

Math 205 Theorems.

1. **Schwarz Lemma.** Let $f : \{z \in \mathbb{C} : |z| < 1\} \rightarrow \mathbb{C}$ be holomorphic and $|f(z)| \leq 1$ for all z , and $f(0) = 0$. Then, $|f(z)| \leq |z|$ and $f'(0) \leq 1$. If for some $z_0 \neq 0$, $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$, then $f(z) = cz$ for some $c \in \mathbb{C}$ with $|c| = 1$.
2. **Theorem.** Let $K \subseteq \mathbb{C}$ compact (write: $K \Subset \mathbb{C}$), $f : K \rightarrow \mathbb{C}$ continuous, f holomorphic on K . Then, $\sup_{z \in K} |f(z)| = \sup_{z \in \partial K} |f(z)|$.
3. **Theorem.** Let $f : \Omega \rightarrow \mathbb{C}$ holomorphic (Ω open & connected), $z_0 \in \Omega$, $|f(z_0)| = \sup_{z \in \Omega} |f(z)|$. Then, f is constant.
4. **Theorem (Horwitz).** Let $\Omega \subseteq \mathbb{C}$ be open & connected, $f : \Omega \rightarrow \mathbb{C}$, $f_n : \Omega \rightarrow \mathbb{C}$, f_n holomorphic, $f_n(\Omega) \subset \mathbb{C} \setminus \{0\}$, $n \in \mathbb{N}$, $\|f_n - f\|_k \rightarrow 0$ for all $K \Subset \Omega$. Then, either $f = 0$ identically or $f(\Omega) \subset \mathbb{C} \setminus \{0\}$.
5. **Theorem.** Let $\Omega \subseteq \mathbb{C}$ be open, \mathcal{F} be a set of holomorphic function $\Omega \rightarrow \mathbb{C}$. Then, TFAE:
 - (a) For every $K \Subset \Omega$, $\sup_{f \in \mathcal{F}} \|f\|_K < \infty$.
 - (b) For every sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{F}$, there exists a subsequence $(f_{n_j})_{j \in \mathbb{N}}$, $n_1 < n_2 < \dots$, such that $(f_{n_j})_{j \in \mathbb{N}}$ is uniformly convergent on compact subsets of Ω .
6. **Lemma.** Let $K \Subset \Omega$, \mathcal{F} family of holomorphic functions $\Omega \rightarrow \mathbb{C}$ so that for every $K \Subset \Omega$, $\sup_{f \in \mathcal{F}} \|f\|_K < \infty$. Given $\varepsilon > 0$, there is a $\delta > 0$ such that $z, z' \in K$ and $|z - z'| < \delta$ imply $|f(z) - f(z')| < \varepsilon$ for every $f \in \mathcal{F}$.
7. **Riemman Mapping Theorem.** Let $\Omega \subset \mathbb{C}$ be open, connected, simply connected, and $\emptyset \neq \Omega \neq \mathbb{C}$. Then, Ω and $\mathbb{D} = \{|z| < 1\}$ are holomorphic and isomorphic (i.e. there exists a holomorphic $f : \Omega \rightarrow \mathbb{D}$ with holomorphic inverse).
8. **Prop.** Let $g \in SL_2(\mathbb{C})$. Then, $T_g \in \text{Aut}(\mathbb{D})$ iff $g \in S \cap (1, 1)$.
9. **Prop.** $\text{Aut}\{\text{Im}z > 0\} = \{T_h \mid h \in SL_2(\mathbb{R})\}$.
10. **Theorem.** Let T_g be a fractional linear transformation and z_1, z_2, z_3, z_4 be distinct points in $\mathbb{C} \cup \{\infty\}$. Then, $(z_1, z_2, z_3, z_4) = (T_g z_1, T_g z_2, T_g z_3, T_g z_4)$.

11. **Lemma.** Let $g \in GL_2(\mathbb{C})$. Then, $\{w \in \mathbb{C} \cup \{\infty\} \mid T_{gw} \in \mathbb{R} \cup \{\infty\}\}$ is a circle on a (straight line) $\cup \{\infty\}$.
12. **Theorem.** Let Ω be an open, connected set so that there is $f : \Omega \rightarrow \mathbb{D}$ that is a holomorphic isomorphism. Then, $\text{iso}(\Omega, \mathbb{D}) \ni g \rightarrow \left(g(z_0), \frac{g'(z_0)}{|g'(z_0)|}\right) \in \mathbb{D} \times \{|z| = 1\}$ is a bijection.
13. **Definition of Jordan Curve.** A Jordan Curve is given by a map $[0, 1] \ni t \rightarrow C(t) \in \mathbb{C}$ which is continuous, 1-1 on $[0, 1)$ and $C(0) = C(1)$ (no self-intersection otherwise).
14. **Jordan Curve Theorem.** If $C : [0, 1] \rightarrow \mathbb{C}$ is a Jordan curve, then $\mathbb{C} \setminus C([0, 1])$ has 2 connected components. One of these is bounded and the other, unbounded. (The bounded component is called the "interior") We shall denote by $|C|$ the set $C([0, 1])$ when $C : [0, 1] \rightarrow \mathbb{C}$.
15. **Caratheodory's Theorem.** Let Γ be a Jordan curve and Ω be the interior region (then $\partial\Omega = |\Gamma|$). Then, if $f : \mathbb{D} \rightarrow \Omega$ is a holomorphic isomorphism, then f extends to a homeomorphism $\overline{\mathbb{D}} \rightarrow \overline{\Omega}$, where $\partial\mathbb{D}$ is mapped to $\partial\Omega = |\Gamma|$.
16. **Rectifiable def.** An arc $\phi : [a, b] \rightarrow \mathbb{C}$ (the map ϕ is 1-1 and continuous) is rectifiable if it has 'length' (bounded variation), that is, if:

