Math 205 Theorems.

1. **Schwarz Lemma.** Let $f : \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C}$ be holomorphic and $|f(z)| \leq 1$ for all z, and $f(0) = 0$. Then, $|f(z)| \leq |z|$ and $f'(0) \leq 1$. If for some $z_0 \neq 0$, $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$, then $f(z) = cz$ for some $c \in \mathbb{C}$ with $|c| = 1$.

2. **Theorem.** Let $K \subseteq \mathbb{C}$ compact (write: $K \Subset \mathbb{C}$), $f : K \to \mathbb{C}$ continuous, f holomorphic on K. Then, $\sup_{z\in K} |f(z)| = \sup_{z\in\partial K} |f(z)|$.

3. **Theorem.** Let $f : \Omega \to \mathbb{C}$ holomorphic (Ω open & connected), $z_0 \in \Omega$, $|f(z_0)| = \sup_{z\in\Omega} |f(z)|$. Then, f is constant.

4. **Theorem (Horwitz).** Let $\Omega \subseteq \mathbb{C}$ be open & connected, $f : \Omega \to \mathbb{C}$, $f_n : \Omega \to \mathbb{C}$, f_n holomorphic, $f_n(\Omega) \subset \mathbb{C} \setminus \{0\}$, $n \in \mathbb{N}$, $||f_n - f||_k \to 0$ for all $K \Subset \Omega$. Then, either $f = 0$ identically or $f(\Omega) \subset \mathbb{C} \setminus \{0\}$.

5. **Theorem.** Let $\Omega \subseteq \mathbb{C}$ be open, $\mathscr{F}$ be a set of holomorphic function $\Omega \to \mathbb{C}$. Then, TFAE:

   (a) For every $K \Subset \Omega$, $\sup_{f\in\mathscr{F}} ||f||_K < \infty$.

   (b) For every sequence $(f_n)_{n\in\mathbb{N}} \subset \mathscr{F}$, there exists a subsequence $(f_{n_j})_{j\in\mathbb{N}}$, $n_1 < n_2 < \ldots$, such that $(f_{n_j})_{j\in\mathbb{N}}$ is uniformly convergent on compact subsets of Ω.

6. **Lemma.** Let $K \Subset \Omega$, $\mathscr{F}$ family of holomorphic functions $\Omega \to \mathbb{C}$ so that for every $K \Subset \Omega$, $\sup_{f\in\mathscr{F}} ||f||_K < \infty$. Given $\epsilon > 0$, there is a $\delta > 0$ such that $z, z' \in K$ and $|z - z'| < \delta$ imply $|f(z) - f(z')| < \epsilon$ for every $f \in \mathscr{F}$.

7. **Riemman Mapping Theorem.** Let $\Omega \subset \mathbb{C}$ be open, connected, simply connected, and $\emptyset \neq \Omega \neq \mathbb{C}$. Then, Ω and $\mathbb{D} = \{|z| < 1\}$ are holomorphic and isomorphic (i.e. there exists a holomorphic $f : \Omega \to \mathbb{D}$ with holomorphic inverse).

8. **Prop.** Let $g \in SL_2(\mathbb{C})$. Then, $T_g \in \mathrm{Aut}(\mathbb{D})$ iff $g \in S \cap (1, 1)$.

9. **Prop.** $\mathrm{Aut}\{\mathrm{Im} z > 0\} = \{T_h \mid h \in SL_2(\mathbb{R}\}$.

10. **Theorem.** Let T_g be a fractional linear transformation and z_1, z_2, z_3, z_4 be distinct points in $\mathbb{C} \cup \{\infty\}$. Then, $(z_1, z_2, z_3, z_4) = (T_g z_1, T_g z_2, T_g z_3, T_g z_4)$.

11. **Lemma.** Let $g \in GL_2(\mathbb{C})$. Then, $\{w \in \mathbb{C} \cup \{\infty\} \mid T_{gw} \in \mathbb{R} \cup \{\infty\}\}$ is a circle on a (straight line) $\cup\{\infty\}$.

12. **Theorem.** Let Ω be an open, connected set so that there is $f : \Omega \to \mathbb{D}$ that is a holomorphic isomorphism. Then, $\mathrm{iso}(\Omega, \mathbb{D}) \ni g \to \left(g(z_0), \frac{g'(z_0)}{|g'(z_0)|}\right) \in \mathbb{D} \times \{|z| = 1\}$ is a bijection.

