

1. 1A. (NOTHING)
2. 1B.
3. **Vector Space.** A vector space V is a set that has scalar multiplication and vector addition defined on it with the following properties:
 - (a) Additive commutativity.
 - (b) Additive associativity of vectors $(u + (v + w) = (u + v) + w)$ and multiplicative associativity for scalars $((ab)v = a(bv))$.
 - (c) Additive identity.
 - (d) Additive inverses.
 - (e) Multiplicative identity.
 - (f) BOTH distributive properties.
4. **V-space (unique additive identity)** A vector space has a unique additive identity.
5. **V-space (unique additive inverses)** Every element in a vector space has a unique additive inverse.
6. 1C.
7. **Subspace.** A subset $U \subseteq V$ is a subspace of V if it is a vector space with the same additive identity, scalar multiplication, and vector addition as defined on V .
8. **Conditions for a Subspace.** A subset $U \subseteq V$ is a subspace of V iff U is closed under vector addition, scalar multiplication, and contains the "zero" element as in V .
9. **Sums of Subspaces.** Let V_1, \dots, V_n be subspaces of V . Then, we have the sum of subspaces as $V_1 + \dots + V_n = \{v_1 + \dots + v_n \mid v_i \in V_i \text{ for all } i\}$.
10. **Smallest subspace containing each subspace** Suppose V_1, \dots, V_n are subspaces of V . Then, $V_1 + \dots + V_n$ is the smallest subspace of V containing V_1, \dots, V_n .
11. **Direct Sum.** Suppose V_1, \dots, V_m are subspaces of V . Then:
 - (a) The sum $V_1 + \dots + V_m$ is direct if each element of $V_1 + \dots + V_m$ can be written uniquely as a sum $v_1 + \dots + v_m$, where $v_i \in V_i$ for all i .
 - (b) If $V_1 + \dots + V_m$ is a direct sum, then we write $V_1 \oplus \dots \oplus V_m$.
12. **Conditions for a direct sum.** Suppose V_1, \dots, V_n are subspaces of V . Then, $V_1 + \dots + V_n$ is direct iff the only way to write 0 from $v_1 + \dots + v_n$ is by taking $v_i = 0$ for all i .
13. **Direct sum of subspaces.** If U, W are subspaces of V , then $U + W$ is direct iff $U \cap W = \{0\}$.
14. 2A.
15. **Span is the smallest containing subspace.** The span of a list of vectors in V is the smallest subspace containing all of the vectors in the list.
16. **Zero polynomial.** The zero polynomial is said to have degree $-\infty$.
17. **Linear Independence.** A list of vectors $v_1, \dots, v_n \in V$ is said to be linearly independent if $a_1 v_1 + \dots + a_n v_n = 0$ implies $a_i = 0$ for all i . Also, the empty list $()$ is said to be linearly independent.
18. **Linear Dependence.** A list of vectors v_1, \dots, v_n is said to be linearly dependent if $a_1 v_1 + \dots + a_n v_n = 0$ implies $a_i \neq 0$ for some i .
19. **Linear Dependence Lemma.** Suppose v_1, \dots, v_m is a linearly dependent list in V . Then, there exists $k \in \{1, \dots, m\}$ such that $v_k \in \text{span}(v_1, \dots, v_{k-1})$. Furthermore, if k satisfies the condition in the previous sentence and the k^{th} term is removed from v_1, \dots, v_m , then the span of the remaining list equals $\text{span}(v_1, \dots, v_m)$.
20. **length of linearly independent list ; length of spanning list.** In a finite-dimensional vector space, the length of every linearly independent list is at most the length of every spanning list of vectors.
