MSBD 5004 HOMEWORK 01

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1. (a) Proof: $1°\ \|\vec{x}\|_\infty = \max_{i\in[1,n]} |x_i| \geq 0$

You need to justify $\|x\|_\infty = 0 \Leftrightarrow X = 0$.

-2

$2°\ \forall \vec{x}\in R^n,\ \alpha\in R,$

$\|\alpha\vec{x}\|_\infty = \max_{i\in[1,n]} |\alpha x_i| = \max_{i\in[1,n]} |\alpha|\cdot|x_i| = |\alpha|\cdot\max_{i\in[1,n]}|x_i| = |\alpha|\cdot\|\vec{x}\|_\infty$

$3°\ \|\vec{x}+\vec{y}\|_\infty = \max_{i\in[1,n]} |x_i+y_i| \leq \max_{i\in[1,n]}|x_i| + \max_{j\in[1,n]}|y_j| = \|\vec{x}\|_\infty + \|\vec{y}\|_\infty$

Over all, $\|\vec{x}\|_\infty = \max_{1\leq i\leq n}|x_i|$ is indeed a norm on R^n.

(b) Proof: As $\forall i\in[1,n]\ (i\in N^+)$, $0\leq |x_i| \leq \max_{i\in[1,n]}|x_i|$

and as $\exists j\in[1,n]\ (j\in N^+)$, $x_j = \max_{i\in[1,n]}|x_i|$

so $\max_{i\in[1,n]}|x_i| \leq \sum_{i=1}^{n}|x_i| \leq \sum_{j=1}^{n}\left(\max_{i\in[1,n]}|x_i|\right) = n\cdot\max_{i\in[1,n]}|x_i|$

i.e. $\underline{\|\vec{x}\|_\infty \leq \|\vec{x}\|_1 \leq n\cdot\|\vec{x}\|_\infty}$

(c) Proof:

$$|\vec{x}^T\vec{y}| = \left|\sum_{i=1}^{n} x_i y_i\right| \leq \sum_{i=1}^{n}|x_i y_i| \leq \sum_{i=1}^{n}\left(|x_i|\cdot\max_{j\in[1,n]}|y_j|\right)$$

$$= \left(\sum_{i=1}^{n}|x_i|\right)\cdot\max_{j\in[1,n]}|y_j|$$

$$= \|\vec{x}\|_1 \cdot \|\vec{y}\|_\infty$$

2. (a) Proof: (I denote the reason "as $\|A\vec{x}\|_2$ is the 2-norm on R^m" as "①")

$1°\ \|A\|_2 = \max_{\vec{x}\in R^n,\|\vec{x}\|_2=1}\|A\vec{x}\|_2 \overset{①}{\geq} 0$

You need to justify $\|A\|_2 = 0 \Leftrightarrow A=0$

-3

$2°\ \|\alpha A\|_2 = \max_{\vec{x}\in R^n,\|\vec{x}\|_2=1}\|(\alpha A)\vec{x}\|_2 = \max_{\vec{x}\in R^n,\|\vec{x}\|_2=1}\|\alpha(A\vec{x})\|_2$

$\overset{①}{=} \max_{\vec{x}\in R^n,\|\vec{x}\|_2=1} \alpha\cdot\|A\vec{x}\|_2$

$= \alpha\cdot\|A\|_2$

$3°$ Assume $B^{m\times n}$,

thus $\|A+B\|_2 = \max_{\vec{x}\in R^n,\|\vec{x}\|_2=1}\|(A+B)\vec{x}\|_2 = \max_{\vec{x}\in R^n,\|\vec{x}\|_2=1}\|A\vec{x}+B\vec{x}\|_2$

$\overset{①}{\leq} \max_{\vec{x}\in R^n,\|\vec{x}\|_2=1}\left(\|A\vec{x}\|_2 + \|B\vec{x}\|_2\right)$

$\leq \max_{\vec{x}\in R^n,\|\vec{x}\|_2=1}\|A\vec{x}\|_2 + \max_{\vec{y}\in R^n,\|\vec{y}\|_2=1}\|B\vec{y}\|_2$

$= \|A\|_2 + \|B\|_2$

Over all, $\|\cdot\|_2$ is indeed a norm on $R^{m\times n}$.

