

Math H110 Theorems.

1. **Lemma.** Let  $F$  be a field,  $\lambda \in F$ ,  $V$  a vector space over  $F$  (denoted by  $V/F$ ),  $v \in V$ . Then, if  $\lambda v = 0$ , then  $\lambda = 0$  or  $v = 0$ .
2. **Lemma.** A vector space over a field is a module over a field.
3. **Theorem.** The intersection of a family of subspaces of a vector space  $V$  is a subspace of  $V$ .
4. **Lemma.** Let  $S = \{v_1, \dots, v_t\}$ . Then the subspace of all linear combinations of the elements of  $S$  is the  $\text{span}S$ .
5. **Theorem.** Let  $L = v_1, \dots, v_n$  be a list of vectors in a vector space  $V$  over a field  $F$  and let  $T : F^n \rightarrow V$  be linear transformation with  $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 v_1 + \dots + \lambda_n v_n$ . Then, we have the following:
  - (a)  $L$  spans  $V$  iff  $T$  is onto.
  - (b)  $L$  is linearly independent iff  $T$  is 1-1 iff  $\text{nul } T = \{0\}$ .
  - (c)  $L$  is a basis iff  $T$  is 1-1 and onto.
6. **Prop.** Consider  $T : F^n \rightarrow V$  with  $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 v_1 + \dots + \lambda_n v_n$ , so  $T(e_i) = v_i$  for all  $i$ . Then,  $T$  is the unique linear map  $F^n \rightarrow V$  that sends  $e_i \mapsto v_i$  for all  $i$ .
7. **Theorem.** Every subspace  $X$  of  $V$  has complement.
8. **Lemma.** If  $v_1, \dots, v_t$  is linearly dependent list, then there is an index  $k$  such that  $v_k \in \text{span}(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_t)$ . Furthermore, the span of the list of length  $t - 1$  gotten by removing  $v_k$  from the list is the same as the span of the original list.
9. **Prop.** In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.
10. **Cor.** Two bases of  $V$  have the same number of elements.
11. **Prop.**  $X + Y$  is direct iff the null space of the sum map is  $\{0\}$ .
12. **Theorem.** Every subspace of a finite-dimensional vector space is finite-dimensional.
13. **Prop.** Every spanning list for a vector space can be pruned down to a basis of the space.
14. **Cor.** Every finite-dimensional vector space has a basis.
15. **Prop.** In a finite-dimensional vector space, every linearly independent list can be extended to a basis of the space.

