is written in rectangular form.

4. **Complex Conjugate.** If $z=a+ib$, then its complex conjugate is $\bar{z}=a-ib$.

5. **Open Sets.** A set $\Omega\subseteq\mathbb{C}$ is called open if for each $z_0\in\mathbb{C}$, there is an $\varepsilon>0$ such that $D_\varepsilon(z_0)\subseteq\Omega$.

6. **Neighborhoods.** An ε-neighborhood of a point z_0 is a set N which contains some open disk $D_\varepsilon(z_0)$.

7. **ε-deleted neighborhoods.** An ε-deleted neighborhood of a point z_0 is a set N which contains a "punctured" open disk $D_\varepsilon(z_0)\setminus\{z_0\}$.

8. **Homeomorphism.** A function is a homeomorphism if it is continuous with a continuous inverse.

9. **Periodic.** $f(z)$ is w-periodic (with $w\in\mathbb{C}$) if $f(z+nw)=f(z)$ for all $z\in\mathbb{C},n\in\mathbb{Z}$.

10. **Limits.** Let $f:\Omega\to\mathbb{C}$ where Ω is an r-deleted neighborhood of a point z_0. Then f has a limit as $z\to z_0$, and write $\lim_{z\to z_0}f(z)=a$. This means that for every $\varepsilon>0$ there exists a $\delta>0$ such that if $z\in\Omega$ has $|z-z_0|<\delta$, then $|f(z)-a|<\varepsilon$.

11. **Continuity.** Let $\Omega\subseteq\mathbb{C}$ be an open set. Then $f:\Omega\to\mathbb{C}$ is continuous at a point $z_0\in\Omega$ if $\lim_{z\to z_0}f(z)=f(z_0)$.

12. **Closed Sets.** A subset $F\subseteq\mathbb{C}$ is called closed if its complement $\mathbb{C}\setminus F$ is open.

13. **Compact.** A subset $K\subseteq\mathbb{C}$ is called compact if every open cover of K has a finite subcover.

14. **Uniform Convergence.** A sequence of functions $f_n:\Omega\to\mathbb{C}$ converges uniformly to a function $f:\Omega\to\mathbb{C}$ if for all $\varepsilon>0$ there exists an $N\in\mathbb{N}$ such that if $n\geq N$, then $|f_n(z)-f(z)|<\varepsilon$ for all $z\in\Omega$.

15. **Real Differentiable.** f is real differentiable at z_0 if the following limit exists: $\lim_{h\to0}\frac{f(z+h)-f(z)-D(f'(z))h}{h}$, where $D(f'(z))$ is the Jacobian matrix.

16. **Derivative.** Let $f:\Omega\to\mathbb{C}$, where Ω is a neighborhood of z_0. The derivative of f at z_0 is the limit $f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$. We call f **complex differentiable** at z_0 if this limit exists. If $f'(z_0)$ exists for all $z_0\in\Omega$, we call f **holomorphic** on Ω.

17. **Branch of the Argument.** This is a choice of interval (here, $[-\pi,\pi)$ or $[a,a+2\pi)$).

18. **Principal Branch of the Logarithm.** Pick a branch $[a,a+2\pi)$. Then, $\log:\mathbb{C}\setminus\{0\}\to\mathbb{R}\times i[a,a+2\pi)$ is defined by $\log z=\log|z|+i\arg z$, where $\arg z\in[a,a+2\pi)$. We call the branch $[-\pi,\pi)$ the principal branch.

19. **Exponentiation of complex numbers a,b.** Choose a branch of log, with $\log:\Omega\to\mathbb{C}$ and $a,b\in\mathbb{C}$. Then, define $a^b:=e^{b\log a}$.

20. **Contour Integral.** Suppose f is continuous on an open set Ω and $\gamma:[a,b]\to\Omega$ is a smooth curve. Then the contour integral of f along γ is defined to be $\int_\gamma f:=\int_\gamma f(z)dz:=\int_a^b f(\gamma(t))\gamma'(t)dt$.

21. **Re-parametrization of γ.** A piecewise smooth $\tilde{\gamma}:[\tilde{a},\tilde{b}]\to\mathbb{C}$ is called a re-parametrization of γ if there exists a continuously differentiable $\alpha:[a,b]\to[\tilde{a},\tilde{b}]$ such that $\alpha(a)=\tilde{a}$ and $\alpha(b)=\tilde{b}$, and $\alpha'(t)>0$ with $\gamma(t)=\tilde{\gamma}(\alpha(t))$.

22. **Primitive.** We say a function $f:\Omega\to\mathbb{C}$ has a primitive on Ω if there exists a holomorphic function $F:\Omega\to\mathbb{C}$ such that $F'(z)=f(z)$ for all $z\in\Omega$.

23. **Path-connected.** We say an open set $\Omega\subseteq\mathbb{C}$ is path-connected if for any pair of points $z_0,z_1\in\Omega$ there exists a continuous path $\gamma:[0,1)\to\mathbb{C}$ such that $z_0=\gamma(0)$ and $z_1=\gamma(1)$, with $\gamma([0,1))\subseteq\Omega$.

24. **Path-independence.** If $z_0,z_1\in\Omega$, then any paths $\gamma,\tilde{\gamma}$ (with shared endpoints $\gamma(0)=\tilde{\gamma}(0)$, $\gamma(1)=\tilde{\gamma}(1)$) have $\int_\gamma f(z)dz=\int_{\tilde{\gamma}}f(z)dz$.

25. **Homotopy.** Let $\gamma_{0,1}:[a,b]\to\mathbb{C}$ be curves with shared endpoints z_a,z_b. A homotopy is a continuous function $H:[a,b]\times[0,1]\to\mathbb{C}$ with $t\times s\to H_s(t)$ such that $H_0(t)=\gamma_0(t)$ and $H_1(t)=\gamma_1(t)$.

26. **Simply-connected.** A set A is called simply-connected if every closed curve (loop) is homotopic to a point in A, with $H_s(t)\in A$ for all s,t. (Note: a point in A is a constant loop).

