MSBD 5004 HOMEWORK 02

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1.

(a) Proof: ① As definition of a norm,

$$\|\vec{x}\| = \|\vec{y}+(\vec{x}-\vec{y})\| \leq \|\vec{y}\| + \|\vec{x}-\vec{y}\|$$

thus, $\|\vec{x}\|-\|\vec{y}\| \leq \|\vec{x}-\vec{y}\|$ (1)

② Similarly,

$$\|\vec{y}\| = \|\vec{x}+(\vec{y}-\vec{x})\| \leq \|\vec{x}\| + \|\vec{y}-\vec{x}\| = \|\vec{x}\| + |-1|\cdot\|\vec{x}-\vec{y}\|$$
$$= \|\vec{x}\| + \|\vec{x}-\vec{y}\|$$

thus, $\|\vec{y}\|-\|\vec{x}\| \leq \|\vec{x}-\vec{y}\|$

i.e. $\|\vec{x}\|-\|\vec{y}\| \geq -\|\vec{x}-\vec{y}\|$ (2)

combine the 2 inequation (1)&(2), i.e. $-\|\vec{x}-\vec{y}\| \leq \|\vec{x}\|-\|\vec{y}\| \leq \|\vec{x}-\vec{y}\|$

i.e. $\big|\|\vec{x}\|-\|\vec{y}\|\big| \leq \|\vec{x}-\vec{y}\|$ (As $\|\vec{a}\| \geq 0$, $\forall \vec{a} \in V$)

(b) Proof:

As definition of a convergent sequence, $\lim_{k\to\infty} \|\vec{x}_k - \vec{x}\| = \vec{0}$ (*)

As we proved in Part (a)

thus when $k\to\infty$, $0 \leq \big|\|\vec{x}_k\| - \|\vec{x}\|\big| \leq \|\vec{x}_k - \vec{x}\| \xrightarrow{(*)} 0$

By the property of absent value

According to the Squeeze Theorem, $\lim_{k\to\infty} \big|\|\vec{x}_k\| - \|\vec{x}\|\big| = 0$

$$\Leftrightarrow \lim_{k\to\infty} \|\vec{x}_k\| - \|\vec{x}\| = 0$$

$$\Leftrightarrow \lim_{k\to\infty} \|\vec{x}_k\| = \|\vec{x}\|$$

2. Proof:

① $\forall \vec{u} = u_1\vec{a}_1 + u_2\vec{a}_2 + \cdots + u_n\vec{a}_n \in V$,

$\langle \vec{u},\vec{u} \rangle = u_1^2 + u_2^2 + \cdots + u_n^2 \geq 0$, and obviously the equality holds if and only if $u_1 = u_2 = \cdots = u_n = 0$

i.e. $\vec{u} = \vec{0}$.

② $\forall \vec{u} = u_1\vec{a}_1 + u_2\vec{a}_2 + \cdots + u_n\vec{a}_n \in V$

$\forall \vec{v} = v_1\vec{a}_1 + v_2\vec{a}_2 + \cdots + v_n\vec{a}_n \in V$

$\forall \vec{y} = c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n \in V$

$\forall \alpha, \beta \in R$, We have :

$$\alpha\vec{u}+\beta\vec{v} = (\alpha u_1+\beta v_1)\vec{a_1} + (\alpha u_2+\beta v_2)\vec{a_2} + \cdots + (\alpha u_n+\beta v_n)\vec{a_n}$$

$$\begin{aligned}\langle \alpha\vec{u}+\beta\vec{v}, \vec{y}\rangle &= (\alpha u_1+\beta v_1)\cdot c_1 + \cdots + (\alpha u_n+\beta v_n)\cdot c_n \\ &= \sum_{i=1}^{n}(\alpha u_i+\beta v_i)\cdot c_i \\ &= \alpha\sum_{i=1}^{n} u_i c_i + \beta\sum_{i=1}^{n} v_i c_i \\ &= \alpha\cdot\langle\vec{u},\vec{y}\rangle + \beta\cdot\langle\vec{v},\vec{y}\rangle, \quad (\forall \alpha,\beta\in\mathbb{R}, \vec{u},\vec{v},\vec{y}\in V)\end{aligned}$$

③ $\langle\vec{u},\vec{v}\rangle = \sum_{i=1}^{n} u_i v_i = \sum_{i=1}^{n} v_i u_i = \langle\vec{v},\vec{u}\rangle$ for $\forall \vec{u},\vec{v}\in V$.

