

1. **Schwarz Lemma.** Let  $f : \{z \in \mathbb{C} : |z| < 1\} \rightarrow \mathbb{C}$  be holomorphic and  $|f(z)| \leq 1$  for all  $z$ , and  $f(0) = 0$ . Then,  $|f(z)| \leq |z|$  and  $f'(0) \leq 1$ . If for some  $z_0 \neq 0$ ,  $|f(z_0)| = |z_0|$  or if  $|f'(0)| = 1$ , then  $f(z) = cz$  for some  $c \in \mathbb{C}$  with  $|c| = 1$ .
2. **Theorem.** Let  $K \subseteq \mathbb{C}$  compact (write:  $K \Subset \mathbb{C}$ ),  $f : K \rightarrow \mathbb{C}$  continuous,  $f$  holomorphic on  $K$ . Then,  $\sup_{z \in K} |f(z)| = \sup_{z \in \partial K} |f(z)|$ .
3. **Theorem.** Let  $f : \Omega \rightarrow \mathbb{C}$  holomorphic ( $\Omega$  open & connected),  $z_0 \in \Omega$ ,  $|f(z_0)| = \sup_{z \in \Omega} |f(z)|$ . Then,  $f$  is constant.
4. **Theorem (Horwitz).** Let  $\Omega \subseteq \mathbb{C}$  be open & connected,  $f : \Omega \rightarrow \mathbb{C}$ ,  $f_n : \Omega \rightarrow \mathbb{C}$ ,  $f_n$  holomorphic,  $f_n(\Omega) \subset \mathbb{C} \setminus \{0\}$ ,  $n \in \mathbb{N}$ ,  $\|f_n - f\|_K \rightarrow 0$  for all  $K \Subset \Omega$ . Then, either  $f = 0$  identically or  $f(\Omega) \subset \mathbb{C} \setminus \{0\}$ .
5. **Theorem.** Let  $\Omega \subseteq \mathbb{C}$  be open,  $\mathcal{F}$  be a set of holomorphic function  $\Omega \rightarrow \mathbb{C}$ . Then, TFAE:
  - (a) For every  $K \Subset \Omega$ ,  $\sup_{f \in \mathcal{F}} \|f\|_K < \infty$ .
  - (b) For every sequence  $(f_n)_{n \in \mathbb{N}} \subset \mathcal{F}$ , there exists a subsequence  $(f_{n_j})_{j \in \mathbb{N}}$ ,  $n_1 < n_2 < \dots$ , such that  $(f_{n_j})_{j \in \mathbb{N}}$  is uniformly convergent on compact subsets of  $\Omega$ .
6. **Lemma.** Let  $K \Subset \Omega$ ,  $\mathcal{F}$  family of holomorphic functions  $\Omega \rightarrow \mathbb{C}$  so that for every  $K \Subset \Omega$ ,  $\sup_{f \in \mathcal{F}} \|f\|_K < \infty$ . Given  $\epsilon > 0$ , there is a  $\delta > 0$  such that  $z, z' \in K$  and  $|z - z'| < \delta$  imply  $|f(z) - f(z')| < \epsilon$  for every  $f \in \mathcal{F}$ .
7. **Riemman Mapping Theorem.** Let  $\Omega \subset \mathbb{C}$  be open, connected, simply connected, and  $\emptyset \neq \Omega \neq \mathbb{C}$ . Then,  $\Omega$  and  $\mathbb{D} = \{|z| < 1\}$  are holomorphic and isomorphic (i.e. there exists a holomorphic  $f : \Omega \rightarrow \mathbb{D}$  with holomorphic inverse).
8. **Prop.** Let  $g \in SL_2(\mathbb{C})$ . Then,  $T_g \in \text{Aut}(\mathbb{D})$  iff  $g \in S \cap (1, 1)$ .
9. **Prop.**  $\text{Aut}\{\text{Im}z > 0\} = \{T_h \mid h \in SL_2(\mathbb{R})\}$ .
10. **Theorem.** Let  $T_g$  be a fractional linear transformation and  $z_1, z_2, z_3, z_4$  be distinct points in  $\mathbb{C} \cup \{\infty\}$ . Then,  $(z_1, z_2, z_3, z_4) = (T_g z_1, T_g z_2, T_g z_3, T_g z_4)$ .
11. **Lemma.** Let  $g \in GL_2(\mathbb{C})$ . Then,  $\{w \in \mathbb{C} \cup \{\infty\} \mid T_{gw} \in \mathbb{R} \cup \{\infty\}\}$  is a circle on a (straight line)  $\cup \{\infty\}$ .
12. **Theorem.** Let  $\Omega$  be an open, connected set so that there is  $f : \Omega \rightarrow \mathbb{D}$  that is a holomorphic isomorphism. Then,  $\text{iso}(\Omega, \mathbb{D}) \ni g \rightarrow \left(g(z_0), \frac{g'(z_0)}{|g'(z_0)|}\right) \in \mathbb{D} \times \{|z| = 1\}$  is a bijection.
13. **Definition of Jordan Curve.** A Jordan Curve is given by a map  $[0, 1] \ni t \rightarrow C(t) \in \mathbb{C}$  which is continuous, 1-1 on  $[0, 1)$  and  $C(0) = C(1)$  (no self-intersection otherwise).