21. **Finite Dimensional subspaces.** Every subspace of a finite-dimensional vector space is finite dimensional.
22. 2B.
23. **Basis.** A basis of V is a list of vectors that is linearly independent and spans V .
24. **Criterion for basis.** A list of vectors $v_1, \dots, v_n \in V$ is a basis of V iff every $v \in V$ can be written uniquely in the form $v = a_1 v_1 + \dots + a_n v_n$, where $a_i \in F$ for all i .
25. **Every spanning list contains a basis.** Every spanning list in a vector space can be reduced to a basis of the vector space.
26. **Basis of finite-dimensional vector space.** Every finite-dimensional vector space has a basis.
27. **Every linearly independent list extends to a basis.** Every linearly independent list in a finite-dimensional vector space can be extended to a basis of the vector space.
28. **Every subspace of V is part of a direct sum equal to V .** Suppose V is finite-dimensional and U is a subspace of V . Then, there is a subspace W of V such that $V = U \oplus W$.
29. 2C.
30. **Basis length does not depend on basis.** Any two bases of a finite-dimensional vector space have the same length.
31. **Dimension of a subspace.** If V is finite-dimensional and U is a subspace of V , then $\dim U \leq \dim V$.
32. **Linearly independent list of the right length is a basis.** Suppose V is finite-dimensional. Then, every linearly independent list of vectors in V (with list length equal to $\dim V$) is a basis of V .
33. **Subspace of full dimension equals the whole space.** Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Then, $U = V$.
34. **Spanning list of the right length is a basis.** Suppose V is finite-dimensional. Then, every spanning list of V of length $\dim V$ is a basis of V .
35. **Dimension of a sum.** If V_1, V_2 are subspaces of a finite-dimensional vector space, then $\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$.
36. 3A.
37. **Set of Linear Maps.** The linear of linear maps from $V \rightarrow W$ is written $\mathcal{L}(V, W)$ and the set of linear maps from $V \rightarrow V$ is written $\mathcal{L}(V)$.
38. **Linear Map lemma.** Suppose v_1, \dots, v_n is a basis of V and $w_1, \dots, w_n \in W$. Then, there exists a unique linear map $T : V \rightarrow W$ such that $T v_k = w_k$ for each k .
39. **Linear maps take 0 to 0.** Suppose $T : V \rightarrow W$ is a linear map. Then, $T(0) = 0$.
40. 3B.
41. **null space is a subspace.** Suppose $T \in \mathcal{L}(V, W)$. Then, T is a subspace of V .
42. **injectivity iff null is 0.** Let $T \in \mathcal{L}(V, W)$. Then, T is 1-1 iff $\text{null } T = \{0\}$.
43. **range is a subspace.** If $T \in \mathcal{L}(V, W)$, then $\text{range } T$ is a subspace of W .
44. **Fundamental Theorem of Linear Maps.** Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then, $\text{range } T$ is finite dimensional and $\dim V = \dim \text{null } T + \dim \text{range } T$.
45. **linear map to a lower-dim space is not 1-1.** Suppose V, W are finite-dimensional vector spaces such that $\dim V > \dim W$. Then, no linear map from $V \rightarrow W$ is 1-1.
46. **linear map to a higher-dim space is not onto.** Suppose V, W are finite-dimensional vector spaces such that $\dim V < \dim W$. Then, no linear map from $V \rightarrow W$ is onto.
47. 3C.
