1. 1A. (NOTHING)

2. 1B.

3. **Vector Space.** A vector space V is a set that has scalar multiplication and vector addition defined on it with the following properties:

   (a) Additive commutativity.

   (b) Additive associativity of vectors $(u+(v+w) = (u+v)+w)$ and multiplicative associativity for scalars $((ab)v = a(bv))$.

   (c) Additive identity.

   (d) Additive inverses.

   (e) Multiplicative identity.

   (f) BOTH distributive properties.

4. **V-space (unique additive identity)** A vector space has a unique additive identity.

5. **V-space (unique additive inverses)** Every element in a vector space has a unique additive inverse.

6. 1C.

7. **Subspace.** A subset $U \subseteq V$ is a subspace of V if it is a vector space with the same additive identity, scalar multiplication, and vector addition as defined on V.

8. **Conditions for a Subspace.** A subset $U \subseteq V$ is a subspace of V iff U is closed under vector addition, scalar multiplication, and contains the "zero" element as in V.

9. **Sums of Subspaces.** Let $V_1, \ldots, V_n$ be subspaces of V. Then, we have the sum of subspaces as $V_1 + \cdots + V_n = \{v_1 + \cdots + v_n \mid v_i \in V_i \text{ for all } i\}$.

10. **Smallest subspace containing each subspace** Suppose $V_1, \ldots, V_n$ are subspaces of V. Then, $V_1 + \cdots + V_n$ is the smallest subspace of V containing $V_1, \ldots, V_n$.

11. **Direct Sum.** Suppose $V_1, \ldots, V_m$ are subspaces of V. Then:

    (a) The sum $V_1 + \cdots + V_m$ is direct if each element of $V_1 + \cdots + V_m$ can be written uniquely as a sum $v_1 + \cdots + v_m$, where $v_i \in V_i$ for all i.

    (b) If $V_1 + \cdots + V_m$ is a direct sum, then we write $V_1 \oplus \cdots \oplus V_m$.

12. **Conditions for a direct sum.** Suppose $V_1, \ldots, V_n$ are subspaces of V. Then, $V_1 + \cdots + V_n$ is direct iff the only way to write 0 from $v_1 + \cdots + v_n$ is by taking $v_i = 0$ for all i.

13. **Direct sum of subspaces.** If U, W are subspaces of V, then $U + W$ is direct iff $U \cap W = \{0\}$.

14. 2A.

15. **Span is the smallest containing subspace.** The span of a list of vectors in V is the smallest subspace containing all of the vectors in the list.

16. **Zero polynomial.** The zero polynomial is said to have degree $-\infty$.

17. **Linear Independence.** A list of vectors $v_1, \ldots, v_n \in V$ is said to be linearly independent if $a_1 v_1 + \cdots + a_n v_n = 0$ implies $a_i = 0$ for all i. Also, the empty list () is said to be linearly independent.

18. **Linear Dependence.** A list of vectors $v_1, \ldots, v_n$ is said to be linearly dependent if $a_1 v_1 + \cdots + a_n v$=0 impies $a_i \neq 0$ for some i.

19. **Linear Dependence Lemma.** Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then, there exists $k \in \{1, \ldots, m\}$ such that $v_k \in \operatorname{span}(v_1, \ldots, v_{k-1})$. Furthermore, if k satisfies the condition in the previous sentence and the k^{th} term is removed from $v_1, \ldots, v_m$, then the span of the remaining list equals $\operatorname{span}(v_1, \ldots, v_m)$.

20. **length of linearly independent list ¡ length of spanning list.** In a finite-dimensional vector space, the length of every linearly independent list is at most the length of every spanning list of vectors.

21. **Finite Dimensional subspaces.** Every subspace of a finite-dimensional vector space is finite dimensional.

22. 2B.

23. **Basis.** A basis of V is a list of vectors that is linearly independent and spans V.

