

Math H110 Theorems.

1. **Lemma.** Let F be a field, $\lambda \in F$, V a vector space over F (denoted by V/F), $v \in V$. Then, if $\lambda v = 0$, then $\lambda = 0$ or $v = 0$.
2. **Lemma.** A vector space over a field is a module over a field.
3. **Theorem.** The intersection of a family of subspaces of a vector space V is a subspace of V .
4. **Lemma.** Let $S = \{v_1, \dots, v_t\}$. Then the subspace of all linear combinations of the elements of S is the $\text{span}S$.
5. **Theorem.** Let $L = v_1, \dots, v_n$ be a list of vectors in a vector space V over a field F and let $T : F^n \rightarrow V$ be linear transformation with $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 v_1 + \dots + \lambda_n v_n$. Then, we have the following:
 - (a) L spans V iff T is onto.
 - (b) L is linearly independent iff T is 1-1 iff $\text{nul } T = \{0\}$.
 - (c) L is a basis iff T is 1-1 and onto.
6. **Prop.** Consider $T : F^n \rightarrow V$ with $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 v_1 + \dots + \lambda_n v_n$, so $T(e_i) = v_i$ for all i . Then, T is the unique linear map $F^n \rightarrow V$ that sends $e_i \mapsto v_i$ for all i .
7. **Theorem.** Every subspace X of V has complement.
8. **Lemma.** If v_1, \dots, v_t is linearly dependent list, then there is an index k such that $v_k \in \text{span}(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_t)$. Furthermore, the span of the list of length $t - 1$ gotten by removing v_k from the list is the same as the span of the original list.
9. **Prop.** In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.
10. **Cor.** Two bases of V have the same number of elements.
11. **Prop.** $X + Y$ is direct iff the null space of the sum map is $\{0\}$.
12. **Theorem.** Every subspace of a finite-dimensional vector space is finite-dimensional.
13. **Prop.** Every spanning list for a vector space can be pruned down to a basis of the space.
14. **Cor.** Every finite-dimensional vector space has a basis.
15. **Prop.** In a finite-dimensional vector space, every linearly independent list can be extended to a basis of the space.

