Portfolio Optimization in Finance: Quantitative Approaches

Introduction to Optimization

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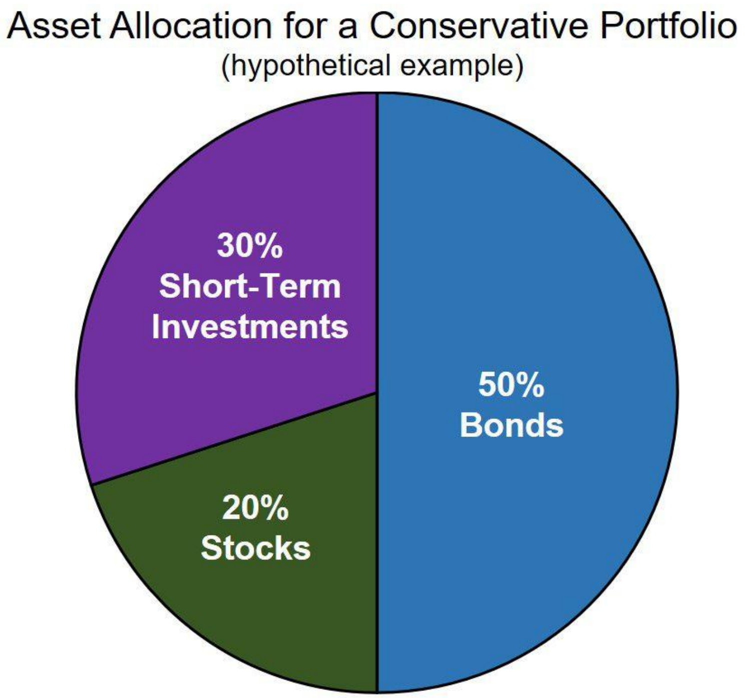
By: Chorobek Sheranov

What is a portfolio?

- A portfolio is a collection or grouping of financial assets owned by an individual, institution, or entity. These assets can include a wide range of investments such as stocks, bonds, mutual funds, exchange-traded funds (ETFs), real estate, commodities, cash equivalents, and other securities.



Portfolio Approach vs Single Asset Approach

Risk vs Return: Portfolio approach seeks to balance risk and return by diversifying investments, while single asset approach may offer higher potential returns but comes with higher risk.

Diversification Benefits: Portfolio approach provides diversification benefits, reducing unsystematic risk, whereas single asset approach lacks

Market Conditions: Market conditions and investor outlook may influence the preference for either approach. During periods of high uncertainty, diversification through portfolio approach may be favored, while in bullish markets, single asset approach may attract investors seeking higher returns.

Quantitative approach to Portfolio Optimization

- The quantitative approach to portfolio optimization involves using mathematical and statistical models to construct portfolios that maximize return or minimize risk based on predefined objectives, constraints, and historical data.
- Quantitative portfolio optimization relies heavily on historical data, including asset returns, volatility, correlations, and other relevant financial metrics.
- Quantitative models incorporate risk management techniques to control portfolio risk exposure and minimize downside volatility.

Quantitative Methods and Approaches

- **Mean Variance Optimization (MPT)**: Balances risk and return by minimizing portfolio volatility for a given level of return.
- **Maximum Sharpe Ratio**: Maximizes risk-adjusted return by optimizing portfolio allocation based on the Sharpe Ratio.
- **Global Minimum Variance**: Seeks to minimize portfolio volatility by allocating assets to achieve the lowest possible overall risk.
- **Risk Parity**: Aims to equalize risk contribution of assets to the overall portfolio risk

Method 1: Mean Variance Optimization

- Mean Variance Optimization (MVO), also known as Modern Portfolio Theory (MPT), is a practical method for selecting investments to maximize their overall returns within an acceptable level of risk. This mathematical framework is used to build a portfolio of investments that maximize the expected return for a given level of risk.
- The result is often described by a plot of an Efficient Frontier, which is a line showing optimal returns for a given level of risk.

Math behind Mean Variance Optimization

$$\min_{\mathbf{w}} \left(\mathbf{w}^T \Sigma \mathbf{w}\right)$$

subject to the constraint:

$$\mathbf{w}^T \mu = \mu_p$$

where:

- $\mathbf{w}$ is the vector of portfolio weights,
- Σ is the covariance matrix of asset returns,
- μ is the vector of expected returns,
- μ_p is the target expected portfolio return.

