Math H110 Midterm 1 CheatSheet

1A. (n/a)

1B.

1. **Vector Space.** A vector space V is a set that has scalar multiplication and vector addition defined on it with the following properties:
   (a) Additive commutativity.
   (b) Additive associativity of vectors ($u+(v+w)=(u+v)+w$) and multiplicative associativity for scalars ($(ab)v=a(bv)$).
   (c) Additive identity.
   (d) Additive inverses.
   (e) Multiplicative identity.
   (f) BOTH distributive properties.
2. **V-space (unique additive identity)** A vector space has a unique additive identity.
3. **V-space (unique additive inverses)** Every element in a vector space has a unique additive inverse.

1C.

1. **Subspace.** A subset $U \subseteq V$ is a subspace of V if it is a vector space with the same additive identity, scalar multiplication, and vector addition as defined on V.
2. **Conditions for a Subspace.** A subset $U \subseteq V$ is a subspace of V iff U is closed under vector addition, scalar multiplication, and contains the "zero" element as in V.
3. **Sums of Subspaces.** Let $V_1, \dots, V_n$ be subspaces of V. Then, we have the sum of subspaces as $V_1 + \cdots + V_n = \{v_1 + \cdots + v_n \mid v_i \in V_i \text{ for all } i\}$.
4. **Smallest subspace containing each subspace** Suppose $V_1, \dots, V_n$ are subspaces of V. Then, $V_1 + \cdots + V_n$ is the smallest subspace of V containing $V_1, \dots, V_n$.
5. **Direct Sum.** Suppose $V_1, \dots, V_m$ are subspaces of V. Then:

(a) The sum $V_1 + \cdots + V_m$ is direct if each element of $V_1 + \cdots + V_m$ can be written uniquely as a sum $v_1 + \cdots + v_m$, where $v_i \in V_i$ for all i.

(b) If $V_1 + \cdots + V_m$ is a direct sum, then we write $V_1 \oplus \cdots \oplus V_m$.

6. **Conditions for a direct sum.** Suppose $V_1, \ldots, V_n$ are subspaces of V. Then, $V_1 + \cdots + V_n$ is direct iff the only way to write 0 from $v_1 + \cdots + v_n$ is by taking $v_i = 0$ for all i.

7. **Direct sum of subspaces.** If U, W are subspaces of V, then $U + W$ is direct iff $U \cap W = \{0\}$.

2A.

1. **Span is the smallest containing subspace.** The span of a list of vectors in V is the smallest subspace containing all of the vectors in the list.

2. **Zero polynomial.** The zero polynomial is said to have degree $-\infty$.

3. **Linear Independence.** A list of vectors $v_1, \ldots, v_n \in V$ is said to be linearly independent if $a_1 v_1 + \cdots + a_n v_n = 0$ implies $a_i = 0$ for all i. Also, the empty list () is said to be linearly independent.

4. **Linear Dependence.** A list of vectors $v_1, \ldots, v_n$ is said to be linearly dependent if $a_1 v_1 + \cdots + a_n v_= 0$ impies $a_i \neq 0$ for some i.

5. **Linear Dependence Lemma.** Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then, there exists $k \in \{1, \ldots, m\}$ such that $v_k \in \operatorname{span}(v_1, \ldots, v_{k-1})$. Furthermore, if k satisfies the condition in the previous sentence and the k^{th} term is removed from $v_1, \ldots, v_m$, then the span of the remaining list equals $\operatorname{span}(v_1, \ldots, v_m)$.

6. **length of linearly independent list ¡ length of spanning list.** In a finite-dimensional vector space, the length of every linearly independent list is at most the length of every spanning list of vectors.

7. **Finite Dimensional subspaces.** Every subspace of a finite-dimensional vector space is finite dimensional.

2B.

1. **Basis.** A basis of V is a list of vectors that is linearly independent and spans V.

2. **Criterion for basis.** A list of vectors $v_1, \dots, v_n \in V$ is a basis of V iff every $v \in V$ can be written uniquely in the form $v = a_1 v_1 + \cdots + a_n v_n$, where $a_i \in F$ for all i.

3. **Every spanning list contains a basis.** Every spanning list in a vector space can be reduced to a basis of the vector space.

4. **Basis of finite-dimensional vector space.** Every finite-dimensional vector space has a basis.

5. **Every linearly independent list extends to a basis.** Every linearly independent list in a finite-dimensional vector space can be extended to a basis of the vector space.

6. **Every subspace of V is part of a direct sum equal to V.** Suppose V is finite-dimensional and U is a subspace of V. Then, there is a subspace W of V such that $V = U \oplus W$.

2C.

1. **Basis length does not depend on basis.** Any two bases of a finite-dimensional vector space have the same length.

2. **Dimension of a subspace.** If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.

3. **Linearly independent list of the right length is a basis.** Suppose V is finite-dimensional. Then, every linearly independent list of vectors in V (with list length equal to $\dim V$) is a basis of V.

4. **Subspace of full dimension equals the whole space.** Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Then, $U = V$.

5. **Spanning list of the right length is a basis.** Suppose V is finite-dimensional. Then, every spanning list of V of length $\dim V$ is a basis of V.

