- 1C.
- 5 — Is R^2 a subspace of the complex vector space C^2? — No, check scalar mutliplication by i.
- 7 — prove/disprove: If U is a nonempty subset of R^2 that is closed under addition and taking additive inverses, then U is a subspace. — No, take $U = \{(x, 0) \mid x \in Q\}$.
- 8 — want an example of a subset U of R^2 that satisfies scalar mutliplication but not vector addition — take x-axis, y-axis.
- 13 — prove that union of three subspaces of V is a subspace iff one contains the other two — suffices to show if W is the union of three of its subspaces, then then one of the three subspaces is contained in the union of the other two, by applying result of prob 1c.12.
- 18 — does the operation of addition on subspaces of V have additive identity? which subspaces have inverses? — add. id. is $\{0\}$ and only $\{0\}$ has additive inverse.
- 19 — prove/disprove: If V_1, V_2, U subspaces of V, then $V_1+U = V_2+U$ implies $V_1 = V_2$ — false. take V_1,V_2,U to be x-axis, y-axis, $y = x$ respectively.
- 23 — prove/disprove: If V_1, V_2, U subspaces of V so that $V = V_1 \oplus U = V_2 \oplus U$, then $V_1 = V_2$. — use same counterexample as in problem above.
- 2A.
- 7b — Show if C is a vector space over C, then $1 + i, 1 - i$ is L.D. — for L.D. relation, choose coeffs $a = i$ and $b = 1$ for $1 + i, 1 - i$, resp.
- 17 — show that V is inf-dim iff there is a sequence $v_1, v_2, \dots \in V$ so that $v_1, \dots, v_m$ l.i. for all positive integers m. — for reverse direction, prove contrapositive.
- 19 — show that real vector space of all cts real values functions on $[0, 1]$ is inf-dim. — look at polynomials.
- 5 — let V be finite-dim and $V = U + W$. show there is a basis of V consisting of vectors from $U \cup W$. — concatenate bases of U and W.
- 2C.
- 8 — let $v_1, \dots, v_m$ be linearly independent in V and $w \in V$. show dim $span(v_1+w, \dots, v_m + w) \geq m - 1$. — look at $v_1 - v_2, \dots, v_1 - v_m$ and it is contained in the dimspan.
- 10 — let m be a positive integer. for $0 \leq k \leq m$, let $p_k = x^k(1 - x)^{m-k}$. show $p_0, \dots, p_m$ is a basis of $P_m(F)$. — set linear dependence relation and choose x-values to substitute.

- 11 — Let U,W be 4-dim subspaces of C^6. show there exist two vectors in $U \cap W$ that aren't scalar multiples of each other. — equivalent to showing $\dim(U \cap W) \geq 2$.
- 16 — let V be finite-dim and U a strict subspace of V. let $n = \dim U$ and $m = \dim V$. show there are $n - m$ subsapces of V each with dim 1 whose intersectin is U — fix a basis of U and extend it to basis of V and construct subspaces that each 'delete' a vector in the extension of U.
- 3A.
- 8 — give a function $\phi : R^2 \to R$ so that $\phi(\lambda v) = \lambda\phi(v)$ but ϕ not linear — take $\phi(x, y) = (x^3 + y^3)^{1/3}$.
- 10 — prove/disprove: if $q \in P(R)$ and $T : P(R) \to P(R)$ by $Tp = q \circ p$, then T is linear — false. take $q = x^2$.
- 11 — let V be finitedim and $T \in L(V)$. show $T = \lambda I$ iff $ST = TS$ for all $S \in L(V)$. — for reverse direction, try contrapositive and look at $\ker T$.
- 12 — let U be a strict subspace of V. Let $S \in L(U, W)$ and $S \neq 0$. Let $T(v) = Sv$ if $v \in U$ and $Tv = 0$ if $v \notin U$. show T is not linear. — pick $v_1 \in U$ and $v_2 \notin U$.
- 17 — let V be finitedim. show the only two-sided ideals of $L(V)$ are $\{0\}$ and $L(V)$. — let w be so that $Tw \neq 0$. let $S_k : V \to V$ that sends v_j to 0 for $j \neq k$ and v_k to w. put R_k so that $R_k(Tw) = v_k$, and look at $R_kTS_kv_j$.
- 3B.
- 15 — Suppose there is a linear map on V so that both null space and range of it are finitedim. show that V is finite dim. — look at basis $Tv_1, \ldots, Tv_n$ for range and $w_1, \ldots, w_k$ for null space.
- 19 — Let W be finitedim and $T \in L(V, W)$. show T is 1-1 iff there exists $S \in L(W, V)$ so that $ST = I$ on V. — letting $T : V \to W$ be 1-1 and looking at $U = \operatorname{range} T$, put $S : U \to V$ as the inverse of T and extend to $S : W \to V$.
- 20 — let W be finite-dim and $T \in L(V, W)$. Show T onto iff there exists $S \in L(W, V)$ so that $TS = I$ on W. — use ontoness of T and look at restriction of T to X, the complement of $nullT$. do isomorphism $X \cong W$ and put $S : W \to X$ so that $TS = I$.
- 3C.
- 5 — Let V, W be finitedim and $T \in L(V, W)$. show there is a basis of V and a basis of W so that in these bases, all entries of $M(T)$ are 0 except those in entries row k col k if $1 \leq k \leq \operatorname{range} T$. — $U = \operatorname{nul} T$ and X is complement to U in V. put bases of X and U. find bases of $\operatorname{range} T$ and complete to get basis of W.

