Math 205 Theorems.

1. **Schwarz Lemma.** Let $f : \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C}$ be holomorphic and $|f(z)| \leq 1$ for all $z$, and $f(0) = 0$. Then, $|f(z)| \leq |z|$ and $f'(0) \leq 1$. If for some $z_0 \neq 0$, $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$, then $f(z) = cz$ for some $c \in \mathbb{C}$ with $|c| = 1$.

2. **Theorem.** Let $K \subseteq \mathbb{C}$ compact (write: $K \Subset \mathbb{C}$), $f : K \to \mathbb{C}$ continuous, $f$ holomorphic on $K$. Then, $\sup_{z\in K} |f(z)| = \sup_{z\in\partial K} |f(z)|$.

3. **Theorem.** Let $f : \Omega \to \mathbb{C}$ holomorphic ($\Omega$ open & connected), $z_0 \in \Omega$, $|f(z_0)| = \sup_{z\in\Omega} |f(z)|$. Then, $f$ is constant.

4. **Theorem (Horwitz).** Let $\Omega \subseteq \mathbb{C}$ be open & connected, $f : \Omega \to \mathbb{C}$, $f_n : \Omega \to \mathbb{C}$, $f_n$ holomorphic, $f_n(\Omega) \subset \mathbb{C} \setminus \{0\}$, $n \in \mathbb{N}$, $||f_n - f||_k \to 0$ for all $K \Subset \Omega$. Then, either $f = 0$ identically or $f(\Omega) \subset \mathbb{C} \setminus \{0\}$.

5. **Theorem.** Let $\Omega \subseteq \mathbb{C}$ be open, $\mathscr{F}$ be a set of holomorphic function $\Omega \to \mathbb{C}$. Then, TFAE:

   (a) For every $K \Subset \Omega$, $\sup_{f\in\mathscr{F}} ||f||_K < \infty$.

   (b) For every sequence $(f_n)_{n\in\mathbb{N}} \subset \mathscr{F}$, there exists a subsequence $(f_{n_j})_{j\in\mathbb{N}}$, $n_1 < n_2 < \ldots$, such that $(f_{n_j})_{j\in\mathbb{N}}$ is uniformly convergent on compact subsets of $\Omega$.

6. **Lemma.** Let $K \Subset \Omega$, $\mathscr{F}$ family of holomorphic functions $\Omega \to \mathbb{C}$ so that for every $K \Subset \Omega$, $\sup_{f\in\mathscr{F}} ||f||_K < \infty$. Given $\varepsilon > 0$, there is a $\delta > 0$ such that $z, z' \in K$ and $|z - z'| < \delta$ imply $|f(z) - f(z')| < \varepsilon$ for every $f \in \mathscr{F}$.

7. **Riemman Mapping Theorem.** Let $\Omega \subset \mathbb{C}$ be open, connected, simply connected, and $\emptyset \neq \Omega \neq \mathbb{C}$. Then, $\Omega$ and $\mathbb{D} = \{|z| < 1\}$ are holomorphic and isomorphic (i.e. there exists a holomorphic $f : \Omega \to \mathbb{D}$ with holomorphic inverse).

8. **Prop.** Let $g \in SL_2(\mathbb{C})$. Then, $T_g \in \mathrm{Aut}(\mathbb{D})$ iff $g \in S \cap (1,1)$.

9. **Prop.** $\mathrm{Aut}\{\mathrm{Im} z > 0\} = \{T_h \mid h \in SL_2(\mathbb{R}\}$.