

Math 205 - HW 4

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Problem: Let $\lambda = (z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_4} / \frac{z_2 - z_3}{z_2 - z_4}$ (the cross ratio) and let σ be a permutation on $\{1, 2, 3, 4\}$. Then, prove that the value of $\sigma(\lambda) := (z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)})$ is one of $\lambda, \frac{1}{\lambda}, 1 - \lambda, \frac{1}{1 - \lambda}, 1 - \frac{1}{\lambda}, \frac{\lambda}{\lambda - 1}$.

Proof: The main idea of this proof is to introduce the operation of swapping elements in a cross ratio and observe its connection to functions of λ , along with their compositions. The first part of this proof requires tedious algebra calculations, which we will omit here. First, notice that swapping z_1 and z_2 in the cross ratio for λ gives $(z_2, z_1, z_3, z_4) = \frac{1}{\lambda}$. Additionally, instead swapping z_3 and z_4 in λ gives $(z_1, z_2, z_4, z_3) = \frac{1}{\lambda}$. Since both of these swaps give the same function of λ , we declare them equivalent. We also consider the other types of swaps that can be made. $z_1 \leftrightarrow z_3$ or $z_2 \leftrightarrow z_4$ give $\lambda \mapsto \frac{\lambda}{\lambda - 1}$ and lastly, $z_1 \leftrightarrow z_4$ or $z_2 \leftrightarrow z_3$ give $\lambda \mapsto 1 - \lambda$. It follows trivially that any permutation of the elements in the cross ratio (z_1, z_2, z_3, z_4) can be formed by composing the swap operations previously listed. Additionally, if σ preserves the positions of all except two of the z_i 's then $\sigma(\lambda)$ is one of $\frac{1}{\lambda}, \frac{\lambda}{\lambda - 1}, 1 - \lambda$. Also, recognize that if a permutation consists of two nonequivalent swaps, then none of z_i 's are in the original placement in $\sigma(\lambda)$. By the definition of a swap (that acts on two elements), it follows that the number of elements that retain their position after $\sigma(\lambda)$ is either 4, 2, or 0. Furthermore, it can be verified (through calculation) that any permutation σ on $\{1, 2, 3, 4\}$ can be obtained by performing at most 3 swaps. If we perform 0 swaps, we have the identity permutation, namely, $(z_1, z_2, z_3, z_4) \xrightarrow{\sigma} (z_1, z_2, z_3, z_4) = \lambda$. START IN THE MIDDLE OF THE 10TH PAGE IN HARD COPY SOLUTION