

Math H110 Theorems.

1. **Lemma.** Let  $F$  be a field,  $\lambda \in F$ ,  $V$  a vector space over  $F$  (denoted by  $V/F$ ),  $v \in V$ . Then, if  $\lambda v = 0$ , then  $\lambda = 0$  or  $v = 0$ .
2. **Lemma.** A vector space over a field is a module over a field.
3. **Theorem.** The intersection of a family of subspaces of a vector space  $V$  is a subspace of  $V$ .
4. **Lemma.** Let  $S = \{v_1, \dots, v_t\}$ . Then the subspace of all linear combinations of the elements of  $S$  is the  $\text{span} S$ .
5. **Theorem.** Let  $L = v_1, \dots, v_n$  be a list of vectors in a vector space  $V$  over a field  $F$  and let  $T : F^n \rightarrow V$  be linear transformation with  $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 v_1 + \dots + \lambda_n v_n$ . Then, we have the following:
  - (a)  $L$  spans  $V$  iff  $T$  is onto.
  - (b)  $L$  is linearly independent iff  $T$  is 1-1 iff  $\text{nul } T = \{0\}$ .
  - (c)  $L$  is a basis iff  $T$  is 1-1 and onto.
6. **Prop.** Consider  $T : F^n \rightarrow V$  with  $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 v_1 + \dots + \lambda_n v_n$ , so  $T(e_i) = v_i$  for all  $i$ . Then,  $T$  is the unique linear map  $F^n \rightarrow V$  that sends  $e_i \mapsto v_i$  for all  $i$ .
7. **Theorem.** Every subspace  $X$  of  $V$  has complement.
8. **Lemma.** If  $v_1, \dots, v_t$  is linearly dependent list, then there is an index  $k$  such that  $v_k \in \text{span}(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_t)$ . Furthermore, the span of the list of length  $t - 1$  gotten by removing  $v_k$  from the list is the same as the span of the original list.
9. **Prop.** In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.
10. **Cor.** Two bases of  $V$  have the same number of elements.
11. **Prop.**  $X + Y$  is direct iff the null space of the sum map is  $\{0\}$ .
12. **Theorem.** Every subspace of a finite-dimensional vector space is finite-dimensional.

13. **Prop.** Every spanning list for a vector space can be pruned down to a basis of the space.
14. **Cor.** Every finite-dimensional vector space has a basis.
15. **Prop.** In a finite-dimensional vector space, every linearly independent list can be extended to a basis of the space.
16. **Major Theorem.** Every subspace of a finite-dimensional vector space has a complement.
17. **Prop.** Let  $X, Y$  be subspaces of a finite-dimensional vector space  $V$ . Then:
  - (a)  $\dim X + \dim Y = \dim V$ .
  - (b)  $X \cap Y = \{0\}$ .
 Then,  $V = X \oplus Y$ .
18. **Prop.**  $\dim(X \oplus Y) = \dim X + \dim Y$ .
19. **Prop.** If  $V$  is a finite-dimensional vector space (with  $\dim V = n$ ), then every subspace has dimension at most  $n$ .
20. **Prop.** Let  $\dim V = n$ . Then, a linearly independent list of vectors of  $V$  with length  $n$  is a basis for  $V$ .
21. **Prop.** Let  $\dim V = n$ . Then, every spanning list for  $V$  of length  $n$  is a basis for  $V$ .
22. **Lemma.** The list  $(x_1, 0), \dots, (x_t, 0); (0, y_1), \dots, (0, y_k)$  of length  $t + k$  is a basis of  $X \times Y$ .
23. **Cor.**  $\dim(X \times Y) = \dim X + \dim Y$ .
24. **Cor.** Let  $T : V \rightarrow W$  be a linear map with  $\dim V = d$ . Then,  $\text{rank } T \leq d$ .
25. **Rank-Nullity Theorem.**  $\dim V = \text{rank } V + \text{nullity } V$ .
26. **Prop.** If  $T : V \rightarrow W$  is 1-1, then  $\text{nullity } T = 0$ .
27. **Cor.** If  $T : V \rightarrow W$  is 1-1 and onto, then  $\dim V = \dim W$ .

28. **Theorem.** The set of linear maps  $V \rightarrow W$  is a vector space  $L \cdot (F^n, W) \rightarrow T \longrightarrow (Te_1, \dots, Te_n) \in W^n$ .
29. **Theorem.**  $\dim(X + Y) = \dim X + \dim Y - \dim(X \cap Y)$ .
30. **Cor.**  $\dim(V/X) = \dim V - \dim X$ .
31. **Theorem.** If  $A$  is a rectangular matrix with elements in a field  $F$ , then row rank  $A =$  column rank  $A$ .
32. **Prop.** Let  $T : V \rightarrow W$  be 1-1. Then,  $\dim W \geq \dim V$ .
33. **Prop.** Let  $T : V \rightarrow W$  be onto. Then,  $\dim V \geq \dim W$ .
34. **Prop.** Let  $T : V \rightarrow W$  and  $\dim V = \dim W$ . Then,  $T$  1-1 iff  $T$  onto iff  $T$  bijective iff  $T$  invertible.
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35. **Lemma.** Let  $V$  be a finite-dimensional vector space and  $U$  a subspace of  $V$ . Then,  $\dim U^\perp = \dim V - \dim U$ .
36. **Theorem.** Every linear functional on a subspace of  $V$  can be extended to  $V$ .
37. **Note.** Annihilator is the dual of the quotient subspace.
38. **Theorem.** Let  $T : V \rightarrow W$  and  $T' : W' \rightarrow V'$ . Then  $\mathcal{M}(T)$  and  $\mathcal{M}(T')$  are transposes of each other.
39. **Lemma.**  $U^\perp$  has dimension  $\dim V - \dim U$ .
40. **Cor.** The annihilator of  $U$  is  $\{0\}$  iff  $U = V$ . The annihilator of  $U$  is  $V$  iff  $U = \{0\}$ .
41. **Prop.** If  $T : V \rightarrow W$  is a linear map, then the null space of  $T'$  is the annihilator of the range of  $T$ . We have  $\text{ann}(\text{range } T) = \{\psi : W \rightarrow F \mid \phi(Tv) = 0 \text{ for all } v \in V, T'(\psi)(v) = 0, T'\psi = 0, \phi \in \text{nul}(T')\}$ .
42. **Cor.** If  $T : V \rightarrow W$  is a linear map between finite-dimensional  $F$ -vector spaces, then  $\dim \text{nul}(T') = \dim \text{nul}(T) + \dim W - \dim V$ .
43. **Cor.** The linear map  $T$  is onto iff  $T'$  is 1-1.

44. **Cor.** If  $T : V \rightarrow W$  is a linear map between finite-dimensional vector spaces, then  $T'$  and  $T$  have equal ranks.
45. **Cor.** We have  $\text{range} T = (\text{nul } T)^0$ .
46. **Theorem.** Let  $F$  be a finite field with  $q = |F|$ . Then,  $a^q = a$  for all  $a \in F$ .
47. **Theorem.** If  $F$  is a finite field, then  $|F| = p^n$  for some prime  $p$  and integer  $n \geq 1$ .