RCSJ model using python3

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This script is based on Tinkham's "Introduction to superconductivity"[1] and Gross' "Applied superconductivity", chapter 3[2]. All simulations were done using `python3` and the `rcsj` module written by Felix[3].

CONTENTS

I. CURRENT BIASING

A. Junction model and ODE

We model the Josephson junction in the simplest way possible: A series network of a resistor R, a capacitor C and a nonlinear inductor. The total current running through the network is

$$I(t) = \quad I_s(t) + I_R(t) + I_C(t) \tag{1}$$

$$I(t) = \quad I_c \sin(\phi) + \tfrac{V}{R} + C\frac{\mathrm{d}V}{\mathrm{d}t} \tag{2}$$

$$V(t) = \quad \tfrac{\hbar}{2e}\frac{\mathrm{d}\phi}{\mathrm{d}t},\ \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\hbar}{2e}\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} \tag{3}$$

$$\to I(t) = I_c \sin(\phi) + \tfrac{\hbar}{2eR}\frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{\hbar C}{2e}\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} \tag{4}$$

Note that for this equation, the resistance in reality depends both on temperature and the voltage across the junction due to quasiparticle tunneling:

$$R(V,T) = \begin{cases} R_{sg}(T) & \text{for } |V| \leq 2\Delta(T)/e \\ R_n & \text{for } |V| \geq 2\Delta(T)/e \end{cases} \tag{5}$$

where typically $R_{sg} \gg R_n$. The characteristic voltage of the junction is accordingly defined as $V_c = I_c R_n$. There are two approaches to simplifying equation 4 in terms of normalization. We here call them the **Tinkham approach**[1] and the **Gross approach**[2].

1. Tinkham approach

The equation is simplified with normalized time via the plasma frequency, $\tau = \omega_p t$:

$$\omega_p = \frac{1}{\tau_p} = \frac{1}{\sqrt{L_c C}} = \sqrt{\frac{2eI_c}{\hbar C}} \tag{6}$$

$$\mathrm{d}t = \frac{1}{\omega_p}\mathrm{d}\tau \rightarrow \frac{\mathrm{d}^n}{\mathrm{d}t^n} = \omega_p^n \frac{\mathrm{d}^n}{\mathrm{d}\tau^n} \tag{7}$$

$$\frac{I}{I_c} - \sin(\phi) = \underbrace{\frac{\hbar}{2eI_cR}\sqrt{\frac{2eI_c}{\hbar C}}}_{\sqrt{\frac{\hbar C}{2eI_c}}\frac{1}{RC}\equiv Q^{-1}} \frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \underbrace{\frac{\hbar C}{2eI_c}\frac{2eI_c}{\hbar C}}_{1}\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} \tag{8}$$

Hence the final ODE is

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} = \frac{I}{I_c} - \sin(\phi) - \frac{1}{Q}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} \tag{9}$$

2. Gross approach

This approach is different in that it seems more intuitive, but the time scale and damping are different: We normalize the time not by the plasma frequency, but the L_c/R_n time constant that yields the characteristic junction frequency:

$$\omega_c = \frac{1}{\tau_c} = \frac{R_n}{L_c} = \frac{2e}{\hbar}I_cR_n = \frac{\Phi_0}{2\pi}V_c \tag{10}$$

$$\tau = \frac{t}{\tau_c},\ \tau_c = \frac{\hbar}{2eI_cR} \tag{11}$$

$$\mathrm{d}t = \tau_c\mathrm{d}\tau \rightarrow \frac{\mathrm{d}^n}{\mathrm{d}t^n} = \frac{1}{\tau_c^n}\frac{\mathrm{d}^n}{\mathrm{d}\tau^n} \tag{12}$$

The above ODE then can be resorted into

$$I - I_c\sin(\phi) = \frac{\hbar\tau_c}{2eR}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \frac{\hbar C\tau^2}{2e}\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} \tag{13}$$

$$i - \sin(\phi) = \underbrace{\frac{\hbar}{2eRI_c}\left(\frac{2eI_cR}{\hbar}\right)}_{1}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \underbrace{\frac{\hbar C}{2e}\left(\frac{2eI_cR}{\hbar}\right)^2}_{\frac{2eI_cR^2C}{\hbar}=\beta_c}\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} \tag{14}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} = \frac{1}{\beta_c}\left(i - \sin(\phi) - \frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right) \tag{15}$$

Note that $\beta_c \equiv Q^2 \equiv \tau_c RC$. For this ODE it is much easier to distinguish between $\beta_c \gg 1$ and $\beta_c \ll 1$, simply because of its form. However, it is important to keep in mind that the timescales are different!

B. Unit discussion

We consider first τ_c. The units of $\hbar/(2e) = \Phi_0/(2\pi)$ are Wb=$\mathrm{kg\,m^2\,s^{-2}\,A^{-1}}$. Critical current is in A, resistance in Ω=$\mathrm{kg\,m^2\,s^{-3}\,A^{-2}}$. Therefore

$$[\tau_c] = \left[\frac{\hbar}{2e}\frac{1}{I_c}\frac{1}{R}\right] = \frac{\mathrm{kg\,m^2}}{\mathrm{s^2\,A}}\frac{1}{\mathrm{A}}\frac{\mathrm{s^3\,A^2}}{\mathrm{kg\,m^2}} = \mathrm{s} \tag{16}$$

$$\rightarrow [\tau] = \left[\frac{t}{\tau_c}\right] = \frac{\mathrm{s}}{\mathrm{s}} = 1 \tag{17}$$

