Final Project: Portfolio Optimization using Quantitative Methods
UGST4090: Introduction to Optimization

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1 Introduction

A *portfolio* is a collection or grouping of financial assets owned by an individual, institution, or entity. These assets can include a wide range of investments such as stocks, bonds, mutual funds, exchange-traded funds (ETFs), real estate, commodities, cash equivalents, and other securities.

1.1 Purpose of Portfolios

Portfolios are essential in finance for several reasons:

- **Diversification**: Portfolios allow investors to spread their investments across different asset classes, industries, and securities, reducing the impact of any single investment's poor performance on the overall portfolio.
- **Risk Management**: By diversifying their holdings, investors can manage risk more effectively, as different assets may react differently to market conditions and economic events.
- **Liquidity Management**: Portfolios can be structured to maintain a balance between liquid and illiquid assets, ensuring investors have access to funds when needed while potentially earning higher returns on longer-term investments.

Instead of solely relying on one stock, it is a better practice to diversify investments to ensure a smoother positive growth with fewer chances of losing everything.

Portfolio optimization is a technique used in finance to construct investment portfolios that maximize returns or minimize risk, or achieve a balance between the two, given certain constraints. The goal is to allocate assets in such a way that the portfolio achieves the investor's objectives, whether it's maximizing return, minimizing risk, or achieving a specific target.

2 Methods and Approaches

For this work, we are considering five different data-driven approaches to portfolio optimization. All methods fall into the process of defining an objective function, minimization of which will bring to the desired allocation of assets within a portfolio. Optimization methods and objective functions are expressed in more detail as follows:

2.1 Mean Variance Optimization (Modern Portfolio Theory)

Mean Variance Optimization (MVO), also known as Modern Portfolio Theory (MPT), is a practical method for selecting investments to maximize their overall returns within an acceptable level of risk. This mathematical framework is used to build a portfolio of investments that maximize the expected return for a given level of risk.

The intuitive approach to program the problem is to define an objective function that minimizes risk given any level of returns. Standard deviation or variance of returns are measures of risk in finance, thus we aim to minimize them.

Mathematically, Mean Variance Optimization can be formulated as follows:

$$\min_{\mathbf{w}} \left(\mathbf{w}^T \Sigma \mathbf{w} \right)$$

subject to the constraint:

$$\mathbf{w}^T \mu = \mu_p$$

where:

- $\mathbf{w}$ is the vector of portfolio weights,
- Σ is the covariance matrix of asset returns,
- μ is the vector of expected returns,
- μ_p is the target expected portfolio return.

This optimization problem aims to minimize the portfolio's variance (risk) subject to achieving the target expected return μ_p.

2.2 Maximum Sharpe Portfolio

The Sharpe Ratio is a widely used measure of the risk-adjusted return of an investment. It calculates the excess return of an investment (above the risk-free rate) per unit of volatility (or risk).

The formula for the Sharpe Ratio is:

$$\text{Sharpe Ratio} = \frac{R_p - R_f}{\sigma_p}$$

where:

- R_p is the expected portfolio return,
- R_f is the risk-free rate,
- σ_p is the standard deviation (volatility) of the portfolio's returns.

The Maximum Sharpe Portfolio seeks to maximize the Sharpe Ratio, indicating the portfolio with the highest risk-adjusted return. Mathematically, it can be formulated as follows:

$$\max_{\mathbf{w}} \left(\frac{\mathbf{w}^T \mu - R_f}{\sqrt{\mathbf{w}^T \Sigma \mathbf{w}}} \right)$$

subject to the constraint:

$$\mathbf{w}^T \mathbf{1} = 1$$

where:

- $\mathbf{w}$ is the vector of portfolio weights,
- μ is the vector of expected returns,
- Σ is the covariance matrix of asset returns,
- R_f is the risk-free rate,
- $\mathbf{1}$ is a vector of ones (ensuring the weights sum up to 1).

This optimization problem aims to maximize the Sharpe Ratio, indicating the portfolio with the highest risk-adjusted return.

2.3 Global Minimum Variance Portfolio

Unlike the previously discussed methods, the Global Minimum Variance (GMV) approach does not require us to have explicit expected returns for the assets. This makes it particularly useful when expected returns are difficult to forecast accurately.

The GMV portfolio aims to allocate weights to assets in a way that minimizes the overall risk of the portfolio. It seeks to construct a portfolio that achieves the lowest possible variance (or standard deviation) of returns, regardless of the expected returns of individual assets.

