MIDTERM 1.

1. 1A. (NOTHING)
2. 1B.
3. **Vector Space.** A vector space V is a set that has scalar multiplication and vector addition defined on it with the following properties:
   (a) Additive commutativity.
   (b) Additive associativity of vectors ($u+(v+w)=(u+v)+w$) and multiplicative associativity for scalars ($(ab)v=a(bv)$).
   (c) Additive identity.
   (d) Additive inverses.
   (e) Multiplicative identity.
   (f) BOTH distributive properties.
4. **V-space (unique additive identity)** A vector space has a unique additive identity.
5. **V-space (unique additive inverses)** Every element in a vector space has a unique additive inverse.
6. 1C.
7. **Subspace.** A subset $U \subseteq V$ is a subspace of V if it is a vector space with the same additive identity, scalar multiplication, and vector addition as defined on V.
8. **Conditions for a Subspace.** A subset $U \subseteq V$ is a subspace of V iff U is closed under vector addition, scalar multiplication, and contains the "zero" element as in V.
9. **Sums of Subspaces.** Let $V_1,\dots,V_n$ be subspaces of V. Then, we have the sum of subspaces as $V_1+\cdots+V_n=\{v_1+\cdots+v_n \mid v_i \in V_i \text{ for all } i\}$.
10. **Smallest subspace containing each subspace** Suppose $V_1,\dots,V_n$ are subspaces of V. Then, $V_1+\cdots+V_n$ is the smallest subspace of V containing $V_1,\dots,V_n$.
11. **Direct Sum.** Suppose $V_1,\dots,V_m$ are subspaces of V. Then:
    (a) The sum $V_1+\cdots+V_m$ is direct if each element of $V_1+\cdots+V_m$ can be written uniquely as a sum $v_1+\cdots+v_m$, where $v_i \in V_i$ for all i.
    (b) If $V_1+\cdots+V_m$ is a direct sum, then we write $V_1\oplus\cdots\oplus V_m$.
12. **Conditions for a direct sum.** Suppose $V_1,\dots,V_n$ are subspaces of V. Then, $V_1+\cdots+V_n$ is direct iff the only way to write 0 from $v_1+\cdots+v_n$ is by taking $v_i=0$ for all i.
13. **Direct sum of subspaces.** If U,W are subspaces of V, then $U+W$ is direct iff $U\cap W=\{0\}$.
14. 2A.
15. **Span is the smallest containing subspace.** The span of a list of vectors in V is the smallest subspace containing all of the vectors in the list.
16. **Zero polynomial.** The zero polynomial is said to have degree $-\infty$.
17. **Linear Independence.** A list of vectors $v_1,\dots,v_n \in V$ is said to be linearly independent if $a_1v_1+\cdots+a_nv_n=0$ implies $a_i=0$ for all i. Also, the empty list () is said to be linearly independent.
18. **Linear Dependence.** A list of vectors $v_1,\dots,v_n$ is said to be linearly dependent if $a_1v_1+\cdots+a_nv_{n}=0$ impies $a_i \neq 0$ for some i.
19. **Linear Dependence Lemma.** Suppose $v_1,\dots,v_m$ is a linearly dependent list in V. Then, there exists $k\in\{1,\dots,m\}$ such that $v_k \in \operatorname{span}(v_1,\dots,v_{k-1})$. Furthermore, if k satisfies the condition in the previous sentence and the k^{th} term is removed from $v_1,\dots,v_m$, then the span of the remaining list equals $\operatorname{span}(v_1,\dots,v_m)$.
20. **length of linearly independent list ¡ length of spanning list.** In a finite-dimensional vector space, the length of every linearly independent list is at most the length of every spanning list of vectors.
21. **Finite Dimensional subspaces.** Every subspace of a finite-dimensional vector space is finite dimensional.
22. 2B.
23. **Basis.** A basis of V is a list of vectors that is linearly independent and spans V.
24. **Criterion for basis.** A list of vectors $v_1,\dots,v_n \in V$ is a basis of V iff every $v\in V$ can be written uniquely in the form $v=a_1v_1+\cdots+a_nv_n$, where $a_i \in F$ for all i.
25. **Every spanning list contains a basis.** Every spanning list in a vector space can be reduced to a basis of the vector space.
26. **Basis of finite-dimensional vector space.** Every finite-dimensional vector space has a basis.
27. **Every linearly independent list extends to a basis.** Every linearly independent list in a finite-dimensional vector space can be extended to a basis of the vector space.
28. **Every subspace of V is part of a direct sum equal to V.** Suppose V is finite-dimensional and U is a subspace of V. Then, there is a subspace W of V such that $V=U\oplus W$.
29. 2C.
30. **Basis length does not depend on basis.** Any two bases of a finite-dimensional vector space have the same length.
31. **Dimension of a subspace.** If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.
32. **Linearly independent list of the right length is a basis.** Suppose V is finite-dimensional. Then, every linearly independent list of vectors in V (with list length equal to $\dim V$) is a basis of V.
33. **Subspace of full dimension equals the whole space.** Suppose V is finite-dimensional and U is a subspace of V such that $\dim U=\dim V$. Then, $U=V$.
34. **Spanning list of the right length is a basis.** Suppose V is finite-dimensional. Then, every spanning list of V of length $\dim V$ is a basis of V.
35. **Dimension of a sum.** If V_1,V_2 are subspaces of a finite-dimensional vector space, then $\dim(V_1+V_2)=\dim V_1+\dim V_2-\dim(V_1\cap V_2)$.
36. 3A.
37. **Set of Linear Maps.** The linear of linear maps from $V\to W$ is written $\mathscr{L}(V,W)$ and the set of linear maps from $V\to V$ is written $\mathscr{L}(V)$.
38. **Linear Map lemma.** Suppose $v_1,\dots,v_n$ is a basis of V and $w_1,\dots,w_n\in W$. Then, there exists a unique linear map $T:V\to W$ such that $Tv_k=w_k$ for each k.
39. **Linear maps take 0 to 0.** Suppose $T:V\to W$ is a linear map. Then, $T(0)=0$.
40. 3B.
41. **null space is a subspace.** Suppose $T\in\mathscr{L}(V,W)$. Then, T is a subspace of V.
42. **injectivity iff null is 0.** Let $T\in\mathscr{L}(V,W)$. Then, T is 1-1 iff $\operatorname{nul} T=\{0\}$.
43. **range is a subspace.** If $T\in\mathscr{L}(V,W)$, then range T is a subspace of W.
44. **Fundamental Theorem of Linear Maps.** Suppose V is finite-dimensional and $T\in\mathscr{L}(V,W)$. Then, range T is finite dimensional and $\dim V=\dim\operatorname{nul}T+\dim\operatorname{range}T$.
45. **linear map to a lower-dim space is not 1-1.** Suppose V,W are finite-dimensional vector spaces such that $\dim V>\dim W$. Then, no linear map from $V\to W$ is 1-1.
46. **linear map to a higher-dim space is not onto.** Suppose V,W are finite-dimensional vector spaces such that $\dim V<\dim W$. Then, no linear map from $V\to W$ is onto.
47. 3C. (NOTHING)
48. 3D.
49. **Theorem.** Let V,W be finite-dimensional vector spaces such that $\dim V=\dim W$ and let $T\in\mathscr{L}(V,W)$. Then, T is invertible iff T is 1-1 iff T is onto.
50. **isomorphism.** An isomorphism is an invertible linear map.
51. **dimension and isomorphic.** Two finite-dimensional vector spaces are isomorphic iff they have the same dimension.
52. **Theorem.** Suppose V and W are finite-dimensional. Then, $\mathscr{L}(V,W)$ is finite-dimensional and $\dim\mathscr{L}(V,W)=(\dim V)(\dim W)$.
53. **ST=I iff TS=I (on vector spaces of the same dimension).** Suppose V and W are finite-dimensional vector spaces of the same dimension, $S\in\mathscr{L}(W,V), T\in\mathscr{L}(V,W)$. Then $ST=I$ iff $TS=I$.
54. **matrix of identity operator with respect to two bases.** Suppose $u_1,\dots,u_n$ and $v_1,\dots,v_n$ are two bases of V. Then, the matrices $\mathscr{M}(I;u_1,\dots,u_n;v_1,\dots,v_n)$ and $\mathscr{M}(I;v_1,\dots,v_n;u_1,\dots,u_n)$ are invertible and are inverses of each other.
55. **Change of basis formula.** Let $T\in\mathscr{L}(V,W)$. Suppose $u_1,\dots,u_n$ and $v_1,\dots,v_n$ are two bases of V. Let $A=\mathscr{M}(T;u_1,\dots,u_n)$ and $B=\mathscr{M}(T;v_1,\dots,v_n)$ and $C=\mathscr{M}(I;u_1,\dots,u_n;v_1,\dots,v_n)$. Then, $A=C^{-1}BC$.
56. Suppose that $v_1,\dots,v_n$ is a basis of V and $T\in\mathscr{L}(V)$ is invertible. Then, $\mathscr{M}(T^{-1})=(\mathscr{M}(T))^{-1}$, where both matrices are with respect to the basis $v_1,\dots,v_n$.
57. 3E.
58. **Product of vector spaces is a vector space.** Suppose $V_1,\dots,V_m$ are vector spaces over $\mathbb{F}$. Then, $V_1\times\cdots\times V_m$ is a vector space over $\mathbb{F}$.
59. **dimension of a product is the sum of the dimensions.** Suppose $V_1,\dots,V_m$ are finite-dimensional vector spaces. Then, $V_1\times\cdots\times V_m$ is finite-dimensional and $\dim(V_1\times\cdots\times V_m)=\dim V_1+\cdots+\dim V_m$.
60. **Products and direct sums.** Suppose $V_1,\dots,V_m$ are subspaces of V. Define a linear map $\Gamma:(V_1\times\cdots\times V_m)\to(V_1+\cdots+V_m)$ by $\Gamma(v_1,\dots,v_m)=v_1+\cdots+v_m$. Then, $V_1+\cdots+V_m$ is direct iff Γ is 1-1.
61. **direct sum iff dimensions add up.** Suppose V is finite-dimensional and $V_1,\dots,V_m$ are subspaces of V. Then, $V_1+\cdots+V_m$ is direct iff $\dim(V_1+\cdots+V_m)=\dim V_1+\cdots+\dim V_m$.
62. **v + U.** Suppose $v\in V$ and $U\subseteq V$. Then, $v+U=\{v+u \mid u\in U\}$.
63. **Translate.** For $v\in V$ and $U\subseteq V$, the set $v+U$ is called a translate of U.
64. **Quotient Space.** Let U be a subspace of V. Then, the quotient space V/U is the set of all translates of U, that is, $V/U=\{v+U \mid v\in V\}$.
65. **two translates of a subspace are either equal or disjoint.** Suppose U is a subspace of V and $v,w\in V$. Then, $v-w\in U$ iff $v+U=w+U$ iff $(v+U)\cap(w+U)\neq\emptyset$.
66. **Addition and scalar multiplication on Quotient space.** Let U be a subspace of V. Then, we have (for all $v,w\in V$, $\lambda\in F$):
    (a) addition on V/U: $(v+U)+(w+U)=(v+w)+U$.
    (b) scalar multiplication on V/U: $\lambda(v+U)=(\lambda v)+U$.
67. **quotient space is a vector space.** Let U be a subspace of V. Then, the quotient space V/U is a subspace of V under the defined scalar multiplication and vector addition.
68. **quotient map.** Let U be a subspace of V. Then, the quotient map $\pi:V\to V/U$ is the linear map defined by $\pi(v)=v+U$ for each $v\in V$.
69. **dimension of quotient space.** Suppose V is finite-dimensional and U is a subspace of V. Then, $\dim(V/U)=\dim V-\dim U$.
70. **Column rank.** The column rank (rank of the column span of a matrix) is $\operatorname{rank} T_A$.
71. **Theorem.** If A is a rectangular matrix of elements in a field F, then row rank A = column rank A.
72. RIBET DEFS.
73. **Endomorphism.** An endomorphism is a group homomorphism from a set to itself (NOTE: does not have to be invertible.)
74. **End V.** The symbol $\operatorname{End}V$ is the set of all endomorphisms on V (and multiplication on $\operatorname{End}V$ is defined to be function composition).
75. **F-Module.** An $F-$module is a generalization of vector spaces over rings.
76. **Linear Map / Linear Transformation.** Let V be a vector space over a field F with $v,w\in V$. Let T be a map on V with $T(v+w)=T(v)+T(w)$ and $T(\lambda v)=\lambda T(v)$ for all $\lambda\in F$. Then, T is called a linear map or linear transformation.
77. **Linear Operator.** If T is a linear transformation on a vector spaces V with $T:V\to V$, then T is linear operator on V.
78. **Spans.** The list $v_1,\dots,v_n$ spans V iff $T:F^n\to V$ is onto.
79. **Finite-dimensional.** V is finite-dimensional if V is spanned by a finite list of vectors.
80. **Direct Sum of Subspaces.** Let $X_1,\dots,X_t$ be subspaces of V. Then, their direct sum, $X_1\oplus\cdots\oplus X_t$, is given by a 1-1 linear map T, with $T:X_1\times\cdots\times X_t\to V$.
81. **Complement of Subspace.** Let X,Y be subspaces of of V. Then, Y is a complementary subspace of X iff $X+Y=V$ and $X+Y=X\oplus Y$.
82. **Rank, Nullity.** The rank of a linear map is the dimension of the range of the linear map. The nullity is the dimension of the null space of the linear map.
83. **Null Space.** The null space is the set of vectors that are mapped to 0.
84. **Isomorphic Vector Spaces.** Two vector spaces V,W are isomorphic if there exists a linear map $T:V\to W$ that is 1-1 and onto.
85. **Quotient Space.** Suppose U is a subspace of V. Then, the quotient space V/U is the set $V/U=\{v+U \mid v\in V\}$.
86. **Column Rank.** The column rank (rank of the column span of a matrix) is defined to be $\operatorname{rank} T_A$.
87. **Conjugation.** Let A be an $n\times n$ matrix (over F) and let Q be an $n\times n$ matrix (over F). Then, the conjugation of A by Q is $Q^{-1}AQ$.
88. RIBET THMS.
89. **Lemma.** Let F be a field, $\lambda\in F$, V a vector space over F (denoted by V/F), $v\in V$. Then, if $\lambda v=0$, then $\lambda=0$ or $v=0$.
90. **Lemma.** A vector space over a field is a module over a field.
91. **Theorem.** The intersection of a family of subspaces of a vector space V is a subspace of V.
92. **Lemma.** Let $S=\{v_1,\dots,v_t\}$. Then the subspace of all linear combinations of the elements of S is the $\operatorname{span}S$.
93. **Theorem.** Let $L=v_1,\dots,v_n$ be a list of vectors in a vector space V over a field F and let $T:F^n:\to V$ be linear transformation with $(\lambda_1,\dots,\lambda_n)\mapsto\lambda_1v_1+\cdots+\lambda_nv_n$. Then, we have the following:

(a) L spans V iff T is onto.
(b) L is linearly independent iff T is 1-1 iff $\operatorname{nul} T = \{0\}$.
(c) L is a basis iff T is 1-1 and onto.

94. **Prop.** Consider $T : F^n \to V$ with $(\lambda_1, \ldots, \lambda_n) \mapsto \lambda_1 v_1 + \cdots + \lambda_n v_n$, so $T(e_i) = v_i$ for all i. Then, T is the unique linear map $F_n \to V$ that sends $e_i \mapsto v_i$ for all i.
95. **Theorem.** Every subspace X of V has complement.
96. **Lemma.** If $v_1, \ldots, v_t$ is linearly dependent list, then there is an index k such that $v_k \in \operatorname{span}(v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_t)$. Furthermore, the span of the list of length $t-1$ gotten by removing v_k from the list is the same as the span of the original list.
97. **Prop.** In a finite-dimensional vector space, the length is of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.
98. **Cor.** Two bases of V have the same number of elements.
99. **Prop.** $X+Y$ is direct iff the null space of the sum map is $\{0\}$.
100. **Theorem.** Every subspace of a finite-dimensional vector space is finite-dimensional.
101. **Prop.** Every spanning list for a vector space can be pruned down to a basis of the space.
102. **Cor.** Every finite-dimensional vector space has a basis.
103. **Prop.** In a finite-dimensional vector space, every linearly independent list can be extended to a basis of the space.
104. **Major Theorem.** Every subspace of a finite-dimensional vector space has a complement.
105. **Prop.** Let X, Y be subspaces of a finite-dimensional vector space V. Then:

   (a) $\dim X + \dim Y = \dim V$.
   (b) $X \cap Y = \{0\}$.

