MSBD5004 HOMEWORK 05

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1.

$$T\begin{bmatrix}8\\3\\7\end{bmatrix} = T\left(2\cdot\begin{bmatrix}1\\0\\-1\end{bmatrix} + 3\begin{bmatrix}2\\1\\3\end{bmatrix}\right)$$

since T is a linear transformation

$$= 2\cdot\left(T\begin{bmatrix}1\\0\\-1\end{bmatrix}\right) + 3\left(T\begin{bmatrix}2\\1\\3\end{bmatrix}\right)$$

$$= 2\cdot\begin{bmatrix}2\\3\end{bmatrix} + 3\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}1\\6\end{bmatrix}$$ ✓

2.

(a) Assume $A = [a_{ij}]_{m\times n} \in R^{m\times n}$, $i=1,\dots,m$, $j=1,\dots,n$.

the entry of A in i-th row and j-th colomn.

denote i-th entry of x (or b) as x_i, b_i., $x\in R^n$, $b\in R^m$

then $f(x) = Ax - b \in R^m$

$f_i(x)$ is the i-th entry of $f(x)$.

$$f_i(x) = \sum_{j=1}^{n} a_{ij}\cdot x_j - b_i$$

Thus, the Jacobian matrix we try to find:

$$J = \left[\frac{\partial f_i(x)}{\partial x_j}\right]_{m\times n} = \left[\frac{\partial\left(\sum_{k=1}^{n} a_{ik}\cdot x_k - b_i\right)}{\partial x_j}\right]_{m\times n} = [a_{ij}]_{m\times n} = A$$ ✓

$i=1,\dots,m,\quad j=1,\dots,n.$

(b). The entry in i-th row and j-th colomn of (xx^T) is:

$x_i x_j$, where x_i means i-th entry of x.

Then i-th enty of $f(x)$ is:

$$f_i(x) = \sum_{j=1}^{n}\left[(x_i x_j)\cdot a_j\right]$$

Thus, the entry in i-th row and j-th column of the Jacobian matrix of $f(x)$ is:

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$$J_{ij} = \frac{\partial f_i(x)}{\partial x_j} = \begin{cases} \dfrac{\partial\left(\sum_{k=1}^{n} x_i x_k a_k\right)}{\partial x_j} = x_i a_j, \; i \neq j \\[2ex] \dfrac{\partial\left(\sum_{k=1}^{n} x_i x_k a_k\right)}{\partial x_i} = 2a_i x_i + \sum_{k=1}^{i-1} x_k a_k + \sum_{k=i+1}^{n} x_k a_k \\ \quad = x_i a_i + \sum_{k=1}^{n} x_k a_k \\ \quad = x_i a_i + x^T a, \\ \quad i = j \end{cases}$$

Then, the Jacobian matrix is :

$$J = [J_{ij}]_{n \times n}, \quad \text{where } J_{ij} = \begin{cases} x_i a_j, & i \neq j \\ x_i a_i + x^T a, & i = j \end{cases}$$

3.

$$h(x,y) = f(g(x,y)) = f(x^2 y, x-y)$$

$$\left.\frac{\partial h}{\partial x}\right|_{\substack{x=1\\y=2}} = \left.\left[\frac{\partial f(x^2y, x-y)}{\partial (x^2 y)} \cdot \frac{\partial (x^2 y)}{\partial x} + \frac{\partial f(x^2y, x-y)}{\partial (x-y)} \cdot \frac{\partial (x-y)}{\partial x}\right]\right|_{\substack{x=1\\y=2}}$$

$$= \left(\left.\frac{\partial f}{\partial x}\right|_{\substack{x=2\\y=-1}}\right) \cdot \left(\left.2yx\right|_{\substack{x=1\\y=2}}\right) + \left(\left.\frac{\partial f}{\partial y}\right|_{\substack{x=2\\y=-1}}\right) \cdot \left.(1)\right|_{\substack{x=1\\y=2}}$$

$$= 3 \times 4 + (-2) \times 1 = 10$$

4.

First we prove that: $f_1(t) * f_2(t) = f_2(t) * f_1(t) = f(t)$ (1)

$$f(t) = f_1(t) * f_2(t) = \int_{-\infty}^{+\infty} f_1(t-s)\, f_2(s)\, ds \overset{\text{let } u=t-s}{\underset{\text{thus } s=t-u}{=}} \int_{+\infty}^{-\infty} f_1(u)\, f_2(t-u)(-du)$$

$$= \int_{-\infty}^{+\infty} f_1(u)\, f_2(t-u)\, du$$

$$= \int_{-\infty}^{+\infty} f_2(t-s)\, f_1(s)\, ds$$

$$= f_2(t) * f_1(t)$$ proved.

