

MIDTERM 1.

1. 1A. (NOTHING)
2. 1B.
3. **Vector Space.** A vector space V is a set that has scalar multiplication and vector addition defined on it with the following properties:
 - (a) Additive commutativity.
 - (b) Additive associativity of vectors $(u + (v + w) = (u + v) + w)$ and multiplicative associativity for scalars $((ab)v = a(bv))$.
 - (c) Additive identity.
 - (d) Additive inverses.
 - (e) Multiplicative identity.
 - (f) BOTH distributive properties.
4. **V-space (unique additive identity)** A vector space has a unique additive identity.
5. **V-space (unique additive inverses)** Every element in a vector space has a unique additive inverse.
6. 1C.
7. **Subspace.** A subset $U \subseteq V$ is a subspace of V if it is a vector space with the same additive identity, scalar multiplication, and vector addition as defined on V .
8. **Conditions for a Subspace.** A subset $U \subseteq V$ is a subspace of V iff U is closed under vector addition, scalar multiplication, and contains the "zero" element as in V .
9. **Sums of Subspaces.** Let V_1, \dots, V_n be subspaces of V . Then, we have the sum of subspaces as $V_1 + \dots + V_n = \{v_1 + \dots + v_n \mid v_i \in V_i \text{ for all } i\}$.
10. **Smallest subspace containing each subspace** Suppose V_1, \dots, V_n are subspaces of V . Then, $V_1 + \dots + V_n$ is the smallest subspace of V containing V_1, \dots, V_n .
11. **Direct Sum.** Suppose V_1, \dots, V_m are subspaces of V . Then:
 - (a) The sum $V_1 + \dots + V_m$ is direct if each element of $V_1 + \dots + V_m$ can be written uniquely as a sum $v_1 + \dots + v_m$, where $v_i \in V_i$ for all i .
 - (b) If $V_1 + \dots + V_m$ is a direct sum, then we write $V_1 \oplus \dots \oplus V_m$.
12. **Conditions for a direct sum.** Suppose V_1, \dots, V_n are subspaces of V . Then, $V_1 + \dots + V_n$ is direct iff the only way to write 0 from $v_1 + \dots + v_n$ is by taking $v_i = 0$ for all i .
13. **Direct sum of subspaces.** If U, W are subspaces of V , then $U + W$ is direct iff $U \cap W = \{0\}$.
14. 2A.
15. **Span is the smallest containing subspace.** The span of a list of vectors in V is the smallest subspace containing all of the vectors in the list.
16. **Zero polynomial.** The zero polynomial is said to have degree $-\infty$.
17. **Linear Independence.** A list of vectors $v_1, \dots, v_n \in V$ is said to be linearly independent if $a_1 v_1 + \dots + a_n v_n = 0$ implies $a_i = 0$ for all i . Also, the empty list $()$ is said to be linearly independent.
18. **Linear Dependence.** A list of vectors v_1, \dots, v_n is said to be linearly dependent if $a_1 v_1 + \dots + a_n v_n = 0$ implies $a_i \neq 0$ for some i .
19. **Linear Dependence Lemma.** Suppose v_1, \dots, v_m is a linearly dependent list in V . Then, there exists $k \in \{1, \dots, m\}$ such that $v_k \in \text{span}(v_1, \dots, v_{k-1})$. Furthermore, if k satisfies the condition in the previous sentence and the k^{th} term is removed from v_1, \dots, v_m , then the span of the remaining list equals $\text{span}(v_1, \dots, v_m)$.
20. **length of linearly independent list ; length of spanning list.** In a finite-dimensional vector space, the length of every linearly independent list is at most the length of every spanning list of vectors.
21. **Finite Dimensional subspaces.** Every subspace of a finite-dimensional vector space is finite dimensional.
22. 2B.
23. **Basis.** A basis of V is a list of vectors that is linearly independent and spans V .
24. **Criterion for basis.** A list of vectors $v_1, \dots, v_n \in V$ is a basis of V iff every $v \in V$ can be written uniquely in the form $v = a_1 v_1 + \dots + a_n v_n$, where $a_i \in F$ for all i .
25. **Every spanning list contains a basis.** Every spanning list in a vector space can be reduced to a basis of the vector space.
26. **Basis of finite-dimensional vector space.** Every finite-dimensional vector space has a basis.
27. **Every linearly independent list extends to a basis.** Every linearly independent list in a finite-dimensional vector space can be extended to a basis of the vector space.
28. **Every subspace of V is part of a direct sum equal to V .** Suppose V is finite-dimensional and U is a subspace of V . Then, there is a subspace W of V such that $V = U \oplus W$.
29. 2C.
30. **Basis length does not depend on basis.** Any two bases of a finite-dimensional vector space have the same length.
31. **Dimension of a subspace.** If V is finite-dimensional and U is a subspace of V , then $\dim U \leq \dim V$.
32. **Linearly independent list of the right length is a basis.** Suppose V is finite-dimensional. Then, every linearly independent list of vectors in V (with list length equal to $\dim V$) is a basis of V .
33. **Subspace of full dimension equals the whole space.** Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Then, $U = V$.
34. **Spanning list of the right length is a basis.** Suppose V is finite-dimensional. Then, every spanning list of V of length $\dim V$ is a basis of V .
35. **Dimension of a sum.** If V_1, V_2 are subspaces of a finite-dimensional vector space, then $\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$.
36. 3A.
37. **Set of Linear Maps.** The linear of linear maps from $V \rightarrow W$ is written $\mathcal{L}(V, W)$ and the set of linear maps from $V \rightarrow V$ is written $\mathcal{L}(V)$.
38. **Linear Map lemma.** Suppose v_1, \dots, v_n is a basis of V and $w_1, \dots, w_n \in W$. Then, there exists a unique linear map $F: V \rightarrow W$ such that $Fv_k = w_k$ for each k .
39. **Linear maps take 0 to 0.** Suppose $T: V \rightarrow W$ is a linear map. Then, $T(0) = 0$.
40. 3B.
41. **null space is a subspace.** Suppose $T \in \mathcal{L}(V, W)$. Then, T is a subspace of V .
42. **injectivity iff nul is 0.** Let $T \in \mathcal{L}(V, W)$. Then, T is 1-1 iff $\text{nul } T = \{0\}$.
43. **range is a subspace.** If $T \in \mathcal{L}(V, W)$, then $\text{range } T$ is a subspace of W .
44. **Fundamental Theorem of Linear Maps.** Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then, $\text{range } T$ is finite dimensional and $\dim V = \dim \text{nul } T + \dim \text{range } T$.
45. **linear map to a lower-dim space is not 1-1.** Suppose V, W are finite-dimensional vector spaces such that $\dim V > \dim W$. Then, no linear map from $V \rightarrow W$ is 1-1.