(b) Proof: $\forall \vec{x} \in R^n$, $\vec{x} = \|\vec{x}\|_2 \cdot \vec{\alpha}$ where $\|\vec{\alpha}\|_2 = 1$ and $\|\vec{x}\|_2 \in R$, $\|\vec{x}\|_2 \geq 0$

thus, $\|A\vec{x}\|_2 = \| A \cdot \|\vec{x}\|_2 \vec{\alpha} \|_2 = \| \|\vec{x}\|_2 \cdot (A\vec{\alpha}) \|_2 = \|\vec{x}\|_2 \cdot \|A\vec{\alpha}\|_2$

$\leq \|\vec{x}\|_2 \cdot \max_{\vec{y} \in R^n, \|\vec{y}\|_2 = 1} \|A\vec{y}\|_2$

$= \|\vec{x}\|_2 \cdot \|A\|_2$

(c) Proof: $\|AB\|_2 = \max_{\vec{x} \in R^p, \|\vec{x}\|_2 = 1} \|(AB)\vec{x}\|_2 = \max_{\vec{x} \in R^p, \|\vec{x}\|_2 = 1} \|A(B\vec{x})\|_2$

$\leq \|A\|_2 \cdot \|B\vec{x}\|_2$, $\vec{x} \in R^p$ and $\|\vec{x}\|_2 = 1$ (as we proved in (b).)

$\leq \|A\|_2 \cdot \|B\|_2 \cdot \|\vec{x}\|_2$

$= \|A\|_2 \cdot \|B\|_2$

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3. Proof:

Part 1: Firstly prove: $\|A\|_2 = \max_{\|\vec{x}\|_2 = 1} \|A\vec{x}\|_2 = \sqrt{\lambda_{max}(A^TA)}$

Given a real symmetric matrix $P^{n\times n}$, there must be an orthogonal matrix O, s.t.

$$O^TPO = O^{-1}PO = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix},$$ where λ_i is a eigenvalue of P and columns of O are eigenvectors of corresponding eigenvalues.

Thus we can prove:

$$\max_{\|\vec{x}\|_2 = 1} \vec{x}^TP\vec{x} \xlongequal[\text{thus } \vec{x} = O\vec{y} \text{ and } \|\vec{y}\|_2 = 1]{\text{let } \vec{y} = O^T\vec{x} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}} \max_{\|\vec{y}\|_2 = 1} \vec{y}^T(O^TPO)\vec{y}$$

$$= \max_{\|\vec{y}\|_2 = 1} \vec{y}^T \Lambda \vec{y}$$

$$= \max_{\|\vec{y}\|_2 = 1} \sum_{i=1}^{n} (\lambda_i \cdot y_i^2)$$

$$\leq \lambda_{max} \cdot \max_{\|\vec{y}\|_2 = 1} \sum_{i=1}^{n} y_i^2$$

$$= \lambda_{max}$$ (λ_{max} is the largest eigenvalue of P)

Therefore, when $\vec{x}$ is an eigenvector of λ_{max}, $\max_{\|\vec{x}\|_2 = 1} \vec{x}^T P \vec{x} = \lambda_{max}$.

As A^TA is a real symmetric matrix from $R^{n\times n}$,

$$\sqrt{\lambda_{max}(A^TA)} = \sqrt{\max_{\|\vec{x}\|_2 = 1} \vec{x}^T(A^TA)\vec{x}} = \sqrt{\max_{\|\vec{x}\|_2 = 1} (A\vec{x})^T(A\vec{x})} = \sqrt{\max_{\|\vec{x}\|_2 = 1} \|A\vec{x}\|_2^2}$$

$$= \max_{\|\vec{x}\|_2 = 1} \|A\vec{x}\|_2 = \|A\|_2$$

Part 2: Through the calculation of A^TA, it's obvious that $\|A\|_F = \sqrt{tr(A^TA)}$

thus $\|A\|_F = \sqrt{tr(A^TA)} = \sqrt{\sum_{i=1}^{n} \lambda_i}$, where each of λ_i is an eigenvalue of A^TA.

since $\lambda_{max}(A^TA) \leq \sum_{i=1}^{n} \lambda_i \leq n \cdot \lambda_{max}(A^TA)$

thus $\sqrt{\lambda_{max}(A^TA)} \leq \sqrt{\sum_{i=1}^{n} \lambda_i} \leq \sqrt{n \cdot \lambda_{max}(A^TA)}$

i.e. $\|A\|_2 \leq \|A\|_F \leq \sqrt{n} \cdot \|A\|_2$

4. (a)

$$M_n \vec{d} = \underbrace{\begin{bmatrix} a_n \\ \vdots \\ a_n \end{bmatrix}}_{n\text{-dim col-vector}}$$

$d^T M_n$?

-2

$$= a_n \cdot \vec{d},$$

thus a_n is an eigenvalue of M_n and $\vec{d}$ is an eigenvector of $\lambda = a_n$.

