

## Math H110 Midterm 1 CheatSheet

1A. (n/a)

1B.

1. **Vector Space.** A vector space  $V$  is a set that has scalar multiplication and vector addition defined on it with the following properties:
  - (a) Additive commutativity.
  - (b) Additive associativity of vectors ( $u + (v + w) = (u + v) + w$ ) and multiplicative associativity for scalars ( $(ab)v = a(bv)$ ).
  - (c) Additive identity.
  - (d) Additive inverses.
  - (e) Multiplicative identity.
  - (f) BOTH distributive properties.
2. **V-space (unique additive identity)** A vector space has a unique additive identity.
3. **V-space (unique additive inverses)** Every element in a vector space has a unique additive inverse.

1C.

1. **Subspace.** A subset  $U \subseteq V$  is a subspace of  $V$  if it is a vector space with the same additive identity, scalar multiplication, and vector addition as defined on  $V$ .
2. **Conditions for a Subspace.** A subset  $U \subseteq V$  is a subspace of  $V$  iff  $U$  is closed under vector addition, scalar multiplication, and contains the "zero" element as in  $V$ .
3. **Sums of Subspaces.** Let  $V_1, \dots, V_n$  be subspaces of  $V$ . Then, we have the sum of subspaces as  $V_1 + \dots + V_n = \{v_1 + \dots + v_n \mid v_i \in V_i \text{ for all } i\}$ .
4. **Smallest subspace containing each subspace** Suppose  $V_1, \dots, V_n$  are subspaces of  $V$ . Then,  $V_1 + \dots + V_n$  is the smallest subspace of  $V$  containing  $V_1, \dots, V_n$ .
5. **Direct Sum.** Suppose  $V_1, \dots, V_m$  are subspaces of  $V$ . Then:

(a) The sum  $V_1 + \cdots + V_m$  is direct if each element of  $V_1 + \cdots + V_m$  can be written uniquely as a sum  $v_1 + \cdots + v_m$ , where  $v_i \in V_i$  for all  $i$ .

(b) If  $V_1 + \cdots + V_m$  is a direct sum, then we write  $V_1 \oplus \cdots \oplus V_m$ .

6. **Conditions for a direct sum.** Suppose  $V_1, \dots, V_n$  are subspaces of  $V$ . Then,  $V_1 + \cdots + V_n$  is direct iff the only way to write 0 from  $v_1 + \cdots + v_n$  is by taking  $v_i = 0$  for all  $i$ .

7. **Direct sum of subspaces.** If  $U, W$  are subspaces of  $V$ , then  $U + W$  is direct iff  $U \cap W = \{0\}$ .

2A.

1. **Span is the smallest containing subspace.** The span of a list of vectors in  $V$  is the smallest subspace containing all of the vectors in the list.

2. **Zero polynomial.** The zero polynomial is said to have degree  $-\infty$ .

3. **Linear Independence.** A list of vectors  $v_1, \dots, v_n \in V$  is said to be linearly independent if  $a_1 v_1 + \cdots + a_n v_n = 0$  implies  $a_i = 0$  for all  $i$ . Also, the empty list  $()$  is said to be linearly independent.

4. **Linear Dependence.** A list of vectors  $v_1, \dots, v_n$  is said to be linearly dependent if  $a_1 v_1 + \cdots + a_n v_n = 0$  implies  $a_i \neq 0$  for some  $i$ .

5. **Linear Dependence Lemma.** Suppose  $v_1, \dots, v_m$  is a linearly dependent list in  $V$ . Then, there exists  $k \in \{1, \dots, m\}$  such that  $v_k \in \text{span}(v_1, \dots, v_{k-1})$ . Furthermore, if  $k$  satisfies the condition in the previous sentence and the  $k^{\text{th}}$  term is removed from  $v_1, \dots, v_m$ , then the span of the remaining list equals  $\text{span}(v_1, \dots, v_m)$ .