Mathematically, the Global Minimum Variance Portfolio can be formulated as follows:

$$\min_{\mathbf{w}} \left(\mathbf{w}^T \Sigma \mathbf{w} \right)$$

subject to the constraint:

$$\mathbf{w}^T \mathbf{1} = 1$$

where:

- $\mathbf{w}$ is the vector of portfolio weights,
- Σ is the covariance matrix of asset returns,
- $\mathbf{1}$ is a vector of ones (ensuring the weights sum up to 1).

2.4 Risk Parity Portfolio

The Risk Parity Portfolio is an approach to portfolio construction that aims to equalize the risk contributions of each asset within the portfolio. This method focuses on balancing the risk exposure rather than the expected returns. It is particularly useful in ensuring that no single asset disproportionately affects the overall risk profile of the portfolio.

Similar to the Global Minimum Variance (GMV) approach, this method does not require expected returns, relying solely on a covariance matrix estimated from historical returns. Risk Parity is a well-renowned method that is widely used in investment management, especially for balancing risk across asset classes.

Mathematically, the Risk Parity Portfolio can be formulated as follows:

1. The risk contribution of an asset to the portfolio can be expressed as:

$$\mathrm{RC}_i = w_i \cdot \left(\frac{\partial \text{Portfolio Risk}}{\partial w_i} \right)$$

   where:

   - RC_i is the risk contribution of asset i,
   - w_i is the weight of asset i in the portfolio,
   - $\frac{\partial \text{Portfolio Risk}}{\partial w_i}$ is the partial derivative of the portfolio risk with respect to the weight of asset i.

2. To achieve equal risk contribution, we set up the following optimization problem:

$$\min_{\mathbf{w}} \sum_{i=1}^{n} \left(\mathrm{RC}_i - \frac{1}{n} \text{Portfolio Risk} \right)^2$$

   subject to the constraint:

$$\mathbf{w}^T \mathbf{1} = 1$$

   where:

- $\mathbf{w}$ is the vector of portfolio weights,
- n is the number of assets in the portfolio,
- $\mathbf{1}$ is a vector of ones (ensuring the weights sum up to 1).

2.5 Equally Weighted Portfolio

The Equally Weighted Portfolio involves no algorithm and simply assigns an equal weight to all assets in the portfolio.

This approach is often used as a benchmark to compare the performance of more sophisticated portfolio optimization models.

Mathematically, the Equally Weighted Portfolio can be represented as follows:

$$\mathbf{w} = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$$

where:

- $\mathbf{w}$ is the vector of portfolio weights,
- n is the total number of assets in the portfolio.

3 Optimization Algorithm for minimizing the negative of the portfolio objective: Sequential Quadratic Programming (SQP)

Sequential Quadratic Programming (SQP) is a widely used optimization algorithm for solving constrained nonlinear optimization problems, such as portfolio optimization. In this section, we present the mathematical formulation of the SQP algorithm for optimizing portfolios by minimizing the negative of the portfolio objective.

Let's consider the following mathematical representation of the portfolio optimization problem:

$$\begin{aligned}
\text{Minimize:} \quad & -R_p = -\sum_{i=1}^{n} w_i r_i \\
\text{Subject to:} \quad & \sum_{i=1}^{n} w_i = 1 \\
& w_i \geq 0 \quad \text{for } i = 1, 2, \ldots, n
\end{aligned}$$

Where:

$$\begin{aligned} R_p &: \text{Expected portfolio return} \\ w_i &: \text{Weight of asset } i \text{ in the portfolio} \\ r_i &: \text{Expected return of asset } i \\ n &: \text{Number of assets in the portfolio} \end{aligned}$$

The objective is to minimize the negative of the expected portfolio return R_p, subject to the constraints that the weights of the assets must sum to 1 and be non-negative.

To apply the SQP algorithm to this problem, we first define the Lagrangian function:

$$L(w, \lambda) = -R_p - \lambda \left(\sum_{i=1}^{n} w_i - 1 \right)$$

Where λ is the Lagrange multiplier associated with the constraint on the sum of the weights.

Next, we iteratively solve a sequence of quadratic subproblems to update the portfolio weights w and Lagrange multiplier λ. At each iteration, we approximate the objective function and constraints using quadratic models and then solve the subproblem to obtain a new iterate.