$$\sup_{a=t_0 < t_1 < \dots < t_k = b} \sum_{j=0}^{k-1} (|\phi(t_{j+1}) - \phi(t_j)|) < \infty$$

where $k \in \mathbb{N}$.

17. **Theorem.** Let Ω, ω be disjoint open regions and Γ a rectifiable arc so that $|\Gamma| \subset \partial\Omega \cap \partial\omega$ and $|\Gamma| \cup \Omega \cup \omega$ open (Γ has no endpoints). Assume also $f : |\Gamma| \cup \Omega \rightarrow \mathbb{C}$, $g : |\Gamma| \cup \omega \rightarrow \mathbb{C}$ are continuous and $f|_{\Omega}, g|_{\omega}$ holomorphic and $f|_{|\Gamma|} = g|_{|\Gamma|}$. Then, $F : \Omega \cup |\Gamma| \cup \omega \rightarrow \mathbb{C}$ is holomorphic.
18. **Theorem (Schwarz Reflection Principle).** Let $\Omega = \Omega^* (= \{\bar{z} \mid z \in \Omega\})$ open region, $\Omega \cap \mathbb{R} \supset (a, b)$ and $\Omega_{\pm} = \Omega \cap \{\pm \text{Im} z > 0\}$. If $f : \Omega_+ \cup (a, b) \rightarrow \mathbb{C}$ continuous, $f|_{(a, b)} \subset \mathbb{R}$, $f|_{\Omega_+}$ holomorphic, then, $F(z)$, with $F(z) = f(z)$ if $z \in \Omega_+ \cup (a, b)$ and $F(z) = \overline{f(\bar{z})}$ if $z \in \Omega_-$ is holomorphic in $\Omega_+ \cup (a, b) \cup \Omega_-$.

19. **Analytic Arc def.** Analytic arc is $\phi : (a, b) \rightarrow \mathbb{C}$ so that there is $f : \omega \rightarrow \mathbb{C}$ univalent with $\omega \supset (a, b)$, $f|_{(a,b)} = \phi$, where we also require ϕ to be holomorphic within some neighborhood containing it.

20. **Theorem.** Let Ω be a region, γ an analytic arc, $|\gamma| \supset \partial\Omega$ from univalent $f : \omega \rightarrow \mathbb{C}$ and assume the following:

- (a) $f(\omega \cap \{\text{Im} z > 0\}) \subset \Omega$.
- (b) $f(\omega \cap \{\text{Im} z < 0\}) \cap \Omega = \emptyset$.
- (c) let $F : \Omega \cup |\gamma| \rightarrow \mathbb{C}$ continuous, and $F|_{\Omega}$ holomorphic with $F(|\gamma|) \subset |\Gamma|$, where Γ is an analytic arc.

Then, there is an open Ω , with $\Omega \supset \Omega \cup |\gamma|$ so that F has a holomorphic extension to Ω .

21. **Theorem (Schwarz-Christoffel Formula).** Let $F : \overline{\mathbb{D}} \rightarrow \overline{\Omega}$ be a homeomorphism (by Caratheodory) which extends the conformal map $F|_{\mathbb{D}} \rightarrow \Omega$ and $F(w_k) = z_k$. Let $\overline{\Omega}$ be a polygon with angles $\alpha_k\pi, \beta_k = 1 - \alpha_k$. Then,

$$F(w) = C \cdot \left(\int_0^w \left(\prod_{k=1}^n (w - w_k)^{-\beta_k} \right) dw \right) + C'.$$

22. **Theorem.** If γ is an analytic arc, then it is automatically rectifiable.

23. **Schwarz-Christoffel Formula for Upper-Half Plane.** If $G : \{\text{Im} u > 0\} \rightarrow \Omega$ is a conformal map, where Ω is the interior of a polygon with outer angles $\beta_1\pi, \dots, \beta_n\pi$ and the point ∞ corresponds to z_n , then:

$$G(u) = C \cdot \left(\int_0^u \left(\prod_{k=1}^{n-1} (u - \xi_k)^{-\beta_k} \right) du \right) + C',$$

where $\xi_k \in \mathbb{R}$. The product has only $n - 1$ factors. The external angle β_n does not appear explicitly. If $\beta_1 + \dots + \beta_{n-1} = 2$, then $\beta_n = 0$.