46. **linear map to a higher-dim space is not onto.** Suppose V, W are finite-dimensional vector spaces such that $\dim V < \dim W$. Then, no linear map from $V \rightarrow W$ is onto.
47. 3C. (NOTHING)
48. 3D.
49. **Theorem.** Let V, W be finite-dimensional vector spaces such that $\dim V = \dim W$ and let $T \in \mathcal{L}(V, W)$. Then, T is invertible iff T is 1-1 iff T is onto.
50. **isomorphism.** An isomorphism is an invertible linear map.
51. **dimension and isomorphic.** Two finite-dimensional vector spaces are isomorphic iff they have the same dimension.
52. **Theorem.** Suppose V and W are finite-dimensional. Then, $\mathcal{L}(V, W)$ is finite-dimensional and $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$.
53. **ST=I iff TS=I (on vector spaces of the same dimension).** Suppose V and W are finite-dimensional vector spaces of the same dimension, $S \in \mathcal{L}(W, V)$, $T \in \mathcal{L}(V, W)$. Then $ST = I$ iff $TS = I$.
54. **matrix of identity operator with respect to two bases.** Suppose u_1, \dots, u_n and v_1, \dots, v_n are two bases of V . Then, the matrices $\mathcal{M}(I; u_1, \dots, u_n; v_1, \dots, v_n)$ and $\mathcal{M}(I; v_1, \dots, v_n; u_1, \dots, u_n)$ are invertible and are inverses of each other.
55. **Change of basis formula.** Let $T \in \mathcal{L}(V, W)$. Suppose u_1, \dots, u_n and v_1, \dots, v_n are two bases of V . Let $A = \mathcal{M}(T; u_1, \dots, u_n)$ and $B = \mathcal{M}(T; v_1, \dots, v_n)$ and $C = \mathcal{M}(I; u_1, \dots, u_n; v_1, \dots, v_n)$. Then, $A = C^{-1}BC$.
56. Suppose that v_1, \dots, v_n is a basis of V and $T \in \mathcal{L}(V)$ is invertible. Then, $\mathcal{M}(T^{-1}) = (\mathcal{M}(T))^{-1}$, where both matrices are with respect to the basis v_1, \dots, v_n .
57. 3E.
58. **Product of vector spaces is a vector space.** Suppose V_1, \dots, V_m are vector spaces over \mathbb{F} . Then, $V_1 \times \dots \times V_m$ is a vector space over \mathbb{F} .
59. **dimension of a product is the sum of the dimensions.** Suppose V_1, \dots, V_m are finite-dimensional vector spaces. Then, $V_1 \times \dots \times V_m$ is finite-dimensional and $\dim(V_1 \times \dots \times V_m) = \dim V_1 + \dots + \dim V_m$.
60. **Products and direct sums.** Suppose V_1, \dots, V_m are subspaces of V . Define a linear map $\Gamma: (V_1 \times \dots \times V_m) \rightarrow (V_1 + \dots + V_m)$ by $\Gamma(v_1, \dots, v_m) = v_1 + \dots + v_m$. Then, $V_1 + \dots + V_m$ is direct iff Γ is 1-1.
61. **direct sum iff dimensions add up.** Suppose V is finite-dimensional and V_1, \dots, V_m are subspaces of V . Then, $V_1 + \dots + V_m$ is direct iff $\dim(V_1 + \dots + V_m) = \dim V_1 + \dots + \dim V_m$.
62. **$v + U$.** Suppose $v \in V$ and $U \subseteq V$. Then, $v + U = \{v + u \mid u \in U\}$.
63. **Translate.** For $v \in V$ and $U \subseteq V$, the set $v + U$ is called a translate of U .
64. **Quotient Space.** Let U be a subspace of V . Then, the quotient space V/U is the set of all translates of U , that is, $V/U = \{v + U \mid v \in V\}$.
65. **two translates of a subspace are either equal or disjoint.** Suppose U is a subspace of V and $v, w \in V$. Then, $v - w \in U$ iff $v + U = w + U$ iff $(v + U) \cap (w + U) \neq \emptyset$.
66. **Addition and scalar multiplication on Quotient space.** Let U be a subspace of V . Then, we have (for all $v, w \in V$, $\lambda \in F$):
 - (a) addition on V/U : $(v + U) + (w + U) = (v + w) + U$.
 - (b) scalar multiplication on V/U : $\lambda(v + U) = (\lambda v) + U$.
67. **quotient space is a vector space.** Let U be a subspace of V . Then, the quotient space V/U is a subspace of V under the defined scalar multiplication and vector addition.
68. **quotient map.** Let U be a subspace of V . Then, the quotient map $\pi: V \rightarrow V/U$ is the linear map defined by $\pi(v) = v + U$ for each $v \in V$.
69. **dimension of quotient space.** Suppose V is finite-dimensional and U is a subspace of V . Then, $\dim(V/U) = \dim V - \dim U$.
70. **Column rank.** The column rank (rank of the column span of a matrix) is $\text{rank } T_A$.
71. **Theorem.** If A is a rectangular matrix of elements in a field F , then $\text{row rank } A = \text{column rank } A$.
72. RIBET DEFS.
73. **Endomorphism.** An endomorphism is a group homomorphism from a set to itself (NOTE: does not have to be invertible.)
74. **End V.** The symbol $\text{End } V$ is the set of all endomorphisms on V (and multiplication on $\text{End } V$ is defined to be function composition).
75. **F-Module.** An F -module is a generalization of vector spaces over rings.
76. **Linear Map / Linear Transformation.** Let V be a vector space over a field F with $v, w \in V$. Let T be a map on V with $T(v + w) = T(v) + T(w)$ and $T(\lambda v) = \lambda T(v)$ for all $\lambda \in F$. Then, T is called a linear map or linear transformation.
77. **Linear Operator.** If T is a linear transformation on a vector spaces V with $T: V \rightarrow V$, then T is linear operator on V .
78. **Spans.** The list v_1, \dots, v_n spans V iff $T: F^n \rightarrow V$ is onto.
79. **Finite-dimensional.** V is finite-dimensional if V is spanned by a finite list of vectors.
80. **Direct Sum of Subspaces.** Let X_1, \dots, X_r be subspaces of V . Then, their direct sum, $X_1 \oplus \dots \oplus X_r$, is given by a 1-1 linear map T , with $T: X_1 \times \dots \times X_r \rightarrow V$.
81. **Complement of Subspace.** Let X, Y be subspaces of V . Then, Y is a complementary subspace of X iff $X + Y = V$ and $X \cap Y = \{0\}$.
82. **Rank, Nullity.** The rank of a linear map is the dimension of the range of the linear map. The nullity is the dimension of the null space of the linear map.
83. **Null Space.** The null space is the set of vectors that are mapped to 0.
84. **Isomorphic Vector Spaces.** Two vector spaces V, W are isomorphic if there exists a linear map $T: V \rightarrow W$ that is 1-1 and onto.
85. **Quotient Space.** Suppose U is a subspace of V . Then, the quotient space V/U is the set $V/U = \{v + U \mid v \in V\}$.
86. **Column Rank.** The column rank (rank of the column span of a matrix) is defined to be $\text{rank } T_A$.
87. **Conjugation.** Let A be an $n \times n$ matrix (over F) and let Q be an $n \times n$ matrix (over F). Then, the conjugation of A by Q is $Q^{-1}AQ$.
88. RIBET THMS.
89. **Lemma.** Let F be a field, $\lambda \in F$, V a vector space over F (denoted by V/F), $v \in V$. Then, if $\lambda v = 0$, then $\lambda = 0$ or $v = 0$.
90. **Lemma.** A vector space over a field is a module over a field.
91. **Theorem.** The intersection of a family of subspaces of a vector space V is a subspace of V .
92. **Lemma.** Let $S = \{v_1, \dots, v_r\}$. Then the subspace of all linear combinations of the elements of S is the $\text{span } S$.
93. **Theorem.** Let $L = v_1, \dots, v_n$ be a list of vectors in a vector space V over a field F and let $T: F^n \rightarrow V$ be linear transformation with $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 v_1 + \dots + \lambda_n v_n$. Then, we have the following:

- (a) L spans V iff T is onto.
 (b) L is linearly independent iff T is 1-1 iff $\text{nul } T = \{0\}$.
 (c) L is a basis iff T is 1-1 and onto.
94. **Prop.** Consider $T : F^n \rightarrow V$ with $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 v_1 + \dots + \lambda_n v_n$, so $T(e_i) = v_i$ for all i . Then, T is the unique linear map $F_n \rightarrow V$ that sends $e_i \mapsto v_i$ for all i .
95. **Theorem.** Every subspace X of V has complement.
96. **Lemma.** If v_1, \dots, v_t is linearly dependent list, then there is an index k such that $v_k \in \text{span}(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_t)$. Furthermore, the span of the list of length $t-1$ gotten by removing v_k from the list is the same as the span of the original list.
97. **Prop.** In a finite-dimensional vector space, the length is of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.
98. **Cor.** Two bases of V have the same number of elements.
99. **Prop.** $X+Y$ is direct iff the null space of the sum map is $\{0\}$.
100. **Theorem.** Every subspace of a finite-dimensional vector space is finite-dimensional.
101. **Prop.** Every spanning list for a vector space can be pruned down to a basis of the space.
102. **Cor.** Every finite-dimensional vector space has a basis.
103. **Prop.** In a finite-dimensional vector space, every linearly independent list can be extended to a basis of the space.
104. **Major Theorem.** Every subspace of a finite-dimensional vector space has a complement.
105. **Prop.** Let X, Y be subspaces of a finite-dimensional vector space V . Then:

- (a) $\dim X + \dim Y = \dim V$.
 (b) $X \cap Y = \{0\}$.

Then, $V = X \oplus Y$.

106. **Prop.** $\dim(X \oplus Y) = \dim X + \dim Y$.
107. **Prop.** If V is a finite-dimensional vector space (with $\dim V = n$), then every subspace has dimension at most n .
108. **Prop.** Let $\dim V = n$. Then, a linearly independent list of vectors of V with length n is a basis for V .
109. **Prop.** Let $\dim V = n$. Then, every spanning list for V of length n is a basis for V .
110. **Lemma.** The list $(x_1, 0), \dots, (x_t, 0); (0, y_1), \dots, (0, y_k)$ of length $t+k$ is a basis of $X \times Y$.
111. **Cor.** $\dim(X \times Y) = \dim X + \dim Y$.
112. **Cor.** Let $T : V \rightarrow W$ be a linear map with $\dim V = d$. Then, $\text{rank } T \leq d$.
113. **Rank-Nullity Theorem.** $\dim V = \text{rank } V + \text{nullity } V$.
114. **Prop.** If $T : V \rightarrow W$ is 1-1, then $\text{nullity } T = 0$.
115. **Cor.** If $T : V \rightarrow W$ is 1-1 and onto, then $\dim V = \dim W$.
116. **Theorem.** The set of linear maps $V \rightarrow W$ is a vector space $L(F^n, W) \rightarrow T \mapsto (Te_1, \dots, Te_n) \in W^n$.
117. **Theorem.** $\dim(X+Y) = \dim X + \dim Y - \dim(X \cap Y)$.
118. **Cor.** $\dim(V/X) = \dim V - \dim X$.
119. **Theorem.** If A is a rectangular matrix with elements in a field F , then $\text{row rank } A = \text{column rank } A$.
120. **Prop.** Let $T : V \rightarrow W$ be 1-1. Then, $\dim W \geq \dim V$.
121. **Prop.** Let $T : V \rightarrow W$ be onto. Then, $\dim V \geq \dim W$.
122. **Prop.** Let $T : V \rightarrow W$ and $\dim V = \dim W$. Then, T 1-1 iff T onto iff T bijective iff T invertible.

MIDTERM 2.