Method 2: Maximum Sharpe

- The Sharpe Ratio is a widely used measure of the risk-adjusted return of an investment. It calculates the excess return of an investment (above the risk-free rate) per unit of volatility (or risk).
- The mathematical formula is:

$$\text{Sharpe Ratio} = \frac{R_p - R_f}{\sigma_p}$$

- R_p is the expected portfolio return,
- R_f is the risk-free rate,
- σ_p is the standard deviation (volatility) of the portfolio's returns.

Math behind Maximum Sharpe Optimization

- The Maximum Sharpe Portfolio seeks to maximize the Sharpe Ratio, indicating the portfolio with the highest risk-adjusted return.

- Mathematically, it can be formulated as follows:

$$\max_{\mathbf{w}} \left(\frac{\mathbf{w}^T \mu - R_f}{\sqrt{\mathbf{w}^T \Sigma \mathbf{w}}} \right)$$

subject to the constraint:

$$\mathbf{w}^T \mathbf{1} = 1$$

where:

- $\mathbf{w}$ is the vector of portfolio weights,
- μ is the vector of expected returns,
- Σ is the covariance matrix of asset returns,
- R_f is the risk-free rate,
- $\mathbf{1}$ is a vector of ones (ensuring the weights sum up to 1).

Method 3: Global Minimum Variance

- Unlike the previously discussed methods, the Global Minimum Variance (GMV) approach does not require us to have explicit expected returns for the assets. This makes it particularly useful when expected returns are difficult to forecast accurately.
- The GMV portfolio aims to allocate weights to assets in a way that minimizes the overall risk of the portfolio.

- Objective function can be expressed as follows:

$$\min_{\mathbf{w}} \left(\mathbf{w}^T \Sigma \mathbf{w} \right)$$

subject to the constraint:

$$\mathbf{w}^T \mathbf{1} = 1$$

where:

- $\mathbf{w}$ is the vector of portfolio weights,
- Σ is the covariance matrix of asset returns,
- $\mathbf{1}$ is a vector of ones (ensuring the weights sum up to 1).

Method 4. Risk Parity Portfolio

- The Risk Parity Portfolio is a portfolio construction approach that aims to equalize the risk contributions of each asset. Widely used in investment management for balancing risk across asset classes, this method does not rely on expected returns.

- Mathematically, it can be expressed by an objective function:

$$\min_{\mathbf{w}} \sum_{i=1}^{n} \left(\mathrm{RC}_i - \frac{1}{n} \text{Portfolio Risk} \right)^2$$

subject to the constraint:

$$\mathbf{w}^T \mathbf{1} = 1$$

where:

-

$$\mathrm{RC}_i = w_i \cdot \left(\frac{\partial \text{Portfolio Risk}}{\partial w_i} \right)$$

is the Risk Contribution of an asset to the Portfolio Risk

- $\mathbf{w}$ is the vector of portfolio weights,
- n is the number of assets in the portfolio,
- $\mathbf{1}$ is a vector of ones (ensuring the weights sum up to 1).