- 6 — Let $v_1, \dots, v_n$ be basis of V and W is finitedim and let $T \in L(V)$. show there is a basis $w_1, \dots, w_m$ so that all entries of $M(T)$, in these bases, are 0 except possibly a 1 in the first row, first col. — first column is Tv_1. consider when $Tv_1 = 0, \neq 0$. put basis $W = span(Tv_1, w_2, \dots, w_m)$.

- 7 — Let $w_1, \dots, w_n$ a basis of W and V finitedim and $T \in L(V, W)$. Show there is a basis $v_1, \dots, v_m$ of V so that all entries in first row of $M(T)$, in these bases, are 0 except possilby a 1 in first row, first col. — Look at $T' : W' \to V'$ and apply 3c.6 result.

- 3D.

- 10 — Let V, W be finite dim and $U \subseteq V$. put $E = \{T \in L(V, W) \mid U \subseteq \operatorname{nul} T\}$. find a formula for $\dim E$ in terms of $\dim V, \dim U, \dim W$. — put $\Phi : L(V, W) \to L(U, W)$ by $\phi(T) = T\mid_U$ and find range and null space.

- 19 — let V be finitedim and $T \in L(V)$. show T has same matrix with respect to every basis of V iff $T = \lambda I$. — fix a matrix of T and for basis $v_1, \dots, v_m$ of V, $v_1, \dots, (1/2)v_k, \dots, v_m$ is also basis; scale and edit.

- 20 — let $q \in P(R)$. show there is a polynomial $p \in P(R)$ so that $q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$. — define $T : P(R) \to P(R)$ by $Tp = (x^2 + x)p''(x) + 2xp'(x) + p(3)$ and show T is 1-1 (and by finitedim of domain,codomain) thus T is onto.

- 3E.

- 9 — Show a nonempty subset A of V is a translate of some subspace of V iff $\lambda v + (1 - \lambda)w \in A$ for all $v, w \in A$, $\lambda \in F$. — for converse, fix $x \in A$ attempt for $A = x + U$, where $U = \{a - x \mid a \in A\}$.

- 17 — Let U be a subspace of V so that $\dim V/U = 1$. show there is $\phi \in L(V, F)$ so that $\operatorname{nul} \phi = U$. — put $T : V/U \to F$ that sends everything to $1 \in F$. let ϕ be composite of $\pi : V \to V/U$ and T.

- 3F.

- 6 — let $\phi, \beta \in V'$. show $\operatorname{nul} \phi \subseteq \operatorname{nul} \beta$ iff there is $c \in F$ so that $\beta = c\phi$. — by a previous problem, there is $S \in L(F)$ so that $\beta = S\phi$.

