1. 1A. (NOTHING)

2. 1B.

3. **Vector Space.** A vector space V is a set that has scalar multiplication and vector addition defined on it with the following properties:

   (a) Additive commutativity.

   (b) Additive associativity of vectors ($u+(v+w)=(u+v)+w$) and multiplicative associativity for scalars ($(ab)v=a(bv)$).

   (c) Additive identity.

   (d) Additive inverses.

   (e) Multiplicative identity.

   (f) BOTH distributive properties.

4. **V-space (unique additive identity)** A vector space has a unique additive identity.

5. **V-space (unique additive inverses)** Every element in a vector space has a unique additive inverse.

6. 1C.

7. **Subspace.** A subset $U \subseteq V$ is a subspace of V if it is a vector space with the same additive identity, scalar multiplication, and vector addition as defined on V.

8. **Conditions for a Subspace.** A subset $U \subseteq V$ is a subspace of V iff U is closed under vector addition, scalar multiplication, and contains the "zero" element as in V.

9. **Sums of Subspaces.** Let $V_1,\ldots,V_n$ be subspaces of V. Then, we have the sum of subspaces as $V_1+\cdots+V_n=\{v_1+\cdots+v_n \mid v_i \in V_i \text{ for all } i\}$.

10. **Smallest subspace containing each subspace** Suppose $V_1,\ldots,V_n$ are subspaces of V. Then, $V_1+\cdots+V_n$ is the smallest subspace of V containing $V_1,\ldots,V_n$.

11. **Direct Sum.** Suppose $V_1,\ldots,V_m$ are subspaces of V. Then:

    (a) The sum $V_1+\cdots+V_m$ is direct if each element of $V_1+\cdots+V_m$ can be written uniquely as a sum $v_1+\cdots+v_m$, where $v_i \in V_i$ for all i.

    (b) If $V_1+\cdots+V_m$ is a direct sum, then we write $V_1 \oplus \cdots \oplus V_m$.

12. **Conditions for a direct sum.** Suppose $V_1,\ldots,V_n$ are subspaces of V. Then, $V_1+\cdots+V_n$ is direct iff the only way to write 0 from $v_1+\cdots+v_n$ is by taking $v_i=0$ for all i.

13. **Direct sum of subspaces.** If U,W are subspaces of V, then $U+W$ is direct iff $U \cap W=\{0\}$.

14. 2A.

15. **Span is the smallest containing subspace.** The span of a list of vectors in V is the smallest subspace containing all of the vectors in the list.

16. **Zero polynomial.** The zero polynomial is said to have degree $-\infty$.

17. **Linear Independence.** A list of vectors $v_1,\ldots,v_n \in V$ is said to be linearly independent if $a_1v_1+\cdots+a_nv_n=0$ implies $a_i=0$ for all i. Also, the empty list () is said to be linearly independent.

18. **Linear Dependence.** A list of vectors $v_1,\ldots,v_n$ is said to be linearly dependent if $a_1v_1+\cdots+a_nv$=0 impies $a_i \neq 0$ for some i.

19. **Linear Dependence Lemma.** Suppose $v_1,\ldots,v_m$ is a linearly dependent list in V. Then, there exists $k \in \{1,\ldots,m\}$ such that $v_k \in \operatorname{span}(v_1,\ldots,v_{k-1})$. Furthermore, if k satisfies the condition in the previous sentence and the k^{th} term is removed from $v_1,\ldots,v_m$, then the span of the remaining list equals $\operatorname{span}(v_1,\ldots,v_m)$.

20. **length of linearly independent list ¡ length of spanning list.** In a finite-dimensional vector space, the length of every linearly independent list is at most the length of every spanning list of vectors.

21. **Finite Dimensional subspaces.** Every subspace of a finite-dimensional vector space is finite dimensional.

22. 2B.

23. **Basis.** A basis of V is a list of vectors that is linearly independent and spans V.

24. **Criterion for basis.** A list of vectors $v_1,\ldots,v_n \in V$ is a basis of V iff every $v \in V$ can be written uniquely in the form $v=a_1v_1+\cdots+a_nv_n$, where $a_i \in F$ for all i.

25. **Every spanning list contains a basis.** Every spanning list in a vector space can be reduced to a basis of the vector space.

26. **Basis of finite-dimensional vector space.** Every finite-dimensional vector space has a basis.

27. **Every linearly independent list extends to a basis.** Every linearly independent list in a finite-dimensional vector space can be extended to a basis of the vector space.

28. **Every subspace of V is part of a direct sum equal to V.** Suppose V is finite-dimensional and U is a subspace of V. Then, there is a subspace W of V such that $V=U \oplus W$.

29. 2C.

30. **Basis length does not depend on basis.** Any two bases of a finite-dimensional vector space have the same length.

31. **Dimension of a subspace.** If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.

32. **Linearly independent list of the right length is a basis.** Suppose V is finite-dimensional. Then, every linearly independent list of vectors in V (with list length equal to $\dim V$) is a basis of V.

33. **Subspace of full dimension equals the whole space.** Suppose V is finite-dimensional and U is a subspace of V such that $\dim U=\dim V$. Then, $U=V$.

34. **Spanning list of the right length is a basis.** Suppose V is finite-dimensional. Then, every spanning list of V of length $\dim V$ is a basis of V.

35. **Dimension of a sum.** If V_1,V_2 are subspaces of a finite-dimensional vector space, then $\dim(V_1+V_2)=\dim V_1+\dim V_2-\dim(V_1\cap V_2)$.

36. 3A.

37. **Set of Linear Maps.** The linear of linear maps from $V \to W$ is written $\mathscr{L}(V,W)$ and the set of linear maps from $V \to V$ is written $\mathscr{L}(V)$.

38. **Linear Map lemma.** Suppose $v_1,\ldots,v_n$ is a basis of V and $w_1,\ldots,w_n \in W$. Then, there exists a unique linear map $T: V \to W$ such that $Tv_k=w_k$ for each k.

39. **Linear maps take 0 to 0.** Suppose $T: V \to W$ is a linear map. Then, $T(0)=0$.