Method 5: (Benchmark) Equally Weighted Portfolio

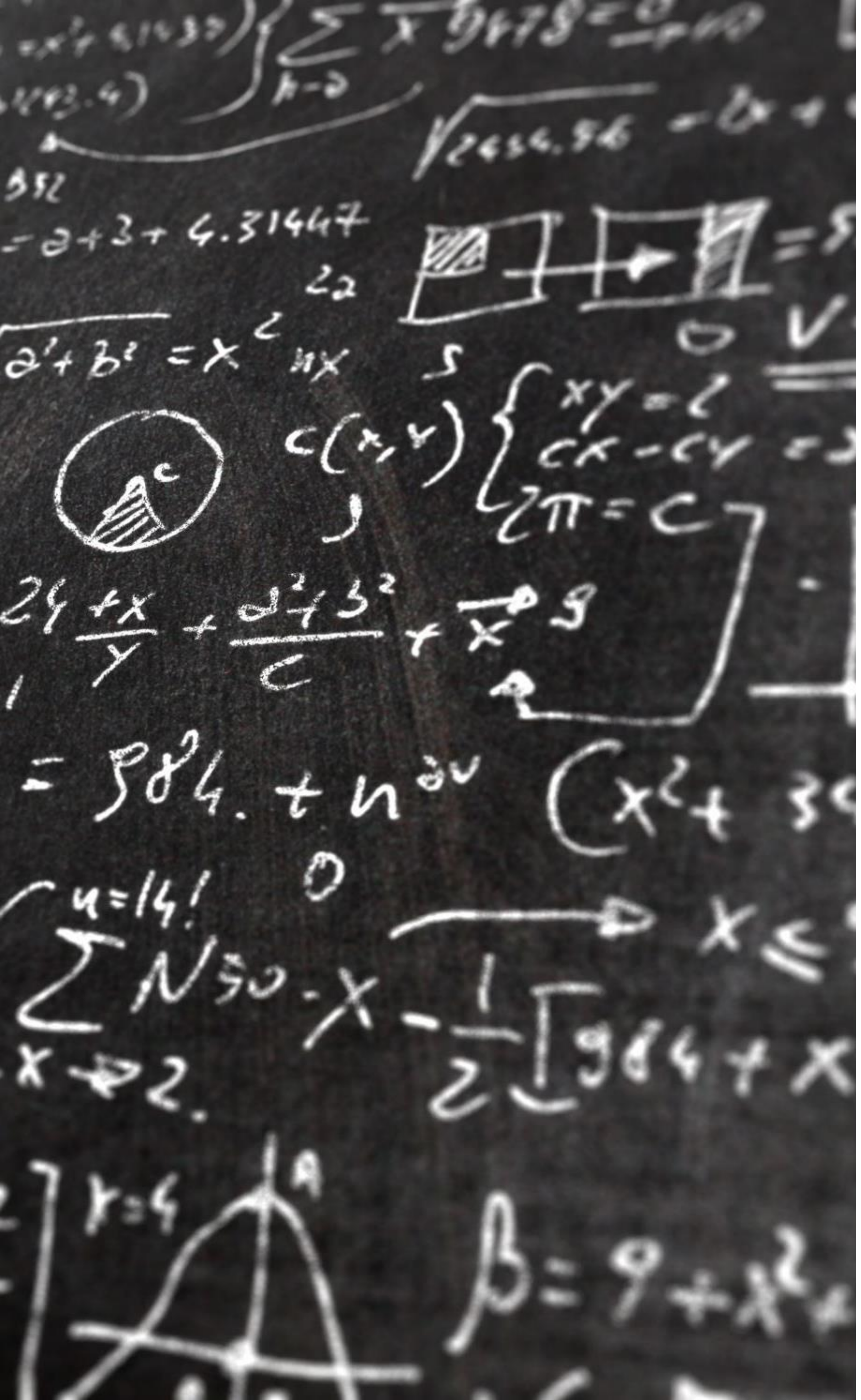
- The Equally Weighted Portfolio involves no algorithm and simply assigns an equal weight to all assets in the portfolio. This approach is often used as a benchmark to compare the performance of more sophisticated portfolio optimization models.

- Mathematically, the Equally Weighted Portfolio can be represented as follows:

$$\mathbf{w} = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right)$$

where:

- $\mathbf{w}$ is the vector of portfolio weights,
- n is the total number of assets in the portfolio.



