Math H110 Midterm 1 CheatSheet

1A. (n/a)

1B.

1. **Vector Space.** A vector space V is a set that has scalar multiplication and vector addition defined on it with the following properties:
   (a) Additive commutativity.
   (b) Additive associativity of vectors ($u+(v+w)=(u+v)+w$) and multiplicative associativity for scalars ($(ab)v=a(bv)$).
   (c) Additive identity.
   (d) Additive inverses.
   (e) Multiplicative identity.
   (f) BOTH distributive properties.
2. **V-space (unique additive identity)** A vector space has a unique additive identity.
3. **V-space (unique additive inverses)** Every element in a vector space has a unique additive inverse.

1C.

1. **Subspace.** A subset $U \subseteq V$ is a subspace of V if it is a vector space with the same additive identity, scalar multiplication, and vector addition as defined on V.
2. **Conditions for a Subspace.** A subset $U \subseteq V$ is a subspace of V iff U is closed under vector addition, scalar multiplication, and contains the ”zero” element as in V.
3. **Sums of Subspaces.** Let $V_1,\ldots,V_n$ be subspaces of V. Then, we have the sum of subspaces as $V_1+\cdots+V_n=\{v_1+\cdots+v_n \mid v_i \in V_i \text{ for all } i\}$.
4. **Smallest subspace containing each subspace** Suppose $V_1,\ldots,V_n$ are subspaces of V. Then, $V_1+\cdots+V_n$ is the smallest subspace of V containing $V_1,\ldots,V_n$.
5. **Direct Sum.** Suppose $V_1,\ldots,V_m$ are subspaces of V. Then:

(a) The sum $V_1 + \cdots + V_m$ is direct if each element of $V_1 + \cdots + V_m$ can be written uniquely as a sum $v_1 + \cdots + v_m$, where $v_i \in V_i$ for all i.

(b) If $V_1 + \cdots + V_m$ is a direct sum, then we write $V_1 \oplus \cdots \oplus V_m$.

6. **Conditions for a direct sum.** Suppose $V_1, \ldots, V_n$ are subspaces of V. Then, $V_1 + \cdots + V_n$ is direct iff the only way to write 0 from $v_1 + \cdots + v_n$ is by taking $v_i = 0$ for all i.

7. **Direct sum of subspaces.** If U, W are subspaces of V, then $U + W$ is direct iff $U \cap W = \{0\}$.

2A.

1. **Span is the smallest containing subspace.** The span of a list of vectors in V is the smallest subspace containing all of the vectors in the list.

2. **Zero polynomial.** The zero polynomial is said to have degree $-\infty$.

3. **Linear Independence.** A list of vectors $v_1, \ldots, v_n \in V$ is said to be linearly independent if $a_1 v_1 + \cdots + a_n v_n = 0$ implies $a_i = 0$ for all i. Also, the empty list () is said to be linearly independent.

4. **Linear Dependence.** A list of vectors $v_1, \ldots, v_n$ is said to be linearly dependent if $a_1 v_1 + \cdots + a_n v_= 0$ impies $a_i \neq 0$ for some i.

5. **Linear Dependence Lemma.** Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then, there exists $k \in \{1, \ldots, m\}$ such that $v_k \in \text{span}(v_1, \ldots, v_{k-1})$. Furthermore, if k satisfies the condition in the previous sentence and the k^{th} term is removed from $v_1, \ldots, v_m$, then the span of the remaining list equals $\text{span}(v_1, \ldots, v_m)$.

6. **length of linearly independent list ¡ length of spanning list.** In a finite-dimensional vector space, the length of every linearly independent list is at most the length of every spanning list of vectors.

7. **Finite Dimensional subspaces.** Every subspace of a finite-dimensional vector space is finite dimensional.

2B.

1. **Basis.** A basis of V is a list of vectors that is linearly independent and spans V.

2. **Criterion for basis.** A list of vectors $v_1, \ldots, v_n \in V$ is a basis of V iff every $v \in V$ can be written uniquely in the form $v = a_1 v_1 + \cdots + a_n v_n$, where $a_i \in F$ for all i.

3. **Every spanning list contains a basis.** Every spanning list in a vector space can be reduced to a basis of the vector space.

4. **Basis of finite-dimensional vector space.** Every finite-dimensional vector space has a basis.

5. **Every linearly independent list extends to a basis.** Every linearly independent list in a finite-dimensional vector space can be extended to a basis of the vector space.

6. **Every subspace of V is part of a direct sum equal to V.** Suppose V is finite-dimensional and U is a subspace of V. Then, there is a subspace W of V such that $V = U \oplus W$.

2C.

1. **Basis length does not depend on basis.** Any two bases of a finite-dimensional vector space have the same length.

2. **Dimension of a subspace.** If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.

3. **Linearly independent list of the right length is a basis.** Suppose V is finite-dimensional. Then, every linearly independent list of vectors in V (with list length equal to $\dim V$) is a basis of V.

4. **Subspace of full dimension equals the whole space.** Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Then, $U = V$.

5. **Spanning list of the right length is a basis.** Suppose V is finite-dimensional. Then, every spanning list of V of length $\dim V$ is a basis of V.

6. **Dimension of a sum.** If V_1, V_2 are subspaces of a finite-dimensional vector space, then $\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$.

3A.

1.