Math 205 - HW 4

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Problem: Let $\lambda = (z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_4} / \frac{z_2 - z_3}{z_2 - z_4}$ (the cross ratio) and let σ be a permutation on $\{1, 2, 3, 4\}$. Then, prove that the value of $\sigma(\lambda) := (z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)})$ is one of $\lambda, \frac{1}{\lambda}, 1 - \lambda, \frac{1}{1-\lambda}, 1 - \frac{1}{\lambda}, \frac{\lambda}{\lambda - 1}$.

Proof: The main idea of this proof is to introduce the operation of swapping elements in a cross ratio and observe its connection to functions of λ, along with their compositions. For brevity, the algebra calculations have been omitted. First, note that swapping z_1 and z_2 in the cross ratio for λ gives $(z_2, z_1, z_3, z_4) = \frac{1}{\lambda}$. Swapping z_3 and z_4 instead gives $(z_1, z_2, z_4, z_3) = \frac{1}{\lambda}$. Since both of these swaps give the same function of λ, we declare them equivalent; we thus have that $z_1 \leftrightarrow z_2$ or $z_3 \leftrightarrow z_4$ gives $\lambda \overset{S_1}{\longmapsto} \frac{1}{\lambda}$. We also consider the other types of swaps that can be made: $z_1 \leftrightarrow z_3$ or $z_2 \leftrightarrow z_4$ gives $\lambda \overset{S_2}{\longmapsto} \frac{\lambda}{\lambda-1}$ and lastly, $z_1 \leftrightarrow z_4$ or $z_2 \leftrightarrow z_3$ gives $\lambda \overset{S_3}{\longmapsto} 1 - \lambda$. It follows trivially that any permutation of the elements in the cross ratio (z_1, z_2, z_3, z_4) can be formed by composing the swap operations previously listed. It can also be verified that if σ preserves the positions of all except two of the z_i's then $\sigma(\lambda)$ is one of $\frac{1}{\lambda}, \frac{\lambda}{\lambda-1}, 1 - \lambda$. Also, recognize that if a permutation consists of two nonequivalent swaps, then none of z_i's are in the original placement in $\sigma(\lambda)$; this can be proved by exhaustion by considering the possibilities of nonequivalent swap pairs and their composition. By the definition of a swap (that acts on two elements), it follows that the number of elements that retain their position after $\sigma(\lambda)$ is even - in this case it is either 4, 2, or 0. Furthermore, it can be verified (by exhaustion) that any permutation σ on $\{1, 2, 3, 4\}$ can be obtained by performing at most 3 swaps. If we perform 0 swaps, we have the identity permutation thus giving the identity map, $(z_1, z_2, z_3, z_4) \overset{\sigma}{\mapsto} (z_1, z_2, z_3, z_4) = \lambda$, so $\lambda \mapsto \lambda$. If we perform 1 swap, then, as mentioned previously, $\sigma(\lambda)$ is one of $\lambda, 1 - \lambda, \frac{\lambda}{\lambda-1}$. If we perform 2 nonequivalent swaps, then $\sigma(\lambda)$ preserves the position of none of the z_i's. It then follows that by the choice of the maps S_1, S_2, S_3 (each given by 2 equivalent swaps), we construct the bijection $A := \{S_1, S_2, S_3\} \longleftrightarrow B := \{\{z_1 \leftrightarrow z_2, z_3 \leftrightarrow z_4\}, \{z_1 \leftrightarrow z_3, z_2 \leftrightarrow z_4\}, \{z_1 \leftrightarrow z_4, z_2 \leftrightarrow z_3\}\}$, with $a_i \longleftrightarrow b_i$ for $i \in \{1, 2, 3\}, a_i \in A, b_i \in B$. Thus, considering the swaps in any b_i as equivalent, we have that any map S_i acting on λ corresponds to a particular swap, and vice-versa. Thus, composing distinct (respectively, indistinct) S_i maps is equivalent to composing nonequivalent (respectively, equivalent) swaps. Now consider the possible compositions of two nonequivalent S_i maps. Permuting the indices in $S_i \circ S_j$ (letting $i, j \in \{1, 2, 3\}$ with $i \neq j$) gives the maps $\lambda \mapsto \frac{1}{1-\lambda}, \lambda \mapsto 1 - \frac{1}{\lambda}$. However, the final case to consider is when σ can only be formed from performing 3 swaps, with nonconsecutive repetition allowed. Then, all possible permutations σ are considered and hence, it can be verified that each of these 3-map compositions gives a value in the list $\lambda, \frac{1}{\lambda}, 1 - \lambda, \frac{1}{1-\lambda}, 1 - \frac{1}{\lambda}, \frac{\lambda}{\lambda-1}$. **Q.E.D.**