40. 3B.

41. **null space is a subspace.** Suppose $T \in \mathscr{L}(V,W)$. Then, T is a subspace of V.

42. **injectivity iff null is 0.** Let $T \in \mathscr{L}(V,W)$. Then, T is 1-1 iff $\operatorname{nul} T=\{0\}$.

43. **range is a subspace.** If $T \in \mathscr{L}(V,W)$, then range T is a subspace of W.

44. **Fundamental Theorem of Linear Maps.** Suppose V is finite-dimensional and $T \in \mathscr{L}(V,W)$. Then, range T is finite dimensional and $\dim V=\dim \operatorname{nul} T+\dim \operatorname{range} T$.

45. **linear map to a lower-dim space is not 1-1.** Suppose V,W are finite-dimensional vector spaces such that $\dim V>\dim W$. Then, no linear map from $V \to W$ is 1-1.

46. **linear map to a higher-dim space is not onto.** Suppose V,W are finite-dimensional vector spaces such that $\dim V<\dim W$. Then, no linear map from $V \to W$ is onto.

47. 3C.

48. **Prop.** $ST=I$ iff $TS=I$ (on vector spaces of the same domain).

49. **Prop.** Let V,W be finite-dimensional with $\dim W=\dim V$. Let $S \in \mathscr{L}(W,V)$, $T \in \mathscr{L}(V,W)$. Then, $ST=I$ iff $TS=I$.

50. 3D.

51. **Theorem.** Let V,W be finite-dimensional vector spaces such that $\dim V=\dim W$ and let $T \in \mathscr{L}(V,W)$. Then, T is invertible iff T is 1-1 iff T is onto.

52. **isomorphism.** An isomorphism is an invertible linear map.

53. **dimension and isomorphic.** Two finite-dimensional vector spaces are isomorphic iff they have the same dimension.

54. **Theorem.** Suppose V and W are finite-dimensional. Then, $\mathscr{L}(V,W)$ is finite-dimensional and $\dim \mathscr{L}(V,W)=(\dim V)(\dim W)$.

55. **ST=I iff TS=I (on vector spaces of the same dimension).** Suppose V and W are finite-dimensional vector spaces of the same dimension, $S \in \mathscr{L}(W,V), T \in \mathscr{L}(V,W)$. Then $ST=I$ iff $TS=I$.

56. **matrix of identity operator with respect to two bases.** Suppose $u_1,\ldots,u_n$ and $v_1,\ldots,v_n$ are two bases of V. Then, the matrices $\mathscr{M}(I;u_1,\ldots,u_n;v_1,\ldots,v_n)$ and $\mathscr{M}(I;v_1,\ldots,v_n;u_1,\ldots,u_n)$ are invertible and are inverses of each other.

57. **Change of basis formula.** Let $T \in \mathscr{L}(V,W)$. Suppose $u_1,\ldots,u_n$ and $v_1,\ldots,v_n$ are two bases of V. Let $A=\mathscr{M}(T;u_1,\ldots,u_n)$ and $B=\mathscr{M}(T;v_1,\ldots,v_n)$ and $C=\mathscr{M}(I;u_1,\ldots,u_n;v_1,\ldots,v_n)$. Then, $A=C^{-1}BC$.

58. Suppose that $v_1,\ldots,v_n$ is a basis of V and $T \in \mathscr{L}(V)$ is invertible. Then, $\mathscr{M}(T^{-1})=(\mathscr{M}(T))^{-1}$, where both matrices are with respect to the basis $v_1,\ldots,v_n$.

59. 3E.

60. **Product of vector spaces is a vector space.** Suppose $V_1,\ldots,V_m$ are vector spaces over $\mathbb{F}$. Then, $V_1 \times \cdots \times V_m$ is a vector space over $\mathbb{F}$.

61. **dimension of a product is the sum of the dimensions.** Suppose $V_1,\ldots,V_m$ are finite-dimensional vector spaces. Then, $V_1 \times \cdots \times V_m$ is finite-dimensional and $\dim(V_1 \times \cdots \times V_m)=\dim V_1+\cdots+\dim V_m$.

62. **Products and direct sums.** Suppose $V_1,\ldots,V_m$ are subspaces of V. Define a linear map $\Gamma:(V_1 \times \cdots \times V_m) \to (V_1+\cdots+V_m)$ by $\Gamma(v_1,\ldots,v_m)=v_1+\cdots+v_m$. Then, $V_1+\cdots+V_m$ is direct iff Γ is 1-1.

63. **direct sum iff dimensions add up.** Suppose V is finite-dimensional and $V_1,\ldots,V_m$ are subspaces of V. Then, $V_1+\cdots+V_m$ is direct iff $\dim(V_1+\cdots+V_m)=\dim V_1+\cdots+\dim V_m$.

64. **v + U.** Suppose $v \in V$ and $U \subseteq V$. Then, $v+U=\{v+u \mid u \in U\}$.

65. **Translate.** For $v \in V$ and $U \subseteq V$, the set $v+U$ is called a translate of U.

66. **Quotient Space.** Let U be a subspace of V. Then, the quotient space V/U is the set of all translates of U, that is, $V/U=\{v+U \mid v \in V\}$.

67. **two translates of a subspace are either equal or disjoint.** Suppose U is a subspace of V and $v,w \in V$. Then, $v-w \in U$ iff $v+U=w+U$ iff $(v+U)\cap(w+U) \neq \emptyset$.

68. **Addition and scalar multiplication on Quotient space.** Let U be a subspace of V. Then, we have (for all $v,w \in V$, $\lambda \in F$):

   (a) addition on V/U: $(v+U)+(w+U)=(v+w)+U$.

   (b) scalar multiplication on V/U: $\lambda(v+U)=(\lambda v)+U$.

69. **quotient space is a vector space.** Let U be a subspace of V. Then, the quotient space V/U is a subspace of V under the defined scalar multiplication and vector addition.

70. **quotient map.** Let U be a subspace of V. Then, the quotient map $\pi : V \to V/U$ is the linear map defined by $\pi(v) = v+U$ for each $v \in V$.