48. **Prop.** $ST = I$ iff $TS = I$ (on vector spaces of the same domain).
49. **Prop.** Let V, W be finite-dimensional with $\dim W = \dim V$. Let $S \in \mathcal{L}(W, V)$, $T \in \mathcal{L}(V, W)$. Then, $ST = I$ iff $TS = I$.
50. 3D.
51. **Theorem.** Let V, W be finite-dimensional vector spaces such that $\dim V = \dim W$ and let $T \in \mathcal{L}(V, W)$. Then, T is invertible iff T is 1-1 iff T is onto.
52. **isomorphism.** An isomorphism is an invertible linear map.
53. **dimension and isomorphic.** Two finite-dimensional vector spaces are isomorphic iff they have the same dimension.
54. **Theorem.** Suppose V and W are finite-dimensional. Then, $\mathcal{L}(V, W)$ is finite-dimensional and $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$.
55. **ST=I iff TS=I (on vector spaces of the same dimension).** Suppose V and W are finite-dimensional vector spaces of the same dimension, $S \in \mathcal{L}(W, V)$, $T \in \mathcal{L}(V, W)$. Then $ST = I$ iff $TS = I$.
56. **matrix of identity operator with respect to two bases.** Suppose u_1, \dots, u_n and v_1, \dots, v_n are two bases of V . Then, the matrices $\mathcal{M}(I; u_1, \dots, u_n; v_1, \dots, v_n)$ and $\mathcal{M}(I; v_1, \dots, v_n; u_1, \dots, u_n)$ are invertible and are inverses of each other.
57. **Change of basis formula.** Let $T \in \mathcal{L}(V, W)$. Suppose u_1, \dots, u_n and v_1, \dots, v_n are two bases of V . Let $A = \mathcal{M}(T; u_1, \dots, u_n)$ and $B = \mathcal{M}(T; v_1, \dots, v_n)$ and $C = \mathcal{M}(I; u_1, \dots, u_n; v_1, \dots, v_n)$. Then, $A = C^{-1} B C$.
58. Suppose that v_1, \dots, v_n is a basis of V and $T \in \mathcal{L}(V)$ is invertible. Then, $\mathcal{M}(T^{-1}) = (\mathcal{M}(T))^{-1}$, where both matrices are with respect to the basis v_1, \dots, v_n .
59. 3E.
60. **Product of vector spaces is a vector space.** Suppose V_1, \dots, V_m are vector spaces over \mathbb{F} . Then, $V_1 \times \dots \times V_m$ is a vector space over \mathbb{F} .
61. **dimension of a product is the sum of the dimensions.** Suppose V_1, \dots, V_m are finite-dimensional vector spaces. Then, $V_1 \times \dots \times V_m$ is finite-dimensional and $\dim(V_1 \times \dots \times V_m) = \dim V_1 + \dots + \dim V_m$.
62. **Products and direct sums.** Suppose V_1, \dots, V_m are subspaces of V . Define a linear map $\Gamma : (V_1 \times \dots \times V_m) \rightarrow (V_1 + \dots + V_m)$ by $\Gamma(v_1, \dots, v_m) = v_1 + \dots + v_m$. Then, $V_1 + \dots + V_m$ is direct iff Γ is 1-1.
63. **direct sum iff dimensions add up.** Suppose V is finite-dimensional and V_1, \dots, V_m are subspaces of V . Then, $V_1 + \dots + V_m$ is direct iff $\dim(V_1 + \dots + V_m) = \dim V_1 + \dots + \dim V_m$.
64. **v + U.** Suppose $v \in V$ and $U \subseteq V$. Then, $v + U = \{v + u \mid u \in U\}$.
65. **Translate.** For $v \in V$ and $U \subseteq V$, the set $v + U$ is called a translate of U .
66. **Quotient Space.** Let U be a subspace of V . Then, the quotient space V/U is the set of all translates of U , that is, $V/U = \{v + U \mid v \in V\}$.
67. **two translates of a subspace are either equal or disjoint.** Suppose U is a subspace of V and $v, w \in V$. Then, $v - w \in U$ iff $v + U = w + U$ iff $(v + U) \cap (w + U) \neq \emptyset$.