13. **Definition of Jordan Curve.** A Jordan Curve is given by a map $[0, 1] \ni t \to C(t) \in \mathbb{C}$ which is continuous, 1-1 on $[0, 1)$ and $C(0) = C(1)$ (no self-intersection otherwise).

14. **Jordan Curve Theorem.** If $C : [0,1] \to \mathbb{C}$ is a Jordan curve, then $\mathbb{C} \setminus C([0,1])$ has 2 connected components. One of these is bounded and the other, unbounded. (The bounded component is called the "interior") We shall denote by $|C|$ the set $C([0,1])$ when $C : [0,1] \to \mathbb{C}$.

15. **Caratheodory's Theorem.** Let Γ be a Jordan curve and Ω be the interior region (then $\partial\Omega = |\Gamma|$). Then, if $f : \mathbb{D} \to \Omega$ is a holomorphic isomorphism, then f extends to a homeomorphism $\overline{\mathbb{D}} \to \overline{\Omega}$, where $\partial\mathbb{D}$ is mapped to $\partial\Omega = |\Gamma|$.

16. **Rectifiable def.** An arc $\phi : [a,b] \to \mathbb{C}$ (the map ϕ is 1-1 and continuous) is rectifiable if it has 'length' (bounded variation), that is, if:

$$\sup_{a=t_0<t_1<\cdots<t_k=b} \sum_{j=0}^{k-1} (|\phi(t_{j+1}) - \phi(t_j)|) < \infty$$

    where $k \in \mathbb{N}$.

17. **Theorem.** Let Ω, ω be disjoint open regions and Γ a rectifiable arc so that $|\Gamma| \subset \partial\Omega \cap \partial\omega$ and $|\Gamma| \cup \Omega \cup \omega$ open (Γ has no endpoints). Assume also $f : |\Gamma| \cup \Omega \to \mathbb{C}$, $g : |\Gamma| \cup \omega \to \mathbb{C}$ are continuous and $f\,|_{\Omega}$, $g\,|_{\Omega}$ holomorphic and $f\,|_{|\Gamma|} = g\,|_{|\Gamma|}$. Then, $F : \Omega \cup |\Gamma| \cup \omega \to \mathbb{C}$ is holomorphic.

18. **Theorem (Schwarz Reflection Principle).** Let $\Omega = \Omega^\star$ $(= \{\overline{z} \mid z \in \Omega\})$ open region, $\Omega \cap \mathbb{R} \supset (a,b)$ and $\Omega_{\pm} = \Omega \cap \{\pm\mathrm{Im} z > 0\}$. If $f : \Omega_+ \cup (a,b) \to \mathbb{C}$ continuous, $f\,|_{(a,b)} \subset \mathbb{R}$, $f\,|_{\Omega_+}$ holomorphic, then, $F(z)$, with $F(z) = f(z)$ if $z \in \Omega_+ \cup (a,b)$ and $F(z) = \overline{f(\overline{z})}$ if $z \in \Omega_-$ is holomorphic in $\Omega_+ \cup (a,b) \cup \Omega_-$.

19. **Analytic Arc def.** Analytic arc is $\phi : (a,b) \to \mathbb{C}$ so that there is $f : \omega \to \mathbb{C}$ univalent with $\omega \supset (a,b)$, $f\,|_{(a,b)} = \phi$, where we also require ϕ to be holomorphic within some neighborhood containing it.

20. **Theorem.** Let Ω be a region, γ an analytic arc, $|\gamma| \supset \partial\Omega$ from univalent $f : \omega \to \mathbb{C}$ and assume the following:

    (a) $f(\omega \cap \{\mathrm{Im} z > 0\}) \subset \Omega$.

    (b) $f(\omega \cap \{\mathrm{Im} z < 0\}) \cap \Omega = \emptyset$.

    (c) let $F : \Omega \cup |\gamma| \to \mathbb{C}$ continuous, and $F\,|_{\Omega}$ holomorphic with $F(|\gamma|) \subset |\Gamma|$, where Γ is an analytic arc.

    Then, there is an open Ω_1, with $\Omega_1 \supset \Omega \cup |\gamma|$ so that F has a holomorphic extension to Ω_1.

