

# Math 205 Theorems.

1. **Schwarz Lemma.** Let  $f : \{z \in \mathbb{C} : |z| < 1\} \rightarrow \mathbb{C}$  be holomorphic and  $|f(z)| \leq 1$  for all  $z$ , and  $f(0) = 0$ . Then,  $|f(z)| \leq |z|$  and  $f'(0) \leq 1$ . If for some  $z_0 \neq 0$ ,  $|f(z_0)| = |z_0|$  or if  $|f'(0)| = 1$ , then  $f(z) = cz$  for some  $c \in \mathbb{C}$  with  $|c| = 1$ .
2. **Theorem.** Let  $K \subseteq \mathbb{C}$  compact (write:  $K \Subset \mathbb{C}$ ),  $f : K \rightarrow \mathbb{C}$  continuous,  $f$  holomorphic on  $K$ . Then,  $\sup_{z \in K} |f(z)| = \sup_{z \in \partial K} |f(z)|$ .
3. **Theorem.** Let  $f : \Omega \rightarrow \mathbb{C}$  holomorphic ( $\Omega$  open & connected),  $z_0 \in \Omega$ ,  $|f(z_0)| = \sup_{z \in \Omega} |f(z)|$ . Then,  $f$  is constant.
4. **Theorem (Horwitz).** Let  $\Omega \subseteq \mathbb{C}$  be open & connected,  $f : \Omega \rightarrow \mathbb{C}$ ,  $f_n : \Omega \rightarrow \mathbb{C}$ ,  $f_n$  holomorphic,  $f_n(\Omega) \subset \mathbb{C} \setminus \{0\}$ ,  $n \in \mathbb{N}$ ,  $\|f_n - f\|_K \rightarrow 0$  for all  $K \Subset \Omega$ . Then, either  $f = 0$  identically or  $f(\Omega) \subset \mathbb{C} \setminus \{0\}$ .
5. **Theorem.** Let  $\Omega \subseteq \mathbb{C}$  be open,  $\mathcal{F}$  be a set of holomorphic function  $\Omega \rightarrow \mathbb{C}$ . Then, TFAE:
  - (a) For every  $K \Subset \Omega$ ,  $\sup_{f \in \mathcal{F}} \|f\|_K < \infty$ .
  - (b) For every sequence  $(f_n)_{n \in \mathbb{N}} \subset \mathcal{F}$ , there exists a subsequence  $(f_{n_j})_{j \in \mathbb{N}}$ ,  $n_1 < n_2 < \dots$ , such that  $(f_{n_j})_{j \in \mathbb{N}}$  is uniformly convergent on compact subsets of  $\Omega$ .
6. **Lemma.** Let  $K \Subset \Omega$ ,  $\mathcal{F}$  family of holomorphic functions  $\Omega \rightarrow \mathbb{C}$  so that for every  $K \Subset \Omega$ ,  $\sup_{f \in \mathcal{F}} \|f\|_K < \infty$ . Given  $\varepsilon > 0$ , there is a  $\delta > 0$  such that  $z, z' \in K$  and  $|z - z'| < \delta$  imply  $|f(z) - f(z')| < \varepsilon$  for every  $f \in \mathcal{F}$ .
7. **Riemman Mapping Theorem.** Let  $\Omega \subset \mathbb{C}$  be open, connected, simply connected, and  $\emptyset \neq \Omega \neq \mathbb{C}$ . Then,  $\Omega$  and  $\mathbb{D} = \{|z| < 1\}$  are holomorphic and isomorphic (i.e. there exists a holomorphic  $f : \Omega \rightarrow \mathbb{D}$  with holomorphic inverse).