71. **dimension of quotient space.** Suppose V is finite-dimensional and U is a subspace of V. Then, $\dim(V/U) = \dim V - \dim U$.

72. **Column rank.** The column rank (rank of the column span of a matrix) is $\operatorname{rank} T_A$.

73. **Theorem.** If A is a rectangular matrix of elements in a field F, then row rank A = column rank A.

74. 3F.

75. **Linear functional.** A linear functional on V is a linear map $\phi : V \to F$.

76. **dual space.** The dual space of V is $V' = \mathscr{L}(V,F)$.

77. **dim space = dim dual space.** Suppose V is finite-dimensional. Then V' is also finite-dimensional and $\dim V = \dim V'$.

78. **dual basis.** If $v_1,\dots,v_n$ is a basis of V, then the dual basis of $v_1,\dots,v_n$ is $\phi_1,\dots,\phi_n$ (elements of V') where $\phi_j(v_k) = 1$ if $k = j$ and $\phi_j(v_k) = 0$ if $k \neq j$.

79. **dual basis gives coefficients for linear combination.** Suppose $v_1,\dots,v_n$ is a basis of V and $\phi_1,\dots,\phi_n$ is dual basis. Then $v = \phi_1(v)v_1 + \cdots + \phi_n(v)v_n$ for each $v \in V$.

80. **dual basis is a basis of dual space.** Suppose V is finite-dimensional. Then the dual basis of V is a basis of V'.

81. **dual map, T'.** Suppose $T \in \mathscr{L}(V,W)$. The dual map of T is $T' \in \mathscr{L}(W',V')$ defined for each $\phi \in W'$ by $T'(\phi) = \phi \circ T$.

82. **algebraic properties of dual maps.** we have $(S+T)' = S'+T'$, $(\lambda S)' = \lambda S'$, $(ST)' = T'S'$.

83. **annihilator.** For $U \subseteq V$, the annihilator of U is $U_0 = \{\phi \in V' \mid \phi(u) = 0 \forall u \in U\}$.

84. **annihilator is a subspace.** If $U \subseteq V$, then $U^0 \subseteq V'$.

85. **dimension of annihilator.** Suppose V is finite-dimensional and $U \subseteq V$. Then $\dim U^0 = \dim V - \dim U$.

86. **condition for annihilator to equal $\{0\}$ or whole space.** Suppose V finite-dimensional and $U \subseteq V$. Then:

   (a) $U^0 = \{0\}$ iff $U = V$.

   (b) $U^0 = V'$ iff $U = \{0\}$.

87. **null space of T'.** Suppose V,W finite-dimensional and $T \in \mathscr{L}(V,W)$. Then:

   (a) $\operatorname{nul} T' = (\operatorname{range} T)^0$.

   (b) $\dim \operatorname{nul} T' = \dim \operatorname{nul} T + \dim W - \dim V$.

88. **T surjective equivalent to T' injective.** Suppose V,W finite-dimensional and $T \in \mathscr{L}(V,W)$. Then T onto iff T' 1-1.

89. **range of T'.** Suppose V,W finite-dim and $T \in \mathscr{L}(V,W)$. Then:

   (a) $\dim \operatorname{range} T' = \dim \operatorname{range} T$.

   (b) $\operatorname{range} T' = (\operatorname{nul} T)^0$.

90. **T injective is equivalent to T' surjective.** Suppose V,W finite-dim and $T \in \mathscr{L}(V,W)$. Then T 1-1 iff T' onto.

91. **matrix of T' is transpose of T.** Suppose V,W finite-dim and $T \in \mathscr{L}(V,W)$. Then $\mathscr{M}(T') = (\mathscr{M}(T))^t$.

92. Ch 4. (NOTHING).

93. 5A.

94. **Invariant subspace.** Suppose $T \in \mathscr{L}(V)$. A subspace $U \subseteq V$ is invariant under T if $Tu \in U$ for all $u \in U$.

95. **Eigenvalue, eigenvector.** Let $T \in \mathscr{L}(V)$. Then $\lambda \in F$ is an eigenvalue of T iff there exists $v \in V$ such that $Tv = \lambda v$ (with $v \neq 0$), where v is eigenvector.

96. **equivalent conditions to be an eigenvalue.** Let V be finite-dim and $T \in \mathscr{L}(V)$ and $\lambda \in F$. Then TFAE:

   (a) λ is an eigenvalue of T.

   (b) $T - \lambda I$ not injective.

   (c) $T - \lambda I$ not surjective.

   (d) $T - \lambda I$ not invertible.

97. **linearly independent eigenvectors.** Let $T \in \mathscr{L}(V)$. Then every list of eigenvectors of T corresponding to different eigenvalues is linearly independent.

98. **operator cannot have more eigenvalues than dimension of space.** Let V be finite-dim. Then each operator on V has at most $\dim V$ distinct eigenvalues.

99. **null space and range of $p(T)$ are invariant under T.** Suppose $T \in \mathscr{L}(V)$ and $p \in \mathscr{P}(F)$. Then $\operatorname{nul} p(T)$ and range $p(T)$ are invariant under T.

100. 5B.

101. **existence of eigenvalues.** Every operator on a finite-dim nonzero complex vector space has an eigenvalue.

102. **existence, uniqueness, and degree of minimal polynomial.** Suppose V finite-dim and let $T \in \mathscr{L}(V)$. Then there is a unique monic polynomial $p \in \mathscr{P}(F)$ of smallest degree such that $p(T) = 0$. Also, $\deg p \leq \dim V$.

103. **minimal polynomial.** Suppose V finite-dim and $T \in \mathscr{L}(V)$. Then the minimal polynomial of T is the unique monic polynomial $p \in \mathscr{P}(F)$ of smallest degree such that $p(T) = 0$.

104. **eigenvalues are the zeros of minimal polynomial.** Let V finite-dim and $T \in L(V)$. Then:

   (a) zeros of the minimal polynomial of T are the eigenvalues of T.

