

Math H110 Definitions.

1. **Endomorphism.** An endomorphism is a group homomorphism from a set to itself (NOTE: does not have to be invertible.)
2. **End  $V$ .** The symbol  $\text{End } V$  is the set of all endomorphisms on  $V$  (and multiplication on  $\text{End } V$  is defined to be function composition).
3. **F-Module.** An  $F$ -module is a generalization of vector spaces over rings.
4. **Subspace.** Let  $V$  be a vector space.  $X$  is a subspace of  $V$  if  $X \subseteq V$  and closed under all relevant operations of  $V$ ,  $X \neq \emptyset$ , and  $X \ni 0$ .
5. **Linear Map / Linear Transformation.** Let  $V$  be a vector space over a field  $F$  with  $v, w \in V$ . Let  $T$  be a map on  $V$  with  $T(v + w) = T(v) + T(w)$  and  $T(\lambda v) = \lambda T(v)$  for all  $\lambda \in F$ . Then,  $T$  is called a linear map or linear transformation.
6. **Linear Operator.** If  $T$  is a linear transformation on a vector spaces  $V$  with  $T : V \rightarrow V$ , then  $T$  is linear operator on  $V$ .
7. **Spans.** The list  $v_1, \dots, v_n$  spans  $V$  iff  $T : F^n \rightarrow V$  is onto.
8. **Linearly Independent.** The list  $v_1, \dots, v_n$  is linearly independent iff  $T : F^n \rightarrow V$  is 1-1. Equivalently, the list  $v_1, \dots, v_n$  is linearly independent if  $\lambda_1 v_1 + \dots + \lambda_n v_n = 0$  implies  $\lambda_i = 0$  for all  $i$ .
9. **Linearly Dependent.** The list  $v_1, \dots, v_n$  is linearly dependent iff  $\lambda_1 v_1 + \dots + \lambda_n v_n = 0$  implies  $\lambda_i \neq 0$  for some  $i$ .
10. **Basis.** The list  $v_1, \dots, v_n$  is a basis of  $V$  if  $\text{span}\{v_1, \dots, v_n\} = V$  and  $v_1, \dots, v_n$  is linearly independent.
11. **Finite-dimensional.**  $V$  is finite-dimensional if  $V$  is spanned by a finite list of vectors.
12. **Sum of Subspaces.** Let  $X_1, \dots, X_t$  be subspaces of  $V$ . Then, we define their sum as  $X_1 + \dots + X_t = \{x_1 + \dots + x_t \mid x_1 \in X_1, \dots, x_t \in X_t\}$ .
13. **Direct Sum of Subspaces.** Let  $X_1, \dots, X_t$  be subspaces of  $V$ . Then, their direct sum,  $X_1 \oplus \dots \oplus X_t$ , is given by a 1-1 linear map  $T$ , with  $T : X_1 \times \dots \times X_t \rightarrow V$ .
14. **Complement of Subspace.** Let  $X, Y$  be subspaces of  $V$ . Then,  $Y$  is a complementary subspace of  $X$  iff  $X + Y = V$  and  $X \cap Y = \{0\}$ .
15. **Rank, Nullity.** The rank of a linear map is the dimension of the range of the linear map. The nullity is the dimension of the null space of the linear map.
16. **Null Space.** The null space is the set of vectors that are mapped to 0.
17. **Isomorphic Vector Spaces.** Two vector spaces  $V, W$  are isomorphic if there exists a linear map  $T : V \rightarrow W$  that is 1-1 and onto.

18. **Quotient Space.** Suppose  $U$  is a subspace of  $V$ . Then, the quotient space  $V/U$  is the set  $V/U = \{v + U \mid v \in V\}$ .
19. **Column Rank.** The column rank (rank of the column span of a matrix) is defined to be  $\text{rank} T_A$ .
20. **Conjugation.** Let  $A$  be an  $n \times n$  matrix (over  $F$ ) and let  $Q$  be an  $n \times n$  matrix (over  $F$ ). Then, the conjugation of  $A$  by  $Q$  is  $Q^{-1}AQ$ .

