Math 205 Theorems.

1. **Schwarz Lemma.** Let $f : \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C}$ be holomorphic and $|f(z)| \leq 1$ for all z, and $f(0) = 0$. Then, $|f(z)| \leq |z|$ and $f'(0) \leq 1$. If for some $z_0 \neq 0$, $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$, then $f(z) = cz$ for some $c \in \mathbb{C}$ with $|c| = 1$.

2. **Theorem.** Let $K \subseteq \mathbb{C}$ compact (write: $K \Subset \mathbb{C}$), $f : K \to \mathbb{C}$ continuous, f holomorphic on K. Then, $\sup_{z \in K} |f(z)| = \sup_{z \in \partial K} |f(z)|$.

3. **Theorem.** Let $f : \Omega \to \mathbb{C}$ holomorphic (Ω open & connected), $z_0 \in \Omega$, $|f(z_0)| = \sup_{z \in \Omega} |f(z)|$. Then, f is constant.

4. **Theorem (Horwitz).** Let $\Omega \subseteq \mathbb{C}$ be open & connected, $f : \Omega \to \mathbb{C}$, $f_n : \Omega \to \mathbb{C}$, f_n holomorphic, $f_n(\Omega) \subset \mathbb{C} \setminus \{0\}$, $n \in \mathbb{N}$, $||f_n - f||_k \to 0$ for all $K \Subset \Omega$. Then, either $f = 0$ identically or $f(\Omega) \subset \mathbb{C} \setminus \{0\}$.

5. **Theorem.** Let $\Omega \subseteq \mathbb{C}$ be open, $\mathscr{F}$ be a set of holomorphic function $\Omega \to \mathbb{C}$. Then, TFAE:

   (a) For every $K \Subset \Omega$, $\sup_{f \in \mathscr{F}} ||f||_K < \infty$.

   (b) For every sequence $(f_n)_{n \in \mathbb{N}} \subset \mathscr{F}$, there exists a subsequence $(f_{n_j})_{j \in \mathbb{N}}$, $n_1 < n_2 < \ldots$, such that $(f_{n_j})_{j \in \mathbb{N}}$ is uniformly convergent on compact subsets of Ω.

6. **Lemma.** Let $K \Subset \Omega$, $\mathscr{F}$ family of holomorphic functions $\Omega \to \mathbb{C}$ so that for every $K \Subset \Omega$, $\sup_{f \in \mathscr{F}} ||f||_K < \infty$. Given $\varepsilon > 0$, there is a $\delta > 0$ such that $z, z' \in K$ and $|z - z'| < \delta$ imply $|f(z) - f(z')| < \varepsilon$ for every $f \in \mathscr{F}$.

7. **Riemman Mapping Theorem.** Let $\Omega \subset \mathbb{C}$ be open, connected, simply connected, and $\emptyset \neq \Omega \neq \mathbb{C}$. Then, Ω and $\mathbb{D} = \{|z| < 1\}$ are holomorphic and isomorphic (i.e. there exists a holomorphic $f : \Omega \to \mathbb{D}$ with holomorphic inverse).

8. **Prop.** Let $g \in SL_2(\mathbb{C})$. Then, $T_g \in \operatorname{Aut}(\mathbb{D})$ iff $g \in S \cap (1,1)$.

9. **Prop.** $\operatorname{Aut}\{\operatorname{Im} z > 0\} = \{T_h \mid h \in SL_2(\mathbb{R}\}$.

10. **Theorem.** Let T_g be a fractional linear transformation and z_1, z_2, z_3, z_4 be distinct points in $\mathbb{C} \cup \{\infty\}$. Then, $(z_1, z_2, z_3, z_4) = (T_g z_1, T_g z_2, T_g z_3, T_g z_4)$.

11. **Lemma.** Let $g \in GL_2(\mathbb{C})$. Then, $\{w \in \mathbb{C} \cup \{\infty\} \mid T_{gw} \in \mathbb{R} \cup \{\infty\}\}$ is a circle on a (straight line) $\cup\{\infty\}$.

12. **Theorem.** Let Ω be an open, connected set so that there is $f : \Omega \to \mathbb{D}$ that is a holomorphic isomorphism. Then, $\text{iso}(\Omega, \mathbb{D}) \ni g \to \left(g(z_0), \frac{g'(z_0)}{|g'(z_0)|}\right) \in \mathbb{D} \times \{|z| = 1\}$ is a bijection.

13. **Definition of Jordan Curve.** A Jordan Curve is given by a map $[0,1] \ni t \to C(t) \in \mathbb{C}$ which is continuous, 1-1 on $[0,1)$ and $C(0) = C(1)$ (no self-intersection otherwise).

14. **Jordan Curve Theorem.** If $C : [0,1] \to \mathbb{C}$ is a Jordan curve, then $\mathbb{C} \setminus C([0,1])$ has 2 connected components. One of these is bounded and the other, unbounded. (The bounded component is called the "interior") We shall denote by $|C|$ the set $C([0,1])$ when $C : [0,1] \to \mathbb{C}$.

15. **Caratheodory's Theorem.** Let Γ be a Jordan curve and Ω be the interior region (then $\partial\Omega = |\Gamma|$). Then, if $f : \mathbb{D} \to \Omega$ is a holomorphic isomorphism, then f extends to a homeomorphism $\overline{\mathbb{D}} \to \overline{\Omega}$, where $\partial\mathbb{D}$ is mapped to $\partial\Omega = |\Gamma|$.

16. **Rectifiable def.** An arc $\phi : [a,b] \to \mathbb{C}$ (the map ϕ is 1-1 and continuous) is rectifiable if it has 'length' (bounded variation), that is, if:

$$\sup_{a=t_0<t_1<\cdots<t_k=b} \sum_{j=0}^{k-1} \left(|\phi(t_{j+1}) - \phi(t_j)|\right) < \infty$$

where $k \in \mathbb{N}$.

17. **Theorem.** Let Ω, ω be disjoint open regions and Γ a rectifiable arc so that $|\Gamma| \subset \partial\Omega \cap \partial\omega$ and $|\Gamma| \cup \Omega \cup \omega$ open (Γ has no endpoints). Assume also $f : |\Gamma| \cup \Omega \to \mathbb{C}$, $g : |\Gamma| \cup \omega \to \mathbb{C}$ are continuous and $f|_\Omega$, $g|_\Omega$ holomorphic and $f|_{|\Gamma|} = g|_{|\Gamma|}$. Then, $F : \Omega \cup |\Gamma| \cup \omega \to \mathbb{C}$ is holomorphic.

18. **Theorem (Schwarz Reflection Principle).** Let $\Omega = \Omega^\star$ $(= \{\bar{z} \mid z \in \Omega\})$ open region, $\Omega \cap \mathbb{R} \supset (a,b)$ and $\Omega_\pm = \Omega \cap \{\pm \text{Im} z > 0\}$. If $f : \Omega_+ \cup (a,b) \to \mathbb{C}$ continuous, $f|_{(a,b)} \subset \mathbb{R}$, $f|_{\Omega_+}$ holomorphic, then, $F(z)$, with $F(z) = f(z)$ if $z \in \Omega_+ \cup (a,b)$ and $F(z) = \overline{f(\bar{z})}$ if $z \in \Omega_-$ is holomorphic in $\Omega_+ \cup (a,b) \cup \Omega_-$.