

- 1C.
- 5 — Is  $R^2$  a subspace of the complex vector space  $C^2$ ? — No, check scalar multiplication by  $i$ .
- 7 — prove/disprove: If  $U$  is a nonempty subset of  $R^2$  that is closed under addition and taking additive inverses, then  $U$  is a subspace. — No, take  $U = \{(x, 0) \mid x \in Q\}$ .
- 8 — want an example of a subset  $U$  of  $R^2$  that satisfies scalar multiplication but not vector addition — take  $x$ -axis,  $y$ -axis.
- 13 — prove that union of three subspaces of  $V$  is a subspace iff one contains the other two — suffices to show if  $W$  is the union of three of its subspaces, then then one of the three subspaces is contained in the union of the other two, by applying result of prob 1c.12.
- 18 — does the operation of addition on subspaces of  $V$  have additive identity? which subspaces have inverses? — add. id. is  $\{0\}$  and only  $\{0\}$  has additive inverse.
- 19 — prove/disprove: If  $V_1, V_2, U$  subspaces of  $V$ , then  $V_1 + U = V_2 + U$  implies  $V_1 = V_2$  — false. take  $V_1, V_2, U$  to be  $x$ -axis,  $y$ -axis,  $y = x$  respectively.
- 23 — prove/disprove: If  $V_1, V_2, U$  subspaces of  $V$  so that  $V = V_1 \oplus U = V_2 \oplus U$ , then  $V_1 = V_2$ . — use same counterexample as in problem above.
- 2A.
- 7b — Show if  $C$  is a vector space over  $C$ , then  $1 + i, 1 - i$  is L.D. — for L.D. relation, choose coeffs  $a = i$  and  $b = 1$  for  $1 + i, 1 - i$ , resp.
- 17 — show that  $V$  is inf-dim iff there is a sequence  $v_1, v_2, \dots \in V$  so that  $v_1, \dots, v_m$  l.i. for all positive integers  $m$ . — for reverse direction, prove contrapositive.
- 19 — show that real vector space of all cts real values functions on  $[0, 1]$  is inf-dim. — look at polynomials.
- 5 — let  $V$  be finite-dim and  $V = U + W$ . show there is a basis of  $V$  consisting of vectors from  $U \cup W$ . — concatenate bases of  $U$  and  $W$ .
- 2C.
- 8 — let  $v_1, \dots, v_m$  be linearly independent in  $V$  and  $w \in V$ . show  $\dim \text{span}(v_1 + w, \dots, v_m + w) \geq m - 1$ . — look at  $v_1 - v_2, \dots, v_1 - v_m$  and it is contained in the dimspan.
- 10 — let  $m$  be a positive integer. for  $0 \leq k \leq m$ , let  $p_k = x^k(1 - x)^{m-k}$ . show  $p_0, \dots, p_m$  is a basis of  $P_m(F)$ . — set linear dependence relation and choose  $x$ -values to substitute.

- 11 — Let  $U, W$  be 4-dim subspaces of  $C^6$ . show there exist two vectors in  $U \cap W$  that aren't scalar multiples of each other. — equivalent to showing  $\dim(U \cap W) \geq 2$ .
- 16 — let  $V$  be finite-dim and  $U$  a strict subspace of  $V$ . let  $n = \dim U$  and  $m = \dim V$ . show there are  $n - m$  subspaces of  $V$  each with  $\dim 1$  whose intersection is  $U$  — fix a basis of  $U$  and extend it to basis of  $V$  and construct subspaces that each 'delete' a vector in the extension of  $U$ .
- 3A.
- 8 — give a function  $\phi : R^2 \rightarrow R$  so that  $\phi(\lambda v) = \lambda\phi(v)$  but  $\phi$  not linear — take  $\phi(x, y) = (x^3 + y^3)^{1/3}$ .
- 10 — prove/disprove: if  $q \in P(R)$  and  $T : P(R) \rightarrow P(R)$  by  $Tp = q \circ p$ , then  $T$  is linear — false. take  $q = x^2$ .
- 11 — let  $V$  be finitedim and  $T \in L(V)$ . show  $T = \lambda I$  iff  $ST = TS$  for all  $S \in L(V)$ . — for reverse direction, try contrapositive and look at  $\ker T$ .
- 12 — let  $U$  be a strict subspace of  $V$ . Let  $S \in L(U, W)$  and  $S \neq 0$ . Let  $T(v) = Sv$  if  $v \in U$  and  $Tv = 0$  if  $v \notin U$ . show  $T$  is not linear. — pick  $v_1 \in U$  and  $v_2 \notin U$ .
- 17 — let  $V$  be finitedim. show the only two-sided ideals of  $L(V)$  are  $\{0\}$  and  $L(V)$ . — let  $w$  be so that  $Tw \neq 0$ . let  $S_k : V \rightarrow V$  that sends  $v_j$  to 0 for  $j \neq k$  and  $v_k$  to  $w$ . put  $R_k$  so that  $R_k(Tw) = v_k$ , and look at  $R_kTS_kv_j$ .
- 3B.
- 15 — Suppose there is a linear map on  $V$  so that both null space and range of it are finitedim. show that  $V$  is finite dim. — look at basis  $Tv_1, \dots, Tv_n$  for range and  $w_1, \dots, w_k$  for null space.
- 19 — Let  $W$  be finitedim and  $T \in L(V, W)$ . show  $T$  is 1-1 iff there exists  $S \in L(W, V)$  so that  $ST = I$  on  $V$ . — letting  $T : V \rightarrow W$  be 1-1 and looking at  $U = \text{range } T$ , put  $S : U \rightarrow V$  as the inverse of  $T$  and extend to  $S : W \rightarrow V$ .
- 20 — let  $W$  be finite-dim and  $T \in L(V, W)$ . Show  $T$  onto iff there exists  $S \in L(W, V)$  so that  $TS = I$  on  $W$ . — use onto-ness of  $T$  and look at restriction of  $T$  to  $X$ , the complement of  $\text{null } T$ . do isomorphism  $X \cong W$  and put  $S : W \rightarrow X$  so that  $TS = I$ .
- 3C.
- 5 — Let  $V, W$  be finitedim and  $T \in L(V, W)$ . show there is a basis of  $V$  and a basis of  $W$  so that in these bases, all entries of  $M(T)$  are 0 except those in entries row  $k$  col  $k$  if  $1 \leq k \leq \text{range } T$ . —  $U = \text{nul } T$  and  $X$  is complement to  $U$  in  $V$ . put bases of  $X$  and  $U$ . find bases of  $\text{range } T$  and complete to get basis of  $W$ .