14. **Jordan Curve Theorem.** If  $C : [0, 1] \rightarrow \mathbb{C}$  is a Jordan curve, then  $\mathbb{C} \setminus C([0, 1])$  has 2 connected components. One of these is bounded and the other, unbounded. (The bounded component is called the "interior") We shall denote by  $|C|$  the set  $C([0, 1])$  when  $C : [0, 1] \rightarrow \mathbb{C}$ .
15. **Caratheodory's Theorem.** Let  $\Gamma$  be a Jordan curve and  $\Omega$  be the interior region (then  $\partial\Omega = |\Gamma|$ ). Then, if  $f : \mathbb{D} \rightarrow \Omega$  is a holomorphic isomorphism, then  $f$  extends to a homeomorphism  $\overline{\mathbb{D}} \rightarrow \overline{\Omega}$ , where  $\partial\mathbb{D}$  is mapped to  $\partial\Omega = |\Gamma|$ .
16. **Rectifiable def.** An arc  $\phi : [a, b] \rightarrow \mathbb{C}$  (the map  $\phi$  is 1-1 and continuous) is rectifiable if it has 'length' (bounded variation), that is, if:

$$\sup_{a=t_0 < t_1 < \dots < t_k=b} \sum_{j=0}^{k-1} (|\phi(t_{j+1}) - \phi(t_j)|) < \infty$$

where  $k \in \mathbb{N}$ .

