

Math 185 Definitions

1. **Complex Numbers, \mathbb{C} .** The set of complex numbers \mathbb{C} is the real vector space \mathbb{R}^2 with the properties $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $a(x_1, y_1) = (ax_1, ay_1)$ and we write $z = x + iy = (x, y)$ for any $z \in \mathbb{C}$.
2. **Polar Form.** Let $z = a + bi \in \mathbb{C}$. Then norm (a.k.a. modulus, absolute value) of $z \in \mathbb{C}$ is written $|z| \in \mathbb{R}$ and is defined by $|z| = \sqrt{a^2 + b^2}$. Then, define the argument of z as $\arg(z) = \theta \in [0, 2\pi)$ as the angle z makes with the real axis. Then, the polar form of z is written as $z = |z|(\cos \theta + i \sin \theta)$.
3. **Rectangular form.** If $z \in \mathbb{C}$ is written as $z = x + iy$, with $x, y \in \mathbb{R}$, then z is written in rectangular form.
4. **Complex Conjugate.** If $z = a + ib$, then its complex conjugate is $\bar{z} = a - ib$.
5. **Open Sets.** A set $\Omega \subseteq \mathbb{C}$ is called open if for each $z_0 \in \Omega$, there is an $\varepsilon > 0$ such that $D_\varepsilon(z_0) \subseteq \Omega$.
6. **Neighborhoods.** An ε -neighborhood of a point z_0 is a set N which contains some open disk $D_\varepsilon(z_0)$.
7. **ε -deleted neighborhoods.** An ε -deleted neighborhood of a point z_0 is a set N which contains a "punctured" open disk $D_\varepsilon(z_0) \setminus \{z_0\}$.
8. **Homeomorphism.** A function is a homeomorphism if it is continuous with a continuous inverse.
9. **Periodic.** $f(z)$ is w -periodic (with $w \in \mathbb{C}$) if $f(z + nw) = f(z)$ for all $z \in \mathbb{C}, n \in \mathbb{Z}$.
10. **Limits.** Let $f : \Omega \rightarrow \mathbb{C}$ where Ω is an r -deleted neighborhood of a point z_0 . Then f has a limit as $z \rightarrow z_0$, and write $\lim_{z \rightarrow z_0} f(z) = a$. This means that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $z \in \Omega$ has $|z - z_0| < \delta$, then $|f(z) - a| < \varepsilon$.
11. **Continuity.** Let $\Omega \subseteq \mathbb{C}$ be an open set. Then $f : \Omega \rightarrow \mathbb{C}$ is continuous at a point $z_0 \in \Omega$ if $\lim_{z \rightarrow z_0} f(z) = f(z_0)$.
12. **Closed Sets.** A subset $F \subseteq \mathbb{C}$ is called closed if its complement $\mathbb{C} \setminus F$ is open.
13. **Compact.** A subset $K \subseteq \mathbb{C}$ is called compact if every open cover of K has a finite subcover.
14. **Uniform Convergence.** A sequence of functions $f_n : \Omega \rightarrow \mathbb{C}$ converges uniformly to a function $f : \Omega \rightarrow \mathbb{C}$ if for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(z) - f(z)| < \varepsilon$ for all $z \in \Omega$.
15. **Derivative.** Let $f : \Omega \rightarrow \mathbb{C}$, where Ω is a neighborhood of z_0 . The derivative of f at z_0 is the limit $f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$. We call f **complex differentiable** at z_0 if this limit exists. If $f'(z_0)$ exists for all $z_0 \in \Omega$, we call f **holomorphic** on Ω .
16. **Branch of the Argument.** This is a choice of interval (here, $[-\pi, \pi)$ or $[a, a + 2\pi)$).
17. **Principal Branch of the Logarithm.** Pick a branch $[a, a + 2\pi)$. Then, $\log : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{R} \times i[a, a + 2\pi)$ is defined by $\log z = \log |z| + i \arg z$, where $\arg z \in [a, a + 2\pi)$. We call the branch $[-\pi, \pi)$ the principal branch.
18. **Exponentiation of complex numbers a, b .** Choose a branch of \log , with $\log : \Omega \rightarrow \mathbb{C}$ and $a, b \in \mathbb{C}$. Then, define $a^b := e^{b \log a}$.

