

1. 1A. (NOTHING)
2. 1B.
3. **Vector Space.** A vector space  $V$  is a set that has scalar multiplication and vector addition defined on it with the following properties:
  - (a) Additive commutativity.
  - (b) Additive associativity of vectors  $(u + (v + w) = (u + v) + w)$  and multiplicative associativity for scalars  $((ab)v = a(bv))$ .
  - (c) Additive identity.
  - (d) Additive inverses.
  - (e) Multiplicative identity.
  - (f) BOTH distributive properties.
4. **V-space (unique additive identity)** A vector space has a unique additive identity.
5. **V-space (unique additive inverses)** Every element in a vector space has a unique additive inverse.
6. 1C.
7. **Subspace.** A subset  $U \subseteq V$  is a subspace of  $V$  if it is a vector space with the same additive identity, scalar multiplication, and vector addition as defined on  $V$ .
8. **Conditions for a Subspace.** A subset  $U \subseteq V$  is a subspace of  $V$  iff  $U$  is closed under vector addition, scalar multiplication, and contains the "zero" element as in  $V$ .
9. **Sums of Subspaces.** Let  $V_1, \dots, V_n$  be subspaces of  $V$ . Then, we have the sum of subspaces as  $V_1 + \dots + V_n = \{v_1 + \dots + v_n \mid v_i \in V_i \text{ for all } i\}$ .
10. **Smallest subspace containing each subspace** Suppose  $V_1, \dots, V_n$  are subspaces of  $V$ . Then,  $V_1 + \dots + V_n$  is the smallest subspace of  $V$  containing  $V_1, \dots, V_n$ .
11. **Direct Sum.** Suppose  $V_1, \dots, V_m$  are subspaces of  $V$ . Then:
  - (a) The sum  $V_1 + \dots + V_m$  is direct if each element of  $V_1 + \dots + V_m$  can be written uniquely as a sum  $v_1 + \dots + v_m$ , where  $v_i \in V_i$  for all  $i$ .
  - (b) If  $V_1 + \dots + V_m$  is a direct sum, then we write  $V_1 \oplus \dots \oplus V_m$ .
12. **Conditions for a direct sum.** Suppose  $V_1, \dots, V_n$  are subspaces of  $V$ . Then,  $V_1 + \dots + V_n$  is direct iff the only way to write 0 from  $v_1 + \dots + v_n$  is by taking  $v_i = 0$  for all  $i$ .
13. **Direct sum of subspaces.** If  $U, W$  are subspaces of  $V$ , then  $U + W$  is direct iff  $U \cap W = \{0\}$ .
14. 2A.
15. **Span is the smallest containing subspace.** The span of a list of vectors in  $V$  is the smallest subspace containing all of the vectors in the list.
16. **Zero polynomial.** The zero polynomial is said to have degree  $-\infty$ .
17. **Linear Independence.** A list of vectors  $v_1, \dots, v_n \in V$  is said to be linearly independent if  $a_1 v_1 + \dots + a_n v_n = 0$  implies  $a_i = 0$  for all  $i$ . Also, the empty list  $()$  is said to be linearly independent.
18. **Linear Dependence.** A list of vectors  $v_1, \dots, v_n$  is said to be linearly dependent if  $a_1 v_1 + \dots + a_n v_n = 0$  implies  $a_i \neq 0$  for some  $i$ .
19. **Linear Dependence Lemma.** Suppose  $v_1, \dots, v_m$  is a linearly dependent list in  $V$ . Then, there exists  $k \in \{1, \dots, m\}$  such that  $v_k \in \text{span}(v_1, \dots, v_{k-1})$ . Furthermore, if  $k$  satisfies the condition in the previous sentence and the  $k^{\text{th}}$  term is removed from  $v_1, \dots, v_m$ , then the span of the remaining list equals  $\text{span}(v_1, \dots, v_m)$ .
20. **length of linearly independent list ; length of spanning list.** In a finite-dimensional vector space, the length of every linearly independent list is at most the length of every spanning list of vectors.
21. **Finite Dimensional subspaces.** Every subspace of a finite-dimensional vector space is finite dimensional.
22. 2B.
23. **Basis.** A basis of  $V$  is a list of vectors that is linearly independent and spans  $V$ .
