

Math H110 Definitions.

1. **Endomorphism.** An endomorphism is a group homomorphism from a set to itself (NOTE: does not have to be invertible.)
2. **End V .** The symbol $\text{End } V$ is the set of all endomorphisms on V (and multiplication on $\text{End } V$ is defined to be function composition).
3. **F-Module.** An F -module is a generalization of vector spaces over rings.
4. **Subspace.** Let V be a vector space. X is a subspace of V if $X \subseteq V$ and closed under all relevant operations of V , $X \neq \emptyset$, and $X \ni 0$.
5. **Linear Map / Linear Transformation.** Let V be a vector space over a field F with $v, w \in V$. Let T be a map on V with $T(v + w) = T(v) + T(w)$ and $T(\lambda v) = \lambda T(v)$ for all $\lambda \in F$. Then, T is called a linear map or linear transformation.
6. **Linear Operator.** If T is a linear transformation on a vector spaces V with $T : V \rightarrow V$, then T is linear operator on V .
7. **Spans.** The list v_1, \dots, v_n spans V iff $T : F^n \rightarrow V$ is onto.
8. **Linearly Independent.** The list v_1, \dots, v_n is linearly independent iff $T : F^n \rightarrow V$ is 1-1. Equivalently, the list v_1, \dots, v_n is linearly independent if $\lambda_1 v_1 + \dots + \lambda_n v_n = 0$ implies $\lambda_i = 0$ for all i .
9. **Linearly Dependent.** The list v_1, \dots, v_n is linearly dependent iff $\lambda_1 v_1 + \dots + \lambda_n v_n = 0$ implies $\lambda_i \neq 0$ for some i .
10. **Basis.** The list v_1, \dots, v_n is a basis of V if $\text{span}\{v_1, \dots, v_n\} = V$ and v_1, \dots, v_n is linearly independent.
11. **Finite-dimensional.** V is finite-dimensional if V is spanned by a finite list of vectors.
12. **Sum of Subspaces.** Let X_1, \dots, X_t be subspaces of V . Then, we define their sum as $X_1 + \dots + X_t = \{x_1 + \dots + x_t \mid x_1 \in X_1, \dots, x_t \in X_t\}$.
13. **Direct Sum of Subspaces.** Let X_1, \dots, X_t be subspaces of V . Then, their direct sum, $X_1 \oplus \dots \oplus X_t$, is given by a 1-1 linear map T , with $T : X_1 \times \dots \times X_t \rightarrow V$.
14. **Complement of Subspace.** Let X, Y be subspaces of V . Then, Y is a complementary subspace of X iff $X + Y = V$ and $X \cap Y = \{0\}$.
15. **Rank, Nullity.** The rank of a linear map is the dimension of the range of the linear map. The nullity is the dimension of the null space of the linear map.
16. **Null Space.** The null space is the set of vectors that are mapped to 0.
17. **Isomorphic Vector Spaces.** Two vector spaces V, W are isomorphic if there exists a linear map $T : V \rightarrow W$ that is 1-1 and onto.

18. **Quotient Space.** Suppose U is a subspace of V . Then, the quotient space V/U is the set $V/U = \{v + U \mid v \in V\}$.
19. **Column Rank.** The column rank (rank of the column span of a matrix) is defined to be $\text{rank} T_A$.
20. **Conjugation.** Let A be an $n \times n$ matrix (over F) and let Q be an $n \times n$ matrix (over F). Then, the conjugation of A by Q is $Q^{-1}AQ$.

21. **Dual Space.** Let V be an F -vector space. Then the dual space of V is $V' = \mathcal{L}(V, F)$ where the elements of V' are called linear functionals.
22. **Annihilator.** For a subspace $U \subseteq V$, we define the annihilator of U to be $U_0 = \{\phi \in V' \mid \phi(u) = 0 \forall u \in U\}$.
23. **Double Dual.** Let V be a finite-dimensional vector space with dual V' . Then the double dual of V is $(V')' = V'' = V$. Also, $\dim V = n = \dim V' = \dim V''$.
24. **Eigenvector / eigenvalue.** Let $T \in \mathcal{L}(V)$. Then an eigenvector of T is a $v \in V$ ($v \neq 0$) such that $Tv = \lambda v$ ($\lambda \in F$ is called an eigenvalue), and v is an eigenvector of T .
25. **Eigenspace.** Let $T \in \mathcal{L}(V)$ and take λ to be an eigenvalue of T . Then, $E(\lambda, T) = \{v \in V \mid Tv = \lambda v\} \neq \emptyset$ is written as V_λ and is called the eigenspace of λ , which is a subspace of V .
26. **Invariant subspace.** E is a T -invariant subspace if $T \in \mathcal{L}(V)$ with $T(E) \subseteq E$.
27. **Idempotent.** If $e = e^2$, then e is called idempotent.
28. **Generalized Eigenvector.** Consider a minimal polynomial $(x - \lambda_1)^{e_1} \cdots (x - \lambda_m)^{e_m}$ on X with $(T - \lambda_1 I)^{e_1} v = 0$. Then, v is called a generalized eigenvector for $\lambda = \lambda_1$.
29. **Characteristic polynomial.** The characteristic polynomial of $T : V \rightarrow V$ (with eigenvalues $\lambda_1, \dots, \lambda_t$) is the polynomial $\prod_{i=1}^t (x - \lambda_i)^{\dim X_i}$, where $V = X_1 \oplus \cdots \oplus X_t$.
30. **Simultaneously diagonalizable.** Operators S and T on V are simultaneously diagonalizable if there is a basis of V that consists of vectors that are eigenvectors for both S and T (i.e. there exists a basis v_1, \dots, v_n of V so that for i , $1 \leq i \leq n$, there are λ_i and μ_i so that $Sw_i = \lambda_i v_i$ and $Tv_i = \mu_i v_i$).

