Math H110 Theorems.

1. **Lemma.** Let F be a field, $\lambda \in F$, V a vector space over F (denoted by V/F), $v \in V$. Then, if $\lambda v = 0$, then $\lambda = 0$ or $v = 0$.

2. **Lemma.** A vector space over a field is a module over a field.

3. **Theorem.** The intersection of a family of subspaces of a vector space V is a subspace of V.

4. **Lemma.** Let $S = \{v_1, \ldots, v_t\}$. Then the subspace of all linear combinations of the elements of S is the span$S$.

5. **Theorem.** Let $L = v_1, \ldots, v_n$ be a list of vectors in a vector space V over a field F and let $T : F^n :\rightarrow V$ be linear transformation with $(\lambda_1, \ldots, \lambda_n) \mapsto \lambda_1 v_1 + \cdots + \lambda_n v_n$. Then, we have the following:

   (a) L spans V iff T is onto.

   (b) L is linearly independent iff T is 1-1 iff $\operatorname{nul} T = \{0\}$.

   (c) L is a basis iff T is 1-1 and onto.

6. **Prop.** Consider $T : F^n \rightarrow V$ with $(\lambda_1, \ldots, \lambda_n) \mapsto \lambda_1 v_1 + \cdots + \lambda_n v_n$, so $T(e_i) = v_i$ for all i. Then, T is the unique linear map $F_n \rightarrow V$ that sends $e_i \mapsto v_i$ for all i.

7. **Theorem.** Every subspace X of V has complement.

8. **Lemma.** If $v_1, \ldots, v_t$ is linearly dependent list, then there is an index k such that $v_k \in \operatorname{span}(v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_t)$. Furthermore, the span of the list of length $t-1$ gotten by removing v_k from the list is the same as the span of the original list.

9. **Prop.** In a finite-dimensional vector space, the length is of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

10. **Cor.** Two bases of V have the same number of elements.

11. **Prop.** $X + Y$ is direct iff the null space of the sum map is $\{0\}$.