- 1C, 2A, 2B, 2C, 3B, 3C, 3D, 3E.
- Direct sum of subspaces.** If U, W are subspaces of V , then $U+W$ is direct iff $U \cap W = \{0\}$.
- Linear Dependence Lemma.** Suppose v_1, \dots, v_m is a linearly dependent list in V . Then, there exists $k \in \{1, \dots, m\}$ such that $v_k \in \text{span}(v_1, \dots, v_{k-1})$. Furthermore, if k satisfies the condition in the previous sentence and the k^{th} term is removed from v_1, \dots, v_m , then the span of the remaining list equals $\text{span}(v_1, \dots, v_m)$.
- Prop.** Let V, W be finite-dimensional with $\dim W = \dim V$. Let $S \in \mathcal{L}(W, V)$, $T \in \mathcal{L}(V, W)$. Then, $ST = I$ iff $TS = I$.
- ST=I iff TS=I (on vector spaces of the same dimension).** Suppose V and W are finite-dimensional vector spaces of the same dimension, $S \in \mathcal{L}(W, V)$, $T \in \mathcal{L}(V, W)$. Then $ST = I$ iff $TS = I$.
- matrix of identity operator with respect to two bases.** Suppose u_1, \dots, u_n and v_1, \dots, v_n are two bases of V . Then, the matrices $\mathcal{M}(I; u_1, \dots, u_n; v_1, \dots, v_n)$ and $\mathcal{M}(I; v_1, \dots, v_n; u_1, \dots, u_n)$ are invertible and are inverses of each other.
- Product of vector spaces is a vector space.** Suppose V_1, \dots, V_m are vector spaces over \mathbb{F} . Then, $V_1 \times \dots \times V_m$ is a vector space over \mathbb{F} .
- dimension of a product is the sum of the dimensions.** Suppose V_1, \dots, V_m are finite-dimensional vector spaces. Then, $V_1 \times \dots \times V_m$ is finite-dimensional and $\dim(V_1 \times \dots \times V_m) = \dim V_1 + \dots + \dim V_m$.
- Products and direct sums.** Suppose V_1, \dots, V_m are subspaces of V . Define a linear map $\Gamma : (V_1 \times \dots \times V_m) \rightarrow (V_1 + \dots + V_m)$ by $\Gamma(v_1, \dots, v_m) = v_1 + \dots + v_m$. Then, $V_1 + \dots + V_m$ is direct iff Γ is 1-1.
- direct sum iff dimensions add up.** Suppose V is finite-dimensional and V_1, \dots, V_m are subspaces of V . Then, $V_1 + \dots + V_m$ is direct iff $\dim(V_1 + \dots + V_m) = \dim V_1 + \dots + \dim V_m$.
- dimension of quotient space.** Suppose V is finite-dimensional and U is a subspace of V . Then, $\dim(V/U) = \dim V - \dim U$.
- 3F.
- Linear functional.** A linear functional on V is a linear map $\phi : V \rightarrow F$.
- dual space.** The dual space of V is $V' = \mathcal{L}(V, F)$.
- dim space = dim dual space.** Suppose V is finite-dimensional. Then V' is also finite-dimensional and $\dim V = \dim V'$.
- dual basis.** If v_1, \dots, v_n is a basis of V , then the dual basis of v_1, \dots, v_n is ϕ_1, \dots, ϕ_n (elements of V') where $\phi_j(v_k) = 1$ if $k = j$ and $\phi_j(v_k) = 0$ if $k \neq j$.
- dual basis gives coefficients for linear combination.** Suppose v_1, \dots, v_n is a basis of V and ϕ_1, \dots, ϕ_n is dual basis. Then $v = \phi_1(v)v_1 + \dots + \phi_n(v)v_n$ for each $v \in V$.
- dual basis is a basis of dual space.** Suppose V is finite-dimensional. Then the dual basis of V is a basis of V' .

19. **dual map, T' .** Suppose $T \in \mathcal{L}(V, W)$. The dual map of T is $T' \in \mathcal{L}(W', V')$ defined for each $\phi \in W'$ by $T'(\phi) = \phi \circ T$.
20. **algebraic properties of dual maps.** we have $(S+T)' = S' + T'$, $(\lambda S)' = \lambda S'$, $(ST)' = T' S'$.
21. **annihilator.** For $U \subseteq V$, the annihilator of U is $U_0 = \{\phi \in V' \mid \phi(u) = 0 \forall u \in U\}$.
22. **annihilator is a subspace.** If $U \subseteq V$, then $U^0 \subseteq V'$.
23. **dimension of annihilator.** Suppose V is finite-dimensional and $U \subseteq V$. Then $\dim U^0 = \dim V - \dim U$.
24. **condition for annihilator to equal $\{0\}$ or whole space.** Suppose V finite-dimensional and $U \subseteq V$. Then:
- (a) $U^0 = \{0\}$ iff $U = V$.
 (b) $U^0 = V'$ iff $U = \{0\}$.
25. **null space of T' .** Suppose V, W finite-dimensional and $T \in \mathcal{L}(V, W)$. Then:
- (a) $\text{nul } T' = (\text{range } T)^0$.
 (b) $\dim \text{nul } T' = \dim \text{nul } T + \dim W - \dim V$.
26. **T surjective equivalent to T' injective.** Suppose V, W finite-dimensional and $T \in \mathcal{L}(V, W)$. Then T onto iff T' 1-1.
27. **range of T' .** Suppose V, W finite-dim and $T \in \mathcal{L}(V, W)$. Then:
- (a) $\dim \text{range } T' = \dim \text{range } T$.
 (b) $\text{range } T' = (\text{nul } T)^0$.
28. **T injective is equivalent to T' surjective.** Suppose V, W finite-dim and $T \in \mathcal{L}(V, W)$. Then T 1-1 iff T' onto.
29. 5A.
30. **equivalent conditions to be an eigenvalue.** Let V be finite-dim and $T \in \mathcal{L}(V)$ and $\lambda \in F$. Then TFAE:
- (a) λ is an eigenvalue of T .
 (b) $T - \lambda I$ not injective.
 (c) $T - \lambda I$ not surjective.
 (d) $T - \lambda I$ not invertible.
31. **linearly independent eigenvectors.** Let $T \in \mathcal{L}(V)$. Then every list of eigenvectors of T corresponding to different eigenvalues is linearly independent.
32. **operator cannot have more eigenvalues than dimension of space.** Let V be finite-dim. Then each operator on V has at most $\dim V$ distinct eigenvalues.
33. **null space and range of $p(T)$ are invariant under T .** Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(F)$. Then $\text{nul } p(T)$ and $\text{range } p(T)$ are invariant under T .
34. 5B.
35. **existence of eigenvalues.** Every operator on a finite-dim nonzero complex vector space has an eigenvalue.
36. **existence, uniqueness, and degree of minimal polynomial.** Suppose V finite-dim and let $T \in \mathcal{L}(V)$. Then there is a unique monic polynomial $p \in \mathcal{P}(F)$ of smallest degree such that $p(T) = 0$. Also, $\deg p \leq \dim V$.
37. **minimal polynomial.** Suppose V finite-dim and $T \in \mathcal{L}(V)$. Then the minimal polynomial of T is the unique monic polynomial $p \in \mathcal{P}(F)$ of smallest degree such that $p(T) = 0$.
38. **eigenvalues are the zeros of minimal polynomial.** Let V finite-dim and $T \in \mathcal{L}(V)$. Then:
- (a) zeros of the minimal polynomial of T are the eigenvalues of T .
 (b) if V is a complex vector space, then minimal polynomial of T has the form $(z - \lambda_1) \cdots (z - \lambda_m)$, where $\lambda_1, \dots, \lambda_m$ is a list of all eigenvalues of T , possibly with repetitions.
39. **$q(T) = 0$ iff q is a polynomial multiple of the minimal polynomial.** Let V finite-dim and $T \in \mathcal{L}(V)$ and $q \in \mathcal{P}(F)$. Then $q(T) = 0$ iff q is a polynomial multiple of the minimal polynomial.
40. **minimal polynomial of a restriction operator.** Let V finite-dim and $T \in \mathcal{L}(V)$ and $U \subseteq V$ that is invariant under T . Then minimal polynomial of T is a polynomial multiple of minimal polynomial of $T|_U$.
41. **T not invertible iff constant term of minimal polynomial of T is 0.** Let V finite-dim and $T \in \mathcal{L}(V)$. Then T is not invertible iff the constant term in the minimal polynomial of T is 0.
42. **even-dimensional null space.** Let $F = \mathbb{R}$ and V finite-dim and $T \in \mathcal{L}(V)$ and $b^2 - 4ac < 0$. Then $\dim(T^2 + bT + cI)$ is an even number.
43. **operators on an odd-dimensional space have eigenvalues.** Every operator on an odd-dimensional vector space has an eigenvalue.
44. 5C.
45. **conditions for upper-triangular matrix.** Suppose $T \in \mathcal{L}(V)$ and v_1, \dots, v_n is a basis of V . Then TFAE:
- (a) the matrix of T with respect to v_1, \dots, v_n is upper-triangular.
 (b) $\text{span}(v_1, \dots, v_k)$ is invariant under T for each $k = 1, 2, \dots, n$.
 (c) $Tv_k \in \text{span}(v_1, \dots, v_k)$ for each $k = 1, \dots, n$.
46. **equation satisfied by operator with upper-triangular matrix.** Suppose $T \in \mathcal{L}(V)$ and V has a basis with respect to which T has an upper-triangular matrix with diagonal entries $\lambda_1, \dots, \lambda_n$. Then $(T - \lambda_1 I) \cdots (T - \lambda_n I) = 0$.
47. **determination of eigenvalues from upper-triangular matrix.** Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V . Then the eigenvalues of T are precisely the entries on the diagonal of that upper-triangular matrix.
48. **necessary and sufficient condition to have an upper-triangular matrix.** Suppose V is finite-dim and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V iff the minimal polynomial of T equals $(z - \lambda_1) \cdots (z - \lambda_n)$ for some $\lambda_i \in F$.
49. **if $F = \mathbb{C}$, then every operator on V has an upper-triangular matrix.** Suppose V is a finite-dim complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V .
50. 5D.
51. **eigenspace, $E(\lambda, T)$.** Suppose $T \in \mathcal{L}(V)$ and $\lambda \in F$. Then the eigenspace of T corresponding to λ is $E(\lambda, T) = \text{nul}(T - \lambda I) = \{v \in V \mid Tv = \lambda v\}$.
52. **sum of eigenspaces is a direct sum.** Suppose $T \in \mathcal{L}(V)$ and $\lambda_1, \dots, \lambda_m$ are the distinct eigenvalues of T . Then $\sum_j E(\lambda_j, T)$ is a direct sum and $\sum_j \dim E(\lambda_j, T) \leq \dim V$.