16. **Major Theorem.** Every subspace of a finite-dimensional vector space has a complement.
17. **Prop.** Let X, Y be subspaces of a finite-dimensional vector space V . Then:
- (a) $\dim X + \dim Y = \dim V$.
 - (b) $X \cap Y = \{0\}$.
- Then, $V = X \oplus Y$.
18. **Prop.** $\dim(X \oplus Y) = \dim X + \dim Y$.
19. **Prop.** If V is a finite-dimensional vector space (with $\dim V = n$), then every subspace has dimension at most n .
20. **Prop.** Let $\dim V = n$. Then, a linearly independent list of vectors of V with length n is a basis for V .
21. **Prop.** Let $\dim V = n$. Then, every spanning list for V of length n is a basis for V .
22. **Lemma.** The list $(x_1, 0), \dots, (x_t, 0); (0, y_1), \dots, (0, y_k)$ of length $t + k$ is a basis of $X \times Y$.
23. **Cor.** $\dim(X \times Y) = \dim X + \dim Y$.
24. **Cor.** Let $T : V \rightarrow W$ be a linear map with $\dim V = d$. Then, $\text{rank} T \leq d$.
25. **Rank-Nullity Theorem.** $\dim V = \text{rank} V + \text{nullity} V$.
26. **Prop.** If $T : V \rightarrow W$ is 1-1, then $\text{nullity} T = 0$.
27. **Cor.** If $T : V \rightarrow W$ is 1-1 and onto, then $\dim V = \dim W$.
28. **Theorem.** The set of linear maps $V \rightarrow W$ is a vector space $L \cdot (F^n, W) \rightarrow T \longrightarrow (Te_1, \dots, Te_n) \in W^n$.
29. **Theorem.** $\dim(X + Y) = \dim X + \dim Y - \dim(X \cap Y)$.
30. **Cor.** $\dim(V/X) = \dim V - \dim X$.
31. **Theorem.** If A is a rectangular matrix with elements in a field F , then $\text{row rank } A = \text{column rank } A$.
32. **Prop.** Let $T : V \rightarrow W$ be 1-1. Then, $\dim W \geq \dim V$.
33. **Prop.** Let $T : V \rightarrow W$ be onto. Then, $\dim V \geq \dim W$.
34. **Prop.** Let $T : V \rightarrow W$ and $\dim V = \dim W$. Then, T 1-1 iff T onto iff T bijective iff T invertible.
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35. **Lemma.** Let V be a finite-dimensional vector space and U a subspace of V . Then, $\dim U_0 = \dim V - \dim U$.
36. **Theorem.** Every linear functional on a subspace of V can be extended to V .
37. **Note.** Annihilator is the dual of the quotient subspace.
38. **Theorem.** Let $T : V \rightarrow W$ and $T' : W' \rightarrow V'$. Then $\mathcal{M}(T)$ and $\mathcal{M}(T')$ are transposes of each other.
39. **Lemma.** U^0 has dimension $\dim V - \dim U$.
40. **Cor.** The annihilator of U is $\{0\}$ iff $U = V$. The annihilator of U is V iff $U = \{0\}$.
41. **Prop.** If $T : V \rightarrow W$ is a linear map, then the null space of T' is the annihilator of the range of T . We have $\text{ann}(\text{range} T) = \{\psi : W \rightarrow F \mid \phi(Tv) = 0 \text{ for all } v \in V, T'(\psi)(v) = 0, T'\psi = 0, \phi \in \text{nul}(T')\}$.
42. **Cor.** If $T : V \rightarrow W$ is a linear map between finite-dimensional F -vector spaces, then $\dim \text{nul}(T') = \dim \text{nul}(T) + \dim W - \dim V$.
43. **Cor.** The linear map T is onto iff T' is 1-1.
44. **Cor.** If $T : V \rightarrow W$ is a linear map between finite-dimensional vector spaces, then T' and T have equal ranks.
45. **Cor.** We have $\text{range} T = (\text{nul } T)^0$.
46. **Theorem.** Let F be a finite field with $q = |F|$. Then, $a^q = a$ for all $a \in F$.
47. **Theorem.** If F is a finite field, then $|F| = p^n$ for some prime p and integer $n \geq 1$.
48. **Theorem.** Take an ideal I in \mathbb{Z} . Then, I is equal to either $\{0\}$ or $m\mathbb{Z}$ (where $m \in \mathbb{Z}_{>0}$).
49. **Theorem.** $F[x]$ is a principal ideal domain; that is, it is an integral domain in which every ideal in $F[x]$ is principal.
50. **Theorem.** Let $T : V \rightarrow V$, V finite-dimensional, and let $\alpha : F[x] \rightarrow \mathcal{L}(V)$, with $f \mapsto f(T)$. Also, we have $\ker \alpha$ to be the principal ideal $(m(x))$. Then, $m(x)$ is the minimal polynomial of T and has degree $\leq n^2$.
51. **Cayley-Hamilton Theorem.** Let $T : V \rightarrow V$, V finite-dimensional, and let $\alpha : F[x] \rightarrow \mathcal{L}(V)$, with $f \mapsto f(T)$. Also, we have $\ker \alpha$ to be the principal ideal $(m(x))$, where $m(x)$ is the minimal polynomial of T . Then, the characteristic polynomial is in $\ker \alpha$; that is, we can plug in the matrix for T into its characteristic polynomial and we get that it is equal to the 0-matrix.
52. **Prop.** For $f(x) \in F[x]$ and $\lambda \in F$, $f(\lambda) = 0$ iff f is divisible by $x - \lambda$, where $x - \lambda$ is an irreducible polynomial.

