

Math H110 Theorems.

1. **Lemma.** Let F be a field, $\lambda \in F$, V a vector space over F (denoted by V/F), $v \in V$. Then, if $\lambda v = 0$, then $\lambda = 0$ or $v = 0$.
2. **Lemma.** A vector space over a field is a module over a field.
3. **Theorem.** The intersection of a family of subspaces of a vector space V is a subspace of V .
4. **Lemma.** Let $S = \{v_1, \dots, v_t\}$. Then the subspace of all linear combinations of the elements of S is the $\text{span} S$.
5. **Theorem.** Let $L = v_1, \dots, v_n$ be a list of vectors in a vector space V over a field F and let $T : F^n \rightarrow V$ be linear transformation with $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 v_1 + \dots + \lambda_n v_n$. Then, we have the following:
 - (a) L spans V iff T is onto.
 - (b) L is linearly independent iff T is 1-1 iff $\text{nul } T = \{0\}$.
 - (c) L is a basis iff T is 1-1 and onto.
6. **Prop.** Consider $T : F^n \rightarrow V$ with $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 v_1 + \dots + \lambda_n v_n$, so $T(e_i) = v_i$ for all i . Then, T is the unique linear map $F^n \rightarrow V$ that sends $e_i \mapsto v_i$ for all i .
7. **Theorem.** Every subspace X of V has complement.
8. **Lemma.** If v_1, \dots, v_t is linearly dependent list, then there is an index k such that $v_k \in \text{span}(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_t)$. Furthermore, the span of the list of length $t - 1$ gotten by removing v_k from the list is the same as the span of the original list.
9. **Prop.** In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.
10. **Cor.** Two bases of V have the same number of elements.
11. **Prop.** $X + Y$ is direct iff the null space of the sum map is $\{0\}$.
12. **Theorem.** Every subspace of a finite-dimensional vector space is finite-dimensional.

13. **Prop.** Every spanning list for a vector space can be pruned down to a basis of the space.
14. **Cor.** Every finite-dimensional vector space has a basis.
15. **Prop.** In a finite-dimensional vector space, every linearly independent list can be extended to a basis of the space.
16. **Major Theorem.** Every subspace of a finite-dimensional vector space has a complement.
17. **Prop.** Let X, Y be subspaces of a finite-dimensional vector space V . Then:
 - (a) $\dim X + \dim Y = \dim V$.
 - (b) $X \cap Y = \{0\}$.
 Then, $V = X \oplus Y$.
18. **Prop.** $\dim(X \oplus Y) = \dim X + \dim Y$.
19. **Prop.** If V is a finite-dimensional vector space (with $\dim V = n$), then every subspace has dimension at most n .
20. **Prop.** Let $\dim V = n$. Then, a linearly independent list of vectors of V with length n is a basis for V .
21. **Prop.** Let $\dim V = n$. Then, every spanning list for V of length n is a basis for V .
22. **Lemma.** The list $(x_1, 0), \dots, (x_t, 0); (0, y_1), \dots, (0, y_k)$ of length $t + k$ is a basis of $X \times Y$.
23. **Cor.** $\dim(X \times Y) = \dim X + \dim Y$.
24. **Cor.** Let $T : V \rightarrow W$ be a linear map with $\dim V = d$. Then, $\text{rank } T \leq d$.
25. **Rank-Nullity Theorem.** $\dim V = \text{rank } V + \text{nullity } V$.
26. **Prop.** If $T : V \rightarrow W$ is 1-1, then $\text{nullity } T = 0$.
27. **Cor.** If $T : V \rightarrow W$ is 1-1 and onto, then $\dim V = \dim W$.

28. **Theorem.** The set of linear maps $V \rightarrow W$ is a vector space $L \cdot (F^n, W) \rightarrow T \longrightarrow (Te_1, \dots, Te_n) \in W^n$.
29. **Theorem.** $\dim(X + Y) = \dim X + \dim Y - \dim(X \cap Y)$.
30. **Cor.** $\dim(V/X) = \dim V - \dim X$.
31. **Theorem.** If A is a rectangular matrix with elements in a field F , then $\text{row rank } A = \text{column rank } A$.