24. **Criterion for basis.** A list of vectors  $v_1, \dots, v_n \in V$  is a basis of  $V$  iff every  $v \in V$  can be written uniquely in the form  $v = a_1 v_1 + \dots + a_n v_n$ , where  $a_i \in F$  for all  $i$ .
25. **Every spanning list contains a basis.** Every spanning list in a vector space can be reduced to a basis of the vector space.
26. **Basis of finite-dimensional vector space.** Every finite-dimensional vector space has a basis.
27. **Every linearly independent list extends to a basis.** Every linearly independent list in a finite-dimensional vector space can be extended to a basis of the vector space.
28. **Every subspace of  $V$  is part of a direct sum equal to  $V$ .** Suppose  $V$  is finite-dimensional and  $U$  is a subspace of  $V$ . Then, there is a subspace  $W$  of  $V$  such that  $V = U \oplus W$ .
29. 2C.
30. **Basis length does not depend on basis.** Any two bases of a finite-dimensional vector space have the same length.
31. **Dimension of a subspace.** If  $V$  is finite-dimensional and  $U$  is a subspace of  $V$ , then  $\dim U \leq \dim V$ .
32. **Linearly independent list of the right length is a basis.** Suppose  $V$  is finite-dimensional. Then, every linearly independent list of vectors in  $V$  (with list length equal to  $\dim V$ ) is a basis of  $V$ .
33. **Subspace of full dimension equals the whole space.** Suppose  $V$  is finite-dimensional and  $U$  is a subspace of  $V$  such that  $\dim U = \dim V$ . Then,  $U = V$ .
34. **Spanning list of the right length is a basis.** Suppose  $V$  is finite-dimensional. Then, every spanning list of  $V$  of length  $\dim V$  is a basis of  $V$ .
35. **Dimension of a sum.** If  $V_1, V_2$  are subspaces of a finite-dimensional vector space, then  $\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$ .
36. 3A.
37. **Set of Linear Maps.** The linear of linear maps from  $V \rightarrow W$  is written  $\mathcal{L}(V, W)$  and the set of linear maps from  $V \rightarrow V$  is written  $\mathcal{L}(V)$ .
38. **Linear Map lemma.** Suppose  $v_1, \dots, v_n$  is a basis of  $V$  and  $w_1, \dots, w_n \in W$ . Then, there exists a unique linear map  $T : V \rightarrow W$  such that  $T v_k = w_k$  for each  $k$ .
39. **Linear maps take 0 to 0.** Suppose  $T : V \rightarrow W$  is a linear map. Then,  $T(0) = 0$ .
40. 3B.
41. **null space is a subspace.** Suppose  $T \in \mathcal{L}(V, W)$ . Then,  $T$  is a subspace of  $V$ .
42. **injectivity iff null is 0.** Let  $T \in \mathcal{L}(V, W)$ . Then,  $T$  is 1-1 iff  $T = \{0\}$ .
43. **range is a subspace.** If  $T \in \mathcal{L}(V, W)$ , then  $\text{range } T$  is a subspace of  $W$ .
44. **Fundamental Theorem of Linear Maps.** Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Then,  $\text{range } T$  is finite dimensional and  $\dim V = \dim \text{nul } T + \dim \text{range } T$ .
45. **linear map to a lower-dim space is not 1-1.** Suppose  $V, W$  are finite-dimensional vector spaces such that  $\dim V > \dim W$ . Then, no linear map from  $V \rightarrow W$  is 1-1.
46. **linear map to a higher-dim space is not onto.** Suppose  $V, W$  are finite-dimensional vector spaces such that  $\dim V < \dim W$ . Then, no linear map from  $V \rightarrow W$  is onto.
47. 3C. (NOTHING)
48. 3D.
49. **Theorem.** Let  $V, W$  be finite-dimensional vector spaces such that  $\dim V = \dim W$  and let  $T \in \mathcal{L}(V, W)$ . Then,  $T$  is invertible iff  $T$  is 1-1 iff  $T$  is onto.
50. **isomorphism.** An isomorphism is an invertible linear map.
51. **dimension and isomorphic.** Two finite-dimensional vector spaces are isomorphic iff they have the same dimension.