(i) Proof:

$$\underbrace{f_1(t-a) * f_2(t)}_{} \overset{(1)}{=} f_2(t) * f_1(t-a)$$

$$= \int_{-\infty}^{+\infty} f_2(t-s) \cdot f_1(s-a)\, ds$$

$$\overset{\text{let } u=s-a}{\underset{\text{then } s=u+a}{=}} \int_{-\infty}^{+\infty} f_2(t-(u+a)) \cdot f_1(u)\, du$$

$$= \int_{-\infty}^{+\infty} f_2((t-u)-a) \cdot f_1(u)\, du$$

$$= \int_{-\infty}^{+\infty} f_2((t-s)-a) \cdot f_1(s)\, ds$$

$$= f_2(t-a) * f_1(t) \overset{(1)}{=} f_1(t) * f_2(t-a)$$

$$f_1(t-a) * f_2(t) = \int_{-\infty}^{+\infty} f_1((t-s)-a) \cdot f_2(s)\, ds = \int_{-\infty}^{+\infty} f_1(\boxed{(t-a)}-s) \cdot f_2(s)\, ds = f(\boxed{t-a})$$

Altogether, we proved: $f_1(t-a) * f_2(t) = f_1(t) * f_2(t-a) = f(t-a)$

(ii) Proof: $f_1(t-a_1) * f_2(t-a_2) = \int_{-\infty}^{+\infty} f_1((t-s)-a_1)\, f_2(s-a_2)\, ds$

$$\overset{\text{let } u=s-a_2}{\underset{\text{then } s=u+a_2}{=}} \int_{-\infty}^{+\infty} f_1(t-u-a_2-a_1)\, f_2(u)\, du$$

$$= \int_{-\infty}^{+\infty} f_1((t-s)-a_2-a_1)\, f_2(s)\, ds$$

$$= f_1(t-a_1-a_2) * f_2(t) = f(t-a_1-a_2)$$

as we have proved in (i)

5. (a) Proof:

① Linear

Proof: $\langle x, S(\alpha y_1 + \beta y_2)\rangle_{V_1} = \langle Tx, \alpha y_1 + \beta y_2\rangle_{V_2}$

$= \alpha\langle Tx, y_1\rangle_{V_2} + \beta\langle Tx, y_2\rangle_{V_2}$

$= \alpha\langle x, Sy_1\rangle_{V_1} + \beta\langle x, Sy_2\rangle_{V_1}$

$= \langle x, \alpha Sy_1 + \beta Sy_2\rangle_{V_1}$ ✓

i.e. $S(\alpha y_1 + \beta y_2) = \alpha Sy_1 + \beta Sy_2$ for $\forall y_1, y_2 \in V_1$, $\forall \alpha, \beta \in R$.

i.e. S is a linear operator.

② bounded

Proof: $\|S\|^2 = \sup_{\|x\|_{V_2}=1} \|Sx\|_{V_1}^2 = \sup_{\|x\|_{V_2}=1} |\langle Sx, Sx\rangle_{V_1}|$

$= \sup_{\|x\|_{V_2}=1} |\langle TSx, x\rangle_{V_2}|$

Cauchy-Schwartz

$\leq \sup_{\|x\|_{V_2}=1} (\|TSx\|_{V_2} \cdot \|x\|_{V_2})$

$= \sup_{\|x\|_{V_2}=1} \|TSx\|_{V_2}$ (*)

since $T \in \mathcal{L}(V_1, V_2)$, $\|Ty\|_{V_2} < +\infty$ for $\forall y \in V_1$

Since $Sx \in V_1$, $(x \in V_2)$ then we have:

$\|S\|^2 \leq \sup_{\|x\|_{V_2}=1} \|TSx\|_{V_2} < +\infty$ ✓

i.e. $\|S\| < +\infty$

i.e. S is bounded.

Altogether, $S \in \mathcal{L}(V_2, V_1)$ i.e. S is a bounded linear operator.

