H110 Lecture Notes - March 12, 2025.

As a starter, take $T : V \to V$, where V is finite-dimensional (where V is a vector space over the field F, and we can consider F to be algebraically closed). Now, consider the following proposition.

Prop. If T is an operator on a finite-dimensional F-vector space, then the minimal polynomial of T has degree at most $\dim V$.

We now give an outline of the proof. As a preliminary matter, consider the map $\alpha : F[x] \to \mathscr{L}(V)$, where $\dim V = n$. Then, α gives the mapping $g(x) \mapsto g(T)$ and $\operatorname{nul} \alpha = (m(x)) \subseteq F[x]$. The proof is done by induction on the dimension. Now look at the base case(s). If $n = 0$, then $V = (0)$, $f(x) = 1$, and $f(T) = T = 0$. If $n = 1$, then $V = F \cdot v$, $Tv = \lambda v$, so $(T - \lambda I)v = 0$, and $f(x) = x - \lambda$. If $n = 2$, we have $V = F^2$, and T corresponds to $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} =: M$. Then, $M^2 - (a+d)M + (ad-bc)M^0 = 0$. Now, consider the case where $n \geq 2$. Then, necessarily, $V \ni v \neq 0$, and let X be the smallest T-invariant subspace of V containing v. When $n = 2$, $X \ni v$, and we do not know if $Tv = \lambda v$ (which would then give $X = F \cdot v$). We set v, Tv to be linearly independent. Let now $X = \operatorname{span}(v, Tv, T^2v, \dots)$. Write $v, Tv, \dots, T^n v$ (which is a list of length $n + 1 > \dim V = n$). Thus, $v, Tv, \dots, T^n v$ is linearly dependent. Say $v, Tv, \dots, T^{m-1}v$ is linearly independent (with $v, Tv, \dots, T^m v$ linearly dependent). Then, $T^m v = a_0 v + a_1 Tv + \cdots + a_{m-1}T^{m-1}v$ is a linear dependence relation. Take $h(x) = x^m - (a_{m-1}x^{m-1} + \cdots + a_0)$. Then, $h(T) \cdot v = 0$, $h(T)Tv = T(h(T)v) = T(0) = 0$. Thus, $h(T) = 0$ on X, h has degree m, and $\dim X = m$. Then, $T^{m+1}v = T(T^m v) = \left(\sum_{i=0}^{m-2} a_i T^{i+1} v\right) + a_m T^m v$. Thus, $X = \operatorname{span}(v, Tv, \dots, T^{m-1}v)$. Our goal now is to show that there exists an $f(x)$ (with $\deg f \leq n$) with $f(T) = 0$ on V. We have shown $0 \subsetneq X \subseteq V$, where X is T-stable (meaning applying T to the list $v, Tv, T^2v, \dots$ results in a shifting of the list) and there exists a polynomial $h(x)$ with $\deg h = \dim X$ such that $h(T) = 0$ on X.

Consider now the question: Is there a $v \in V$ such that $X = \operatorname{span}(v, Tv, \dots, T^{m-1}v)$ is all of V? Take now $T = I$. Then, $v = Tv = \cdots = v$. Now, look at V/X. Since X is T-stable, we have V/X has an induced action of T, meaning that we have $T_{V/X} : V/X \to V/X$ (with $v + X \mapsto Tv + X$) and $T|_X : X \to X$ (so $h(T|_X) = 0$). We have $\dim X \geq 1$, while it could be that $\dim X \leq n$. However, $\dim(V/X) < n$, since $\dim(V/X) = \dim V - \dim X$. The induction hypothesis gives $\dim(V/X) < n$. So there exists a polynomial $g \in X$ with $\deg g \leq \dim(V/X)$ such that $g(T|_{V/X}) = 0$ (denote this last equation as $*$). Now consider $g(T) : V \to V$, so $*$ gives $\operatorname{range} g(T) \subseteq X$. Define $f = gh = hg$. Then, $\deg f = \deg g + \deg h$. We have $\deg h = \dim X$ and $\deg g \leq \dim(V/x) = n - m$. Thus, $\deg f \leq n$. Now, take $(hg)(T) = h(T) \circ g(T) = 0$, and we are done. $\square$

Prop. If T is upper-triangular with respect to some basis of V, and if the diagonal entries of an upper-triangular matrix representation of T are $\lambda_1, \dots, \lambda_n$, then $(T - \lambda_1 I) \cdot \cdots \cdot (T - \lambda_n I) = 0$.

Take an upper-triangular matrix with respect to the basis $v_1, \ldots, v_n$. Then, $Tv_1 = \lambda_1 v_1$, $Tv_2 = \lambda_2 v_2 + a_{12} v_1$, $Tv_3 = \lambda_3 v_3 + (a_{13} v_1 + a_{23} v_2)$. Thus, $(T - \lambda_1 I) \cdot \cdots \cdot (T - \lambda_n I) = 0$ iff $((T - \lambda_1 I) \cdot \cdots \cdot (T - \lambda_n I))v_j$ (where v_j is a basis vector). So, $(T - \lambda_1 I)v_1 = 0$, $(T - \lambda_2 I)v_2 = a_{12} v_1$, and $(T - \lambda_1 I)(T - \lambda_2 I)v_2 = 0$. Now consider $(x - \lambda_1) \cdot \cdots \cdot (x - \lambda_n) =: f(x)$ (where f is the characteristic polynomial). Then, $f(T) = 0$, so the minimal polynomial must divide f. Consider the following examples:

1. Take $T = I_n$. Then, $f(x) = (x-1)^n$ and $m(x) = x - 1$.
2. Take $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then, $f(x) = (x-1)^2$, but $m(x) = (x-1)^2$.

So far, we know that the eigenvalues of T are the roots of the minimal polynomial. Thus, every eigenvalue of T is one of the diagonal elements λ_i. We explore the converse next class.