17. **Theorem.** Let  $\Omega, \omega$  be disjoint open regions and  $\Gamma$  a rectifiable arc so that  $|\Gamma| \subset \partial\Omega \cap \partial\omega$  and  $|\Gamma| \cup \Omega \cup \omega$  open ( $\Gamma$  has no endpoints). Assume also  $f : |\Gamma| \cup \Omega \rightarrow \mathbb{C}$ ,  $g : |\Gamma| \cup \omega \rightarrow \mathbb{C}$  are continuous and  $f|_{\Omega}$ ,  $g|_{\omega}$  holomorphic and  $f|_{|\Gamma|} = g|_{|\Gamma|}$ . Then,  $F : \Omega \cup |\Gamma| \cup \omega \rightarrow \mathbb{C}$  is holomorphic.
18. **Theorem (Schwarz Reflection Principle).** Let  $\Omega = \Omega^* (= \{\bar{z} \mid z \in \Omega\})$  open region,  $\Omega \cap \mathbb{R} \supset (a, b)$  and  $\Omega_{\pm} = \Omega \cap \{\pm \text{Im} z > 0\}$ . If  $f : \Omega_+ \cup (a, b) \rightarrow \mathbb{C}$  continuous,  $f|_{(a,b)} \subset \mathbb{R}$ ,  $f|_{\Omega_+}$  holomorphic, then,  $F(z)$ , with  $F(z) = f(z)$  if  $z \in \Omega_+ \cup (a, b)$  and  $F(z) = \overline{f(\bar{z})}$  if  $z \in \Omega_-$  is holomorphic in  $\Omega_+ \cup (a, b) \cup \Omega_-$ .
19. **Analytic Arc def.** Analytic arc is  $\phi : (a, b) \rightarrow \mathbb{C}$  so that there is  $f : \omega \rightarrow \mathbb{C}$  univalent with  $\omega \supset (a, b)$ ,  $f|_{(a,b)} = \phi$ , where we also require  $\phi$  to be holomorphic within some neighborhood containing it.
20. **Theorem.** Let  $\Omega$  be a region,  $\gamma$  an analytic arc,  $|\gamma| \supset \partial\Omega$  from univalent  $f : \omega \rightarrow \mathbb{C}$  and assume the following:
- (a)  $f(\omega \cap \{\text{Im} z > 0\}) \subset \Omega$ .
  - (b)  $f(\omega \cap \{\text{Im} z < 0\}) \cap \Omega = \emptyset$ .
  - (c) let  $F : \Omega \cup |\gamma| \rightarrow \mathbb{C}$  continuous, and  $F|_{\Omega}$  holomorphic with  $F(|\gamma|) \subset |\Gamma|$ , where  $\Gamma$  is an analytic arc.

Then, there is an open  $\Omega_1$ , with  $\Omega_1 \supset \Omega \cup |\gamma|$  so that  $F$  has a holomorphic extension to  $\Omega_1$ .

21. **Theorem (Schwarz-Christoffel Formula).** Let  $F : \overline{\mathbb{D}} \rightarrow \overline{\Omega}$  be a homeomorphism (by Caratheodory) which extends the conformal map  $F|_{\mathbb{D}} \rightarrow \Omega$  and  $F(w_k) = z_k$ . Let  $\overline{\Omega}$  be a polygon with angles  $\alpha_k\pi$ ,  $\beta_k = 1 - \alpha_k$ . Then,

$$F(w) = C \cdot \left( \int_0^w \left( \prod_{k=1}^n (w - w_k)^{-\beta_k} \right) dw \right) + C'.$$

