Involved Integrals - 3

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$$\int \frac{1}{x^3 - 1}\, dx$$

Step 1: Partial Fractions

The denominator of the integrand is a difference of cubes and can be rewritten as such.

$$a^3 - b^3 = (a - b)\left(a^2 + ab + b^2\right)$$

$$x^3 - 1 = (x - 1)\left(x^2 + x + 1\right)$$

$$\int \frac{1}{(x-1)\left(x^2 + x + 1\right)}\, dx$$

We can now write the partial fraction decomposition of the integrand.

$$\frac{A}{x-1} + \frac{Bx + C}{x^2 + x + 1} = \frac{1}{(x-1)\left(x^2 + x + 1\right)}$$

A, B, and C are all constants. Since the original factors did not consist of any duplicate factors, we can write A as the numerator for $x - 1$, as it trails $x - 1$ by an x-degree of 1. Likewise, the $Bx + C$ term trails the $x^2 + x + 1$ term by a single degree. Now, multiply both sides of the equation by $(x-1)\left(x^2 + x + 1\right)$, which would make the right-hand side of the equation equal 1. Then, distribute and simplify.

$$A\left(x^2 + x + 1\right) + (Bx + C)(x - 1) = 1$$

$$Ax^2 + Ax + A + Bx^2 - Bx + Cx - C = 1$$

$$(A+B)x^2 + (A-B+C)x + (A-C) = 1$$

We can now use the equating coefficients method to solve for A, B, and C.

$$(A+B)x^2 + (A-B+C)x + (A-C) = (0)x^2 + (0)x + (1)$$

The number of x^2 terms on the right-hand side of the equation is 0, so therefore, $A+B=0$. Likewise, the number of x terms on the right is 0, so $A-B+C=0$. Finally, $A-C=1$ for the constant terms. Here is the linear system of equations that we need to solve.

$$\begin{cases} A+B=0 \\ A-B+C=0 \\ A-C=1 \end{cases}$$

Solving the system yields: $A=\frac{1}{3}$, $B=-\frac{1}{3}$, and $C=-\frac{2}{3}$. Now, plug in these values into the new integrand.

$$\begin{aligned} \int \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}\,dx &= \int \frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}x-\frac{2}{3}}{x^2+x+1}\,dx \\ &= \frac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+2}{x^2+x+1}\,dx \end{aligned}$$

Now, examine $\int \frac{x+2}{x^2+x+1}\,dx$. We can start off by splitting the integral and solving each one separately.

$$\int \frac{x+2}{x^2+x+1}\,dx = \int \frac{x}{x^2+x+1}\,dx + \int \frac{2}{x^2+x+1}\,dx$$

Step 2: Trigonometric Substitution

We can solve $\int \frac{x}{x^2+x+1}\,dx$ by first completing the square in the denominator of the integrand.

$$\int \frac{x}{x^2+x+1}\,dx = \int \frac{x}{\frac{3}{4} + \left(x+\frac{1}{2}\right)^2}\,dx$$

Now, perform the following trigonometric substitution.

$$x + \frac{1}{2} = \frac{\sqrt{3}}{2}\tan(\theta), \quad x = \frac{\sqrt{3}}{2}\tan(\theta) - \frac{1}{2}$$

$$dx = \frac{\sqrt{3}}{2} \sec^2(\theta)\, d\theta$$

$$\int \frac{\left(\frac{\sqrt{3}}{2}\tan(\theta) - \frac{1}{2}\right)}{\frac{3}{4} + \frac{3}{4}\tan^2(\theta)} \cdot \frac{\sqrt{3}}{2}\sec^2(\theta)\, d\theta$$

$$\frac{\sqrt{3}}{2} \int \frac{\left(\frac{\sqrt{3}}{2}\tan(\theta) - \frac{1}{2}\right)}{\frac{3}{4}\left(1 + \tan^2(\theta)\right)} \cdot \sec^2(\theta)\, d\theta$$

Recall the trigonometric identity: $1 + \tan^2(\theta) = \sec^2(\theta)$

$$\frac{\sqrt{3}}{2} \int \frac{\left(\frac{\sqrt{3}}{2}\tan(\theta) - \frac{1}{2}\right)}{\frac{3}{4}\sec^2(\theta)} \cdot \sec^2(\theta)\, d\theta$$

$$\frac{2\sqrt{3}}{3} \int \left(\frac{\sqrt{3}}{2}\tan(\theta) - \frac{1}{2}\right) d\theta$$

