

Math H110 Theorems.

1. **Lemma.** Let F be a field, $\lambda \in F$, V a vector space over F (denoted by V/F), $v \in V$. Then, if $\lambda v = 0$, then $\lambda = 0$ or $v = 0$.
2. **Lemma.** A vector space over a field is a module over a field.
3. **Theorem.** The intersection of a family of subspaces of a vector space V is a subspace of V .
4. **Lemma.** Let $S = \{v_1, \dots, v_t\}$. Then the subspace of all linear combinations of the elements of S is the $\text{span}S$.
5. **Theorem.** Let $L = v_1, \dots, v_n$ be a list of vectors in a vector space V over a field F and let $T : F^n \rightarrow V$ be linear transformation with $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 v_1 + \dots + \lambda_n v_n$. Then, we have the following:
 - (a) L spans V iff T is onto.
 - (b) L is linearly independent iff T is 1-1 iff $\text{nul } T = \{0\}$.
 - (c) L is a basis iff T is 1-1 and onto.
6. **Prop.** Consider $T : F^n \rightarrow V$ with $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 v_1 + \dots + \lambda_n v_n$, so $T(e_i) = v_i$ for all i . Then, T is the unique linear map $F^n \rightarrow V$ that sends $e_i \mapsto v_i$ for all i .
7. **Theorem.** Every subspace X of V has complement.
8. **Lemma.** If v_1, \dots, v_t is linearly dependent list, then there is an index k such that $v_k \in \text{span}(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_t)$. Furthermore, the span of the list of length $t - 1$ gotten by removing v_k from the list is the same as the span of the original list.
9. **Prop.** In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.
10. **Cor.** Two bases of V have the same number of elements.
11. **Prop.** $X + Y$ is direct iff the null space of the sum map is $\{0\}$.
12. **Theorem.** Every subspace of a finite-dimensional vector space is finite-dimensional.
13. **Prop.** Every spanning list for a vector space can be pruned down to a basis of the space.
14. **Cor.** Every finite-dimensional vector space has a basis.
15. **Prop.** In a finite-dimensional vector space, every linearly independent list can be extended to a basis of the space.