16. **Major Theorem.** Every subspace of a finite-dimensional vector space has a complement.
17. **Prop.** Let  $X, Y$  be subspaces of a finite-dimensional vector space  $V$ . Then:
- (a)  $\dim X + \dim Y = \dim V$ .
  - (b)  $X \cap Y = \{0\}$ .
- Then,  $V = X \oplus Y$ .
18. **Prop.**  $\dim(X \oplus Y) = \dim X + \dim Y$ .
19. **Prop.** If  $V$  is a finite-dimensional vector space (with  $\dim V = n$ ), then every subspace has dimension at most  $n$ .
20. **Prop.** Let  $\dim V = n$ . Then, a linearly independent list of vectors of  $V$  with length  $n$  is a basis for  $V$ .
21. **Prop.** Let  $\dim V = n$ . Then, every spanning list for  $V$  of length  $n$  is a basis for  $V$ .
22. **Lemma.** The list  $(x_1, 0), \dots, (x_t, 0); (0, y_1), \dots, (0, y_k)$  of length  $t + k$  is a basis of  $X \times Y$ .
23. **Cor.**  $\dim(X \times Y) = \dim X + \dim Y$ .
24. **Cor.** Let  $T : V \rightarrow W$  be a linear map with  $\dim V = d$ . Then,  $\text{rank} T \leq d$ .
25. **Rank-Nullity Theorem.**  $\dim V = \text{rank} V + \text{nullity} V$ .
26. **Prop.** If  $T : V \rightarrow W$  is 1-1, then  $\text{nullity} T = 0$ .
27. **Cor.** If  $T : V \rightarrow W$  is 1-1 and onto, then  $\dim V = \dim W$ .
28. **Theorem.** The set of linear maps  $V \rightarrow W$  is a vector space  $L \cdot (F^n, W) \rightarrow T \longrightarrow (Te_1, \dots, Te_n) \in W^n$ .
29. **Theorem.**  $\dim(X + Y) = \dim X + \dim Y - \dim(X \cap Y)$ .
30. **Cor.**  $\dim(V/X) = \dim V - \dim X$ .
31. **Theorem.** If  $A$  is a rectangular matrix with elements in a field  $F$ , then  $\text{row rank } A = \text{column rank } A$ .
32. **Prop.** Let  $T : V \rightarrow W$  be 1-1. Then,  $\dim W \geq \dim V$ .
33. **Prop.** Let  $T : V \rightarrow W$  be onto. Then,  $\dim V \geq \dim W$ .
34. **Prop.** Let  $T : V \rightarrow W$  and  $\dim V = \dim W$ . Then,  $T$  1-1 iff  $T$  onto iff  $T$  bijective iff  $T$  invertible.
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35. **Lemma.** Let  $V$  be a finite-dimensional vector space and  $U$  a subspace of  $V$ . Then,  $\dim U^0 = \dim V - \dim U$ .
36. **Theorem.** Every linear functional on a subspace of  $V$  can be extended to  $V$ .
37. **Note.** Annihilator is the dual of the quotient subspace.
38. **Theorem.** Let  $T : V \rightarrow W$  and  $T' : W' \rightarrow V'$ . Then  $\mathcal{M}(T)$  and  $\mathcal{M}(T')$  are transposes of each other.
39. **Lemma.**  $U^0$  has dimension  $\dim V - \dim U$ .
40. **Cor.** The annihilator of  $U$  is  $\{0\}$  iff  $U = V$ . The annihilator of  $U$  is  $V$  iff  $U = \{0\}$ .
41. **Prop.** If  $T : V \rightarrow W$  is a linear map, then the null space of  $T'$  is the annihilator of the range of  $T$ . We have  $\text{ann}(\text{range } T) = \{\psi : W \rightarrow F \mid \phi(Tv) = 0 \text{ for all } v \in V, T'(\psi)(v) = 0, T'\psi = 0, \phi \in \text{nul}(T')\}$ .
42. **Cor.** If  $T : V \rightarrow W$  is a linear map between finite-dimensional  $F$ -vector spaces, then  $\dim \text{nul}(T') = \dim \text{nul}(T) + \dim W - \dim V$ .
43. **Cor.** The linear map  $T$  is onto iff  $T'$  is 1-1.
44. **Cor.** If  $T : V \rightarrow W$  is a linear map between finite-dimensional vector spaces, then  $T'$  and  $T$  have equal ranks.
45. **Cor.** We have  $\text{range } T = (\text{nul } T)^0$ .
46. **Theorem.** Let  $F$  be a finite field with  $q = |F|$ . Then,  $a^q = a$  for all  $a \in F$ .
47. **Theorem.** If  $F$  is a finite field, then  $|F| = p^n$  for some prime  $p$  and integer  $n \geq 1$ .
48. **Theorem.** Take an ideal  $I$  in  $\mathbb{Z}$ . Then,  $I$  is equal to either  $\{0\}$  or  $m\mathbb{Z}$  (where  $m \in \mathbb{Z}_{>0}$ ).
49. **Theorem.**  $F[x]$  is a principal ideal domain; that is, it is an integral domain in which every ideal in  $F[x]$  is principal.
50. **Theorem.** Let  $T : V \rightarrow V$ ,  $V$  finite-dimensional, and let  $\alpha : F[x] \rightarrow \mathcal{L}(V)$ , with  $f \mapsto f(T)$ . Also, we have  $\ker \alpha$  to be the principal ideal  $(m(x))$ . Then,  $m(x)$  is the minimal polynomial of  $T$  and has degree  $\leq n^2$ .
51. **Cayley-Hamilton Theorem.** Let  $T : V \rightarrow V$ ,  $V$  finite-dimensional, and let  $\alpha : F[x] \rightarrow \mathcal{L}(V)$ , with  $f \mapsto f(T)$ . Also, we have  $\ker \alpha$  to be the principal ideal  $(m(x))$ , where  $m(x)$  is the minimal polynomial of  $T$ . Then, the characteristic polynomial is in  $\ker \alpha$ ; that is, we can plug in the matrix for  $T$  into its characteristic polynomial and we get that it is equal to the 0-matrix.
52. **Prop.** For  $f(x) \in F[x]$  and  $\lambda \in F$ ,  $f(\lambda) = 0$  iff  $f$  is divisible by  $x - \lambda$ , where  $x - \lambda$  is an irreducible polynomial.