- 6 — Let  $v_1, \dots, v_n$  be basis of  $V$  and  $W$  is finitedim and let  $T \in L(V)$ . show there is a basis  $w_1, \dots, w_m$  so that all entries of  $M(T)$ , in these bases, are 0 except possibly a 1 in the first row, first col. — first column is  $Tv_1$ . consider when  $Tv_1 = 0, \neq 0$ . put basis  $W = \text{span}(Tv_1, w_2, \dots, w_m)$ .
- 7 — Let  $w_1, \dots, w_n$  a basis of  $W$  and  $V$  finitedim and  $T \in L(V, W)$ . Show there is a basis  $v_1, \dots, v_m$  of  $V$  so that all entries in first row of  $M(T)$ , in these bases, are 0 except possibly a 1 in first row, first col. — Look at  $T' : W' \rightarrow V'$  and apply 3c.6 result.
- 3D.
- 10 — Let  $V, W$  be finite dim and  $U \subseteq V$ . put  $E = \{T \in L(V, W) \mid U \subseteq \text{nul } T\}$ . find a formula for  $\dim E$  in terms of  $\dim V, \dim U, \dim W$ . — put  $\Phi : L(V, W) \rightarrow L(U, W)$  by  $\phi(T) = T|_U$  and find range and null space.
- 19 — let  $V$  be finitedim and  $T \in L(V)$ . show  $T$  has same matrix with respect to every basis of  $V$  iff  $T = \lambda I$ . — fix a matrix of  $T$  and for basis  $v_1, \dots, v_m$  of  $V$ ,  $v_1, \dots, (1/2)v_k, \dots, v_m$  is also basis; scale and edit.
- 20 — let  $q \in P(R)$ . show there is a polynomial  $p \in P(R)$  so that  $q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ . — define  $T : P(R) \rightarrow P(R)$  by  $Tp = (x^2 + x)p''(x) + 2xp'(x) + p(3)$  and show  $T$  is 1-1 (and by finitedim of domain, codomain) thus  $T$  is onto.
- 3E.
- 9 — Show a nonempty subset  $A$  of  $V$  is a translate of some subspace of  $V$  iff  $\lambda v + (1 - \lambda)w \in A$  for all  $v, w \in A, \lambda \in F$ . — for converse, fix  $x \in A$  attempt for  $A = x + U$ , where  $U = \{a - x \mid a \in A\}$ .
- 17 — Let  $U$  be a subspace of  $V$  so that  $\dim V/U = 1$ . show there is  $\phi \in L(V, F)$  so that  $\text{nul } \phi = U$ . — put  $T : V/U \rightarrow F$  that sends everything to  $1 \in F$ . let  $\phi$  be composite of  $\pi : V \rightarrow V/U$  and  $T$ .
- 3F.
- 6 — let  $\phi, \beta \in V'$ . show  $\text{nul } \phi \subseteq \text{nul } \beta$  iff there is  $c \in F$  so that  $\beta = c\phi$ . — by a previous problem, there is  $S \in L(F)$  so that  $\beta = S\phi$ .
- 26 — let  $V$  be finitedim and  $\Omega$  be a subspace of  $V'$ . show  $\Omega = \{v \in V \mid \phi(v) = 0 \forall \phi \in \Omega\}^0 = U^0$ . — show  $U = \bigcap_{i=1}^m (\text{nul } \phi_i)$ .
- 5A.
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