

1. **Complex Numbers, \mathbb{C} .** The set of complex numbers \mathbb{C} is the real vector space \mathbb{R}^2 with the properties $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $a(x_1, y_1) = (ax_1, ay_1)$ and we write $z = x + iy = (x, y)$ for any $z \in \mathbb{C}$.
2. **Polar Form.** Let $z = a + bi \in \mathbb{C}$. Then norm (a.k.a. modulus, absolute value) of $z \in \mathbb{C}$ is written $|z| \in \mathbb{R}$ and is defined by $|z| = \sqrt{a^2 + b^2}$. Then, define the argument of z as $\arg(z) = \theta \in [0, 2\pi)$ as the angle z makes with the real axis. Then, the polar form of z is written as $z = |z|(\cos \theta + i \sin \theta)$.
3. **Rectangular form.** If $z \in \mathbb{C}$ is written as $z = x + iy$, with $x, y \in \mathbb{R}$, then z is written in rectangular form.
4. **Complex Conjugate.** If $z = a + ib$, then its complex conjugate is $\bar{z} = a - ib$.
5. **Open Sets.** A set $\Omega \subseteq \mathbb{C}$ is called open if for each $z_0 \in \Omega$, there is an $\varepsilon > 0$ such that $D_\varepsilon(z_0) \subseteq \Omega$.
6. **Neighborhoods.** An ε -neighborhood of a point z_0 is a set N which contains some open disk $D_\varepsilon(z_0)$.
7. **ε -deleted neighborhoods.** An ε -deleted neighborhood of a point z_0 is a set N which contains a "punctured" open disk $D_\varepsilon(z_0) \setminus \{z_0\}$.
8. **Homeomorphism.** A function is a homeomorphism if it is continuous with a continuous inverse.
9. **Periodic.** $f(z)$ is w -periodic (with $w \in \mathbb{C}$) if $f(z + nw) = f(z)$ for all $z \in \mathbb{C}, n \in \mathbb{Z}$.
10. **Limits.** Let $f : \Omega \rightarrow \mathbb{C}$ where Ω is an r -deleted neighborhood of a point z_0 . Then f has a limit as $z \rightarrow z_0$, and write $\lim_{z \rightarrow z_0} f(z) = a$. This means that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $z \in \Omega$ has $|z - z_0| < \delta$, then $|f(z) - a| < \varepsilon$.
11. **Continuity.** Let $\Omega \subseteq \mathbb{C}$ be an open set. Then $f : \Omega \rightarrow \mathbb{C}$ is continuous at a point $z_0 \in \Omega$ if $\lim_{z \rightarrow z_0} f(z) = f(z_0)$.
12. **Closed Sets.** A subset $F \subseteq \mathbb{C}$ is called closed if its complement $\mathbb{C} \setminus F$ is open.
13. **Compact.** A subset $K \subseteq \mathbb{C}$ is called compact if every open cover of K has a finite subcover.
14. **Uniform Convergence.** A sequence of functions $f_n : \Omega \rightarrow \mathbb{C}$ converges uniformly to a function $f : \Omega \rightarrow \mathbb{C}$ if for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(z) - f(z)| < \varepsilon$ for all $z \in \Omega$.
15. **Real Differentiable.** f is real differentiable at z_0 if the following limit exists: $\lim_{h \rightarrow 0} \frac{f(z+h) - f(z) - D(f'(z))h}{h}$, where $D(f'(z))$ is the Jacobian matrix.
16. **Derivative.** Let $f : \Omega \rightarrow \mathbb{C}$, where Ω is a neighborhood of z_0 . The derivative of f at z_0 is the limit $f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$. We call f **complex differentiable** at z_0 if this limit exists. If $f'(z_0)$ exists for all $z_0 \in \Omega$, we call f **holomorphic** on Ω .
17. **Branch of the Argument.** This is a choice of interval (here, $[-\pi, \pi)$ or $[a, a + 2\pi)$).
18. **Principal Branch of the Logarithm.** Pick a branch $[a, a + 2\pi)$. Then, $\log : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{R} \times i[a, a + 2\pi)$ is defined by $\log z = \log |z| + i \arg z$, where $\arg z \in [a, a + 2\pi)$. We call the branch $[-\pi, \pi)$ the principal branch.
19. **Exponentiation of complex numbers a, b .** Choose a branch of log, with $\log : \Omega \rightarrow \mathbb{C}$ and $a, b \in \mathbb{C}$. Then, define $a^b := e^{b \log a}$.
20. **Contour Integral.** Suppose f is continuous on an open set Ω and $\gamma : [a, b] \rightarrow \Omega$ is a smooth curve. Then the contour integral of f along γ is defined to be $\int_\gamma f := \int_\gamma f(z) dz := \int_a^b f(\gamma(t)) \gamma'(t) dt$.
21. **Re-parametrization of γ .** A piecewise smooth $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \rightarrow \mathbb{C}$ is called a re-parametrization of γ if there exists a continuously differentiable $\alpha : [a, b] \rightarrow [\tilde{a}, \tilde{b}]$ such that $\alpha(a) = \tilde{a}$ and $\alpha(b) = \tilde{b}$, and $\alpha'(t) > 0$ with $\gamma(t) = \tilde{\gamma}(\alpha(t))$.
22. **Primitive.** We say a function $f : \Omega \rightarrow \mathbb{C}$ has a primitive on Ω if there exists a holomorphic function $F : \Omega \rightarrow \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in \Omega$.
23. **Path-connected.** We say an open set $\Omega \subseteq \mathbb{C}$ is path-connected if for any pair of points $z_0, z_1 \in \Omega$ there exists a continuous path $\gamma : [0, 1] \rightarrow \Omega$ such that $z_0 = \gamma(0)$ and $z_1 = \gamma(1)$, with $\gamma([0, 1]) \subseteq \Omega$.
24. **Path-independence.** If $z_0, z_1 \in \Omega$, then any paths $\gamma, \tilde{\gamma}$ (with shared endpoints $\gamma(0) = \tilde{\gamma}(0)$, $\gamma(1) = \tilde{\gamma}(1)$) have $\int_\gamma f(z) dz = \int_{\tilde{\gamma}} f(z) dz$.
25. **Homotopy.** Let $\gamma_{0,1} : [a, b] \rightarrow \mathbb{C}$ be curves with shared endpoints z_a, z_b . A homotopy is a continuous function $H : [a, b] \times [0, 1] \rightarrow \mathbb{C}$ with $t \times s \rightarrow H_s(t)$ such that $H_0(t) = \gamma_0(t)$ and $H_1(t) = \gamma_1(t)$.
26. **Simply-connected.** A set A is called simply-connected if every closed curve (loop) is homotopic to a point in A , with $H_s(t) \in A$ for all s, t . (Note: a point in A is a constant loop).
27. **Winding Number.** Let γ be a loop in \mathbb{C} and $z_0 \in \mathbb{C}$ but not on γ . Then the winding number of γ (with respect to z_0) is $I(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} dz$.
28. **Entire.** We call a function $f : \mathbb{C} \rightarrow \mathbb{C}$ entire if it is holomorphic on \mathbb{C} .
29. **Closure.** The closure of a set A , written \bar{A} , is $\bar{A} = \{\text{limit points of } A\}$.