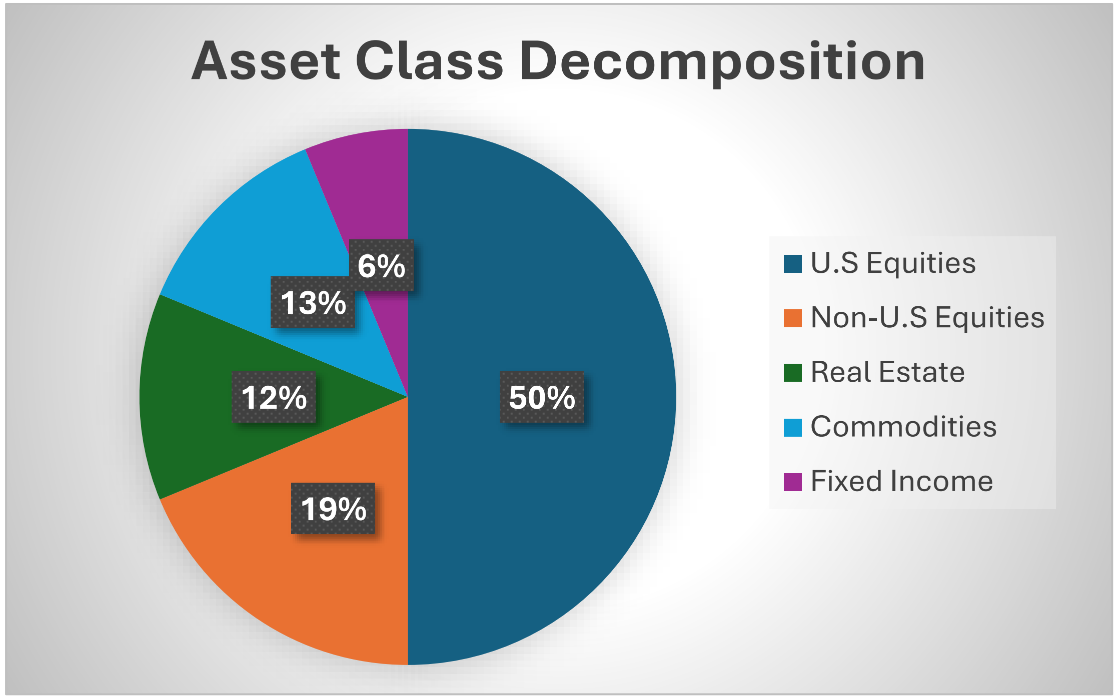
Mathematical Optimization Algorithm for Investment Objective Functions: Sequential Quadratic Programming (SQP)

- Sequential Quadratic Programming (SQP) is a widely used optimization algorithm for solving constrained nonlinear optimization problems, such as portfolio optimization.
- If the problem has only equality constraints (our case), then the method is equivalent to applying Newton’s method to the first-order optimality conditions of the problem.

Implementation: Portfolio Assets

Portfolio Assets:

- **U.S. Equities**: SPDR S&P 500 ETF Trust (SPY), Apple Inc. (AAPL), Microsoft Corporation (MSFT), Johnson & Johnson (JNJ), JPMorgan Chase & Co. (JPM), Boeing Company (BA), Nike Inc. (NKE), Amazon.com Inc. (AMZN).
- **Non-U.S. Equities**: iShares Core MSCI Emerging Markets ETF (IEMG), iShares Latin America 40 ETF (ILF), Deutsche Börse AG (DB1.DE).
- **Real Estate**: iShares U.S. Real Estate ETF (IYR), iShares Mortgage Real Estate ETF (REM).
- **Commodities**: SPDR Gold Shares (GLD), WTI Crude Oil (CL=F).
- **Fixed Income:** iShares 1-5 Year USD Bond ETF (ISTB),



Dataset

- Returns data during the January 1st 2014 – January 1st 2024 period
- Train set (Portfolio being optimized on): January 1st 2014 - January 1st 2021

Train set is a 7 year long dataset which incorporates historical behavior of assets, their performance during periods of growth as well as crisis (Covid - 19).