27. **Winding Number.** Let γ be a loop in $\mathbb{C}$ and $z_0\in\mathbb{C}$ but not on γ. Then the winding number of γ (with respect to z_0) is $I(\gamma,z_0)=\frac{1}{2\pi i}\int_\gamma\frac{1}{z-z_0}dz$.

28. **Entire.** We call a function $f:\mathbb{C}\to\mathbb{C}$ entire if it is holomorphic on $\mathbb{C}$.

29. **Closure.** The closure of a set A, written $\overline{A}$, is $\overline{A}=\{\text{limit points of } A\}$.

30. **Boundary of A.** The boundary ∂A of a set $A\subseteq\mathbb{C}$ is $\partial A=\overline{A}\cap\overline{(\mathbb{C}\setminus A)}$.

31. **Reflection:** $\tilde{z}=\frac{R^2}{\bar{z}}$. $\tilde{z}=\frac{R^2}{\bar{z}}$ is the reflection over the line the circle $|\xi|=R$.

32. **Analytic.** A function $f:\Omega\to\mathbb{C}$ is analytic at $z_0\in\Omega$ if there is a neighborhood $\mathscr{U}$ of z_0 on which $f(z)=\sum_{k=0}^\infty a_k(z-z_0)^k$ (for all $z\in\mathscr{U}$) where the RHS is a convergent power series.

33. **Pole of Order m.** If f is holomorphic on a deleted neighborhood $\mathscr{U}\setminus z_0$, we say f has a pole of order m if $\frac{1}{f}$ has a zero of order m.

34. **Simple Pole.** If f has a pole of order 1 at $z=z_0$, then f has a simple pole at z_0.

35. **Residue of f at $z=z_0$.** Consider the principal part of the Laurent expansion of f at $z=z_0$. Then, the coefficient a_{-1} is the residue of f at z_0 and we write it as $\mathrm{Res}_{z_0}(f)=a_{-1}$.

36. **Meromorphic.** A function $f:\Omega\to\mathbb{C}$ is meromorphic if it is holomorphic on all of Ω except at a discrete set of poles.

37. **Essential Singularity.** Let f be holomorphic on Ω except at a point z_0. We call z_0 an essential singularity if z_0 is neither a pole nor a removable singularity.

38. **Alternative Definition of Essential Singularity.** Let f be holomorphic except possibly at a point z_0. Let $C_1=\{z_0\}$ and $C_2=\partial D_r(z_0)$. Then, z_0 is an essential singularity if there are infinitely many a_{-n} in the Laurent series of f, where still $\mathrm{Res}_{z_0}(f)=a_{-1}$.

39. **Holomorphic at Infinity.** Let $\mathscr{U}\subseteq\mathbb{C}$ be an open set containing $\mathbb{C}\setminus\overline{D_R(0)}$. A function $f:\mathscr{U}\to\mathbb{C}$ is holomorphic at infinity if $g(z)=f(1/z)$ has a removable singularity at 0, which in that case, we define $f(\infty)=g(0)$.

40. **Zero (respectively, pole) of order at Infinity.** Let $\mathscr{U}\subseteq\mathbb{C}$ be an open set containing $\mathbb{C}\setminus\overline{D_R(0)}$. We say that $f:\mathscr{U}\to\mathbb{C}$ has a zero (respectively, pole) of order at ∞ if $g(z)=f(1/z)$ has a zero (respectively, pole) at $z=0$ (of order m).

41. **Logarithmic Derivative.** Let $f:\Omega\to\mathbb{C}$ be meromorphic. Then, the logarithmic derivative of f is f'/f.

42. **Locally injective.** We call a function $f:\Omega\to\mathbb{C}$ locally injective near z_0 if there exists a neighborhood $\mathscr{U}$ of z_0 such that $f:\mathscr{U}\to\mathbb{C}$ is injective.

43. **Infinite Product.** Suppose the sequence of finite products $P_N:=\prod_{n=1}^N c_n=c_1c_2\ldots c_N$ converges to a finite number (where $c_i\in\mathbb{C}$ for all i). We define $\prod_{n=1}^\infty c_n=\lim_{N\to\infty}\prod_{n=1}^N c_n$ to be the infinite product, and say that this infinite product converges.

44. **Riemann-Zeta Function.** We define the Riemann-Zeta function to be $\zeta(z)=\sum_{n=1}^\infty\frac{1}{n^z}$.

45. **Conformal Maps.** Smooth, invertible maps $f:\mathbb{R}^2\to\mathbb{R}^2$ whose Jacobian at a point can be factored as (scaling)$\cdot$(rotation) are called conformal maps. (note: by Cauchy-Riemann equations, we have that $\begin{pmatrix}a&-b\\b&a\end{pmatrix}=|a+bi|^2\cdot\frac{1}{|a+bi|^2}\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}$, where the $|a+bi|^2$ factor represents scaling and the other factors together represent the orthogonal scaling matrix).

46. **Conformal Maps (2nd definition).** Let Ω and Ω' be open connected regions in $\mathbb{C}$. We say that a map $g:\Omega\to\Omega'$ is conformal if it is holomorphic and invertible with g^{-1} holomorphic.

47. **Conformally Equivalent.** We call Ω,Ω' conformally equivalent (write: $\Omega\sim\Omega'$) if there exists a conformal map $g:\Omega\to\Omega'$.

48. **Set of holomorphic functions / meromorphic functions.** Let $\mathscr{H}(\Omega)=\{\text{holomorphic functions } f:\Omega\to\mathbb{C}\}$ and $\mathscr{M}(\Omega)=\{\text{meromorphic functions } f:\Omega\to\mathbb{C}\}$.

49. **Conformal Automorphisms.** We call a conformal map $g:\Omega\to\Omega$ a conformal automorphism and write $\mathrm{Aut}(\Omega)$ to denote the collection of automorphisms (strictly conformal) on Ω.

50. **Prop.** Let $z_1,z_2\in\mathbb{C}$. Then $|z_1z_2|=|z_1||z_2|$ and $\arg(z_1z_2)=\arg(z_1)+\arg(z_2)\pmod{2\pi}$.

51. **Theorem.** $\mathbb{C}$ is a field.

52. **De Moivre's Formula.** If $z=r(\cos\theta+i\sin\theta)$ and $n\in\mathbb{Z}_{>0}$, then $z^n=r^n(\cos n\theta+i\sin n\theta)$.

