Math H110 Definitions.

1. **Endomorphism.** An endomorphism is a group homomorphism from a set to itself (NOTE: does not have to be invertible.)

2. **End V.** The symbol $\operatorname{End} V$ is the set of all endomorphisms on $V$ (and multiplication on $\operatorname{End} V$ is defined to be function composition).

3. **F-Module.** An $F-$module is a generalization of vector spaces over rings.

4. **Subspace.** Let $V$ be a vector space. $X$ is a subspace of $V$ if $X \subseteq V$ and closed under all relevant operations of $V$, $X \neq \emptyset$, and $X \ni 0$.

5. **Linear Map / Linear Transformation.** Let $V$ be a vector space over a field $F$ with $v, w \in V$. Let $T$ be a map on $V$ with $T(v+w) = T(v) + T(w)$ and $T(\lambda v) = \lambda T(v)$ for all $\lambda \in F$. Then, $T$ is called a linear map or linear transformation.

6. **Linear Operator.** If $T$ is a linear transformation on a vector spaces $V$ with $T : V \to V$, then $T$ is linear operator on $V$.

7. **Spans.** The list $v_1, \ldots, v_n$ spans $V$ iff $T : F^n \to V$ is onto.

8. **Linearly Independent.** The list $v_1, \ldots, v_n$ is linearly independent iff $T : F^n \to V$ is 1-1. Equivalently, the list $v_1, \ldots, v_n$ is linearly independent if $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$ implies $\lambda_i = 0$ for all $i$.

9. **Linearly Dependent.** The list $v_1, \ldots, v_n$ is linearly dependent iff $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$ implies $\lambda_i \neq 0$ for some $i$.

10. **Basis.** The list $v_1, \ldots, v_n$ is a basis of $V$ if $\operatorname{span}\{v_1, \ldots, v_n\} = V$ and $v_1, \ldots, v_n$ is linearly independent.

11. **Finite-dimensional.** $V$ is finite-dimensional if $V$ is spanned by a finite list of vectors.

12. **Sum of Subspaces.** Let $X_1, \ldots, X_t$ be subspaces of $V$. Then, we define their sum as $X_1 + \cdots + X_t = \{x_1 + \cdots + x_t \mid x_1 \in X_1, \ldots, x_t \in X_t\}$.

13. **Direct Sum of Subspaces.** Let $X_1, \ldots, X_t$ be subspaces of $V$. Then, their direct sum, $X_1 \oplus \cdots \oplus X_t$, is given by a 1-1 linear map $T$, with $T : X_1 \times \cdots \times X_t \to V$.

14. **Complement of Subspace.** Let $X, Y$ be subspaces of of $V$. Then, $Y$ is a complementary subspace of $X$ iff $X + Y = V$ and $X + Y = X \oplus Y$.

15. **Rank, Nullity.** The rank of a linear map is the dimension of the range of the linear map. The nullity is the dimension of the null space of the linear map.

16. **Null Space.** The null space is the set of vectors that are mapped to 0.

17. **Isomorphic Vector Spaces.** Two vector spaces $V, W$ are isomorphic if there exists a linear map $T : V \to W$ that is 1-1 and onto.

18. **Quotient Space.** Suppose $U$ is a subspace of $V$. Then, the quotient space $V/U$ is the set $V/U = \{v + U \mid v \in V\}$.

19. **Column Rank.** The column rank (rank of the column span of a matrix) is defined to be $\text{rank}T_A$.

20. **Conjugation.** Let $A$ be an $n \times n$ matrix (over $F$) and let $Q$ be an $n \times n$ matrix (over $F$). Then, the conjugation of $A$ by $Q$ is $Q^{-1}AQ$.

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21. **Dual Space.** Let $V$ be an $F$-vector space. Then the dual space of $V$ is $V' = \mathscr{L}(V, F)$ where the elements of $V'$ are called linear functionals.

22. **Annihilator.** For a subspace $U \subseteq V$, we define the annihilator of $U$ to be $U_0 = \{\phi \in V' \mid \phi(u) = 0 \forall u \in U\}$.

23. **Double Dual.** Let $V$ be a finite-dimensional vector space with dual $V'$. Then the double dual of $V$ is $(V')' = V'' = V$. Also, $\dim V = n = \dim V' = \dim V''$.

24. **Eigenvector / eigenvalue.** Let $T \in \mathscr{L}(V)$. Then an eigenvector of $T$ is a $v \in V$ $(v \neq 0)$ such that $Tv = \lambda v$ ($\lambda \in F$ is called an eigenvalue), and $v$ is an eigenvector of $T$.

25. **Eigenspace.** Let $T \in \mathscr{L}(V)$ and take $\lambda$ to be an eigenvalue of $T$. Then, $E(\lambda, T) = \{v \in V \mid Tv = \lambda v\} \neq \emptyset$ is written as $V_\lambda$ and is called the eigenspace of $\lambda$, which is a subspace of $V$.

26. **Invariant subspace.** $E$ is a $T$-invariant subspace if $T \in \mathscr{L}(V)$ with $T(E) \subseteq E$.

27. textbfIdempotent. If $e = e^2$, then $e$ is called idempotent.

28. **Generalized Eigenvector.** Consider a minimal polynomial $(x - \lambda_1)^{e_1} \cdot \dots \cdot (x - \lambda_m)^{e_m}$ on $X$ with $(T - \lambda_1 I)^{e_1} v = 0$. Then, $v$ is called a generalized eigenvector for $\lambda = \lambda_1$.