30. **Boundary of A .** The boundary ∂A of a set $A \subseteq \mathbb{C}$ is $\partial A = \bar{A} \cap \overline{(\mathbb{C} \setminus A)}$.
31. **Reflection:** $\tilde{z} = \frac{R^2}{\bar{z}}$. $\tilde{z} = \frac{R^2}{\bar{z}}$ is the reflection over the line the circle $|\xi| = R$.
32. **Analytic.** A function $f : \Omega \rightarrow \mathbb{C}$ is analytic at $z_0 \in \Omega$ if there is a neighborhood \mathcal{U} of z_0 on which $f(z) = \sum_{k=0}^\infty a_k(z - z_0)^k$ (for all $z \in \mathcal{U}$) where the RHS is a convergent power series.
33. **Pole of Order m .** If f is holomorphic on a deleted neighborhood $\mathcal{U} \setminus z_0$, we say f has a pole of order m if $\frac{1}{f}$ has a zero of order m .
34. **Simple Pole.** If f has a pole of order 1 at $z = z_0$, then f has a simple pole at z_0 .
35. **Residue of f at $z = z_0$.** Consider the principal part of the Laurent expansion of f at $z = z_0$. Then, the coefficient a_{-1} is the residue of f at z_0 and we write it as $\text{Res}_{z_0}(f) = a_{-1}$.
36. **Meromorphic.** A function $f : \Omega \rightarrow \mathbb{C}$ is meromorphic if it is holomorphic on all of Ω except at a discrete set of poles.
37. **Essential Singularity.** Let f be holomorphic on Ω except at a point z_0 . We call z_0 an essential singularity if z_0 is neither a pole nor a removable singularity.
38. **Alternative Definition of Essential Singularity.** Let f be holomorphic except possibly at a point z_0 . Let $C_1 = \{z_0\}$ and $C_2 = \partial D_r(z_0)$. Then, z_0 is an essential singularity if there are infinitely many a_{-n} in the Laurent series of f , where still $\text{Res}_{z_0}(f) = a_{-1}$.
39. **Holomorphic at Infinity.** Let $\mathcal{U} \subseteq \mathbb{C}$ be an open set containing $\mathbb{C} \setminus \overline{D_R(0)}$. A function $f : \mathcal{U} \rightarrow \mathbb{C}$ is holomorphic at infinity if $g(z) = f(1/z)$ has a removable singularity at 0, which in that case, we define $f(\infty) = g(0)$.
40. **Zero (respectively, pole) of order at Infinity.** Let $\mathcal{U} \subseteq \mathbb{C}$ be an open set containing $\mathbb{C} \setminus \overline{D_R(0)}$. We say that $f : \mathcal{U} \rightarrow \mathbb{C}$ has a zero (respectively, pole) of order at ∞ if $g(z) = f(1/z)$ has a zero (respectively, pole) at $z = 0$ (of order m).
41. **Logarithmic Derivative.** Let $f : \Omega \rightarrow \mathbb{C}$ be meromorphic. Then, the logarithmic derivative of f is f'/f .
42. **Locally injective.** We call a function $f : \Omega \rightarrow \mathbb{C}$ locally injective near z_0 if there exists a neighborhood \mathcal{U} of z_0 such that $f : \mathcal{U} \rightarrow \mathbb{C}$ is injective.
43. **Infinite Product.** Suppose the sequence of finite products $P_N := \prod_{n=1}^N c_n = c_1 c_2 \dots c_N$ converges to a finite number (where $c_i \in \mathbb{C}$ for all i). We define $\prod_{n=1}^\infty c_n = \lim_{N \rightarrow \infty} \prod_{n=1}^N c_n$ to be the infinite product, and say that this infinite product converges.
44. **Riemann-Zeta Function.** We define the Riemann-Zeta function to be $\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z}$.
45. **Conformal Maps.** Smooth, invertible maps $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ whose Jacobian at a point can be factored as (scaling) \cdot (rotation) are called conformal maps. (note: by Cauchy-Riemann equations, we have that $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = |a + bi|^2 \cdot \frac{1}{|a + bi|^2} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$, where the $|a + bi|^2$ factor represents scaling and the other factors together represent the orthogonal scaling matrix).
46. **Conformal Maps (2nd definition).** Let Ω and Ω' be open connected regions in \mathbb{C} . We say that a map $g : \Omega \rightarrow \Omega'$ is conformal if it is holomorphic and invertible with g^{-1} holomorphic.
47. **Conformally Equivalent.** We call Ω, Ω' conformally equivalent (write: $\Omega \sim \Omega'$) if there exists a conformal map $g : \Omega \rightarrow \Omega'$.
48. **Set of holomorphic functions / meromorphic functions.** Let $\mathcal{H}(\Omega) = \{\text{holomorphic functions } f : \Omega \rightarrow \mathbb{C}\}$ and $\mathcal{M}(\Omega) = \{\text{meromorphic functions } f : \Omega \rightarrow \mathbb{C}\}$.
49. **Conformal Automorphisms.** We call a conformal map $g : \Omega \rightarrow \Omega$ a conformal automorphism and write $\text{Aut}(\Omega)$ to denote the collection of automorphisms (strictly conformal) on Ω .
50. **Prop.** Let $z_1, z_2 \in \mathbb{C}$. Then $|z_1 z_2| = |z_1| |z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \pmod{2\pi}$.
51. **Theorem.** \mathbb{C} is a field.
52. **De Moivre's Formula.** If $z = r(\cos \theta + i \sin \theta)$ and $n \in \mathbb{Z}_{>0}$, then $z^n = r^n(\cos n\theta + i \sin n\theta)$.
53. **Prop.** Let $z, w \in \mathbb{C}$. Then:
 - (a) $\overline{z + w} = \bar{z} + \bar{w}$.
 - (b) $\overline{zw} = \bar{z} \cdot \bar{w}$.
 - (c) $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$ (with $w \neq 0$).
 - (d) $z\bar{z} = |z|^2$. If $z \neq 0$, then $z^{-1} = \frac{\bar{z}}{|z|^2}$.
 - (e) If $z = \bar{z}$, then $z \in \mathbb{R}$ and so $z = \text{Re}(z)$.
 - (f) $\text{Re}(z) = \frac{z + \bar{z}}{2}$ and $\text{Im}(z) = \frac{z - \bar{z}}{2i}$.
 - (g) $\bar{\bar{z}} = z$.
54. **Prop.** Let $z, w \in \mathbb{C}$. Then:
 - (a) $|z| \geq 0$ and if $|z| = 0$, then $z = 0$.
 - (b) $|zw| = |z| |w|$.
 - (c) If $w \neq 0$, then $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$.
 - (d) $|\text{Re}(z)| \leq |z|$ and $|\text{Im}(z)| \leq |z|$.
 - (e) $|\bar{z}| = |z|$.
 - (f) $|z + w| \leq |z| + |w|$.
 - (g) $||z| - |w|| \leq |z - w|$.
 - (h) $|z_1 w_1 + \dots + z_n w_n| \leq \sqrt{|z_1|^2 + \dots + |z_n|^2} \sqrt{|w_1|^2 + \dots + |w_n|^2}$.
55. **Prop.** Fix $r > 0$ and $z \in \mathbb{C}$. The open disk $D_\varepsilon(z_0)$ is an open set.
56. **Prop.** The following are true:
 - (a) \mathbb{C} is open.
 - (b) The empty set \emptyset is open.
 - (c) The union of open sets is open.
 - (d) The intersection of finitely many open sets is open.