21. **Theorem (Schwarz-Christoffel Formula).** Let $F : \overline{\mathbb{D}} \to \overline{\Omega}$ be a homeomorphism (by Caratheodory) which extends the conformal map $F\,|_{\mathbb{D}} \to \Omega$ and $F(w_k) = z_k$. Let $\overline{\Omega}$ be a polygon with angles $\alpha_k\pi, \beta_k = 1 - \alpha_k$. Then,

$$F(w) = C \cdot \left(\int_0^w \left(\prod_{k=1}^{n} (w - w_k)^{-\beta_k} \right) dw \right) + C'.$$

22. **Theorem.** If γ is an analytic arc, then it is automatically rectifiable.

23. **Schwarz-Christoffel Formula for Upper-Half Plane.** If $G : \{\mathrm{Im} u > 0\} \to \Omega$ is a conformal map, where Ω is the interior of a polygon with outer angles $\beta_1\pi, \dots, \beta_k\pi$ and the point ∞ corresponds to z_n, then:

$$G(u) = C \cdot \left(\int_0^u \left(\prod_{k=1}^{n-1} (u - \xi_k)^{-\beta_k} \right) du \right) + C',$$

where $\xi_k \in \mathbb{R}$. The product has only $n-1$ factors. The external angle β_n does not appear explicitly. If $\beta_1 + \cdots + \beta_{n-1} = 2$, then $\beta_n = 0$.

24. **Schwarzian Derivative.** For a function f, the Schwarzian derivative of f is defined as:
$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

25. **A formula using Schwarzian derivative.** $S(f \circ g) = (S(f) \circ g)(g')^2 + S(g)$.

26. **Cor.** Now let $f(z) = \frac{az+b}{cz+d}$ be a fractional linear transformation. Then, $S(f) = 0$. Also, we get that $S(f \circ g) = S(g)$. Thus, we conclude that $S(g)$ is invariant under composition with a fractional linear transformation, under the Schwarzian derivative operator.

27. Γ **Free Group def.** This is defined to be $\Gamma := \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$, with the two listed matrices as its generators. Also, we have that Γ is a subgroup of $SL_2(\mathbb{Z})$.

28. **Prop.** Let y_1, y_2 be two linearly independent solutions to $y'' + py = 0$. Then, $u = \frac{y_1}{y_2}$ is so that $S(u) = 2p$, where S is the Schwarzian derivative operator.

29. **Modular Function def.** Consider the free group Γ as defined two items above. Now, consider Γ except now, $b \equiv c \equiv 0 \pmod 2$. Define a function $\lambda : S \to \mathbb{H}$, where λ takes $0, 1, \infty$ to $1, \infty, 0$, respectively (here, S refers to domain from class based on the conformal mapping operated on the sides of the non-Euclidean triangle; namely, $S = \{0 < \mathrm{Re} z < 1\} \setminus \{\frac{1}{2} + z \mid |z| < \frac{1}{2}\}$).

30. **Picard's Theorem.** Let $g : \mathbb{C} \to \mathbb{C}$ be entire. If there exists at least two points in $\mathbb{C} \setminus \mathrm{range}(g)$, tehn g is constant.

31. **Mittag-Leffler Theorem.** Given $b_n \in \mathbb{C}$, $n \in \mathbb{N}$, $\lim_{n\to\infty} |b_n| = \infty$ and principal parts $P_n = \sum_{k=-N_m}^{-1} c_k^{(n)} (z - b_n)^k$ with $c_{-N_m} \neq 0$. Then, there is a meromorphic function on $\mathbb{C}$ with poles $(b_n)_{n\in\mathbb{N}}$ and principal parts P_n of the Laurent expansions at the poles.

32. **Formula.** $\frac{\pi^2}{(\sin(\pi z))^2} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$.

33. **Formula.** $\lim_{N\to+\infty} \sum_{|n|\leq N} \frac{1}{z-n} = \pi \cdot \cot(\pi z)$.

34. **Infinite product convergence def.** $\prod_{k\geq 1} z_k$ converges iff $\lim_{k\to\infty}\prod_{i=1}^{k} z_i$ exists and is nonzero.

35. **Notation.** Let $\log z := \{a \in \mathbb{C} \mid e^a = z\}$. Let $\operatorname{Log} z := a + i(-\pi,\pi]$, with $a \in \mathbb{R}$. Similarly, let $\arg z := \operatorname{Im}\log z$ and let $\operatorname{Arg} z := \operatorname{Im}\operatorname{Log} z$.