- 26 — let V be finitedim and Ω be a subspace of V'. show $\Omega = \{v \in V \mid \phi(v) = 0 \forall \phi \in \Omega\}^0 = U^0$. — show $U = \cap_{i=1}^m (\operatorname{nul} \phi_i)$.

- 5A.

- 15 — Let V be finitedim, $T \in L(V)$, and $\lambda \in F$. show λ is an eigenvalue of T iff λ is an eigenvalue of T' — use chain of if and only ifs.

- 20 — let $S \in L(F^\infty)$ be backwards shift by $S(z_1, z_2, \dots) = (z_2, z_3, \dots)$. show each $f \in F$ is an eigenvalue and find all eigenvectors. — look at $(1, \lambda, \lambda^2, \dots)$.

- 28 — let V be finitedim and $T \in L(V)$. show T has at most $1 + \dim \operatorname{range} T$ distinct eigenvalues. — put distinct eigenvalues/vectors and for nonzero eigenvalues, look at $v_i = T((1\lambda_i)v_i)$ and linear independence and range.

- 39 — Let V be finitedim and $T \in L(V)$. show T has eigenvalue iff there is a subspace of V of $\dim V - 1$ that is T-invariant. — one direction: use fact eigenvalues of $T_{V/U}$ are eigenvalues of T. other direction: if λ eigenvalue, then $T - \lambda I$ noninvertible so its range has dim $< \dim V$. if $X = \operatorname{range} T$, every subspace W of V with $X \subseteq W \subseteq V$ is T-invariant.

- 5B.

- 2 — let V be a complex vector space and $T \in L(V)$ have no eigenvalues. show every subspcae of V invariant under T is $\{0\}$ or infinite-dim. — Take instead a finitedim $X \subseteq V$; it has an eigenvector.

- 3 — let $n \in \mathbb{Z}_{>0}$ and $T \in L(F^n)$ by $T(x_1, \ldots, x_n) = (x_1 + \cdots + x_n, \ldots, x_1 + \cdots + x_n)$. find all eigenvalues/vectors and minimal polynomial of T — $\operatorname{range} T = \{(a, \ldots, a)\}$.

- 4 — let $F = C$, $T \in L(V)$, $p \in P(C)$ is a nonconstant polynomial and $\alpha \in C$. show α is eigenvalue of $p(T)$ iff $\alpha = p(\lambda)$ for some eigenvalue λ of T — one direction: $p(T)v = p(\lambda)v$. other direction: T is upper-triangular in some basis of V; look at diagonal and look at $p(T)$.

- 5 — for above question, find an example where instead $V = R^2$. — $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

- 7 — show if V finitedim and $S, T \in L(V)$, then if at least one of S, T invertible, then minimal poly of ST equals that of TS. — first show $Sp(T)S^{-1} = p(STS^{-1})$. then T, STS^{-1} have same minimal poly. replace T by TS.

- 10 — let V be finitedim and $T \in L(V)$. show $span(v, Tv, \ldots, T^m v) = span(v, Tv, \ldots, T^{\dim V - 1} v)$ for all $m \geq \dim V - 1$. — note $v, Tv, \ldots, T^m v$ has dim m.

- 19 — let V be finitedim and $T \in L(V)$. let $\epsilon = \{q(T) \mid q \in P(F)\}$. show $\dim \epsilon$ = degree of minimal poly of T — observe $F[x]/(\operatorname{nul} \alpha)$, algebra.

- 25 — V finitedim, $T \in L(V)$, $U \subseteq V$ invariant under T. show minimal poly of T is poly multiple of minimal poly of $T_{V/U}$. also show (min poly of $T|_U$) x (min poly of $T_{V/U}$) is poly multiple of min poly of T. — first part: if m is min poly of T, then $m(T|_U)$ is poly multiple of min poly of $T|_U$, similary for $T_{V/U}$. second part: let g be min poly of $T_{V/U}$ and f be min poly of $T|_U$ and show $(fg)(T) = 0$. $g(T)$ is 0 map on V/U and $f(T)$ maps U to $\{0\}$.