21. **Dual Space.** Let  $V$  be an  $F$ -vector space. Then the dual space of  $V$  is  $V' = \mathcal{L}(V, F)$  where the elements of  $V'$  are called linear functionals.
22. **Annihilator.** For a subspace  $U \subseteq V$ , we define the annihilator of  $U$  to be  $U_0 = \{\phi \in V' \mid \phi(u) = 0 \forall u \in U\}$ .
23. **Double Dual.** Let  $V$  be a finite-dimensional vector space with dual  $V'$ . Then the double dual of  $V$  is  $(V')' = V'' = V$ . Also,  $\dim V = n = \dim V' = \dim V''$ .
24. **Eigenvector / eigenvalue.** Let  $T \in \mathcal{L}(V)$ . Then an eigenvector of  $T$  is a  $v \in V$  ( $v \neq 0$ ) such that  $Tv = \lambda v$  ( $\lambda \in F$  is called an eigenvalue), and  $v$  is an eigenvector of  $T$ .
25. **Eigenspace.** Let  $T \in \mathcal{L}(V)$  and take  $\lambda$  to be an eigenvalue of  $T$ . Then,  $E(\lambda, T) = \{v \in V \mid Tv = \lambda v\} \neq \emptyset$  is written as  $V_\lambda$  and is called the eigenspace of  $\lambda$ , which is a subspace of  $V$ .
26. **Invariant subspace.**  $E$  is a  $T$ -invariant subspace if  $T \in \mathcal{L}(V)$  with  $T(E) \subseteq E$ .
27. **Idempotent.** If  $e = e^2$ , then  $e$  is called idempotent.
28. **Generalized Eigenvector.** Consider a minimal polynomial  $(x - \lambda_1)^{e_1} \cdots (x - \lambda_m)^{e_m}$  on  $X$  with  $(T - \lambda_1 I)^{e_1} v = 0$ . Then,  $v$  is called a generalized eigenvector for  $\lambda = \lambda_1$ .
29. **Characteristic polynomial.** The characteristic polynomial of  $T : V \rightarrow V$  (with eigenvalues  $\lambda_1, \dots, \lambda_t$ ) is the polynomial  $\prod_{i=1}^t (x - \lambda_i)^{\dim X_i}$ , where  $V = X_1 \oplus \cdots \oplus X_t$ .
30. **Simultaneously diagonalizable.** Operators  $S$  and  $T$  on  $V$  are simultaneously diagonalizable if there is a basis of  $V$  that consists of vectors that are eigenvectors for both  $S$  and  $T$  (i.e. there exists a basis  $v_1, \dots, v_n$  of  $V$  so that for  $i$ ,  $1 \leq i \leq n$ , there are  $\lambda_i$  and  $\mu_i$  so that  $Sw_i = \lambda_i v_i$  and  $Tv_i = \mu_i v_i$ ).

31. **Inner product.** Let  $V$  be a vector space over  $\mathbb{R}$ , possibly infinite-dim. An inner product on  $V$  is a bilinear function  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$  such that  $\langle x, x \rangle \geq 0$  for all  $x \in V$  and  $\langle x, x \rangle = 0$  iff  $x = 0$ .