The SQP algorithm proceeds as follows:

1. Initialize the portfolio weights w and Lagrange multiplier λ.
2. At each iteration:
   (a) Compute the gradient and Hessian of the Lagrangian function with respect to w and λ.
   (b) Solve the quadratic programming subproblem to obtain a new iterate for w and λ.
   (c) Update the portfolio weights and Lagrange multiplier.
3. Repeat until convergence criteria are met.

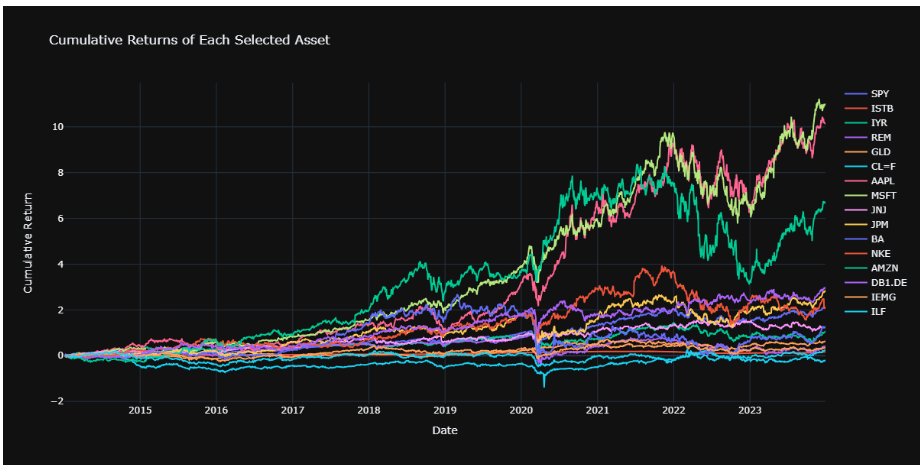
The SQP algorithm provides an efficient method for optimizing portfolios while satisfying the constraints on the weights of the assets. By iteratively updating the portfolio weights and Lagrange multiplier, the algorithm converges to a locally optimal solution that minimizes the negative of the expected portfolio return subject to the constraints.

4 Implementation and Backtest

We selected 16 assets across diverse markets and industries, including U.S. Equities, Fixed Income, Real Estate, Commodities, Emerging Markets, and European Equities.

Our choice aimed for a diversified pool of assets, and we avoided looklahead biases by simply doing random selections which resulted in the following assets:

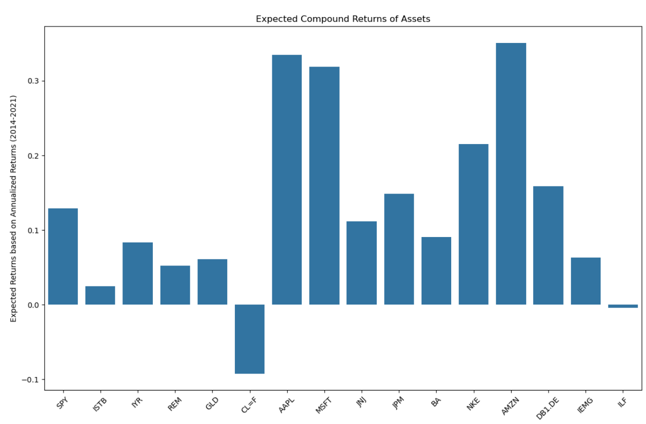
S&P 500 ETF Trust, iShares 1-5 Year USD Bond ETF, iShares U.S. Real Estate ETF, iShares Mortgage Real Estate ETF, SPDR Gold Shares, WTI Crude Oil, Apple Inc., Microsoft Corporation, Johnson & Johnson, JPMorgan Chase & Co., Boeing Company, Nike Inc., Amazon.com Inc., Deutsche Börse AG, iShares Core MSCI Emerging Markets ETF, and iShares Latin America 40 ETF.



As to individual assets performances over 2014-2024, we see that Microsoft Corporation (MSFT), Apple Inc. (AAPL) and Amazon.com Inc. (AMZN) have significantly outperformed the other assets, especially during the post covid-period. The pandemic accelerated the global shift towards digitalization, leading to increased demand for technology solutions, including cloud computing, personal devices, and e-commerce platforms. As businesses and consumers adapted to remote work and online learning, these companies saw a surge in demand for their products and services.