53. **Prop.** Let $z,w\in\mathbb{C}$. Then:

    (a) $\overline{z+w}=\bar{z}+\bar{w}$.

    (b) $\overline{zw}=\bar{z}\cdot\bar{w}$.

    (c) $\overline{\left(\frac{z}{w}\right)}=\frac{\bar{z}}{\bar{w}}$ (with $w\neq0$).

    (d) $z\bar{z}=|z|^2$. If $z\neq0$, then $z^{-1}=\frac{\bar{z}}{|z|^2}$.

    (e) If $z=\bar{z}$, then $z\in\mathbb{R}$ and so $z=\mathrm{Re}(z)$.

    (f) $\mathrm{Re}(z)=\frac{z+\bar{z}}{2}$ and $\mathrm{Im}(z)=\frac{z-\bar{z}}{2i}$.

    (g) $\bar{\bar{z}}=z$.

54. **Prop.** Let $z,w\in\mathbb{C}$. Then:

    (a) $|z|\geq0$ and if $|z|=0$, then $z=0$.

    (b) $|zw|=|z||w|$.

    (c) If $w\neq0$, then $\left|\frac{z}{w}\right|=\frac{|z|}{|w|}$.

    (d) $|\mathrm{Re}(z)|\leq|z|$ and $|\mathrm{Im}(z)|\leq|z|$.

    (e) $|\bar{z}|=|z|$.

    (f) $|z+w|\leq|z|+|w|$.

    (g) $||z|-|w||\leq|z-w|$.

    (h) $|z_1w_1+\cdots+z_nw_n|\leq\sqrt{|z_1|^2+\cdots+|z_n|^2}\sqrt{|w_1|^2+\cdots+|w_n|^2}$.

55. **Prop.** Fix $r>0$ and $z\in\mathbb{C}$. The open disk $D_\varepsilon(z_0)$ is an open set.

56. **Prop.** The following are true:

    (a) $\mathbb{C}$ is open.

    (b) The empty set ϕ is open.

    (c) The union of open sets is open.

    (d) The intersection of finitely many open sets is open.

57. **Prop.** Limits are unique (if they exist).

58. **Prop.** If $\lim_{z\to z_0} f(z) = a$ and $\lim_{z\to z_0} g(z) = b$, then:

   (a) $\lim_{z\to z_0}(f(z)+g(z)) = a+b$.

   (b) $\lim_{z\to z_0}((f(z)g(z))) = ab$.

   (c) $\lim_{z\to z_0}\left(\frac{f(z)}{g(z)}\right) = \frac{a}{b}$ (with $b \neq 0$).

59. **Prop.** The following are true:

   (a) If $\lim_{z\to z_0} f(z) = a$ and h is continuous at a, then $\lim_{z\to z_0} h(f(z)) = h(a)$.

   (b) If f is continuous on an open set $\Omega \subseteq \mathbb{C}$, and h is continuous on $f(\Omega)$, then $h\circ f$ is continuous on Ω, with $(h\circ f)(z) = h(f(z))$.

60. **Prop.** The following are true:

   (a) The empty set ϕ is closed.

   (b) $\mathbb{C}$ is closed.

   (c) The intersection of a collection of closed sets is closed.

   (d) The union of finitely many closed sets is closed.

61. **Prop.** A set F is closed iff whenever $z_1, z_2, z_3, \ldots$ is a sequence of points in F converging to $\lim_{k\to\infty} z_k = w$, then $w \in F$.

62. **Prop.** If $f : \mathbb{C} \to \mathbb{C}$, TFAE:

   (a) f is continuous.

   (b) If $F \subseteq \mathbb{C}$ is closed, then $f^{-1}(F)$ is closed.

   (c) If Ω is open, then $f^{-1}(\Omega)$ is also open.

63. **Prop. (Heine-Borel + Sequential Compactness).** For $K \subseteq \mathbb{C}$, TFAE:

   (a) K is compact.

   (b) K is closed and bounded.

   (c) Every sequence of points in K has a convergent subsequence converging in K (sequentially compact).

64. **Prop.** If K is compact and $f : K \to \mathbb{C}$ is continuous, then the image $f(K)$ is compact.

65. **Theorem (Extreme Value Theorem).** If K is compact and $f : K \to \mathbb{R}$ is continuous, then f attains its minimum and maximum.

66. **Stereographic Projection / Riemann Sphere.** Identify the plane $\overline{\mathbb{C}} = S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$. If $f : \mathbb{C} \to S^2 \setminus \{N\}$, then we have $(u,v) \mapsto \frac{1}{1+u^2+v^2}(2u, 2v, -1+u^2+v^2)$ is a homeomorphism (is continuous with continuous inverse) $f^{-1} : S^2 \setminus \{N\} \to \mathbb{C}$ with $(x,y,z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$.

67. **Prop. (Uniform Convergence).** If $f_n \to f$ uniformly and each f_n is continuous, then f is continuous.

68. **Euler's Formula.** $e^{iz} = \cos z + i \sin z$ for all $z \in \mathbb{C}$.

69. **Theorem.** $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$.

70. **Properties of e.** Let $x, y \in \mathbb{R}$ and $z, w \in \mathbb{C}$. Then:

   (a) $e^{z+w} = e^z e^w$.

   (b) $|e^{x+iy}| = |e^x e^{iy}| = |e^x||e^{iy}| = e^x$.

   (c) $\arg(e^{x+iy}) = y \pmod{2\pi}$.

   (d) $e^z \neq 0$ for all $z \in \mathbb{C}$.

   (e) $e^z = 1$ iff $z = 2\pi i n$ for some $n \in \mathbb{Z}$.

   (f) $e^z = e^{z+2\pi n i}$.

71. **Prop. (Chain Rule).** Let $\Omega, A \subseteq \mathbb{C}$ be open sets, and let $f : \Omega \to A$, $g : A \to \mathbb{C}$ be holomorphic functions. Then, $g\circ f : \Omega \to \mathbb{C}$ is holomorphic and $\frac{d}{dz}(f\circ g)(z) = \frac{dg}{df}(f(z)) \cdot \frac{df}{dz}(z)$.