57. **Prop.** Limits are unique (if they exist).

58. **Prop.** If $\lim_{z \rightarrow z_0} f(z) = a$ and $\lim_{z \rightarrow z_0} g(z) = b$, then:

- (a) $\lim_{z \rightarrow z_0} (f(z) + g(z)) = a + b$.
- (b) $\lim_{z \rightarrow z_0} ((f(z)g(z))) = ab$.
- (c) $\lim_{z \rightarrow z_0} \left(\frac{f(z)}{g(z)} \right) = \frac{a}{b}$ (with $b \neq 0$).

59. **Prop.** The following are true:

- (a) If $\lim_{z \rightarrow z_0} f(z) = a$ and h is continuous at a , then $\lim_{z \rightarrow z_0} h(f(z)) = h(a)$.
- (b) If f is continuous on an open set $\Omega \subseteq \mathbb{C}$, and h is continuous on $f(\Omega)$, then $h \circ f$ is continuous on Ω , with $(h \circ f)(z) = h(f(z))$.

60. **Prop.** The following are true:

- (a) The empty set \emptyset is closed.
- (b) \mathbb{C} is closed.
- (c) The intersection of a collection of closed sets is closed.
- (d) The union of finitely many closed sets is closed.

61. **Prop.** A set F is closed iff whenever z_1, z_2, z_3, \dots is a sequence of points in F converging to $\lim_{k \rightarrow \infty} z_k = w$, then $w \in F$.

62. **Prop.** If $f: \mathbb{C} \rightarrow \mathbb{C}$, TFAE:

- (a) f is continuous.
- (b) If $F \subseteq \mathbb{C}$ is closed, then $f^{-1}(F)$ is closed.
- (c) If Ω is open, then $f^{-1}(\Omega)$ is also open.

63. **Prop. (Heine-Borel + Sequential Compactness).** For $K \subseteq \mathbb{C}$, TFAE:

- (a) K is compact.
- (b) K is closed and bounded.
- (c) Every sequence of points in K has a convergent subsequence converging in K (sequentially compact).

64. **Prop.** If K is compact and $f: K \rightarrow \mathbb{C}$ is continuous, then the image $f(K)$ is compact.

65. **Theorem (Extreme Value Theorem).** If K is compact and $f: K \rightarrow \mathbb{R}$ is continuous, then f attains its minimum and maximum.

