Math H110 Theorems.

1. **Lemma.** Let $F$ be a field, $\lambda \in F$, $V$ a vector space over $F$ (denoted by $V/F$), $v \in V$. Then, if $\lambda v = 0$, then $\lambda = 0$ or $v = 0$.

2. **Lemma.** A vector space over a field is a module over a field.

3. **Theorem.** The intersection of a family of subspaces of a vector space $V$ is a subspace of $V$.

4. **Lemma.** Let $S = \{v_1, \ldots, v_t\}$. Then the subspace of all linear combinations of the elements of $S$ is the span$S$.

5. **Theorem.** Let $L = v_1, \ldots, v_n$ be a list of vectors in a vector space $V$ over a field $F$ and let $T : F^n :\to V$ be linear transformation with $(\lambda_1, \ldots, \lambda_n) \mapsto \lambda_1 v_1 + \cdots + \lambda_n v_n$. Then, we have the following:

   (a) $L$ spans $V$ iff $T$ is onto.

   (b) $L$ is linearly independent iff $T$ is 1-1 iff $\operatorname{nul} T = \{0\}$.

   (c) $L$ is a basis iff $T$ is 1-1 and onto.

6. **Prop.** Consider $T : F^n \to V$ with $(\lambda_1, \ldots, \lambda_n) \mapsto \lambda_1 v_1 + \cdots + \lambda_n v_n$, so $T(e_i) = v_i$ for all $i$. Then, $T$ is the unique linear map $F_n \to V$ that sends $e_i \mapsto v_i$ for all $i$.

7. **Theorem.** Every subspace $X$ of $V$ has complement.

8. **Lemma.** If $v_1, \ldots, v_t$ is linearly dependent list, then there is an index $k$ such that $v_k \in \operatorname{span}(v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_t)$. Furthermore, the span of the list of length $t - 1$ gotten by removing $v_k$ from the list is the same as the span of the original list.

9. **Prop.** In a finite-dimensional vector space, the length is of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

10. **Cor.** Two bases of $V$ have the same number of elements.

11. **Prop.** $X + Y$ is direct iff the null space of the sum map is $\{0\}$.

12. **Theorem.** Every subspace of a finite-dimensional vector space is finite-dimensional.

13. **Prop.** Every spanning list for a vector space can be pruned down to a basis of the space.

14. **Cor.** Every finite-dimensional vector space has a basis.

15. **Prop.** In a finite-dimensional vector space, every linearly independent list can be extended to a basis of the space.

16. **Major Theorem.** Every subspace of a finite-dimensional vector space has a complement.

17. **Prop.** Let $X, Y$ be subspaces of a finite-dimensional vector space $V$. Then:

    (a) $\dim X + \dim Y = \dim V$.

    (b) $X \cap Y = \{0\}$.

    Then, $V = X \oplus Y$.

18. **Prop.** $\dim(X \oplus Y) = \dim X + \dim Y$.

19. **Prop.** If $V$ is a finite-dimensional vector space (with $\dim V = n$), then every subspace has dimension at most $n$.

20. **Prop.** Let $\dim V = n$. Then, a linearly independent list of vectors of $V$ with length $n$ is a basis for $V$.

21. **Prop.** Let $\dim V = n$. Then, every spanning list for $V$ of length $n$ is a basis for $V$.

22. **Lemma.** The list $(x_1, 0), \ldots, (x_t, 0); (0, y_1), \ldots, (0, y_k)$ of length $t + k$ is a basis of $X \times Y$.

23. **Cor.** $\dim(X \times Y) = \dim X + \dim Y$.

24. **Cor.** Let $T : V \to W$ be a linear map with $\dim V = d$. Then, $\text{rank} T \leq d$.

25. **Rank-Nullity Theorem.** $\dim V = \text{rank} V + \text{nullity} V$.

26. **Prop.** If $T : V \to W$ is 1-1, then $\text{nullity} T = 0$.

27. **Cor.** If $T : V \to W$ is 1-1 and onto, then $\dim V = \dim W$.

28. **Theorem.** The set of linear maps $V \to W$ is a vector space $L \cdot (F^n, W) \to T \longrightarrow (Te_1, \ldots, Te_n) \in W^n$.

29. **Theorem.** $\dim(X + Y) = \dim X + \dim Y - \dim(X \cap Y)$.

30. **Cor.** $\dim(V/X) = \dim V - \dim X$.

31. **Theorem.** If $A$ is a rectangular matrix with elements in a field $F$, then row rank $A$ = column rank $A$.

32. **Prop.** Let $T : V \to W$ be 1-1. Then, $\dim W \geq \dim V$.

33. **Prop.** Let $T : V \to W$ be onto. Then, $\dim V \geq \dim W$.

34. **Prop.** Let $T : V \to W$ and $\dim V = \dim W$. Then, $T$ 1-1 iff $T$ onto iff $T$ bijective iff $T$ invertible.