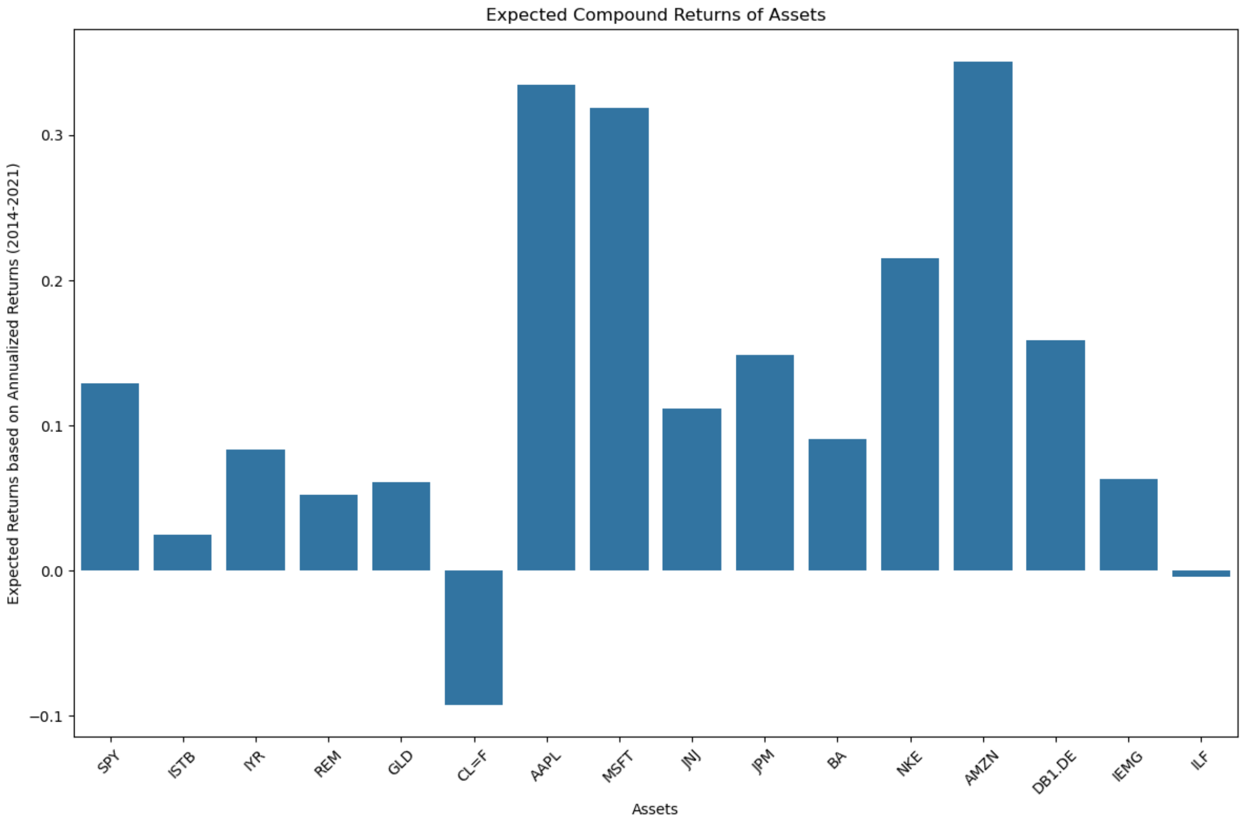
Backtest will be implemented on data from: January 1st 2021 – January 1st 2024.