66. **Stereographic Projection / Riemann Sphere.** Identify the plane $\overline{\mathbb{C}} = S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. If $f: \mathbb{C} \rightarrow S^2 \setminus \{N\}$, then we have $(u, v) \mapsto \frac{1}{1+u^2+v^2} (2u, 2v, -1+u^2+v^2)$ is a homeomorphism (is continuous with continuous inverse) $f^{-1}: S^2 \setminus \{N\} \rightarrow \mathbb{C}$ with $(x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z} \right)$.

67. **Prop. (Uniform Convergence).** If $f_n \rightarrow f$ uniformly and each f_n is continuous, then f is continuous.

68. **Euler's Formula.** $e^{iz} = \cos z + i \sin z$ for all $z \in \mathbb{C}$.

69. **Theorem.** $\cos z = \frac{e^{iz} + e^{-iz}}{2}$ and $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$.

70. **Properties of e .** Let $x, y \in \mathbb{R}$ and $z, w \in \mathbb{C}$. Then:

- (a) $e^{z+w} = e^z e^w$.
- (b) $|e^{x+iy}| = |e^x e^{iy}| = |e^x| |e^{iy}| = e^x$.
- (c) $\arg(e^{x+iy}) = y \pmod{2\pi}$.
- (d) $e^z \neq 0$ for all $z \in \mathbb{C}$.
- (e) $e^z = 1$ iff $z = 2\pi i n$ for some $n \in \mathbb{Z}$.
- (f) $e^z = e^{z+2\pi i n}$.

71. **Prop. (Chain Rule).** Let $\Omega, A \subseteq \mathbb{C}$ be open sets, and let $f: \Omega \rightarrow A, g: A \rightarrow \mathbb{C}$ be holomorphic functions. Then, $g \circ f: \Omega \rightarrow \mathbb{C}$ is holomorphic and $\frac{dg}{dz}(f \circ g)(z) = \frac{dg}{df}(f(z)) \cdot \frac{df}{dz}(z)$.

72. **Prop.** Let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic. Then $f: \Omega \rightarrow \mathbb{R}^2$ is real differentiable at all $(x, y) \in \Omega$.

73. **Cauchy-Riemann Equations.** Let Ω be an open set in \mathbb{C} and let $f: \Omega \rightarrow \mathbb{C}$ be given by $f(x, y) = u(x, y) + iv(x, y)$. Then:

- (a) $f'(z)$ exists at $z \in \Omega$ iff f is real differentiable and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (these are the Cauchy-Riemann equations).
- (b) $f(z)$ is holomorphic on Ω iff partials are continuous and satisfy the CR equations.
- (c) If $f'(z_0)$ exists, then $f'(z_0) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial y} - i \frac{\partial v}{\partial y} = \frac{1}{i} \frac{\partial f}{\partial y}$.

74. **Inverse Function Theorem for \mathbb{R}^2 .** If $f: \Omega \rightarrow \mathbb{R}^2$ is continuously differentiable and the Jacobian $Df(z_0)$ has $\det(Df(z_0)) \neq 0$, then there are neighborhoods $U \ni z_0$ and $V \ni f(z_0)$ such that $f: U \rightarrow V$ is bijective with continuously differentiable $f^{-1}: V \rightarrow U$ such that $Df^{-1}(z_0) = [Df(z_0)]^{-1}$, which is the inverse matrix of $Df(z_0)$.

75. **Inverse Function Theorem for \mathbb{C} .** Let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic (with continuous $f'(z_0)$), and $f'(z) \neq 0$ for some $z_0 \in \Omega$. Then there exists a neighborhood $U \ni z_0$ and $V \ni f(z_0)$ such that $f: U \rightarrow V$ is bijective with holomorphic inverse $f^{-1}: V \rightarrow U$ such that for all $z_0 \in U, \frac{d}{dw} f^{-1}(w) = \frac{1}{f'(w)}$ with $w = f(z)$.

76. **Prop.** Pick a branch $[a, a + 2\pi)$. Then $\log z: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{R} \times i[a, a + 2\pi)$ is the inverse of $\exp: \mathbb{R} \times i[a, a + 2\pi) \rightarrow \mathbb{C}$.

77. **Prop.** $\log: \mathbb{C} \setminus \mathbb{R}_{\leq 0} \rightarrow \mathbb{R} \times i(-\pi, \pi)$ is holomorphic with $\frac{d}{dz} \log z = \frac{1}{z}$.

78. **Prop.** If $z_1, z_2 \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, then $\log(z_1 z_2) = \log z_1 + \log z_2 \pmod{2\pi i}$.

79. **Prop.** By choosing different branches of log, we have the following:

- (a) a^b is independent of the branch iff $b \in \mathbb{Z}$.
- (b) a^b takes on exactly q different values iff $b \in \mathbb{Q}$, so $b = \frac{p}{q}$ (with p, q coprime).
- (c) a^b takes on infinitely many values iff b is irrational or $\text{Im}(b) \neq 0$.

80. **Cor.** Choose a branch of log. Then the n th root function is given by $z^{1/n} = e^{\log(z)/n}$, where the n th root function has n branches.

81. **Prop.** Let $a, b \in \mathbb{C}$. Then:

- (a) For any choice of branch of log, the function $z \mapsto a^z$ is holomorphic on \mathbb{C} , and $z \mapsto (\log a) a^z$.
- (b) Choose a branch of log. Then the function $z \mapsto z^b$ is holomorphic on the domain of log with derivative $z \mapsto b z^{b-1}$.

82. **Prop. (Re-parametrization).** If $\tilde{\gamma}$ is a re-parametrization of γ , then $\int_{\tilde{\gamma}} f = \int_{\gamma} f$ for any continuous f on Ω .

83. **Fundamental Theorem of Line Integrals.** Let $F: \Omega \rightarrow \mathbb{C}$ be holomorphic on an open Ω and let $\gamma: [0, 1] \rightarrow \Omega$ be piecewise smooth. Then, $\int_{\gamma} F'(z) dz = F(\gamma(1)) - F(\gamma(0))$.

