1. 1C, 2A, 2B, 2C, 3B, 3C, 3D, 3E.
2. **Direct sum of subspaces.** If U,W are subspaces of V, then $U+W$ is direct iff $U \cap W = \{0\}$.
3. **Linear Dependence Lemma.** Suppose $v_1,\dots,v_m$ is a linearly dependent list in V. Then, there exists $k \in \{1,\dots,m\}$ such that $v_k \in \operatorname{span}(v_1,\dots,v_{k-1})$. Furthermore, if k satisfies the condition in the previous sentence and the k^{th} term is removed from $v_1,\dots,v_m$, then the span of the remaining list equals $\operatorname{span}(v_1,\dots,v_m)$.
4. **Prop.** Let V,W be finite-dimensional with $\dim W = \dim V$. Let $S \in \mathscr{L}(W,V), T \in \mathscr{L}(V,W)$. Then, $ST = I$ iff $TS = I$.
5. **ST=I iff TS=I (on vector spaces of the same dimension).** Suppose V and W are finite-dimensional vector spaces of the same dimension, $S \in \mathscr{L}(W,V), T \in \mathscr{L}(V,W)$. Then $ST = I$ iff $TS = I$.
6. **matrix of identity operator with respect to two bases.** Suppose $u_1,\dots,u_n$ and $v_1,\dots,v_n$ are two bases of V. Then, the matrices $\mathscr{M}(I;u_1,\dots,u_n;v_1,\dots,v_n)$ and $\mathscr{M}(I;v_1,\dots,v_n;u_1,\dots,u_n)$ are invertible and are inverses of each other.
7. **Product of vector spaces is a vector space.** Suppose $V_1,\dots,V_m$ are vector spaces over $\mathbb{F}$. Then, $V_1 \times \cdots \times V_m$ is a vector space over $\mathbb{F}$.
8. **dimension of a product is the sum of the dimensions.** Suppose $V_1,\dots,V_m$ are finite-dimensional vector spaces. Then, $V_1 \times \cdots \times V_m$ is finite-dimensional and $\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m$.
9. **Products and direct sums.** Suppose $V_1,\dots,V_m$ are subspaces of V. Define a linear map $\Gamma : (V_1 \times \cdots \times V_m) \to (V_1 + \cdots + V_m)$ by $\Gamma(v_1,\dots,v_m) = v_1 + \cdots + v_m$. Then, $V_1 + \cdots + V_m$ is direct iff Γ is 1-1.
10. **direct sum iff dimensions add up.** Suppose V is finite-dimensional and $V_1,\dots,V_m$ are subspaces of V. Then, $V_1 + \cdots + V_m$ is direct iff $\dim(V_1 + \cdots + V_m) = \dim V_1 + \cdots + \dim V_m$.
11. **dimension of quotient space.** Suppose V is finite-dimensional and U is a subspace of V. Then, $\dim(V/U) = \dim V - \dim U$.
12. 3F.
13. **Linear functional.** A linear functional on V is a linear map $\phi : V \to F$.
14. **dual space.** The dual space of V is $V' = \mathscr{L}(V,F)$.
15. **dim space = dim dual space.** Suppose V is finite-dimensional. Then V' is also finite-dimensional and $\dim V = \dim V'$.
16. **dual basis.** If $v_1,\dots,v_n$ is a basis of V, then the dual basis of $v_1,\dots,v_n$ is $\phi_1,\dots,\phi_n$ (elements of V') where $\phi_j(v_k) = 1$ if $k = j$ and $\phi_j(v_k) = 0$ if $k \neq j$.
17. **dual basis gives coefficients for linear combination.** Suppose $v_1,\dots,v_n$ is a basis of V and $\phi_1,\dots,\phi_n$ is dual basis. Then $v = \phi_1(v)v_1 + \cdots + \phi_n(v)v_n$ for each $v \in V$.
18. **dual basis is a basis of dual space.** Suppose V is finite-dimensional. Then the dual basis of V is a basis of V'.
19. **dual map, T'.** Suppose $T \in \mathscr{L}(V,W)$. The dual map of T is $T' \in \mathscr{L}(W',V')$ defined for each $\phi \in W'$ by $T'(\phi) = \phi \circ T$.
20. **algebraic properties of dual maps.** we have $(S+T)' = S' + T', (\lambda S)' = \lambda S', (ST)' = T'S'$.
