Math H110 Theorems.

1. **Lemma.** Let F be a field, $\lambda \in F$, V a vector space over F (denoted by V/F), $v \in V$. Then, if $\lambda v = 0$, then $\lambda = 0$ or $v = 0$.

2. **Lemma.** A vector space over a field is a module over a field.

3. **Theorem.** The intersection of a family of subspaces of a vector space V is a subspace of V.

4. **Lemma.** Let $S = \{v_1, \dots, v_t\}$. Then the subspace of all linear combinations of the elements of S is the span$S$.

5. **Theorem.** Let $L = v_1, \dots, v_n$ be a list of vectors in a vector space V over a field F and let $T : F^n :\to V$ be linear transformation with $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 v_1 + \cdots + \lambda_n v_n$. Then, we have the following:

   (a) L spans V iff T is onto.

   (b) L is linearly independent iff T is 1-1 iff $\operatorname{nul} T = \{0\}$.

   (c) L is a basis iff T is 1-1 and onto.

6. **Prop.** Consider $T : F^n \to V$ with $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 v_1 + \cdots + \lambda_n v_n$, so $T(e_i) = v_i$ for all i. Then, T is the unique linear map $F_n \to V$ that sends $e_i \mapsto v_i$ for all i.

7. **Theorem.** Every subspace X of V has complement.

8. **Lemma.** If $v_1, \dots, v_t$ is linearly dependent list, then there is an index k such that $v_k \in \operatorname{span}(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_t)$. Furthermore, the span of the list of length $t-1$ gotten by removing v_k from the list is the same as the span of the original list.

9. **Prop.** In a finite-dimensional vector space, the length is of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

10. **Cor.** Two bases of V have the same number of elements.

11. **Prop.** $X + Y$ is direct iff the null space of the sum map is $\{0\}$.

12. **Theorem.** Every subspace of a finite-dimensional vector space is finite-dimensional.

13. **Prop.** Every spanning list for a vector space can be pruned down to a basis of the space.

14. **Cor.** Every finite-dimensional vector space has a basis.

15. **Prop.** In a finite-dimensional vector space, every linearly independent list can be extended to a basis of the space.

16. **Major Theorem.** Every subspace of a finite-dimensional vector space has a complement.

17. **Prop.** Let X, Y be subspaces of a finite-dimensional vector space V. Then:

    (a) $\dim X + \dim Y = \dim V$.

    (b) $X \cap Y = \{0\}$.

    Then, $V = X \oplus Y$.

18. **Prop.** $\dim(X \oplus Y) = \dim X + \dim Y$.

19. **Prop.** If V is a finite-dimensional vector space (with $\dim V = n$), then every subspace has dimension at most n.

20. **Prop.** Let $\dim V = n$. Then, a linearly independent list of vectors of V with length n is a basis for V.

21. **Prop.** Let $\dim V = n$. Then, every spanning list for V of length n is a basis for V.

22. **Lemma.** The list $(x_1, 0), \ldots, (x_t, 0); (0, y_1), \ldots, (0, y_k)$ of length $t + k$ is a basis of $X \times Y$.

23. **Cor.** $\dim(X \times Y) = \dim X + \dim Y$.

24. **Cor.** Let $T : V \to W$ be a linear map with $\dim V = d$. Then, $\text{rank}T \leq d$.

25. **Rank-Nullity Theorem.** $\dim V = \text{rank}V + \text{nullity}V$.

26. **Prop.** If $T : V \to W$ is 1-1, then $\text{nullity}T = 0$.

27. **Cor.** If $T : V \to W$ is 1-1 and onto, then $\dim V = \dim W$.

28. **Theorem.** The set of linear maps $V \to W$ is a vector space $L \cdot (F^n, W) \to T \longrightarrow (Te_1, \ldots, Te_n) \in W^n$.

29. **Theorem.** $\dim(X + Y) = \dim X + \dim Y - \dim(X \cap Y)$.

30. **Cor.** $\dim(V/X) = \dim V - \dim X$.

31. **Theorem.** If A is a rectangular matrix with elements in a field F, then row rank A = column rank A.

32. **Prop.** Let $T : V \to W$ be 1-1. Then, $\dim W \geq \dim V$.

33. **Prop.** Let $T : V \to W$ be onto. Then, $\dim V \geq \dim W$.

34. **Prop.** Let $T : V \to W$ and $\dim V = \dim W$. Then, T 1-1 iff T onto iff T bijective iff T invertible.

---

35. **Lemma.** Let V be a finite-dimensional vector space and U a subspace of V. Then, $\dim U_0 = \dim V - \dim U$.

