

Math 205 Theorems.

1. **Schwarz Lemma.** Let $f : \{z \in \mathbb{C} : |z| < 1\} \rightarrow \mathbb{C}$ be holomorphic and $|f(z)| \leq 1$ for all z , and $f(0) = 0$. Then, $|f(z)| \leq |z|$ and $f'(0) \leq 1$. If for some $z_0 \neq 0$, $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$, then $f(z) = cz$ for some $c \in \mathbb{C}$ with $|c| = 1$.
2. **Theorem.** Let $K \subseteq \mathbb{C}$ compact (write: $K \Subset \mathbb{C}$), $f : K \rightarrow \mathbb{C}$ continuous, f holomorphic on K . Then, $\sup_{z \in K} |f(z)| = \sup_{z \in \partial K} |f(z)|$.
3. **Theorem.** Let $f : \Omega \rightarrow \mathbb{C}$ holomorphic (Ω open & connected), $z_0 \in \Omega$, $|f(z_0)| = \sup_{z \in \Omega} |f(z)|$. Then, f is constant.
4. **Theorem (Horwitz).** Let $\Omega \subseteq \mathbb{C}$ be open & connected, $f : \Omega \rightarrow \mathbb{C}$, $f_n : \Omega \rightarrow \mathbb{C}$, f_n holomorphic, $f_n(\Omega) \subset \mathbb{C} \setminus \{0\}$, $n \in \mathbb{N}$, $\|f_n - f\|_k \rightarrow 0$ for all $K \Subset \Omega$. Then, either $f = 0$ identically or $f(\Omega) \subset \mathbb{C} \setminus \{0\}$.
5. **Theorem.** Let $\Omega \subseteq \mathbb{C}$ be open, \mathcal{F} be a set of holomorphic function $\Omega \rightarrow \mathbb{C}$. Then, TFAE:
 - (a) For every $K \Subset \Omega$, $\sup_{f \in \mathcal{F}} \|f\|_K < \infty$.
 - (b) For every sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{F}$, there exists a subsequence $(f_{n_j})_{j \in \mathbb{N}}$, $n_1 < n_2 < \dots$, such that $(f_{n_j})_{j \in \mathbb{N}}$ is uniformly convergent on compact subsets of Ω .
6. **Lemma.** Let $K \Subset \Omega$, \mathcal{F} family of holomorphic functions $\Omega \rightarrow \mathbb{C}$ so that for every $K \Subset \Omega$, $\sup_{f \in \mathcal{F}} \|f\|_K < \infty$. Given $\varepsilon > 0$, there is a $\delta > 0$ such that $z, z' \in K$ and $|z - z'| < \delta$ imply $|f(z) - f(z')| < \varepsilon$ for every $f \in \mathcal{F}$.
7. **Riemman Mapping Theorem.** Let $\Omega \subset \mathbb{C}$ be open, connected, simply connected, and $\emptyset \neq \Omega \neq \mathbb{C}$. Then, Ω and $\mathbb{D} = \{|z| < 1\}$ are holomorphic and isomorphic (i.e. there exists a holomorphic $f : \Omega \rightarrow \mathbb{D}$ with holomorphic inverse).
8. **Prop.** Let $g \in SL_2(\mathbb{C})$. Then, $T_g \in \text{Aut}(\mathbb{D})$ iff $g \in S \cap (1, 1)$.
9. **Prop.** $\text{Aut}\{\text{Im}z > 0\} = \{T_h \mid h \in SL_2(\mathbb{R})\}$.
10. **Theorem.** Let T_g be a fractional linear transformation and z_1, z_2, z_3, z_4 be distinct points in $\mathbb{C} \cup \{\infty\}$. Then, $(z_1, z_2, z_3, z_4) = (T_g z_1, T_g z_2, T_g z_3, T_g z_4)$.

11. **Lemma.** Let $g \in GL_2(\mathbb{C})$. Then, $\{w \in \mathbb{C} \cup \{\infty\} \mid T_{gw} \in \mathbb{R} \cup \{\infty\}\}$ is a circle on a (straight line) $\cup \{\infty\}$.
12. **Theorem.** Let Ω be an open, connected set so that there is $f : \Omega \rightarrow \mathbb{D}$ that is a holomorphic isomorphism. Then, $\text{iso}(\Omega, \mathbb{D}) \ni g \rightarrow \left(g(z_0), \frac{g'(z_0)}{|g'(z_0)|}\right) \in \mathbb{D} \times \{|z| = 1\}$ is a bijection.