21. **annihilator.** For $U \subseteq V$, the annihilator of U is $U_0 = \{\phi \in V' \mid \phi(u) = 0 \forall u \in U\}$.
22. **annihilator is a subspace.** If $U \subseteq V$, then $U^0 \subseteq V'$.
23. **dimension of annihilator.** Suppose V is finite-dimensional and $U \subseteq V$. Then $\dim U^0 = \dim V - \dim U$.
24. **condition for annihilator to equal $\{0\}$ or whole space.** Suppose V finite-dimensional and $U \subseteq V$. Then:
    (a) $U^0 = \{0\}$ iff $U = V$.
    (b) $U^0 = V'$ iff $U = \{0\}$.
25. **null space of T'.** Suppose V,W finite-dimensional and $T \in \mathscr{L}(V,W)$. Then:
    (a) $\operatorname{nul} T' = (\operatorname{range} T)^0$.
    (b) $\dim \operatorname{nul} T' = \dim \operatorname{nul} T + \dim W - \dim V$.
26. **T surjective equivalent to T' injective.** Suppose V,W finite-dimensional and $T \in \mathscr{L}(V,W)$. Then T onto iff T' 1-1.
27. **range of T'.** Suppose V,W finite-dim and $T \in \mathscr{L}(V,W)$. Then:
    (a) $\dim \operatorname{range} T' = \dim \operatorname{range} T$.
    (b) $\operatorname{range} T' = (\operatorname{nul} T)^0$.
28. **T injective is equivalent to T' surjective.** Suppose V,W finite-dim and $T \in \mathscr{L}(V,W)$. Then T 1-1 iff T' onto.
29. 5A.
30. **equivalent conditions to be an eigenvalue.** Let V be finite-dim and $T \in \mathscr{L}(V)$ and $\lambda \in F$. Then TFAE:
    (a) λ is an eigenvalue of T.
    (b) $T - \lambda I$ not injective.
    (c) $T - \lambda I$ not surjective.
    (d) $T - \lambda I$ not invertible.
31. **linearly independent eigenvectors.** Let $T \in \mathscr{L}(V)$. Then every list of eigenvectors of T corresponding to different eigenvalues is linearly independent.
32. **operator cannot have more eigenvalues than dimension of space.** Let V be finite-dim. Then each operator on V has at most $\dim V$ distinct eigenvalues.
33. **null space and range of $p(T)$ are invariant under T.** Suppose $T \in \mathscr{L}(V)$ and $p \in \mathscr{P}(F)$. Then $\operatorname{nul} p(T)$ and $\operatorname{range} p(T)$ are invariant under T.
34. 5B.
35. **existence of eigenvalues.** Every operator on a finite-dim nonzero complex vector space has an eigenvalue.
36. **existence, uniqueness, and degree of minimal polynomial.** Suppose V finite-dim and let $T \in \mathscr{L}(V)$. Then there is a unique monic polynomial $p \in \mathscr{P}(F)$ of smallest degree such that $p(T) = 0$. Also, $\deg p \leq \dim V$.
37. **minimal polynomial.** Suppose V finite-dim and $T \in \mathscr{L}(V)$. Then the minimal polynomial of T is the unique monic polynomial $p \in \mathscr{P}(F)$ of smallest degree such that $p(T) = 0$.
38. **eigenvalues are the zeros of minimal polynomial.** Let V finite-dim and $T \in L(V)$. Then:
    (a) zeros of the minimal polynomial of T are the eigenvalues of T.
    (b) if V is a complex vector space, then minimal polynomial of T has the form $(z-\lambda_1)\cdots\cdots(z-\lambda_m)$, where $\lambda_1,\dots,\lambda_m$ is a list of all eigenvalues of T, possibly with repetitions.
39. **$q(T) = 0$ iff q is a polynomial multiple of the minimal polynomial.** Let V finite-dim and $T \in L(V)$ and $q \in P(F)$. Then $q(T) = 0$ iff q is a polynomial multiple of the minimal polynomial.