36. **Theorem.** Every linear functional on a subspace of V can be extended to V.

37. **Note.** Annihilator is the dual of the quotient subspace.

38. **Theorem.** Let $T : V \to W$ and $T' : W' \to V'$. THen $\mathscr{M}(T)$ and $\mathscr{M}(T')$ are transposes of each other.

39. **Lemma.** U^0 has dimension $\dim V - \dim U$.

40. **Cor.** The annihilator of U is $\{0\}$ iff $U = V$. The annihilator of U is V iff $U = \{0\}$.

41. **Prop.** If $T : V \to W$ is a linear map, then the null space of T' is the annihilator of the range of T. We have $\text{ann}(\text{range}T) = \{\psi : W \to F \mid \phi(Tv) = 0 \text{ for all } v \in V, T'(\psi)(v) = 0, T'\psi = 0, \phi \in \text{nul}(T')\}$.

42. **Cor.** If $T : V \to W$ is a linear map between finite-dimensional F-vector spaces, then $\dim \text{nul}(T') = \dim \text{nul}(T) + \dim W - \dim V$.

43. **Cor.** The linear map T is onto iff T' is 1-1.

44. **Cor.** If $T : V \to W$ is a linear map between finite-dimensional vector spaces, then T' and T have equal ranks.

45. **Cor.** We have $\text{range}T = (\text{nul}\, T)^0$.

46. **Theorem.** Let F be a finite field with $q = |F|$. Then, $a^q = a$ for all $a \in F$.

47. **Theorem.** If F is a finite field, then $|F| = p^n$ for some prime p and integer $n \geq 1$.

48. **Theorem.** Take an ideal I in $\mathbb{Z}$. Then, I is equal to either $\{0\}$ or $m\mathbb{Z}$ (where $m \in \mathbb{Z}_{>0}$).

49. **Theorem.** $F[x]$ is a principal ideal domain; that is, it is an integral domain in which every ideal in $F[x]$ is principal.

50. **Theorem.** Let $T : V \to V$, V finite-dimensional, and let $\alpha : F[x] \to \mathscr{L}(V)$, with $f \mapsto f(T)$. Also, we have $\ker \alpha$ to be the principal ideal $(m(x))$. Then, $m(x)$ is the minimal polynomial of T and has degree $\leq n^2$.

51. **Cayley-Hamilton Theorem.** Let $T : V \to V$, V finite-dimensional, and let $\alpha : F[x] \to \mathscr{L}(V)$, with $f \mapsto f(T)$. Also, we have $\ker \alpha$ to be the principal ideal $(m(x))$, where $m(x)$ is the minimal polynomial of T. Then, the characteristic polynomial is in $\ker \alpha$; that is, we can plug in the matrix for T into its characteristic polynomial and we get that it is equal to the 0-matrix.

52. **Prop.** For $f(x) \in F[x]$ and $\lambda \in F$, $f(\lambda) = 0$ iff f is divisible by $x - \lambda$, where $x - \lambda$ is an irreducible polynomial.

53. **Cor.** A polynomial of degree n can have at most n roots.

54. **Cor.** A polynomial with infinitely many roots is identically the zero polynomial.

55. **Lemma.** Let $f \in \mathbb{R}[x]$ be a real polynomial. If λ is a complex root of f, so is $\overline{\lambda}$, which is the complex conjugate of λ.

56. **Prop.** A scalar λ is an eigenvalue of $T : V \to V$ iff $T - \lambda I$ is not 1-1.

57. **Cor.** The map $T : V \to V$ is invertible iff 0 is not an eigenvalue of T.

58. **Key lemma.** Every list of eigenvectors of T that corresponds to distinct eigenvalues of T is a linearly independent list.

59. **Cor.** Let $\lambda_1, \dots, \lambda_t$ be distinct eigenvalues and take $E_i = E(\lambda_i, T) = \{v \in V \mid Tv = \lambda_i v\} \subseteq V$. Now, take $E_1 \times \cdots \times E_t$. Then there exists a summation map $E_1 \times \cdots \times E_t \xrightarrow{\text{sum}} V$ with $(v_1, \dots, v_t) \mapsto v_1 + \cdots + v_t$. Then, the sum map is 1-1.

60. **Cor.** Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

61. **Prop.** Suppose T is an operator on an F-vector space V. If $f \in F[x]$ is a polynomial satisfied by T (meaning $f(T) = 0$), then every eigenvalue of T on V is a root of f.

62. **Cor.** Suppose λ is an eigenvalue of operator T on a finite-dimensional F-vector space. Then λ is a root of the minimal polynomial of T.