53. **conditions equivalent to diagonalizability.** Suppose V finite-dim and $T \in L(V)$. Let $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of T . Then TFAE:

- T is diagonalizable.
- V has a basis consisting of eigenvectors of T .
- $V = \bigoplus_i E(\lambda_i, T)$
- $\dim V = \sum_i \dim E(\lambda_i, T)$.

54. **enough eigenvalues implies diagonalizability.** Let V be finite-dim and $T \in L(V)$ has $\dim V$ distinct eigenvalues. Then T is diagonalizable.

55. **necessary and sufficient condition for diagonalizability.** Suppose V finite-dim and $T \in L(V)$. Then T diagonalizable iff the minimal polynomial of T equals $(z - \lambda_1) \cdots (z - \lambda_m)$ for some distinct $\lambda_1, \dots, \lambda_m \in F$.

56. **restriction of diagonalizable operator to invariant subspace.** Suppose $T \in L(V)$ and U is a T -invariant subspace of V . Then $T|_U$ is a diagonalizable operator on U .

57. 5E.

58. **commuting operators correspond to commuting matrices.** Suppose $S, T \in L(V)$ and v_1, \dots, v_n is a basis of V . Then S and T commute iff $M(S, (v_1, \dots, v_n))$ and $M(T, (v_1, \dots, v_n))$ commute.

59. **eigenspace is invariant under commuting operators.** Suppose $S, T \in L(V)$ commute and $\lambda \in F$. Then $E(\lambda, S)$ is invariant under T .

60. **simultaneous diagonalizability iff commutativity.** Two diagonalizable operators on the same vector space have diagonal matrices with respect to the same basis iff the two operators commute.

61. **common eigenvector for commuting operators.** every pair of commuting operators on a finite-dim nonzero complex vector space has a common eigenvector.

62. **commuting operators are simultaneously upper-triangularizable.** Suppose V is a finite-dim nonzero complex vector space and S, T are commuting operators on V . Then there is a basis of V with respect to which both S, T have upper-triangular matrices.

63. **eigenvalues of sum and product of commuting operators.** Suppose V is a finite-dim complex vector space and S, T are commuting operators on V . Then:

- every eigenvalue of $S + T$ is an eigenvalue of S plus an eigenvalue of T .
- every eigenvalue of ST is an eigenvalue of S times an eigenvalue of T .

64. 8A.

65. **sequence of increasing null spaces.** Let $T \in L(V)$. Then $\{0\} = \text{nul } T^0 \subseteq \text{nul } T_1 \subseteq \cdots \subseteq \text{nul } T^k \subseteq \cdots$

66. **equality in the sequence of null spaces.** Let $T \in L(V)$ and m is a nonnegative integer such that $\text{nul } T^m = \text{nul } T^{m+1}$. Then $\text{nul } T^m = \text{nul } T^{m+1} = \cdots$

67. **null spaces stop growing.** Let $T \in L(V)$. Then $\text{nul } T^{\dim V} = \text{nul } T^{\dim V+1} = \cdots$

68. **V is the direct sum of $\text{nul } T^{\dim V}$ and $\text{range } T^{\dim V}$.** Let $T \in L(V)$. Then $V = \text{nul } T^{\dim V} \oplus \text{range } T^{\dim V}$.

69. **generalized eigenvector.** Let $T \in L(V)$ and λ be an eigenvalue of T . A vector $v \in V$ ($v \neq 0$) is called a generalized eigenvector of T corresponding to λ if $(T - \lambda I)^k v = 0$ for some $k \in \mathbb{Z}_{>0}$.