84. **Path-independence and Primitives Theorem.** Let $f: \Omega \rightarrow \mathbb{C}$ be continuous and Ω is open and connected. Then, TFAE:

- (a) (path-independence) if $z_0, z_1 \in \Omega$, then any paths $\gamma, \tilde{\gamma}$ with shared endpoints $\gamma(0) = \tilde{\gamma}(0)$ and $\gamma(1) = \tilde{\gamma}(1)$ have $\int_{\gamma} f(z) dz = \int_{\tilde{\gamma}} f(z) dz$.
- (b) (integral along loops is 0) if Γ is a loop, with $\Gamma(1) = \Gamma(0)$, then $\int_{\Gamma} f(z) dz = 0$.
- (c) (f has a primitive) There is a primitive F for f on Ω .

85. **Cauchy-Goursat Theorem.** Let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic on Ω , simply connected, and open. Then for any loop $\Gamma \subseteq \Omega$, $\int_{\Gamma} f(z) dz = 0$.

86. **Green's Theorem.** Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a vector field and let γ be a loop, and A a region in the loop γ . Let $f(x, y) = (P(x, y), Q(x, y))$. Then, $\int_{\gamma} P(x, y) dx + Q(x, y) dy = \iint \text{curl } F dA = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$.

87. **Prop.** If $f(x + iy) = u(x, y) + iv(x, y)$, then $\int_{\gamma} f = \int_{\gamma} u dx - v dy + i \int_{\gamma} v dx + u dy$.

88. **Cauchy-Goursat Theorem (Weaker Version).** Let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic with $f'(z)$ continuous and $\gamma: [0, 1] \rightarrow \mathbb{C}$ a simple closed curve and Ω an open & simply connected set. Then, $\int_{\gamma} f = 0$.

89. **Cauchy-Goursat Theorem (for rectangles).** Let R be a rectangle with R and its interior are contained in an open set Ω . Let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic. Then, $\int_R f = 0$.

90. **Cauchy-Goursat Theorem (for disks).** Suppose $f: D \rightarrow \mathbb{C}$ is holomorphic on an open disk $D := D_{\rho}(z_0)$. Then:

- (a) f has a primitive F on D .
- (b) if Γ is any loop in D , then $\int_{\Gamma} f = 0$.

91. **Deformation Theorem.** Suppose f is holomorphic on an open set Ω and γ_0, γ_1 are piecewise continuously differentiable. Then there are continuously differentiable curves in Ω . Then:

- (a) If γ_0, γ_1 are paths from z_0 to z_1 , which are homotopic in Ω , then $\int_{\gamma_0} F = \int_{\gamma_1} F$.
- (b) If γ_0, γ_1 are loops homotpic in Ω , then $\int_{\gamma_0} F = \int_{\gamma_1} F$. (note: this also works for constant loops, where constant loops are just points)

92. **Cauchy-Goursat Theorem (restated).** Let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic with Ω open and a loop (let γ) be homotopic to a point in Ω . Then, $\int_{\gamma} f dz = 0$.

93. **Cor.** If Ω is simply connected, then every loop γ has $\int_{\gamma} f dz = 0$.

94. **Cor.** Let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic on a simply connected oen set Ω . Then, f has a primitive F on Ω (unique up to constants).

95. **Winding number (as an index).** Let $\gamma: [a, b] \rightarrow \mathbb{C}$ (a piecewise continuous) loop and $z \notin \gamma([a, b])$. Then, the winding number of γ around z_0 is an integer.

96. **Cauchy's Integral Formula.** Let f be holomorphic on Ω and γ a loop in Ω hommotopic to a point. Let $z_0 \in \Omega$ but $z_0 \notin \gamma$. Then,

$$f(z_0) \cdot I(\gamma, z_0) = \frac{1}{2\pi i} \cdot \int_{\gamma} \frac{f(z)}{z - z_0} dz.$$

97. **Cauchy's Integral Formula for Derivatives.** Let f be holomorphic on Ω . Then f is infinitely differentiable (complex) and if γ is a loop homotpic to a point (simple loop) $I(\gamma, z_0) = 1$, then:

$$f^{(n)}(z_0) = \frac{n}{2\pi i} \int_{\gamma} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi.$$

98. **Cor. Cauchy-Type Integrals.** Let γ be a loop $\gamma: [a, b] \rightarrow \mathbb{C}$ and g a continuous function on γ . Set $\tilde{g}(z) := \int_{\gamma} \frac{g(\xi)}{\xi - z} d\xi$. Then, $\tilde{g}(z)$ is holomorphic inside γ and so $\tilde{g}(z)$ is infinitely differentiable.

99. **Prop. (Cauchy Inequalities).** Let f be holomorphic on Ω and let $\overline{D_R(z_0)} \subseteq \Omega$ with boundary γ . Suppose $f(z)$ is bounded above $|f(z)| \leq M$ for all $z \in \gamma$. Then for all $k = 1, 2, \dots$, the k th derivative is also upper bounded with $|f^{(k)}(z_0)| \leq \frac{k!}{R^k} M$.

100. **Louisville's Theorem.** If f is entire and bounded (i.e. there exists an $M \in \mathbb{R}_{>0}$ with $|f(z)| \leq M$ for all $z \in \mathbb{C}$), then f is constant.

101. **Fundamental Theorem of Algebra.** Let $a_0, \dots, a_n \in \mathbb{C}$ with $a_i \neq 0$ for $n \geq 1$. Then the polynomial $p(z) = a_n z^n + \dots + a_0$ has a zero (root) where $z_0 \in \mathbb{C}$ with $p(z_0) = 0$.