(b) Assume $M_n = [m_{ij}]$, $M_n^2 = [c_{ij}]$

thus $c_{ij} = \sum_{k=1}^{n} m_{ik} m_{kj}$

for any row of M_n^2, the sum of the row is:

$$\sum_{j=1}^{n}\left(\sum_{k=1}^{n} m_{ik} m_{kj}\right) = \sum_{k=1}^{n}\left(m_{ik} \cdot \sum_{j=1}^{n} m_{kj}\right)$$
$$= \sum_{k=1}^{n}(m_{ik} \cdot a_n)$$
$$= a_n \cdot a_n = a_n^2$$

Similarly, for any column of M_n^2, the sum of the column is

$$\sum_{i=1}^{n}\left(\sum_{k=1}^{n} m_{ik} m_{kj}\right) = \sum_{k=1}^{n}\left(m_{kj} \cdot \sum_{i=1}^{n} m_{ik}\right)$$
$$= \sum_{k=1}^{n}(m_{kj} \cdot a_n) = a_n^2$$

Proved.

(c) Proof: According to Schur decomposition, $\forall A \in R^{n\times n}$, A can be decomposed as $A = QHQ^T$, where Q is an orthogonal matrix and H is a triangular matrix.

As $M_n \in R^{n\times n}$, thus we assume $M_n = QHQ^T = QHQ^{-1}$, where $Q^{-1} = Q^T$ and H is triangular.

so $H = Q^T M_n Q \quad (1)$, we assume that $H = [h_{ij}]$, $Q = [q_{ij}]$, $M_n = [m_{ij}]$, $Q^T M_n = [z_{ij}]$

by calculating (1), we have

$$h_{ij} = \sum_{l=1}^{n} z_{il} \cdot q_{lj}$$
$$= \sum_{l=1}^{n}\left[q_{lj} \cdot \left(\sum_{k=1}^{n} q_{ki} \cdot m_{kl}\right)\right]$$
$$= \sum_{l=1}^{n}\sum_{k=1}^{n}(q_{lj} \cdot q_{ki} \cdot m_{kl})$$

thus, $|h_{ij}| = \left| \sum_{l=1}^{n} \sum_{k=1}^{n} (q_{lj} \cdot q_{ki} \cdot m_{kl}) \right|$

$$\leq \sum_{l} \sum_{k} \left(m_{kl} \cdot |q_{lj} \cdot q_{ki}| \right) \quad (m_{kl} > 0)$$

$$\leq \frac{1}{2} \sum_{l} \sum_{k} \left[m_{kl} \cdot (q_{lj}^2 + q_{ki}^2) \right]$$

$$= \frac{1}{2} \left[\left(\sum_{l} q_{lj}^2 \cdot \sum_{k} m_{kl} \right) + \left(\sum_{k} q_{ki}^2 \cdot \sum_{k} m_{kl} \right) \right]$$

$$= \frac{1}{2} \left[\left(\sum_{l} q_{lj}^2 \cdot a_n \right) + \left(\sum_{k} q_{ki}^2 \cdot a_n \right) \right]$$

i.e. $|h_{ij}| \leq a_n$

$$= \frac{a_n}{2} \left(\sum_{l} q_{lj}^2 + \sum_{k} q_{ki}^2 \right) = \frac{a_n}{2} \times (1+1) = a_n \text{ --- denoted as } (2)$$

Because $M_n \sim H$, so M_n and H share the same eigenvalues, and as H is triangular, diagonal entries of H are eigenvalues of H.

So diagonal entries of H are eigenvalues of M_n.

As we have proved in (2), all entries of $H \leq a_n$, and as we showed in problem (a), a_n is an eigenvalue of M_n, thus we can infer that a_n is the largest eigenvalue of M_n.

As we proved in question 3, we have $\|M_n\|_2 = \sqrt{\lambda_{max}(M_n^T M_n)}$

$$= \sigma_{max}(M_n) \quad \text{why ?}$$

largest eigenvalue

$$= a_n.$$

5. Proof: Order the m numbers from smallest to largest, and denote them as $a_{s_1}, a_{s_2}, \cdots, a_{s_m}$.

smallest   largest

Part 1: If we assume the number which minimizes this summation is in the range of $(-\infty, a_{s_1}) \cup (a_{s_m}, +\infty)$, and denote the summation in this situation as $d \Rightarrow$ Then we can always find that the 2 endpoints of the interval $[a_{s_1}, a_{s_m}]$, i.e. a_{s_1} & a_{s_m}, can make the summation smaller than d.