6. **length of linearly independent list ; length of spanning list.** In a finite-dimensional vector space, the length of every linearly independent list is at most the length of every spanning list of vectors.

7. **Finite Dimensional subspaces.** Every subspace of a finite-dimensional vector space is finite dimensional.

2B.

1. **Basis.** A basis of  $V$  is a list of vectors that is linearly independent and spans  $V$ .

2. **Criterion for basis.** A list of vectors  $v_1, \dots, v_n \in V$  is a basis of  $V$  iff every  $v \in V$  can be written uniquely in the form  $v = a_1v_1 + \dots + a_nv_n$ , where  $a_i \in F$  for all  $i$ .
3. **Every spanning list contains a basis.** Every spanning list in a vector space can be reduced to a basis of the vector space.
4. **Basis of finite-dimensional vector space.** Every finite-dimensional vector space has a basis.
5. **Every linearly independent list extends to a basis.** Every linearly independent list in a finite-dimensional vector space can be extended to a basis of the vector space.
6. **Every subspace of  $V$  is part of a direct sum equal to  $V$ .** Suppose  $V$  is finite-dimensional and  $U$  is a subspace of  $V$ . Then, there is a subspace  $W$  of  $V$  such that  $V = U \oplus W$ .

2C.

1. **Basis length does not depend on basis.** Any two bases of a finite-dimensional vector space have the same length.
2. **Dimension of a subspace.** If  $V$  is finite-dimensional and  $U$  is a subspace of  $V$ , then  $\dim U \leq \dim V$ .
3. **Linearly independent list of the right length is a basis.** Suppose  $V$  is finite-dimensional. Then, every linearly independent list of vectors in  $V$  (with list length equal to  $\dim V$ ) is a basis of  $V$ .
4. **Subspace of full dimension equals the whole space.** Suppose  $V$  is finite-dimensional and  $U$  is a subspace of  $V$  such that  $\dim U = \dim V$ . Then,  $U = V$ .
5. **Spanning list of the right length is a basis.** Suppose  $V$  is finite-dimensional. Then, every spanning list of  $V$  of length  $\dim V$  is a basis of  $V$ .
6. **Dimension of a sum.** If  $V_1, V_2$  are subspaces of a finite-dimensional vector space, then  $\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$ .

3A.

1. **Set of Linear Maps.** The linear of linear maps from  $V \rightarrow W$  is written  $\mathcal{L}(V, W)$  and the set of linear maps from  $V \rightarrow V$  is written  $\mathcal{L}(V)$ .
2. **Linear Map lemma.** Suppose  $v_1, \dots, v_n$  is a basis of  $V$  and  $w_1, \dots, w_n \in W$ . Then, there exists a unique linear map  $T : V \rightarrow W$  such that  $Tv_k = w_k$  for each  $k$ .
3. **Linear maps take 0 to 0.** Suppose  $T : V \rightarrow W$  is a linear map. Then,  $T(0) = 0$ .

3B.

1. **null space is a subspace.** Suppose  $T \in \mathcal{L}(V, W)$ . Then,  $T$  is a subspace of  $V$ .
2. **injectivity iff null is 0.** Let  $T \in \mathcal{L}(V, W)$ . Then,  $T$  is 1-1 iff  $T = \{0\}$ .
3. **range is a subspace.** If  $T \in \mathcal{L}(V, W)$ , then  $\text{range } T$  is a subspace of  $W$ .
4. **Fundamental Theorem of Linear Maps.** Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Then,  $\text{range } T$  is finite dimensional and  $\dim V = \dim T + \dim \text{null } T$ .
5. **linear map to a lower-dim space is not 1-1.** Suppose  $V, W$  are finite-dimensional vector spaces such that  $\dim V > \dim W$ . Then, no linear map from  $V \rightarrow W$  is 1-1.
6. **linear map to a higher-dim space is not onto.** Suppose  $V, W$  are finite-dimensional vector spaces such that  $\dim V < \dim W$ . Then, no linear map from  $V \rightarrow W$  is onto.