102. **Cor.** A degree n complex polynomial has exactly n roots, counting multiplicity.

103. **Morera's Theorem. (partial converse to Cauchy-Goursat)** Let f continuous on an open Ω and suppose that $\int_\gamma f = 0$ for every loop in Ω . Then, f is holomorphic on Ω and f has a primitive F on Ω .

104. **Cor. to Morera's Theorem (Removable Singularities Theorem).** Let f be continuous on an open Ω in \mathbb{C} and holomorphic on $\Omega \setminus \{z_0\}$, with $z_0 \in \mathbb{C}$. Then, f is holomorphic on Ω .

105. **Another Cor. to Morera's Theorem.** If \tilde{f} is holomorphic on $\Omega \setminus \{z_0\}$ and bounded on a neighborhood of z_0 , there is unique holomorphic extension \hat{f} of f to γ defined by $\hat{f}(z) = f(z)$ if $z \neq z_0$ and $\hat{f}(z) = \lim_{z \rightarrow z_0} f(z)$ if $z = z_0$.

106. **Mean Value Property.** Let f be holomorphic on $\overline{D_R(z_0)}$. Then $f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$.

107. **Maximum Principle, local version.** Let f be holomorphic on a neighborhood Ω of z_0 , and suppose that $|f|$ has a relative max at z_0 . Then, f is constant on some neighborhood U of z_0 .

108. **Prop.** The following are true:

- (a) $A \subseteq \bar{A}$.
- (b) \bar{A} is closed.
- (c) A is closed iff $A = \bar{A}$.
- (d) If $A \subseteq C$ and C closed, then $\bar{A} \subseteq C$.

109. **Maximum Modulus Principle.** Let A be an open, connected, bounded set in \mathbb{C} and suppose $f: \bar{A} \rightarrow \mathbb{C}$ is holomorphic on A and continuous on \bar{A} . Then $|f|$ has a finite maximum value on \bar{A} which is achieved on ∂A . If $|f|$ is attained in A , then f is constant.

110. **Prop.** Let $u: \Omega \rightarrow \mathbb{R}$ be an twice-continuous harmonic function on an open set $\Omega \subseteq \mathbb{C}$. Then u is infinitely differentiable, so u is C^∞ , and in the neighborhood U of $z_0 \in \Omega$, there exists a holomorphic function $f: U \rightarrow \mathbb{C}$ such that $u = \text{Re}(f)$.

111. **Dirichlet Problem.** $\Delta u = 0$, $u|_{\partial\Omega}(\theta) = g(\theta)$.

112. **Prop.** Let u, \tilde{u} solve the Dirichlet Problem. Then, $u = \tilde{u}$, so the solution to the Dirichlet Problem is unique.

113. **Solution to the Dirichlet Problem.** This is given by:

$$u(re^{i\phi}) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \cdot \frac{R^2 - r^2}{|Re^{i\theta} - re^{i\phi}|} d\theta.$$

114. **Analytic Convergence Theorem.** Let $f_n: \Omega \rightarrow \mathbb{C}$ be a sequence of holomorphic functions. If $f_n \rightarrow f$ uniformly on every closed disk in Ω , then:

- (a) f is holomorphic on Ω .
- (b) f'_n converges to f' uniformly on every closed disk, and pointwise on Ω .

115. **Prop.** Let $\gamma: [a, b] \rightarrow \Omega$ be a contour and $f_n: \gamma([a, b]) \rightarrow \mathbb{C}$ be a sequence of continuous functions. If $f_n \rightarrow f$ uniformly on $\gamma([a, b])$, then $\int_\gamma f_n \rightarrow \int_\gamma f$.

116. **Power Series Convergence Theorem.** Let $\sum_{n=0}^\infty a_n(z - z_0)^n$ be a power series. Then there is a unique $R \geq 0$, possibly $R = \infty$, such that:

- (a) If $|z - z_0| < R$, the series converges pointwise.
- (b) If $|z - z_0| \leq R - \epsilon$, the series converges uniformly.
- (c) If $|z - z_0| > R$, the series diverges.
- (d) If $|z - z_0| = R$, need to check.

117. **Cor.** A power series is analytic on its disk of convergence and so, holomorphic by the analytic convergence theorem.

118. **Cor 2.** We can apply term-by-term differentiation to the power series of an analytic function.

119. **Cor 3.** Power series expansions around some center z_0 are unique.

120. **Taylor Series Theorem.** Let f be holomorphic on a region Ω , and let $D_r(z_0) \subseteq \Omega$ with $r > 0$. Then for every $z \in D_r(z_0)$, the power series $f(z) = \sum_{n=0}^\infty \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$ converges on $D_r(z_0)$ and equal to $f(z)$. We call this the Taylor series of f centered at z_0 .

121. **Theorem.** f is analytic iff f is holomorphic.

122. **Prop.** If f is holomorphic on an open, connected set Ω and the zero set $\{z \in \Omega \mid f(z) = 0\}$ contains a limit point, then $f = 0$ on Ω .

123. **Cor. (Identity Theorem).** Let f, g be holomorphic on an open, connected Ω and $f(z) = g(z)$ for a set of z with a limit point in Ω . Then, $f = g$ on Ω .

124. **Cor. (Zeros are Isolated).** If f is holomorphic on Ω , and not identically zero on Ω , then for any zero z_0 of f , there is a deleted neighborhood $U \setminus \{z_0\}$ on which $f(z) \neq 0$ for all $z \in U \setminus \{z_0\}$.

125. **Cor. (Analytic Continuation).** If f is holomorphic on an open, connected set Ω and f_+ is holomorphic on an open connected $\Omega_+ \supseteq \Omega$ with $f_+ = f$ on Ω , then f_+ is the unique such extension, i.e. if there exists another such extension, \tilde{f}_+ , then $\tilde{f}_+ = f_+$.

