Schwarz Lemma. Let $f : \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C}$ be holomorphic and $|f(z)| \leq 1$ for all z, $f(0) = 0$. Then, $|f(z)| \leq |z|$, $|f'(0)| \leq 1$. If for some $z_0 \neq 0$, $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$, then $f(z) = cz$ for some $c \in \mathbb{C}$ with $|c| = 1$.

Proof. Consider the function $g(z)$ defined to be $f(z)/z$ when $z \neq 0$ and $f'(0)$ when $z = 0$ with $g : \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C}$. Then, since z is holomorphic and when $z \neq 0$, it follows that g is holomorphic. By the removable singularities theorem (a corollary to Morrera's theorem), we have that since g is holomorphic on $D_1(0)$ and not on the single point $0 \in D_1(0)$, thus, we get that g is holomorphic on all of $D_1(0)$. Since g is holomorphic (therefore continuous) on $D_1(0)$, then it is continuous on the compact set $\overline{D_r(0)}$ where $0 < r < 1$. Then, g is uniformly continuous and bounded on $\overline{D_r(0)}$. Then, by the maximum modulus principle, the supremum of g is attained on the boundary, so $\sup_{|z| \leq r} |g(z)| = \sup_{|z|=r} |g(z)| = \sup_{|z|=r} \frac{|f(z)|}{|z|} \leq \frac{1}{r}$. Thus, it follows that $|f(z)| \leq r|z|$ for all $z \in \overline{D_r(0)}$ and since $r < 1$ we have that $r|z| < |z|$, and so, $|f(z)| < |z|$ and so $|f(z)| \leq |z|$. By the definition of g, we also get that $|f'(0)| \leq 1$. Now, we prove the second sentence of the theorem. By assumption, $|g(z_0)| = 1$ for some nonzero $z_0 \in D_1(0)$. By the chain of inequalities 4 lines above, it follows that that supremum of $|g|$ is attained inside $D_1(0)$, since $0 < |z_0| < 1$, so by the second sentence of the maximum modulus principle, g is constant, so put $g = c$ on $D_1(0)$ with $|c| = 1$, by the chain of inequalities mentioned in the beginning of this sentence. Thus, on $D_1(0)$, $\frac{f(z)}{z} = c$, so $f(z) = cz$ with $|c| = 1$, as mentioned previously. $\square$