16. **Major Theorem.** Every subspace of a finite-dimensional vector space has a complement.
 17. **Prop.** Let X, Y be subspaces of a finite-dimensional vector space V . Then:
 - (a) $\dim X + \dim Y = \dim V$.
 - (b) $X \cap Y = \{0\}$.
 Then, $V = X \oplus Y$.
 18. **Prop.** $\dim(X \oplus Y) = \dim X + \dim Y$.
 19. **Prop.** If V is a finite-dimensional vector space (with $\dim V = n$), then every subspace has dimension at most n .
 20. **Prop.** Let $\dim V = n$. Then, a linearly independent list of vectors of V with length n is a basis for V .
 21. **Prop.** Let $\dim V = n$. Then, every spanning list for V of length n is a basis for V .
 22. **Lemma.** The list $(x_1, 0), \dots, (x_t, 0); (0, y_1), \dots, (0, y_k)$ of length $t + k$ is a basis of $X \times Y$.
 23. **Cor.** $\dim(X \times Y) = \dim X + \dim Y$.
 24. **Cor.** Let $T : V \rightarrow W$ be a linear map with $\dim V = d$. Then, $\text{rank} T \leq d$.
 25. **Rank-Nullity Theorem.** $\dim V = \text{rank} V + \text{nullity} V$.
 26. **Prop.** If $T : V \rightarrow W$ is 1-1, then $\text{nullity} T = 0$.
 27. **Cor.** If $T : V \rightarrow W$ is 1-1 and onto, then $\dim V = \dim W$.
 28. **Theorem.** The set of linear maps $V \rightarrow W$ is a vector space $L \cdot (F^n, W) \rightarrow T \longrightarrow (Te_1, \dots, Te_n) \in W^n$.
 29. **Theorem.** $\dim(X + Y) = \dim X + \dim Y - \dim(X \cap Y)$.
 30. **Cor.** $\dim(V/X) = \dim V - \dim X$.
 31. **Theorem.** If A is a rectangular matrix with elements in a field F , then $\text{row rank } A = \text{column rank } A$.
 32. **Prop.** Let $T : V \rightarrow W$ be 1-1. Then, $\dim W \geq \dim V$.
 33. **Prop.** Let $T : V \rightarrow W$ be onto. Then, $\dim V \geq \dim W$.
 34. **Prop.** Let $T : V \rightarrow W$ and $\dim V = \dim W$. Then, T 1-1 iff T onto iff T bijective iff T invertible.
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35. **Lemma.** Let V be a finite-dimensional vector space and U a subspace of V . Then, $\dim U_0 = \dim V - \dim U$.
36. **Theorem.** Every linear functional on a subspace of V can be extended to V .
37. **Note.** Annihilator is the dual of the quotient subspace.
38. **Theorem.** Let $T : V \rightarrow W$ and $T' : W' \rightarrow V'$. Then $\mathcal{M}(T)$ and $\mathcal{M}(T')$ are transposes of each other.
39. **Lemma.** U^0 has dimension $\dim V - \dim U$.
40. **Cor.** The annihilator of U is $\{0\}$ iff $U = V$. The annihilator of U is V iff $U = \{0\}$.
41. **Prop.** If $T : V \rightarrow W$ is a linear map, then the null space of T' is the annihilator of the range of T . We have $\text{ann}(\text{range } T) = \{\psi : W \rightarrow F \mid \phi(Tv) = 0 \text{ for all } v \in V, T'(\psi)(v) = 0, T'\psi = 0, \phi \in \text{nul}(T')\}$.
42. **Cor.** If $T : V \rightarrow W$ is a linear map between finite-dimensional F -vector spaces, then $\dim \text{nul}(T') = \dim \text{nul}(T) + \dim W - \dim V$.
43. **Cor.** The linear map T is onto iff T' is 1-1.
44. **Cor.** If $T : V \rightarrow W$ is a linear map between finite-dimensional vector spaces, then T' and T have equal ranks.
45. **Cor.** We have $\text{range } T = (\text{nul } T)^0$.
46. **Theorem.** Let F be a finite field with $q = |F|$. Then, $a^q = a$ for all $a \in F$.
47. **Theorem.** If F is a finite field, then $|F| = p^n$ for some prime p and integer $n \geq 1$.
48. **Theorem.** Take an ideal I in \mathbb{Z} . Then, I is equal to either $\{0\}$ or $m\mathbb{Z}$ (where $m \in \mathbb{Z}_{>0}$).
49. **Theorem.** $F[x]$ is a principal ideal domain; that is, it is an integral domain in which every ideal in $F[x]$ is principal.
50. **Theorem.** Let $T : V \rightarrow V$, V finite-dimensional, and let $\alpha : F[x] \rightarrow \mathcal{L}(V)$, with $f \mapsto f(T)$. Also, we have $\ker \alpha$ to be the principal ideal $(m(x))$. Then, $m(x)$ is the minimal polynomial of T and has degree $\leq n^2$.
51. **Cayley-Hamilton Theorem.** Let $T : V \rightarrow V$, V finite-dimensional, and let $\alpha : F[x] \rightarrow \mathcal{L}(V)$, with $f \mapsto f(T)$. Also, we have $\ker \alpha$ to be the principal ideal $(m(x))$, where $m(x)$ is the minimal polynomial of T . Then, the characteristic polynomial is in $\ker \alpha$; that is, we can plug in the matrix for T into its characteristic polynomial and we get that it is equal to the 0-matrix.
52. **Prop.** For $f(x) \in F[x]$ and $\lambda \in F$, $f(\lambda) = 0$ iff f is divisible by $x - \lambda$, where $x - \lambda$ is an irreducible polynomial.

53. **Cor.** A polynomial of degree n can have at most n roots.
54. **Cor.** A polynomial with infinitely many roots is identically the zero polynomial.
55. **Lemma.** Let $f \in \mathbb{R}[x]$ be a real polynomial. If λ is a complex root of f , so is $\bar{\lambda}$, which is the complex conjugate of λ .