126. **Lemma.** Let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic, not identically 0, with a zero z_0 . Then in a neighborhood U of z_0 , we may write $f(z) = (z - z_0)^m g(z)$ for all $z \in U$, where $g(z) \neq 0$ and m is unique.

127. **Lemma.** A function f has a pole of order m at z_0 iff there is a neighborhood U of z_0 on which $f(z) = (z - z_0)^{-m} g(z)$ for all $z \in U \setminus \{z_0\}$ with a nonzero $g(z)$ and g holomorphic on U .

128. **Theorem.** If f has a pole of order m at z_0 , then it can be represented uniquely as:

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \dots + \frac{a_{-1}}{z - z_0} + G(z)$$

where $G(z)$ is holomorphic on a neighborhood U of z_0 and $a_{-m}, \dots, a_{-1} \in \mathbb{C}$ with $a_{-m} \neq 0$.

129. **Residue Theorem, Simple Version.** Let f be holomorphic on a set $\Omega \supseteq \overline{D_R(z_0)}$, $\gamma = \partial D_R(z_0)$ except at z_0 . Then $\int_\gamma f(z) dz = 2\pi i \text{Res}_{z_0}(f)$.

130. **Residue Theorem, Simple Closed Loops.** Let Ω be open, connected and γ a simple loop homotopic to a point in Ω . Let f be a function $f: \Omega \rightarrow \mathbb{C}$ be holomorphic except at a finite set of points z_1, \dots, z_N inside γ . Then, $\int_\gamma f(z) dz = 2\pi i \sum_{k=1}^N \text{Res}_{z_k}(f)$.

131. **Laurent Series Theorem.** Let C_1, C_2 be two circles centered at z_0 (it is fine if $C_1 = \{z_0\}$ and C_2 "encloses" \mathbb{C}). Call R the region the annulus between C_1 and C_2 . Let f be holomorphic on R . Then f can be expanded uniquely as a (absolutely) convergent power series in R by:

$$f(z) = \sum_{n=1}^\infty \frac{a_{-n}}{(z - z_0)^n} + \sum_{n=0}^\infty a_n (z - z_0)^n.$$

where the first infinite series is called the principal part and second one is called the Taylor series / holomorphic part.

132. **Casorati-Weierstrauss Theorem.** If f is holomorphic in a deleted $D_r(z_0) \setminus \{z_0\}$ and has an essential singularity at z_0 , then the image of $f(D_r(z_0) \setminus \{z_0\})$ is dense in \mathbb{C} .

133. **Prop. (Fourier Transform).** Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a real function. The Fourier Transform of f is the function $\hat{f}: \mathbb{R} \rightarrow \mathbb{R}$ given by $\hat{f}(k) = \int_{-\infty}^\infty f(x) \cdot e^{-ikx} dx$.

134. **Jordan's Lemma.** $\int_0^\pi e^{-R \sin \theta} d\theta \leq \frac{\pi}{R}$.

135. **Cauchy Principal Value.** Take the real integral symmetrically, so we can find an indefinite integral (with discontinuity in the interval) by approaching "the same way" from both sides of the discontinuity.

136. **Argument Principle.** Let $f: \Omega \rightarrow \mathbb{C}$ be meromorphic and γ a simple loop in Ω bounding a simply connected region R_γ , with $\bar{R}_\gamma \subseteq \Omega$. Let f have no zeros or poles on γ . Then:

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = N - P$$

where N is the number of zeros of f inside R_γ (counting multiplicity) and P is the number of poles of f inside R_γ (counting multiplicity).

137. **Rouche's Theorem.** Let $f, g: \Omega \rightarrow \mathbb{C}$ be holomorphic and let γ be a simple loop bounding a simply connected open U , with $\bar{U} \subseteq \Omega$. If $|f(z)| > |g(z)|$ on γ , then f and $f + g$ have the same number of zeros inside U .

138. **Open-Mapping Theorem.** Any nonconstant holomorphic function is an open map, meaning it maps open sets to open sets.

139. **Lemma. (Local Injectivity).** Let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic and $z_0 \in \Omega$. If $f'(z_0) \neq 0$, then f is locally injective near z .

140. **Theorem.** If $g: \Omega \rightarrow \mathbb{C}$ is holomorphic and Ω is simply connected, and $g \neq 0$, there exists a holomorphic function $F: \Omega \rightarrow \mathbb{C}$ satisfying $e^{F(z)} = g(z)$, where $F(z)$ is unique up to $2\pi i k$, with $k \in \mathbb{Z}$.

141. **Theorem (Local description of holomorphic).** Let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic, Ω open. Let $z_0 \in \Omega$ and let $k \geq 1$ denote the order of the zero $f(z) - f(z_0)$ at z_0 . Then, there exists an open neighborhood U of z_0 (and $r > 0$) and a function $\phi: U \rightarrow D_r(z_0)$ such that:

- (a) ϕ is holomorphic with a holomorphic inverse.
- (b) $\phi(z_0) = 0$.
- (c) We have $f(z) = f(z_0) + (\phi(z))^k$ with $z \in U$.

142. **Prop.** Let $\{a_n\}_{n=1}^\infty$ be a sequence of complex numbers. If $\sum_{n=1}^\infty |a_n| < \infty$, then $\prod_{n=1}^\infty (1 + a_n)$ converges and its value is 0 iff one of the $1 + a_n$ factors is zero.

143. **Prop.** Let $\{f_n\}_{n=1}^\infty$ be a sequence of holomorphic functions on Ω . If $\sum_{n=1}^\infty |f_n|$ converges uniformly on compact subsets of Ω , then so does $\prod_{n=1}^\infty (1 + f_n(z))$. Moreover, the limiting function is holomorphic and nonzero everywhere except at points z such that $1 + f_n(z) = 0$ (for some n).

