Involved Integrals - 1

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$$\int \cos(x)\sqrt{9 + 25\sin^2(x)}\, dx$$

Step 1: U-Substitution

Firstly, it is a good idea to break down the entire expression into something a little more manageable. Outside the radical, we notice the function $\cos(x)$. Inside the radical, there exists a term containing $\sin(x)$ to some nth power. This is important, since we can implement a u-substitution to cancel out the $\cos(x)$.

$$u = \sin(x)$$

$$\frac{du}{dx} = \cos(x)$$

$$dx = \frac{du}{\cos(x)}$$

We can now write u in for $\sin(x)$ and the respective dx substitution.

$$\int \cos(x)\sqrt{9 + 25u^2}\, \frac{du}{\cos(x)}$$

Now, the $\cos(x)$ factors in the numerator and denominator will cancel.

$$\int \sqrt{9 + 25u^2}\, du$$

Step 2: Trigonometric Substitution

At this point, inside the radical, we have the sum of two individual squares. This is now of the form $\sqrt{a^2 + (bu)^2}$, where a and b are constants. Let us now factor out a^2, all inside the radical.

$$\int \sqrt{a^2 \left(1 + \frac{(bu)^2}{a^2}\right)}\, du$$

Now, write in the respective values for a and b and simplify.

$$\int \sqrt{3^2 \left(1 + \frac{(5u)^2}{3^2}\right)}\, du$$

$$3 \int \sqrt{1 + \left(\frac{5u}{3}\right)^2}\, du$$

Now, perform the following trigonometric substitution.

$$\frac{5u}{3} = \tan(\theta)$$

$$\frac{5}{3}\, du = \sec^2(\theta)\, d\theta$$

$$du = \frac{3}{5}\sec^2(\theta)\, d\theta$$

$$3 \int \sqrt{1 + \tan^2(\theta)} \cdot \frac{3}{5}\sec^2(\theta)\, d\theta$$

Recall the trigonometric identity: $1 + \tan^2(\theta) = \sec^2(\theta)$

$$\frac{9}{5} \int \sqrt{\sec^2(\theta)} \cdot \sec^2(\theta)\, d\theta$$

$$\frac{9}{5} \int |\sec(\theta)| \cdot \sec^2(\theta)\, d\theta$$

Since we have $|\sec(\theta)|$, we cannot simply write it as $\sec(\theta)$. We must first set a domain for θ that would make $\sec(\theta)$ positive. This is because it is simpler to deal with positive version of functions rather than their negative counterparts. So, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, making $\sec(\theta) > 0$. Note that we could have also chosen to write $-\sec(\theta)$ from the absolute value, thus resulting in a different domain for θ.

Now we can rewrite $|\sec(\theta)|$ to $\sec(\theta)$.

$$\frac{9}{5}\int \sec^3(\theta)\ d\theta$$

Step 3: Integration by Parts

We can now continue solving $\int \sec^3(\theta)\ d\theta$ using the technique of integration by parts. NOTE: The variable, u, used in this section is entirely different to the u used during the u-substitution process.

$$u = \sec(\theta), \quad du = \sec(\theta)\tan(\theta)\ d\theta$$

$$dv = \sec^2(\theta)\ d\theta, \quad v = \tan(\theta)$$

$$\begin{aligned}
\int \sec^3(\theta)\ d\theta &= \sec(\theta)\tan(\theta) - \int \sec(\theta)\tan^2(\theta)\ d\theta \\
&= \sec(\theta)\tan(\theta) - \int \sec(\theta)\left(\sec^2(\theta) - 1\right)\ d\theta \\
&= \sec(\theta)\tan(\theta) - \int \sec^3(\theta) - \sec(\theta)\ d\theta \\
&= \sec(\theta)\tan(\theta) - \int \sec^3(\theta)\ d\theta + \int \sec(\theta)\ d\theta
\end{aligned}$$

Now, let us focus on $\int \sec(\theta)\ d\theta$. We can cleverly multiply $\sec(\theta)$ by 1 and perform a t-substitution.