31. **Inner product.** Let V be a vector space over \mathbb{R} , possibly infinite-dim. An inner product on V is a bilinear function $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ such that $\langle x, x \rangle \geq 0$ for all $x \in V$ and $\langle x, x \rangle = 0$ iff $x = 0$.

32. **Complex Inner product.** Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be in \mathbb{C}^n . We have $\langle z, w \rangle = \sum_i z_i \overline{w_i}$, with $\langle w, z \rangle = \overline{\langle z, w \rangle}$ and $\langle \alpha z, w \rangle = \alpha \langle z, w \rangle$ and $\langle w, \alpha z \rangle = \overline{\alpha} \langle z, w \rangle$.
33. **Norm.** Norm of $v \in V$ is $\|v\| = \sqrt{\langle v, v \rangle}$.
34. **Orthogonal.** Two vectors are orthogonal if $\langle v, w \rangle = 0$, where $v, w \in V$.
35. **Orthogonal Complement.** Take U to be a subset of V . Then, $U^\perp = \{v \in V \mid \langle v, u \rangle = 0 \forall u \in U\} = \{v \in V \mid \langle u, v \rangle = 0 \forall u \in U\} = \{v \in V \mid \phi_v = 0 \text{ in } U\} = \{v \in V \mid \phi_v \in U^0\}$.
36. **Adjoint.** Let $T : V \rightarrow W$. Then the adjoint of T is $T^* : W \rightarrow V$ such that $T^* = \alpha_V^{-1} \circ T' \circ \alpha_W$, where $\alpha_V : V \rightarrow V'$ and $\alpha_W : W \rightarrow W'$.
37. **Self-adjoint.** An operator $T : V \rightarrow V$ is self-adjoint if $T = T^*$.
38. **Symmetric.** T is symmetric if $M(T) = M(T)^t$.
39. **Normal.** T is a normal operator if $TT^* = T^*T$.
40. **Alternating.** A bilinear form $\psi : V \times V \rightarrow F$ is alternating if $\psi(v, v) = 0$ for $v \in V$.
41. **Anti-symmetric.** Let $\psi : V \times V \rightarrow F$ be a bilinear form. Then ψ is anti-symmetric if $\psi(x, y) = -\psi(y, x)$.
42. **Positive operator.** T is a positive operator on an inner product space if $T = T^*$ and $\langle Tv, v \rangle \geq 0$.
43. **Square root of an operator.** A square root of an operator T is an operator R such that $R^2 = T$.
44. **Isometry.** An operator $S \in L(V)$ is an isometry if it preserves distance, i.e. $\|Sv\| = \|v\| \forall v \in V$, i.e. $S^* = S^{-1}$. In the real case, they are called orthogonal operators. In the complex case, they are called unitary operators.
45. **Singular values.** The singular values of T are the eigenvalues of $\sqrt{T^*T}$.
46. **Def 1 of Determinants.** (cofactor expansion). $\det A = \sum_{j=1}^n (-1)^{j+1} \cdot a_{ij} \det(A_{ij})$.
47. **Def 2 of Determinants.** If $T \in L(V)$, where V is over \mathbb{C} , then the determinant of T is the product of the eigenvalues $\lambda_1, \dots, \lambda_n$ (with multiplicity).
48. **Def 3 of Determinants.** The determinant of an operator $T \in L(V)$ is $(-1)^n$ times the constant term of its characteristic polynomial $(z - \lambda_1) \cdots (z - \lambda_n)$.
49. **Permutation.** A permutation $m = (m_1, \dots, m_r)$ is a list containing $1, \dots, n$ exactly once each, and write S_n to denote the set of n -element permutations.
50. **Inversion.** An inversion (in a permutation) is a pair $\{i, j\}$ such that $m_i > m_j$ where $i < j$.

51. **Def 4 of Determinants.** Let A be an $n \times n$ matrix. Then, $\det A = \sum_{m \in S_n} (\text{sgn}(m) \cdot a_{m_1,1} \cdots a_{m_n,n})$.
52. **Box.** The box defined by $v_1, \dots, v_n \in \mathbb{R}^n$ is $\{a_1 v_1 + \cdots + a_n v_n \mid 0 \leq a_i \leq 1, i = 1, \dots, n\}$ and as volume, $\det(v_1 \ \dots \ v_n)$, where v_1, \dots, v_n are the columns of a matrix A .
53. **Norm of T .** This is defined to be $\sup_{v \in V \setminus \{0\}} \frac{\|Tv\|}{\|v\|}$.