22. **Theorem.** If  $\gamma$  is an analytic arc, then it is automatically rectifiable.
23. **Schwarz-Christoffel Formula for Upper-Half Plane.** If  $G : \{\text{Im} u > 0\} \rightarrow \Omega$  is a conformal map, where  $\Omega$  is the interior of a polygon with outer angles  $\beta_1\pi, \dots, \beta_k\pi$  and the point  $\infty$  corresponds to  $z_n$ , then:
- $$G(u) = C \cdot \left( \int_0^u \left( \prod_{k=1}^{n-1} (u - \xi_k)^{-\beta_k} \right) du \right) + C',$$
- where  $\xi_k \in \mathbb{R}$ . The product has only  $n - 1$  factors. The external angle  $\beta_n$  does not appear explicitly. If  $\beta_1 + \dots + \beta_{n-1} = 2$ , then  $\beta_n = 0$ .
24. **Schwarzian Derivative.** For a function  $f$ , the Schwarzian derivative of  $f$  is defined as:
- $$S(f) = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2 = \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2.$$
25. **A formula using Schwarzian derivative.**  $S(f \circ g) = (S(f) \circ g)(g')^2 + S(g)$ .
26. **Cor.** Now let  $f(z) = \frac{az+b}{cz+d}$  be a fractional linear transformation. Then,  $S(f) = 0$ . Also, we get that  $S(f \circ g) = S(g)$ . Thus, we conclude that  $S(g)$  is invariant under composition with a fractional linear transformation, under the Schwarzian derivative operator.
27.  **$\Gamma$  Free Group def.** This is defined to be  $\Gamma := \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$ , with the two listed matrices as its generators. Also, we have that  $\Gamma$  is a subgroup of  $SL_2(\mathbb{Z})$ .
28. **Prop.** Let  $y_1, y_2$  be two linearly independent solutions to  $y'' + py = 0$ . Then,  $u = \frac{y_1}{y_2}$  is so that  $S(u) = 2p$ , where  $S$  is the Schwarzian derivative operator.
29. **Modular Function def.** Consider the free group  $\Gamma$  as defined two items above. Now, consider  $\Gamma$  except now,  $b \equiv c \equiv 0 \pmod{2}$ . Define a function  $\lambda : S \rightarrow \mathbb{H}$ , where  $\lambda$  takes  $0, 1, \infty$  to  $1, \infty, 0$ , respectively (here,  $S$  refers to domain from class based on the conformal mapping operated on the sides of the non-Euclidean triangle; namely,  $S = \{0 < \text{Re} z < 1\} \setminus \{\frac{1}{2} + z \mid |z| < \frac{1}{2}\}$ ).
30. **Picard's Theorem.** Let  $g : \mathbb{C} \rightarrow \mathbb{C}$  be entire. If there exists at least two points in  $\mathbb{C} \setminus \text{range}(g)$ , then  $g$  is constant.
31. **Mittag-Leffler Theorem.** Given  $b_n \in \mathbb{C}$ ,  $n \in \mathbb{N}$ ,  $\lim_{n \rightarrow \infty} |b_n| = \infty$  and principal parts  $P_n = \sum_{k=-N_n}^{-1} c_k^{(n)} (z - b_n)^k$  with  $c_{-N_n} \neq 0$ . Then, there is a meromorphic function on  $\mathbb{C}$  with poles  $(b_n)_{n \in \mathbb{N}}$  and principal parts  $P_n$  of the Laurent expansions at the poles.
32. **Formula.**  $\frac{\pi^2}{(\sin(\pi z))^2} = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}$ .
33. **Formula.**  $\lim_{N \rightarrow +\infty} \sum_{|n| \leq N} \frac{1}{z-n} = \pi \cdot \cot(\pi z)$ .

34. **Infinite product convergence def.**  $\prod_{k \geq 1} z_k$  converges iff  $\lim_{k \rightarrow \infty} \prod_{i=1}^k z_i$  exists and is nonzero.

35. **Notation.** Let  $\log z := \{a \in \mathbb{C} \mid e^a = z\}$ . Let  $\text{Log } z := a + i(-\pi, \pi]$ , with  $a \in \mathbb{R}$ . Similarly, let  $\arg z := \text{Im } \log z$  and let  $\text{Arg } z := \text{Im } \text{Log } z$ .

36. **Theorem.**  $\prod_{k \geq 1} z_k$  converges iff  $\sum_{k \geq 1} \text{Log } z_k$  converges.

37. **Theorem.**  $\prod_{k \geq 1} z_k$  converges implies  $z_k \rightarrow 1$ .

38. **Infinite product absolute convergence def.**  $\prod_{k \geq 1} z_k$  is absolutely convergent iff  $\sum_{k \geq 1} |\text{Log } z_k| < \infty$ .

39. **Theorem.**  $\prod_{k \geq 1} z_k$  is absolutely convergent iff  $\sum_{k \geq 1} |z_k - 1| < \infty$ .

40. **Weierstrass Theorem.** Given  $a_n \in \mathbb{C}$ ,  $|a_n| \rightarrow \infty$ ,  $a_n \neq 0$ , and  $n \geq 0$  an integer, there exists an entire function with multiplicity of 0 at zero 0 and other zeros at  $a_n$  (multiplicities by repetition). Every function with these zeros is of the form

$$f(z) = z^m \cdot e^{g(z)} \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \cdot e^{\frac{z}{a_n} + \dots + \frac{z^{k_n}}{a_n^{k_n}} \cdot \frac{1}{k_n}}$$

for some  $k_n \geq 0$ ,  $g$  entire, and the infinite product uniformly absolutely convergent on compact subsets of  $\mathbb{C}$ .

41. **Cor.** If  $f$  is meromorphic on  $\mathbb{C}$ , then there are  $f_1, f_2$  entire functions so that  $f = \frac{f_1}{f_2}$ .