68. **Addition and scalar multiplication on Quotient space.** Let U be a subspace of V . Then, we have (for all $v, w \in V, \lambda \in F$):

- (a) addition on V/U : $(v+U) + (w+U) = (v+w) + U$.
- (b) scalar multiplication on V/U : $\lambda(v+U) = (\lambda v) + U$.

69. **quotient space is a vector space.** Let U be a subspace of V . Then, the quotient space V/U is a subspace of V under the defined scalar multiplication and vector addition.

70. **quotient map.** Let U be a subspace of V . Then, the quotient map $\pi: V \rightarrow V/U$ is the linear map defined by $\pi(v) = v+U$ for each $v \in V$.

71. **dimension of quotient space.** Suppose V is finite-dimensional and U is a subspace of V . Then, $\dim(V/U) = \dim V - \dim U$.

72. **Column rank.** The column rank (rank of the column span of a matrix) is $\text{rank} T_A$.

73. **Theorem.** If A is a rectangular matrix of elements in a field F , then $\text{row rank } A = \text{column rank } A$.

74. 3F.

75. **Linear functional.** A linear functional on V is a linear map $\phi: V \rightarrow F$.

76. **dual space.** The dual space of V is $V' = \mathcal{L}(V, F)$.

77. **dim space = dim dual space.** Suppose V is finite-dimensional. Then V' is also finite-dimensional and $\dim V = \dim V'$.

78. **dual basis.** If v_1, \dots, v_n is a basis of V , then the dual basis of v_1, \dots, v_n is ϕ_1, \dots, ϕ_n (elements of V') where $\phi_j(v_k) = 1$ if $k = j$ and $\phi_j(v_k) = 0$ if $k \neq j$.

79. **dual basis gives coefficients for linear combination.** Suppose v_1, \dots, v_n is a basis of V and ϕ_1, \dots, ϕ_n is dual basis. Then $v = \phi_1(v)v_1 + \dots + \phi_n(v)v_n$ for each $v \in V$.

80. **dual basis is a basis of dual space.** Suppose V is finite-dimensional. Then the dual basis of V is a basis of V' .

81. **dual map, T' .** Suppose $T \in \mathcal{L}(V, W)$. The dual map of T is $T' \in \mathcal{L}(W', V')$ defined for each $\phi \in W'$ by $T'(\phi) = \phi \circ T$.

82. **algebraic properties of dual maps.** we have $(S+T)' = S' + T', (\lambda S)' = \lambda S', (ST)' = T'S'$.

83. **annihilator.** For $U \subseteq V$, the annihilator of U is $U_0 = \{\phi \in V' \mid \phi(u) = 0 \forall u \in U\}$.

84. **annihilator is a subspace.** If $U \subseteq V$, then $U^0 \subseteq V'$.

85. **dimension of annihilator.** Suppose V is finite-dimensional and $U \subseteq V$. Then $\dim U^0 = \dim V - \dim U$.

86. **condition for annihilator to equal $\{0\}$ or whole space.** Suppose V finite-dimensional and $U \subseteq V$. Then:

- (a) $U^0 = \{0\}$ iff $U = V$.
- (b) $U^0 = V'$ iff $U = \{0\}$.

87. **null space of T' .** Suppose V, W finite-dimensional and $T \in \mathcal{L}(V, W)$. Then:

- (a) $\text{nul } T' = (\text{range } T)^0$.
- (b) $\dim \text{nul } T' = \dim \text{nul } T + \dim W - \dim V$.

88. **T surjective equivalent to T' injective.** Suppose V, W finite-dimensional and $T \in \mathcal{L}(V, W)$. Then T onto iff T' 1-1.

89. **range of T' .** Suppose V, W finite-dim and $T \in \mathcal{L}(V, W)$. Then:

- (a) $\dim \text{range } T' = \dim \text{range } T$.
- (b) $\text{range } T' = (\text{nul } T)^0$.

90. **T injective is equivalent to T' surjective.** Suppose V, W finite-dim and $T \in \mathcal{L}(V, W)$. Then T 1-1 iff T' onto.

91. **matrix of T' is transpose of T .** Suppose V, W finite-dim and $T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(T') = (\mathcal{M}(T))^t$.

92. Ch 4. (NOTHING).

93. 5A.

95. RIBET DEFS MT1.

96. **Endomorphism.** An endomorphism is a group homomorphism from a set to itself (NOTE: does not have to be invertible.)

97. **End V.** The symbol $\text{End } V$ is the set of all endomorphisms on V (and multiplication on $\text{End } V$ is defined to be function composition).

98. **F-Module.** An F -module is a generalization of vector spaces over rings.

99. **Linear Map / Linear Transformation.** Let V be a vector space over a field F with $v, w \in V$. Let T be a map on V with $T(v+w) = T(v) + T(w)$ and $T(\lambda v) = \lambda T(v)$ for all $\lambda \in F$. Then, T is called a linear map or linear transformation.

100. **Linear Operator.** If T is a linear transformation on a vector spaces V with $T: V \rightarrow V$, then T is linear operator on V .

101. **Spans.** The list v_1, \dots, v_n spans V iff $T: F^n \rightarrow V$ is onto.

102. **Finite-dimensional.** V is finite-dimensional if V is spanned by a finite list of vectors.