144. **Prop. (Partial fractions expansion for log derivatives).** Same assumptions as the above proposition. Then, the log derivative of product = sum of log derivatives. i.e.

$$\frac{(\prod_{n=1}^\infty (1 + f_n))'}{\prod_{n=1}^\infty (1 + f_n)} = \sum_{n=1}^\infty \frac{f'_n}{1 + f_n}.$$

145. **Infinite products formula for sine.** $\sin(\pi z) = \pi z \cdot \prod_{n=1}^\infty (1 - \frac{z}{n})(1 + \frac{z}{n})$, for $z \in \mathbb{C}$.

146. **Prop.** Fix $z \in \mathbb{C} \setminus \mathbb{Z}$ and a large positive $N \in \mathbb{Z}$. By the residue theorem, the integral $I_N(z) := \int_{\gamma_N} \frac{\pi \cot(\pi z)}{(w+z)^2} dw$.

147. **Cor.** $\cos(\pi z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\left(n - \frac{1}{2}\right)^2}\right)$, $e^z - 1 = z e^{z/2} \cdot \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right)$, and $\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$ (for the contangent expression, $z \in \mathbb{C} \setminus \mathbb{Z}$).

148. **Theorem (Schwarz Reflection Principle).** Let A be a region in the upper-half plane with $\partial A \cap \mathbb{R}$ nonempty and containing $[a, b] \subseteq \mathbb{R}$. Let f be holomorphic on A and continuous on $\partial A \cap [a, b]$ and real on $[a, b]$. Then f can be uniquely extended to a holomorphic function on $A \cup (a, b) \cup A_{\text{ref}}$, where $A_{\text{ref}} = \{\bar{z} \mid z \in A\}$ with $f(z) = \overline{f(\bar{z})}$ for all $z \in A_{\text{ref}}$.

149. **Theorem (Gamma Function).** There is a unique $\Gamma(s)$ with the following:

- (a) $\Gamma(s)$ is meromorphic.
- (b) (Factorial). $\Gamma(n+1) = n!$ for $n = 0, 1, 2, \dots$
- (c) (Special Value). $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
- (d) (Integral Representation). $\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1} dx$ (for $\text{Re}(s) > 0$).
- (e) (Infinite Product Representation). $\Gamma(s) = s^{-1} e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right)^{-1} e^{s/n}$ where $\gamma = \lim_{n \rightarrow \infty} \left(\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n} - \log n\right) \approx 0.58\dots$ (Euler-Mascheroni constant) (for $s \in \mathbb{C}$, except poles).
- (f) (Limit of finite products). $\Gamma(s) = \lim_{n \rightarrow \infty} \frac{n! n^s}{s(s+1)\dots(s+n)}$ (for $s \in \mathbb{C}$, except poles).
- (g) (Zeros). $\Gamma(s)$ has no zeros.
- (h) (Poles). $\Gamma(s)$ has poles at nonpositive integers $s = 0, -1, -2, \dots$ and is holomorphic everywhere else. At $s = -n$, the pole is simple and $\text{Res}_{-n}(\Gamma) = -\frac{1}{n!}$.
- (i) (Functional Equation). $\Gamma(s+1) = s\Gamma(s)$ (for $s \in \mathbb{C}$, except at poles).
- (j) (Reflection Formula). $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$ (for $s \in \mathbb{C}$, except at poles).

150. **Theorem.** The conformal equivalence, which is a relation, is an equivalence relation.

151. **Cor.** Holomorphic f is locally injective iff $f'(z_0) \neq 0$.

152. **Conformal Equivalence Classes.** These are the following:

- (a) Complex plane, \mathbb{C} .
- (b) Punctured plane, $\mathbb{C} \setminus \{0\}$.
- (c) Unit disk, $D_1(0)$.
- (d) Upper-half plane, $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$.
- (e) Riemann sphere*, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (not a subset of \mathbb{C} but we can still talk about holomorphic/meromorphic functions on it).
- (f) The slit plane, $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.
- (g) Strip (similar to critical region from Riemann hypothesis).
- (h) Rectangle.
- (i) Annulus.
- (j) Blob

153. **Lemma.** If $\Omega \sim \Omega'$, then Ω and Ω' are homeomorphic (a.k.a. there is a continuous map $g : \Omega \rightarrow \Omega'$ such that g^{-1} is defined and also continuous).

154. **Lemma.** If $\Omega \sim \Omega'$, then they are homeomorphic.

155. **Prop.** If $g : \Omega \rightarrow \Omega'$ is holomorphic and invertible, then g^{-1} is holomorphic (i.e. g is conformal).

156. **Lemma.** $\text{Aut}(\Omega)$ is a group, with function composition. In other words, let $f, g, h \in \text{Aut}(\Omega)$. Then:

- (a) $(g \circ f) \circ h = g \circ (f \circ h)$.
- (b) If $g \in \text{Aut}(\Omega)$, then $g^{-1} \in \text{Aut}(\Omega)$.
- (c) There is an identity map $\text{id} \in \text{Aut}(\Omega)$ such that $\text{id} \circ g(z) = g(z) = g \circ \text{id}(z) = g(z)$.

157. **Theorem.** Let $g : \mathbb{C} \rightarrow \Omega$ be a conformal map between \mathbb{C} and a region Ω . Then, $\Omega = \mathbb{C}$ and $g(z)$ is a conformal automorphism of the form $g(z) = az + b$, with $a \neq 0$ and $b \in \mathbb{C}$.

158. **Theorem (Riemann Sphere).** Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. If $g : \hat{\mathbb{C}} \rightarrow \Omega$ is a conformal map, then $\Omega = \hat{\mathbb{C}}$ and g is a conformal automorphism, with $g(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{C}$ (i.e. g is a Möbius transformation).