24. **Criterion for basis.** A list of vectors $v_1, \ldots, v_n \in V$ is a basis of V iff every $v \in V$ can be written uniquely in the form $v = a_1 v_1 + \cdots + a_n v_n$, where $a_i \in F$ for all i.

25. **Every spanning list contains a basis.** Every spanning list in a vector space can be reduced to a basis of the vector space.

26. **Basis of finite-dimensional vector space.** Every finite-dimensional vector space has a basis.

27. **Every linearly independent list extends to a basis.** Every linearly independent list in a finite-dimensional vector space can be extended to a basis of the vector space.

28. **Every subspace of V is part of a direct sum equal to V.** Suppose V is finite-dimensional and U is a subspace of V. Then, there is a subspace W of V such that $V = U \oplus W$.

29. 2C.

30. **Basis length does not depend on basis.** Any two bases of a finite-dimensional vector space have the same length.

31. **Dimension of a subspace.** If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.

32. **Linearly independent list of the right length is a basis.** Suppose V is finite-dimensional. Then, every linearly independent list of vectors in V (with list length equal to $\dim V$) is a basis of V.

33. **Subspace of full dimension equals the whole space.** Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Then, $U = V$.

34. **Spanning list of the right length is a basis.** Suppose V is finite-dimensional. Then, every spanning list of V of length $\dim V$ is a basis of V.

35. **Dimension of a sum.** If V_1, V_2 are subspaces of a finite-dimensional vector space, then $\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$.

36. 3A.

37. **Set of Linear Maps.** The linear of linear maps from $V \to W$ is written $\mathscr{L}(V, W)$ and the set of linear maps from $V \to V$ is written $\mathscr{L}(V)$.

38. **Linear Map lemma.** Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then, there exists a unique linear map $T : V \to W$ such that $Tv_k = w_k$ for each k.

39. **Linear maps take 0 to 0.** Suppose $T : V \to W$ is a linear map. Then, $T(0) = 0$.

40. 3B.

41. **null space is a subspace.** Suppose $T \in \mathscr{L}(V, W)$. Then, T is a subspace of V.

42. **injectivity iff null is 0.** Let $T \in \mathscr{L}(V, W)$. Then, T is 1-1 iff $T = \{0\}$.

43. **range is a subspace.** If $T \in \mathscr{L}(V, W)$, then range T is a subspace of W.

44. **Fundamental Theorem of Linear Maps.** Suppose V is finite-dimensional and $T \in \mathscr{L}(V, W)$. Then, range T is finite dimensional and $\dim V = \dim \operatorname{nul} T + \dim \operatorname{range} T$.

45. **linear map to a lower-dim space is not 1-1.** Suppose V, W are finite-dimensional vector spaces such that $\dim V > \dim W$. Then, no linear map from $V \to W$ is 1-1.

46. **linear map to a higher-dim space is not onto.** Suppose V, W are finite-dimensional vector spaces such that $\dim V < \dim W$. Then, no linear map from $V \to W$ is onto.

47. 3C. (NOTHING)

48. 3D.

49. **Theorem.** Let V, W be finite-dimensional vector spaces such that $\dim V = \dim W$ and let $T \in \mathscr{L}(V, W)$. Then, T is invertible iff T is 1-1 iff T is onto.

50. **isomorphism.** An isomorphism is an invertible linear map.

51. **dimension and isomorphic.** Two finite-dimensional vector spaces are isomorphic iff they have the same dimension.

52. **Theorem.** Suppose V and W are finite-dimensional. Then, $\mathscr{L}(V, W)$ is finite-dimensional and $\dim \mathscr{L}(V, W) = (\dim V)(\dim W)$.

53. **matrix of identity operator with respect to two bases.** Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are two bases of V. Then, the matrices $\mathscr{M}(I; u_1, \ldots, u_n; v_1, \ldots, v_n)$ and $\mathscr{M}(I; v_1, \ldots, v_n; u_1, \ldots, u_n)$ are invertible and are inverses of each other.