70. **a basis of generalized eigenvectors.** Let $F = \mathbb{C}$ and $T \in L(V)$. Then there is a basis of V consisting of generalized eigenvectors of T .

71. **generalized eigenvector corresponds to a unique eigenvalue.** Let $T \in L(V)$. Then each generalized eigenvector of T corresponds to only one eigenvalue of T .

72. **linearly independent generalized eigenvectors.** Let $T \in L(V)$. Then every list of generalized eigenvectors of T corresponding to distinct eigenvalues of T is linearly independent.

73. **nilpotent operator raised to dimension of domain is 0.** Let $T \in L(V)$ be nilpotent. Then $T^{\dim V} = 0$.

74. **eigenvalues of nilpotent operator.** Let $T \in L(V)$. Then:

- if T is nilpotent then 0 is an eigenvalue of T and T has no other eigenvalues.
- if $F = \mathbb{C}$ and 0 is the only eigenvalue of T , then T is nilpotent.

75. **minimal polynomial & upper-triangular matrix of nilpotent operator.** Let $T \in L(V)$. Then TFAE:

- T is nilpotent.
- minimal polynomial of T is z^m for some positive integer m .
- there is a basis of V with respect to which the matrix of T has the form

$$\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$

76. 8B.

77. **generalized eigenspace.** Suppose $T \in L(V)$ and $\lambda \in F$. The generalized eigenspace of T corresponding to λ is $G(\lambda, T) = \{v \in V \mid (T - \lambda I)^k v = 0 \text{ for some } k \in \mathbb{Z}_{>0}\}$, which is the set of generalized eigenvectors of T corresponding to λ , including the 0-vector.

78. **description of generalized eigenspaces.** Suppose $T \in L(V)$ and $\lambda \in F$. Then $G(\lambda, T) = \text{nul}(T - \lambda I)^{\dim V}$.

79. **generalized eigenspace decomposition.**

80. Suppose $F = \mathbb{C}$ and $T \in L(V)$. Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T . Then:

- $G(\lambda_k, T)$ is invariant under T for each $k = 1, \dots, m$.
- $(T - \lambda_k I)|_{G(\lambda_k, T)}$ is nilpotent for each $k = 1, \dots, m$.
- $V = \bigoplus_i G(\lambda_i, T)$.

81. **multiplicity.** Let $T \in L(V)$. The multiplicity of an eigenvalue λ of T is defined to be the dimension of the corresponding generalized eigenspace $G(\lambda, T)$, so multiplicity of λ is $\dim \text{nul}(T - \lambda I)^{\dim V}$.

82. **sum of the multiplicities equals $\dim V$.** Suppose $F = \mathbb{C}$ and $T \in L(V)$. Then the sum of all the multiplicities of all the eigenvalues of T equals $\dim V$.

83. **characteristic polynomial.** Let $F = \mathbb{C}$ and $T \in L(V)$. Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T , with multiplicities d_1, \dots, d_m . Then the polynomial $(z - \lambda_1)^{d_1} \cdots (z - \lambda_m)^{d_m}$ is called the characteristic polynomial of T .

84. **degree and zeros of the characteristic polynomial.** Let $F = \mathbb{C}$ and $T \in L(V)$. Then:

- characteristic polynomial of T has degree $\dim V$.
- zeros of the characteristic polynomial are the eigenvalues of T .

85. **Cayley-Hamilton theorem.** Let $F = \mathbb{C}$, $T \in L(V)$ and q be the characteristic polynomial of T . Then $q(T) = 0$.

86. **characteristic polynomial is a multiple of minimal polynomial.** Let $F = \mathbb{C}$ and $T \in L(V)$. Then characteristic polynomial of T is a polynomial multiple of the minimal polynomial of T .

87. **multiplicity of an eigenvalue equals number of times on diagonal.** Let $F = \mathbb{C}$ and $T \in L(V)$. Let v_1, \dots, v_n be a basis of V such that $M(T, (v_1, \dots, v_n))$ is upper-triangular. The number of times the eigenvalue λ appears on the diagonal of $M(T, (v_1, \dots, v_n))$ equals the multiplicity of λ as an eigenvalue of T .

88. **block diagonal matrix with upper-triangular blocks.** Let $F = \mathbb{C}$ and $T \in L(V)$. Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T with multiplicities d_1, \dots, d_m . Then there is a basis of V with respect to which T has a block diagonal matrix of the form

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix}$$

, where each A_k is a d_k -by- d_k upper-triangular matrix of the form

$$\begin{pmatrix} \lambda_k & & * \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix}$$

89. 8C.

90. **Jordan basis.** Let $T \in L(V)$. A basis of V is called a Jordan basis for T if with respect to this basis T has a block diagonal matrix

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_p \end{pmatrix}$$

in which each A_k is an upper-triangular matrix of the form

$$\begin{pmatrix} \lambda_k & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & \ddots & 1 \\ & & & \lambda_k \end{pmatrix}$$

91. **every nilpotent operator has a Jordan basis.** Let $T \in L(V)$ be nilpotent. Then there is a basis for V that is a Jordan basis for T .

92. **Jordan form.** Let $F = \mathbb{C}$ and $T \in L(V)$. Then there is a basis of V that is a Jordan basis.

93. 6A.

94. **inner product.** An inner product on V is a function that takes an ordered pair (u, v) of elements of V to a number $\langle u, v \rangle \in F$ so that

- positivity: $\langle v, v \rangle \geq 0 \forall v \in V$.
- definiteness: $\langle v, v \rangle = 0$ iff $v = 0$.
- additivity in the first slot: $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$.
- homogeneity in the first slot: $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$.
- conjugate symmetry: $\langle u, v \rangle = \overline{\langle v, u \rangle}$.

95. **basis properties of inner product spaces.**

- $\langle u, v \rangle$ is a linear map from V to F for a fixed $u \in V$.
- $\langle 0, v \rangle = 0$ for all $v \in V$.
- $\langle v, 0 \rangle = 0$ for all $v \in V$.
- $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$.
- $\langle u, \lambda v \rangle = \overline{\lambda} \langle u, v \rangle$.

96. **basic properties of norm.**

- $\|v\| = 0$ iff $v = 0$.
- $\|\lambda v\| = |\lambda| \cdot \|v\|$.

97. **pythagorean theorem.** if $u, v \in V$ with u, v orthogonal, then $\|u + v\|^2 = \|u\|^2 + \|v\|^2$.

98. **orthogonal decomposition.** let $u, v \in V$ with $v \neq 0$. set $c = \frac{\langle u, v \rangle}{\|v\|^2}$ and $w = u - \frac{\langle u, v \rangle}{\|v\|^2} v$. then $u = cv + w$ and $\langle v, w \rangle = 0$.