   Then, $V = X \oplus Y$.
106. **Prop.** $\dim(X \oplus Y) = \dim X + \dim Y$.
107. **Prop.** If V is a finite-dimensional vector space (with $\dim V = n$), then every subspace has dimension at most n.
108. **Prop.** Let $\dim V = n$. Then, a linearly independent list of vectors of V with length n is a basis for V.
109. **Prop.** Let $\dim V = n$. Then, every spanning list for V of length n is a basis for V.
110. **Lemma.** The list $(x_1, 0), \ldots, (x_t, 0); (0, y_1), \ldots, (0, y_k)$ of length $t+k$ is a basis of $X \times Y$.
111. **Cor.** $\dim(X \times Y) = \dim X + \dim Y$.
112. **Cor.** Let $T : V \to W$ be a linear map with $\dim V = d$. Then, $\operatorname{rank} T \leq d$.
113. **Rank-Nullity Theorem.** $\dim V = \operatorname{rank} V + \operatorname{nullity} V$.
114. **Prop.** If $T : V \to W$ is 1-1, then $\operatorname{nullity} T = 0$.
115. **Cor.** If $T : V \to W$ is 1-1 and onto, then $\dim V = \dim W$.
116. **Theorem.** The set of linear maps $V \to W$ is a vector space $L \cdot (F^n, W) \to T \longrightarrow (Te_1, \ldots, Te_n) \in W^n$.
117. **Theorem.** $\dim(X+Y) = \dim X + \dim Y - \dim(X \cap Y)$.
118. **Cor.** $\dim(V/X) = \dim V - \dim X$.
119. **Theorem.** If A is a rectangular matrix with elements in a field F, then row rank A = column rank A.
120. **Prop.** Let $T : V \to W$ be 1-1. Then, $\dim W \geq \dim V$.
121. **Prop.** Let $T : V \to W$ be onto. Then, $\dim V \geq \dim W$.
122. **Prop.** Let $T : V \to W$ and $\dim V = \dim W$. Then, T 1-1 iff T onto iff T bijective iff T invertible.

MIDTERM 2.

1. 1C, 2A, 2B, 2C, 3B, 3C, 3D, 3E.
2. **Direct sum of subspaces.** If U, W are subspaces of V, then $U+W$ is direct iff $U \cap W = \{0\}$.
3. **Linear Dependence Lemma.** Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then, there exists $k \in \{1, \ldots, m\}$ such that $v_k \in \operatorname{span}(v_1, \ldots, v_{k-1})$. Furthermore, if k satisfies the condition in the previous sentence and the k^{th} term is removed from $v_1, \ldots, v_m$, then the span of the remaining list equals $\operatorname{span}(v_1, \ldots, v_m)$.
4. **Prop.** Let V, W be finite-dimensional with $\dim W = \dim V$. Let $S \in \mathscr{L}(W, V), T \in \mathscr{L}(V, W)$. Then, $ST = I$ iff $TS = I$.
5. **ST=I iff TS=I (on vector spaces of the same dimension).** Suppose V and W are finite-dimensional vector spaces of the same dimension, $S \in \mathscr{L}(W, V), T \in \mathscr{L}(V, W)$. Then $ST = I$ iff $TS = I$.
6. **matrix of identity operator with respect to two bases.** Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are two bases of V. Then, the matrices $\mathscr{M}(I; u_1, \ldots, u_n; v_1, \ldots, v_n)$ and $\mathscr{M}(I; v_1, \ldots, v_n; u_1, \ldots, u_n)$ are invertible and are inverses of each other.
7. **Product of vector spaces is a vector space.** Suppose $V_1, \ldots, V_m$ are vector spaces over $\mathbb{F}$. Then, $V_1 \times \cdots \times V_m$ is a vector space over $\mathbb{F}$.
8. **dimension of a product is the sum of the dimensions.** Suppose $V_1, \ldots, V_m$ are finite-dimensional vector spaces. Then, $V_1 \times \cdots \times V_m$ is finite-dimensional and $\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m$.
9. **Products and direct sums.** Suppose $V_1, \ldots, V_m$ are subspaces of V. Define a linear map $\Gamma : (V_1 \times \cdots \times V_m) \to (V_1 + \cdots + V_m)$ by $\Gamma(v_1, \ldots, v_m) = v_1 + \cdots + v_m$. Then, $V_1 + \cdots + V_m$ is direct iff Γ is 1-1.
10. **direct sum iff dimensions add up.** Suppose V is finite-dimensional and $V_1, \ldots, V_m$ are subspaces of V. Then, $V_1 + \cdots + V_m$ is direct iff $\dim(V_1 + \cdots + V_m) = \dim V_1 + \cdots + \dim V_m$.
11. **dimension of quotient space.** Suppose V is finite-dimensional and U is a subspace of V. Then, $\dim(V/U) = \dim V - \dim U$.
12. 3F.
13. **Linear functional.** A linear functional on V is a linear map $\phi : V \to F$.
14. **dual space.** The dual space of V is $V' = \mathscr{L}(V, F)$.
15. **dim space = dim dual space.** Suppose V is finite-dimensional. Then V' is also finite-dimensional and $\dim V = \dim V'$.
16. **dual basis.** If $v_1, \ldots, v_n$ is a basis of V, then the dual basis of $v_1, \ldots, v_n$ is $\phi_1, \ldots, \phi_n$ (elements of V') where $\phi_j(v_k) = 1$ if $k = j$ and $\phi_j(v_k) = 0$ if $k \neq j$.
17. **dual basis gives coefficients for linear combination.** Suppose $v_1, \ldots, v_n$ is a basis of V and $\phi_1, \ldots, \phi_n$ is dual basis. Then $v = \phi_1(v) v_1 + \cdots + \phi_n(v) v_n$ for each $v \in V$.
18. **dual basis is a basis of dual space.** Suppose V is finite-dimensional. Then the dual basis of V is a basis of V'.
19. **dual map, T'.** Suppose $T \in \mathscr{L}(V, W)$. The dual map of T is $T' \in \mathscr{L}(W', V')$ defined for each $\phi \in W'$ by $T'(\phi) = \phi \circ T$.
20. **algebraic properties of dual maps.** we have $(S+T)' = S' + T', (\lambda S)' = \lambda S', (ST)' = T'S'$.
21. **annihilator.** For $U \subseteq V$, the annihilator of U is $U_0 = \{\phi \in V' \mid \phi(u) = 0 \forall u \in U\}$.
22. **annihilator is a subspace.** If $U \subseteq V$, then $U^0 \subseteq V'$.
23. **dimension of annihilator.** Suppose V is finite-dimensional and $U \subseteq V$. Then $\dim U^0 = \dim V - \dim U$.
24. **condition for annihilator to equal $\{0\}$ or whole space.** Suppose V finite-dimensional and $U \subseteq V$. Then:

   (a) $U^0 = \{0\}$ iff $U = V$.
   (b) $U^0 = V'$ iff $U = \{0\}$.
25. **null space of T'.** Suppose V, W finite-dimensional and $T \in \mathscr{L}(V, W)$. Then:

   (a) $\operatorname{nul} T' = (\operatorname{range} T)^0$.
   (b) $\dim \operatorname{nul} T' = \dim \operatorname{nul} T + \dim W - \dim V$.
26. **T surjective equivalent to T' injective.** Suppose V, W finite-dimensional and $T \in \mathscr{L}(V, W)$. Then T onto iff T' 1-1.
27. **range of T'.** Suppose V, W finite-dim and $T \in \mathscr{L}(V, W)$. Then:

   (a) $\dim \operatorname{range} T' = \dim \operatorname{range} T$.
   (b) $\operatorname{range} T' = (\operatorname{nul} T)^0$.
28. **T injective is equivalent to T' surjective.** Suppose V, W finite-dim and $T \in \mathscr{L}(V, W)$. Then T 1-1 iff T' onto.
29. 5A.
30. **equivalent conditions to be an eigenvalue.** Let V be finite-dim and $T \in \mathscr{L}(V)$ and $\lambda \in F$. Then TFAE:

   (a) λ is an eigenvalue of T.
   (b) $T - \lambda I$ not injective.
   (c) $T - \lambda I$ not surjective.
   (d) $T - \lambda I$ not invertible.
31. **linearly independent eigenvectors.** Let $T \in \mathscr{L}(V)$. Then every list of eigenvectors of T corresponding to different eigenvalues is linearly independent.
32. **operator cannot have more eigenvalues than dimension of space.** Let V be finite-dim. Then each operator on V has at most $\dim V$ distinct eigenvalues.
33. **null space and range of $p(T)$ are invariant under T.** Suppose $T \in \mathscr{L}(V)$ and $p \in \mathscr{P}(F)$. Then $\operatorname{nul} p(T)$ and range $p(T)$ are invariant under T.
34. 5B.
35. **existence of eigenvalues.** Every operator on a finite-dim nonzero complex vector space has an eigenvalue.
36. **existence, uniqueness, and degree of minimal polynomial.** Suppose V finite-dim and let $T \in \mathscr{L}(V)$. Then there is a unique monic polynomial $p \in \mathscr{P}(F)$ of smallest degree such that $p(T) = 0$. Also, $\deg p \leq \dim V$.
37. **minimal polynomial.** Suppose V finite-dim and $T \in \mathscr{L}(V)$. Then the minimal polynomial of T is the unique monic polynomial $p \in \mathscr{P}(F)$ of smallest degree such that $p(T) = 0$.
38. **eigenvalues are the zeros of minimal polynomial.** Let V finite-dim and $T \in L(V)$. Then:

   (a) zeros of the minimal polynomial of T are the eigenvalues of T.
   (b) if V is a complex vector space, then minimal polynomial of T has the form $(z - \lambda_1) \cdot \cdots \cdot (z - \lambda_m)$, where $\lambda_1, \ldots, \lambda_m$ is a list of all eigenvalues of T, possibly with repetitions.
39. **$q(T) = 0$ iff q is a polynomial multiple of the minimal polynomial.** Let V finite-dim and $T \in L(V)$ and $q \in P(F)$. Then $q(T) = 0$ iff q is a polynomial multiple of the minimal polynomial.
40. **minimal polynomial of a restriction operator.** Let V finite-dim and $T \in L(V)$ and $U \subseteq V$ that is invariant under T. Then minimal polynomial of T is a polynomial multiple of minimal polynomial of $T |_U$.
41. **T not invertible iff constant term of minimal polynomial of T is 0.** Let V finite-dim and $T \in L(V)$. Then T is not invertible iff the constant term in the minimal polynomial of T is 0.
42. **even-dimensional null space.** Let $F = \mathbb{R}$ and V finite-dim and $T \in L(V)$ and $b^2 - 4ac < 0$. Then $\dim(T^2 + bT + cI)$ is an even number.
43. **operators on an odd-dimensional space have eigenvalues.** Every operator on an odd-dimensional vector space has an eigenvalue.
44. 5C.
45. **conditions for upper-triangular matrix.** Suppose $T \in L(V)$ and $v_1, \ldots, v_n$ is a basis of V. Then TFAE:

   (a) the matrix of T with respect to $v_1, \ldots, v_n$ is upper-triangular.
   (b) $\operatorname{span}(v_1, \ldots, v_k)$ is invariant under T for each $k = 1, 2, \ldots, n$.
   (c) $Tv_k \in \operatorname{span}(v_1, \ldots, v_k)$ for each $k = 1, \ldots, n$.
46. **equation satisfied by operator with upper-triangular matrix.** Suppose $T \in L(V)$ and V has a basis with respect to which T has an upper-triangular matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. Then $(T - \lambda_1 I) \cdot \cdots \cdot (T - \lambda_n I) = 0$.
47. **determination of eigenvalues from upper-triangular matrix.** Suppose $T \in L(V)$ has an upper-triangular matrix with respect to some basis of V. Then the eigenvalues of T are precisely the entries on the diagonal of that upper-triangular matrix.
48. **necessary and sufficient condition to have an upper-triangular matrix.** Suppose V is finite-dim and $T \in L(V)$. Then T has an upper-triangular matrix with respect to some basis of V iff the minimal polynomial of T equals $(z - \lambda_1) \cdot \cdots \cdot (z - \lambda_n)$ for some $\lambda_i \in F$.
49. **if $F = \mathbb{C}$, then every operator on V has an upper-triangular matrix.** Suppose V is a finite-dim complex vector space and $T \in L(V)$. Then T has an upper-triangular matrix with respect to some basis of V.
50. 5D.
51. **eigenspace, $E(\lambda, T)$.** Suppose $T \in L(V)$ and $\lambda \in F$. Then the eigenspace of T corresponding to λ is $E(\lambda, T) = \operatorname{nul}(T - \lambda I) = \{v \in V \mid Tv = \lambda v\}$.
52. **sum of eigenspaces is a direct sum.** Suppose $T \in L(V)$ and $\lambda_1, \ldots, \lambda_m$ are the distinct eigenvalues of T. Then $\sum_i E(\lambda_i, T)$ is a direct sum and $\sum_i \dim E(\lambda_i, T) \leq \dim V$.

53. **conditions equivalent to diagonalizability.** Suppose V finite-dim and $T \in L(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T. Then TFAE:

(a) T is diagonalizable.
(b) V has a basis consisting of eigenvectors of T.
(c) $V = \oplus_i E(\lambda_i, T)$
(d) $\dim V = \sum_i \dim E(\lambda_i, T)$.

54. **enough eigenvalues implies diagonalizability.** Let V be finite-dim and $T \in L(V)$ has $\dim V$ distinct eigenvalues. Then T is diagonalizable.

55. **necessary and sufficient condition for diagonalizability.** Suppose V finite-dim and $T \in L(V)$. Then T diagonalizable iff the minimal polynomial of T equals $(z-\lambda_1)\cdots(z-\lambda_m)$ for some distinct $\lambda_1, \ldots, \lambda_i \in F$.

56. **restriction of diagonalizable operator to invariant subspace.** Suppose $T \in L(V)$ and U is a T-invariant subspace of V. Then $T|_U$ is a diagonalizable operator on U.

57. 5E.

58. **commuting operators correspond to commuting matrices.** Suppose $S, T \in L(V)$ and $v_1, \ldots, v_n$ is a basis of V. Then S and T commute iff $M(S, (v_1, \ldots, v_n))$ and $M(T, (v_1, \ldots, v_n))$ commute.

59. **eigenspace is invariant under commuting operators.** Suppose $S, T \in L(V)$ commute and $\lambda \in F$. Then $E(\lambda, S)$ is invariant under T.

60. **simultaneous diagonalizability iff commutativity.** Two diagonalizable operators on the same vector space have diagonal matrices with respect to the same basis iff the two operators commute.

61. **common eigenvector for commuting operators.** every pair of commuting operators on a finite-dim nonzero complex vector space has a common eigenvector.

62. **commuting operators are simultaneously upper-triangularizable.** Suppose V is a finite-dim nonzero complex vector space and S, T are commuting operators on V. Then there is a basis of V with respect to which both S, T have upper-triangular matrices.

63. **eigenvalues of sum and product of commuting operators.** Suppose V is a finite-dim complex vector space and S, T are commuting operators on V. Then:

(a) every eigenvalue of $S+T$ is an eigenvalue of S plus an eigenvalue of T.
(b) every eigenvalue of ST is an eigenvalue of S times an eigenvalue of T.

64. 8A.