72. **Prop.** Let $f : \Omega \to \mathbb{C}$ be holomorphic. Then $f : \Omega \to \mathbb{R}^2$ is real differentiable at all $(x,y) \in \Omega$.

73. **Cauchy-Riemann Equations.** Let Ω be an open set in $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be given by $f(x,y) = u(x,y) + iv(x,y)$. Then:

   (a) $f'(z)$ exists at $z \in \Omega$ iff f is real differentiable and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (these are the Cauchy-Riemann equations).

   (b) $f(z)$ is holomorphic on Ω iff partials are continuous and satisfy the CR equations.

   (c) If $f'(z_0)$ exists, then $f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y} = \frac{1}{i}\frac{\partial f}{\partial y}$.

74. **Inverse Function Theorem for $\mathbb{R}^2$.** If $f : \Omega \to \mathbb{R}^2$ is continuously differentiable and the Jacobian $Df(z_0)$ has $\det(Df(z_0)) \neq 0$, then there are neighborhoods $U \ni z_0$ and $V \ni f(z_0)$ such that $f : U \to V$is bijective with continuously differentiable $f^{-1} : V \to U$ such that $Df^{-1}(z_0) = [Df(z_0)]^{-1}$, which is the inverse matrix of $Df(z_0)$.

75. **Inverse Function Theorem for $\mathbb{C}$.** Let $f : \Omega \to \mathbb{C}$ be holomorphic (with continuous $f'(z_0)$), and $f'(z) \neq 0$ for some $z_0 \in \Omega$. Then there exists a neighborhood $U \ni z_0$ and $V \ni f(z_0)$ such that $f : U \to V$ is bijective with holomorphic inverse $f^{-1} : V \to U$ such that for all $z_0 \in U$, $\frac{d}{dw} f^{-1}(w) = \frac{1}{f'(w)}$ with $w = f(z)$.

76. **Prop.** Pick a branch $[a, a+2\pi)$. Then $\log z : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times i[a, a+2\pi)$ is the inverse of $\exp : \mathbb{R} \times i[a, a+2\pi) \to \mathbb{C}$.

77. **Prop.** $\log : \mathbb{C} \setminus \mathbb{R}_{\leq 0} \to \mathbb{R} \times i(-\pi, \pi)$ is holomorphic with $\frac{d}{dz}\log z = \frac{1}{z}$.

78. **Prop.** If $z_1, z_2 \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, then $\log(z_1 z_2) = \log z_1 + \log z_2 \pmod{2\pi i}$.

79. **Prop.** By choosing different branches of log, we have the following:

   (a) a^b is independent of the branch iff $b \in \mathbb{Z}$.

   (b) a^b takes on exactly q different values iff $b \in \mathbb{Q}$, so $b = \frac{p}{q}$ (with p, q coprime).

   (c) a^b takes on infinitely many values iff b is irrational or $\operatorname{Im}(b) \neq 0$.

80. **Cor.** Choose a branch of log. Then the nth root function is given by $z^{1/n} = e^{\log(z/n)}$, where the nth root function has n branches.

81. **Prop.** Let $a, b \in \mathbb{C}$. Then:

   (a) For any choice of branch of log, the function $\mapsto a^z$ is holomorphic on $\mathbb{C}$, and $z \mapsto (\log a)a^z$.

   (b) Choose a branch of log. Then the function $z \mapsto z^b$ is holomorphic on the domain of log with derivative $z \mapsto bz^{b-1}$.

82. **Prop. (Re-parametrization).** If $\tilde{\gamma}$ is a re-parametrization of γ, then $\int_\gamma f = \int_{\tilde{\gamma}} f$ for any continuous f on Ω.

83. **Fundamental Theorem of Line Integrals.** Let $F : \Omega \to \mathbb{C}$ be holomorphic on an open Ω and let $\gamma : [0,1] \to \Omega$ be piecewise smooth. Then, $\int_\gamma F'(z)dz = F(\gamma(1)) - F(\gamma(0))$.

84. **Path-independence and Primitives Theorem.** Let $f : \Omega \to \mathbb{C}$ be continuous and Ω is open and connected. Then, TFAE:

   (a) (path-independence) if $z_0, z_1 \in \Omega$, then any paths $\gamma, \tilde{\gamma}$ with shared endpoints $\gamma(0) = \tilde{\gamma}(0)$ and $\gamma(1) = \tilde{\gamma}(1)$ have $\int_\gamma f(z)dz = \int_{\tilde{\gamma}} f(z)dz$.

   (b) (integral along loops is 0) if Γ is a loop, with $\Gamma(1) = \Gamma(0)$, then $\int_\Gamma f(z)dz = 0$.

   (c) (f has a primitive) There is a primitive F for f on Ω.

85. **Cauchy-Goursat Theorem.** Let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω, simply connected, and open. Then for any loop $\Gamma \subseteq \Omega$, $\int_\Gamma f(z)dz = 0$.

86. **Green's Theorem.** Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field and let γ be a loop, and A a region in the loop γ. Let $f(x,y) = (P(x,y), Q(x,y))$. Then, $\int_\gamma P(x,y)dx + Q(x,y)dy = \iint \operatorname{curl} F dA = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$.

87. **Prop.** If $f(x+iy) = u(x,y) + iv(x,y)$, then $\int_\gamma f = \int_\gamma udx - vdy + i\int_\gamma udx + vdy$.

88. **Cauchy-Goursat Theorem (Weaker Version).** Let $f : \Omega \to \mathbb{C}$ be holomorphic with $f'(z)$ continuous and $\gamma : [0,1] \to \mathbb{C}$ a simple closed curve and Ω an open & simply connected set. Then, $\int_\gamma f = 0$.

89. **Cauchy-Goursat Theorem (for rectangles).** Let R be a rectangle with R and its interior are contained in an open set Ω. Let $f : \Omega \to \mathbb{C}$ be holomorphic. Then, $\int_R f = 0$.

90. **Cauchy-Goursat Theorem (for disks).** Suppose $f : D \to \mathbb{C}$ is holomorphic on an open disk $D := D_\rho(z_0)$. Then:

   (a) f has a primitive F on D.