53. **Cor.** A polynomial of degree n can have at most n roots.
54. **Cor.** A polynomial with infinitely many roots is identically the zero polynomial.
55. **Lemma.** Let $f \in \mathbb{R}[x]$ be a real polynomial. If λ is a complex root of f , so is $\bar{\lambda}$, which is the complex conjugate of λ .
56. **Prop.** A scalar λ is an eigenvalue of $T : V \rightarrow V$ iff $T - \lambda I$ is not 1-1.
57. **Cor.** The map $T : V \rightarrow V$ is invertible iff 0 is not an eigenvalue of T .
58. **Key lemma.** Every list of eigenvectors of T that corresponds to distinct eigenvalues of T is a linearly independent list.
59. **Cor.** Let $\lambda_1, \dots, \lambda_t$ be distinct eigenvalues and take $E_i = E(\lambda_i, T) = \{v \in V \mid Tv = \lambda_i v\} \subseteq V$. Now, take $E_1 \times \dots \times E_t$. Then there exists a summation map $E_1 \times \dots \times E_t \xrightarrow{\text{sum}} V$ with $(v_1, \dots, v_t) \mapsto v_1 + \dots + v_t$. Then, the sum map is 1-1.
60. **Cor.** Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.
61. **Prop.** Suppose T is an operator on an F -vector space V . If $f \in F[x]$ is a polynomial satisfied by T (meaning $f(T) = 0$), then every eigenvalue of T on V is a root of f .
62. **Cor.** Suppose λ is an eigenvalue of operator T on a finite-dimensional F -vector space. Then λ is a root of the minimal polynomial of T .
63. **Prop.** Let T be an operator on a finite-dimensional vector space. Suppose λ is a root of the minimal polynomial. Then λ is an eigenvalue of T .
64. **Theorem.** All operators on a nonzero finite-dimensional vector space over an algebraically closed field have at least one eigenvalue.
65. **Prop.** Assume that $F = \mathbb{R}$ and that $f(x) := x^2 + bx + c$ is an irreducible polynomial. If $T \in \mathcal{L}(V)$ and V is finite-dimensional, then the null space of $f(T)$ is even-dimensional.
66. **Prop (honors version).** Let T be an operator on a finite-dimensional vector space over F . If p is an irreducible polynomial over F , then the dimension of the null space of $p(T)$ is a multiple of the degree of p .
67. **Prop.** $F[x]/(p)$ (where p is irreducible) is a field.
68. **Formula.** $\dim_F V = [K : F] \cdot \dim_K V = \dim_F K \cdot \dim_K V$.
69. **Cor.** Every operator on an odd-dimensional \mathbb{R} -vector space has an eigenvalue.
70. **Prop.** If T is an operator on a finite-dimensional F -vector space, then the minimal polynomial of T has degree at most $\dim V$.

71. **Prop.** If T is upper-triangular with respect to some basis of V , and if the diagonal entries of an upper-triangular matrix representation of T are $\lambda_1, \dots, \lambda_n$, then $(T - \lambda_1 I) \cdots (T - \lambda_n I) = 0$.
72. **Prop.** Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V)$ and let $\lambda_1, \dots, \lambda_m$ be the eigenvalues of T . Then, $V = \oplus E(\lambda_i, T)$ iff T is diagonalizable.
73. **Prop.** TFAE.
- (a) T is diagonalizable.
 - (b) V has a basis consisting of eigenvectors.
 - (c) The direct sum $\bigoplus_i V_{\lambda_i}$ is all of V .
 - (d) $\dim \left(\bigoplus_i V_{\lambda_i} \right) = \dim V$.
74. **Prop.** If $T : V \rightarrow V$ has $\dim V$ different eigenvalues, then T is diagonalizable.
75. **Prop.** The operator $T : V \rightarrow V$ is diagonalizable iff its minimal polynomial splits completely as a product of distinct linear factors of the form $x - r$.
76. **Jordan Canonical Form.** X can be written as a direct sum of Jordan blocks, where $\sum \dim(\text{block}) = \dim X$.
77. **Lemma.** Let $X = \bigoplus \text{span}(U_i v)$ for $i \in \{0, \dots, k_1\}$. If Z is a subspace of X' that is U' -invariant, then $\text{ann}(Z) =: Y$ is U -invariant.
78. **Lemma.** Suppose S and T are commuting operators on V . If λ is an eigenvalue for T on V , then the eigenspace $E(\lambda, T)$ is S -invariant.
79. **Theorem.** The diagonalize operators on the same finite-dimensional vector space are simultaneously diagonalizable iff they commute with each other.
80. **Theorem.** Every pair of commuting operators on a finite-dimensional nonzero complex vector space has a common eigenvector.
81. **Prop.** Two commuting operators on a finite-dimensional nonzero complex vector space can be simultaneously upper-triangularized.
82. **Prop.** We have:
- (a) Every eigenvalue of $S+T$ is the sum of an eigenvalue of S and an eigenvalue of T .
 - (b) Every eigenvalue of ST is the product of an eigenvalue of S and an eigenvalue of T .