   (b) if V is a complex vector space, then minimal polynomial of T has the form $(z-\lambda_1)\cdot\dots\cdot(z-\lambda_m)$, where $\lambda_1,\dots,\lambda_m$ is a list of all eigenvalues of T, possibly with repetitions.

105. **$q(T) = 0$ iff q is a polynomial multiple of the minimal polynomial.** Let V finite-dim and $T \in L(V)$ and $q \in P(F)$. Then $q(T) = 0$ iff q is a polynomial multiple of the minimal polynomial.

106. **minimal polynomial of a restriction operator.** Let V finite-dim and $T \in L(V)$ and $U \subseteq V$ that is invariant under T. Then minimal polynomial of T is a polynomial multiple of minimal polynomial of $T|_U$.

107. **T not invertible iff constant term of minimal polynomial of T is 0.** Let V finite-dim and $T \in L(V)$. Then T is not invertible iff the constant term in the minimal polynomial of T is 0.

108. **even-dimensional null space.** Let $F = \mathbb{R}$ and V finite-dim and $T \in L(V)$ and $b^2 - 4ac < 0$. Then $\dim(T^2 + bT + cI)$ is an even number.

109. **operators on an odd-dimensional space have eigenvalues.** Every operator on an odd-dimensional vector space has an eigenvalue.

110. 5C.

111. **upper triangular.** A matrix is called upper-triangular if all entries below the main diagonal are zero.

112. **conditions for upper-triangular matrix.** Suppose $T \in L(V)$ and $v_1,\dots,v_n$ is a basis of V. Then TFAE:

   (a) the matrix of T with respect to $v_1,\dots,v_n$ is upper-triangular.

   (b) $\operatorname{span}(v_1,\dots,v_k)$ is invariant under T for each $k = 1,2,\dots,n$.

   (c) $Tv_k \in \operatorname{span}(v_1,\dots,v_k)$ for each $k = 1,\dots,n$.

113. **equation satisfied by operator with upper-triangular matrix.** Suppose $T \in L(V)$ and V has a basis with respect to which T has an upper-triangular matrix with diagonal entries $\lambda_1,\dots,\lambda_n$. Then $(T-\lambda_1 I)\cdot\dots\cdot(T-\lambda_n I) = 0$.

114. **determination of eigenvalues from upper-triangular matrix.** Suppose $T \in L(V)$ has an upper-triangular matrix with respect to some basis of V. Then the eigenvalues of T are precisely the entries on the diagonal of that upper-triangular matrix.

115. **necessary and sufficient condition to have an upper-triangular matrix.** Suppose V is finite-dim and $T \in L(V)$. Then T has an upper-triangular matrix with respect to some basis of V iff the minimal polynomial of T equals $(z-\lambda_1)\cdot\dots\cdot(z-\lambda_n)$ for some $\lambda_i \in F$.

116. **if $F = \mathbb{C}$, then every operator on V has an upper-triangular matrix.** Suppose V is a finite-dim complex vector space and $T \in L(V)$. Then T has an upper-triangular matrix with respect to some basis of V.

117. 5D.

118. **eigenspace, $E(\lambda,T)$.** Suppose $T \in L(V)$ and $\lambda \in F$. Then the eigenspace of T corresponding to λ is $E(\lambda,T) = \operatorname{nul}(T-\lambda I) = \{v \in V \mid Tv = \lambda v\}$.

119. **sum of eigenspaces is a direct sum.** Suppose $T \in L(V)$ and $\lambda_1,\dots,\lambda_m$ are the distinct eigenvalues of T. Then $\sum_i E(\lambda_i,T)$ is a direct sum and $\sum_i \dim E(\lambda_i,T) \leq \dim V$.

120. **conditions equivalent to diagonalizability.** Suppose V finite-dim and $T \in L(V)$. Let $\lambda_1,\dots,\lambda_m$ denote the distinct eigenvalues of T. Then TFAE:

   (a) T is diagonalizable.

   (b) V has a basis consisting of eigenvectors of T.

   (c) $V = \oplus_i E(\lambda_i,T)$

   (d) $\dim V = \sum_i \dim E(\lambda_i,T)$.

121. **enough eigenvalues implies diagonalizability.** Let V be finite-dim and $T \in L(V)$ has $\dim V$ distinct eigenvalues. Then T is diagonalizable.

122. **necessary and sufficient condition for diagonalizability.** Suppose V finite-dim and $T \in L(V)$. Then T diagonalizable iff the minimal polynomial of T equals $(z-\lambda_1)\cdots(z-\lambda_m)$ for some distinct $\lambda_1,\dots,\lambda_i \in F$.

123. **restriction of diagonalizable operator to invariant subspace.** Suppose $T \in L(V)$ and U is a T-invariant subspace of V. Then $T|_U$ is a diagonalizable operator on U.

124. 5E.

125. **commuting operators correspond to commuting matrices.** Suppose $S,T \in L(V)$ and $v_1,\dots,v_n$ is a basis of V. Then S and T commute iff $M(S,(v_1,\dots,v_n))$ and $M(T,(v_1,\dots,v_n))$ commute.

126. **eigenspace is invariant under commuting operators.** Suppose $S,T \in L(V)$ commute and $\lambda \in F$. Then $E(\lambda,S)$ is invariant under T.

127. **simultaneous diagonalizability iff commutativity.** Two diagonalizable operators on the same vector space have diagonal matrices with respect to the same basis iff the two operators commute.

128. **common eigenvector for commuting operators.** every pair of commuting operators on a finite-dim nonzero complex vector space has a common eigenvector.

129. **commuting operators are simultaneously upper-triangularizable.** Suppose V is a finite-dim nonzero complex vector space and S,T are commuting operators on V. Then there is a basis of V with respect to which both S,T have upper-triangular matrices.

130. **eigenvalues of sum and product of commuting operators.** Suppose V is a finite-dim complex vector space and S,T are commuting operators on V. Then:

    (a) every eigenvalue of $S+T$ is an eigenvalue of S plus an eigenvalue of T.

    (b) every eigenvalue of ST is an eigenvalue of S times an eigenvalue of T.

131. 8A.