   (b) if Γ is any loop in D, then $\int_\Gamma = 0$.

91. **Deformation Theorem.** Suppose f is holomorphic on an open set Ω and γ_0, γ_1 are piecewise continuously differentiable. Then there are continuously differentiable curves in Ω. Then:

   (a) If γ_0, γ_1 are paths from z_0 to z_1, which are homotopic in Ω, then $\int_{\gamma_0} F = \int_{\gamma_1} F$.

   (b) If γ_0, γ_1 are loops homtopic in Ω, then $\int_{\gamma_0} F = \int_{\gamma_1} F$. (note: this also works for constant loops, where constant loops are just points)

92. **Cauchy-Goursat Theorem (restated).** Let $f : \Omega \to \mathbb{C}$ be holomorphic with Ω open and a loop (let γ) be homotopic to a point in Ω. Then, $\int_\gamma f dz = 0$.

93. **Cor.** If Ω is simply connected, then every loop γ has $\int_\gamma f dz = 0$.

94. **Cor.** Let $f : \Omega \to \mathbb{C}$ be holomorphic on a simply connected oen set Ω. Then, f has a primitive F on Ω (unique up to constants).

95. **Winding number (as an index).** Let $\gamma : [a,b] \to \mathbb{C}$ (a piecewise continuous) loop and $z \notin \gamma([a,b])$. Then, the winding number of γ around z_0 is an integer.

96. **Cauchy's Integral Formula.** Let f be holomorphic on Ω and γ a loop in Ω hommotopic to a point. Let $z_0 \in \Omega$ but $z_0 \notin \gamma$. Then,

$$f(z_0) \cdot I(\gamma, z_0) = \frac{1}{2\pi i} \cdot \int_\gamma \frac{f(z)}{z - z_0} dz.$$

97. **Cauchy's Integral Formula for Derivatives.** Let f be holomorphic on Ω. Then f is infinitely differentiable (complex) and if γ is a loop homotopic to a point (simple loop) $I(\gamma, z_0) = 1$, then:

$$f^{(n)}(z_0) = \frac{n}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi.$$

98. **Cor. Cauchy-Type Integrals.** Let γ be a loop $\gamma : [a,b] \to \mathbb{C}$ and g a continuous function on γ. Set $\tilde{g}(z) := \int_\gamma \frac{g(\xi)}{\xi - z} d\xi$. Then, $\tilde{g}(z)$ is holomorphic inside γ and so $\tilde{g}(z)$ is infinitely differentiable.

99. **Prop. (Cauchy Inequalities).** Let f be holomorphic on Ω and let $\overline{D_R(z_0)} \subseteq \Omega$ with boundary γ. Suppose $f(z)$ is bounded above $|f(z)| \leq M$ for all $z \in \gamma$. Then for all $k = 1, 2, \dots$, the kth derivative is also upper bounded with $|f^{(k)}(z_0)| \leq \frac{k_!}{R^k} M$.

100. **Louisville's Theorem.** If f is entire and bounded (i.e. there exists an $M \in \mathbb{R}_{>0}$ with $|f(z)| \leq M$ for all $z \in \mathbb{C}$), then f is constant.

101. **Fundamental Theorem of Algebra.** Let $a_0, \dots, a_n \in \mathbb{C}$ with $a_i \neq 0$ for $n \geq 1$. Then the polynomial $p(z) = a_n z^n + \cdots + a_0$ has a zero (root) where $z_0 \in \mathbb{C}$ with $p(z_0) = 0$.

102. **Cor.** A degree n complex polynomial has exactly n roots, counting multiplicity.

103. **Morera's Theorem. (partial converse to Cauchy-Goursat)** Let f continuous on an open Ω and suppose that $\int_\gamma f = 0$ for every loop in Ω. Then, f is holomorphic on Ω and f has a primitive F on Ω.

104. **Cor. to Morera's Theorem (Removable Singularities Theorem).** Let f be continuous on an open Ω in $\mathbb{C}$ and holomorphic on $\Omega \setminus \{z_0\}$, with $z_0 \in \mathbb{C}$. Then, f is holomorphic on Ω.

105. **Another Cor. to Morera's Theorem.** If f is holomorphic on $\Omega \setminus \{z_0\}$ and bounded on a neighborhood of z_0, there is unique holomorphic extension $\tilde{f}$ of f to γ defined by $\tilde{f}(z) = f(z)$ if $z \neq z_0$ and $\tilde{f}(z) = \lim_{z \to z_0} f(z)$ if $z = z_0$.

106. **Mean Value Property.** Let f be holomorphic on $\overline{D_R(z_0)}$. Then $f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$.

107. **Maximum Principle, local version.** Let f be holomorphic on a neighborhood Ω of z_0, and suppose that $|f|$ has a relative max at z_0. Then, f is constant on some neighborhood U of z_0.

108. **Prop.** The following are true:

   (a) $A \subseteq \overline{A}$.

   (b) $\overline{A}$ is closed.

   (c) A is closed iff $A = \overline{A}$.

   (d) If $A \subseteq C$ and C closed, then $\overline{A} \subseteq C$.

109. **Maximum Modulus Principle.** Let A be an open, connected, bounded set in $\mathbb{C}$ and suppose $f : \overline{A} \to \mathbb{C}$ is holomorphic on A and continuous on $\overline{A}$. Then $|f|$ has a finite maximum value on $\overline{A}$ which is achieved on ∂A. IF $|f|$ is attained in A, then f is constant.

110. **Minimum Modulus Principle.** Let f be holomorphic on D, an open connected set in $\mathbb{C}$. Then, if z_0 is a point in D such that $0 < |f(z_0)| \leq |f(z)|$ for all z in some neighborhood about z_0, then f is constant on D (we get this result by applying the maximum modulus principle to $1/f$).

111. **Prop.** Let $u : \Omega \to \mathbb{R}$ be an twice-continuous harmonic function on an open set $\Omega \subseteq \mathbb{C}$. Then u is infinitely differentiable, so u is C^∞, and in the neighborhood U of $z_0 \in \Omega$, there exists a holomorphic function $f : U \to \mathbb{C}$ such that $u = \mathrm{Re}(f)$.