65. **sequence of increasing null spaces.** Let $T \in L(V)$. Then $\{0\} = \operatorname{nul} T^0 \subseteq \operatorname{nul} T_1 \subseteq \cdots \subseteq \operatorname{nul} T^k \subseteq \ldots$.

66. **equality in the sequence of null spaces.** Let $T \in L(V)$ and m is a nonnegative integer such that $\operatorname{nul} T^m = \operatorname{nul} T^{m+1}$. Then $\operatorname{nul} T^m = \operatorname{nul} T^{m+1} = \ldots$.

67. **null spaces stop growing.** Let $T \in L(V)$. Then $\operatorname{nul} T^{\dim V} = \operatorname{nul} T^{\dim V+1} = \ldots$.

68. **V is the direct sum of $\operatorname{nul} T^{\dim V}$ and $\operatorname{range} T^{\dim V}$.** Let $T \in L(V)$. Then $V = \operatorname{nul} T^{\dim V} \oplus \operatorname{range} T^{\dim V}$.

69. **generalized eigenvector.** Let $T \in L(V)$ and λ be an eigenvalue of T. A vector $v \in V$ $(v \neq 0)$ is called a generalized eigenvector of T corresponding to λ if $(T-\lambda I)^k v = 0$ for some $k \in \mathbb{Z}_{>0}$.

70. **a basis of generalized eigenvectors.** Let $F = \mathbb{C}$ and $T \in L(V)$. Then there is a basis of V consisting of generalized eigenvectors of T.

71. **generalized eigenvector corresponds to a unique eigenvalue.** Let $T \in L(V)$. Then each generalized eigenvector of T corresponds to only one eigenvalue of T.

72. **linearly independent generalized eigenvectors.** Let $T \in L(V)$. Then every list of generalized eigenvectors of T corresponding to distinct eigenvalues of T is linearly independent.

73. **nilpotent operator raised to dimension of domain is 0.** Let $T \in L(V)$ be nilpotent. Then $T^{\dim V} = 0$.

74. **eigenvalues of nilpotent operator.** Let $T \in L(V)$. Then:

(a) if T is nilpotent then 0 is an eigenvalue of T and T has no other eigenvalues.
(b) if $F = \mathbb{C}$ and 0 is the only eigenvalue of T, then T is nilpotent.

75. **minimal polynomial & upper-triangular matrix of nilpotent operator.** Let $T \in L(V)$. Then TFAE:

(a) T is nilpotent.
(b) minimal polynomial of T is z^m for some positive integer m.
(c) there is a basis of V with respect to which the matrix of T has the form

$$\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$

.

76. 8B.

77. **generalized eigenspace.** Suppose $T \in L(V)$ and $\lambda \in F$. The generalized eigenspace of T corresponding to λ is $G(\lambda, T) = \{v \in V \mid (T-\lambda I)^k v = 0 \text{ for some } k \in \mathbb{Z}_{>0}\}$, which is the set of generalized eigenvectors of T corresponding to λ, including the 0-vector.

78. **description of generalized eigenspaces.** Suppose $T \in L(V)$ and $\lambda \in F$. Then $G(\lambda, T) = \operatorname{nul}(T-\lambda I)^{\dim V}$.

79. **generalized eigenspace decomposition.**

80. Suppose $F = \mathbb{C}$ and $T \in L(V)$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T. Then:

(a) $G(\lambda_k, T)$ is invariant under T for each $k = 1, \ldots, m$.
(b) $(T-\lambda_k I)|_{G(\lambda_k, T)}$ is nilpotent for each $k = 1, \ldots, m$.
(c) $V = \oplus_i G(\lambda_i, T)$.

81. **multiplicity.** Let $T \in L(V)$. The multiplicity of an eigenvalue λ of T is defined to be the dimension of the corresponding generalized eigenspace $G(\lambda, T)$, so multiplicity of λ is $\dim \operatorname{nul}(T-\lambda I)^{\dim V}$.

82. **sum of the multiplicities equals $\dim V$.** Suppose $F = \mathbb{C}$ and $T \in L(V)$. Then the sum of all the multiplicities of all the eigenvalues of T equals $\dim V$.

83. **characteristic polynomial.** Let $F = \mathbb{C}$ and $T \in L(V)$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T, with multiplicities $d_1, \ldots, d_m$. Then the polynomial $(z-\lambda_1)^{d_1}\cdots(z-\lambda_m)^{d_m}$ is called the characteristic polynomial of T.

84. **degree and zeros of the characteristic polynomial.** Let $F = \mathbb{C}$ and $T \in L(V)$. Then:

(a) characteristic polynomial of T has degree $\dim V$.
(b) zeros of the characterisit polynomial are the eigenvalues of T.

85. **Cayley-Hamilton theorem.** Let $F = \mathbb{C}$, $T \in L(V)$ and q be the characteristic polynomial of T. Then $q(T) = 0$.

86. **characteristic polynomial is a multiple of minimal polynomial.** Let $F = \mathbb{C}$ and $T \in L(V)$. Then characteristic polynomial of T is a polynomial multiple of the minimal polynomial of T.

87. **multiplicity of an eigenvalue equals number of times on diagonal.** Let $F = \mathbb{C}$ and $T \in L(V)$. Let $v_1, \ldots, v_n$ be a basis of V such that $M(T, (v_1, \ldots, v_n))$ is upper-triangular. The number of times the eigenvalue λ ppears on the diagonal of $M(T, (v_1, \ldots, v_n))$ equals the multiplicity of λ as an eigenvalue of T.

88. **block diagonal matrix with upper-triangular blocks.** Let $F = \mathbb{C}$ and $T \in L(V)$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T with multiplicities $d_1, \ldots, d_m$. Then there is a basis of V with respect to which T has a block diagonal matrix of the form

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix}$$

, where each A_k is a d_k-by-d_k upper-triangular matrix of the form

$$\begin{pmatrix} \lambda_k & & * \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix}$$

.

89. 8C.

90. **jordan basis.** Let $T \in L(V)$. A basis of V is called a Jordan basis for T if with respect to this basis T has a block diagonal matrix

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_p \end{pmatrix}$$

in which each A_k is an upper-triangular matrix of the form

$$\begin{pmatrix} \lambda_k & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_k \end{pmatrix}$$

.

91. **every nilpotent operator has a jordan basis.** Let $T \in L(V)$ be nilpotent. Then there is a basis for V that is a Jordan basis for T.

92. **Jordan form.** Let $F = \mathbb{C}$ and $T \in L(V)$. Then there is a basis of V that is a Jordan basis.

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93. 6A.

94. **inner product.** An inner product on V is a function that takes an ordered pair (u, v) of elements of V to a number $\langle u, v \rangle \in F$ so that

(a) positivity: $\langle v, v \rangle \geq 0 \forall v \in V$.
(b) definiteness: $\langle v, v \rangle = 0$ iff $v = 0$.
(c) additivity in the first slot: $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$.
(d) homogeneity in the first slot: $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$.
(e) conjugate symmetry: $\langle u, v \rangle = \overline{\langle v, u \rangle}$.

95. **basis properties of inner product spaces.**

(a) $\langle u, v \rangle$ is a linear map from V to F for a fixed $u \in V$.
(b) $\langle 0, v \rangle = 0$ for all $v \in V$.
(c) $\langle v, 0 \rangle = 0$ for all $v \in V$.
(d) $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$.
(e) $\langle u, \lambda v \rangle = \overline{\lambda} \langle u, v \rangle$.

96. **basic properties of norm.**

(a) $||v|| = 0$ iff $v = 0$.
(b) $||\lambda v|| = |\lambda| \cdot ||v||$.

97. **pythagorean theorem.** if $u, v \in V$ with u, v orthogonal, then $||u+v||^2 = ||u||^2 + ||v||^2$.

98. **orthogonal decomposition.** let $u, v \in V$ with $v \neq 0$. set $c = \frac{\langle u, v \rangle}{||v||^2}$ and $w = u - \frac{\langle u, v \rangle}{||v||^2}$. then $u = cv + w$ and $\langle v, w \rangle = 0$.