132. **sequence of increasing null spaces.** Let $T \in L(V)$. Then $\{0\} = \operatorname{nul} T^0 \subseteq \operatorname{nul} T_1 \subseteq \cdots \subseteq \operatorname{nul} T^k \subseteq \ldots$.

133. **equality in the sequence of null spaces.** Let $T \in L(V)$ and m is a nonnegative integer such that $\operatorname{nul} T^m = \operatorname{nul} T^{m+1}$. Then $\operatorname{nul} T^m = \operatorname{nul} T^{m+1} = \ldots$.

134. **null spaces stop growing.** Let $T \in L(V)$. Then $\operatorname{nul} T^{\dim V} = \operatorname{nul} T^{\dim V+1} = \ldots$.

135. **V is the direct sum of $\operatorname{nul} T^{\dim V}$ and $\operatorname{range} T^{\dim V}$.** Let $T \in L(V)$. Then $V = \operatorname{nul} T^{\dim V} \oplus \operatorname{range} T^{\dim V}$.

136. **generalized eigenvector.** Let $T \in L(V)$ and λ be an eigenvalue of T. A vector $v \in V$ ($v \neq 0$) is called a generalized eigenvector of T corresponding to λ if $(T-\lambda I)^k v = 0$ for some $k \in \mathbb{Z}_{>0}$.

137. **a basis of generalized eigenvectors.** Let $F = \mathbb{C}$ and $T \in L(V)$. Then there is a basis of V consisting of generalized eigenvectors of T.

138. **generalized eigenvector corresponds to a unique eigenvalue.** Let $T \in L(V)$. Then each generalized eigenvector of T corresponds to only one eigenvalue of T.

139. **linearly independent generalized eigenvectors.** Let $T \in L(V)$. Then every list of generalized eigenvectors of T corresponding to distinct eigenvalues of T is linearly independent.

140. **nilpotent.** An operator is called nilpotent if some power of it is 0.

141. **nilpotent operator raised to dimension of domain is 0.** Let $T \in L(V)$ be nilpotent. Then $T^{\dim V} = 0$.

142. **eigenvalues of nilpotent operator.** Let $T \in L(V)$. Then:

    (a) if T is nilpotent then 0 is an eigenvalue of T and T has no other eigenvalues.

    (b) if $F = \mathbb{C}$ and 0 is the only eigenvalue of T, then T is nilpotent.

143. **minimal polynomial & upper-triangular matrix of nilpotent operator.** Let $T \in L(V)$. Then TFAE:

    (a) T is nilpotent.

    (b) minimal polynomial of T is z^m for some positive integer m.

    (c) there is a basis of V with respect to which the matrix of T has the form

$$\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$

.

144. 8B.

145. **generalized eigenspace.** Suppose $T \in L(V)$ and $\lambda \in F$. The generalized eigenspace of T corresponding to λ is $G(\lambda,T) = \{v \in V \mid (T-\lambda I)^k \text{ for some } k \in \mathbb{Z}_{>0}\}$, which is the set of generalized eigenvectors of T corresponding to λ, including the 0-vector.

146. **description of generalized eigenspaces.** Suppose $T \in L(V)$ and $\lambda \in F$. Then $G(\lambda,T) = \operatorname{nul}(T-\lambda I)^{\dim V}$.

147. **generalized eigenspace decomposition.**

148. Suppose $F = \mathbb{C}$ and $T \in L(V)$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T. Then:

    (a) $G(\lambda_k, T)$ is invariant under T for each $k = 1, \ldots, m$.

    (b) $(T-\lambda_k I)\,|_{G(\lambda_k,T)}$ is nilpotent for each $k = 1, \ldots, m$.

    (c) $V = \oplus_i G(\lambda_i, T)$.

149. **multiplicity.** Let $T \in L(V)$. The multiplicity of an eigenvalue λ of T is defined to be the dimension of the corresponding generalized eigenspace $G(\lambda,T)$, so multiplicity of λ is $\dim \operatorname{nul}(T-\lambda I)^{\dim V}$.

150. **sum of the multiplicities equals $\dim V$.** Suppose $F = \mathbb{C}$ and $T \in L(V)$. Then the sum of all the multiplicities of all the eigenvalues of T equals $\dim V$.

151. **characteristic polynomial.** Let $F = \mathbb{C}$ and $T \in L(V)$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T, with multiplicities $d_1, \ldots, d_m$. Then the polynomial $(z-\lambda_1)^{d_1} \cdots (z-\lambda_m)^{d_m}$ is called the characteristic polynomial of T.

152. **degree and zeros of the characteristic polynomial.** Let $F = \mathbb{C}$ and $T \in L(V)$. Then:

    (a) characteristic polynomial of T has degree $\dim V$.

    (b) zeros of the characterisit polynomial are the eigenvalues of T.

153. **Cayley-Hamilton theorem.** Let $F = \mathbb{C}$, $T \in L(V)$ and q be the characteristic polynomial of T. Then $q(T) = 0$.

154. **characteristic polynomial is a multiple of minimal polynomial.** Let $F = \mathbb{C}$ and $T \in L(V)$. Then characteristic polynomial of T is a polynomial multiple of the minimal polynomial of T.

155. **multiplicity of an eigenvalue equals number of times on diagonal.** Let $F = \mathbb{C}$ and $T \in L(V)$. Let $v_1, \ldots, v_n$ be a basis of V such that $M(T,(v_1,\ldots,v_n))$ is upper-triangular. The number of times the eigenvalue λ ppears on the diagonal of $M(T,(v_1,\ldots,v_n))$ equals the multiplicity of λ as an eigenvalue of T.

156. **block diagonal matrix with upper-triangular blocks.** Let $F = \mathbb{C}$ and $T \in L(V)$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T with multiplicities $d_1, \ldots, d_m$. Then there is a basis of V with respect to which T has a block diagonal matrix of the form

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix}$$

, where each A_k is a d_k-by-d_k upper-triangular matrix of the form

$$\begin{pmatrix} \lambda_k & & * \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix}$$

.

157. 8C.

