Math H110 Definitions.

1. **Endomorphism.** An endomorphism is a group homomorphism from a set to itself (NOTE: does not have to be invertible.)

2. **End V.** The symbol $\mathrm{End}\,V$ is the set of all endomorphisms on V (and multiplication on $\mathrm{End}\,V$ is defined to be function composition).

3. **F-Module.** An $F-$module is a generalization of vector spaces over rings.

4. **Subspace.** Let V be a vector space. X is a subspace of V if $X \subseteq V$ and closed under all relevant operations of V, $X \neq \emptyset$, and $X \ni 0$.

5. **Linear Map / Linear Transformation.** Let V be a vector space over a field F with $v, w \in V$. Let T be a map on V with $T(v+w) = T(v) + T(w)$ and $T(\lambda v) = \lambda T(v)$ for all $\lambda \in F$. Then, T is called a linear map or linear transformation.

6. **Linear Operator.** If T is a linear transformation on a vector spaces V with $T : V \to V$, then T is linear operator on V.

7. **Spans.** The list $v_1, \ldots, v_n$ spans V iff $T : F^n \to V$ is onto.

8. **Linearly Independent.** The list $v_1, \ldots, v_n$ is linearly independent iff $T : F^n \to V$ is 1-1. Equivalently, the list $v_1, \ldots, v_n$ is linearly independent if $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$ implies $\lambda_i = 0$ for all i.

9. **Linearly Dependent.** The list $v_1, \ldots, v_n$ is linearly dependent iff $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$ implies $\lambda_i \neq 0$ for some i.

10. **Basis.** The list $v_1, \ldots, v_n$ is a basis of V if $\mathrm{span}\{v_1, \ldots, v_n\} = V$ and $v_1, \ldots, v_n$ is linearly independent.

11. **Finite-dimensional.** V is finite-dimensional if V is spanned by a finite list of vectors.

12. **Sum of Subspaces.** Let $X_1, \ldots, X_t$ be subspaces of V. Then, we define their sum as $X_1 + \cdots + X_t = \{x_1 + \cdots + x_t \mid x_1 \in X_1, \ldots, x_t \in X_t\}$.

13. **Direct Sum of Subspaces.** Let $X_1, \ldots, X_t$ be subspaces of V. Then, their direct sum, $X_1 \oplus \cdots \oplus X_t$, is given by a 1-1 linear map T, with $T : X_1 \times \cdots \times X_t \to V$.

14. **Complement of Subspace.** Let X, Y be subspaces of of V. Then, Y is a complementary subspace of X iff $X + Y = V$ and $X + Y = X \oplus Y$.

15. **Rank, Nullity.** The rank of a linear map is the dimension of the range of the linear map. The nullity is the dimension of the null space of the linear map.

16. **Null Space.** The null space is the set of vectors that are mapped to 0.

17. **Isomorphic Vector Spaces.** Two vector spaces V, W are isomorphic if there exists a linear map $T : V \to W$ that is 1-1 and onto.

18. **Quotient Space.** Suppose U is a subspace of V. Then, the quotient space V/U is the set $V/U = \{v + U \mid v \in V\}$.

19. **Column Rank.** The column rank (rank of the column span of a matrix) is defined to be rankT_A.

20. **Conjugation.** Let A be an $n \times n$ matrix (over F) and let Q be an $n \times n$ matrix (over F). Then, the conjugation of A by Q is $Q^{-1}AQ$.

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21. **Dual Space.** Let V be an F-vector space. Then the dual space of V is $V' = \mathscr{L}(V, F)$ where the elements of V' are called linear functionals.

22. **Annihilator.** For a subspace $U \subseteq V$, we define the annihilator of U to be $U_0 = \{\phi \in V' \mid \phi(u) = 0 \forall u \in U\}$.

23. **Double Dual.** Let V be a finite-dimensional vector space with dual V'. Then the double dual of V is $(V')' = V'' = V$. Also, $\dim V = n = \dim V' = \dim V''$.

24. **Eigenvector / eigenvalue.** Let $T \in \mathscr{L}(V)$. Then an eigenvector of T is a $v \in V$ $(v \neq 0)$ such that $Tv = \lambda v$ ($\lambda \in F$ is called an eigenvalue), and v is an eigenvector of T.