63. **Prop.** Let T be an operator on a finite-dimensinoal vector space. Suppose λ is a root of the minimal polynomial. Then λ is an eigenvalue of T.

64. **Theorem.** All operators on a nonzero finite-dimensional vector space over an algebraically closed field have at least one eigenvalue.

65. **Prop.** Assume that $F = \mathbb{R}$ and that $f(x) := x^2 + bx + c$ is an irreducible polynomial. If $T \in \mathscr{L}(V)$ and V is finite-dimensional, then the null space of $f(T)$ is even-dimensional.

66. **Prop (honors version).** Let T be an operator on a finite-dimensional vector space over F. If p is an irreducible polynomial over F, then the dimension of the null space of $p(T)$ is a multiple of the degree of p.

67. **Prop.** $F[x]/(p)$ (where p is irreducible) is a field.

68. **Formula.** $\dim_F V = [K : F] \cdot \dim_K V = \dim_F K \cdot \dim_K V$.

69. **Cor.** Every operator on an odd-dimensional $\mathbb{R}$-vector space has an eigenvalue.

70. **Prop.** If T is an operator on a finite-dimensional F-vector space, then the minimal polynomial of T has degree at most $\dim V$.

71. **Prop.** If T is upper-triangular with respect to some basis of V, and if the diagonal entries of an upper-triangular matrix representation of T are $\lambda_1, \dots, \lambda_n$, then $(T - \lambda_1 I) \cdot \dots \cdot (T - \lambda_n I) = 0$.

72. **Prop.** Let V be a finite-dimensional vector space and $T \in \mathscr{L}(V)$ and let $\lambda_1, \dots, \lambda_m$ be the eigenvalues of T. Then, $V = \oplus E(\lambda_i, T)$ iff T is diagonalizable.

73. **Prop.** TFAE.

    (a) T is diagonalizable.

    (b) V has a basis consisting of eigenvectors.

    (c) The direct sum $\bigoplus\limits_i V_{\lambda_i}$ is all of V.

    (d) $\dim\left(\bigoplus\limits_i V_{\lambda_i}\right) = \dim V$.

74. **Prop.** If $T : V \to V$ has $\dim V$ different eigenvalues, then T is diagonalizable.

75. **Prop.** The operator $T : V \to V$ is diagonalizable iff its minimal polynomial splits completely as a product of distinct linear factors of the form $x - r$.

76. **Jordan Canonical Form.** X can be written as a direct sum of Jordan blocks, where $\sum \dim(\text{block}) = \dim X$.

77. **Lemma.** Let $X = \oplus \operatorname{span}(U_i v)$ for $i \in \{0, \dots, k_1\}$. If Z is a subspace of X' that is U'-invariant, then $\operatorname{ann}(Z) =: Y$ is U-invariant.

78. **Lemma.** Suppose S and T are commuting operators on V. If λ is an eigenvalue for T on V, then the eigenspace $E(\lambda, T)$ is S-invariant.

79. **Theorem.** The diagonalize operatosr on the same finite-dimensional vector space are simulateneously diagonalizable iff they commute with each other.

80. **Theorem.** Every pair of commuting operators on a finite-dimensional nonzero complex vector spcae has a common eigenvector.

81. **Prop.** Two commuting operators on a finite-dimensional nonzero complex vector space can be simultaneously upper-triangularized.

82. **Prop.** We have:

    (a) Every eigenvalue of $S+T$ is the sum of an eigenvalue of S and an eigenvalue of T.

    (b) Every eigenvalue of ST is the product of an eigenvalue of S and an eigenvalue of T.

---

83. **Formula.** $\langle v + w, v + w \rangle = \langle v, v \rangle + \langle w, w \rangle + 2\langle v, w \rangle$.

84. **Lemma.** If u, v are orthogonal, then $||u + v||^2 = ||u||^2 + ||v||^2$.

85. **Lemma.** If $v \in V$ and $v \neq 0$, then every $u \in V$ is the sum of a multiple of v and a vector orthogonal to v.

86. **Prop (Cauchy-Schwarz).** For $u, v \in V$, we have $|\langle u, v\rangle| \leq ||u|| \cdot ||v||$.

87. **Prop (Triangle inequality).** For $u, v \in V$, $||u + v|| \leq ||u|| + ||v||$.

88. **Prop.** The triangle inequality is equality iff one of u, v is a nonnegative (real) multiple of the other.