83. **Formula.** $\langle v + w, v + w \rangle = \langle v, v \rangle + \langle w, w \rangle + 2\langle v, w \rangle$.

84. **Lemma.** If u, v are orthogonal, then $\|u + v\|^2 = \|u\|^2 + \|v\|^2$.

85. **Lemma.** If $v \in V$ and $v \neq 0$, then every $u \in V$ is the sum of a multiple of v and a vector orthogonal to v .
86. **Prop (Cauchy-Schwarz).** For $u, v \in V$, we have $|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$.
87. **Prop (Triangle inequality).** For $u, v \in V$, $\|u + v\| \leq \|u\| + \|v\|$.
88. **Prop.** The triangle inequality is equality iff one of u, v is a nonnegative (real) multiple of the other.
89. **Prop.** Let $\alpha : V \rightarrow V'$ with $v \mapsto \phi_v$, where $\phi_v : V \rightarrow F$ such that $\phi_v(x) = \langle x, v \rangle$. Then, $\alpha(\lambda v) = \bar{\lambda}(\alpha(v))$ for $\lambda \in F$.
90. **Prop.** If V is finite-dim then $\alpha : V \rightarrow V'$ is an invertible linear map of \mathbb{R} -vector spaces. It is an isomorphism of F -vector spaces if $F = \mathbb{R}$ and a conjugate-linear bijection if $F = \mathbb{C}$.
91. **Riesz Representation Theorem.** Let V be a finite-dim inner product space over F (which is \mathbb{R} or \mathbb{C}). If ϕ is a linear functional on V , there is a unique $v \in V$ such that $\phi(x) = \langle x, v \rangle$ for all $x \in V$.
92. **Prop.** An orthogonal list that consists of nonzero vectors is linearly independent.
93. **Cor.** An orthonormal list is linearly independent.
94. **Prop.** If v_1, \dots, v_m is orthonormal and a_1, \dots, a_m are elements of F , then $\|a_1 v_1 + \dots + a_m v_m\|^2 = |a_1|^2 + \dots + |a_m|^2$.
95. **Prop.** If $v = a_1 v_1 + \dots + a_m v_m$ and v_1, \dots, v_m orthogonal, then $a_k = \langle v, v_k \rangle$, $k = 1, \dots, m$. If v_1, \dots, v_m is orthonormal basis of V then $v = \langle v, v_1 \rangle v_1 + \dots + \langle v, v_m \rangle v_m$.
96. **Prop.** If V is a finite-dim inner product space, then V has an orthonormal basis.
97. **Prop.** Suppose V is finite-dim. Then every orthonormal list of vectors in V can be extended to an orthonormal basis of V .
98. **Schur's Theorem.** Let T be an operator on a finite-dim inner product space V . If T is upper-triangular with respect to some basis, then it is upper-triangular with respect to some orthonormal basis of V .
99. **Prop.** If $U \subseteq W$, then $W^\perp \subseteq U^\perp$.
100. **Prop.** If U is a subset of V , then $U \cap U^\perp \subseteq \{0\}$.
101. **Lemma.** If U is a finite-dim subspace of V , then $V = U \oplus U^\perp$.
102. **Formula.** Assume V is finite-dim and U is a subspace of V . Then $\dim U^\perp = \dim V - \dim U$.
103. **Theorem.** If U is a finite-dim subspace of V , then $(U^\perp)^\perp = U$.