54. **Change of basis formula.** Let $T \in \mathscr{L}(V, W)$. Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are two bases of V. Let $A = \mathscr{M}(T; u_1, \ldots, u_n)$ and $B = \mathscr{M}(T; v_1, \ldots, v_n)$ and $C = \mathscr{M}(I; u_1, \ldots, u_n; v_1, \ldots, v_n)$. Then, $A = C^{-1}BC$.

55. Suppose that $v_1, \ldots, v_n$ is a basis of V and $T \in \mathscr{L}(V)$ is invertible. Then, $\mathscr{M}(T^{-1}) = (\mathscr{M}(T))^{-1}$, where both matrices are with respect to the basis $v_1, \ldots, v_n$.

56. 3E.

57. **Product of vector spaces is a vector space.** Suppose $V_1, \ldots, V_m$ are vector spaces over $\mathbb{F}$. Then, $V_1 \times \cdots \times V_m$ is a vector space over $\mathbb{F}$.

58. **dimension of a product is the sum of the dimensions.** Suppose $V_1, \ldots, V_m$ are finite-dimensional vector spaces. Then, $V_1 \times \cdots \times V_m$ is finite-dimensional and $\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m$.

59. **Products and direct sums.** Suppose $V_1, \ldots, V_m$ are subspaces of V. Define a linear map $\Gamma : (V_1 \times \cdots \times V_m) \to (V_1 + \cdots + V_m)$ by $\Gamma(v_1, \ldots, v_m) = v_1 + \cdots + v_m$. Then, $V_1 + \cdots + V_m$ is direct iff Γ is 1-1.

60. **direct sum iff dimensions add up.** Suppose V is finite-dimensional and $V_1, \ldots, V_m$ are subspaces of V. Then, $V_1 + \cdots + V_m$ is direct iff $\dim(V_1 + \cdots + V_m) = \dim V_1 + \cdots + \dim V_m$.

61. **v + U.** Suppose $v \in V$ and $U \subseteq V$. Then, $v + U = \{v + u \mid u \in U\}$.

62. **Translate.** For $v \in V$ and $U \subseteq V$, the set $v + U$ is called a translate of U.

63. **Quotient Space.** Let U be a subspace of V. Then, the quotient space V/U is the set of all translates of U, that is, $V/U = \{v + U \mid v \in V\}$.

64. **two translates of a subspace are either equal or disjoint.** Suppose U is a subspace of V and $v, w \in V$. Then, $v - w \in U$ iff $v + U = w + U$ iff $(v + U) \cap (w + U) \neq \emptyset$.

65. **Addition and scalar multiplication on Quotient space.** Let U be a subspace of V. Then, we have (for all $v, w \in V$, $\lambda \in F$):

    (a) addition on V/U: $(v + U) + (w + U) = (v + w) + U$.

    (b) scalar multiplication on V/U: $\lambda(v + U) = (\lambda v) + U$.

66. **quotient space is a vector space.** Let U be a subspace of V. Then, the quotient space V/U is a subspace of V under the defined scalar multiplication and vector addition.

67. **quotient map.** Let U be a subspace of V. Then, the quotient map $\pi : V \to V/U$ is the linear map defined by $\pi(v) = v + U$ for each $v \in V$.

68. **dimension of quotient space.** Suppose V is finite-dimensional and U is a subspace of V. Then, $\dim(V/U) = \dim V - \dim U$.

69. **Column rank.** The column rank (rank of the column span of a matrix) is $\operatorname{rank} T_A$.

70. **Theorem.** If A is a rectangular matrix of elements in a field F, then row rank A = column rank A.

71. RIBET DEFS.

72. **Endomorphism.** An endomorphism is a group homomorphism from a set to itself (NOTE: does not have to be invertible.)

73. **End V.** The symbol V is the set of all endomorphisms on V (and multiplication on V is defined to be function composition).