(b) Proof:

As we have showed in (a):

$$\begin{cases} ① \ S \in \mathcal{L}(V_2, V_1) \\ ② \ \forall x \in V_1, y \in V_2, \ \langle Tx, y \rangle_{V_2} = \langle x, Sy \rangle_{V_1} \end{cases}$$

$\Rightarrow S = T^*$

$T \in \mathcal{L}(V_1, V_2)$

$$\begin{cases} ③ \ T \in \mathcal{L}(V_1, V_2) \\ ④ \ \langle Sy, x \rangle_{V_1} = \langle x, Sy \rangle_{V_1} = \langle Tx, y \rangle_{V_2} = \langle y, Tx \rangle_{V_2}, \end{cases}$$

for $\forall x \in V_1, y \in V_2$

③, ④ $\Rightarrow S^* = T$

Altogether, we have $(T^*)^* = S^* = T$. proved.

(c) Proof:

$\boxed{\text{Part 0:}}$ Prove: $\boxed{\|Tx\|_{V_2} \le \|T\| \cdot \|x\|_{V_1}, \ \forall x \in V_1}$ (1)

Proof: If $x = 0$, $\|T0\|_{V_2} = \|0\|_{V_2} = 0 = \|T\| \cdot \|0\|_{V_1}$

If $x \neq 0$, $\|T\| = \sup_{\substack{y \in V_1 \\ y \neq 0}} \frac{\|Ty\|_{V_2}}{\|y\|_{V_1}} \geq \frac{\|Tx\|_{V_2}}{\|x\|_{V_1}}$

$\Rightarrow \|Tx\|_{V_2} \le \|T\| \cdot \|x\|_{V_1}$

Altogether, $\|Tx\|_{V_2} \le \|T\| \cdot \|x\|_{V_1}$

Prove: $\boxed{\|Sx\|_{V_1} \le \|S\| \cdot \|x\|_{V_2}, \ \forall x \in V_2}$ (2)

Proof: If $x = 0$, $\|S0\|_{V_1} = \|0\|_{V_1} = 0 = \|S\| \cdot \|0\|_{V_2}$

If $x \neq 0$, $\|S\| = \sup_{\substack{y \in V_2 \\ y \neq 0}} \frac{\|Sy\|_{V_1}}{\|y\|_{V_2}} \geq \frac{\|Sx\|_{V_1}}{\|x\|_{V_2}} \Rightarrow \|Sx\|_{V_1} \le \|S\| \cdot \|x\|_{V_2}$

$\boxed{\text{Part 1:}}$ Prove: $\|S\| \le \|T\|$

Proof: As we have showed in 5(a)-(*),

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$$\|S\|^2 \le \sup_{\|x\|_{V_2}=1} \|TSx\|_{V_2} \underset{(1) \text{ in } \boxed{\text{Part 0}}}{\le} \sup_{\|x\|_{V_2}=1} \|T\| \cdot \|Sx\|_{V_1}$$

$$= \|T\| \cdot \sup_{\|x\|_{V_2}=1} \|Sx\|_{V_1}$$

$$= \|T\| \cdot \|S\|$$

If $\|S\|=0$, then $\|S\| = 0 \le \|T\|$ (since $\|T\| \ge 0$ as a norm)

If $\|S\| \ne 0$, then $\|S\|^2 \le \|T\| \cdot \|S\| \Rightarrow \|S\| \le \|T\|$

So we proved $\|S\| \le \|T\|$.

$\boxed{\text{Part 2:}}$ Prove: $\|T\| \le \|S\|$

Proof: $\|T\|^2 = \sup_{\|x\|_{V_1}=1} \|Tx\|_{V_2}^2 = \sup_{\|x\|_{V_1}=1} |\langle Tx, Tx \rangle_{V_2}|$

$$= \sup_{\|x\|_{V_1}=1} |\langle x, STx \rangle_{V_1}|$$

Cauchy-Schwartz

$$\le \sup_{\|x\|_{V_1}=1} \left(\|x\|_{V_1} \cdot \|STx\|_{V_1}\right)$$

$$= \sup_{\|x\|_{V_1}=1} \|STx\|_{V_1}$$

(2) in $\boxed{\text{Part 0}}$

$$\le \sup_{\|x\|_{V_1}=1} \left(\|S\| \cdot \|Tx\|_{V_2}\right)$$

$$= \|S\| \cdot \sup_{\|x\|_{V_1}=1} \|Tx\|_{V_2}$$

$$= \|S\| \cdot \|T\|$$

If $\|T\| = 0$, then $\|T\| = 0 \le \|S\|$ (Since $\|S\| \ge 0$ as a norm)

If $\|T\| \ne 0$, then $\|T\|^2 \le \|S\| \cdot \|T\| \Rightarrow \|T\| \le \|S\|$

So we proved $\|T\| \le \|S\|$.