89. **Prop.** Let $\alpha : V \to V'$ with $v \mapsto \phi_v$, where $\phi_v : V \to F$ such that $\phi_v(x) = \langle x, v\rangle$. Then, $\alpha(\lambda v) = \overline{\lambda}(\alpha(v))$ for $\lambda \in F$.

90. **Prop.** If V is finite-dim then $\alpha : V \to V'$ is an invertible linear map of $\mathbb{R}$-vector spaces. It is an isomorphism of F-vector spaces if $F = \mathbb{R}$ and a congugate-linear bijection if $F = \mathbb{C}$.

91. **Riesz Representation Theorem.** Let V be a finite-dim inner product space over F (which is $\mathbb{R}$ or $\mathbb{C}$). If ϕ is a linear functional on V, there is a unique $v \in V$ such that $\phi(x) = \langle x, v\rangle$ for all $x \in V$.

92. **Prop.** An orthogonal list that consists of nonzero vectors is linearly independent.

93. **Cor.** An orthonormal list is linearly independent.

94. **Prop.** If $v_1, \ldots, v_m$ is orthonormal and $a_1, \ldots, a_m$ are elements of F, then $||a_1v_1 + \cdots + a_mv_m||^2 = |a_1|^2 + \ldots |a_m|^2$.

95. **Prop.** If $v = a_1v_1 + \cdots + a_mv_m$ and $v_1, \ldots, v_m$ orthogonal, then $a_k = \langle v, v_k\rangle$, $k = 1, \ldots, m$. If $v_1, \ldots, v_m$ is orthonormal basis of V then $v = \langle v, v_1\rangle v_1 + \cdots + \langle v, v_m\rangle v_m$.

96. **Prop.** If V is a finite-dim inner product space, then V has an orthonormal basis.

97. **Prop.** Suppose V is finite-dim. Then every orthonormal list of vectors in V can be extended to an orthonormal basis of V.

98. **Schur's Theorem.** Let T be an operator on a finite-dim inner product space V. If T is upper-triangular with respect to some basis, then it is upper-triangular with respect to some orthonormal basis of V.

99. **Prop.** If $U \subseteq W$, then $W^\perp \subseteq U^\perp$.

100. **Prop.** If U is a subset of V, then $U \cap U^\perp \subseteq \{0\}$.

101. **Lemma.** If U is a finite-dim subspace of V, then $V = U \oplus U^\perp$.

102. **Formula.** Assume V is finite-dim and U is a subspace of V. Then $\dim U^\perp = \dim V - \dim U$.

103. **Theorem.** If U is a finite-dim subspace of V, then $(U^\perp)^\perp = U$.

104. **Prop.** Suppose U is generated by a single nonzero vector w. Then $P_U(v) = \frac{\langle v,w \rangle}{\langle w,w \rangle} w$.

105. **Prop.** If $e_1, \ldots, e_d$ is an orthonormal basis of U¡ then $P_U(v) = \langle v, e_1 \rangle e_1 + \cdots + \langle , e_d \rangle e_d$.

106. **Formula.** $\alpha_V T^* = T' \circ \alpha_W$, where $T : V \to W$ and $T^* : W \to V$.

107. **Formula.** Let $T : V \to W$. Then $(T' \alpha_W(w))(v) = \langle Tv, w \rangle$.

108. **Prop.** $\langle Tv, w \rangle_W = \langle v, T^* w \rangle_V$.

109. **Lemma.** If $T : V \to W$ is a linear map between finite-dim inner product spaces, then $(T^*)^* = T$.

110. **Lemma.** If $T : V \to W$ is a linear map betwen finite-dim inner product spaces, then if $a \in F$, then $(aT)^* = \overline{a} T^*$.

111. **Prop.** If $M(T) = (a_{ij})$, then $M(T^*) = (\overline{a_{ij}})^t$.

112. **Prop.**

    (a) $I^* = I$.
    (b) $(S+T)^* = S^* + T^*$.
    (c) $(ST)^* = T^* S^*$.
    (d) $(T^{-1})^* = (T^*)^{-1}$.

113. **Prop.** The matrix of T^* is the conjugate transpose of the matrix of T if the same orthonormal bases of V and W are used to compute the matrices.

114. **Formula.** $\overline{a_{ij}} = \langle T^* w_i, v_j \rangle_V$ iff $a_{-j} = \langle v_j, T^* w_i \rangle_V$.

115. **Prop.** $\mathrm{nul}(T^*) = (\mathrm{range}\, T)^\perp$.

116. **Formula.**

    (a) $\mathrm{range}(T^*) = (\mathrm{nul}\, T)^\perp$.
    (b) $\mathrm{nul}\, T = (\mathrm{range}(T^*))^\perp$.
    (c) $\mathrm{range}\, T = (\mathrm{nul}(T^*))^\perp$.

