Math 205 Theorems.

1. **Schwarz Lemma.** Let $f : \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C}$ be holomorphic and $|f(z)| \leq 1$ for all z, and $f(0) = 0$. Then, $|f(z)| \leq |z|$ and $f'(0) \leq 1$. If for some $z_0 \neq 0$, $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$, then $f(z) = cz$ for some $c \in \mathbb{C}$ with $|c| = 1$.

2. **Theorem.** Let $K \subseteq \mathbb{C}$ compact (write: $K \Subset \mathbb{C}$), $f : K \to \mathbb{C}$ continuous, f holomorphic on K. Then, $\sup_{z\in K} |f(z)| = \sup_{z\in\partial K} |f(z)|$.

3. **Theorem.** Let $f : \Omega \to \mathbb{C}$ holomorphic (Ω open & connected), $z_0 \in \Omega$, $|f(z_0)| = \sup_{z\in\Omega} |f(z)|$. Then, f is constant.

4. **Theorem (Horwitz).** Let $\Omega \subseteq \mathbb{C}$ be open & connected, $f : \Omega \to \mathbb{C}$, $f_n : \Omega \to \mathbb{C}$, f_n holomorphic, $f_n(\Omega) \subset \mathbb{C} \setminus \{0\}$, $n \in \mathbb{N}$, $||f_n - f||_k \to 0$ for all $K \Subset \Omega$. Then, either $f = 0$ identically or $f(\Omega) \subset \mathbb{C} \setminus \{0\}$.

5. **Theorem.** Let $\Omega \subseteq \mathbb{C}$ be open, $\mathscr{F}$ be a set of holomorphic function $\Omega \to \mathbb{C}$. Then, TFAE:

   (a) For every $K \Subset \Omega$, $\sup_{f\in\mathscr{F}} ||f||_K < \infty$.

   (b) For every sequence $(f_n)_{n\in\mathbb{N}} \subset \mathscr{F}$, there exists a subsequence $(f_{n_j})_{j\in\mathbb{N}}$, $n_1 < n_2 < \dots$, such that $(f_{n_j})_{j\in\mathbb{N}}$ is uniformly convergent on compact subsets of Ω.

6. **Lemma.** Let $K \Subset \Omega$, $\mathscr{F}$ family of holomorphic functions $\Omega \to \mathbb{C}$ so that for every $K \Subset \Omega$, $\sup_{f\in\mathscr{F}} ||f||_K < \infty$. Given $\epsilon > 0$, there is a $\delta > 0$ such that $z, z' \in K$ and $|z - z'| < \delta$ imply $|f(z) - f(z')| < \epsilon$ for every $f \in \mathscr{F}$.

7. **Riemman Mapping Theorem.** Let $\Omega \subset \mathbb{C}$ be open, connected, simply connected, and $\emptyset \neq \Omega \neq \mathbb{C}$. Then, Ω and $\mathbb{D} = \{|z| < 1\}$ are holomorphic and isomorphic (i.e. there exists a holomorphic $f : \Omega \to \mathbb{D}$ with holomorphic inverse).

8. **Prop.** Let $g \in SL_2(\mathbb{C})$. Then, $T_g \in \text{Aut}(\mathbb{D})$ iff $g \in S \cap (1, 1)$.

9. **Prop.** $\text{Aut}\{\text{Im} z > 0\} = \{T_h \mid h \in SL_2(\mathbb{R}\}$.

10. **Theorem.** Let T_g be a fractional linear transformation and z_1, z_2, z_3, z_4 be distinct points in $\mathbb{C} \cup \{\infty\}$. Then, $(z_1, z_2, z_3, z_4) = (T_g z_1, T_g z_2, T_g z_3, T_g z_4)$.

11. **Lemma.** Let $g \in GL_2(\mathbb{C})$. Then, $\{w \in \mathbb{C} \cup \{\infty\} \mid T_{gw} \in \mathbb{R} \cup \{\infty\}\}$ is a circle on a (straight line) $\cup\{\infty\}$.

12. **Theorem.** Let Ω be an open, connected set so that there is $f : \Omega \to \mathbb{D}$ that is a holomorphic isomorphism. Then, $\text{iso}(\Omega, \mathbb{D}) \ni g \to \left(g(z_0), \frac{g'(z_0)}{|g'(z_0)|}\right) \in \mathbb{D} \times \{|z| = 1\}$ is a bijection.

13. **Definition of Jordan Curve.** A Jordan Curve is given by a map $[0, 1] \ni t \to C(t) \in \mathbb{C}$ which is continuous, 1-1 on $[0, 1)$ and $C(0) = C(1)$ (no self-intersection otherwise).

14. **Jordan Curve Theorem.** If $C : [0,1] \to \mathbb{C}$ is a Jordan curve, then $\mathbb{C} \setminus C([0,1])$ has 2 connected components. One of these is bounded and the other, unbounded. (The bounded component is called the "interior") We shall denote by $|C|$ the set $C([0,1])$ when $C : [0,1] \to \mathbb{C}$.

15. **Caratheodory's Theorem.** Let Γ be a Jordan curve and Ω be the interior region (then $\partial\Omega = |\Gamma|$). Then, if $f : \mathbb{D} \to \Omega$ is a holomorphic isomorphism, then f extends to a homeomorphism $\overline{\mathbb{D}} \to \overline{\Omega}$, where $\partial\mathbb{D}$ is mapped to $\partial\Omega = |\Gamma|$.