103. **Direct Sum of Subspaces.** Let X_1, \dots, X_t be subspaces of V . Then, their direct sum, $X_1 \oplus \dots \oplus X_t$, is given by a 1-1 linear map T , with $T: X_1 \times \dots \times X_t \rightarrow V$.

104. **Complement of Subspace.** Let X, Y be subspaces of V . Then, Y is a complementary subspace of X iff $X+Y = V$ and $X \cap Y = \{0\}$.

105. **Rank, Nullity.** The rank of a linear map is the dimension of the range of the linear map. The nullity is the dimension of the null space of the linear map.

106. **Null Space.** The null space is the set of vectors that are mapped to 0.

107. **Isomorphic Vector Spaces.** Two vector spaces V, W are isomorphic if there exists a linear map $T: V \rightarrow W$ that is 1-1 and onto.

108. **Quotient Space.** Suppose U is a subspace of V . Then, the quotient space V/U is the set $V/U = \{v+U \mid v \in V\}$.

109. **Column Rank.** The column rank (rank of the column span of a matrix) is defined to be $\text{rank} T_A$.

110. **Conjugation.** Let A be an $n \times n$ matrix (over F) and let Q be an $n \times n$ matrix (over F). Then, the conjugation of A by Q is $Q^{-1}AQ$.

111. RIBET DEFS MT2.

112. **Dual Space.** Let V be an F -vector space. Then the dual space of V is $V' = \mathcal{L}(V, F)$ where the elements of V' are called linear functionals.

113. **Annihilator.** For a subspace $U \subseteq V$, we define the annihilator of U to be $U_0 = \{\phi \in V' \mid \phi(u) = 0 \forall u \in U\}$.

114. **Double Dual.** Let V be a finite-dimensional vector space with dual V' . Then the double dual of V is $(V')' = V'' = V$. Also, $\dim V = n = \dim V' = \dim V''$.

115. **Eigenvector / eigenvalue.** Let $T \in \mathcal{L}(V)$. Then an eigenvector of T is a $v \in V$ ($v \neq 0$) such that $Tv = \lambda v$ ($\lambda \in F$ is called an eigenvalue), and v is an eigenvector of T .

116. **Eigenspace.** Let $T \in \mathcal{L}(V)$ and take λ to be an eigenvalue of T . Then, $E(\lambda, T) = \{v \in V \mid Tv = \lambda v\} \neq \emptyset$ is written as V_λ and is called the eigenspace of λ , which is a subspace of V .

117. **Invariant subspace.** E is a T -invariant subspace if $T \in \mathcal{L}(V)$ with $T(E) \subseteq E$.

118. **textbf{idempotent.** If $e = e^2$, then e is called idempotent.

119. **Generalized Eigenvector.** Consider a minimal polynomial $(x - \lambda_1)^{e_1} \cdots (x - \lambda_m)^{e_m}$ on X with $(T - \lambda_1 I)^{e_1} v = 0$. Then, v is called a generalized eigenvector for $\lambda = \lambda_1$.

120. **Characteristic polynomial.** The characteristic polynomial of $T: V \rightarrow V$ (with eigenvalues $\lambda_1, \dots, \lambda_t$) is the polynomial $\prod_{i=1}^t (x - \lambda_i)^{\dim X_i}$, where $V = X_1 \oplus \dots \oplus X_t$.

121. **Simultaneously diagonalizable.** Operators S and T on V are simultaneously diagonalizable if there is a basis of V that consts of vectors that are eigenvectors for both S and T (i.e. there exists a basis v_1, \dots, v_n of V so that for $i, 1 \leq i \leq n$, there are λ_i and μ_i so that $Sv_i = \lambda_i v_i$ and $Tv_i = \mu_i v_i$).

122. RIBET THMS MT1.

123. **Lemma.** Let F be a field, $\lambda \in F, V$ a vector space over F (denoted by V/F), $v \in V$. Then, if $\lambda v = 0$, then $\lambda = 0$ or $v = 0$.

124. **Lemma.** A vector space over a field is a module over a field.

125. **Theorem.** The intersection of a family of subspaces of a vector space V is a subspace of V .