104. **Prop.** Suppose U is generated by a single nonzero vector w . Then $P_U(v) = \frac{\langle v, w \rangle}{\langle w, w \rangle} w$.
105. **Prop.** If e_1, \dots, e_d is an orthonormal basis of U then $P_U(v) = \langle v, e_1 \rangle e_1 + \dots + \langle v, e_d \rangle e_d$.
106. **Formula.** $\alpha_V T^* = T' \circ \alpha_W$, where $T : V \rightarrow W$ and $T^* : W \rightarrow V$.
107. **Formula.** Let $T : V \rightarrow W$. Then $(T' \alpha_W(w))(v) = \langle Tv, w \rangle$.
108. **Prop.** $\langle Tv, w \rangle_W = \langle v, T^* w \rangle_V$.
109. **Lemma.** If $T : V \rightarrow W$ is a linear map between finite-dim inner product spaces, then $(T^*)^* = T$.
110. **Lemma.** If $T : V \rightarrow W$ is a linear map between finite-dim inner product spaces, then if $a \in F$, then $(aT)^* = \overline{a} T^*$.
111. **Prop.** If $M(T) = (a_{ij})$, then $M(T^*) = (\overline{a_{ij}})^t$.
112. **Prop.**
- (a) $I^* = I$.
 - (b) $(S + T)^* = S^* + T^*$.
 - (c) $(ST)^* = T^* S^*$.
 - (d) $(T^{-1})^* = (T^*)^{-1}$.
113. **Prop.** The matrix of T^* is the conjugate transpose of the matrix of T if the same orthonormal bases of V and W are used to compute the matrices.
114. **Formula.** $\overline{a_{ij}} = \langle T^* w_i, v_j \rangle_V$ iff $a_{-j} = \langle v_j, T^* w_i \rangle_V$.
115. **Prop.** $\text{nul}(T^*) = (\text{range } T)^\perp$.
116. **Formula.**
- (a) $\text{range}(T^*) = (\text{nul } T)^\perp$.
 - (b) $\text{nul } T = (\text{range}(T^*))^\perp$.
 - (c) $\text{range } T = (\text{nul}(T^*))^\perp$.
117. **Theorem.** If T is symmetric, then T is orthonormal diagonalizable.
118. **Theorem.** Every eigenvalue of a self-adjoint operator is real.
119. **Lemma.** Let T be an operator on a complex inner product space. Suppose $\langle Tv, v \rangle = 0$ for all $v \in V$. Then $T = 0$.
120. **Cor.** If T is an operator on a complex inner product space, then $\langle Tv, v \rangle \in \mathbb{R}$ for all $v \in V$ iff T is self-adjoint.
121. **Prop.** Alternating implies anti-symmetric.