Our investment horizon spans from January 2021 to January 2024 (3 years). For estimating returns and obtaining the covariance matrix, we used historical returns data from 2014 to 2021. Given the extensive timeframe which includes the crisis period of COVID-19 as well as growth periods, we believe the timeframe effectively represents the overall behavior of the assets. Typically, expected returns are challenging to be accurately predicted, and there are some more advanced methodologies like Capital Asset Pricing Model (CAPM) and Dividend Discount Model (DDM) for forecasting them. However, for the sake of simplicity, we are sticking with historical returns as approximations of future returns.

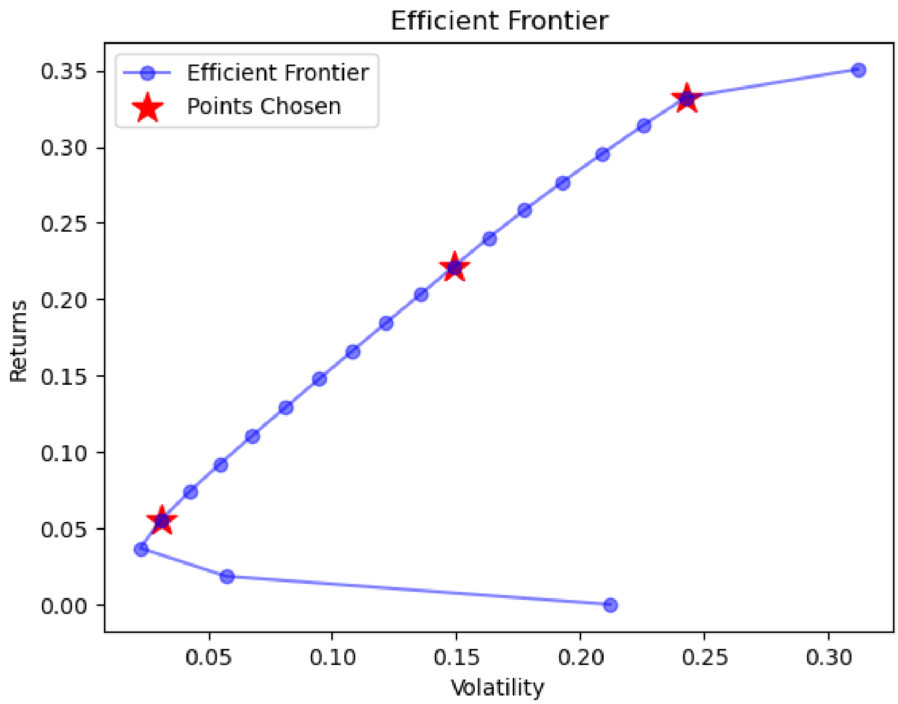
We can also observe that U.S Equities perform the best from the absolute returns generation perspective, however, as expected, they tend to be more volatile or risky as a downturn.



From the figure above, we can observe the expected returns of our assets approximated by historical means over the train period (2014-2021).

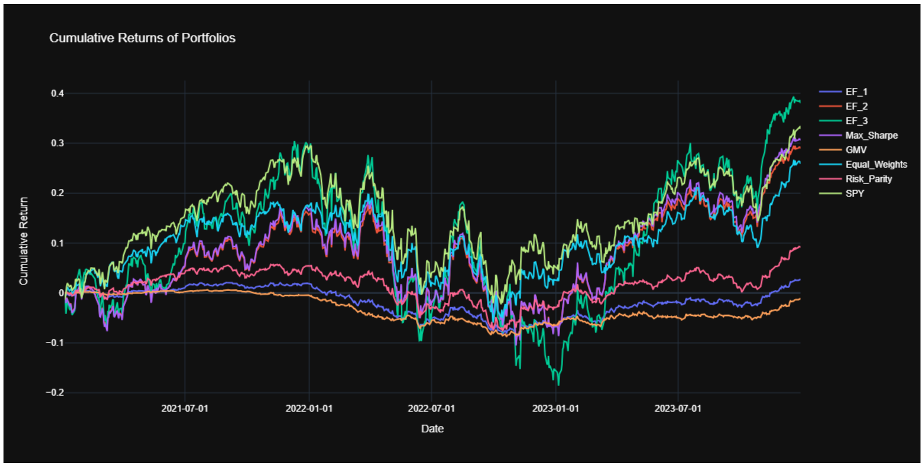
Now, we refer to actual performances of our strategies over the backtest period (2021-2024):

The efficient frontier is a result of a Mean Variance Optimization. Based on the given graph, we can find our optimal allocations: minimum possible risk for a given level of return.