74. **F-Module.** An $F-$module is a generalization of vector spaces over rings.

75. **Subspace.** Let V be a vector space. X is a subspace of V if $X \subseteq V$ and closed under all relevant operations of V, $X \neq \emptyset$, and $X \ni 0$.

76. **Linear Map / Linear Transformation.** Let V be a vector space over a field F with $v, w \in V$. Let T be a map on V with $T(v+w) = T(v) + T(w)$ and $T(\lambda v) = \lambda T(v)$ for all $\lambda \in F$. Then, T is called a linear map or linear transformation.

77. **Linear Operator.** If T is a linear transformation on a vector spaces V with $T : V \to V$, then T is linear operator on V.

78. **Spans.** The list $v_1, \ldots, v_n$ spans V iff $T : F^n \to V$ is onto.

79. **Linearly Independent.** The list $v_1, \ldots, v_n$ is linearly independent iff $T : F^n \to V$ is 1-1. Equivalently, the list $v_1, \ldots, v_n$ is linearly independent if $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$ implies $\lambda_i = 0$ for all i.

80. **Linearly Dependent.** The list $v_1, \ldots, v_n$ is linearly dependent iff $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$ implies $\lambda_i \neq 0$ for some i.

81. **Basis.** The list $v_1, \ldots, v_n$ is a basis of V if $\operatorname{span}\{v_1, \ldots, v_n\} = V$ and $v_1, \ldots, v_n$ is linearly independent.

82. **Finite-dimensional.** V is finite-dimensional if V is spanned by a finite list of vectors.

83. **Sum of Subspaces.** Let $X_1, \ldots, X_t$ be subspaces of V. Then, we define their sum as $X_1 + \cdots + X_t = \{x_1 + \cdots + x_t \mid x_1 \in X_1, \ldots, x_t \in X_t\}$.

84. **Direct Sum of Subspaces.** Let $X_1, \ldots, X_t$ be subspaces of V. Then, their direct sum, $X_1 \oplus \cdots \oplus X_t$, is given by a 1-1 linear map T, with $T : X_1 \times \cdots \times X_t \to V$.

85. **Complement of Subspace.** Let X, Y be subspaces of of V. Then, Y is a complementary subspace of X iff $X + Y = V$ and $X + Y = X \oplus Y$.

86. **Rank, Nullity.** The rank of a linear map is the dimension of the range of the linear map. The nullity is the dimension of the null space of the linear map.

87. **Null Space.** The null space is the set of vectors that are mapped to 0.

88. **Isomorphic Vector Spaces.** Two vector spaces V, W are isomorphic if there exists a linear map $T : V \to W$ that is 1-1 and onto.

89. **Quotient Space.** Suppose U is a subspace of V. Then, the quotient space V/U is the set $V/U = \{v + U \mid v \in V\}$.

90. **Column Rank.** The column rank (rank of the column span of a matrix) is defined to be $\operatorname{rank} T_A$.

91. **Conjugation.** Let A be an $n \times n$ matrix (over F) and let Q be an $n \times n$ matrix (over F). Then, the conjugation of A by Q is $Q^{-1}AQ$.

92. RIBET THMS.

93. **Lemma.** Let F be a field, $\lambda \in F$, V a vector space over F (denoted by V/F), $v \in V$. Then, if $\lambda v = 0$, then $\lambda = 0$ or $v = 0$.

94. **Lemma.** A vector space over a field is a module over a field.

95. **Theorem.** The intersection of a family of subspaces of a vector space V is a subspace of V.

96. **Lemma.** Let $S = \{v_1, \ldots, v_t\}$. Then the subspace of all linear combinations of the elements of S is the $\operatorname{span} S$.

97. **Theorem.** Let $L = v_1, \ldots, v_n$ be a list of vectors in a vector space V over a field F and let $T : F^n :\to V$ be linear transformation with $(\lambda_1, \ldots, \lambda_n) \mapsto \lambda_1 v_1 + \cdots + \lambda_n v_n$. Then, we have the following:

    (a) L spans V iff T is onto.