40. **minimal polynomial of a restriction operator.** Let V finite-dim and $T \in L(V)$ and $U \subseteq V$ that is invariant under T. Then minimal polynomial of T is a polynomial multiple of minimal polynomial of $T\mid_U$.
41. **T not invertible iff constant term of minimal polynomial of T is 0.** Let V finite-dim and $T \in L(V)$. Then T is not invertible iff the constant term in the minimal polynomial of T is 0.
42. **even-dimensional null space.** Let $F = \mathbb{R}$ and V finite-dim and $T \in L(V)$ and $b^2 - 4ac < 0$. Then $\dim(T^2 + bT + cI)$ is an even number.
43. **operators on an odd-dimensional space have eigenvalues.** Every operator on an odd-dimensional vector space has an eigenvalue.
44. 5C.
45. **conditions for upper-triangular matrix.** Suppose $T \in L(V)$ and $v_1,\dots,v_n$ is a basis of V. Then TFAE:
    (a) the matrix of T with respect to $v_1,\dots,v_n$ is upper-triangular.
    (b) $\operatorname{span}(v_1,\dots,v_k)$ is invariant under T for each $k = 1,2,\dots,n$.
    (c) $Tv_k \in \operatorname{span}(v_1,\dots,v_k)$ for each $k = 1,\dots,n$.
46. **equation satisfied by operator with upper-triangular matrix.** Suppose $T \in L(V)$ and V has a basis with respect to which T has an upper-triangular matrix with diagonal entries $\lambda_1,\dots,\lambda_n$. Then $(T-\lambda_1 I)\cdots\cdots(T-\lambda_n I) = 0$.
47. **determination of eigenvalues from upper-triangular matrix.** Suppose $T \in L(V)$ has an upper-triangular matrix with respect to some basis of V. Then the eigenvalues of T are precisely the entries on the diagonal of that upper-triangular matrix.
48. **necessary and sufficient condition to have an upper-triangular matrix.** Suppose V is finite-dim and $T \in L(V)$. Then T has an upper-triangular matrix with respect to some basis of V iff the minimal polynomial of T equals $(z-\lambda_1)\cdots\cdots(z-\lambda_n)$ for some $\lambda_i \in F$.
49. **if $F = \mathbb{C}$, then every operator on V has an upper-triangular matrix.** Suppose V is a finite-dim complex vector space and $T \in L(V)$. Then T has an upper-triangular matrix with respect to some basis of V.
50. 5D.
51. **eigenspace, $E(\lambda,T)$.** Suppose $T \in L(V)$ and $\lambda \in F$. Then the eigenspace of T corresponding to λ is $E(\lambda,T) = \operatorname{nul}(T-\lambda I) = \{v \in V \mid Tv = \lambda v\}$.
52. **sum of eigenspaces is a direct sum.** Suppose $T \in L(V)$ and $\lambda_1,\dots,\lambda_m$ are the distinct eigenvalues of T. Then $\sum_i E(\lambda_i,T)$ is a direct sum and $\sum_i \dim E(\lambda_i,T) \leq \dim V$.
53. **conditions equivalent to diagonalizability.** Suppose V finite-dim and $T \in L(V)$. Let $\lambda_1,\dots,\lambda_m$ denote the distinct eigenvalues of T. Then TFAE:
    (a) T is diagonalizable.
    (b) V has a basis consisting of eigenvectors of T.
    (c) $V = \oplus_i E(\lambda_i,T)$
    (d) $\dim V = \sum_i \dim E(\lambda_i,T)$.
54. **enough eigenvalues implies diagonalizability.** Let V be finite-dim and $T \in L(V)$ has $\dim V$ distinct eigenvalues. Then T is diagonalizable.
55. **necessary and sufficient condition for diagonalizability.** Suppose V finite-dim and $T \in L(V)$. Then T diagonalizable iff the minimal polynomial of T equals $(z-\lambda_1)\cdots(z-\lambda_m)$ for some distinct $\lambda_1,\dots,\lambda_i \in F$.