99. **cauchy-schwarz.** if $u, v \in V$, then $|\langle u, v \rangle| \leq ||u|| \cdot ||v||$, with equality iff one of u, v is a scalar multiple of the other.

100. **triangle inequality.** if $u, v \in V$, then $||u+v|| \leq ||u|| + ||v||$, with equality iff one of u, v is a nonnegative real multiple of the other.

101. **parallelogram equality.** if $u, v \in V$, then $||u+v||^2 + ||u-v||^2 = 2(||u||^2 + ||v||^2)$.

102. 6B.

103. **norm of an orthonormal linear combination.** let $e_1, \ldots, e_m$ be an orthonormal list in V. then $||a_1 e_1 + \cdots + a_m e_m||^2 = |a_1|^2 + \cdots + |a_m|^2$.

104. **bessel's inequality.** let $e_1, \ldots, e_m$ be an orthonormal list in V. then if $v \in V$, then $|\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2 \leq ||v||^2$.

105. **writing a vector as a linear combination of an orthonormal basis.** let $e_1, \ldots, e_m$ be an orthonormal basis of V and $u, v \in V$. then

(a) $v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_m \rangle e_m$.

(b) $||v||^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$.

(c) $\langle u, v\rangle = \langle u, e_1\rangle\overline{\langle v, e_1\rangle} + \cdots + \langle u, e_m\rangle\overline{\langle v, e_m\rangle}$.

106. **gram-schmidt.** let $v_1, \ldots, v_m$ be L.I. in V. let $f_1 = v_1$. for $k > 2$, define $f_k = v_k - \frac{\langle v_k, f_1\rangle}{||f_1||^2} f_1 - \cdots - \frac{\langle v_k, f_{k-1}\rangle}{||f_{k-1}||^2} f_{k-1}$. then, $f_1, \ldots, f_m$ is an orthogonal basis of $span(v_1, \ldots, v_m)$.

107. **existence of orthonormal basis.** every finitedim inner product space has an orthonormal basis.

108. **every orthonormal list extends to an orthonormal basis.** if V is finitedim, then every orthonormal list in V can be extended to an orthonormal basis of V.

109. **upper-triangular matrix with respect to some orthonormal basis.** let V be finitedim and $T \in L(V)$. then T has an upper-triangular matrix with respect to some orthonormal basis of V iff min poly of T equals $(z-\lambda_1)\cdots(z-\lambda_m)$ for some m.

110. **schur's theorem.** every operator on a finitedim complex inner product space has an upper-triangular matrix with respect to some orthonormal basis.

111. **riesz representation theorem.** let V be finitedim and ϕ is a linear functional on V. then there is a unique $v \in V$ so that $\phi(u) = \langle u, v\rangle$ for every $u \in V$.

112. 6C.

113. **properties of orthogonal complement.**

(a) if U is a subset of V, then $U^\perp$ is a subspace of V.

(b) $\{0\}^\perp = V$.

(c) $V^\perp = \{0\}$.

(d) if U is a subset of V, then $U \cap U^\perp \subseteq \{0\}$.

(e) if G, H subsets of V and $G \subseteq H$, then $H^\perp \subseteq G^\perp$.

114. **direct sum of a subspace and its orthogonal complement.** let U be a finitedim subspace of V. then $V = U \oplus U^\perp$.

115. **dimension of orthogonal complement.** let V be finitedim and U be a subspace of V. then $\dim U^\perp = \dim V - \dim U$.

116. **orthogonal complement of orthogonal complement.** let U be a finitedim subspace of V. then $(U^\perp)^\perp = U$.

117. **prop.** let U be a finitedim subspace of V. then $U^\perp = \{0\}$ iff $U = V$.

118. **orthogonal projection operator.** let U be a finitedim subspace of V. orthogonal projection of V onto U is $P_U \in L(V)$, where if $v \in V$, write $v = u + w$, where $u \in U$ and $w \in U^\perp$, and get $P_U v = u$.

119. **properties of orthogonal projection operator.** let U be a finitedim subspace of V.

(a) $P_U \in L(V)$.

(b) $P_U u = u \forall u \in U$.

(c) $P_U w = 0 \forall w \in U^\perp$.

(d) $\operatorname{range} P_U = U$.

(e) $\operatorname{nul} P_U = U^\perp$.

(f) $v - P_U v \in U^\perp$ for all $v \in V$.

(g) $P_U^2 = P_U$.

(h) $||P_U v|| \le ||v||$ for all $v \in V$.

(i) if $e_1, \ldots, e_m$ is an orthonormal basis of V and $v \in V$, then $P_U v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m$.

120. **riesz representation theorem (again).** let V be finitedim. for each $v \in V$ define $\phi_v \in V'$ so that $\phi_v(u) = \langle u, v\rangle$ for each $u \in V$. then $v \mapsto \phi_v$ is a 1-1 function $V \to V'$.

121. 7A.

122. **adjoint.** let $T \in L(V, W)$. the adjoint of T is $T^* : W \to V$ so that $\langle Tv, w\rangle = \langle v, T^* w\rangle$ for all $v \in V, w \in W$.

123. **adjoint of a linear map is a linear map.** if $T \in L(V, W)$, then $T^* \in L(W, V)$.

124. **properties of adjoint.** Let $T \in L(V, W)$.

(a) $(S+T)^* = S^* + T^*$.

(b) $(\lambda T)^* = \overline{\lambda} T^*$.

(c) $(T^*)^* = T$.

(d) $(ST)^* = T^* S^*$ for all $S \in L(W, U)$, where U is a finitedim inner product space.

(e) $I^* = I$.

(f) if T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

125. **null space and range of T^*.** Let $T \in L(V, W)$. then

(a) $\operatorname{nul} T^* = (\operatorname{range} T)^\perp$.

(b) $\operatorname{range} T^* = (\operatorname{nul} T)^\perp$.

(c) $\operatorname{nul} T = (\operatorname{range} T^*)^\perp$.

(d) $\operatorname{range} T = (\operatorname{nul} T^*)^\perp$.

126. **matrix of T^* is the conjugate transpose of T.** let $T \in L(V, W)$. let $e_1, \ldots, e_n$ be an orthonormal basis of V and $f_1, \ldots, f_m$ be an orthonormal basis of W. then $M(T^*) = \overline{M(T)^t}$.

127. **self-adjoint.** let $T \in L(V)$. T is self-adjoint if $T = T^*$.

128. **eigenvalues of self-adjoint operators.** every eigenvalue of a self-adjoint operator is real.

129. **prop.** Suppose V is a complex inner product space and $T \in L(V)$. then $\langle Tv, v\rangle = 0 \forall v \in V$ iff $T = 0$.

130. **prop.** suppose V is a complex inner product space and $T \in L(V)$. then T is self-adjoint iff $\langle Tv, v\rangle \in R \forall v \in V$.

131. **prop.** let T be a self-adjoint operator on V. then $\langle Tv, v\rangle = 0 \forall v \in V$ iff $T = 0$.

132. **normal.** $T \in L(V)$ is normal if $TT^* = T^*T$.

133. **prop.** let $T \in L(V)$. then T is normal iff $||Tv|| = ||T^*v|| \forall v \in V$.

134. **range, null space, eigenvectors of a normal operator.** let $T \in L(V)$ be normal. then

(a) $\operatorname{nul} T = \operatorname{nul} T^*$.

(b) $\operatorname{range} T = \operatorname{range} T^*$.

(c) $V = \operatorname{nul} T \oplus \operatorname{range} T$.

(d) $T - \lambda I$ is normal for all $\lambda in F$.

(e) if $v \in V$ and $\lambda \in F$, then $Tv = \lambda v$ iff $T^* v = \overline{\lambda} v$.

135. **orthogonal eigenvectors for normal operators.** let $T \in L(V)$ be normal. then eigenvectors of T corresponding to different eigenvalues are orthogonal.

136. **T normal iff real/imaginary parts of T commute.** let $F = C$ and $T \in L(V)$. then T is normal iff there exist commuting self-adjoint operators A, B so that $T = A + iB$.

137. 7B.