53. **Cor.** A polynomial of degree  $n$  can have at most  $n$  roots.
54. **Cor.** A polynomial with infinitely many roots is identically the zero polynomial.
55. **Lemma.** Let  $f \in \mathbb{R}[x]$  be a real polynomial. If  $\lambda$  is a complex root of  $f$ , so is  $\bar{\lambda}$ , which is the complex conjugate of  $\lambda$ .
56. **Prop.** A scalar  $\lambda$  is an eigenvalue of  $T : V \rightarrow V$  iff  $T - \lambda I$  is not 1-1.
57. **Cor.** The map  $T : V \rightarrow V$  is invertible iff 0 is not an eigenvalue of  $T$ .
58. **Key lemma.** Every list of eigenvectors of  $T$  that corresponds to distinct eigenvalues of  $T$  is a linearly independent list.
59. **Cor.** Let  $\lambda_1, \dots, \lambda_t$  be distinct eigenvalues and take  $E_i = E(\lambda_i, T) = \{v \in V \mid Tv = \lambda_i v\} \subseteq V$ . Now, take  $E_1 \times \dots \times E_t$ . Then there exists a summation map  $E_1 \times \dots \times E_t \xrightarrow{\text{sum}} V$  with  $(v_1, \dots, v_t) \mapsto v_1 + \dots + v_t$ . Then, the sum map is 1-1.
60. **Cor.** Suppose  $V$  is finite-dimensional. Then each operator on  $V$  has at most  $\dim V$  distinct eigenvalues.
61. **Prop.** Suppose  $T$  is an operator on an  $F$ -vector space  $V$ . If  $f \in F[x]$  is a polynomial satisfied by  $T$  (meaning  $f(T) = 0$ ), then every eigenvalue of  $T$  on  $V$  is a root of  $f$ .
62. **Cor.** Suppose  $\lambda$  is an eigenvalue of operator  $T$  on a finite-dimensional  $F$ -vector space. Then  $\lambda$  is a root of the minimal polynomial of  $T$ .
63. **Prop.** Let  $T$  be an operator on a finite-dimensional vector space. Suppose  $\lambda$  is a root of the minimal polynomial. Then  $\lambda$  is an eigenvalue of  $T$ .
64. **Theorem.** All operators on a nonzero finite-dimensional vector space over an algebraically closed field have at least one eigenvalue.
65. **Prop.** Assume that  $F = \mathbb{R}$  and that  $f(x) := x^2 + bx + c$  is an irreducible polynomial. If  $T \in \mathcal{L}(V)$  and  $V$  is finite-dimensional, then the null space of  $f(T)$  is even-dimensional.
66. **Prop (honors version).** Let  $T$  be an operator on a finite-dimensional vector space over  $F$ . If  $p$  is an irreducible polynomial over  $F$ , then the dimension of the null space of  $p(T)$  is a multiple of the degree of  $p$ .
67. **Prop.**  $F[x]/(p)$  (where  $p$  is irreducible) is a field.
68. **Formula.**  $\dim_F V = [K : F] \cdot \dim_K V = \dim_F K \cdot \dim_K V$ .
69. **Cor.** Every operator on an odd-dimensional  $\mathbb{R}$ -vector space has an eigenvalue.
70. **Prop.** If  $T$  is an operator on a finite-dimensional  $F$ -vector space, then the minimal polynomial of  $T$  has degree at most  $\dim V$ .

71. **Prop.** If  $T$  is upper-triangular with respect to some basis of  $V$ , and if the diagonal entries of an upper-triangular matrix representation of  $T$  are  $\lambda_1, \dots, \lambda_n$ , then  $(T - \lambda_1 I) \cdots (T - \lambda_n I) = 0$ .
72. **Prop.** Let  $V$  be a finite-dimensional vector space and  $T \in \mathcal{L}(V)$  and let  $\lambda_1, \dots, \lambda_m$  be the eigenvalues of  $T$ . Then,  $V = \oplus E(\lambda_i, T)$  iff  $T$  is diagonalizable.
73. **Prop.** TFAE.
- (a)  $T$  is diagonalizable.
  - (b)  $V$  has a basis consisting of eigenvectors.
  - (c) The direct sum  $\bigoplus_i V_{\lambda_i}$  is all of  $V$ .
  - (d)  $\dim \left( \bigoplus_i V_{\lambda_i} \right) = \dim V$ .
74. **Prop.** If  $T : V \rightarrow V$  has  $\dim V$  different eigenvalues, then  $T$  is diagonalizable.
75. **Prop.** The operator  $T : V \rightarrow V$  is diagonalizable iff its minimal polynomial splits completely as a product of distinct linear factors of the form  $x - r$ .
76. **Jordan Canonical Form.**  $X$  can be written as a direct sum of Jordan blocks, where  $\sum \dim(\text{block}) = \dim X$ .
77. **Lemma.** Let  $X = \bigoplus \text{span}(U_i v)$  for  $i \in \{0, \dots, k_1\}$ . If  $Z$  is a subspace of  $X'$  that is  $U'$ -invariant, then  $\text{ann}(Z) =: Y$  is  $U$ -invariant.
78. **Lemma.** Suppose  $S$  and  $T$  are commuting operators on  $V$ . If  $\lambda$  is an eigenvalue for  $T$  on  $V$ , then the eigenspace  $E(\lambda, T)$  is  $S$ -invariant.
79. **Theorem.** The diagonalize operators on the same finite-dimensional vector space are simultaneously diagonalizable iff they commute with each other.
80. **Theorem.** Every pair of commuting operators on a finite-dimensional nonzero complex vector space has a common eigenvector.
81. **Prop.** Two commuting operators on a finite-dimensional nonzero complex vector space can be simultaneously upper-triangularized.
82. **Prop.** We have:
- (a) Every eigenvalue of  $S+T$  is the sum of an eigenvalue of  $S$  and an eigenvalue of  $T$ .
  - (b) Every eigenvalue of  $ST$  is the product of an eigenvalue of  $S$  and an eigenvalue of  $T$ .