56. **restriction of diagonalizable operator to invariant subspace.** Suppose $T \in L(V)$ and U is a T-invariant subspace of V. Then $T\mid_U$ is a diagonalizable operator on U.
57. 5E.
58. **commuting operators correspond to commuting matrices.** Suppose $S,T \in L(V)$ and $v_1,\dots,v_n$ is a basis of V. Then S and T commute iff $M(S,(v_1,\dots,v_n))$ and $M(T,(v_1,\dots,v_n))$ commute.
59. **eigenspace is invariant under commuting operators.** Suppose $S,T \in L(V)$ commute and $\lambda \in F$. Then $E(\lambda,S)$ is invariant under T.
60. **simultaneous diagonalizability iff commutativity.** Two diagonalizable operators on the same vector space have diagonal matrices with respect to the same basis iff the two operators commute.
61. **common eigenvector for commuting operators.** every pair of commuting operators on a finite-dim nonzero complex vector space has a common eigenvector.
62. **commuting operators are simultaneously upper-triangularizable.** Suppose V is a finite-dim nonzero complex vector space and S,T are commuting operators on V. Then there is a basis of V with respect to which both S,T have upper-triangular matrices.
63. **eigenvalues of sum and product of commuting operators.** Suppose V is a finite-dim complex vector space and S,T are commuting operators on V. Then:
    (a) every eigenvalue of $S+T$ is an eigenvalue of S plus an eigenvalue of T.
    (b) every eigenvalue of ST is an eigenvalue of S times an eigenvalue of T.
64. 8A.
65. **sequence of increasing null spaces.** Let $T \in L(V)$. Then $\{0\} = \operatorname{nul} T^0 \subseteq \operatorname{nul} T_1 \subseteq \cdots \subseteq \operatorname{nul} T^k \subseteq \dots$.
66. **equality in the sequence of null spaces.** Let $T \in L(V)$ and m is a nonnegative integer such that $\operatorname{nul} T^m = \operatorname{nul} T^{m+1}$. Then $\operatorname{nul} T^m = \operatorname{nul} T^{m+1} = \dots$.
67. **null spaces stop growing.** Let $T \in L(V)$. Then $\operatorname{nul} T^{\dim V} = \operatorname{nul} T^{\dim V + 1} = \dots$.
68. **V is the direct sum of $\operatorname{nul} T^{\dim V}$ and $\operatorname{range} T^{\dim V}$.** Let $T \in L(V)$. Then $V = \operatorname{nul} T^{\dim V} \oplus \operatorname{range} T^{\dim V}$.
69. **generalized eigenvector.** Let $T \in L(V)$ and λ be an eigenvalue of T. A vector $v \in V$ ($v \neq 0$) is called a generalized eigenvector of T corresponding to λ if $(T-\lambda I)^k v = 0$ for some $k \in \mathbb{Z}_{>0}$.
70. **a basis of generalized eigenvectors.** Let $F = \mathbb{C}$ and $T \in L(V)$. Then there is a basis of V consisting of generalized eigenvectors of T.
71. **generalized eigenvector corresponds to a unique eigenvalue.** Let $T \in L(V)$. Then each generalized eigenvector of T corresponds to only one eigenvalue of T.
72. **linearly independent generalized eigenvectors.** Let $T \in L(V)$. Then every list of generalized eigenvectors of T corresponding to distinct eigenvalues of T is linearly independent.
73. **nilpotent operator raised to dimension of domain is 0.** Let $T \in L(V)$ be nilpotent. Then $T^{\dim V} = 0$.
74. **eigenvalues of nilpotent operator.** Let $T \in L(V)$. Then:
    (a) if T is nilpotent then 0 is an eigenvalue of T and T has no other eigenvalues.
    (b) if $F = \mathbb{C}$ and 0 is the only eigenvalue of T, then T is nilpotent.