122. **Prop.** Let $x^2 + bx + c$ be an irreducible quadratic over \mathbb{R} . Then the operator $T^2 + bT + cI$ is injective on V .
123. **Real Spectral Theorem.** Suppose T is a self-adjoint operator on a real inner product space ($T = T^*$). Then T is diagonal in an orthonormal basis of V .
124. **Complex Spectral Theorem.** Suppose T is a normal operator on a complex inner product space ($T^*T = TT^*$). Then T is diagonal in an orthonormal basis of V .
125. **Theorem.** An operator T is normal iff $\|Tv\| = \|T^*v\|$ for all $v \in V$.
126. **Prop.** If T is normal and $Tv = \lambda v$, then $T^*v = \bar{\lambda}v$.
127. **Prop.** Suppose T is normal and v, w eigenvectors for T with different eigenvalues. Then the vectors v, w are orthogonal.
128. **Theorem.** If T is normal and $F = \mathbb{C}$, then T is diagonal in an orthonormal basis of V .
129. **Prop.** Let $T : V \rightarrow V$ be a symmetric (self-adjoint) operator on a nonzero finite-dim inner product space. Then T has an eigenvalue.
130. **Prop.** If T is self-adjoint, then it is diagonalizable in the real and complex case.
131. **Prop.** Nilpotent 2×2 operators (nonzero) have no square root.
132. **Prop.** The identity operator has infinitely many square roots if $\dim V \geq 2$.
133. **Theorem.** Let T be an operator on a finite-dim inner product space. Then TFAE:
- (a) T is positive.
 - (b) T is self-adjoint and has only nonnegative eigenvalues.
 - (c) T has a positive square root.
 - (d) T has a self-adjoint square root.
 - (e) $T = R^*R$ for some operator R on V .
134. **Theorem.** Let T be an operator on a finite-dim inner product space V . Then TFAE:
- (a) S is an isometry.
 - (b) S preserves $\langle \cdot, \cdot \rangle$, i.e. $\langle Su, Sv \rangle = \langle u, v \rangle$ for all $u, v \in V$.
 - (c) If e_1, \dots, e_m is an orthonormal list of vectors in V , then so is Se_1, \dots, Se_m .
 - (d) There exists an orthonormal basis e_1, \dots, e_m of V such that Se_1, \dots, Se_m is also an orthonormal basis of V .
 - (e) $S^*S = I$.

- (f) $SS^* = I$.
 - (g) S^* is an isometry.
 - (h) S is invertible and $S^* = S^{-1}$.
135. **Prop.** The operator S is an isometry iff V has an orthonormal basis of eigenvectors for which the corresponding eigenvalues have absolute value 1.
136. **Theorem.** If $T \in L(V)$, there is an isometry $S \in L(V)$ so that $T = S\sqrt{T^*T} = (\text{isometry}) \cdot (\text{positive operator})$.
137. **Observation.**
- (a) T^*T is a positive operator.
 - (b) $\text{nul}(T^*T) = \text{nul } T$.
 - (c) $\text{range}(T^*T) = \text{range}(T^*)$.
 - (d) $\dim \text{range } T = \dim \text{range } T^*$.
138. **Formula.** The characteristic polynomial of a 2×2 matrix is $z^2 - \text{tr}(A)z + \det A$.
139. **Prop.** The determinant of the matrix of an operator is independent of basis.
140. **Properties.** Let A, B be $n \times n$ matrices. Then
- (a) $\det(AB) = \det A \cdot \det B$.
 - (b) $\det A = 0$ iff A is not invertible.
 - (c) If $m \in S_n$, then $\det(A_{m_1} \dots A_{m_n}) = \text{sgn}(m) \det A$.
 - (d) $\det(A_1 \dots \alpha \cdot A_k \dots A_n) = \alpha \det A$.
141. **Prop.** Let $S : V \rightarrow W$ be a linear map between inner product spaces. Then S is an isometry iff all singular values of S are 1.
142. **Theorem.** If the positive singular values of $T : V \rightarrow W$ are s_1, \dots, s_m , then there are orthogonal lists $e_1, \dots, e_m \in V$ and $f_1, \dots, f_m \in W$ so that $Tv = s_1\langle v, e_1 \rangle f_1 + \dots + s_m\langle v, e_m \rangle f_m$ for all $v \in V$.
143. **Prop.** Let $T : V \rightarrow W$ be a linear map. If s_1 is the largest singular value of T , then $\|Tv\| \leq s_1\|v\|$ for all $v \in V$.