6. **Dimension of a sum.** If V_1, V_2 are subspaces of a finite-dimensional vector space, then $\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$.

3A.

1. **Set of Linear Maps.** The linear of linear maps from $V \to W$ is written $\mathscr{L}(V,W)$ and the set of linear maps from $V \to V$ is written $\mathscr{L}(V)$.

2. **Linear Map lemma.** Suppose $v_1,\ldots,v_n$ is a basis of V and $w_1,\ldots,w_n \in W$. Then, there exists a unique linear map $T : V \to W$ such that $Tv_k = w_k$ for each k.

3. **Linear maps take 0 to 0.** Suppose $T : V \to W$ is a linear map. Then, $T(0) = 0$.

3B.

1. **null space is a subspace.** Suppose $T \in \mathscr{L}(V,W)$. Then, T is a subspace of V.

2. **injectivity iff null is 0.** Let $T \in \mathscr{L}(V,W)$. Then, T is 1-1 iff $T = \{0\}$.

3. **range is a subspace.** If $T \in \mathscr{L}(V,W)$, then range T is a subspace of W.

4. **Fundamental Theorem of Linear Maps.** Suppose V is finite-dimensional and $T \in \mathscr{L}(V,W)$. Then, range T is finite dimensional and $\dim V = \dim T + \dim \operatorname{range} T$.

5. **linear map to a lower-dim space is not 1-1.** Suppose V,W are finite-dimensional vector spaces such that $\dim V > \dim W$. Then, no linear map from $V \to W$ is 1-1.

6. **linear map to a higher-dim space is not onto.** Suppose V,W are finite-dimensional vector spaces such that $\dim V < \dim W$. Then, no linear map from $V \to W$ is onto.

3C. n/a.
3D.

1. **Theorem.** Let V,W be finite-dimensional vector spaces such that $\dim V = \dim W$ and let $T \in \mathscr{L}(V,W)$. Then, T is invertible iff T is 1-1 iff T is onto.

2. **isomorphism.** An isomorphism is an invertible linear map.

3. **dimension and isomorphic.** Two finite-dimensional vector spaces are isomorphic iff they have the same dimension.

4. **Theorem.** Suppose V and W are finite-dimensional. Then, $\mathscr{L}(V,W)$ is finite-dimensional and $\dim \mathscr{L}(V,W) = (\dim V)(\dim W)$.

3E.

1. **Product of vector spaces is a vector space.** Suppose $V_1, \ldots, V_m$ are vector spaces over $\mathbb{F}$. Then, $V_1 \times \cdots \times V_m$ is a vector space over $\mathbb{F}$.

2. **dimension of a product is the sum of the dimensions.** Suppose $V_1, \ldots, V_m$ are finite-dimensional vector spaces. Then, $V_1 \times \cdots \times V_m$ is finite-dimensional and $\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m$.

3. **Products and direct sums.** Suppose $V_1, \ldots, V_m$ are subspaces of V. Define a linear map $\Gamma : (V_1 \times \cdots \times V_m) \to (V_1 + \cdots + V_m)$ by $\Gamma(v_1, \ldots, v_m) = v_1 + \cdots + v_m$. Then, $V_1 + \cdots + V_m$ is direct iff Γ is 1-1.

4. **direct sum iff dimensions add up.** Suppose V is finite-dimensional and $V_1, \ldots, V_m$ are subspaces of V. Then, $V_1 + \cdots + V_m$ is direct iff $\dim(V_1 + \cdots + V_m) = \dim V_1 + \cdots + \dim V_m$.

5. **v + U.** Suppose $v \in V$ and $U \subseteq V$. Then, $v + U = \{v + u \mid u \in U\}$.

6. **Translate.** For $v \in V$ and $U \subseteq V$, the set $v + U$ is called a translate of U.

7. **Quotient Space.** Let U be a subspace of V. Then, the quotient space V/U is the set of all translates of U, that is, $V/U = \{v + U \mid v \in V\}$.

8. **two translates of a subspace are either equal or disjoint.** Suppose U is a subspace of V and $v, w \in V$. Then, $v - w \in U$ iff $v + U = w + U$ iff $(v + U) \cap (w + U) \neq \emptyset$.

9. **Addition and scalar multiplication on Quotient space.** Let U be a subspace of V. Then, we have (for all $v, w \in V$, $\lambda \in F$):

   (a) addition on V/U: $(v + U) + (w + U) = (v + w) + U$.

   (b) scalar multiplication on V/U: $\lambda(v + U) = (\lambda v) + U$.

10. **quotient space is a vector space.** Let U be a subspace of V. Then, the quotient space V/U is a subspace of V under the defined scalar multiplication and vector addition.

11. **quotient map.** Let U be a subspace of V. Then, the quotient map $\pi : V \to V/U$ is the linear map defined by $\pi(v) = v + U$ for each $v \in V$.

12. **dimension of quotient space.** Suppose V is finite-dimensional and U is a subspace of V. Then, $\dim(V/U) = \dim V - \dim U$.

13. **Column rank.** The column rank (rank of the column span of a matrix) is $\text{rank} T_A$.

14. **Theorem.** If A is a rectangular matrix of elements in a field F, then row rank A = column rank A.