75. **minimal polynomial & upper-triangular matrix of nilpotent operator.** Let $T \in L(V)$. Then TFAE:

(a) T is nilpotent.
(b) minimal polynomial of T is z^m for some positive integer m.
(c) there is a basis of V with respect to which the matrix of T has the form

$$\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$

.

76. 8B.

77. **generalized eigenspace.** Suppose $T \in L(V)$ and $\lambda \in F$. The generalized eigenspace of T corresponding to λ is $G(\lambda, T) = \{v \in V \mid (T-\lambda I)^k \text{ for some } k \in \mathbb{Z}_{>0}\}$, which is the set of generalized eigenvectors of T corresponding to λ, including the 0-vector.

78. **description of generalized eigenspaces.** Suppose $T \in L(V)$ and $\lambda \in F$. Then $G(\lambda, T) = \text{nul}(T-\lambda I)^{\dim V}$.

79. **generalized eigenspace decomposition.**

80. Suppose $F = \mathbb{C}$ and $T \in L(V)$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T. Then:

(a) $G(\lambda_k, T)$ is invariant under T for each $k = 1, \ldots, m$.
(b) $(T-\lambda_k I)\mid_{G(\lambda_k,T)}$ is nilpotent for each $k = 1, \ldots, m$.
(c) $V = \oplus_i G(\lambda_i, T)$.

81. **multiplicity.** Let $T \in L(V)$. The multiplicity of an eigenvalue λ of T is defined to be the dimension of the corresponding generalized eigenspace $G(\lambda, T)$, so multiplicity of λ is $\dim \text{nul}(T-\lambda I)^{\dim V}$.

82. **sum of the multiplicities equals** $\dim V$**.** Suppose $F = \mathbb{C}$ and $T \in L(V)$. Then the sum of all the multiplicities of all the eigenvalues of T equals $\dim V$.

83. **characteristic polynomial.** Let $F = \mathbb{C}$ and $T \in L(V)$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T, with multiplicities $d_1, \ldots, d_m$. Then the polynomial $(z-\lambda_1)^{d_1} \cdots (z-\lambda_m)^{d_m}$ is called the characteristic polynomial of T.

84. **degree and zeros of the characteristic polynomial.** Let $F = \mathbb{C}$ and $T \in L(V)$. Then:

(a) characteristic polynomial of T has degree $\dim V$.
(b) zeros of the characterisit polynomial are the eigenvalues of T.

85. **Cayley-Hamilton theorem.** Let $F = \mathbb{C}$, $T \in L(V)$ and q be the characteristic polynomial of T. Then $q(T) = 0$.

86. **characteristic polynomial is a multiple of minimal polynomial.** Let $F = \mathbb{C}$ and $T \in L(V)$. Then characteristic polynomial of T is a polynomial multiple of the minimal polynomial of T.

87. **multiplicity of an eigenvalue equals number of times on diagonal.** Let $F = \mathbb{C}$ and $T \in L(V)$. Let $v_1, \ldots, v_n$ be a basis of V such that $M(T, (v_1, \ldots, v_n))$ is upper-triangular. The number of times the eigenvalue λ ppears on the diagonal of $M(T, (v_1, \ldots, v_n))$ equals the multiplicity of λ as an eigenvalue of T.

88. **block diagonal matrix with upper-triangular blocks.** Let $F = \mathbb{C}$ and $T \in L(V)$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T with multiplicities $d_1, \ldots, d_m$. Then there is a basis of V with respect to which T has a block diagonal matrix of the form

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix}$$

, where each A_k is a d_k-by-d_k upper-triangular matrix of the form

$$\begin{pmatrix} \lambda_k & & * \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix}$$

.

89. 8C.

90. **jordan basis.** Let $T \in L(V)$. A basis of V is called a Jordan basis for T if with respect to this basis T has a block diagonal matrix

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_p \end{pmatrix}$$

in which each A_k is an upper-triangular matrix of the form

$$\begin{pmatrix} \lambda_k & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_k \end{pmatrix}$$

.

91. **every nilpotent operator has a jordan basis.** Let $T \in L(V)$ be nilpotent. Then there is a basis for V that is a Jordan basis for T.

92. **Jordan form.** Let $F = \mathbb{C}$ and $T \in L(V)$. Then there is a basis of V that is a Jordan basis.

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93. RIBET DEFS MT2.

94. **Double Dual.** Let V be a finite-dimensional vector space with dual V'. Then the double dual of V is $(V')' = V'' = V$. Also, $\dim V = n = \dim V' = \dim V''$.