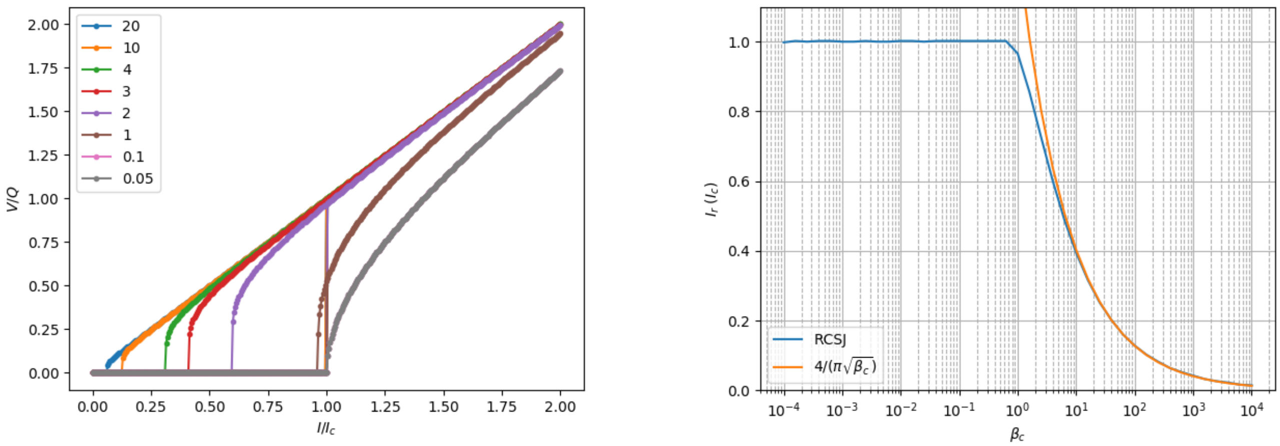

Figure 1. (a) Current-voltage curves for various damping cases. As predicted by theory, high damping ($Q \ll 1$) corresponds to a square-root like IVC with no hysteresis, while low damping ($Q \gg 1$) results in strong hystersis with a almost linear retrapping branch. (b) Hysteresis as a function of damping in terms of the Steeqart-McCumber parameter. For low damping, $\beta_c \gg 1$, the hysteresis can be approximated by $4/(\pi Q)$.

Now the same for the plasma frequency: Capacitance is given in units of $\mathrm{F}{=}\mathrm{s}^4\,\mathrm{A}^2\,\mathrm{m}^{-2}\,\mathrm{kg}^{-1}$

$$[\omega_p] = \left[\sqrt{\frac{2e}{\hbar}\frac{I_c}{C}}\right] = \sqrt{\frac{\mathrm{s}^2\,\mathrm{A}}{\mathrm{kg}\,\mathrm{m}^2}\mathrm{A}\frac{\mathrm{m}^2\,\mathrm{kg}}{\mathrm{s}^4\,\mathrm{A}^2}} = \sqrt{\frac{1}{\mathrm{s}^2}} = \frac{1}{\mathrm{s}} \tag{18}$$

$$\rightarrow [\tau] = [\omega_p t] = \frac{\mathrm{s}}{\mathrm{s}} = 1 \tag{19}$$

C. Damping and hysteresis

We consider eq. 15 for simplicity. For **very weak damping**, i.e. $\beta_c \gg 1$, the left hand side of the ODE can be set to zero. The result is a linear IVC on the retrapping branch, with $I_r \leq I_c$:

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} = 0 \rightarrow \langle V(t)\rangle = IR_n \tag{20}$$

$$\frac{I_r}{I_c} \approx \frac{4}{\pi\sqrt{\beta_c}} \equiv \frac{4}{\pi Q} \tag{21}$$

An open question is why Gross (and others) report significant hysteresis $I_r/I_c \approx 0.85$ already for $\beta_c = 1$, while we achieve at least $I_r/I_c = 0.95$ for the implemented python simulations.

For **very strong damping**, i.e. $\beta_c \ll 1$, the right hand side of the ODE is zero:

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = i - \sin(\phi) \rightarrow \langle V(t)\rangle = I_c R\sqrt{(I/I_c)^2 - 1} \tag{22}$$

without hysteresis. See also Fig. 1 for the resulting simulations of the two cases.

D. Frequency analysis

E. Notes on simulation

Some things I noted during simulations:

- The initial conditions for each iteration are extremely important!

- Currently (December 13, 2017), after letting the system evolve for a certain number of time steps (adjusted for each Q), the voltage is calculated by averaging over the last cycle. If there is no voltage cycle the voltage gets set to zero. The initial condition of the next run is the final state of the system from the previous run.
- Another option would be to set the maximum phase change of the previous run as initial condition. If we do this, the system jumps to the normal branch much earlier than at $i = 1$. In frequency space, this leads to fancy wave patterns.
- Probably the most correct approach would be to set the maximum phase change of the last cycle as initial condition. Or should it be the minimum? Does this have influence on ramping speed as well?

• Q versus β_c

- The package is now optimized for the use of Q for damping. Using β_c is finally also implemented, but not yet tested. **TODO!!**

• Subgap resistance

- **TODO!!**

II. VOLTAGE BIASING

The system equation only slightly changes. What remains to be seen is the normalization and the exact values needed to achieve voltage biasing.

REFERENCES

[1] M. Tinkham, *Introduction to Superconductivity: Second Edition (Dover Books on Physics) (Vol i)*, 2nd ed. (Dover Publications, 2004).
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[3] F. E. Schmidt, "RCSJ model," `https://github.com/feschmidt/rcsj` (2017).