117. **Theorem.** If T is symmetric, then T is orthonormal diagonalizable.

118. **Theorem.** Every eigenvalue of a self-adjoint operator is real.

119. **Lemma.** Let T be an operator on a complex inner product space. Suppose $\langle Tv, v \rangle = 0$ for all $v \in V$. Then $T = 0$.

120. **Cor.** If T is an operator on a complex inner product space, then $\langle Tv, v \rangle \in \mathbb{R}$ for all $v \in V$ iff T is self-adjoint.

121. **Prop.** Alternating implies anti-symmetric.

122. **Prop.** Let $x^2 + bx + c$ be an irreducible quadratic over $\mathbb{R}$. Then the operator $T^2 + bT + cI$ is injective on V.

123. **Real Spectral Theorem.** Supppose T is a self-adjoint operator on a real inner product space ($T = T^*$). Then T is diagonal in an orthonormal basis of V.

124. **Complex Spectral Theorem.** Suppose T is a normal operator on a complex product inner product spcae ($T^*TT = TT^*$). Then T is diagonal in an orthonormal basis of V.

125. **Theorem.** An operator T is normal iff $||Tv|| = ||T^*v||$ for all $v \in V$.

126. **Prop.** If T is normal and $Tv = \lambda v$, then $T^*v = \overline{\lambda}v$.

127. **Prop.** Suppose T is normal and v, w eigenvectors for T with different eigenvalues. Then the vectors v, w are orthogonal.

128. **Theorem.** If T is normal and $F = \mathbb{C}$, then T is diagonal in an orthonormal basis of V.

129. **Prop.** Let $T : V \to V$ be a symmetric (self-adjoint) operator on a nonzero finite-dim inner product space. Then T has an eigenvalue.

130. **Prop.** If T is self-adjoint, then it is diagonalizable in the real and complex case.

131. **Prop.** Nilpotent $2x2$ operators (nonzero) have no square root.

132. **Prop.** The identity operator has infinitely many square roots if $\dim V \geq 2$.

133. **Theorem.** Let T be an operator on a finite-dim inner product space. Then TFAE:

    (a) T is positive.
    (b) T is self-adjoint and has only nonnegative eigenvalues.
    (c) T has a positive square root.
    (d) T has a self-adjoint square root.
    (e) $T = R^*R$ for some operator R on V.

134. **Theorem.** Let T be an operator on a finite-dim inner product space V. Then TFAE:

    (a) S is an isometry.
    (b) S preserves $\langle \cdot, \cdot \rangle$, i.e. $\langle Su, Sv \rangle = \langle u, v \rangle$ for all $u, v \in V$.
    (c) If $e_1, \ldots, e_m$ is an orthonormal list of vectors in V, then so is $Se_1, \ldots, Se_m$.
    (d) There exists an orthonormal basis $e_1, \ldots, e_m$ of V such that $Se_1, \ldots, Se_m$ is also an orthonormal basis of V.
    (e) $S^*S = I$.

(f) $SS^* = I$.

(g) S^* is an isometry.

(h) S is invertible and $S^* = S^{-1}$.

135. **Prop.** The operator S is an isometry iff V has an orthonormal basis of eigenvectors for which the corresponding eigenvalues have aboslute value 1.

136. **Theorem.** If $T \in L(V)$, there is an isometry $S \in L(V)$ so that $T = S\sqrt{T^*T} =$ (isometry) $\cdot$ (positive operator).

137. **Observation.**

(a) T^*T is a positive operator.

(b) $\operatorname{nul}(T^*T) = \operatorname{nul} T$.

(c) $\operatorname{range}(T^*T) = \operatorname{range}(T^*)$.

(d) $\dim \operatorname{range} T = \dim \operatorname{range} T^*$.

138. **Formula.** The characteristic polynomial of a 2×2 matrix is $z^2 - \operatorname{tr}(A)z + \det A$.

139. **Prop.** The determinant of the matrix of an operator is independent of basis.

140. **Properties.** Let A, B be $n \times n$ matrices. Then

(a) $\det(AB) = \det A \cdot \det B$.

(b) $\det A = 0$ iff A is not invertible.

(c) If $m \in S_n$, then $\det \begin{pmatrix} A_{m_1} & \dots & A_{m_n} \end{pmatrix} = \operatorname{sgn}(m) \det A$.

(d) $\det \begin{pmatrix} A_1 & \dots & \alpha \cdot A_k & \dots & A_n \end{pmatrix} = \alpha \det A$.