As ①②③ we have discussed above, $\langle\vec{u},\vec{v}\rangle = u_1v_1+\cdots+u_nv_n$ is an inner product on V.

3. (a) Proof:

$$f(\vec{x},\vec{x}) = \frac{1}{2}\left(\|\vec{x}+\vec{x}\|^2 - \|\vec{x}\|^2 - \|\vec{x}\|^2\right)$$

$$= \frac{1}{2}\left(4\|\vec{x}\|^2 - 2\|\vec{x}\|^2\right) = \|\vec{x}\|^2 \geqslant 0$$

by the definition of a norm.

And $f(\vec{x},\vec{x}) = \|\vec{x}\|^2 = 0 \Longleftrightarrow \|\vec{x}\| = 0 \Longleftrightarrow \vec{x} = \vec{0}$

by the property of square operation

(b) Proof:

$\forall \vec{x},\vec{y}\in V$, we have

$$\begin{aligned}f(\vec{x},\vec{y}) &= \frac{1}{2}\left(\|\vec{x}+\vec{y}\|^2 - \|\vec{x}\|^2 - \|\vec{y}\|^2\right) \\ &= \frac{1}{2}\left(\|\vec{y}+\vec{x}\|^2 - \|\vec{y}\|^2 - \|\vec{x}\|^2\right) \\ &= f(\vec{y},\vec{x})\end{aligned}$$

proved.

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(c) $f(\vec{x},\vec{y}) = \frac{1}{2}\left(\|\vec{x}+\vec{y}\|^2 - \|\vec{x}\|^2 - \|\vec{y}\|^2\right) \overset{\text{denoted as}}{=\!=\!=} A$

$\overset{\text{apply the parallelogram}}{\underset{\text{indentity}}{=\!=\!=}} \frac{1}{2}\left(\|\vec{x}\|^2 + \|\vec{y}\|^2 - \|\vec{x}-\vec{y}\|^2\right) \overset{\text{denoted as}}{=\!=\!=} B.$

$$\boxed{\text{i.e. } f(\vec{x},\vec{y}) = A = B = \frac{1}{2}(A+B) = \frac{1}{4}\left(\|\vec{x}+\vec{y}\|^2 - \|\vec{x}-\vec{y}\|^2\right) . \quad (*)}$$

By the parallelogram indentity,

$$\begin{cases} \|\vec{x}+\vec{z}+\vec{y}\|^2 + \|\vec{x}+\vec{z}-\vec{y}\|^2 = 2\|\vec{x}+\vec{z}\|^2 + 2\|\vec{y}\|^2 \\ \|\vec{y}+\vec{z}+\vec{x}\|^2 + \|\vec{y}+\vec{z}-\vec{x}\|^2 = 2\|\vec{y}+\vec{z}\|^2 + 2\|\vec{x}\|^2 \end{cases}$$

then we have: $\begin{cases} \|\vec{x}+\vec{y}+\vec{z}\|^2 = 2\|\vec{x}+\vec{z}\|^2 + 2\|\vec{y}\|^2 - \|\vec{x}-\vec{y}+\vec{z}\|^2 \overset{\text{denoted as}}{=\!=\!=} D \\ \|\vec{x}+\vec{y}+\vec{z}\|^2 = 2\|\vec{y}+\vec{z}\|^2 + 2\|\vec{x}\|^2 - \|\vec{y}-\vec{x}+\vec{z}\|^2 \overset{\text{denoted as}}{=\!=\!=} E \end{cases}$

thus, $\|\vec{x}+\vec{y}+\vec{z}\|^2 = D = E = \frac{1}{2}(D+E)$

$$= \|\vec{x}\|^2 + \|\vec{y}\|^2 + \|\vec{x}+\vec{z}\|^2 + \|\vec{y}+\vec{z}\|^2 - \frac{1}{2}\|\vec{x}-\vec{y}+\vec{z}\|^2 - \frac{1}{2}\|\vec{y}-\vec{x}+\vec{z}\|^2 \quad ①$$