Cumulative returns of Portfolio Assets

Cumulative Returns of Portfolio Classes of Selected Assets

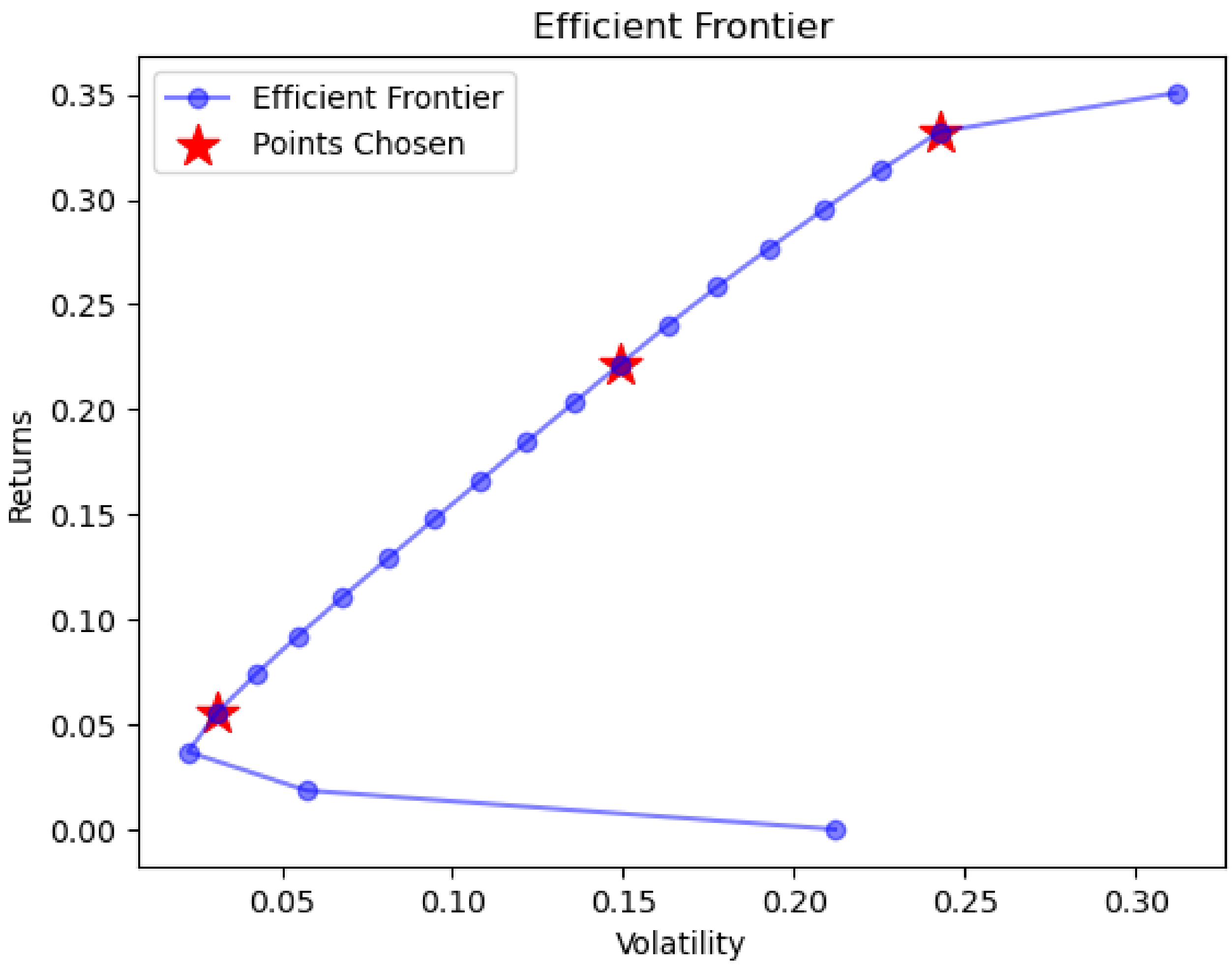
Approximation of Expected Returns

- As approximation, we took the annualized returns over the train set period: 2014 – 2021.
- It is a naïve and logical approach, but with the period covering a broad set of outcomes, the estimates should be more or less accurate.