19. **Contour Integral.** Suppose f is continuous on an open set Ω and $\gamma : [a, b] \rightarrow \Omega$ is a smooth curve. Then the contour integral of f along γ is defined to be $\int_{\gamma} f := \int_a^b f(\gamma(t))\gamma'(t)dt$.
20. **Re-parametrization of γ .** A piecewise smooth $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \rightarrow \mathbb{C}$ is called a re-parametrization of γ if there exists a continuously differentiable $\alpha : [a, b] \rightarrow [\tilde{a}, \tilde{b}]$ such that $\alpha(a) = \tilde{a}$ and $\alpha(b) = \tilde{b}$, and $\alpha'(t) > 0$ with $\gamma(t) = \tilde{\gamma}(\alpha(t))$.
21. **Primitive.** We say a function $f : \Omega \rightarrow \mathbb{C}$ has a primitive on Ω if there exists a holomorphic function $F : \Omega \rightarrow \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in \Omega$.
22. **Path-connected.** We say an open set $\Omega \subseteq \mathbb{C}$ is path-connected if for any pair of points $z_0, z_1 \in \Omega$ there exists a continuous path $\gamma : [0, 1] \rightarrow \Omega$ such that $z_0 = \gamma(0)$ and $z_1 = \gamma(1)$, with $\gamma([0, 1]) \subseteq \Omega$.
23. **Path-independence.** If $z_0, z_1 \in \Omega$, then any paths $\gamma, \tilde{\gamma}$ (with shared endpoints $\gamma(0) = \tilde{\gamma}(0)$, $\gamma(1) = \tilde{\gamma}(1)$) have $\int_{\gamma} f(z)dz = \int_{\tilde{\gamma}} f(z)dz$.
24. **Homotopy.** Let $\gamma_{0,1} : [a, b] \rightarrow \mathbb{C}$ be curves with shared endpoints z_a, z_b . A homotopy is a continuous function $H : [a, b] \times [0, 1] \rightarrow \mathbb{C}$ with $t \times s \rightarrow H_s(t)$ such that $H_0(t) = \gamma_0(t)$ and $H_1(t) = \gamma_1(t)$.
25. **Simply-connected.** A set A is called simply-connected if every closed curve (loop) is homotopic to a point in A , with $H_s(t) \in A$ for all s, t . (Note: a point in A is a constant loop).
26. **Winding Number.** Let γ be a loop in \mathbb{C} and $z_0 \in \mathbb{C}$ but not on γ . Then the winding number of γ (with respect to z_0) is $I(\gamma, z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - z_0} dz$.
27. **Entire.** We call a function $f : \mathbb{C} \rightarrow \mathbb{C}$ entire if it is holomorphic on \mathbb{C} .
28. **Closure.** The closure of a set A , written \bar{A} , is $\bar{A} = \{\text{limit points of } A\}$.
29. **Boundary of A .** The boundary ∂A of a set $A \subseteq \mathbb{C}$ is $\partial A = \bar{A} \cap \overline{(\mathbb{C} \setminus A)}$.
30. **Reflection:** $\tilde{z} = \frac{R^2}{\bar{z}}$. $\tilde{z} = \frac{R^2}{\bar{z}}$ is the reflection over the line the circle $|\xi| = R$.
31. **Analytic.** A function $f : \Omega \rightarrow \mathbb{C}$ is analytic at $z_0 \in \Omega$ if there is a neighborhood \mathcal{U} of z_0 on which $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$ (for all $z \in \mathcal{U}$) where the RHS is a convergent power series.
32. **Pole of Order m .** If f is holomorphic on a deleted neighborhood $\mathcal{U} \setminus z_0$, we say f has a pole of order m if $\frac{1}{f}$ has a zero of order m .
33. **Simple Pole.** If f has a pole of order 1 at $z = z_0$, then f has a simple pole at z_0 .
34. **Residue of f at $z = z_0$.** Consider the principal part of the Laurent expansion of f at $z = z_0$. Then, the coefficient a_{-1} is the residue of f at z_0 and we write it as $\text{Res}_{z_0}(f) = a_{-1}$.
35. **Meromorphic.** A function $f : \Omega \rightarrow \mathbb{C}$ is meromorphic if it is holomorphic on all of Ω except at a discrete set of poles.
36. **Essential Singularity.** Let f be holomorphic on Ω except at a point z_0 . We call z_0 an essential singularity if z_0 is neither a pole nor a removable singularity.
37. **Alternative Definition of Essential Singularity.** Let f be holomorphic except possibly at a point z_0 . Let $C_1 = \{z_0\}$ and $C_2 = \partial D_r(z_0)$. Then, z_0 is an essential singularity if there are infinitely many a_{-n} in the Laurent series of f , where still $\text{Res}_{z_0}(f) = a_{-1}$.
38. **Holomorphic at Infinity.** Let $\mathcal{U} \subseteq \mathbb{C}$ be an open set containing $\mathbb{C} \setminus \overline{D_R(0)}$. A function $f : \mathcal{U} \rightarrow \mathbb{C}$ is holomorphic at infinity if $g(z) = f(1/z)$ has a removable singularity at 0, which in that case, we define $f(\infty) = g(0)$.

39. **Zero (respectively, pole) of order at Infinity.** Let $\mathcal{U} \subseteq \mathbb{C}$ be an open set containing $\mathbb{C} \setminus \overline{D_R(0)}$. We say that $f: \mathcal{U} \rightarrow \mathbb{C}$ has a zero (respectively, pole) of order at ∞ if $g(z) = f(1/z)$ has a zero (respectively, pole) at $z = 0$ (of order m).
40. **Logarithmic Derivative.** Let $f: \Omega \rightarrow \mathbb{C}$ be meromorphic. Then, the logarithmic derivative of f is f'/f .
41. **Locally injective.** We call a function $f: \Omega \rightarrow \mathbb{C}$ locally injective near z_0 if there exists a neighborhood \mathcal{U} of z_0 such that $f: \mathcal{U} \rightarrow \mathbb{C}$ is injective.
42. **Infinite Product.** Suppose the sequence of finite products $P_N := \prod_{n=1}^N c_n = c_1 c_2 \dots c_N$ converges to a finite number (where $c_i \in \mathbb{C}$ for all i). We define $\prod_{n=1}^{\infty} c_n = \lim_{N \rightarrow \infty} \prod_{n=1}^N c_n$ to be the infinite product, and say that this infinite product converges.
43. **Riemann-Zeta Function.** We define the Riemann-Zeta function to be $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$.
44. RESUME DEFINITIONS FOR NOVEMBER 18 LECTURE ONWARDS