by using $-\vec{z}$ to replace $\vec{z}$, we have

$$\|\vec{x}+\vec{y}-\vec{z}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 + \|\vec{x}-\vec{z}\|^2 + \|\vec{y}-\vec{z}\|^2 - \frac{1}{2}\|\vec{x}-\vec{y}-\vec{z}\|^2 - \frac{1}{2}\|\vec{y}-\vec{x}-\vec{z}\|^2$$

$$\overset{\text{(by definition of norm)}}{\underset{(\|\vec{a}\|=\|-\vec{a}\|)}{=\!=\!=}} \|\vec{x}\|^2 + \|\vec{y}\|^2 + \|\vec{x}-\vec{z}\|^2 + \|\vec{y}-\vec{z}\|^2 - \frac{1}{2}\|\vec{y}-\vec{x}+\vec{z}\|^2 - \frac{1}{2}\|\vec{x}-\vec{y}+\vec{z}\|^2 \quad ②$$

$$f(\vec{x}+\vec{y},\vec{z}) \overset{(*)}{=} \frac{1}{4}\left(\|\vec{x}+\vec{y}+\vec{z}\|^2 - \|\vec{x}+\vec{y}-\vec{z}\|\right)$$

$$\overset{\frac{1}{4}(①-②)}{=} \frac{1}{4}\left(\|\vec{x}+\vec{z}\|^2 - \|\vec{x}-\vec{z}\|^2 + \|\vec{y}+\vec{z}\|^2 - \|\vec{y}-\vec{z}\|^2\right)$$

$$= \frac{1}{4}\left(\|\vec{x}+\vec{z}\|^2 - \|\vec{x}-\vec{z}\|^2\right) + \frac{1}{4}\left(\|\vec{y}+\vec{z}\|^2 - \|\vec{y}-\vec{z}\|^2\right)$$

$$\overset{(*)}{=} f(\vec{x},\vec{z}) + f(\vec{y},\vec{z}), \text{ for } \forall\, \vec{x},\vec{y},\vec{z} \in V$$

proved.

(d) As the equation-(*) we have proved in part-(c),

$$f(\vec{x},\vec{y}) = \frac{1}{4}\left(\|\vec{x}+\vec{y}\|^2 - \|\vec{x}-\vec{y}\|^2\right) . \quad (*)$$

thus, $f(-\vec{x},\vec{y}) \overset{(*)}{=} \frac{1}{4}\left(\|-\vec{x}+\vec{y}\|^2 - \|-\vec{x}-\vec{y}\|^2\right)$

$$\overset{\|\vec{a}\|=\|-\vec{a}\|}{=\!=\!=} \frac{1}{4}\left(\|\vec{x}-\vec{y}\|^2 - \|\vec{x}+\vec{y}\|^2\right) \text{ (in next page)}$$

$$= -\frac{1}{4}\left(\|\vec{x}+\vec{y}\| - \|\vec{x}-\vec{y}\|^2\right)$$

$$\overset{(*)}{=} \; -f(\vec{x},\vec{y}), \quad \text{for } \forall \vec{x},\vec{y} \in V.$$

proved

(e) Proof:

$$f(\vec{x},\vec{y}) = \frac{1}{2}\left(\|\vec{x}+\vec{y}\|^2 - \|\vec{x}\|^2 - \|\vec{y}\|^2\right)$$

Triangular Inequality

$$\le \frac{1}{2}\left(\left(\|\vec{x}\|+\|\vec{y}\|\right)^2 - \|\vec{x}\|^2 - \|\vec{y}\|^2\right)$$

$$= \frac{1}{2}\left(\|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\|\vec{x}\|\|\vec{y}\| - \|\vec{x}\|^2 - \|\vec{y}\|^2\right)$$

$$= \frac{1}{2}\cdot 2\|\vec{x}\|\|\vec{y}\|$$

$$= \|\vec{x}\|\|\vec{y}\|$$

As we have proved in part-(a),

$$f(\vec{x},\vec{x}) = \|\vec{x}\|^2, \quad f(\vec{y},\vec{y}) = \|\vec{y}\|^2$$

thus, $\left(f(\vec{x},\vec{y})\right)^2 \le \|\vec{x}\|^2\|\vec{y}\|^2 = f(\vec{x},\vec{x})\, f(\vec{y},\vec{y})$

for $\forall \vec{x},\vec{y} \in V$.