- 5C.

- 7 — V finitedim, $T \in L(V)$, and $v \in V$. show there is unique monic poly p_v of smallest degree so that $p_v(T)v = 0$. also show min poly of T is a poly mult of p_v. — first part: $I = \{f(x) \mid f(T)v = 0\}$, it contains 0 and closed under addition, and 'external multiplication' and use well-ordering.

- 5D.
- 2 — let $T \in L(V)$ have diagonal matrix A corresponding to some basis of V. show that if $\lambda \in F$, then λ appears on diag of A exactly $\dim E(\lambda, T)$ times. — $E(\lambda, T) = \text{nul}(T - \lambda I)$ and look at matrix multiplication.
- 3 — V finitedim, $T \in L(V)$ diagonalizable. show $V = \text{nul}\, T \oplus \text{range}\, T$. — look at eigenvalues that are 0 and eigenvalues that are nonzero.
- 5 — V finitedim complex vector space, $T \in L(V)$ and $V = \text{nul}(T - \lambda I) \oplus \text{range}(T - \lambda I)$ for all $\lambda \in C$. show T diagonalizable. — do induction on $\dim V$.
- 19 — prove/disprove: if $T \in L(V)$ and $U \subseteq V$ is invariant under T so that $T\mid_U$ and $T_{V/U}$ are diagonalizable, then T diagonalizable. — false: take $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
- 5E.
- 2 — let ϵ be subset of V where every $T \in \epsilon$ is diagonalizable. show there is a basis of V with respect to which every $T \in \epsilon$ has diag matrix iff every pair $S, T \in \epsilon$ commutes. — converse: look at direct sum of operators, eigenspaces, and restrictions.
- 6 — V finitedim nonzero complex vector space and $ST = TS$. show there exist $\alpha, \lambda \in C$ so that $\text{range}(S - \alpha I) + \text{range}(T - \lambda I) \neq V$. — look at 2 upper triangular matrices, one with α in bottom left corner and another with λ in bottom left corner.
- 10 — want commuting operators S, T so that $S + T$ has an eigenvalue that is not sum of eigenvalue of S and eigenvalue of T, and similarly for ST. — let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = -S$.
- 6A.
- 1 — prove/disprove: if $v_1, \ldots, v_m \in V$ then $\sum_{j=1}^m \sum_{k=1}^m \langle v_j, v_k \rangle \geq 0$. — true, do induction and apply formula for $||v_1 + \cdots + v_m||^2$.
- 4 — let $T \in L(V)$ so that $||Tv|| \leq ||v|| \forall v \in V$. show $T - \sqrt{2}I$ is injective. — do by contradiction and use triangle inequality.
- 8 — let $a, b, c, x, y \in R$ so that $a^2 + b^2 + c^2 + x^2 + y^2 \leq 1$. show $a + b + c + 4x + 9y \leq 10$. — vectors: $(a, b, c, x, y), (1, 1, 1, 4, 9)$ and apply cauchy-schwarz.
- 9 — let $u, v \in V$ so that $||u|| = 1 = ||v||$ and $\langle u, v \rangle = 1$. show $u = v$. — use cauchy-schwarz.
- 12 — let $a, b, c, d > 0$. show $(a + b + c + d)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}) \geq 16$, and for which a, b, c, d do we get equality? — first part: show it for the square root of the entire inequality, second part hint: equality iff scalar multiple.

- apollonius problem — take initially a as left edge, d as middle edge, b as right edge, c has bottom edge. put $y = a$ and $x = \frac{1}{2}c$, and apply parallelogram identiy.

- 6B.

- 1 — let $e_1, \ldots, e_m \in V$ so that $||a_1e_1 + \cdots + a_me_m||^2 = |a_1|^2 + \cdots + |a_m|^2$. show $e_1, \ldots, e_m$ is orthonormal — to show orthogonal, we have $||e_a||^2 \leq 1 + |a|^2 = ||e_a + ae_b||^2$.