42. **Canonical product, genus def.** If we have that the sum in the exponential of  $e$  in the infinite product component of  $f$  (as in Weierstrass theorem) has last term that is raised to a fixed exponent  $h$ , we say  $f$  is the canonical product and say that  $f$  has genus  $h$ . Equivalently, we say if  $f : \mathbb{C} \rightarrow \mathbb{C}$  entire, we say  $f$  has finite genus if  $f = e^{g(z)} \cdot P(z)$ , where  $P(z)$  is a canonical product and  $g$  is a polynomial.

43. **Order of growth def.** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be holomorphic. Then the order of growth of  $f$  is

$$\rho = \limsup_{R \rightarrow \infty} \frac{\log(\log \|f\|_{R\mathbb{D}})}{\log R} = \inf\{m \geq 0 \mid |f(z)| \leq C e^{c|z|^m}\}.$$

44. **Hadamard's Theorem.** If  $\rho$  and  $h$  are the order of growth and the genus of an entire function of finite genus respectively, then  $h \leq \rho \leq h + 1$ .

45. **Cor.** If  $\rho$  is fractional, the entire function takes every value infinitely many times.

46.  **$\gamma$ , Euler-Mascheroni Constant.**  $\gamma = \lim_{N \rightarrow \infty} (-\log N + (1 + \dots + \frac{1}{N}))$ .

47. **Formula.**  $\frac{\pi}{\sin \pi z} = \Gamma(z)\Gamma(1-z)$ .

48. **Equivalent definition of  $\Gamma$ -function.**  $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$ .