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4. Assume $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2$, then we construct:

$$\phi(\vec{x}) = \begin{pmatrix} \boxed{1} \\ \boxed{\begin{matrix} x_1 \\ x_2 \end{matrix}} \\ \boxed{\begin{matrix} \frac{x_1^2}{\sqrt{2!}} \\ \frac{\sqrt{2}x_1x_2}{\sqrt{2!}} \\ \frac{x_2^2}{\sqrt{2!}} \end{matrix}} \\ \boxed{\begin{matrix} \frac{x_1^3}{\sqrt{3!}} \\ \frac{\sqrt{3}x_1^2x_2}{\sqrt{3!}} \\ \frac{\sqrt{3}x_1x_2^2}{\sqrt{3!}} \\ \frac{x_2^3}{\sqrt{3!}} \end{matrix}} \\ \vdots \end{pmatrix} \in l_2 \text{ with inner product } \langle \vec{a}, \vec{b} \rangle = \sum_{i=1}^{\infty} a_i b_i$$

Because $\|\phi(\vec{x})\|_2 = \sqrt{1 + x_1^2 + x_2^2 + \frac{x_1^4}{2} + \frac{2x_1^2x_2^2}{2} + \cdots}$

$= \sqrt{e^{\vec{x}^T\vec{x}}} < +\infty$ for $\forall \vec{x} \in R^2$

(as we discussed in Ch3, l_2 is a Hilbert space with:

inner product: $\langle \vec{a}, \vec{b} \rangle = \sum_{i=1}^{\infty} a_i b_i$

& induced norm which is the 2-norm actually: $\|\vec{a}\|_2 = \sqrt{\sum_{i=1}^{\infty} a_i^2}$.)

s.t. $\langle \phi(\vec{x}), \phi(\vec{y}) \rangle$

You need to express each term of $\phi(X)$

$= \sum_{i=1}^{\infty} \phi^{(i)}(\vec{x}) \phi^{(i)}(\vec{y})$ (we use $\vec{a}^{(i)}$ to denote i-th entry of $\vec{a}$.)

$$= \boxed{1} + \boxed{x_1y_1 + x_2y_2} + \boxed{\frac{x_1^2y_1^2}{2!} + \frac{\sqrt{2}x_1x_2\sqrt{2}y_1y_2}{2!} + \frac{x_2^2y_2^2}{2!}}$$
$$+ \boxed{\frac{x_1^3y_1^3}{3!} + \frac{\sqrt{3}x_1^2x_2 \cdot \sqrt{3}y_1^2y_2}{3!} + \frac{\sqrt{3}x_1x_2^2 \cdot \sqrt{3}y_1y_2^2}{3!} + \frac{x_2^3y_2^3}{3!}}$$
$$+ \cdots$$

$$= \boxed{1} + \boxed{x_1y_1 + x_2y_2} + \boxed{\frac{(x_1y_1)^2 + 2x_1y_1x_2y_2 + (x_2y_2)^2}{2!}} + \boxed{\frac{(x_1y_1)^3 + 3x_1^2x_2y_1^2y_2 + 3x_1x_2^2y_1y_2^2 + (x_2y_2)^3}{3!}}$$
$$+ \cdots$$

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$$= 1 + \frac{x_1y_1 + x_2y_2}{1!} + \frac{(x_1y_1 + x_2y_2)^2}{2!} + \frac{(x_1y_1 + x_2y_2)^3}{3!} + \cdots$$

$$= 1 + \frac{\vec{x}^T\vec{y}}{1!} + \frac{(\vec{x}^T\vec{y})^2}{2!} + \frac{(\vec{x}^T\vec{y})^2}{3!} + \cdots$$

$$= \sum_{i=0}^{+\infty} \frac{(\vec{x}^T\vec{y})^i}{i!}$$

$$\underset{\text{Taylor's expansion}}{=\!=} e^{\vec{x}^T\vec{y}},$$ which statisfies the requirement of the problem.