Integrating $\tan(\theta)$ may initially be unobvious, but we can rewrite the integrand using the definition of $\tan(\theta)$.

$$\int \tan(\theta)\, d\theta = \int \frac{\sin(\theta)}{\cos(\theta)}\, d\theta$$

Perform the following u-substitution.

$$u = \cos(\theta)$$

$$\frac{du}{d\theta} = -\sin(\theta)$$

$$d\theta = \frac{du}{-\sin(\theta)}$$

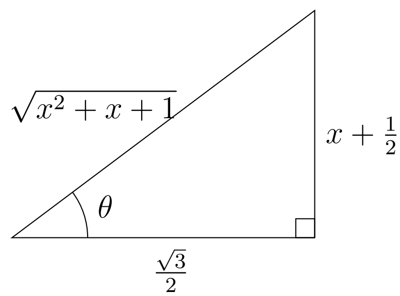
$$\int \frac{\sin(\theta)}{u} \cdot \frac{du}{-\sin(\theta)}$$

$$\begin{aligned}
\int \frac{\sin(\theta)}{\cos(\theta)}\, d\theta &= \int \frac{\sin(\theta)}{u} \cdot \frac{du}{-\sin(\theta)} \\
&= -\int \frac{1}{u}\, du \\
&= -\ln|u| \\
&= -\ln|\cos(\theta)|
\end{aligned}$$

We can now continue solving the integral.

$$\begin{aligned}
\frac{2\sqrt{3}}{3}\int\left(\frac{\sqrt{3}}{2}\tan(\theta)-\frac{1}{2}\right)\,d\theta &= \frac{2\sqrt{3}}{3}\left(-\frac{\sqrt{3}}{2}\ln|\cos(\theta)|-\frac{1}{2}\theta\right)\\
&= -\ln|\cos(\theta)|-\frac{\sqrt{3}}{3}\theta
\end{aligned}$$

Now, we need to substitute x back into this result based off of the definition of the trigonometric substitution we made.



$$x+\frac{1}{2}=\frac{\sqrt{3}}{2}\tan(\theta), \qquad \tan(\theta)=\frac{x+\frac{1}{2}}{\left(\frac{\sqrt{3}}{2}\right)}=\frac{2x+1}{\sqrt{3}}$$

$$\begin{aligned}
\int\frac{x}{x^2+x+1}\,dx &= -\ln|\cos(\theta)|-\frac{\sqrt{3}}{3}\theta\\
&= -\ln\left|\frac{\sqrt{3}}{2\sqrt{x^2+x+1}}\right|-\frac{\sqrt{3}}{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)
\end{aligned}$$

Step 3: t-Substitution

Now, let us solve the other integral when we split the original after the partial fraction decomposition. We can start with complete the square again, as seen in Step 2.

$$\int\frac{2}{x^2+x+1}\,dx=2\int\frac{1}{\frac{3}{4}+\left(x+\frac{1}{2}\right)^2}\,dx$$

If we factor out a $\frac{3}{4}$ from the denominator of the integrand, we would still have a linear term being squared, which could be integrated easily using the inverse tangent function.

$$2\int \frac{1}{\frac{3}{4}+\left(x+\frac{1}{2}\right)^2}\,dx = 2\int \frac{1}{\frac{3}{4}\left(1+\left(\frac{2x+1}{\sqrt{3}}\right)^2\right)}\,dx$$
$$= \frac{8}{3}\int \frac{1}{1+\left(\frac{2x+1}{\sqrt{3}}\right)^2}\,dx$$

Now, perform the following t-substitution.

$$t = \frac{2x+1}{\sqrt{3}}$$

$$\frac{dt}{dx} = \frac{2}{\sqrt{3}}$$

$$dx = \frac{\sqrt{3}\,dt}{2}$$

$$\frac{8}{3}\int \frac{1}{1+t^2}\cdot\frac{\sqrt{3}\,dt}{2}$$

$$\frac{4\sqrt{3}}{3}\arctan(t) = \frac{4\sqrt{3}}{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)$$

Step 4: Wrapping Up

Our final step is to write the final answer based off of the sub integrals we solved.

$$\int \frac{1}{x^3-1}\,dx$$

$$= \boxed{\frac{1}{3}\ln|x-1| - \frac{1}{3}\left[\frac{4\sqrt{3}}{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) - \ln\left|\frac{\sqrt{3}}{2\sqrt{x^2+x+1}}\right| - \frac{\sqrt{3}}{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)\right] + D}$$