52. **Theorem.** Suppose  $V$  and  $W$  are finite-dimensional. Then,  $\mathcal{L}(V, W)$  is finite-dimensional and  $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$ .
53. **matrix of identity operator with respect to two bases.** Suppose  $u_1, \dots, u_n$  and  $v_1, \dots, v_n$  are two bases of  $V$ . Then, the matrices  $\mathcal{M}(I; u_1, \dots, u_n; v_1, \dots, v_n)$  and  $\mathcal{M}(I; v_1, \dots, v_n; u_1, \dots, u_n)$  are invertible and are inverses of each other.
54. **Change of basis formula.** Let  $T \in \mathcal{L}(V, W)$ . Suppose  $u_1, \dots, u_n$  and  $v_1, \dots, v_n$  are two bases of  $V$ . Let  $A = \mathcal{M}(T; u_1, \dots, u_n)$  and  $B = \mathcal{M}(T; v_1, \dots, v_n)$  and  $C = \mathcal{M}(I; u_1, \dots, u_n; v_1, \dots, v_n)$ . Then,  $A = C^{-1} B C$ .
55. Suppose that  $v_1, \dots, v_n$  is a basis of  $V$  and  $T \in \mathcal{L}(V)$  is invertible. Then,  $\mathcal{M}(T^{-1}) = (\mathcal{M}(T))^{-1}$ , where both matrices are with respect to the basis  $v_1, \dots, v_n$ .
56. 3E.
57. **Product of vector spaces is a vector space.** Suppose  $V_1, \dots, V_m$  are vector spaces over  $\mathbb{F}$ . Then,  $V_1 \times \dots \times V_m$  is a vector space over  $\mathbb{F}$ .
58. **dimension of a product is the sum of the dimensions.** Suppose  $V_1, \dots, V_m$  are finite-dimensional vector spaces. Then,  $V_1 \times \dots \times V_m$  is finite-dimensional and  $\dim(V_1 \times \dots \times V_m) = \dim V_1 + \dots + \dim V_m$ .
59. **Products and direct sums.** Suppose  $V_1, \dots, V_m$  are subspaces of  $V$ . Define a linear map  $\Gamma : (V_1 \times \dots \times V_m) \rightarrow (V_1 + \dots + V_m)$  by  $\Gamma(v_1, \dots, v_m) = v_1 + \dots + v_m$ . Then,  $V_1 + \dots + V_m$  is direct iff  $\Gamma$  is 1-1.
60. **direct sum iff dimensions add up.** Suppose  $V$  is finite-dimensional and  $V_1, \dots, V_m$  are subspaces of  $V$ . Then,  $V_1 + \dots + V_m$  is direct iff  $\dim(V_1 + \dots + V_m) = \dim V_1 + \dots + \dim V_m$ .
61.  **$v + U$ .** Suppose  $v \in V$  and  $U \subseteq V$ . Then,  $v + U = \{v + u \mid u \in U\}$ .
62. **Translate.** For  $v \in V$  and  $U \subseteq V$ , the set  $v + U$  is called a translate of  $U$ .
63. **Quotient Space.** Let  $U$  be a subspace of  $V$ . Then, the quotient space  $V/U$  is the set of all translates of  $U$ , that is,  $V/U = \{v + U \mid v \in V\}$ .
64. **two translates of a subspace are either equal or disjoint.** Suppose  $U$  is a subspace of  $V$  and  $v, w \in V$ . Then,  $v - w \in U$  iff  $v + U = w + U$  iff  $(v + U) \cap (w + U) \neq \emptyset$ .
65. **Addition and scalar multiplication on Quotient space.** Let  $U$  be a subspace of  $V$ . Then, we have (for all  $v, w \in V, \lambda \in F$ ):
  - (a) addition on  $V/U$ :  $(v + U) + (w + U) = (v + w) + U$ .
  - (b) scalar multiplication on  $V/U$ :  $\lambda(v + U) = (\lambda v) + U$ .
66. **quotient space is a vector space.** Let  $U$  be a subspace of  $V$ . Then, the quotient space  $V/U$  is a subspace of  $V$  under the defined scalar multiplication and vector addition.

