Problem 5.
Show that if $f(z)$ has an essential singularity at z_0, then $e^{f(z)}$ also has an essential singularity at z_0.

Solution: We aim to show that $e^{f(z)}$ has neither a pole nor a removable singularity at z_0. Put $g(z) = e^{f(z)}$. First, suppose g has a pole at z_0. We utilize the following result.

Theorem 1. z_0 is a pole of f if and only if there is a punctured disc $D_r(z_0)\backslash\{z_0\}$ on which $|f|$ is bounded below and unbounded above.

Then, by Theorem 1, there is a punctured disc $D_r(z_0)\backslash\{z_0\}$ on which f is bounded above and unbounded below. Choose a point $z_1 \in D_r(z_0)\backslash\{z_0\}$ with $|z_0 - z_1|$ sufficiently small. Then, since Thus, z_0 is not a pole of g. Now, suppose g has a removable singularity at z_0. From the hint, we have that a singularity z_0 is removable if and only if $|f|$ is bounded on some $D_r(z_0)$. By assumption, $|g|$ is bounded on some $D_r(z_0)$. Then,