- 3 — let $e_1, \ldots, e_m$ be orthonormal in $V \ni v$. show $||v||^2 = |\langle v, e_1, \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2$ iff $v \in span(e_1, \ldots, e_m)$. — for forward direction, set $x = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_m \rangle e_m$ - look at $\langle x, v \rangle$ and $||x - v||^2$.

- 6 — let $e_1, \ldots, e_n$ be an ON basis of V. (1) show if $v_1, \ldots, v_n \in V$ so that $||e_i - v_i|| \leq \frac{1}{\sqrt{n}}$, then the list of $v_i's$ is a basis of V. (2) show there are $v_1, \ldots, v_n \in V$ so that $||e_i - v_i|| \leq \frac{1}{\sqrt{n}}$ but v_i's are L.D. — (1): show linear independence and observe $|a_1|^2 + \cdots + |a_n|^2 = ||a_1e_1 + \cdots + a_ne_n||^2 = ||(a_1e_1 + \cdots + a_ne_n) - (a_1v_1 + \cdots + a_nv_n)||^2$, apply triangle, C-S inequalities. (2): put $v_i := e_i - \frac{1}{n}(e_1 + \cdots + e_n)$.

- 9 — let $e_1, \ldots, e_m$ be the result of applying GPS to L.I. list $v_1, \ldots, v_n \in V$. show $\langle v_k, e_k \rangle > 0 \forall k$. — for case when $v_1, \ldots, v_n$ not orthogonal, show contrapositive and note $||v_a||^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v_a, e_m \rangle|^2$.

- 17 — let $F = C$ and V finitedim. show if T is an operator on V so that 1 is only eigenvalue of T and $||Tv|| \leq ||v|| \forall v \in V$, then $T = I$. — use schur's theorem; then diagonal entries are all 1. then write Te_k as a linear combo of the e_i's via matrix entries, upper bound coefficients to 0, so coefficients are 0, so $T = I$.

- 6C.

- 9 — V finitedim. let $P \in L(V)$ so that $P^2 = P$ and every vector in $\operatorname{nul} P$ is orthogonal to every vector in $\operatorname{range} P$. show $\exists U \subseteq V$ so that $P = P_U$. — look at direct sums, use range, null space perp identies, put $U := \operatorname{range} P$.

- 7A.

- 2 — if $T \in L(V, W)$, show $T = 0 \iff T^* = 0 \iff T^*T = 0 \iff TT^* = 0$ — show $(a) \iff (b), (a) \iff (c), (b) \iff (d)$.

- 3 — let $T \in L(V)$ and $\lambda \in F$. show λ eigenvalue of T iff λ eigenvalue of T^* — use chain of iffs and identities.

- 5 — let $T \in L(V, W)$. let $e_1, \ldots, e_n$ be ON basis of V and $f_1, \ldots, f_m$ be ON basis of W. show $||Te_1||^2 + \cdots + ||Te_n||^2 = ||T^*f_1||^2 + \cdots + ||T^*f_m||^2$. — note $\sum ||Te_i||^2 = \sum \sum |\langle Te_i, f_j \rangle|^2$ and use inner product properties.

- 8 — let $A_{m \times n}$. Show row rank A equals col rank A. — note $M(T^*)$ is the conjugate transpose of $M(T)$. let $v_1, \ldots, v_m$ be a basis of col $M(T)$. show $\overline{v_1}, \ldots, \overline{v_m}$ is L.I. by applying conjugations. use also $\dim \operatorname{range} T = \dim \operatorname{range} T^*$.

- 27 — let $T \in L(V)$ be normal. show $\operatorname{nul} T^k = \operatorname{nul} T$, $\operatorname{range} T^k = \operatorname{range} T$. — first part: induction & by normality, look at $\langle T^* T^n v, T^{n-1} v\rangle$ and inner product calculations. second part: use identities and first part.

- 29 — prove/disprove: if $T \in L(V)$, there is an ON basis $e_1, \ldots, e_n$ so that $||Te_i|| = ||T^* e_i|| \forall i$. — false: take $T = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$.