36. **Theorem.** $\prod_{k\geq 1} z_k$ converges iff $\sum_{k\geq 1}\operatorname{Log} z_k$ converges.

37. **Theorem.** $\prod_{k\geq 1} z_k$ converges implies $z_k \to 1$.

38. **Infinite proudct absolute convergence def.** $\prod_{k\geq 1} z_k$ is absolutely convergent iff $\sum_{k\geq 1}|\operatorname{Log} z_k| < \infty$.

39. **Theorem.** $\prod_{k\geq 1} z_k$ is absolutely convergent iff $\sum_{k\geq 1}|z_k - 1| < \infty$.

40. **Weierstrass Theorem.** Given $a_n \in \mathbb{C}$, $|a_n| \to \infty$, $a_n \neq 0$, and $n \geq 0$ an integer, there exists an entire function with multiplicity of 0 at zero 0 and other zeros at a_n (multiplicities by repetition). Every function with these zeros is of the form
$$f(z) = z^m \cdot e^{g(z)} \cdot \prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right) \cdot e^{\frac{z}{a_n}+\cdots+\frac{z^{k_n}}{a_n^{k_n}}\cdot\frac{1}{k_n}}$$
for some $k_n \geq 0$, g entire, and the infinite product uniformly absolutely convergent on compact subsets of $\mathbb{C}$.

41. **Cor.** If f is meromorphic on $\mathbb{C}$, then there are f_1, f_2 entire functions so that $f = \frac{f_1}{f_2}$.

42. **Canonical product, genus def.** If we have that the sum in the exponential of e in the infinite product component of f (as in Weierstrass theorem) has last term that is raised to a fixed exponent h, we say f is the canonical product and say that f has genus h. Equivalently, we say if $f : \mathbb{C} \to \mathbb{C}$ entire, we say f has finite genus if $f = e^{g(z)} \cdot P(z)$, where $P(z)$ is a canonical product and g is a polynomial.

43. **Order of growth def.** Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. Then the order of growth of f is
$$\rho = \limsup_{R\to\infty}\frac{\log(\log||f||_{R\mathbb{D}})}{\log R} = \inf\{m \geq 0 \mid |f(z)| \leq Ce^{c|z|^m}\}.$$

44. **Hadamard's Theorem.** If ρ and h are the order of growth and the genus of an entire function of finite genus respectively, then $hleq\rho \leq h+1$.

45. **Cor.** If ρ is fractional, the entire function takes every value infinitely many times.

46. γ**, Euler-Mascheroni Constant.** $\gamma = \lim_{N\to\infty}\left(-\log N + (1 + \cdots + \frac{1}{N})\right)$.

47. **Formula.** $\frac{\pi}{\sin \pi z} = \Gamma(z)\Gamma(1-z)$.

48. **Equivalent definition of** Γ**-function.** $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$.

49. **Stirling's Formula.** $\Gamma(z) = \sqrt{2\pi} \cdot z^{z-\frac{1}{2}} \cdot e^{-z} \cdot e^{\mathscr{J}(z)}$ ($\mathrm{Re} z > 0$), where $\mathscr{J}(z) = \frac{1}{\pi} \cdot \int_0^\infty \frac{z}{\eta^2+z^2} \cdot \log\left(\frac{1}{1-e^{-2\pi\eta}}\right) d\eta$.

50. **Fourier transform def.** The Fourier transform of f (on the real line) at $x \in \mathbb{R}$ is $\mathscr{F}f(x) = \int_{\mathbb{R}} f(t) \cdot e^{ixt} dt$.

51. **Mellin transform def.** This is a Fourier transform on $((0,\infty),\cdot)$ $\int_0^\infty \lambda^z \cdot f(\lambda) \cdot \frac{d\lambda}{\lambda} = \int_0^\infty \lambda^{z-1} \cdot f(\lambda) \cdot d\lambda$. To get $I(z)$, one takes $f(\lambda) = e^{-\lambda}$, for $\lambda \in (0,\infty)$.

52. **Lemma.** $\int_0^\infty t^z e^{-\lambda t} \frac{dt}{t} = \lambda^{-z}\Gamma(z)$, for $\lambda > 0$ and $\mathrm{Re} z > 0$.

