HW 2.

Problem: Show $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges uniformly on compact subsets of $A := \{z \in \mathbb{C} \mid \mathrm{Re} z > 1\}$ (where $n^{-z} = e^{-z \log n}$).

Proof. Let $K \Subset A$. By Heine-Borel, since K is compact, then K is closed & bounded. Then, for all $z \in K \subsetneq A$, we have that when z is written as $z = a + bi$, thus $a > 1$. Since K is bounded, there exists an $R \in \mathbb{R}$ such that $|z_1 - z_2| < R$ for all $z_1, z_2 \in K$. Since $z \in K \subsetneq A$, we get that $|z| > 1$ for all $z \in K$. Now, we show uniform convergence. Show that for each $\varepsilon > 0$ there exists an $M \in \mathbb{R}_{>0}$ such that if $n \geq M$, then $|f_n(z) - f(z)| < \varepsilon$ for all $z \in K$. Note that $|f_n(z) - f(z)| = \left|\left(\sum_{n=1}^{M} n^{-z}\right) - \left(\sum_{n=1}^{\infty} n^{-z}\right)\right| = \left|\sum_{n=M+1}^{\infty} n^{-z}\right| \leq \sum_{n=M+1}^{\infty} |n^{-z}| \leq \sum_{n=M+1}^{\infty} |e^{-z \log n}|$. If we write $z = a + bi$, then $\sum_{n=M+1}^{\infty} |e^{-z \log n}| \leq \sum_{n=M+1}^{\infty} |e^{-(a+bi)\log n}| \leq \sum_{n=M+1}^{\infty} |e^{(-a \log n) + (-b \log n)i}|$ (note: $|e^{x+iy}| = |e^x \cdot e^{iy}| = |e^x||e^{iy}| = e^x$). Then, we get $\sum_{n=M+1}^{\infty} |e^{(-a \log n) + (-b \log n)i}| \leq \sum_{n=M+1}^{\infty} |e^{-a \log n}| \leq \sum_{n=M+1}^{\infty} \frac{1}{e^{a \log n}} \leq \sum_{n=M+1}^{\infty} \frac{1}{n^a} < \varepsilon$, since $\sum \frac{1}{n^a}$ is a convergent p-series (from calculus) with $p = a > 1$, since $z \in K \subsetneq A$, so thus we apply the Cauchy criterion to $\sum \frac{1}{n^a}$ to get $\sum_{n=M+1}^{\infty} \frac{1}{n^a} < \varepsilon$. Thus, we conclude that $\sum_{n=1}^{\infty} n^{-z}$ converges uniformly on compact subsets of $A = \{z \in \mathbb{C} \mid \mathrm{Re} > 1\}$.