99. **cauchy-schwarz.** if $u, v \in V$, then $|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$, with equality iff one of u, v is a scalar multiple of the other.

100. **triangle inequality.** if $u, v \in V$, then $\|u + v\| \leq \|u\| + \|v\|$, with equality iff one of u, v is a nonnegative real multiple of the other.

101. **parallelogram equality.** if $u, v \in V$, then $\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$.

102. 6B.

103. **norm of an orthonormal linear combination.** let e_1, \dots, e_m be an orthonormal list in V . then $\|a_1 e_1 + \cdots + a_m e_m\|^2 = |a_1|^2 + \cdots + |a_m|^2$.

104. **bessel's inequality.** let e_1, \dots, e_m be an orthonormal list in V . then if $v \in V$, then $|\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2 \leq \|v\|^2$.

105. **writing a vector as a linear combination of an orthonormal basis.** let e_1, \dots, e_m be an orthonormal basis of V and $u, v \in V$. then

$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_m \rangle e_m.$$

- (b) $\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2$.
- (c) $\langle u, v \rangle = \langle u, e_1 \rangle \overline{\langle v, e_1 \rangle} + \cdots + \langle u, e_m \rangle \overline{\langle v, e_m \rangle}$.
106. **gram-schmidt.** let v_1, \dots, v_m be L.I. in V . let $f_1 = v_1$. for $k > 2$, define $f_k = v_k - \frac{\langle v_k, f_1 \rangle}{\|f_1\|^2} f_1 - \cdots - \frac{\langle v_k, f_{k-1} \rangle}{\|f_{k-1}\|^2} f_{k-1}$. then, f_1, \dots, f_m is an orthogonal basis of $\text{span}(v_1, \dots, v_m)$.
107. **existence of orthonormal basis.** every finitdim inner product space has an orthonormal basis.
108. **every orthonormal list extends to an orthonormal basis.** if V is finitdim, then every orthonormal list in V can be extended to an orthonormal basis of V .
109. **upper-triangular matrix with respect to some orthonormal basis.** let V be finitdim and $T \in L(V)$. then T has an upper-triangular matrix with respect to some orthonormal basis of V iff min poly of T equals $(z - \lambda_1) \cdots (z - \lambda_m)$ for some m .
110. **schur's theorem.** every operator on a finitdim complex inner product space has an upper-triangular matrix with respect to some orthonormal basis.
111. **riesz representation theorem.** let V be finitdim and ϕ is a linear functional on V . then there is a unique $v \in V$ so that $\phi(u) = \langle u, v \rangle$ for every $u \in V$.
112. 6C.
113. **properties of orthogonal complement.**
- (a) if U is a subset of V , then U^\perp is a subspace of V .
- (b) $\{0\}^\perp = V$.
- (c) $V^\perp = \{0\}$.
- (d) if U is a subset of V , then $U \cap U^\perp \subseteq \{0\}$.
- (e) if G, H subsets of V and $G \subseteq H$, then $H^\perp \subseteq G^\perp$.
114. **direct sum of a subspace and its orthogonal complement.** let U be a finitdim subspace of V . then $V = U \oplus U^\perp$.
115. **dimension of orthogonal complement.** let V be finitdim and U be a subspace of V . then $\dim U^\perp = \dim V - \dim U$.
116. **orthogonal complement of orthogonal complement.** let U be a finitdim subspace of V . then $(U^\perp)^\perp = U$.
117. **prop.** let U be a finitdim subspace of V . then $U^\perp = \{0\}$ iff $U = V$.
118. **orthogonal projection operator.** let U be a finitdim subspace of V . orthogonal projection of V onto U is $P_U \in L(V)$, where if $v \in V$, write $v = u + w$, where $u \in U$ and $w \in U^\perp$, and get $P_U v = u$.
119. **properties of orthogonal projection operator.** let U be a finitdim subspace of V .
- (a) $P_U \in L(V)$.
- (b) $P_U u = u \forall u \in U$.
- (c) $P_U w = 0 \forall w \in U^\perp$.
- (d) $\text{range } P_U = U$.
- (e) $\text{nul } P_U = U^\perp$.
- (f) $v - P_U v \in U^\perp$ for all $v \in V$.
- (g) $P_U^2 = P_U$.
- (h) $\|P_U v\| \leq \|v\|$ for all $v \in V$.
- (i) if e_1, \dots, e_m is an orthonormal basis of V and $v \in V$, then $P_U v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_m \rangle e_m$.
120. **riesz representation theorem (again).** let V be finitdim. for each $v \in V$ define $\phi_v \in V'$ so that $\phi_v(u) = \langle u, v \rangle$ for each $u \in V$. then $v \mapsto \phi_v$ is a 1-1 function $V \rightarrow V'$.

121. RIBET DEFS MT2.
122. **Double Dual.** Let V be a finite-dimensional vector space with dual V' . Then the double dual of V is $(V')' = V'' = V$. Also, $\dim V = n = \dim V' = \dim V''$.
123. **Eigenspace.** Let $T \in \mathcal{L}(V)$ and take λ to be an eigenvalue of T . Then, $E(\lambda, T) = \{v \in V \mid Tv = \lambda v\} \neq \emptyset$ is written as V_λ and is called the eigenspace of λ , which is a subspace of V .
124. **Generalized Eigenvector.** Consider a minimal polynomial $(x - \lambda_1)^{e_1} \cdots (x - \lambda_m)^{e_m}$ on X with $(T - \lambda_1 I)^{e_1} v = 0$. Then, v is called a generalized eigenvector for $\lambda = \lambda_1$.
125. **Characteristic polynomial.** The characteristic polynomial of $T: V \rightarrow V$ (with eigenvalues $\lambda_1, \dots, \lambda_t$) is the polynomial $\prod_{i=1}^t (x - \lambda_i)^{\dim X_{\lambda_i}}$, where $V = X_1 \oplus \cdots \oplus X_t$.
126. **Simultaneously diagonalizable.** Operators S and T on V are simultaneously diagonalizable if there is a basis of V that consists of vectors that are eigenvectors for both S and T (i.e. there exists a basis v_1, \dots, v_n of V so that for i , $1 \leq i \leq n$, there are λ_i and μ_i so that $Sv_i = \lambda_i v_i$ and $Tv_i = \mu_i v_i$).

127. RIBET THMS MT1.
128. **Theorem.** The intersection of a family of subspaces of a vector space V is a subspace of V .
129. **Theorem.** Every subspace X of V has complement.
130. **Prop.** Let X, Y be subspaces of a finite-dimensional vector space V . Then:

- (a) $\dim X + \dim Y = \dim V$.
- (b) $X \cap Y = \{0\}$.