53. **Formula.** Take $g(t) = \sum_n c_n e^{\lambda_n t}$ with $\lambda_n \to \infty$, with $\lambda_n > 0$. Then, the Mellin transform of $g(t)$ is $\int_0^\infty t^z (\sum_n c_n e^{-\lambda_n t}) \frac{dt}{t} = \Gamma(z) \cdot \sum_n c_n \lambda_n^{-z}$.

54. **Formula.** $\frac{1}{\Gamma(z)} = \lim_{n\to\infty} \frac{1}{n!} \left(1 - \frac{\log n}{n} z\right)^n \cdot \prod_{m=0}^n (z+m)$.

55. **Formula.** $\theta(t) = \frac{1}{\sqrt{t}} \cdot \theta(\frac{1}{t})$ if $t > 0$, where $\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}$.

56. **Formula.** The Poisson Summation Formula is roughly that for "good f", $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \mathscr{F}f(n)$.

57. **Prop.** If $f : \mathbb{R} \to \mathbb{C}$ so that f is continuous $\sum_{n\in\mathbb{Z}} ||f^{(k)}||_{[n,n+1]} < \infty$ for $k = 0, 1, 2, \ldots$, then the Poisson Summation Formula $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \mathscr{F}f(n)$ holds.

58. **Cor.** If $f : \mathbb{R} \to \mathbb{C}$, C^2 and $|f|(\lambda) \le C(1+t^2)^{-1}$, $|f'|(t) \le C(1+t^2)^{-1}$, $|f''(t)| \le C(1+t^2)^{-2}$, then the Poisson Summation formula holds for f.

59. **Cor.** Let $\lambda > 0$. Then the Poisson Summation Formula holds for $f(t) = e^{-\lambda t^2}$.

60. **Lemma.** $\int_{-\infty}^\infty e^{-\pi t^2} dt = 1$, which is the Fourier transform of $e^{-\lambda t^2}$.

61. **Prop.** If $\mathrm{Re} z > \frac{1}{2}$, then $\xi(2z)\Gamma(z) \cdot \pi^{-z} = \int_1^\infty \left(t^{z-1} + t^{-z-\frac{1}{2}}\right) \cdot \psi(t) dt + \frac{1}{2z(2z-1)}$.

62. **Prop.** $\pi^{-\frac{w}{2}} \cdot \xi(w) \cdot \Gamma(\frac{w}{w}) = \pi^{-\frac{1-w}{2}} \cdot \xi(1-w) \cdot \Gamma(\frac{1-w}{2})$.

63. **Euler Product Formula.** $\zeta(z) = \prod_{n\ge1} (1 - p_n^{-z})^{-1}$, where $p_1, p_2, \ldots$ is the sequence of primes.

64. **Euler-Beta Function.** This is the function $\beta(z,w) = \int_0^1 (1-t)^z \cdot t^{w-1} dt$.

65. **Characteristic Jacobi Θ-Function def.** $\Theta_{a,b}(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i(n+a)^2\tau + 2\pi i(n+a)(z+b)}$.

66. **Special operators.** We define S_b and T_a to be linear operators on the space $\mathscr{F}(\mathbb{C} \to \mathbb{C})$ to be $(S_b f)(z) = f(z+b)$ and $(T_a f)(z) = f(z + a\tau) \cdot e^{\pi a^2 \tau + 2\pi i a z}$.

67. **Lemma.** $\Theta(z_0,\tau) = 0$ iff $z_0 \in (\mathbb{Z} + \frac{1}{2})\tau + \mathbb{Z}$ and are simple zeros.

68. **Cor.** $\Theta_{a,b}(z,\tau)$ has simple zeros which are located at $\frac{-2a+1}{2}\tau + \frac{-2b+1}{2} + (\mathbb{Z} + \tau\mathbb{Z})$.

69. **Formula.** $S_b T_a = e^{2\pi i a b} T_a S_b$.

70. **Heisenberg group.** This is the group $N = \{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid (x, y, z) \in \mathbb{R}^3 \}$, with matrix multiplication as the group operation. We also define the subgroup $\Gamma \subseteq N$ to be $\Gamma = \{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid (x, y, z) \in \mathbb{Z}^3 \}$. We write elements of the Heisenberg group as $[x, y, z]$, for brevity.