32. **Complex Inner product.** Let  $z = (z_1, \dots, z_n)$  and  $w = (w_1, \dots, w_n)$  be in  $\mathbb{C}^n$ . We have  $\langle z, w \rangle = \sum_i z_i \overline{w_i}$ , with  $\langle w, z \rangle = \overline{\langle z, w \rangle}$  and  $\langle \alpha z, w \rangle = \alpha \langle z, w \rangle$  and  $\langle w, \alpha z \rangle = \overline{\alpha} \langle z, w \rangle$ .
33. **Norm.** Norm of  $v \in V$  is  $\|v\| = \sqrt{\langle v, v \rangle}$ .
34. **Orthogonal.** Two vectors are orthogonal if  $\langle v, w \rangle = 0$ , where  $v, w \in V$ .
35. **Orthogonal Complement.** Take  $U$  to be a subset of  $V$ . Then,  $U^\perp = \{v \in V \mid \langle v, u \rangle = 0 \forall u \in U\} = \{v \in V \mid \langle u, v \rangle = 0 \forall u \in U\} = \{v \in V \mid \phi_v = 0 \text{ in } U\} = \{v \in V \mid \phi_v \in U^0\}$ .
36. **Adjoint.** Let  $T : V \rightarrow W$ . Then the adjoint of  $T$  is  $T^* : W \rightarrow V$  such that  $T^* = \alpha_V^{-1} \circ T' \circ \alpha_W$ , where  $\alpha_V : V \rightarrow V'$  and  $\alpha_W : W \rightarrow W'$ .
37. **Self-adjoint.** An operator  $T : V \rightarrow V$  is self-adjoint if  $T = T^*$ .
38. **Symmetric.**  $T$  is symmetric if  $M(T) = M(T)^t$ .
39. **Normal.**  $T$  is a normal operator if  $TT^* = T^*T$ .
40. **Alternating.** A bilinear form  $\psi : V \times V \rightarrow F$  is alternating if  $\psi(v, v) = 0$  for  $v \in V$ .
41. **Anti-symmetric.** Let  $\psi : V \times V \rightarrow F$  be a bilinear form. Then  $\psi$  is anti-symmetric if  $\psi(x, y) = -\psi(y, x)$ .
42. **Positive operator.**  $T$  is a positive operator on an inner product space if  $T = T^*$  and  $\langle Tv, v \rangle \geq 0$ .
43. **Square root of an operator.** A square root of an operator  $T$  is an operator  $R$  such that  $R^2 = T$ .
44. **Isometry.** An operator  $S \in L(V)$  is an isometry if it preserves distance, i.e.  $\|Sv\| = \|v\| \forall v \in V$ , i.e.  $S^* = S^{-1}$ . In the real case, they are called orthogonal operators. In the complex case, they are called unitary operators.
45. **Singular values.** The singular values of  $T$  are the eigenvalues of  $\sqrt{T^*T}$ .
46. **Def 1 of Determinants.** (cofactor expansion).  $\det A = \sum_{j=1}^n (-1)^{j+1} \cdot a_{ij} \det(A_{ij})$ .
47. **Def 2 of Determinants.** If  $T \in L(V)$ , where  $V$  is over  $\mathbb{C}$ , then the determinant of  $T$  is the product of the eigenvalues  $\lambda_1, \dots, \lambda_n$  (with multiplicity).
48. **Def 3 of Determinants.** The determinant of an operator  $T \in L(V)$  is  $(-1)^n$  times the constant term of its characteristic polynomial  $(z - \lambda_1) \cdots (z - \lambda_n)$ .
49. **Permutation.** A permutation  $m = (m_1, \dots, m_r)$  is a list containing  $1, \dots, n$  exactly once each, and write  $S_n$  to denote the set of  $n$ -element permutations.
50. **Inversion.** An inversion (in a permutation) is a pair  $\{i, j\}$  such that  $m_i > m_j$  where  $i < j$ .

51. **Def 4 of Determinants.** Let  $A$  be an  $n \times n$  matrix. Then,  $\det A = \sum_{m \in S_n} (\operatorname{sgn}(m) \cdot a_{m_1,1} \cdots a_{m_n,n})$ .
52. **Box.** The box defined by  $v_1, \dots, v_n \in \mathbb{R}^n$  is  $\{a_1 v_1 + \cdots + a_n v_n \mid 0 \leq a_i \leq 1, i = 1, \dots, n\}$  and as volume,  $\det (v_1 \ \dots \ v_n)$ , where  $v_1, \dots, v_n$  are the columns of a matrix  $A$ .