126. **Lemma.** Let $S = \{v_1, \dots, v_t\}$. Then the subspace of all linear combinations of the elements of S is the $\text{span} S$.

127. **Theorem.** Let $L = v_1, \dots, v_n$ be a list of vectors in a vector space V over a field F and let $T: F^n \rightarrow V$ be linear transformation with $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 v_1 + \dots + \lambda_n v_n$. Then, we have the following:

- (a) L spans V iff T is onto.
- (b) L is linearly independent iff T is 1-1 iff $\text{nul } T = \{0\}$.
- (c) L is a basis iff T is 1-1 and onto.

128. **Prop.** Consider $T: F^n \rightarrow V$ with $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 v_1 + \dots + \lambda_n v_n$, so $T(e_i) = v_i$ for all i . Then, T is the unique linear map $F^n \rightarrow V$ that sends $e_i \mapsto v_i$ for all i .

129. **Theorem.** Every subspace X of V has complement.

130. **Lemma.** If v_1, \dots, v_t is linearly dependent list, then there is an index k such that $v_k \in \text{span}(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_t)$. Furthermore, the span of the list of length $t-1$ gotten by removing v_k from the list is the same as the span of the original list.

131. **Prop.** In a finite-dimensional vector space, the length is of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

132. **Cor.** Two bases of V have the same number of elements.

133. **Prop.** $X + Y$ is direct iff the null space of the sum map is $\{0\}$.

134. **Theorem.** Every subspace of a finite-dimensional vector space is finite-dimensional.

135. **Prop.** Every spanning list for a vector space can be pruned down to a basis of the space.

136. **Cor.** Every finite-dimensional vector space has a basis.

137. **Prop.** In a finite-dimensional vector space, every linearly independent list can be extended to a basis of the space.

138. **Major Theorem.** Every subspace of a finite-dimensional vector space has a complement.

139. **Prop.** Let X, Y be subspaces of a finite-dimensional vector space V . Then:

(a) $\dim X + \dim Y = \dim V$.

(b) $X \cap Y = \{0\}$.

Then, $V = X \oplus Y$.

140. **Prop.** $\dim(X \oplus Y) = \dim X + \dim Y$.

141. **Prop.** If V is a finite-dimensional vector space (with $\dim V = n$), then every subspace has dimension at most n .

142. **Prop.** Let $\dim V = n$. Then, a linearly independent list of vectors of V with length n is a basis for V .

143. **Prop.** Let $\dim V = n$. Then, every spanning list for V of length n is a basis for V .

144. **Lemma.** The list $(x_1, 0), \dots, (x_t, 0); (0, y_1), \dots, (0, y_k)$ of length $t + k$ is a basis of $X \times Y$.

145. **Cor.** $\dim(X \times Y) = \dim X + \dim Y$.

146. **Cor.** Let $T : V \rightarrow W$ be a linear map with $\dim V = d$. Then, $\text{rank } T \leq d$.

147. **Rank-Nullity Theorem.** $\dim V = \text{rank } V + \text{nullity } V$.

148. **Prop.** If $T : V \rightarrow W$ is 1-1, then $\text{nullity } T = 0$.

149. **Cor.** If $T : V \rightarrow W$ is 1-1 and onto, then $\dim V = \dim W$.

150. **Theorem.** The set of linear maps $V \rightarrow W$ is a vector space $L \cdot (F^n, W) \rightarrow T \longrightarrow (Te_1, \dots, Te_n) \in W^n$.

151. **Theorem.** $\dim(X + Y) = \dim X + \dim Y - \dim(X \cap Y)$.

152. **Cor.** $\dim(V/X) = \dim V - \dim X$.

153. **Theorem.** If A is a rectangular matrix with elements in a field F , then $\text{row rank } A = \text{column rank } A$.

154. **Prop.** Let $T : V \rightarrow W$ be 1-1. Then, $\dim W \geq \dim V$.

155. **Prop.** Let $T : V \rightarrow W$ be onto. Then, $\dim V \geq \dim W$.

156. **Prop.** Let $T : V \rightarrow W$ and $\dim V = \dim W$. Then, T 1-1 iff T onto iff T bijective iff T invertible.