29. **Characteristic polynomial.** The characteristic polynomial of $T : V \to V$ (with eigenvalues $\lambda_1, \dots, \lambda_t$) is the polynomial $\prod_{i=1}^{t}(x - \lambda_i)^{\dim X_i}$, where $V = X_1 \oplus \dots \oplus X_t$.

30. **Simultaneously diagonalizable.** Operators $S$ and $T$ on $V$ are simulatenously diagonalizable if there is a basis of $V$ that consts of vectors that are eigenvectors for both $S$ and $T$ (i.e. there exists a basis $v_1, \dots, v_n$ of $V$ so that for $i$, $1 \leq i \leq n$, there are $\lambda_i$ and $\mu_i$ so that $Sv_i = \lambda_i v_i$ and $Tv_i = \mu_i v_i$).

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31. **Inner product.** Let $V$ be a vector space over $\mathbb{R}$, possibly infinite-dim. An inner product on $V$ is a bilinear functon $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that $\langle x, x \rangle \geq 0$ for all $x \in V$ and $\langle x, x \rangle = 0$ iff $x = 0$.

32. **Complex Inner product.** Let $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ be in $\mathbb{C}^n$. We have $\langle z, w \rangle = \sum_i z_i \overline{w_i}$, with $\langle w, z \rangle = \overline{\langle z, w \rangle}$ and $\langle \alpha z, w \rangle = \alpha \langle z, w \rangle$ and $\langle w, \alpha z \rangle = \overline{\alpha} \langle z, w \rangle$.

33. **Norm.** Norm of $v \in V$ is $||v|| = \sqrt{\langle v, v \rangle}$.

34. **Orthogonal.** Two vectors are orthogonal if $\langle v, w \rangle = 0$, where $v, w \in V$.

35. **Orthogonal Complement.** Take $U$ to be a subset of $V$. Then, $U^\perp = \{v \in V \mid \langle v, u \rangle = 0 \forall u \in U\} = \{v \in V \mid \langle u, v \rangle = 0 \forall u \in U\} = \{v \in V \mid \phi_v = 0 \text{ in } U\} = \{v \in V \mid \phi_v \in U^0\}$.

36. **Adjoint.** Let $T : V \to W$. Then the adjoint of $T$ is $T^* : W \to V$ such that $T^* = \alpha_V^{-1} \circ T' \circ \alpha_W$, where $\alpha_V : V \to V'$ and $\alpha_W : W \to W'$.

37. **Self-adjoint.** An operator $T : V \to V$ is self-adjoint if $T = T^*$.

38. **Symmetric.** $T$ is symmetric if $M(T) = M(T)^t$.

39. **Normal.** $T$ is a normal operator if $TT^* = T^*T$.

40. **Alternating.** A bilinear form $\psi : V \times V \to F$ is alternating if $psi(v, v) = 0$. for $v \in V$.

41. **Anti-symmetric.** Let $\psi : V \times V \to F$ be a bilinear form. Then $\psi$ is anti-symmetric if $\psi(x, y) = -\psi(y, x)$.

42. **Positive operator.** $T$ is a positive operator on an inner product space if $T = T^*$ and $\langle Tv, v \rangle \geq 0$.

43. **Square root of an operator.** A square root of an operator $T$ is an operator $R$ such that $R^2 = T$.

44. **Isometry.** An operator $S \in L(V)$ is an isometry if it preserves distance, i.e. $||Sv|| = ||v|| \forall v \in V$, i.e. $S^* = S^{-1}$. In the real case, they are called orthogonal operators. In the complex case, they are called unitary operators.

45. **Singular values.** The singular values of $T$ are the eigenvalues of $\sqrt{T^*T}$.

46. **Def 1 of Determinants.** (cofactor expansion). $\det A = \sum_{j=1}^n (-1)^{j+1} \cdot a_{ij} \det(A_{ij})$.

47. **Def 2 of Determinants.** If $T \in L(V)$, where $V$ is over $\mathbb{C}$, then the determinant of $T$ is the product of the eigenvalues $\lambda_1, \ldots, \lambda_n$ (with multiplicity).

48. **Def 3 of Determinants.** The determinant of an operator $T \in L(V)$ is $(-1)^n$ times the constant term of its characteristic polynomial $(z - \lambda_1) \cdots (z - \lambda_n)$.

49. **Permutation.** A permutation $m = (m_1, \ldots, m_r)$ is a list containing $1, \ldots, n$ exactly once each, and write $S_n$ to denote the set of $n$-element permutations.

50. **Inversion.** An inversion (in a permutation) is a pair $\{i, j\}$ such that $m_i > m_j$ where $i < j$.

51. **Def 4 of Determinants.** Let $A$ be an $n \times n$ matrix. Then, $\det A = \sum_{m \in S_n} (\operatorname{sgn}(m) \cdot a_{m_1,1} \cdots a_{m_n,n})$.

52. **Box.** The box defined by $v_1, \ldots, v_n \in \mathbb{R}^n$ is $\{a_1 v_1 + \cdots + a_n v_n \mid 0 \leq a_i \leq 1, i = 1, \ldots, n\}$ and as volume, $\det \begin{pmatrix} v_1 & \ldots & v_n \end{pmatrix}$, where $v_1, \ldots, v_n$ are the columns of a matrix $A$.

53. **Norm of $T$.** This is defined to be $\sup_{v \in V \setminus \{0\}} \frac{||Tv||}{||v||}$.

54. **Tensor product.** $V \otimes W := \operatorname{Bil}(V', W')$.