Thus, we can deny the assumption above and infer that the number we try to find is in the range of $[a_{s_1}, a_{s_m}]$.

Part 2: When the target number is in $[a_{s_1}, a_{s_m}]$, the summation can be expressed as:

$$\sum_{i=1}^{m} |a_i - b| = \sum_{i=1}^{m} |a_{s_i} - b| = \left(|a_{s_1} - b| + |a_{s_m} - b| \right) + \sum_{i=2}^{m-1} |a_{s_i} - b|$$

$$= (a_{s_m} - a_{s_1}) + \sum_{i=2}^{m-1} |a_{s_i} - b|, \quad (b \in [a_{s_1}, a_{s_m}])$$

where $(a_{sm} - a_{s1})$ is constant and the optimization problem is changed into minimizing the summation : $\sum_{i=2}^{m-1} |a_{si} - b|$

Part 3: Exactly the same as the discussion above, we can infer that the target number is not in the range of $[a_{s1}, a_{s2}) \cup (a_{sm-1}, a_{sn}]$, i.e. we narrow the potienal interval into $[a_{s2}, a_{sm-1}]$..

If we do this procedure iteratively, we can finally determine that the number we want is located in the range denoted as T:

$$T = \begin{cases} [a_{s,\frac{m}{2}},\ a_{s,\frac{m}{2}+1}], & \text{if } m \text{ is even} \\ [a_{s,\lfloor\frac{m}{2}\rfloor},\ a_{s,\lceil\frac{m}{2}\rceil+1}], & \text{if } m \text{ is odd} \end{cases}$$

Part 4:

1° if m is even,

$\forall b \in T$, b can minimize $\sum_{i=1}^{m} |a_i - b|$ as $\sum_{i=1}^{\frac{m}{2}} (a_{s,m+1-i} - a_{si})$

let $b = \dfrac{a_{s,\frac{m}{2}} + a_{s,\frac{m}{2}+1}}{2}$ i.e. b is the median.

2° if m is odd,

$$\forall b \in T, \quad \sum_{i=1}^{m} |a_i - b| = \underbrace{\sum_{i=1}^{\lfloor\frac{m}{2}\rfloor} (a_{s,m+1-i} - a_{si})}_{\text{constant}} + |a_{s,\lfloor\frac{m}{2}\rfloor+1} - b|$$

obviously if we let $b = a_{s,\lfloor\frac{m}{2}\rfloor+1}$ i.e. b is the median, then the summation is minimized.

Over all, a median of $a_1, a_2, \cdots a_m$ minimizes $\sum_{i=1}^{m} |a_i - b|$ over all $b \in \mathbb{R}$.

6. (a) Because in the K-means algorithm, we calculate $\vec{z_j}$ using

$$\begin{pmatrix} z_{j1} \\ z_{j2} \\ \vdots \\ z_{jn} \end{pmatrix} = \frac{1}{|G_j|} \cdot \sum_{i \in G_j} \begin{pmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{in} \end{pmatrix}$$

since $x_{i1}, x_{i2}, \cdots, x_{in}$ are all nonnegative and $|G_j| \geq 1$,

we can know $z_{j1}, z_{j2}, \cdots, z_{jn}$ are all nonnegative, i.e. all $\vec{z_j}$ are also nonnegative.

(b) Suppose $\vec{x_i} = \begin{pmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{in} \end{pmatrix}$ where $\sum_{k=1}^{n} x_{ik} = 1$

since $\vec{z_j} = \frac{1}{|G_j|} \sum_{i \in G_j} \vec{x_i} = \begin{pmatrix} z_{j1} \\ z_{j2} \\ \vdots \\ z_{jn} \end{pmatrix}$

thus $z_{jk} = \frac{1}{|G_j|} \sum_{i \in G_j} x_{ik} \quad (k = 1, 2, \cdots, n)$

thus $\sum_{k=1}^{n} z_{jk} = \sum_{k=1}^{n} \left(\frac{1}{|G_j|} \sum_{i \in G_j} x_{ik} \right) = \frac{1}{|G_j|} \left(\sum_{i \in G_j} \sum_{k=1}^{n} x_{ik} \right)$

$$= \frac{1}{|G_j|} \left(\sum_{i \in G_j} 1 \right)$$

$$= \frac{1}{|G_j|} \cdot |G_j| = 1.$$

As we have explained in problem (a), because $\vec{x_i}$ are nonnegative, $\vec{z_j}$ are also nonnegative.

In conclusion, all $\vec{z_j}$ are also proportions.

(c) $(\vec{z_j})_i$ means the proportion of vectors which has a value of 1 in i-th entry in all vectors of Group j.