112. **Dirichlet Problem.** $\Delta u = 0$, $u \mid_{\partial\Omega} (\theta) = g(\theta)$.

113. **Prop.** Let $u, \tilde{u}$ solve the Dirichlet Problem. Then, $u = \tilde{u}$, so the solution to the Dirichlet Problem is unique.

114. **Solution to the Dirichlet Problem.** This is given by:

$$u(re^{i\phi}) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \cdot \frac{R^2 - r^2}{|Re^{i\theta} - re^{i\phi}|} d\theta.$$

115. **Poisson's Formula.** Another way to write the solution to the Dirichlet Problem is with the following:

$$u(re^{i\phi}) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{u(Re^{i\theta})}{R^2 - 2rR\cos(\theta - \phi) + r^2} d\theta.$$

116. **Analytic Convergence Theorem.** Let $f_n : \Omega \to \mathbb{C}$ be a sequence of holomorphic functions. If $f_n \to f$ uniformly on every closed disk in Ω, then:

   (a) f is holomorphic on Ω.

   (b) f_n' converges to f' uniformly on every closed disk, and pointwise on Ω.

117. **Prop.** Let $\gamma : [a,b] \to \Omega$ be a contour and $f_n : \gamma([a,b]) \to \mathbb{C}$ be a sequence of continuous functions. If $f_n \to f$ uniformly on $\gamma([a,b])$, then $\int_\gamma f_n \to \int_\gamma f$.

118. **Power Series Convergence Theorem.** Let $\sum_{n=0}^\infty a_n (z - z_0)^n$ be a power series. Then there is a unique $R \geq 0$, possibly $R = \infty$, such that:

   (a) If $|z - z_0| < R$, the series converges pointwise.

   (b) If $|z - z_0| \leq R - \varepsilon$, the series converges uniformly.

   (c) If $|z - z_0| > R$, the series diverges.

   (d) If $|z - z_0| = R$, need to check.

119. **Cor.** A power series is analytic on its disk of convergence and so, holomorphic by the analytic convergence theorem.

120. **Cor 2.** We can apply term-by-term differentiation to the power series of an analytic function.

121. **Cor 3.** Power series expansions around some center z_0 are unique.

122. **Taylor Series Theorem.** Let f be holomorphic on a region Ω, and let $D_r(z_0) \subseteq \Omega$ with $r > 0$. Then for every $z \in D_r(z_0)$, the power series $f(z) = \sum_{n=0}^\infty \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$ converges on $D_r(z_0)$ and equal to $f(z)$. We call this the Taylor series of f centered at z_0.

123. **Theorem.** f is analytic iff f is holomorphic.

124. **Prop.** If f is holomorphic on an open, connected set Ω and the zero set $\{z \in \Omega \mid f(z) = 0\}$ contains a limit point, then $f = 0$ on Ω.

125. **Cor. (Identity Theorem).** Let f, g be holomorphic on an open, connected Ω and $f(z) = g(z)$ for a set of z with a limit point in Ω. Then, $f = g$ on Ω.

126. **Cor. (Zeros are Isolated).** If f is holomorphic on Ω, and not identically zero on Ω, then for any zero z_0 of f, there is a deleted neighborhood $U \setminus \{z_0\}$ on which $f(z) \neq 0$ for all $z \in U \setminus \{z_0\}$.

127. **Cor. (Analytic Continuation).** If f is holomorphic on an open, connected set Ω and f_+ is holomorphic on an open connected $\Omega_+ \supseteq \Omega$ with $f_+ = f$ on Ω, then f_+ is the unique such extension, i.e. if there exists another such extension, $\tilde{f}_+$, then $\tilde{f}_+ = f_+$.

128. **Lemma.** Let $f : \Omega \to \mathbb{C}$ be holomorphic, not identically 0, with a zero z_0. Then in a neighborhood U of z_0, we may write $f(z) = (z - z_0)^m g(z)$ for all $z \in U$, where $g(z) \neq 0$ and m is unique.

129. **Lemma.** A function f has a pole of order m at z_0 iff there is a neighborhood U of z_0 on which $f(z) = (z - z_0)^{-m} g(z)$ for all $z \in U \setminus \{z_0\}$ with a nonzero $g(z)$ and g holomorphic on U.

130. **Theorem.** If f has a pole of order m at z_0, then it can be represented uniquely as:

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z - z_0} + G(z)$$

where $G(z)$ is holomorphic on a neighborhood U of z_0 and $a_{-m}, \dots, a_{-1} \in \mathbb{C}$ with $a_{-m} \neq 0$.

131. **Residue Theorem, Simple Version.** Let f be holomorphic on a set $\Omega \supseteq \overline{D_R(z_0)}$, $\gamma = \partial\overline{D_R(z_0)}$ except at z_0. Then $\int_\gamma f(z) dz = 2\pi i \, \mathrm{Res}_{z_0}(f)$.

132. **Residue Theorem, Simple Closed Loops.** Let Ω be open, connected and γ a simple loop homotopic to a point in Ω. Let f be a function $f : \Omega \to \mathbb{C}$ be holomorphic except at a finite set of points $z_1, \dots, z_N$ inside γ. Then, $\int_\gamma f(z) dz = 2\pi i \sum_{k=1}^N \mathrm{Res}_{z_k}(f)$.

133. **Laurent Series Theorem.** Let C_1, C_2 be two circles centered at z_0 (it is fine if $C_1 = \{z_0\}$ and C_2 "encloses" $\mathbb{C}$). Call R the region the annulus between C_1 and C_2. Let f be holomorphic on R. Then f can be expanded uniquely as a (absolutely) convergent power series in R by:

$$f(z) = \sum_{n=1}^\infty \frac{a_{-n}}{(z - z_0)^n} + \sum_{n=0}^\infty a_n (z - z_0)^n.$$

where the first infinite series is called the principal part and second one is called the Taylor series / holomorphic part.

134. **Casorati-Weierstrauss Theorem.** If f is holomorphic in a deleted $D_r(z_0) \setminus \{z_0\}$ and has an essential singularity at z_0, then the image of $f(D_r(z_0) \setminus \{z_0\})$ is dense in $\mathbb{C}$.