138. **inveritble quadratic expressions.** let $T \in L(V)$ be self-adjoint and $b, c \in R$ so that $b^2 < 4c$. then $T^2 + bT + cI$ is an invertible operator.

139. **minimal polynomial of self-adjoint operator.** let $T \in L(V)$ self-adjoint. then the min poly of T equals $(z-\lambda_1)\cdots(z-\lambda_m)$ for some m.

140. **Real Spectral Theorem.** let $F = R$ and $T \in L(V)$. TFAE:

(a) T is self-adjoint.

(b) T has a diagonal matrix with respect to some orthonormal basis of V.

(c) V has an orthonormal basis consisting of eigenvectors of T.

141. **Complex Spectral Theorem.** let $F = C$ and $T \in L(V)$. TFAE:

(a) T is normal.

(b) T has a diagonal matrix with respect to some orthonormal basis of V.

(c) V has an orthonormal basis consisting of eigenvectors of T.

142. 7C.

143. **positive operator.** an operator $T \in L(V)$ is called positive if it's self-adjoint and $\langle Tv, v\rangle \ge 0 \forall v \in V$.

144. **square root.** an operator R is called a square root of T if $R^2 = T$.

145. **characterizations of positive operators.** let $T \in L(V)$. TFAE:

(a) T is a positive operator.

(b) T is self-adjoint and all eigenvalues of T are nonnegative.

(c) with respect to some orthonormal basis of T, the matrix of T is diagonal with only nonnegative numbers on diagonal.

(d) T has a positive square root.

(e) T has a self-adjoint square root.

(f) $T = R^* R$ for some $R \in L(V)$.

146. **prop.** each positive operator on V has a unique positive square root.

147. **$\sqrt{T}$.** let $\sqrt{T}$ denote the unique positive square root of a positive operator T.

148. **prop.** let T be a positive operator on V and $v \in V$ so that $\langle Tv, v\rangle = 0$. then $Tv = 0$.

149. 7D.

150. **isometry.** a linear map $S \in L(V, W)$ is an isometry if $||Sv|| = ||v|| \forall v \in V$.

151. **characterizations of isometries.** let $S \in L(V, W)$. let $e_1, \ldots, e_n$ be an orthonormal basis of V and $f_1, \ldots, f_m$ be an orthonormal basis of W. TFAE:

(a) S is an isometry.

(b) $S^* S = I$.

(c) $\langle Su, Sv\rangle = \langle u, v\rangle \forall u, v \in V$.

(d) $Se_1, \ldots, Se_n$ is an orthonormal list in W.

(e) the columns of $M(S, (e_1, \ldots, e_n), (f_1, \ldots, f_m))$ form an orthonormal list in F^m with respect to the Euclidean inner product.

152. **unitary operator.** An operator $S \in L(V)$ is called unitary if it is an invertible isometry.

153. **characterizations of unitary operators.** let $S \in L(V)$ and $e_1, \ldots, e_n$ be an orthonormal basis of V. TFAE:

(a) S is a unitary operator.

(b) $S^* S = SS^* = I$.

(c) S is invertible and $S^{-1} = S^*$.

(d) $Se_1, \ldots, Se_n$ is an orthonormal basis of V.

(e) the rows of $M(S, (e_1, \ldots, e_n))$ form an orthonormal basis of F^n with respect to the Euclidean inner product.

(f) S^* is a unitary operator.

154. **eigenvalues of unitary operators have absolute value 1.** let λ be an eigenvalue of a unitary operator. then $|\lambda| = 1$.

155. **description of unitary operators on complex inner product spaces.** let $F = C$ and $S \in L(V)$. TFAE:

(a) S is a unitary operator.

(b) there is an orthonormal basis of V consisting of eigenvectors of S whose corresponding eigenvalues all have absolute value 1.

156. 7E.

157. **properties of T^*T.** Let $T \in L(V, W)$. then

(a) T^*T is a positive operator on V.

(b) $\operatorname{nul} T^*T = \operatorname{nul} T$.

(c) $\operatorname{range} T^*T = \operatorname{range} T^*$.

(d) $\dim \operatorname{range} T = \dim \operatorname{range} T^* = \dim \operatorname{range} T^*T$.

158. **singular values.** let $T \in L(V, W)$. the singular values of T are the nonnegative square roots of eigenvalues of T^*T, listed in decreasing order, each included as many times as the dimension of the corresponding eigenspace of T^*T.

159. **role of positive singular values.** let $T \in L(V, W)$. then

(a) T is 1-1 iff 0 is not a singular value of T.

(b) number of positive singular values of T equals $\dim \operatorname{range} T$.

(c) T is onto iff number of positive singular values of T equals $\dim W$.

160. **isometries characterized by having all singular values equal 1.** let $S \in L(V,W)$. then S is an isometry iff all singular values of S equal 1.

161. **singular value decomposition.** let $T \in L(V,W)$ and positive singular values of T are $s_1,\dots,s_m$. then there exist orthonormal lists $e_1,\dots,e_m \in V$ and $f_1,\dots,f_m \in W$ so that $Tv = s_1\langle v,e_1\rangle f_1 + \cdots + s_m\langle v,e_m\rangle f_m$ for all $v \in V$.

162. **matrix version of SVD.** let A be a $p\times n$ matrix with rank$A \geq 1$. then there exist a $p\times m$ matrix B with orthonormal columns, an $m\times m$ matrix D with positive numbers on diagonal, and an $n\times m$ matrix C with orthonormal columns so that $A = BDC^*$.

163. 7F.

164. **upper bound for** $||Tv||$**.** let $T \in L(V,W)$. let s_1 be the largest singular value for T. then $||Tv|| \leq s_1||v|| \forall v \in V$.

165. **norm of a linear map** $||\cdot||$**.** let $T \in L(V,W)$. then define norm of T has $||T|| = \max\{||Tv|| \mid v \in V, ||v|| \leq 1\}$.

166. **basis properties of norms of linear maps.** let $T \in L(V,W)$. then

    (a) $||T|| \geq 0$.
    (b) $||T|| = 0$ iff $T = 0$.
    (c) $||\lambda T|| = |\lambda|\cdot||T||$.
    (d) $||S+T|| \leq ||S|| + ||T|| \forall S \in L(V,W)$.

167. **alternative formulas for** $||T||$**.** let $T \in L(V,W)$. then

    (a) $||T||$ is the largest singular value of T.
    (b) $||T|| = \max\{||Tv|| \mid v \in V, ||v|| = 1\}$.
    (c) $||T||$ is the smallest number c so that $||Tv|| \leq c||v||$ for all $v \in V$.

168. **norm of adjoint.** let $T \in L(V,W)$. then $||T^*|| = ||T||$.

169. **best approximation by linear map whose range has dimension** $\leq k$**.** let $T \in L(V,W)$ and $s_1 \geq \cdots \geq s_m$ are the positive singular values of T. let $1 \leq k \leq m$. then $\min\{||T-S|| \mid S \in L(V,W), \dim\operatorname{range} S \leq k\} = s_{k+1}$. Also, if $Tv = s_1\langle v,e_1\rangle f_1 + \cdots + s_m\langle v,e_m\rangle f_m$ is a singular value decomposition of T and $T_k \in L(V,W)$ is defined by $T_k v = s_1\langle v,e_1\rangle f_1 + \cdots + s_k\langle v,e_k\rangle f_k$ for each $v \in V$, then $\dim\operatorname{range} T_k = k$ and $||T - T_k|| = s_{k+1}$.

170. **polar decomposition.** let $T \in L(V)$. then there exists a unitary operator $S \in L(V)$ so that $T = S\sqrt{T^*T}$.

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171. RIBET DEFS MT2.

172. **Double Dual.** Let V be a finite-dimensional vector space with dual V'. Then the double dual of V is $(V')' = V'' = V$. Also, $\dim V = n = \dim V' = \dim V''$.

173. **Eigenspace.** Let $T \in \mathscr{L}(V)$ and take λ to be an eigenvalue of T. Then, $E(\lambda,T) = \{v \in V \mid Tv = \lambda v\} \neq \emptyset$ is written as V_λ and is called the eigenspace of λ, which is a subspace of V.