3C. n/a.

3D.

1. **Theorem.** Let  $V, W$  be finite-dimensional vector spaces such that  $\dim V = \dim W$  and let  $T \in \mathcal{L}(V, W)$ . Then,  $T$  is invertible iff  $T$  is 1-1 iff  $T$  is onto.
2. **isomorphism.** An isomorphism is an invertible linear map.
3. **dimension and isomorphic.** Two finite-dimensional vector spaces are isomorphic iff they have the same dimension.

4. **Theorem.** Suppose  $V$  and  $W$  are finite-dimensional. Then,  $\mathcal{L}(V, W)$  is finite-dimensional and  $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$ .

3E.

1. **Product of vector spaces is a vector space.** Suppose  $V_1, \dots, V_m$  are vector spaces over  $\mathbb{F}$ . Then,  $V_1 \times \dots \times V_m$  is a vector space over  $\mathbb{F}$ .
2. **dimension of a product is the sum of the dimensions.** Suppose  $V_1, \dots, V_m$  are finite-dimensional vector spaces. Then,  $V_1 \times \dots \times V_m$  is finite-dimensional and  $\dim(V_1 \times \dots \times V_m) = \dim V_1 + \dots + \dim V_m$ .
3. **Products and direct sums.** Suppose  $V_1, \dots, V_m$  are subspaces of  $V$ . Define a linear map  $\Gamma : (V_1 \times \dots \times V_m) \rightarrow (V_1 + \dots + V_m)$  by  $\Gamma(v_1, \dots, v_m) = v_1 + \dots + v_m$ . Then,  $V_1 + \dots + V_m$  is direct iff  $\Gamma$  is 1-1.
4. **direct sum iff dimensions add up.** Suppose  $V$  is finite-dimensional and  $V_1, \dots, V_m$  are subspaces of  $V$ . Then,  $V_1 + \dots + V_m$  is direct iff  $\dim(V_1 + \dots + V_m) = \dim V_1 + \dots + \dim V_m$ .
5.  **$v + U$ .** Suppose  $v \in V$  and  $U \subseteq V$ . Then,  $v + U = \{v + u \mid u \in U\}$ .
6. **Translate.** For  $v \in V$  and  $U \subseteq V$ , the set  $v + U$  is called a translate of  $U$ .
7. **Quotient Space.** Let  $U$  be a subspace of  $V$ . Then, the quotient space  $V/U$  is the set of all translates of  $U$ , that is,  $V/U = \{v + U \mid v \in V\}$ .
8. **two translates of a subspace are either equal or disjoint.** Suppose  $U$  is a subspace of  $V$  and  $v, w \in V$ . Then,  $v - w \in U$  iff  $v + U = w + U$  iff  $(v + U) \cap (w + U) \neq \emptyset$ .
9. **Addition and scalar multiplication on Quotient space.** Let  $U$  be a subspace of  $V$ . Then, we have (for all  $v, w \in V, \lambda \in F$ ):
  - (a) addition on  $V/U$ :  $(v + U) + (w + U) = (v + w) + U$ .
  - (b) scalar multiplication on  $V/U$ :  $\lambda(v + U) = (\lambda v) + U$ .
10. **quotient space is a vector space.** Let  $U$  be a subspace of  $V$ . Then, the quotient space  $V/U$  is a subspace of  $V$  under the defined scalar multiplication and vector addition.

11. **quotient map.** Let  $U$  be a subspace of  $V$ . Then, the quotient map  $\pi : V \rightarrow V/U$  is the linear map defined by  $\pi(v) = v + U$  for each  $v \in V$ .
12. **dimension of quotient space.** Suppose  $V$  is finite-dimensional and  $U$  is a subspace of  $V$ . Then,  $\dim(V/U) = \dim V - \dim U$ .