$$\int \sec(\theta)\ d\theta = \int \sec(\theta)\left(\frac{\sec(\theta) + \tan(\theta)}{\sec(\theta) + \tan(\theta)}\right)\ d\theta$$

$$= \int \frac{\sec^2(\theta) + \sec(\theta)\tan(\theta)}{\sec(\theta) + \tan(\theta)}\ d\theta$$

$$
\begin{aligned}
t &= \sec(\theta) + \tan(\theta) \\
\frac{dt}{d\theta} &= \sec^2(\theta) + \sec(\theta)\tan(\theta) \\
d\theta &= \frac{dt}{\sec^2(\theta) + \sec(\theta)\tan(\theta)}
\end{aligned}
$$

Now we can write in t in for $\sec(\theta) + \tan(\theta)$.

$$
\int \frac{\sec^2(\theta) + \sec(\theta)\tan(\theta)}{t} \cdot \frac{dt}{\sec^2(\theta) + \sec(\theta)\tan(\theta)}
$$

$$
\begin{aligned}
\int \frac{1}{t}\, dt &= \ln|t| \\
&= \ln|\sec(\theta)\tan(\theta)| \\
&= \int \sec(\theta)\, d\theta
\end{aligned}
$$

Now we can continue solving for $\frac{9}{5}\int \sec^3(\theta)\, d\theta$.

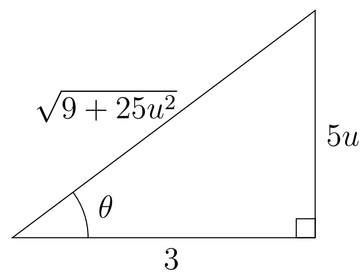
$$
\begin{aligned}
\frac{9}{5}\int \sec^3(\theta)\, d\theta &= \frac{9}{5}\left[\sec(\theta)\tan(\theta) - \int \sec^3(\theta)\, d\theta - \ln|\sec(\theta) + \tan(\theta)|\right] \\
&= \frac{9}{5}\sec(\theta)\tan(\theta) - \frac{9}{5}\int \sec^3(\theta)\, d\theta + \frac{9}{5}\ln|\sec(\theta) + \tan(\theta)|
\end{aligned}
$$

Solving for $\frac{9}{5}\int \sec^3(\theta)\, d\theta$ is now simple.

$$
\begin{aligned}
\frac{18}{5}\int \sec^3(\theta)\, d\theta &= \frac{9}{5}\sec(\theta)\tan(\theta) + \frac{9}{5}\ln|\sec(\theta) + \tan(\theta)| \\
\frac{9}{5}\int \sec^3(\theta)\, d\theta &= \frac{9}{10}\sec(\theta)\tan(\theta) + \frac{9}{10}\ln|\sec(\theta) + \tan(\theta)|
\end{aligned}
$$

Step 4: Wrapping Up

Next, we must write everything in terms of u (the same variable from Step 1). We can do this by going back to how we made the trigonometric substitution in the first place.



$$\tan(\theta) = \frac{5u}{3}, \qquad \sec(\theta) = \frac{\sqrt{9+25u^2}}{3}$$

We know the values of $\tan(\theta)$ and $\sec(\theta)$ from our trigonometric substitution and from the triangle above. Now, we can rewrite the integral, now in terms of u, and then write everything in terms of x, as $u = \sin(x)$.

$$\frac{9}{5}\int \sec^3(\theta)\, d\theta = \frac{9}{10}\left(\frac{5u\sqrt{9+25u^2}}{9} + \ln\left|\frac{\sqrt{9+25u^2}}{3} + \frac{5u}{3}\right|\right) + C$$

$$\int \cos(x)\sqrt{9+25\sin^2(x)}\, dx = \frac{9}{5}\int \sec^3(\theta)\, d\theta$$

$$\boxed{= \frac{9}{10}\left(\frac{5\sin(x)\sqrt{9+25\sin^2(x)}}{9} + \ln\left|\frac{\sqrt{9+25\sin^2(x)}}{3} + \frac{5\sin(x)}{3}\right|\right) + C}$$