Math H110 Theorems.

1. **Lemma.** Let F be a field, $\lambda \in F$, V a vector space over F (denoted by V/F), $v \in V$. Then, if $\lambda v = 0$, then $\lambda = 0$ or $v = 0$.

2. **Lemma.** A vector space over a field is a module over a field.

3. **Theorem.** The intersection of a family of subspaces of a vector space V is a subspace of V.

4. **Lemma.** Let $S = \{v_1, \ldots, v_t\}$. Then the subspace of all linear combinations of the elements of S is the span$S$.

5. **Theorem.** Let $L = v_1, \ldots, v_n$ be a list of vectors in a vector space V over a field F and let $T : F^n :\to V$ be linear transformation with $(\lambda_1, \ldots, \lambda_n) \mapsto \lambda_1 v_1 + \cdots + \lambda_n v_n$. Then, we have the following:

   (a) L spans V iff T is onto.

   (b) L is linearly independent iff T is 1-1 iff $\mathrm{nul}\, T = \{0\}$.

   (c) L is a basis iff T is 1-1 and onto.

6. **Prop.** Consider $T : F^n \to V$ with $(\lambda_1, \ldots, \lambda_n) \mapsto \lambda_1 v_1 + \cdots + \lambda_n v_n$, so $T(e_i) = v_i$ for all i. Then, T is the unique linear map $F_n \to V$ that sends $e_i \mapsto v_i$ for all i.

7. **Theorem.** Every subspace X of V has complement.

8. **Lemma.** If $v_1, \ldots, v_t$ is linearly dependent list, then there is an index k such that $v_k \in \mathrm{span}(v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_t)$. Furthermore, the span of the list of length $t - 1$ gotten by removing v_k from the list is the same as the span of the original list.

9. **Prop.** In a finite-dimensional vector space, the length is of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

10. **Cor.** Two bases of V have the same number of elements.

11. **Prop.** $X + Y$ is direct iff the null space of the sum map is $\{0\}$.

12. **Theorem.** Every subspace of a finite-dimensional vector space is finite-dimensional.

13. **Prop.** Every spanning list for a vector space can be pruned down to a basis of the space.

14. **Cor.** Every finite-dimensional vector space has a basis.

15. **Prop.** In a finite-dimensional vector space, every linearly independent list can be extended to a basis of the space.

16. **Major Theorem.** Every subspace of a finite-dimensional vector space has a complement.

17. **Prop.** Let X, Y be subspaces of a finite-dimensional vector space V. Then:

    (a) $\dim X + \dim Y = \dim V$.

    (b) $X \cap Y = \{0\}$.

    Then, $V = X \oplus Y$.

18. **Prop.** $\dim(X \oplus Y) = \dim X + \dim Y$.

19. **Prop.** If V is a finite-dimensional vector space (with $\dim V = n$), then every subspace has dimension at most n.

20. **Prop.** Let $\dim V = n$. Then, a linearly independent list of vectors of V with length n is a basis for V.

21. **Prop.** Let $\dim V = n$. Then, every spanning list for V of length n is a basis for V.

22. **Lemma.** The list $(x_1, 0), \ldots, (x_t, 0); (0, y_1), \ldots, (0, y_k)$ of length $t + k$ is a basis of $X \times Y$.

23. **Cor.** $\dim(X \times Y) = \dim X + \dim Y$.

24. **Cor.** Let $T : V \to W$ be a linear map with $\dim V = d$. Then, $\text{rank} T \leq d$.

25. **Rank-Nullity Theorem.** $\dim V = \text{rank} V + \text{nullity} V$.

26. **Prop.** If $T : V \to W$ is 1-1, then $\text{nullity} T = 0$.

27. **Cor.** If $T : V \to W$ is 1-1 and onto, then $\dim V = \dim W$.

28. **Theorem.** The set of linear maps $V \to W$ is a vector space $L \cdot (F^n, W) \to T \longrightarrow (Te_1, \ldots, Te_n) \in W^n$.

29. **Theorem.** $\dim(X+Y) = \dim X + \dim Y - \dim(X \cap Y)$.

30. **Cor.** $\dim(V/X) = \dim V - \dim X$.

31. **Theorem.** If A is a rectangular matrix with elements in a field F, then row rank A = column rank A.

32. **Prop.** Let $T : V \to W$ be 1-1. Then, $\dim W \geq \dim V$.

33. **Prop.** Let $T : V \to W$ be onto. Then, $\dim V \geq \dim W$.

34. **Prop.** Let $T : V \to W$ and $\dim V = \dim W$. Then, T 1-1 iff T onto iff T bijective iff T invertible.

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35. **Lemma.** Let V be a finite-dimensional vector space and U a subspace of V. Then, $\dim U_0 = \dim V - \dim U$.

36. **Theorem.** Every linear functional on a subspace of V can be extended to V.

37. **Note.** Annihilator is the dual of the quotient subspace.

38. **Theorem.** Let $T : V \to W$ and $T' : W' \to V'$. THen $\mathscr{M}(T)$ and $\mathscr{M}(T')$ are transposes of each other.

39. **Lemma.** U^0 has dimension $\dim V - \dim U$.

40. **Cor.** The annihilator of U is $\{0\}$ iff $U = V$. The annihilator of U is V iff $U = \{0\}$.

41. **Prop.** If $T : V \to W$ is a linear map, then the null space of T' is the annihilator of the range of T. We have $\text{ann}(\text{range}T) = \{\psi : W \to F \mid \phi(Tv) = 0 \text{ for all } v \in V, T'(\psi)(v) = 0, T'\psi = 0, \phi \in \text{nul}(T')\}$