158. **jordan basis.** Let $T \in L(V)$. A basis of V is called a Jordan basis for T if with respect to this basis T has a block diagonal matrix

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_p \end{pmatrix}$$

in which each A_k is an upper-triangular matrix of the form

$$\begin{pmatrix} \lambda_k & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_k \end{pmatrix}$$

.

159. **every nilpotent operator has a jordan basis.** Let $T \in L(V)$ be nilpotent. Then there is a basis for V that is a Jordan basis for T.

160. **Jordan form.** Let $F = \mathbb{C}$ and $T \in L(V)$. Then there is a basis of V that is a Jordan basis.

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161. RIBET DEFS MT1.

162. **Endomorphism.** An endomorphism is a group homomorphism from a set to itself (NOTE: does not have to be invertible.)

163. **End V.** The symbol $\operatorname{End} V$ is the set of all endomorphisms on V (and multiplication on $\operatorname{End} V$ is defined to be function composition).

164. **F-Module.** An $F-$module is a generalization of vector spaces over rings.

165. **Linear Map / Linear Transformation.** Let V be a vector space over a field F with $v, w \in V$. Let T be a map on V with $T(v+w) = T(v) + T(w)$ and $T(\lambda v) = \lambda T(v)$ for all $\lambda \in F$. Then, T is called a linear map or linear transformation.

166. **Linear Operator.** If T is a linear transformation on a vector spaces V with $T: V \to V$, then T is linear operator on V.

167. **Spans.** The list $v_1, \ldots, v_n$ spans V iff $T: F^n \to V$ is onto.

168. **Finite-dimensional.** V is finite-dimensional if V is spanned by a finite list of vectors.

169. **Direct Sum of Subspaces.** Let $X_1, \ldots, X_t$ be subspaces of V. Then, their direct sum, $X_1 \oplus \cdots \oplus X_t$, is given by a 1-1 linear map T, with $T: X_1 \times \cdots \times X_t \to V$.

170. **Complement of Subspace.** Let X, Y be subspaces of of V. Then, Y is a complementary subspace of X iff $X + Y = V$ and $X + Y = X \oplus Y$.

171. **Rank, Nullity.** The rank of a linear map is the dimension of the range of the linear map. The nullity is the dimension of the null space of the linear map.

172. **Null Space.** The null space is the set of vectors that are mapped to 0.

173. **Isomorphic Vector Spaces.** Two vector spaces V, W are isomorphic if there exists a linear map $T: V \to W$ that is 1-1 and onto.

174. **Quotient Space.** Suppose U is a subspace of V. Then, the quotient space V/U is the set $V/U = \{v+U \mid v \in V\}$.

175. **Column Rank.** The column rank (rank of the column span of a matrix) is defined to be rankT_A.

176. **Conjugation.** Let A be an $n \times n$ matrix (over F) and let Q be an $n \times n$ matrix (over F). Then, the conjugation of A by Q is $Q^{-1}AQ$.

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177. RIBET DEFS MT2.

178. **Dual Space.** Let V be an F-vector space. Then the dual space of V is $V' = \mathscr{L}(V,F)$ where the elements of V' are called linear functionals.

179. **Annihilator.** For a subspace $U \subseteq V$, we define the annihilator of U to be $U_0 = \{\phi \in V' \mid \phi(u) = 0 \forall u \in U\}$.

180. **Double Dual.** Let V be a finite-dimensional vector space with dual V'. Then the double dual of V is $(V')' = V'' = V$. Also, $\dim V = n = \dim V' = \dim V''$.

181. **Eigenvector / eigenvalue.** Let $T \in \mathscr{L}(V)$. Then an eigenvector of T is a $v \in V$ ($v \neq 0$) such that $Tv = \lambda v$ ($\lambda \in F$ is called an eigenvalue), and v is an eigenvector of T.

182. **Eigenspace.** Let $T \in \mathscr{L}(V)$ and take λ to be an eigenvalue of T. Then, $E(\lambda,T) = \{v \in V \mid Tv = \lambda v\} \neq \emptyset$ is written as V_λ and is called the eigenspace of λ, which is a subspace of V.

183. **Invariant subspace.** E is a T-invariant subspace if $T \in \mathscr{L}(V)$ with $T(E) \subseteq E$.

184. textbfIdempotent. If $e = e^2$, then e is called idempotent.

185. **Generalized Eigenvector.** Consider a minimal polynomial $(x-\lambda_1)^{e_1} \cdot \cdots \cdot (x-\lambda_m)^{e_m}$ on X with $(T-\lambda_1 I)^{e_1} v = 0$. Then, v is called a generalized eigenvector for $\lambda = \lambda_1$.

186. **Characteristic polynomial.** The characteristic polynomial of $T : V \to V$ (with eigenvalues $\lambda_1,\ldots,\lambda_t$) is the polynomial $\prod_{i=1}^{t}(x-\lambda_i)^{\dim X_i}$, where $V = X_1 \oplus \cdots \oplus X_t$.

187. **Simultaneously diagonalizable.** Operators S and T on V are simulatenously diagonalizable if there is a basis of V that consts of vectors that are eigenvectors for both S and T (i.e. there exists a basis $v_1,\ldots,v_n$ of V so that for i, $1 \leq i \leq n$, there are λ_i and μ_i so that $Sv_i = \lambda_i v_i$ and $Tv_i = \mu_i v_i$).

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188. RIBET THMS MT1.

189. **Lemma.** Let F be a field, $\lambda \in F$, V a vector space over F (denoted by V/F), $v \in V$. Then, if $\lambda v = 0$, then $\lambda = 0$ or $v = 0$.

190. **Lemma.** A vector space over a field is a module over a field.

191. **Theorem.** The intersection of a family of subspaces of a vector space V is a subspace of V.

192. **Lemma.** Let $S = \{v_1,\ldots,v_t\}$. Then the subspace of all linear combinations of the elements of S is the span$S$.

193. **Theorem.** Let $L = v_1,\ldots,v_n$ be a list of vectors in a vector space V over a field F and let $T : F^n :\to V$ be linear transformation with $(\lambda_1,\ldots,\lambda_n) \mapsto \lambda_1 v_1 + \cdots + \lambda_n v_n$. Then, we have the following:

    (a) L spans V iff T is onto.