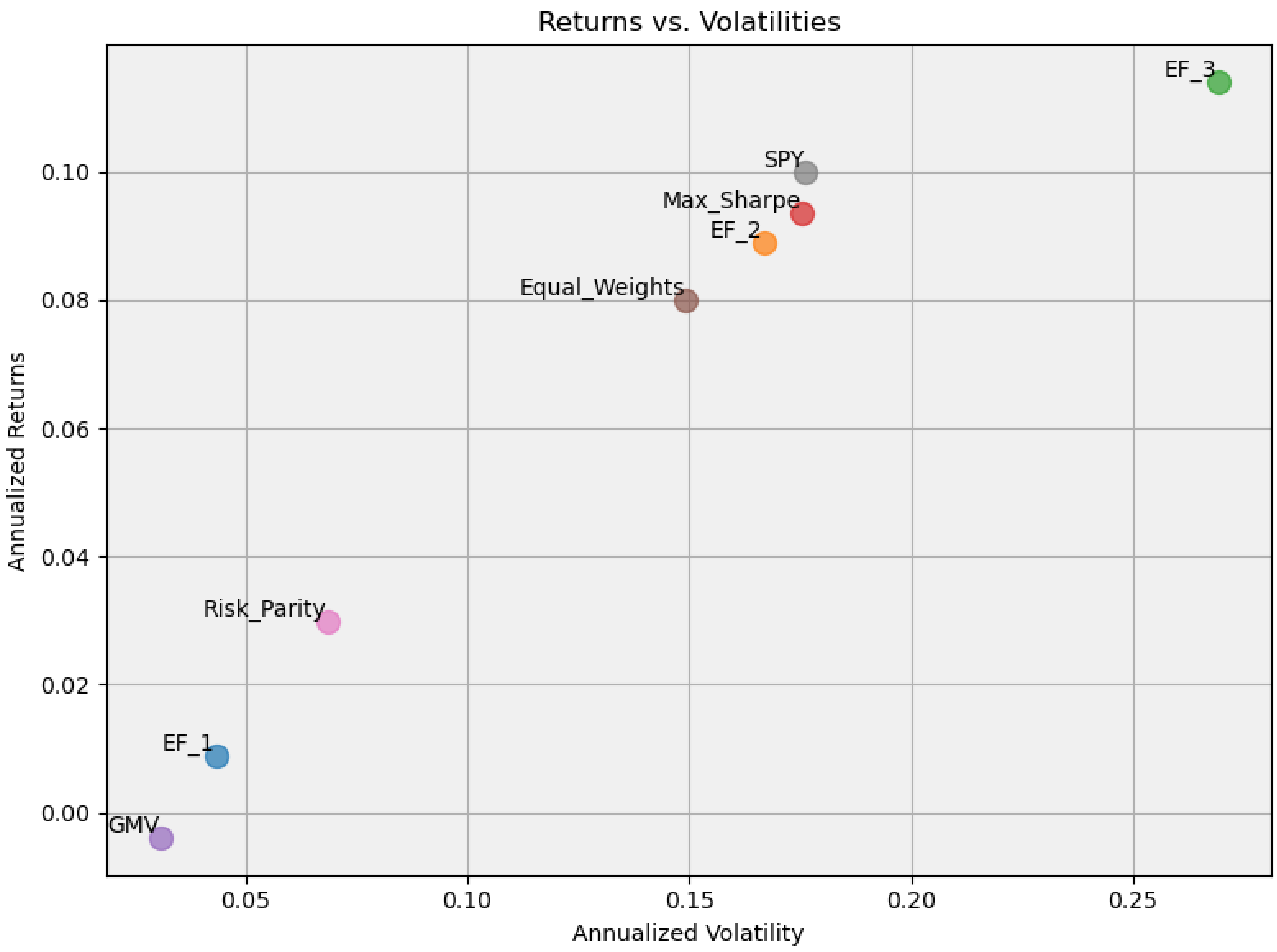
Optimization and Results: Efficient Frontier

- Efficient frontier, as mentioned, is the plot of an optimal line which is an answer to the question of minimal risk for a given level of return (Mean Variance Optimization).
- Taking all combinations of points and backtesting them would not make sense, and since we are imagining a real-world case, we pick three points which are highlighted by stars.