Efficient Frontier contains a lot of optimal points but we decided to pick three sample points marked as red stars and take them as potential selections from the results of Mean Variance Optimization.

For other portfolio optimization strategies the solution is unique, therefore we can proceed to check the performances of our portfolios by plotting their cumulative returns over the test period:

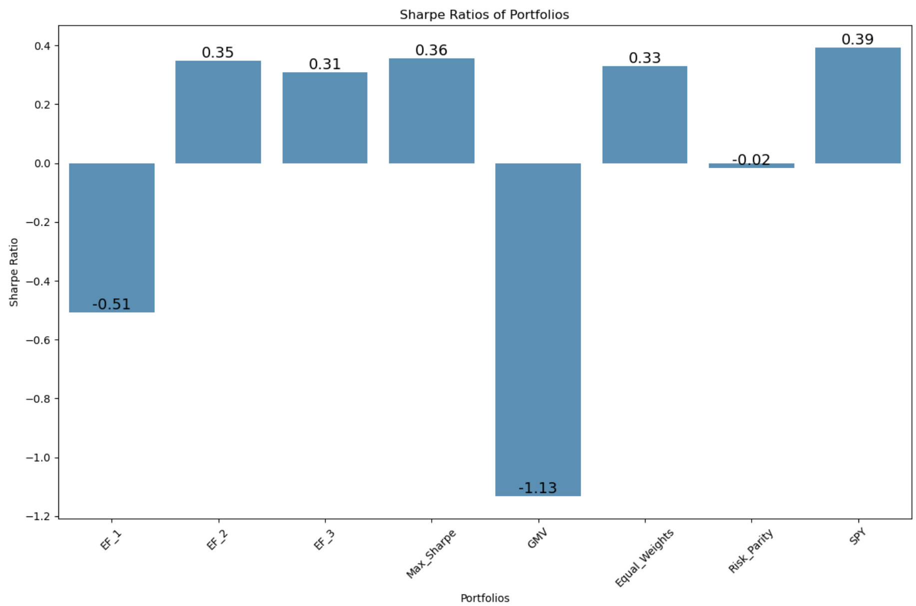


From the above plot, we see that the weights referring to the highest attainable return from the efficient frontier performed the best, achieving a 38.6 % return over three years, or an annualized return of 11%.

The backtest period encompasses the bullish market following COVID-19, hence, the strong performance of the Efficient Frontier Portfolio can be attributed to favorable market conditions and the economic recovery.

In contrast, the S&P 500 comes in second overall in terms of cumulative returns, with a 33.3% return over three years, or around 10% annualized. This strong performance of the market benchmark also reflects the favorable investment horizon, supported by monetary policies that stimulated growth as well as a broad economic recovery.

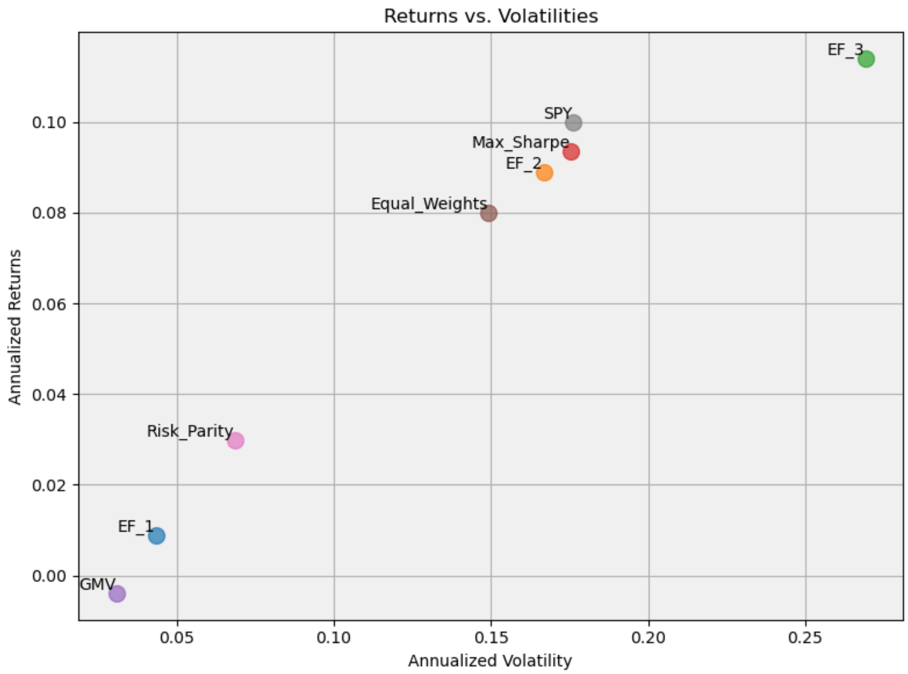
High returns are not always the best indicator of a good investment. It is important to consider risks associated with an investment. Therefore, we also compare Sharpe ratios, a measure of a risk-adjusted return, across our portfolios.