    (b) L is linearly independent iff T is 1-1 iff nul$\,T = \{0\}$.

    (c) L is a basis iff T is 1-1 and onto.

194. **Prop.** Consider $T : F^n \to V$ with $(\lambda_1,\ldots,\lambda_n) \mapsto \lambda_1 v_1 + \cdots + \lambda_n v_n$, so $T(e_i) = v_i$ for all i. Then, T is the unique linear map $F_n \to V$ that sends $e_i \mapsto v_i$ for all i.

195. **Theorem.** Every subspace X of V has complement.

196. **Lemma.** If $v_1,\ldots,v_t$ is linearly dependent list, then there is an index k such that $v_k \in \text{span}(v_1,\ldots,v_{k-1},v_{k+1},\ldots,v_t)$. Furthermore, the span of the list of length $t-1$ gotten by removing v_k from the list is the same as the span of the original list.

197. **Prop.** In a finite-dimensional vector space, the length is of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

198. **Cor.** Two bases of V have the same number of elements.

199. **Prop.** $X+Y$ is direct iff the null space of the sum map is $\{0\}$.

200. **Theorem.** Every subspace of a finite-dimensional vector space is finite-dimensional.

201. **Prop.** Every spanning list for a vector space can be pruned down to a basis of the space.

202. **Cor.** Every finite-dimensional vector space has a basis.

203. **Prop.** In a finite-dimensional vector space, every linearly independent list can be extended to a basis of the space.

204. **Major Theorem.** Every subspace of a finite-dimensional vector space has a complement.

205. **Prop.** Let X,Y be subspaces of a finite-dimensional vector space V. Then:

    (a) $\dim X + \dim Y = \dim V$.

    (b) $X \cap Y = \{0\}$.

    Then, $V = X \oplus Y$.

206. **Prop.** $\dim(X \oplus Y) = \dim X + \dim Y$.

207. **Prop.** If V is a finite-dimensional vector space (with $\dim V = n$), then every subspace has dimension at most n.

208. **Prop.** Let $\dim V = n$. Then, a linearly independent list of vectors of V with length n is a basis for V.

209. **Prop.** Let $\dim V = n$. Then, every spanning list for V of length n is a basis for V.

210. **Lemma.** The list $(x_1,0),\ldots,(x_t,0);(0,y_1),\ldots,(0,y_k)$ of length $t+k$ is a basis of $X \times Y$.

211. **Cor.** $\dim(X \times Y) = \dim X + \dim Y$.

212. **Cor.** Let $T : V \to W$ be a linear map with $\dim V = d$. Then, rank$T \leq d$.

213. **Rank-Nullity Theorem.** $\dim V = \text{rank}V + \text{nullity}V$.

214. **Prop.** If $T : V \to W$ is 1-1, then nullity$T = 0$.

215. **Cor.** If $T : V \to W$ is 1-1 and onto, then $\dim V = \dim W$.

216. **Theorem.** The set of linear maps $V \to W$ is a vector space $L \cdot (F^n, W) \to T \longrightarrow (Te_1,\ldots,Te_n) \in W^n$.

217. **Theorem.** $\dim(X+Y) = \dim X + \dim Y - \dim(X \cap Y)$.

218. **Cor.** $\dim(V/X) = \dim V - \dim X$.

219. **Theorem.** If A is a rectangular matrix with elements in a field F, then row rank A = column rank A.

220. **Prop.** Let $T : V \to W$ be 1-1. Then, $\dim W \geq \dim V$.

221. **Prop.** Let $T : V \to W$ be onto. Then, $\dim V \geq \dim W$.

222. **Prop.** Let $T : V \to W$ and $\dim V = \dim W$. Then, T 1-1 iff T onto iff T bijective iff T invertible.

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223. RIBET THMS MT2

224. **Lemma.** Let V be a finite-dimensional vector space and U a subspace of V. Then, $\dim U_0 = \dim V - \dim U$.

225. **Theorem.** Every linear functional on a subspace of V can be extended to V.

226. **Note.** Annihilator is the dual of the quotient subspace.

227. **Theorem.** Let $T : V \to W$ and $T' : W' \to V'$. THen $\mathscr{M}(T)$ and $\mathscr{M}(T')$ are transposes of each other.

228. **Lemma.** U^0 has dimension $\dim V - \dim U$.

229. **Cor.** The annihilator of U is $\{0\}$ iff $U = V$. The annihilator of U is V iff $U = \{0\}$.

230. **Prop.** If $T : V \to W$ is a linear map, then the null space of T' is the annihilator of the range of T. We have ann(range$T$) $= \{\psi : W \to F \mid \phi(Tv) = 0 \text{ for all } v \in V, T'(\psi)(v) = 0, T'\psi = 0, \phi \in \text{nul}(T')\}$.

231. **Cor.** If $T : V \to W$ is a linear map between finite-dimensional F-vector spaces, then $\dim \text{nul}(T') = \dim \text{nul}(T) + \dim W - \dim V$.

232. **Cor.** The linear map T is onto iff T' is 1-1.

233. **Cor.** If $T : V \to W$ is a linear map between finite-dimensional vector spaces, then T' and T have equal ranks.

234. **Cor.** We have range$T = (\text{nul}\,T)^0$.

235. **Theorem.** Let F be a finite field with $q = |F|$. Then, $a^q = a$ for all $a \in F$.

236. **Theorem.** If F is a finite field, then $|F| = p^n$ for some prime p and integer $n \geq 1$.

237. **Theorem.** Take an ideal I in $\mathbb{Z}$. Then, I is equal to either $\{0\}$ or $m\mathbb{Z}$ (where $m \in \mathbb{Z}_{>0}$).

238. **Theorem.** $F[x]$ is a principal ideal domain; that is, it is an integral domain in which every ideal in $F[x]$ is principal.