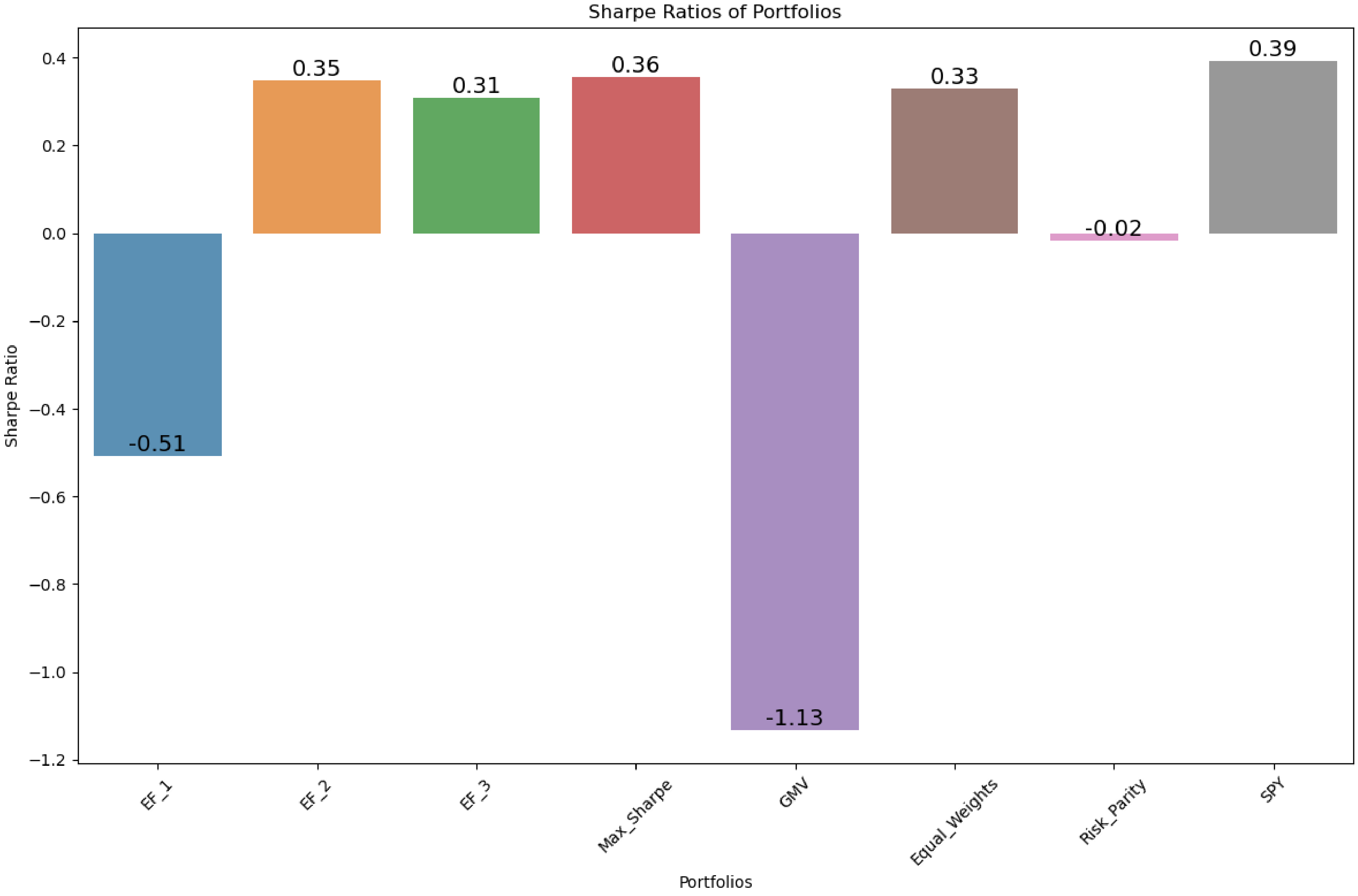


Cumulative Returns of Optimization Strategies over the backtest period

Evaluation: Returns and Volatilities of Optimization Startegies



Evaluation: Sharpe Ratios



Efficient Frontier Point 1

- EF_1 Total: **2.7 %**
- SPY Total: **32.9%**
- Equal Total: **25.9%**

Efficient Frontier Point 2:

- EF_2 Total: **29.0%**
- SPY Total: **32.9%**
- Equal Total: **25.9%**

Efficient Frontier Point 3

- EF_3 Total: **38.0%**
- SPY Total: **32.9%**
- Equal Total: **25.9%**

Maximum Sharpe Ratio Portfolio

- Max Sharpe Total: **30.6%**
- SPY Total: **32.9%**
- Equal Total: **25.9%**

Global Minimum Variance Portfolio

- GMV Total: **-1.1%**
- SPY Total: **32.9%**
- Equal Total: **25.9%**

Risk Parity Portfolio

- Risk Parity Total: **9.1%**
- SPY Total: **32.9%**
- Equal Total: **25.9%**

Conclusion

- Mathematical optimization techniques find widespread applications across various industries. In this study, we have demonstrated their effectiveness in the realm of finance, specifically in the optimization of investment portfolios.
- In conclusion, while portfolio optimization offers a structured and analytical approach to asset allocation, its effectiveness is limited when applied in isolation. The financial markets are influenced by a myriad of factors, many of which are difficult to capture through quantitative models alone, especially for longer periods.
- To achieve consistent outperformance, it is essential to complement optimization techniques with comprehensive market research, qualitative insights, and advanced forecasting methods.