Math 205 - HW 4

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Problem: Let $\lambda = (z_1, z_2, z_3, z_4) = \frac{z_1-z_3}{z_1-z_4}/\frac{z_2-z_3}{z_2-z_4}$ (the cross ratio) and let σ be a permutation on $\{1,2,3,4\}$. Then, prove that the value of $\sigma(\lambda) := (z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)})$ is one of $\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, 1-\frac{1}{\lambda}, \frac{\lambda}{\lambda-1}$.

Proof: The main idea of this proof is to introduce the operation of swapping elements in a cross ratio and observe its connection to functions of λ, along with their compositions. The first part of this proof requires tedious algebra calculations, which we will omit here. First, notice that swapping z_1 and z_2 in the cross ratio for λ gives $(z_2, z_1, z_3, z_4) = \frac{1}{\lambda}$. Additionally, instead swapping z_3 and z_4 in λ gives $(z_1, z_2, z_4, z_3) = \frac{1}{\lambda}$. Since both of these swaps give the same function of λ, we declare them equivalent; we thus have the denotion $z_1 \leftrightarrow z_2$ or $z_3 \leftrightarrow z_4$ gives $\lambda \overset{S_1}{\longmapsto} \frac{1}{\lambda}$. We also consider the other types of swaps that can be made (with the following denotions). $z_1 \leftrightarrow z_3$ or $z_2 \leftrightarrow z_4$ gives $\lambda \overset{S_2}{\longmapsto} \frac{\lambda}{\lambda-1}$ and lastly, $z_1 \leftrightarrow z_4$ or $z_2 \leftrightarrow z_3$ gives $\lambda \overset{S_3}{\longmapsto} 1-\lambda$. It follows trivially that any permutation of the elements in the cross ration (z_1, z_2, z_3, z_4) can be formed by composing the swap operations previously listed. Additionally, if σ preserves the positions of all except two of the z_i's then $\sigma(\lambda)$ is one of $\frac{1}{\lambda}, \frac{\lambda}{\lambda-1}, 1-\lambda$. Also, recognize that if a permutation consists of two nonequivalent swaps, then none of z_i's are in the original placement in $\sigma(\lambda)$. By the definition of a swap (that acts on two elements), it follows that the number of elements that retain their position after $\sigma(\lambda)$ is either 4, 2, or 0. Furthermore, it can be verified (through calculation) that any permutation σ on $\{1,2,3,4\}$ can be obtained by performing at most 3 swaps. If we perform 0 swaps, we have the identity permutation, namely, $(z_1, z_2, z_3, z_4) \overset{\sigma}{\mapsto} (z_1, z_2, z_3, z_4) = \lambda$, so $\lambda \mapsto \lambda$. If we perform 1 swap, then, as shown previously, $\sigma(\lambda)$ is one of $\lambda, 1-\lambda, \frac{\lambda}{\lambda-1}$. If we perform 2 nonequivalent swaps, then $\sigma(\lambda)$ preserves the position of none of the z_i's. It then follows that by the choice of the maps S_1, S_2, S_3 (each given by 2 equivalent swaps), we construct the bijection $A := \{S_1, S_2, S_3\} \leftrightarrow B := \{\{z_1 \leftrightarrow z_2, z_3 \leftrightarrow z_4\}, \{z_1 \leftrightarrow z_3, z_2 \leftrightarrow z_4\}, \{z_1 \leftrightarrow z_4, z_2 \leftrightarrow z_3\}\}$, with $a_i \leftrightarrow b_i$ for $i \in \{1,2,3\}$. Thus, thinking the swaps in any b_i as equivalent, we have that any map S_i acting on λ corresponds to a particular swap, and conversely. Thus, composing distinct S_i maps (respectively, indistinct) is equivalent to composing nonequivalent swaps (respectively, equivalent). Now consider the possible compositions of two nonequivalent S_i maps. Permuting the indices in $S_i \circ S_j$ (letting $i, j \in \{1,2,3\}$ with $i \neq j$) gives the maps $\lambda \mapsto \frac{1}{1-\lambda}, \lambda \mapsto 1-\frac{1}{\lambda}$. However, the final case to consider is when σ can only be formed from performing 3 swaps, with repetition allowed, but not consecutively. Then, all possible permutations σ are considered and hence, it can be verified that each of these 3-compositions gives a value in the list $\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, 1-\frac{1}{\lambda}, \frac{\lambda}{\lambda-1}$. $\square$.