95. **Eigenspace.** Let $T \in \mathscr{L}(V)$ and take λ to be an eigenvalue of T. Then, $E(\lambda, T) = \{v \in V \mid Tv = \lambda v\} \neq \emptyset$ is written as V_λ and is called the eigenspace of λ, which is a subspace of V.

96. **Generalized Eigenvector.** Consider a minimal polynomial $(x-\lambda_1)^{e_1} \cdot \cdots \cdot (x-\lambda_m)^{e_m}$ on X with $(T-\lambda_1 I)^{e_1} v = 0$. Then, v is called a generalized eigenvector for $\lambda = \lambda_1$.

97. **Characteristic polynomial.** The characteristic polynomial of $T: V \to V$ (with eigenvalues $\lambda_1, \ldots, \lambda_t$) is the polynomial $\prod_{i=1}^t (x-\lambda_i)^{\dim X_i}$, where $V = X_1 \oplus \cdots \oplus X_t$.

98. **Simultaneously diagonalizable.** Operators S and T on V are simulatenously diagonalizable if there is a basis of V that consts of vectors that are eigenvectors for both S and T (i.e. there exists a basis $v_1, \ldots, v_n$ of V so that for i, $1 \le i \le n$, there are λ_i and μ_i so that $Sv_i = \lambda_i v_i$ and $Tv_i = \mu_i v_i$).

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99. RIBET THMS MT1.

100. **Theorem.** The intersection of a family of subspaces of a vector space V is a subspace of V.

101. **Theorem.** Every subspace X of V has complement.

102. **Prop.** Let X, Y be subspaces of a finite-dimensional vector space V. Then:

(a) $\dim X + \dim Y = \dim V$.
(b) $X \cap Y = \{0\}$.

Then, $V = X \oplus Y$.

103. **Prop.** $\dim(X \oplus Y) = \dim(X \times Y) = \dim X + \dim Y$.

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104. RIBET THMS MT2

105. **Theorem.** Every linear functional on a subspace of V can be extended to V.

106. **Note.** Annihilator is the dual of the quotient subspace.

107. **Cor.** The annihilator of U is $\{0\}$ iff $U = V$. The annihilator of U is V iff $U = \{0\}$.

108. **Prop.** If $T: V \to W$ is a linear map, then the null space of T' is the annihilator of the range of T. We have $\text{ann}(\text{range}T) = \{\psi : W \to F \mid \phi(Tv) = 0 \text{ for all } v \in V, T'(\psi)(v) = 0, T'\psi = 0, \phi \in \text{nul}(T')\}$.

109. **Cor.** If $T: V \to W$ is a linear map between finite-dimensional F-vector spaces, then $\dim \text{nul}(T') = \dim \text{nul}(T) + \dim W - \dim V$.

110. **Cor.** The linear map T is onto iff T' is 1-1.

111. **Cor.** If $T: V \to W$ is a linear map between finite-dimensional vector spaces, then T' and T have equal ranks.

112. **Cor.** We have $\text{range}T = (\text{nul}\, T)^0$.

113. **Theorem.** Let F be a finite field with $q = |F|$. Then, $a^q = a$ for all $a \in F$.

114. **Theorem.** If F is a finite field, then $|F| = p^n$ for some prime p and integer $n \ge 1$.

115. **Theorem.** Let $T: V \to V$, V finite-dimensional, and let $\alpha : F[x] \to \mathscr{L}(V)$, with $f \mapsto f(T)$. Also, we have $\ker \alpha$ to be the principal ideal $(m(x))$. Then, $m(x)$ is the minimal polynomial of T and has degree $\le n^2$.

116. **Cayley-Hamilton Theorem.** Let $T: V \to V$, V finite-dimensional, and let $\alpha : F[x] \to \mathscr{L}(V)$, with $f \mapsto f(T)$. Also, we have $\ker \alpha$ to be the principal ideal $(m(x))$, where $m(x)$ is the minimal polynomial of T. Then, the characteristic polynomial is in $\ker \alpha$; that is, we can plug in the matrix for T into its characteristic polynomial and we get that it is equal to the 0-matrix.

117. **Lemma.** Let $f \in \mathbb{R}[x]$ be a real polynomial. If λ is a complex root of f, so is $\overline{\lambda}$, which is the complex conjugate of λ.