135. **Prop. (Fourier Transform).** Let $f : \mathbb{R} \to \mathbb{R}$ be a real function. The Fourier Transform of f is the function $\hat{f} : \mathbb{R} \to \mathbb{R}$ given by $\hat{f}(k) = \int_{-\infty}^\infty f(x) \cdot e^{-ikx} dx$.

136. **Jordan's Lemma.** $\int_0^\pi e^{-R\sin\theta} d\theta \leq \frac{\pi}{R}$.

137. **Cauchy Principal Value.** Take the real integral symmetrically, so we can find an indefinite integral (with discontinuity in the interval) by approaching "the same way" from both sides of the discontinuity.

138. **Argument Principle.** Let $f : \Omega \to \mathbb{C}$ be meromorphic and γ a simple loop in Ω bounding a simply connected region R_γ, with $\overline{R_\gamma} \subseteq \Omega$. Let f have no zeros or poles on γ. Then:

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = N - P$$

where N is the number of zeros of f inside R_γ (counting multiplicity) and P is the number of poles of f inside R_γ (counting multiplicity).

139. **Rouche's Theorem.** Let $f, g : \Omega \to \mathbb{C}$ be holomorphic and let γ be a simple loop bounding a simply connected open U, with $\overline{U} \subseteq \Omega$. If $|f(z)| > |g(z)|$ on γ, then f and $f + g$ have the same number of zeros inside U.

140. **Open-Mapping Theorem.** Any nonconstant holomorphic function is an open map, meaning it maps open sets to open sets.

141. **Lemma. (Local Injectivity).** Let $f : \Omega \to \mathbb{C}$ be holomorphic and $z_0 \in \Omega$. If $f'(z_0) \neq 0$, then f is locally injective near z.

142. **Theorem.** If $g : \Omega \to \mathbb{C}$ is holomorphic and Ω is simply connected, and $g \neq 0$, there exists a holomorphic function $F : \Omega \to \mathbb{C}$ satisfying $e^{F(z)} = g(z)$, where $F(z)$ is unique up to $2\pi i k$, with $k \in \mathbb{Z}$.

143. **Theorem (Local description of holomorphic).** Let $f : \Omega \to \mathbb{C}$ be holomorphic, Ω open. Let $z_0 \in \Omega$ and let $k \geq 1$ denote the order of the zero $f(z) - f(z_0)$ at z_0. Then, there exists an open neighborhood U of z_0 (and $r > 0$) and a function $\phi : U \to D_r(z_0)$ such that:

   (a) ϕ is holomorphic with a holomorphic inverse.

   (b) $\phi(z_0) = 0$.

   (c) We have $f(z) = f(z_0) + (\phi(z))^k$ with $z \in U$.

144. **Prop.** Let $\{a_n\}_{n=1}^\infty$ be a sequence of complex numbers. If $\sum_{n=1}^\infty |a_n| < \infty$, then $\prod_{n=1}^\infty (1 + a_n)$ converges and its value is 0 iff one of the $1 + a_n$ factors is zero.

145. **Prop.** Let $\{f_n\}_{n=1}^\infty$ be a sequence of holomorphic functions on Ω. If $\sum_{n=1}^\infty |f_n|$ converges uniformly on compact subsets of Ω, then so does $\prod_{n=1}^\infty (1 + f_n(z))$. Moreover, the limiting function is holomorphic and nonzero everywhere except at points z such that $1 + f_n(z) = 0$ (for some n).

146. **Prop. (Partial fractions expansion for log derivatives).** Same assumptions as the above proposition. Then, the log derivative of product = sum of log derivatives. i.e.

$$\frac{\left(\prod_{n=1}^{\infty}(1+f_n)\right)'}{\prod_{n=1}^{\infty}(1+f_n)} = \sum_{n=1}^{\infty} \frac{f_n'}{1+f_n}.$$

147. **Infinite products formula for sine.** $\sin(\pi z) = \pi z \cdot \prod_{n=1}^{\infty}(1-\frac{z}{n})(1+\frac{z}{n})$, for $z \in \mathbb{C}$.

148. **Prop.** Fix $z \in \mathbb{C} \setminus \mathbb{Z}$ and a large positive $N \in \mathbb{Z}$. By the residue theorem, the integral $I_N(z) := \int_{\gamma_N} \frac{\pi \cot(\pi z)}{(w+z)^2} dw$.

149. **Cor.** $\cos(\pi z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\left(n-\frac{1}{2}\right)^2}\right)$, $e^z - 1 = ze^{z/2} \cdot \prod_{n=1}^{\infty}\left(1+\frac{z^2}{4\pi^2 n^2}\right)$, and $\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}$ (for the contangent expression, $z \in \mathbb{C} \setminus \mathbb{Z}$).

150. **Theorem (Schwarz Reflection Principle).** Let A be a region in the upper-half plane with $\partial A \cap \mathbb{R}$ nonempty and containing $[a,b] \subseteq \mathbb{R}$. Let f be holomorphic on A and continuous on $\partial A \cap [a,b]$ and real on $[a,b]$. Then f can be uniquely extended to a holomorphic function on $A \cup (a,b) \cup A_{\text{ref}}$, where $A_{\text{ref}} = \{\bar{z} \mid z \in A\}$ with $f(z) = \overline{f(\bar{z})}$ for all $z \in A_{\text{ref}}$.

151. **Theorem (Gamma Function).** There is a unique $\Gamma(s)$ with the following:

    (a) $\Gamma(s)$ is meromorphic.

    (b) (Factorial). $\Gamma(n+1) = n!$ for $n = 0,1,2,\ldots$.

    (c) (Special Value). $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

    (d) (Integral Representation). $\Gamma(s) = \int_0^{\infty} e^{-x}x^{s-1}dx$ (for $\text{Re}(s) > 0$).

    (e) (Infinite Product Representation). $\Gamma(s) = s^{-1}e^{\gamma s}\prod_{n=1}^{\infty}\left(1+\frac{s}{s}\right)^{-1}e^{s/n}$ where $\gamma = \lim_{n\to\infty}\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right) \approx 0.58...$ (Euler-Mascheroni constant) (for $s \in \mathbb{C}$, except poles).