「Remark: If we expand $(\vec{x}^T\vec{y})^i = (x_1y_1 + x_2y_2)^i$ into a polynomial, then each addition of the polynomial (with a corresponding Taylor coefficient) forms each entry of $\phi(\vec{x})$ and $\phi(\vec{y})$, where every symmetric part of each addition will be assigned to either $\phi(\vec{x})$ or $\phi(\vec{y})$..

additions

ex. $(x_1y_1 + x_2y_2)^3 = x_1^3 y_1^3 + \sqrt{3}x_1^2x_2 \sqrt{3}y_1^2y_2 + \sqrt{3}x_1x_2^2 \sqrt{3}y_1y_2^2 + x_2^3 y_2^3$

assigned to construct an entry of $\phi(\vec{x})$

assigned to construct an entry of $\phi(\vec{y})$

$\boxed{\#}$ 」

5. Proof: As $\phi(\vec{x}) = (x_1^2, x_2^2, \sqrt{2}x_1x_2)$, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2$,
thus $K(\vec{x}, \vec{y}) = \langle \phi(\vec{x}), \phi(\vec{y}) \rangle = x_1^2y_1^2 + x_2^2y_2^2 + 2x_1x_2y_1y_2 = (x_1y_1 + x_2y_2)^2$.

Then we construct a matrix
$$K_m = \begin{bmatrix} K(\vec{y}_1, \vec{y}_1) & K(\vec{y}_1, \vec{y}_2) & \cdots & K(\vec{y}_1, \vec{y}_n) \\ K(\vec{y}_2, \vec{y}_1) & K(\vec{y}_2, \vec{y}_2) & \cdots & K(\vec{y}_2, \vec{y}_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(\vec{y}_n, \vec{y}_1) & K(\vec{y}_n, \vec{y}_2) & \cdots & K(\vec{y}_n, \vec{y}_n) \end{bmatrix},$$

where $\vec{y}_1, \cdots \vec{y}_n \in R^2$ are arbitrary and $n \in R$ is arbitrary.
thus the entry in i-th row and j-th column of K_m is :

$$k_{ij} = (y_{i1}y_{j1} + y_{i2}y_{j2})^2$$

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① — positive semi-definite

so for $\forall \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in R^n$, we have

$$\vec{c}^T K_m \vec{c} = \sum_{j=1}^{n} c_j \sum_{i=1}^{n} c_i k_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j (y_{i1} y_{j1} + y_{i2} y_{j2})^2$$

$$= \left\langle \sum_{i=1}^{n} c_i \begin{pmatrix} y_{i1}^2 \\ y_{i2}^2 \\ \sqrt{2} y_{i1} y_{i2} \end{pmatrix}, \sum_{j=1}^{n} c_j \begin{pmatrix} y_{j1}^2 \\ y_{j2}^2 \\ \sqrt{2} y_{j1} y_{j2} \end{pmatrix} \right\rangle$$

(where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in R^3.)

which is actually $\phi(\vec{y_i})$

$$= \left\langle \sum_{i=1}^{n} c_i \begin{pmatrix} y_{i1}^2 \\ y_{i2}^2 \\ \sqrt{2} y_{i1} y_{i2} \end{pmatrix}, \sum_{i=1}^{n} c_i \begin{pmatrix} y_{i1}^2 \\ y_{i2}^2 \\ \sqrt{2} y_{i1} y_{i2} \end{pmatrix} \right\rangle$$

$\geqslant 0$ (as $\langle \vec{x}, \vec{x} \rangle \geqslant 0$ for $\forall \vec{x} \in R^3$) ✓

i.e. K_m is positive semi-definite.

② — symmetric.

$$K(\vec{x}, \vec{y}) = (x_1 y_1 + x_2 y_2)^2 = (y_1 x_1 + y_2 x_2)^2 = K(\vec{y}, \vec{x})$$

According to ①, ② we have discussed above,

$K(\vec{x}, \vec{y})$ is symmetric positive semi-definite ✓

According to Mercer's theorem we can conclude:

$\exists \phi$ s.t. $K(\vec{x}, \vec{y}) = \langle \phi(\vec{x}), \phi(\vec{y}) \rangle$

i.e. $K(\vec{x}, \vec{y})$ is indeed a kernel function.

$\boxtimes$