49. **Stirling's Formula.**  $\Gamma(z) = \sqrt{2\pi} \cdot z^{z-\frac{1}{2}} \cdot e^{-z} \cdot e^{\mathcal{J}(z)}$  ( $\operatorname{Re} z > 0$ ), where  $\mathcal{J}(z) = \frac{1}{\pi} \cdot \int_0^\infty \frac{z}{\eta^2 + z^2} \cdot \log\left(\frac{1}{1-e^{-2\pi\eta}}\right) d\eta$ .
50. **Fourier transform def.** The Fourier transform of  $f$  (on the real line) at  $x \in \mathbb{R}$  is  $\mathcal{F}f(x) = \int_{\mathbb{R}} f(t) \cdot e^{ixt} dt$ .
51. **Mellin transform def.** This is a Fourier transform on  $((0, \infty), \cdot)$   $\int_0^\infty \lambda^z \cdot f(\lambda) \cdot \frac{d\lambda}{\lambda} = \int_0^\infty \lambda^{z-1} \cdot f(\lambda) \cdot d\lambda$ . To get  $I(z)$ , one takes  $f(\lambda) = e^{-\lambda}$ , for  $\lambda \in (0, \infty)$ .
52. **Lemma.**  $\int_0^\infty t^z e^{-\lambda t} \frac{dt}{t} = \lambda^{-z} \Gamma(z)$ , for  $\lambda > 0$  and  $\operatorname{Re} z > 0$ .
53. **Formula.** Take  $g(t) = \sum_n c_n e^{\lambda_n t}$  with  $\lambda_n \rightarrow \infty$ , with  $\lambda_n > 0$ . Then, the Mellin transform of  $g(t)$  is  $\int_0^\infty t^z (\sum_n c_n e^{-\lambda_n t}) \frac{dt}{t} = \Gamma(z) \cdot \sum_n c_n \lambda_n^{-z}$ .
54. **Formula.**  $\frac{1}{\Gamma(z)} = \lim_{n \rightarrow \infty} \frac{1}{n!} \left(1 - \frac{\log n}{n} z\right)^n \cdot \prod_{m=0}^n (z + m)$ .
55. **Formula.**  $\theta(t) = \frac{1}{\sqrt{t}} \cdot \theta\left(\frac{1}{t}\right)$  if  $t > 0$ , where  $\theta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t}$ .
56. **Formula.** The Poisson Summation Formula is roughly that for “good  $f$ ”,  $\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \mathcal{F}f(n)$ .
57. **Prop.** If  $f : \mathbb{R} \rightarrow \mathbb{C}$  so that  $f$  is continuous  $\sum_{n \in \mathbb{Z}} \|f^{(k)}\|_{[n, n+1]} < \infty$  for  $k = 0, 1, 2, \dots$ , then the Poisson Summation Formula  $\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \mathcal{F}f(n)$  holds.
58. **Cor.** If  $f : \mathbb{R} \rightarrow \mathbb{C}$ ,  $C^2$  and  $|f|(\lambda) \leq C(1+t^2)^{-1}$ ,  $|f'|(\lambda) \leq C(1+t^2)^{-1}$ ,  $|f''(\lambda)| \leq C(1+t^2)^{-2}$ , then the Poisson Summation formula holds for  $f$ .
59. **Cor.** Let  $\lambda > 0$ . Then the Poisson Summation Formula holds for  $f(t) = e^{-\lambda t^2}$ .
60. **Lemma.**  $\int_{-\infty}^\infty e^{-\pi t^2} dt = 1$ , which is the Fourier transform of  $e^{-\lambda t^2}$ .
61. **Prop.** If  $\operatorname{Re} z > \frac{1}{2}$ , then  $\xi(2z)\Gamma(z) \cdot \pi^{-z} = \int_1^\infty \left(t^{z-1} + t^{-z-\frac{1}{2}}\right) \cdot \psi(t) dt + \frac{1}{2z(2z-1)}$ .
62. **Prop.**  $\pi^{-\frac{w}{2}} \cdot \xi(w) \cdot \Gamma\left(\frac{w}{2}\right) = \pi^{-\frac{1-w}{2}} \cdot \xi(1-w) \cdot \Gamma\left(\frac{1-w}{2}\right)$ .
63. **Euler Product Formula.**  $\zeta(z) = \prod_{n \geq 1} (1 - p_n^{-z})^{-1}$ , where  $p_1, p_2, \dots$  is the sequence of primes.
64. **Euler-Beta Function.** This is the function  $\beta(z, w) = \int_0^1 (1-t)^z \cdot t^{w-1} dt$ .
65. **Characteristic Jacobi  $\Theta$ -Function def.**  $\Theta_{a,b}(z, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i(n+a)^2 \tau + 2\pi i(n+a)(z+b)}$ .
66. **Special operators.** We define  $S_b$  and  $T_a$  to be linear operators on the space  $\mathcal{F}(\mathbb{C} \rightarrow \mathbb{C})$  to be  $(S_b f)(z) = f(z+b)$  and  $(T_a f)(z) = f(z+a\tau) \cdot e^{\pi a^2 \tau + 2\pi i a z}$ .
67. **Lemma.**  $\Theta(z_0, \tau) = 0$  iff  $z_0 \in (\mathbb{Z} + \frac{1}{2})\tau + \mathbb{Z}$  and are simple zeros.
68. **Cor.**  $\Theta_{a,b}(z, \tau)$  has simple zeros which are located at  $\frac{-2a+1}{2}\tau + \frac{-2b+1}{2} + (\mathbb{Z} + \tau\mathbb{Z})$ .
69. **Formula.**  $S_b T_a = e^{2\pi i a b} T_a S_b$ .

70. **Heisenberg group.** This is the group  $N = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid (x, y, z) \in \mathbb{R}^3 \right\}$ , with matrix multiplication as the group operation. We also define the subgroup  $\Gamma \subseteq N$  to be  $\Gamma = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid (x, y, z) \in \mathbb{Z}^3 \right\}$ . We write elements of the Heisenberg group as  $[x, y, z]$ , for brevity.
71. **Representation of Heisenberg Group.** Put  $\rho([a, b, c]) : \mathcal{F}(\mathbb{C}) \rightarrow \mathcal{F}(\mathbb{C})$  by  $\rho([a, b, c])(f) = e^{2\pi i c} T_b S_a f$ .