Then, $V = X \oplus Y$.

131. **Prop.** $\dim(X \oplus Y) = \dim(X \times Y) = \dim X + \dim Y$.

132. RIBET THMS MT2
133. **Theorem.** Every linear functional on a subspace of V can be extended to V .
134. **Note.** Annihilator is the dual of the quotient subspace.
135. **Cor.** The annihilator of U is $\{0\}$ iff $U = V$. The annihilator of U is V iff $U = \{0\}$.
136. **Prop.** If $T: V \rightarrow W$ is a linear map, then the null space of T' is the annihilator of the range of T . We have $\text{ann}(\text{range } T) = \{\psi: W \rightarrow F \mid \phi(Tv) = 0 \text{ for all } v \in V, T'(\psi)(v) = 0, T'\psi = 0, \phi \in \text{nul}(T')\}$.
137. **Cor.** If $T: V \rightarrow W$ is a linear map between finite-dimensional F -vector spaces, then $\dim \text{nul}(T') = \dim \text{nul}(T) + \dim W - \dim V$.
138. **Cor.** The linear map T is onto iff T' is 1-1.
139. **Cor.** If $T: V \rightarrow W$ is a linear map between finite-dimensional vector spaces, then T' and T have equal ranks.
140. **Cor.** We have $\text{range } T = (\text{nul } T')^0$.
141. **Theorem.** Let F be a finite field with $q = |F|$. Then, $a^q = a$ for all $a \in F$.
142. **Theorem.** If F is a finite field, then $|F| = p^n$ for some prime p and integer $n \geq 1$.
143. **Theorem.** Let $T: V \rightarrow V$, V finite-dimensional, and let $\alpha: F[x] \rightarrow \mathcal{L}(V)$, with $f \mapsto f(T)$. Also, we have $\ker \alpha$ to be the principal ideal $(m(x))$. Then, $m(x)$ is the minimal polynomial of T and has degree $\leq n^2$.
144. **Cayley-Hamilton Theorem.** Let $T: V \rightarrow V$, V finite-dimensional, and let $\alpha: F[x] \rightarrow \mathcal{L}(V)$, with $f \mapsto f(T)$. Also, we have $\ker \alpha$ to be the principal ideal $(m(x))$, where $m(x)$ is the minimal polynomial of T . Then, the characteristic polynomial is in $\ker \alpha$; that is, we can plug in the matrix for T into its characteristic polynomial and we get that it is equal to the 0-matrix.
145. **Lemma.** Let $f \in \mathbb{R}[x]$ be a real polynomial. If λ is a complex root of f , so is $\bar{\lambda}$, which is the complex conjugate of λ .
146. **Cor.** Let $\lambda_1, \dots, \lambda_t$ be distinct eigenvalues and take $E_i = E(\lambda_i, T) = \{v \in V \mid Tv = \lambda_i v\} \subseteq V$. Now, take $E_1 \times \cdots \times E_t$. Then there exists a summation map $E_1 \times \cdots \times E_t \xrightarrow{\text{sum}} V$ with $(v_1, \dots, v_t) \mapsto v_1 + \cdots + v_t$. Then, the sum map is 1-1.
147. **Cor.** Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.
148. **Prop.** Suppose T is an operator on an F -vector space V . If $f \in F[x]$ is a polynomial satisfied by T (meaning $f(T) = 0$), then every eigenvalue of T on V is a root of f .
149. **Cor.** Suppose λ is an eigenvalue of operator T on a finite-dimensional F -vector space. Then λ is a root of the minimal polynomial of T iff λ is an eigenvalue of T .
150. **Theorem.** All operators on a nonzero finite-dimensional vector space over an algebraically closed field have at least one eigenvalue.
151. **Prop.** Assume that $F = \mathbb{R}$ and that $f(x) := x^2 + bx + c$ is an irreducible polynomial. If $T \in \mathcal{L}(V)$ and V is finite-dimensional, then the null space of $f(T)$ is even-dimensional.
152. **Prop (honors version).** Let T be an operator on a finite-dimensional vector space over F . If p is an irreducible polynomial over F , then the dimension of the null space of $p(T)$ is a multiple of the degree of p .
153. **Prop.** $F[x]/(p)$ (where p is irreducible) is a field.
154. **Formula.** $\dim_F V = [K : F] \cdot \dim_K V = \dim_F K \cdot \dim_K V$.
155. **Cor.** Every operator on an odd-dimensional \mathbb{R} -vector space has an eigenvalue.
156. **Prop.** If T is an operator on a finite-dimensional F -vector space, then the minimal polynomial of T has degree at most $\dim V$.
157. **Prop.** If T is upper-triangular with respect to some basis of V , and if the diagonal entries of an upper-triangular matrix representation of T are $\lambda_1, \dots, \lambda_n$, then $(T - \lambda_1 I) \cdots (T - \lambda_n I) = 0$.
158. **Prop.** Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V)$ and let $\lambda_1, \dots, \lambda_m$ be the eigenvalues of T . Then, $V = \oplus E(\lambda_i, T)$ iff T is diagonalizable.
159. **Prop.** TFAE.

- (a) T is diagonalizable.
- (b) V has a basis consisting of eigenvectors.
- (c) The direct sum $\oplus_i V_{\lambda_i}$ is all of V .

$$(d) \dim \left(\oplus_i V_{\lambda_i} \right) = \dim V.$$

160. **Prop.** If $T: V \rightarrow V$ has $\dim V$ different eigenvalues, then T is diagonalizable.
161. **Jordan Canonical Form.** X can be written as a direct sum of Jordan blocks, where $\sum \dim(\text{block}) = \dim X$.
162. **Lemma.** Let $X = \oplus \text{span}(U_i v)$ for $i \in \{0, \dots, k_1\}$. If Z is a subspace of X' that is U' -invariant, then $\text{ann}(Z) := Y$ is U -invariant.
163. **Lemma.** Suppose S and T are commuting operators on V . If λ is an eigenvalue for T on V , then the eigenspace $E(\lambda, T)$ is S -invariant.
164. **Theorem.** The diagonalize operators on the same finite-dimensional vector space are simultaneously diagonalizable iff they commute with each other.
165. **Theorem.** Every pair of commuting operators on a finite-dimensional nonzero complex vector space has a common eigenvector.
166. **Prop.** Two commuting operators on a finite-dimensional nonzero complex vector space can be simultaneously upper-triangularized.
167. **Prop.** We have:

- (a) Every eigenvalue of $S + T$ is the sum of an eigenvalue of S and an eigenvalue of T .
- (b) Every eigenvalue of ST is the product of an eigenvalue of S and an eigenvalue of T .