118. **Cor.** Let $\lambda_1, \ldots, \lambda_t$ be distinct eigenvalues and take $E_i = E(\lambda_i, T) = \{v \in V \mid Tv = \lambda_i v\} \subseteq V$. Now, take $E_1 \times \cdots \times E_t$. Then there exists a summation map $E_1 \times \cdots \times E_t \xrightarrow{\text{sum}} V$ with $(v_1, \ldots, v_t) \mapsto v_1 + \cdots + v_t$. Then, the sum map is 1-1.

119. **Cor.** Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

120. **Prop.** Suppose T is an operator on an F-vector space V. If $f \in F[x]$ is a polynomial satisfied by T (meaning $f(T) = 0$), then every eigenvalue of T on V is a root of f.

121. **Cor.** Suppose λ is an eigenvalue of operator T on a finite-dimensional F-vector space. Then λ is a root of the minimal polynomial of T iff λ is an eigenvalue of T.

122. **Theorem.** All operators on a nonzero finite-dimensional vector space over an algebraically closed field have at least one eigenvalue.

123. **Prop.** Assume that $F = \mathbb{R}$ and that $f(x) := x^2 + bx + c$ is an irreducible polynomial. If $T \in \mathscr{L}(V)$ and V is finite-dimensional, then the null space of $f(T)$ is even-dimensional.

124. **Prop (honors version).** Let T be an operator on a finite-dimensional vector space over F. If p is an irreducible polynomial over F, then the dimension of the null space of $p(T)$ is a multiple of the degree of p.

125. **Prop.** $F[x]/(p)$ (where p is irreducible) is a field.

126. **Formula.** $\dim_F V = [K : F] \cdot \dim_K V = \dim_F K \cdot \dim_K V$.

127. **Cor.** Every operator on an odd-dimensional $\mathbb{R}$-vector space has an eigenvalue.

128. **Prop.** If T is an operator on a finite-dimensional F-vector space, then the minimal polynomial of T has degree at most $\dim V$.

129. **Prop.** If T is upper-triangular with respect to some basis of V, and if the diagonal entries of an upper-triangular matrix representation of T are $\lambda_1, \ldots, \lambda_n$, then $(T-\lambda_1 I) \cdot \cdots \cdot (T-\lambda_n I) = 0$.

130. **Prop.** Let V be a finite-dimensional vector space and $T \in \mathscr{L}(V)$ and let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of T. Then, $V = \oplus E(\lambda_i, T)$ iff T is diagonalizable.

131. **Prop.** TFAE.

(a) T is diagonalizable.
(b) V has a basis consisting of eigenvectors.
(c) The direct sum $\bigoplus_i V_{\lambda_i}$ is all of V.
(d) $\dim \left(\bigoplus_i V_{\lambda_i} \right) = \dim V$.

132. **Prop.** If $T: V \to V$ has $\dim V$ different eigenvalues, then T is diagonalizable.

133. **Jordan Canonical Form.** X can be written as a direct sum of Jordan blocks, where $\sum$ dim(block) = $\dim X$.

134. **Lemma.** Let $X = \oplus \text{span}(U_i v)$ for $i \in \{0, \ldots, k_1\}$. If Z is a subspace of X' that is U'-invariant, then $\text{ann}(Z) =: Y$ is U-invariant.

135. **Lemma.** Suppose S and T are commuting operators on V. If λ is an eigenvalue for T on V, then the eigenspace $E(\lambda, T)$ is S-invariant.

136. **Theorem.** The diagonalize operatosr on the same finite-dimensional vector space are simulateneously diagonalizable iff they commute with each other.

137. **Theorem.** Every pair of commuting operators on a finite-dimensional nonzero complex vector spcae has a common eigenvector.

138. **Prop.** Two commuting operators on a finite-dimensional nonzero complex vector space can be simultaneously upper-triangularized.

139. **Prop.** We have:

(a) Every eigenvalue of $S+T$ is the sum of an eigenvalue of S and an eigenvalue of T.
(b) Every eigenvalue of ST is the product of an eigenvalue of S and an eigenvalue of T.