67. **quotient map.** Let  $U$  be a subspace of  $V$ . Then, the quotient map  $\pi : V \rightarrow V/U$  is the linear map defined by  $\pi(v) = v + U$  for each  $v \in V$ .
68. **dimension of quotient space.** Suppose  $V$  is finite-dimensional and  $U$  is a subspace of  $V$ . Then,  $\dim(V/U) = \dim V - \dim U$ .
69. **Column rank.** The column rank (rank of the column span of a matrix) is  $\text{rank} T_A$ .
70. **Theorem.** If  $A$  is a rectangular matrix of elements in a field  $F$ , then  $\text{row rank } A = \text{column rank } A$ .
71. RIBET DEFS.
72. **Endomorphism.** An endomorphism is a group homomorphism from a set to itself (NOTE: does not have to be invertible.)
73. **End V.** The symbol  $\text{End } V$  is the set of all endomorphisms on  $V$  (and multiplication on  $\text{End } V$  is defined to be function composition).
74. **F-Module.** An  $F$ -module is a generalization of vector spaces over rings.
75. **Linear Map / Linear Transformation.** Let  $V$  be a vector space over a field  $F$  with  $v, w \in V$ . Let  $T$  be a map on  $V$  with  $T(v+w) = T(v) + T(w)$  and  $T(\lambda v) = \lambda T(v)$  for all  $\lambda \in F$ . Then,  $T$  is called a linear map or linear transformation.
76. **Linear Operator.** If  $T$  is a linear transformation on a vector spaces  $V$  with  $T : V \rightarrow V$ , then  $T$  is linear operator on  $V$ .
77. **Spans.** The list  $v_1, \dots, v_n$  spans  $V$  iff  $T : F^n \rightarrow V$  is onto.
78. **Finite-dimensional.**  $V$  is finite-dimensional if  $V$  is spanned by a finite list of vectors.
79. **Direct Sum of Subspaces.** Let  $X_1, \dots, X_t$  be subspaces of  $V$ . Then, their direct sum,  $X_1 \oplus \dots \oplus X_t$ , is given by a 1-1 linear map  $T$ , with  $T : X_1 \times \dots \times X_t \rightarrow V$ .
80. **Complement of Subspace.** Let  $X, Y$  be subspaces of  $V$ . Then,  $Y$  is a complementary subspace of  $X$  iff  $X + Y = V$  and  $X \cap Y = \{0\}$ .
81. **Rank, Nullity.** The rank of a linear map is the dimension of the range of the linear map. The nullity is the dimension of the null space of the linear map.
82. **Null Space.** The null space is the set of vectors that are mapped to 0.
83. **Isomorphic Vector Spaces.** Two vector spaces  $V, W$  are isomorphic if there exists a linear map  $T : V \rightarrow W$  that is 1-1 and onto.
84. **Quotient Space.** Suppose  $U$  is a subspace of  $V$ . Then, the quotient space  $V/U$  is the set  $V/U = \{v + U \mid v \in V\}$ .
85. **Column Rank.** The column rank (rank of the column span of a matrix) is defined to be  $\text{rank} T_A$ .
86. **Conjugation.** Let  $A$  be an  $n \times n$  matrix (over  $F$ ) and let  $Q$  be an  $n \times n$  matrix (over  $F$ ). Then, the conjugation of  $A$  by  $Q$  is  $Q^{-1}AQ$ .
87. RIBET THMS.
88. **Lemma.** Let  $F$  be a field,  $\lambda \in F$ ,  $V$  a vector space over  $F$  (denoted by  $V/F$ ),  $v \in V$ . Then, if  $\lambda v = 0$ , then  $\lambda = 0$  or  $v = 0$ .
89. **Lemma.** A vector space over a field is a module over a field.
90. **Theorem.** The intersection of a family of subspaces of a vector space  $V$  is a subspace of  $V$ .
91. **Lemma.** Let  $S = \{v_1, \dots, v_t\}$ . Then the subspace of all linear combinations of the elements of  $S$  is the  $\text{span} S$ .
92. **Theorem.** Let  $L = v_1, \dots, v_n$  be a list of vectors in a vector space  $V$  over a field  $F$  and let  $T : F^n \rightarrow V$  be linear transformation with  $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 v_1 + \dots + \lambda_n v_n$ . Then, we have the following:
- $L$  spans  $V$  iff  $T$  is onto.