239. **Theorem.** Let $T : V \to V$, V finite-dimensional, and let $\alpha : F[x] \to \mathscr{L}(V)$, with $f \mapsto f(T)$. Also, we have $\ker\alpha$ to be the principal ideal $(m(x))$. Then, $m(x)$ is the minimal polynomial of T and has degree $\leq n^2$.

240. **Cayley-Hamilton Theorem.** Let $T : V \to V$, V finite-dimensional, and let $\alpha : F[x] \to \mathscr{L}(V)$, with $f \mapsto f(T)$. Also, we have $\ker\alpha$ to be the principal ideal $(m(x))$, where $m(x)$ is the minimal polynomial of T. Then, the characteristic polynomial is in $\ker\alpha$; that is, we can plug in the matrix for T into its characteristic polynomial and we get that it is equal to the 0-matrix.

241. **Prop.** For $f(x) \in F[x]$ and $\lambda \in F$, $f(\lambda) = 0$ iff f is divisible by $x-\lambda$, where $x-\lambda$ is an irreducible polynomial.

242. **Cor.** A polynomial of degree n can have at most n roots.

243. **Cor.** A polynomial with infinitely many roots is identically the zero polynomial.

244. **Lemma.** Let $f \in \mathbb{R}[x]$ be a real polynomial. If λ is a complex root of f, so is $\overline{\lambda}$, which is the complex conjugate of λ.

245. **Prop.** A scalar λ is an eigenvalue of $T : V \to V$ iff $T - \lambda I$ is not 1-1.

246. **Cor.** The map $T : V \to V$ is invertible iff 0 is not an eigenvalue of T.

247. **Key lemma.** Every list of eigenvectors of T that corresponds to distinct eigenvalues of T is a linearly independent list.

248. **Cor.** Let $\lambda_1, \dots, \lambda_t$ be distinct eigenvalues and take $E_i = E(\lambda_i, T) = \{v \in V \mid Tv = \lambda_i v\} \subseteq V$. Now, take $E_1 \times \cdots \times E_t$. Then there exists a summation map $E_1 \times \cdots \times E_t \xrightarrow{\text{sum}} V$ with $(v_1, \dots, v_t) \mapsto v_1 + \cdots + v_t$. Then, the sum map is 1-1.

249. **Cor.** Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

250. **Prop.** Suppose T is an operator on an F-vector space V. If $f \in F[x]$ is a polynomial satisfied by T (meaning $f(T) = 0$), then every eigenvalue of T on V is a root of f.

251. **Cor.** Suppose λ is an eigenvalue of operator T on a finite-dimensional F-vector space. Then λ is a root of the minimal polynomial of T.

252. **Prop.** Let T be an operator on a finite-dimensinoal vector space. Suppose λ is a root of the minimal polynomial. Then λ is an eigenvalue of T.

253. **Theorem.** All operators on a nonzero finite-dimensional vector space over an algebraically closed field have at least one eigenvalue.

254. **Prop.** Assume that $F = \mathbb{R}$ and that $f(x) := x^2 + bx + c$ is an irreducible polynomial. If $T \in \mathscr{L}(V)$ and V is finite-dimensional, then the null space of $f(T)$ is even-dimensional.

255. **Prop (honors version).** Let T be an operator on a finite-dimensional vector space over F. If p is an irreducible polynomial over F, then the dimension of the null space of $p(T)$ is a multiple of the degree of p.

256. **Prop.** $F[x]/(p)$ (where p is irreducible) is a field.

257. **Formula.** $\dim_F V = [K : F] \cdot \dim_K V = \dim_F K \cdot \dim_K V$.

258. **Cor.** Every operator on an odd-dimensional $\mathbb{R}$-vector space has an eigenvalue.

259. **Prop.** If T is an operator on a finite-dimensional F-vector space, then the minimal polynomial of T has degree at most $\dim V$.

260. **Prop.** If T is upper-triangular with respect to some basis of V, and if the diagonal entries of an upper-triangular matrix representation of T are $\lambda_1, \dots, \lambda_n$, then $(T - \lambda_1 I) \cdot \cdots \cdot (T - \lambda_n I) = 0$.

261. **Prop.** Let V be a finite-dimensional vector space and $T \in \mathscr{L}(V)$ and let $\lambda_1, \dots, \lambda_m$ be the eigenvalues of T. Then, $V = \oplus E(\lambda_i, T)$ iff T is diagonalizable.

262. **Prop.** TFAE.

   (a) T is diagonalizable.

   (b) V has a basis consisting of eigenvectors.

   (c) The direct sum $\bigoplus_i V_{\lambda_i}$ is all of V.

   (d) $\dim\left(\bigoplus_i V_{\lambda_i}\right) = \dim V$.

263. **Prop.** If $T : V \to V$ has $\dim V$ different eigenvalues, then T is diagonalizable.

264. **Prop.** The operator $T : V \to V$ is diagonalizable iff its minimal polynomial splits completely as a product of distinct linear factors of the form $x - r$.

265. **Jordan Canonical Form.** X can be written as a direct sum of Jordan blocks, where $\sum$ dim(block) = $\dim X$.

266. **Lemma.** Let $X = \oplus \operatorname{span}(U_i v)$ for $i \in \{0, \dots, k_1\}$. If Z is a subspace of X' that is U'-invariant, then $\operatorname{ann}(Z) =: Y$ is U-invariant.

267. **Lemma.** Suppose S and T are commuting operators on V. If λ is an eigenvalue for T on V, then the eigenspace $E(\lambda, T)$ is S-invariant.

268. **Theorem.** The diagonalize operatosr on the same finite-dimensional vector space are simulateneously diagonalizable iff they commute with each other.

269. **Theorem.** Every pair of commuting operators on a finite-dimensional nonzero complex vector spcae has a common eigenvector.

270. **Prop.** Two commuting operators on a finite-dimensional nonzero complex vector space can be simultaneously upper-triangularized.

271. **Prop.** We have:

   (a) Every eigenvalue of $S + T$ is the sum of an eigenvalue of S and an eigenvalue of T.

   (b) Every eigenvalue of ST is the product of an eigenvalue of S and an eigenvalue of T.