174. **Generalized Eigenvector.** Consider a minimal polynomial $(x-\lambda_1)^{e_1}\cdot\cdots\cdot(x-\lambda_m)^{e_m}$ on X with $(T-\lambda_1 I)^{e_1}v = 0$. Then, v is called a generalized eigenvector for $\lambda = \lambda_1$.

175. **Characteristic polynomial.** The characteristic polynomial of $T: V \to V$ (with eigenvalues $\lambda_1,\dots,\lambda_t$) is the polynomial $\prod_{i=1}^t (x-\lambda_i)^{\dim X_i}$, where $V = X_1 \oplus \cdots \oplus X_t$.

176. **Simultaneously diagonalizable.** Operators S and T on V are simulatenously diagonalizable if there is a basis of V that consts of vectors that are eigenvectors for both S and T (i.e. there exists a basis $v_1,\dots,v_n$ of V so that for i, $1 \leq i \leq n$, there are λ_i and μ_i so that $Sv_i = \lambda_i v_i$ and $Tv_i = \mu_i v_i$).

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177. RIBET THMS MT1.

178. **Theorem.** The intersection of a family of subspaces of a vector space V is a subspace of V.

179. **Theorem.** Every subspace X of V has complement.

180. **Prop.** Let X,Y be subspaces of a finite-dimensional vector space V. Then:

    (a) $\dim X + \dim Y = \dim V$.
    (b) $X \cap Y = \{0\}$.

    Then, $V = X \oplus Y$.

181. **Prop.** $\dim(X \oplus Y) = \dim(X \times Y) = \dim X + \dim Y$.

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182. RIBET THMS MT2

183. **Theorem.** Every linear functional on a subspace of V can be extended to V.

184. **Note.** Annihilator is the dual of the quotient subspace.

185. **Cor.** The annihilator of U is $\{0\}$ iff $U = V$. The annihilator of U is V iff $U = \{0\}$.

186. **Prop.** If $T: V \to W$ is a linear map, then the null space of T' is the annihilator of the range of T. We have $\operatorname{ann}(\operatorname{range} T) = \{\psi : W \to F \mid \phi(Tv) = 0 \text{ for all } v \in V, T'(\psi)(v) = 0, T'\psi = 0, \phi \in \operatorname{nul}(T')\}$.

187. **Cor.** If $T: V \to W$ is a linear map between finite-dimensional F-vector spaces, then $\dim\operatorname{nul}(T') = \dim\operatorname{nul}(T) + \dim W - \dim V$.

188. **Cor.** The linear map T is onto iff T' is 1-1.

189. **Cor.** If $T: V \to W$ is a linear map between finite-dimensional vector spaces, then T' and T have equal ranks.

190. **Cor.** We have range$T = (\operatorname{nul} T)^0$.

191. **Theorem.** Let F be a finite field with $q = |F|$. Then, $a^q = a$ for all $a \in F$.

192. **Theorem.** If F is a finite field, then $|F| = p^n$ for some prime p and integer $n \geq 1$.

193. **Theorem.** Let $T: V \to V$, V finite-dimensional, and let $\alpha: F[x] \to \mathscr{L}(V)$, with $f \mapsto f(T)$. Also, we have $\ker\alpha$ to be the principal ideal $(m(x))$. Then, $m(x)$ is the minimal polynomial of T and has degree $\leq n^2$.

194. **Cayley-Hamilton Theorem.** Let $T: V \to V$, V finite-dimensional, and let $\alpha: F[x] \to \mathscr{L}(V)$, with $f \mapsto f(T)$. Also, we have $\ker\alpha$ to be the principal ideal $(m(x))$, where $m(x)$ is the minimal polynomial of T. Then, the characteristic polynomial is in $\ker\alpha$; that is, we can plug in the matrix for T into its characteristic polynomial and we get that it is equal to the 0-matrix.

195. **Lemma.** Let $f \in \mathbb{R}[x]$ be a real polynomial. If λ is a complex root of f, so is $\overline{\lambda}$, which is the complex conjugate of λ.

196. **Cor.** Let $\lambda_1,\dots,\lambda_t$ be distinct eigenvalues and take $E_i = E(\lambda_i,T) = \{v \in V \mid Tv = \lambda_i v\} \subseteq V$. Now, take $E_1 \times \cdots \times E_t$. Then there exists a summation map $E_1 \times \cdots \times E_t \xrightarrow{\text{sum}} V$ with $(v_1,\dots,v_t) \mapsto v_1 + \cdots + v_t$. Then, the sum map is 1-1.

197. **Cor.** Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

198. **Prop.** Suppose T is an operator on an F-vector space V. If $f \in F[x]$ is a polynomial satisfied by T (meaning $f(T) = 0$), then every eigenvalue of T on V is a root of f.

199. **Cor.** Suppose λ is an eigenvalue of operator T on a finite-dimensional F-vector space. Then λ is a root of the minimal polynomial of T iff λ is an eigenvalue of T.

200. **Theorem.** All operators on a nonzero finite-dimensional vector space over an algebraically closed field have at least one eigenvalue.

201. **Prop.** Assume that $F = \mathbb{R}$ and that $f(x) := x^2 + bx + c$ is an irreducible polynomial. If $T \in \mathscr{L}(V)$ and V is finite-dimensional, then the null space of $f(T)$ is even-dimensional.

202. **Prop (honors version).** Let T be an operator on a finite-dimensional vector space over F. If p is an irreducible polynomial over F, then the dimension of the null space of $p(T)$ is a multiple of the degree of p.

203. **Prop.** $F[x]/(p)$ (where p is irreducible) is a field.

204. **Formula.** $\dim_F V = [K:F]\cdot\dim_K V = \dim_F K \cdot \dim_K V$.

205. **Cor.** Every operator on an odd-dimensional $\mathbb{R}$-vector space has an eigenvalue.

206. **Prop.** If T is an operator on a finite-dimensional F-vector space, then the minimal polynomial of T has degree at most $\dim V$.

207. **Prop.** If T is upper-triangular with respect to some basis of V, and if the diagonal entries of an upper-triangular matrix representation of T are $\lambda_1,\dots,\lambda_n$, then $(T-\lambda_1 I)\cdot\cdots\cdot(T-\lambda_n I) = 0$.

208. **Prop.** Let V be a finite-dimensional vector space and $T \in \mathscr{L}(V)$ and let $\lambda_1,\dots,\lambda_m$ be the eigenvalues of T. Then, $V = \oplus E(\lambda_i,T)$ iff T is diagonalizable.

209. **Prop.** TFAE.

    (a) T is diagonalizable.
    (b) V has a basis consisting of eigenvectors.
    (c) The direct sum $\bigoplus_i V_{\lambda_i}$ is all of V.
    (d) $\dim\left(\bigoplus_i V_{\lambda_i}\right) = \dim V$.

210. **Prop.** If $T: V \to V$ has $\dim V$ different eigenvalues, then T is diagonalizable.

211. **Jordan Canonical Form.** X can be written as a direct sum of Jordan blocks, where $\sum$ dim(block) = $\dim X$.

212. **Lemma.** Let $X = \oplus\operatorname{span}(U_i v)$ for $i \in \{0,\dots,k_1\}$. If Z is a subspace of X' that is U'-invariant, then $\operatorname{ann}(Z) =: Y$ is U-invariant.

213. **Lemma.** Suppose S and T are commuting operators on V. If λ is an eigenvalue for T on V, then the eigenspace $E(\lambda,T)$ is S-invariant.

214. **Theorem.** The diagonalize operatosr on the same finite-dimensional vector space are simulateneously diagonalizable iff they commute with each other.

215. **Theorem.** Every pair of commuting operators on a finite-dimensional nonzero complex vector spcae has a common eigenvector.

216. **Prop.** Two commuting operators on a finite-dimensional nonzero complex vector space can be simultaneously upper-triangularized.

217. **Prop.** We have:

    (a) Every eigenvalue of $S+T$ is the sum of an eigenvalue of S and an eigenvalue of T.
    (b) Every eigenvalue of ST is the product of an eigenvalue of S and an eigenvalue of T.