As to Sharpe ratios, we see that the S&P 500 has the highest ratio of 0.39 surpassing Maximum Sharpe and 2nd point from the Efficient Frontier by only a few points.

On the other hand, our most successful portfolio in terms of cumulative returns EF3 has a Sharpe ratio of 0.31, which is lower than the above mentioned three. This again, means that higher rewards come with greater risk and returns are not the only assessment metrics for investments.

Furthermore, we cannot ignore the strong performance of the Equal Weights portfolio, which just assigns equal weights to all assets within a portfolio. This again speaks about favorable market conditions, stimulating growth across assets over the backtest period.



Based on the plot above, which portrays volatilites of our portfolios against returns, we can notice that there is almost perfect positive linear relationship between risk and returns. This again proves the point that higher returns require risks.

The S&P 500 index and Equal Weights Portfolios are both commonly used as benchmarks for comparative analysis of portfolio performance. While one of our portfolios was able to outperform the market index in terms of returns, the S&P 500 had the highest Sharpe Ratio. This underscores the hypothesis of Efficient Markets: beating the market consistently is indeed a challenging task.

Quantitative approaches to portfolio selection, such as those we've explored, offer significant benefits but should not be used in isolation for decision-making. To achieve optimal results, these approaches should be complemented with qualitative and market analysis, stock picking, and more sophisticated methods for approximating future returns.

Below we plot returns performance of each portfolio alongside benchmarks: the Equal Weights portfolio and the S&P 500 to examine things closer.


EF_1 vs SPY vs Equal Weights
EF_1
SPY (Benchmark)
Equal Weights
Return
Date
Jul 2021
Jan 2022
Jul 2022
Jan 2023
Jul 2023


EF_2 vs SPY vs Equal Weights
EF_2
SPY (Benchmark)
Equal Weights
Return
Date
Jul 2021
Jan 2022
Jul 2022
Jan 2023
Jul 2023


EF_3 vs SPY vs Equal Weights
EF_3
SPY (Benchmark)
Equal Weights
0.4
0.3
0.2
0.1
0
−0.1
−0.2
Return
Jul 2021
Jan 2022
Jul 2022
Jan 2023
Jul 2023
Date


Max_Sharpe vs SPY vs Equal Weights
Max_Sharpe
SPY (Benchmark)
Equal Weights
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−0.05
−0.1
Return
Jul 2021
Jan 2022
Jul 2022
Jan 2023
Jul 2023
Date

From the figures above, where each portfolio is plotted alongside benchmarks, we see that our portfolios are struggling to outperform them. While the portfolio optimization techniques we tested are data-driven and free from bias or emotion, it is crucial to integrate them with proper qualitative and market analysis, stock picking, and more robust methods for estimating expected returns. The true power of these optimization methods can only be fully harnessed when used in conjunction with these additional strategies.

5 Conclusion

Mathematical optimization techniques find widespread applications across various industries. In this study, we have demonstrated their effectiveness in the realm of finance, specifically in the optimization of investment portfolios.

In conclusion, while portfolio optimization offers a structured and analytical approach to asset allocation, its effectiveness can be limited when applied in isolation. The financial markets are influenced by a myriad of factors, many of which are difficult to capture through quantitative models alone, especially over a long time period. To achieve consistent outperformance, it is essential to complement optimization techniques with comprehensive market research, qualitative insights, and advanced forecasting methods. By combining these approaches, investors can better navigate the complexities of the market and enhance the potential for superior returns. As financial markets continue to evolve, the integration of quantitative methods remains a valuable tool for optimizing investment portfolios and realizing financial goals.