71. **Representation of Heisenberg Group.** Put $\rho([a, b, c]) : \mathscr{F}(\mathbb{C}) \to \mathscr{F}(\mathbb{C})$ by $\rho([a, b, c])(f) = e^{2\pi i c} T_b S_a f$.

72. **Lemma.** $S_1 \Theta(\cdot, \tau) = T_1 \Theta(\cdot, \tau) = \Theta(\cdot, \tau)$.

73. **Fact.** If $f \in \mathrm{Hol}(\mathbb{C})$, then $\rho(\Gamma) f = f$ iff $f \in \mathbb{C}\Theta(\cdot, \tau)$ (the set of multiples of the Jacobi Theta-function).

74. **Product formula for Jacobi Theta-function.**

$$\Theta(z, \tau) = \left(\prod_{n \geq 1} (1 - q^{2n}) \right) \cdot \left(\prod_{m \geq 1} (1 + q^{2m-1} p^2)(1 + q^{2m-1} p^{-2}) \right)$$

where $q = e^{\pi i \tau}$ and $p = e^{\pi i z}$.

75. **Jacobi Derivative Formula.**

$$\frac{\partial}{\partial z} \Theta_{1,1}(z, \tau) \mid_{z=0} = -\pi \cdot \Theta_{0,0}(0, \tau) \cdot Theta_{0,1}(0, \tau) \cdot \Theta_{1,0}(0, \tau).$$

76. **Lemma.**

$$\Theta(\frac{z}{\tau}, -\frac{1}{\tau}) = (-ie)^{1/2} \cdot e^{\pi i \cdot \frac{z^2}{\tau}} \cdot \Theta(z, \tau).$$

77. **Fact.** $\Theta(z, t)$ is the unique holomorphic function $f(z; \tau)$ on $\mathbb{C} \times H$ so that

   (a) $f(z + 1, \tau) = f(z, \tau)$.
   (b) $f(z + \tau, \tau) = \exp(-\pi i \tau - 2\pi i z) \cdot f(z, \tau)$.
   (c) $f(z + \frac{1}{z}, \tau + 1 = f(z, \tau)$.
   (d) $f(z/\tau, -1/\tau) = (-i\tau)^{1/2} \exp(\pi i \frac{z^2}{\tau}) \cdot f(z, \tau)$.
   (e) for all $z \in \mathbb{C}$, $\lim_{\mathrm{Im}\tau \to +\infty} f(z, \tau) = 1$.

78. **Fact.** If $a_1, \ldots, a_k, b_1, \ldots,_k \in \mathbb{C}$ are so that $\sum a_j = \sum b_j$ and $\{a_1, \ldots, a_k\} \cap \{b_1, \ldots, b_k\} = \emptyset$, then

$$\prod_{1 \leq j \leq k} \frac{\Theta(z - a_j, \tau)}{\Theta(z - b_j, \tau)}$$

(where τ is fixed) is a meromorphic function that is doubly periodic (with periods $1, \tau$) with poles at $b_j + \frac{1}{2}(1 + \tau)$ and zeros at $a_j + \frac{1}{2}(1 + \tau)$ mod $\mathbb{Z} + \tau\mathbb{Z}$ (the poles and zeros are simple).

79. r_k **generating series.** If $m \in \mathbb{Z}_{\geq 0}$, define $r_k(m) = \#\{(n_1, \ldots, n_k) \in \mathbb{Z}^k \mid n_1^2 + \ldots n_k^2 = m\}$.

80. **Formula.**

$$\Theta(0,\tau)^k = \left(\sum_{n\in\mathbb{Z}} q^{n^2}\right)^k = \left(\sum_{n_1\in\mathbb{Z}} q^{n_1^2}\right)\cdots\left(\sum_{n_k\in\mathbb{Z}} q^{n_k^2}\right).$$

81. **Theorem.** If $d_1(n)$ is the number of divisors of n which are congruent to $1 \pmod 4$ and $d_3(n)$ which are congruent to $3 \pmod 4$, then if $n \geq 1$, then $r_2(n) = 4(d_1(n) - d_3(n))$.

82. **Theorem.** $r_4(n) = 8\sum_{d|n} d$ if $n \equiv 1 \pmod 2$ and $r_4(n) = 24\sum_{d|n, d \text{ odd}} d$ if $n \equiv 0 \pmod 2$.