    (f) (Limit of finite products). $\Gamma(s) = \lim_{n\to\infty} \frac{n!n^s}{s(s+1)\ldots(s+n)}$ (for $s \in \mathbb{C}$, except poles).

    (g) (Zeros). $\Gamma(s)$ has no zeros.

    (h) (Poles). $\Gamma(s)$ has poles at nonpositive integers $s = 0,-1,-2,\ldots$ and is holomorphic everywhere else. At $s = -n$, the pole is simple and $\text{Res}_{-n}(\Gamma) = -\frac{1}{n!}$.

    (i) (Functional Equation). $\Gamma(s+1) = s\Gamma(s)$ (for $s \in \mathbb{C}$, except at poles).

    (j) (Reflection Formula). $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$ (for $s \in \mathbb{C}$, except at poles).

152. **Theorem.** The conformal equivalence, which is a relation, is an equivalence relation.

153. **Cor.** Holomorphic f is locally injective iff $f'(z_0) \neq 0$.

154. **Conformal Equivalence Classes.** These are the following:

    (a) Complex plane, $\mathbb{C}$.

    (b) Punctured plane, $\mathbb{C} \setminus \{0\}$.

    (c) Unit disk, $D_1(0)$.

    (d) Upper-half plane, $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$.

    (e) Riemann sphere*, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (not a subset of $\mathbb{C}$ but we can still talk about holomorphic/meromorphic functions on it).

    (f) The slit plane, $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

    (g) Strip (similar to critical region from Riemann hypothesis).

    (h) Rectangle.

    (i) Annulus.

    (j) Blob

155. **Lemma.** If $\Omega \sim \Omega'$, then Ω and Ω' are homeomorphic (a.k.a. there is a continuous map $g : \Omega \to \Omega'$ such that g^{-1} is defined and also continuous).

156. **Lemma.** If $\Omega \sim \Omega'$, then they are homeomorphic.

157. **Prop.** If $g : \Omega \to \Omega'$ is holomorphic and invertible, then g^{-1} is holomorphic (i.e. g is conformal).

158. **Lemma.** $\text{Aut}(\Omega)$ is a group, with function composition. In other words, let $f,g,h \in \text{Aut}(\Omega)$. Then:

    (a) $(g \circ f) \circ h = g \circ (f \circ h)$.

    (b) If $g \in \text{Aut}(\Omega)$, then $g^{-1} \in \text{Aut}(\Omega)$.

    (c) There is an identity map $\text{id} \in \text{Aut}(\Omega)$ such that $\text{id} \circ g(z) = g(z) = g \circ \text{id}(z) = g(z)$.

159. **Theorem.** Let $g : \mathbb{C} \to \Omega$ be a conformal map between $\mathbb{C}$ and a region Ω. Then, $\Omega = \mathbb{C}$ and $g(z)$ is a conformal automorphism of the form $g(z) = az+b$, with $a \neq 0$ and $b \in \mathbb{C}$.

160. **Theorem (Riemann Sphere).** Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. If $g : \hat{\mathbb{C}} \to \Omega$ is a conformal map, then $\Omega = \hat{\mathbb{C}}$ and g is a conformal automorphism, with $g(z) = \frac{az+b}{cz+d}$ with $a,b,c,d \in \mathbb{C}$ (i.e. g is a Möbius transformation).

161. **Riemann Mapping Theorem (Simplified).** Let Ω, Ω' be simply connected, open subsets of $\mathbb{C}$ with $\Omega, \Omega' \neq \mathbb{C}$. Then Ω, Ω' are conformally equivalent.

162. **Cor.** Ω and Ω' are homeomorphic.

163. **Fact.** If Ω is conformally equivalent to Ω', then as groups, $\text{Aut}(\Omega) \cong \text{Aut}(\Omega')$.

164. **Schwarz-Lemma.** Let $g \in \text{Aut}(D_1(0))$ and $g(0) = 0$, i.e. g fixes the origin 0. Then:

    (a) $|g(z)| \leq |z|$ for all $z \in D_1(0)$.

    (b) If $|g(z)| = |z|$ for some $z \neq 0$, then $g(z)$ is a rotation.

    (c) $|g'(0)| \leq 1$.

    (d) If $|g'(0)| = 1$, then g is a rotation.

165. **Cor.** The automorphisms $g : D_1(0) \to D_1(0)$ which fix 0 are precisely the rotations.

166. **Lemma 3.9R.** ϕ_w is an automorphism of $D_1(0)$, where $\phi_w(z) = \frac{w-z}{1-\bar{w}z}$, with $w \in D_1(0)$. Then:

    (a) $\phi_w(0) = w$.

    (b) $\phi_w(w) = 0$.

    (c) $\phi_w^{-1} = \phi_w$.

167. **Theorem 3.10.** A function $g : D_1(0) \to D_1(0)$ is a conformal automorphism iff it is of the form $g(z) = e^{i\theta} \cdot \frac{w-z}{1-\bar{w}z}$ and $0 \in [0,2\pi)$ and $w \in D_1(0)$. Also, the part of (θ, w) is unique.

168. **Cor. (Alternative form of $g \in \text{Aut}(D_1(0))$).** $g(z) = \frac{\mu z+\nu}{\bar{\nu}z+\bar{\mu}}$ for $\mu,\nu \in \mathbb{C}$ with $|\mu|^2 - |\nu|^2 = 1$.

169. **Upper-Half Plane.** We define the upper-half plane as $\mathbb{H} := \{z : \text{Im}(z) > 0\}$.

170. **Theorem 3.13. (Conformal automorphisms of $\mathbb{H}$).** $g : \mathbb{H} \to \mathbb{H}$ is a conformal automorphism iff it is of the form $g(z) = \frac{az+b}{cz+d}$ for <u>*real*</u> numbers $a,b,c,d \in \mathbb{R}$ such that $ad - bc = 1$. (These numbers are unique up to sign, so $(a,b,c,d) = \pm(a',b',c',d')$).

171. **Theorem.**

$$\text{Aut}(\mathbb{H}) = \{z \mapsto \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{R} \wedge ad-bc = 1\}.$$