    (b) L is linearly independent iff T is 1-1 iff $\operatorname{nul} T = \{0\}$.

    (c) L is a basis iff T is 1-1 and onto.

98. **Prop.** Consider $T : F^n \to V$ with $(\lambda_1, \ldots, \lambda_n) \mapsto \lambda_1 v_1 + \cdots + \lambda_n v_n$, so $T(e_i) = v_i$ for all i. Then, T is the unique linear map $F_n \to V$ that sends $e_i \mapsto v_i$ for all i.

99. **Theorem.** Every subspace X of V has complement.

100. **Lemma.** If $v_1, \ldots, v_t$ is linearly dependent list, then there is an index k such that $v_k \in \operatorname{span}(v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_t)$. Furthermore, the span of the list of length $t-1$ gotten by removing v_k from the list is the same as the span of the original list.

101. **Prop.** In a finite-dimensional vector space, the length is of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

102. **Cor.** Two bases of V have the same number of elements.

103. **Prop.** $X + Y$ is direct iff the null space of the sum map is $\{0\}$.

104. **Theorem.** Every subspace of a finite-dimensional vector space is finite-dimensional.

105. **Prop.** Every spanning list for a vector space can be pruned down to a basis of the space.

106. **Cor.** Every finite-dimensional vector space has a basis.

107. **Prop.** In a finite-dimensional vector space, every linearly independent list can be extended to a basis of the space.

108. **Major Theorem.** Every subspace of a finite-dimensional vector space has a complement.

109. **Prop.** Let X, Y be subspaces of a finite-dimensional vector space V. Then:

    (a) $\dim X + \dim Y = \dim V$.

    (b) $X \cap Y = \{0\}$.

    Then, $V = X \oplus Y$.

110. **Prop.** $\dim(X \oplus Y) = \dim X + \dim Y$.

111. **Prop.** If V is a finite-dimensional vector space (with $\dim V = n$), then every subspace has dimension at most n.

112. **Prop.** Let $\dim V = n$. Then, a linearly independent list of vectors of V with length n is a basis for V.

113. **Prop.** Let $\dim V = n$. Then, every spanning list for V of length n is a basis for V.

114. **Lemma.** The list $(x_1, 0), \ldots, (x_t, 0); (0, y_1), \ldots, (0, y_k)$ of length $t + k$ is a basis of $X \times Y$.

115. **Cor.** $\dim(X \times Y) = \dim X + \dim Y$.

116. **Cor.** Let $T : V \to W$ be a linear map with $\dim V = d$. Then, $\operatorname{rank} T \leq d$.

117. **Rank-Nullity Theorem.** $\dim V = \operatorname{rank} V + \operatorname{nullity} V$.

118. **Prop.** If $T : V \to W$ is 1-1, then $\operatorname{nullity} T = 0$.

119. **Cor.** If $T : V \to W$ is 1-1 and onto, then $\dim V = \dim W$.

120. **Theorem.** The set of linear maps $V \to W$ is a vector space $L \cdot (F^n, W) \to T \longrightarrow (Te_1, \ldots, Te_n) \in W^n$.

121. **Theorem.** $\dim(X + Y) = \dim X + \dim Y - \dim(X \cap Y)$.

122. **Cor.** $\dim(V/X) = \dim V - \dim X$.

123. **Theorem.** If A is a rectangular matrix with elements in a field F, then row rank A = column rank A.

124. **Prop.** Let $T : V \to W$ be 1-1. Then, $\dim W \geq \dim V$.

125. **Prop.** Let $T : V \to W$ be onto. Then, $\dim V \geq \dim W$.

126. **Prop.** Let $T : V \to W$ and $\dim V = \dim W$. Then, T 1-1 iff T onto iff T bijective iff T invertible.