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35. **Lemma.** Let $V$ be a finite-dimensional vector space and $U$ a subspace of $V$. Then, $\dim U_0 = \dim V - \dim U$.

36. **Theorem.** Every linear functional on a subspace of $V$ can be extended to $V$.

37. **Note.** Annihilator is the dual of the quotient subspace.

38. **Theorem.** Let $T : V \to W$ and $T' : W' \to V'$. THen $\mathscr{M}(T)$ and $\mathscr{M}(T')$ are transposes of each other.

39. **Lemma.** $U^0$ has dimension $\dim V - \dim U$.

40. **Cor.** The annihilator of $U$ is $\{0\}$ iff $U = V$. The annihilator of $U$ is $V$ iff $U = \{0\}$.

41. **Prop.** If $T : V \to W$ is a linear map, then the null space of $T'$ is the annihilator of the range of $T$. We have $\operatorname{ann}(\operatorname{range} T) = \{\psi : W \to F \mid \phi(Tv) = 0 \text{ for all } v \in V, T'(\psi)(v) = 0, T'\psi = 0, \phi \in \operatorname{nul}(T')\}$.

42. **Cor.** If $T : V \to W$ is a linear map between finite-dimensional $F$-vector spaces, then $\dim \operatorname{nul}(T') = \dim \operatorname{nul}(T) + \dim W - \dim V$.

43. **Cor.** The linear map $T$ is onto iff $T'$ is 1-1.

44. **Cor.** If $T : V \to W$ is a linear map between finite-dimensional vector spaces, then $T'$ and $T$ have equal ranks.

45. **Cor.** We have $\operatorname{range} T = (\operatorname{nul} T)^0$.

46. **Theorem.** Let $F$ be a finite field with $q = |F|$. Then, $a^q = a$ for all $a \in F$.

47. **Theorem.** If $F$ is a finite field, then $|F| = p^n$ for some prime $p$ and integer $n \geq 1$.

48. **Theorem.** Take an ideal $I$ in $\mathbb{Z}$. Then, $I$ is equal to either $\{0\}$ or $m\mathbb{Z}$ (where $m \in \mathbb{Z}_{>0}$).

49. **Theorem.** $F[x]$ is a principal ideal domain; that is, it is an integral domain in which every ideal in $F[x]$ is principal.

50. **Theorem.** Let $T : V \to V$, $V$ finite-dimensional, and let $\alpha : F[x] \to \mathscr{L}(V)$, with $f \mapsto f(T)$. Also, we have $\ker \alpha$ to be the principal ideal $(m(x))$. Then, $m(x)$ is the minimal polynomial of $T$ and has degree $\leq n^2$.

51. **Cayley-Hamilton Theorem.** Let $T : V \to V$, $V$ finite-dimensional, and let $\alpha : F[x] \to \mathscr{L}(V)$, with $f \mapsto f(T)$. Also, we have $\ker \alpha$ to be the principal ideal $(m(x))$, where $m(x)$ is the minimal polynomial of $T$. Then, the characteristic polynomial is in $\ker \alpha$; that is, we can plug in the matrix for $T$ into its characteristic polynomial and we get that it is equal to the 0-matrix.

52. **Prop.** For $f(x) \in F[x]$ and $\lambda \in F$, $f(\lambda) = 0$ iff $f$ is divisible by $x - \lambda$, where $x - \lambda$ is an irreducible polynomial.

53. **Cor.** A polynomial of degree $n$ can have at most $n$ roots.

54. **Cor.** A polynomial with infinitely many roots is identically the zero polynomial.

55. **Lemma.** Let $f \in \mathbb{R}[x]$ be a real polynomial. If $\lambda$ is a complex root of $f$, so is $\overline{\lambda}$, which is the complex conjugate of $\lambda$.

56. **Prop.** A scalar $\lambda$ is an eigenvalue of $T : V \to V$ iff $T - \lambda I$ is not 1-1.

57. **Cor.** The map $T : V \to V$ is invertible iff 0 is not an eigenvalue of $T$.

58. **Key lemma.** Every list of eigenvectors of $T$ that corresponds to distinct eigenvalues of $T$ is a linearly independent list.

59. **Cor.** Let $\lambda_1, \ldots, \lambda_t$ be distinct eigenvalues and take $E_i = E(\lambda_i, T) = \{v \in V \mid Tv = \lambda_i v\} \subseteq V$. Now, take $E_1 \times \cdots \times E_t$. Then there exists a summation map $E_1 \times \cdots \times E_t \xrightarrow{\text{sum}} V$ with $(v_1, \ldots, v_t) \mapsto v_1 + \cdots + v_t$. Then, the sum map is 1-1.

60. **Cor.** Suppose $V$ is finite-dimensional. Then each operator on $V$ has at most $\dim V$ distinct eigenvalues.