157. RIBET THMS MT2

158. **Lemma.** Let V be a finite-dimensional vector space and U a subspace of V . Then, $\dim U_0 = \dim V - \dim U$.

159. **Theorem.** Every linear functional on a subspace of V can be extended to V .

160. **Note.** Annihilator is the dual of the quotient subspace.

161. **Theorem.** Let $T : V \rightarrow W$ and $T' : W' \rightarrow V'$. Then $\mathcal{M}(T)$ and $\mathcal{M}(T')$ are transposes of each other.

162. **Lemma.** U^0 has dimension $\dim V - \dim U$.

163. **Cor.** The annihilator of U is $\{0\}$ iff $U = V$. The annihilator of U is V iff $U = \{0\}$.

164. **Prop.** If $T : V \rightarrow W$ is a linear map, then the null space of T' is the annihilator of the range of T . We have $\text{ann}(\text{range } T) = \{\psi : W \rightarrow F \mid \phi(Tv) = 0 \text{ for all } v \in V, T'(\psi)(v) = 0, T'\psi = 0, \phi \in \text{nul}(T')\}$.

165. **Cor.** If $T : V \rightarrow W$ is a linear map between finite-dimensional F -vector spaces, then $\dim \text{nul}(T') = \dim \text{nul}(T) + \dim W - \dim V$.

166. **Cor.** The linear map T is onto iff T' is 1-1.

167. **Cor.** If $T : V \rightarrow W$ is a linear map between finite-dimensional vector spaces, then T' and T have equal ranks.

168. **Cor.** We have $\text{range } T = (\text{nul } T)^0$.

169. **Theorem.** Let F be a finite field with $q = |F|$. Then, $a^q = a$ for all $a \in F$.

170. **Theorem.** If F is a finite field, then $|F| = p^n$ for some prime p and integer $n \geq 1$.

171. **Theorem.** Take an ideal I in \mathbb{Z} . Then, I is equal to either $\{0\}$ or $m\mathbb{Z}$ (where $m \in \mathbb{Z}_{>0}$).

172. **Theorem.** $F[x]$ is a principal ideal domain; that is, it is an integral domain in which every ideal in $F[x]$ is principal.

173. **Theorem.** Let $T : V \rightarrow V$, V finite-dimensional, and let $\alpha : F[x] \rightarrow \mathcal{L}(V)$, with $f \mapsto f(T)$. Also, we have $\ker \alpha$ to be the principal ideal $(m(x))$. Then, $m(x)$ is the minimal polynomial of T and has degree $\leq n^2$.

174. **Cayley-Hamilton Theorem.** Let $T : V \rightarrow V$, V finite-dimensional, and let $\alpha : F[x] \rightarrow \mathcal{L}(V)$, with $f \mapsto f(T)$. Also, we have $\ker \alpha$ to be the principal ideal $(m(x))$, where $m(x)$ is the minimal polynomial of T . Then, the characteristic polynomial is in $\ker \alpha$; that is, we can plug in the matrix for T into its characteristic polynomial and we get that it is equal to the 0-matrix.

175. **Prop.** For $f(x) \in F[x]$ and $\lambda \in F$, $f(\lambda) = 0$ iff f is divisible by $x - \lambda$, where $x - \lambda$ is an irreducible polynomial.

176. **Cor.** A polynomial of degree n can have at most n roots.

177. **Cor.** A polynomial with infinitely many roots is identically the zero polynomial.

178. **Lemma.** Let $f \in \mathbb{R}[x]$ be a real polynomial. If λ is a complex root of f , so is $\overline{\lambda}$, which is the complex conjugate of λ .

179. **Prop.** A scalar λ is an eigenvalue of $T : V \rightarrow V$ iff $T - \lambda I$ is not 1-1.

180. **Cor.** The map $T : V \rightarrow V$ is invertible iff 0 is not an eigenvalue of T .

181. **Key lemma.** Every list of eigenvectors of T that corresponds to distinct eigenvalues of T is a linearly independent list.