16. **Rectifiable def.** An arc $\phi : [a,b] \to \mathbb{C}$ (the map ϕ is 1-1 and continuous) is rectifiable if it has 'length' (bounded variation), that is, if:

$$\sup_{a=t_0<t_1<\cdots<t_k=b} \sum_{j=0}^{k-1} (|\phi(t_{j+1}) - \phi(t_j)|) < \infty$$

where $k \in \mathbb{N}$.

17. **Theorem.** Let Ω, ω be disjoint open regions and Γ a rectifiable arc so that $|\Gamma| \subset \partial\Omega \cap \partial\omega$ and $|\Gamma| \cup \Omega \cup \omega$ open (Γ has no endpoints). Assume also $f : |\Gamma| \cup \Omega \to \mathbb{C}$, $g : |\Gamma| \cup \omega \to \mathbb{C}$ are continuous and $f\ |_{\Omega}$, $g\ |_{\Omega}$ holomorphic and $f\ |_{|\Gamma|} = g\ |_{|\Gamma|}$. Then, $F : \Omega \cup |\Gamma| \cup \omega \to \mathbb{C}$ is holomorphic.

18. **Theorem (Schwarz Reflection Principle).** Let $\Omega = \Omega^\star$ $(= \{\overline{z} \mid z \in \Omega\})$ open region, $\Omega \cap \mathbb{R} \supset (a,b)$ and $\Omega_\pm = \Omega \cap \{\pm\text{Im}z > 0\}$. If $f : \Omega_+ \cup (a,b) \to \mathbb{C}$ continuous, $f\ |_{(a,b)} \subset \mathbb{R}$, $f\ |_{\Omega_+}$ holomorphic, then, $F(z)$, with $F(z) = f(z)$ if $z \in \Omega_+ \cup (a,b)$ and $F(z) = \overline{f(\overline{z})}$ if $z \in \Omega_-$ is holomorphic in $\Omega_+ \cup (a,b) \cup \Omega_-$.

19. **Analytic Arc def.** Analytic arc is $\phi : (a,b) \to \mathbb{C}$ so that there is $f : \omega \to \mathbb{C}$ univalent with $\omega \supset (a,b)$, $f\ |_{(a,b)} = \phi$, where we also require ϕ to be holomorphic within some neighborhood containing it.

20. **Theorem.** Let Ω be a region, γ an analytic arc, $|\gamma| \supset \partial\Omega$ from univalent $f : \omega \to \mathbb{C}$ and assume the following:

    (a) $f(\omega \cap \{\text{Im}z > 0\}) \subset \Omega$.

    (b) $f(\omega \cap \{\text{Im}z < 0\}) \cap \Omega = \emptyset$.

    (c) let $F : \Omega \cup |\gamma| \to \mathbb{C}$ continuous, and $F\ |_{\Omega}$ holomorphic with $F(|\gamma|) \subset |\Gamma|$, where Γ is an analytic arc.

    Then, there is an open Ω_1, with $\Omega_1 \supset \Omega \cup |\gamma|$ so that F has a holomorphic extension to Ω_1.

21. **Theorem (Schwarz-Christoffel Formula).** Let $F : \overline{\mathbb{D}} \to \overline{\Omega}$ be a homeomorphism (by Caratheodory) which extends the conformal map $F\ |_{\mathbb{D}} \to \Omega$ and $F(w_k) = z_k$. Let $\overline{\Omega}$ be a polygon with angles $\alpha_k\pi, \beta_k = 1 - \alpha_k$. Then,

$$F(w) = C \cdot \left(\int_0^w \left(\prod_{k=1}^{n} (w - w_k)^{-\beta_k} \right) dw \right) + C'.$$

22. **Theorem.** If γ is an analytic arc, then it is automatically rectifiable.

23. **Schwarz-Christoffel Formula for Upper-Half Plane.** If $G : \{\mathrm{Im} u > 0\} \to \Omega$ is a conformal map, where Ω is the interior of a polygon with outer angles $\beta_1\pi, \ldots, \beta_k\pi$ and the point ∞ corresponds to z_n, then:

$$G(u) = C \cdot \left(\int_0^u \left(\prod_{k=1}^{n-1} (u - \xi_k)^{-\beta_k} \right) du \right) + C',$$

where $\xi_k \in \mathbb{R}$. The product has only $n-1$ factors. The external angle β_n does not appear explicitly. If $\beta_1 + \cdots + \beta_{n-1} = 2$, then $\beta_n = 0$.

24. **Schwarzian Derivative.** For a function f, the Schwarzian derivative of f is defined as:

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

25. **A formula using Schwarzian derivative.** $S(f \circ g) = (S(f) \circ g)(g')^2 + S(g)$.

26. **Cor.** Now let $f(z) = \frac{az+b}{cz+d}$ be a fractional linear transformation. Then, $S(f) = 0$. Also, we get that $S(f \circ g) = S(g)$. Thus, we conclude that $S(g)$ is invariant under composition with a fractional linear transformation, under the Schwarzian derivative operator.