- 7B.

- 5 — prove/disprove: if $T \in L(C^3)$ is diagonalizable, then T normal — false: take $v_1 = (1, 0, 0), v_2 = (0, 1, 0), v_3 = (1, 0, 1)$ and put $Tv_1 = v_1, Tv_2 = v_2, Tv_3 = 3v_3$.

- 6 — V complex inner product space and $T \in L(V)$ normal and $T^9 = T^8$. show T self-adjoint and $T^2 = T$. — look at orthonormal basis of V of eigenvectors and see eigenvalues in $\{0, 1\}$. then by prev problem, $T = P_U$ for some $U \subseteq V$.

- 8 — $F = C, T \in L(V)$, show T normal iff each eigenvector of T is eigenvector of T^*. — reverse direction: by class, $\exists$ ON basis of V where T is upper-triangular, observe matrices, apply complex spectral theorem.

- 18 — V inner product space. want $T \in L(V)$ so that $T^2 + bT + cI$ noninvertible with $b^2 < 4c$. — take $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

- 7C.

- 5 — let $T \in L(V)$ self-adjoint. show T positive iff for every ON basis $e_1, \ldots, e_n$ of V, all entries on diagonal of $M(T, (e_1, \ldots, e_n))$ are nonnegative. — forward: use thm 'writing a vector as a linear combo of ON basis' reverse: spectral theorem and equivalent statement to T positive.

- 7 — $S \in L(V)$ invertible & positive and $T \in L(V)$ positive. show $S + T$ invertible — first show X positive & invertible $\iff \rangle Xv, v\rangle \forall v \in V \setminus \{0\}$, then apply.

- 15 — $T \in L(V)$ self-adjoint. show $\exists A, B \in L(V)$ so that $T = A - B, \sqrt{T^*T} = A + B, AB = BA = 0$. — spectral theorem, and only real eigenvalues $\lambda_1, \ldots, \lambda_n$. put $\alpha_i = \lambda_i$ if $\lambda_i \geq 0$, else, 0. put $\beta_i = -\lambda_i$ if $\lambda_i \leq 0$, else, 0. put $Ae_k = \alpha_k e_k$, B similarly.

- 18 — $S, T \in L(V)$, both positive. show ST positive $\iff ST = TS$. — forward: prf by contradiction gives $ST \neq (ST)^*$, so ST not self-adjoint, contradiction reverse: there is ON basis $e_1, \ldots, e_n$ of eigenvectors of S, T, so $Se_i = \mu_i e_i$, $Te_i = \lambda_i e_i$ with $\lambda_i, \mu_i \geq 0 \forall i$.

- 7D.

- 1 — $\dim V \geq 2$ and $S \in L(V,W)$. show S isometry iff Se_1, Se_2 ON list in W for all ON list e_1, e_2 in V. — forward: put $U := span(e_1, e_2)$ and look $S\mid_U$ & apply equivalence thm from axler. reverse: fix ON basis of V and look at equivalence thm from axler.

- 2 — $T \in L(V,W)$. show $T = \lambda I$ iff T preserves orthogonality. — reverse: fix ON basis of V, look at $\langle u+v, u-v\rangle = ||u||^2 - ||v||^2$ and apply to pairs in ON basis, put $\lambda := ||Te_i||$ & do cases, $\lambda = 0, \neq 0$.

- 4 — $F = C$ and A, B self-adjoint. show $A+iB$ unitary iff $AB = BA$, $A^2+B^2 = I$. — forward: look at $||(A+ib)v||^2$ and $SS^* = I$ and inner products.

- 5 — $S \in L(V)$. show TFAE: (1) S self-adjoint & unitary; (2) $S = 2P - I$ for some orthogonal proj P; (3) $\exists U \subseteq V$ so that $Su = u \forall u \in U, Sv = -v \forall v \in U^{\perp}$. — $(a) \implies (b)$: put $P = (1/2)(S+I)$. $(c) \implies (a)$: show $S^2 = S$.