182. **Cor.** Let $\lambda_1, \dots, \lambda_t$ be distinct eigenvalues and take $E_t = E(\lambda_t, T) = \{v \in V \mid Tv = \lambda_tv\} \subseteq V$. Now, take $E_1 \times \dots \times E_t$. Then there exists a summation map $E_1 \times \dots \times E_t \xrightarrow{\text{sum}} V$ with $(v_1, \dots, v_t) \mapsto v_1 + \dots + v_t$. Then, the sum map is 1-1.

183. **Cor.** Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

184. **Prop.** Suppose T is an operator on an F -vector space V . If $f \in F[x]$ is a polynomial satisfied by T (meaning $f(T) = 0$), then every eigenvalue of T on V is a root of f .

185. **Cor.** Suppose λ is an eigenvalue of operator T on a finite-dimensional F -vector space. Then λ is a root of the minimal polynomial of T .

186. **Prop.** Let T be an operator on a finite-dimensioal vector space. Suppose λ is a root of the minimal polynomial. Then λ is an eigenvalue of T .

187. **Theorem.** All operators on a nonzero finite-dimensional vector space over an algebraically closed field have at least one eigenvalue.

188. **Prop.** Assume that $F = \mathbb{R}$ and that $f(x) := x^2 + bx + c$ is an irreducible polynomial. If $T \in \mathcal{L}(V)$ and V is finite-dimensional, then the null space of $f(T)$ is even-dimensional.

189. **Prop (honors version).** Let T be an operator on a finite-dimensional vector space over F . If p is an irreducible polynomial over F , then the dimension of the null space of $p(T)$ is a multiple of the degree of p .

190. **Prop.** $F[x]/(p)$ (where p is irreducible) is a field.

191. **Formula.** $\dim_F V = [K : F] \cdot \dim_K V = \dim_F K \cdot \dim_K V$.

192. **Cor.** Every operator on an odd-dimensional \mathbb{R} -vector space has an eigenvalue.

193. **Prop.** If T is an operator on a finite-dimensional F -vector space, then the minimal polynomial of T has degree at most $\dim V$.

194. **Prop.** If T is upper-triangular with respect to some basis of V , and if the diagonal entries of an upper-triangular matrix representation of T are $\lambda_1, \dots, \lambda_n$, then $(T - \lambda_1 I) \cdots (T - \lambda_n I) = 0$.

195. **Prop.** Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V)$ and let $\lambda_1, \dots, \lambda_m$ be the eigenvalues of T . Then, $V = \oplus E(\lambda_i, T)$ iff T is diagonalizable.

196. **Prop.** TFAE.

(a) T is diagonalizable.

(b) V has a basis consisting of eigenvectors.

(c) The direct sum $\oplus_i V_{\lambda_i}$ is all of V .

(d) $\dim \left(\oplus_i V_{\lambda_i} \right) = \dim V$.

197. **Prop.** If $T : V \rightarrow V$ has $\dim V$ different eigenvalues, then T is diagonalizable.

198. **Prop.** The operator $T : V \rightarrow V$ is diagonalizable iff its minimal polynomial splits completely as a product of distinct linear factors of the form $x - r$.

199. **Jordan Canonical Form.** X can be written as a direct sum of Jordan blocks, where $\sum \dim(\text{block}) = \dim X$.

200. **Lemma.** Let $X = \oplus \text{span}(U_i v)$ for $i \in \{0, \dots, k_1\}$. If Z is a subspace of X' that is U' -invariant, then $\text{ann}(Z) =: Y$ is U -invariant.

201. **Lemma.** Suppose S and T are commuting operators on V . If λ is an eigenvalue for T on V , then the eigenspace $E(\lambda, T)$ is S -invariant.

202. **Theorem.** The diagonalize operators on the same finite-dimensional vector space are simultaneously diagonalizable iff they commute with each other.

203. **Theorem.** Every pair of commuting operators on a finite-dimensional nonzero complex vector spcae has a common eigenvector.

204. **Prop.** Two commuting operators on a finite-dimensional nonzero complex vector space can be simultaneously upper-triangularized.

205. **Prop.** We have:

(a) Every eigenvalue of $S + T$ is the sum of an eigenvalue of S and an eigenvalue of T .

(b) Every eigenvalue of ST is the product of an eigenvalue of S and an eigenvalue of T .