25. **Eigenspace.** Let $T \in \mathscr{L}(V)$ and take λ to be an eigenvalue of T. Then, $E(\lambda, T) = \{v \in V \mid Tv = \lambda v\} \neq \emptyset$ is written as V_λ and is called the eigenspace of λ, which is a subspace of V.

26. **Invariant subspace.** E is a T-invariant subspace if $T \in \mathscr{L}(V)$ with $T(E) \subseteq E$.

27. textbfIdempotent. If $e = e^2$, then e is called idempotent.

28. **Generalized Eigenvector.** Consider a minimal polynomial $(x - \lambda_1)^{e_1} \cdot \cdots \cdot (x - \lambda_m)^{e_m}$ on X with $(T - \lambda_1 I)^{e_1} v = 0$. Then, v is called a generalized eigenvector for $\lambda = \lambda_1$.

29. **Characteristic polynomial.** The characteristic polynomial of $T : V \to V$ (with eigenvalues $\lambda_1, \dots, \lambda_t$) is the polynomial $\prod_{i=1}^{t}(x - \lambda_i)^{\dim X_i}$, where $V = X_1 \oplus \cdots \oplus X_t$.

30. **Simultaneously diagonalizable.** Operators S and T on V are simulatenously diagonalizable if there is a basis of V that consts of vectors that are eigenvectors for both S and T (i.e. there exists a basis $v_1, \dots, v_n$ of V so that for i, $1 \leq i \leq n$, there are λ_i and μ_i so that $Sv_i = \lambda_i v_i$ and $Tv_i = \mu_i v_i$).

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31. **Inner product.** Let V be a vector space over $\mathbb{R}$, possibly infinite-dim. An inner product on V is a bilinear functon $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that $\langle x, x \rangle \geq 0$ for all $x \in V$ and $\langle x, x \rangle = 0$ iff $x = 0$.

32. **Complex Inner product.** Let $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ be in $\mathbb{C}^n$. We have $\langle z, w \rangle = \sum_i z_i \overline{w_i}$, with $\langle w, z \rangle = \overline{\langle z, w \rangle}$ and $\langle \alpha z, w \rangle = \alpha \langle z, w \rangle$ and $\langle w, \alpha z \rangle = \overline{\alpha} \langle z, w \rangle$.

33. **Norm.** Norm of $v \in V$ is $||v|| = \sqrt{\langle v, v \rangle}$.

34. **Orthogonal.** Two vectors are orthogonal if $\langle v, w \rangle = 0$, where $v, w \in V$.

35. **Orthogonal Complement.** Take U to be a subset of V. Then, $U^\perp = \{v \in V \mid \langle v, u \rangle = 0 \forall u \in U\} = \{v \in V \mid \langle u, v \rangle = 0 \forall u \in U\} = \{v \in V \mid \phi_v = 0 \text{ in } U\} = \{v \in V \mid \phi_v \in U^0\}$.

36. **Adjoint.** Let $T : V \to W$. Then the adjoint of T is $T^* : W \to V$ such that $T^* = \alpha_V^{-1} \circ T' \circ \alpha_W$, where $\alpha_V : V \to V'$ and $\alpha_W : W \to W'$.

37. **Self-adjoint.** An operator $T : V \to V$ is self-adjoint if $T = T^*$.

38. **Symmetric.** T is symmetric if $M(T) = M(T)^t$.

39. **Normal.** T is a normal operator if $TT^* = T^*T$.

40. **Alternating.** A bilinear form $\psi : V \times V \to F$ is alternating if $psi(v, v) = 0$. for $v \in V$.

41. **Anti-symmetric.** Let $\psi : V \times V \to F$ be a bilinear form. Then ψ is anti-symmetric if $\psi(x, y) = -\psi(y, x)$.

42. **Positive operator.** T is a positive operator on an inner product space if $T = T^*$ and $\langle Tv, v \rangle \geq 0$.

43. **Square root of an operator.** A square root of an operator T is an operator R such that $R^2 = T$.

44. **Isometry.** An operator $S \in L(V)$ is an isometry if it preserves distance, i.e. $||Sv|| = ||v|| \forall v \in V$, i.e. $S^* = S^{-1}$. In the real case, they are called orthogonal operators. In the complex case, they are called unitary operators.