  - $L$  is linearly independent iff  $T$  is 1-1 iff  $\text{nul } T = \{0\}$ .
  - $L$  is a basis iff  $T$  is 1-1 and onto.
93. **Prop.** Consider  $T : F^n \rightarrow V$  with  $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 v_1 + \dots + \lambda_n v_n$ , so  $T(e_i) = v_i$  for all  $i$ . Then,  $T$  is the unique linear map  $F^n \rightarrow V$  that sends  $e_i \mapsto v_i$  for all  $i$ .
94. **Theorem.** Every subspace  $X$  of  $V$  has complement.
95. **Lemma.** If  $v_1, \dots, v_t$  is linearly dependent list, then there is an index  $k$  such that  $v_k \in \text{span}(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_t)$ . Furthermore, the span of the list of length  $t - 1$  gotten by removing  $v_k$  from the list is the same as the span of the original list.
96. **Prop.** In a finite-dimensional vector space, the length is of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.
97. **Cor.** Two bases of  $V$  have the same number of elements.
98. **Prop.**  $X + Y$  is direct iff the null space of the sum map is  $\{0\}$ .
99. **Theorem.** Every subspace of a finite-dimensional vector space is finite-dimensional.
100. **Prop.** Every spanning list for a vector space can be pruned down to a basis of the space.
101. **Cor.** Every finite-dimensional vector space has a basis.
102. **Prop.** In a finite-dimensional vector space, every linearly independent list can be extended to a basis of the space.
103. **Major Theorem.** Every subspace of a finite-dimensional vector space has a complement.
104. **Prop.** Let  $X, Y$  be subspaces of a finite-dimensional vector space  $V$ . Then:
- $\dim X + \dim Y = \dim V$ .
  - $X \cap Y = \{0\}$ .
- Then,  $V = X \oplus Y$ .
105. **Prop.**  $\dim(X \oplus Y) = \dim X + \dim Y$ .
106. **Prop.** If  $V$  is a finite-dimensional vector space (with  $\dim V = n$ ), then every subspace has dimension at most  $n$ .
107. **Prop.** Let  $\dim V = n$ . Then, a linearly independent list of vectors of  $V$  with length  $n$  is a basis for  $V$ .
108. **Prop.** Let  $\dim V = n$ . Then, every spanning list for  $V$  of length  $n$  is a basis for  $V$ .
109. **Lemma.** The list  $(x_1, 0), \dots, (x_t, 0); (0, y_1), \dots, (0, y_k)$  of length  $t + k$  is a basis of  $X \times Y$ .
110. **Cor.**  $\dim(X \times Y) = \dim X + \dim Y$ .
111. **Cor.** Let  $T : V \rightarrow W$  be a linear map with  $\dim V = d$ . Then,  $\text{rank } T \leq d$ .
112. **Rank-Nullity Theorem.**  $\dim V = \text{rank } V + \text{nullity } V$ .
113. **Prop.** If  $T : V \rightarrow W$  is 1-1, then  $\text{nullity } T = 0$ .
114. **Cor.** If  $T : V \rightarrow W$  is 1-1 and onto, then  $\dim V = \dim W$ .
115. **Theorem.** The set of linear maps  $V \rightarrow W$  is a vector space  $L \cdot (F^n, W) \rightarrow T \longrightarrow (Te_1, \dots, Te_n) \in W^n$ .
116. **Theorem.**  $\dim(X + Y) = \dim X + \dim Y - \dim(X \cap Y)$ .
117. **Cor.**  $\dim(V/X) = \dim V - \dim X$ .
118. **Theorem.** If  $A$  is a rectangular matrix with elements in a field  $F$ , then  $\text{row rank } A = \text{column rank } A$ .
119. **Prop.** Let  $T : V \rightarrow W$  be 1-1. Then,  $\dim W \geq \dim V$ .
120. **Prop.** Let  $T : V \rightarrow W$  be onto. Then,  $\dim V \geq \dim W$ .
121. **Prop.** Let  $T : V \rightarrow W$  and  $\dim V = \dim W$ . Then,  $T$  1-1 iff  $T$  onto iff  $T$  bijective iff  $T$  invertible.