Altogether, since $\|T\| \le \|S\|$ and $\|S\| \le \|T\|$, we can infer that $\|T\| = \|S\|$ i.e. $\|T\| = \|T^*\|$

6.

(a)

proof: for $\forall \alpha, \beta \in R$, $\forall \{x_n\}_{n\in N}, \{z_n\}_{n\in N} \in \ell_\infty$,

Denote $T(\alpha\{x_n\}_{n\in N} + \beta\{z_n\}_{n\in N})$ as $\{w_n\}_{n\in N}$

By definition of T, the n-th entry of $\{w_n\}$ $(n=0,1,\cdots)$ is:

$$w_n = \alpha x_{n+1} + \beta z_{n+1} \quad (*)$$

By definition of T, $\alpha T(\{x_n\}_{n\in N}) + \beta T(\{z_n\}_{n\in N})$

$$= \alpha\{y_n\}_{n\in N} + \beta\{v_n\}_{n\in N}, \text{ where } y_n = x_{n+1}, v_n = z_{n+1}$$

$$= \{\alpha y_n + \beta v_n\}_{n\in N}, \text{ where } y_n = x_{n+1}, v_n = z_{n+1}$$

$$\overset{(*)}{=} \{w_n\}_{n\in N}.$$

i.e. $T(\alpha\{x_n\}_{n\in N} + \beta\{z_n\}_{n\in N}) = \alpha T(\{x_n\}_{n\in N}) + \beta T(\{z_n\}_{n\in N})$

for $\forall \alpha, \beta \in R$,

$\forall \{x_n\}_{n\in N}, \{z_n\}_{n\in N} \in \ell_\infty$.

i.e T is a linear operator.

(b)

proof: $\forall \{x_n\}_{n\in N} \in \ell_\infty$,

by definition of T and $\|\cdot\|_\infty$

$$0 \le \|T(\{x_n\}_{n\in N})\|_\infty = \sup_{n\in N} |y_n| \quad (\text{where } y_n = x_{n+1})$$

$$= \sup_{n\in N} |x_{n+1}|$$

since $\{x_n\}_{n\in N} \in \ell_\infty$

$$\le \sup_{n\in N} |x_n| = \|\{x_n\}_{n\in N}\|_\infty < +\infty$$

i.e. $0 \le \|T(\{x_n\}_{n\in N})\|_\infty \overset{(*)}{\le} \|\{x_n\}_{n\in N}\|_\infty < +\infty$, and $\|T\| = \sup_{\|\{x_n\}_{n\in N}\|_\infty = 1} \|T(\{x_n\}_{n\in N})\|_\infty$

$\le \sup_{\|\{x_n\}_{n\in N}\|_\infty = 1} \|\{x_n\}_{n\in N}\|_\infty = 1$,

so T is a bounded operator.

(c)

Proof:

$$\|T\| = \sup_{\|\{x_n\}_{n\in N}\|_\infty = 1} \|T(\{x_n\}_{n\in N})\|_\infty \overset{(*)\text{ in } 6(b)}{\leq} \sup_{\|\{x_n\}_{n\in N}\|_\infty = 1} \|\{x_n\}_{n\in N}\|_\infty = 1 \quad ①$$

we choose a special $\{x_n\}_{n\in N} \in l_\infty$:

$x_n = 1$ for $\forall n \in N$.

Then $T(\{x_n\}_{n\in N}) = \{y_n\}_{n\in N}$, $y_n = x_{n+1}$,

so $y_n = 1$ for $\forall n \in N$. i.e. $\{y_n\}_{n\in N} = \{x_n\}_{n\in N}$.

so in this special case:

$$\|\{x_n\}_{n\in N}\|_\infty = \sup_{n\in N} |x_n| = 1$$

and $\|T(\{x_n\}_{n\in N})\|_\infty = \|\{y_n\}_{n\in N}\|_\infty = \|\{x_n\}_{n\in N}\|_\infty = 1$

so. we have found a special $\{x_n\}_{n\in N}$ such that the supremum in the inequality ① is attained.

so $\|T\| = 1$.