61. **Prop.** Suppose $T$ is an operator on an $F$-vector space $V$. If $f \in F[x]$ is a polynomial satisfied by $T$ (meaning $f(T) = 0$), then every eigenvalue of $T$ on $V$ is a root of $f$.

62. **Cor.** Suppose $\lambda$ is an eigenvalue of operator $T$ on a finite-dimensional $F$-vector space. Then $\lambda$ is a root of the minimal polynomial of $T$.

63. **Prop.** Let $T$ be an operator on a finite-dimensinoal vector space. Suppose $\lambda$ is a root of the minimal polynomial. Then $\lambda$ is an eigenvalue of $T$.

64. **Theorem.** All operators on a nonzero finite-dimensional vector space over an algebraically closed field have at least one eigenvalue.

65. **Prop.** Assume that $F = \mathbb{R}$ and that $f(x) := x^2 + bx + c$ is an irreducible polynomial. If $T \in \mathscr{L}(V)$ and $V$ is finite-dimensional, then the null space of $f(T)$ is even-dimensional.

66. **Prop (honors version).** Let $T$ be an operator on a finite-dimensional vector space over $F$. If $p$ is an irreducible polynomial over $F$, then the dimension of the null space of $p(T)$ is a multiple of the degree of $p$.

67. **Prop.** $F[x]/(p)$ (where $p$ is irreducible) is a field.

68. **Formula.** $\dim_F V = [K : F] \cdot \dim_K V = \dim_F K \cdot \dim_K V$.

69. **Cor.** Every operator on an odd-dimensional $\mathbb{R}$-vector space has an eigenvalue.

70. **Prop.** If $T$ is an operator on a finite-dimensional $F$-vector space, then the minimal polynomial of $T$ has degree at most $\dim V$.

71. **Prop.** If $T$ is upper-triangular with respect to some basis of $V$, and if the diagonal entries of an upper-triangular matrix representation of $T$ are $\lambda_1, \ldots, \lambda_n$, then $(T - \lambda_1 I) \cdot \cdots \cdot (T - \lambda_n I) = 0$.

72. **Prop.** Let $V$ be a finite-dimensional vector space and $T \in \mathscr{L}(V)$ and let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of $T$. Then, $V = \oplus E(\lambda_i, T)$ iff $T$ is diagonalizable.

73. **Prop.** TFAE.

    (a) $T$ is diagonalizable.

    (b) $V$ has a basis consisting of eigenvectors.

    (c) The direct sum $\underset{i}{\oplus} V_{\lambda_i}$ is all of $V$.

    (d) $\dim \left( \underset{i}{\oplus} V_{\lambda_i} \right) = \dim V$.

74. **Prop.** If $T : V \to V$ has $\dim V$ different eigenvalues, then $T$ is diagonalizable.

75. **Prop.** The operator $T : V \to V$ is diagonalizable iff its minimal polynomial splits completely as a product of distinct linear factors of the form $x - r$.

76. **Jordan Canonical Form.** $X$ can be written as a direct sum of Jordan blocks, where $\sum \dim(\text{block}) = \dim X$.

77. **Lemma.** Let $X = \oplus \text{span}(U_i v)$ for $i \in \{0, \ldots, k_1\}$. If $Z$ is a subspace of $X'$ that is $U'$-invariant, then $\text{ann}(Z) =: Y$ is $U$-invariant.

78. **Lemma.** Suppose $S$ and $T$ are commuting operators on $V$. If $\lambda$ is an eigenvalue for $T$ on $V$, then the eigenspace $E(\lambda, T)$ is $S$-invariant.

79. **Theorem.** The diagonalize operatosr on the same finite-dimensional vector space are simulateneously diagonalizable iff they commute with each other.

80. **Theorem.** Every pair of commuting operators on a finite-dimensional nonzero complex vector spcae has a common eigenvector.

81. **Prop.** Two commuting operators on a finite-dimensional nonzero complex vector space can be simultaneously upper-triangularized.

82. **Prop.** We have:

    (a) Every eigenvalue of $S+T$ is the sum of an eigenvalue of $S$ and an eigenvalue of $T$.

    (b) Every eigenvalue of $ST$ is the product of an eigenvalue of $S$ and an eigenvalue of $T$.

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83. **Formula.** $\langle v + w, v + w \rangle = \langle v, v \rangle + \langle w, w \rangle + 2\langle v, w \rangle$.

84. **Lemma.** If $u, v$ are orthogonal, then $||u + v||^2 = ||u||^2 + ||v||^2$.

85. **Lemma.** If $v \in V$ and $v \neq 0$, then every $u \in V$ is the sum of a multiple of $v$ and a vector orthogonal to $v$.

86. **Prop (Cauchy-Schwarz).** For $u, v \in V$, we have $|\langle u, v\rangle| \leq ||u|| \cdot ||v||$.