MSBD5004 Mathematical Methods for Data Analysis

Homework 2

Due date: Thursday, October 17

1. Let $(V, \|\cdot\|)$ be a normed vector space.
   (a) Prove that, for all $\boldsymbol{x}, \boldsymbol{y} \in V$,
   $$\big|\|\boldsymbol{x}\| - \|\boldsymbol{y}\|\big| \le \|\boldsymbol{x} - \boldsymbol{y}\|.$$
   (b) Let $\{\boldsymbol{x}_k\}_{k\in\mathbb{N}}$ be a convergent sequence in V with limit $\boldsymbol{x} \in V$. Prove that
   $$\lim_{k\to\infty} \|\boldsymbol{x}_k\| = \|\boldsymbol{x}\|.$$
   (*Hint: Use part (a).*)

2. Let V be a vector space and $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}$ be a basis of V. If $\boldsymbol{u} = u_1\boldsymbol{a}_1 + \cdots + u_n\boldsymbol{a}_n$ and $\boldsymbol{v} = v_1\boldsymbol{a}_1 + \cdots + v_n\boldsymbol{a}_n$ are two vectors in V, define
   $$\langle \boldsymbol{u}, \boldsymbol{v}\rangle = u_1v_1 + \cdots + u_nv_n.$$
   Show that this is an inner product on V.

3. Let V be a vector space with a norm $\|\cdot\|$ that satisfies the parallelogram identity
   $$\|\boldsymbol{x}+\boldsymbol{y}\|^2 + \|\boldsymbol{x}-\boldsymbol{y}\|^2 = 2\|\boldsymbol{x}\|^2 + 2\|\boldsymbol{y}\|^2, \qquad \forall\, \boldsymbol{x}, \boldsymbol{y} \in V.$$
   Note that we don't have an inner product on V so far. For any $\boldsymbol{x}, \boldsymbol{y} \in V$, define
   $$f(\boldsymbol{x}, \boldsymbol{y}) := \frac{1}{2}\left(\|\boldsymbol{x}+\boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2\right)$$
   (a) Prove $f(\boldsymbol{x}, \boldsymbol{x}) \ge 0$ for any $\boldsymbol{x} \in V$, and $f(\boldsymbol{x}, \boldsymbol{x}) = 0$ if and only if $\boldsymbol{x} = \boldsymbol{0}$.
   (b) Prove $f(\boldsymbol{x}, \boldsymbol{y}) = f(\boldsymbol{y}, \boldsymbol{x})$ for all $\boldsymbol{x}, \boldsymbol{y} \in V$.
   (c) Prove $f(\boldsymbol{x}+\boldsymbol{y}, \boldsymbol{z}) = f(\boldsymbol{x}, \boldsymbol{z}) + f(\boldsymbol{y}, \boldsymbol{z})$ for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V$.
   (d) Prove $f(-\boldsymbol{x}, \boldsymbol{y}) = -f(\boldsymbol{x}, \boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in V$.
   (e) Prove $(f(\boldsymbol{x}, \boldsymbol{y}))^2 \le f(\boldsymbol{x}, \boldsymbol{x})f(\boldsymbol{y}, \boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in V$.

   (c)(d)(e) together with some other technique can show that $f(\alpha\boldsymbol{x} + \beta\boldsymbol{y}, \boldsymbol{z}) = \alpha f(\boldsymbol{x}, \boldsymbol{z}) + \beta f(\boldsymbol{y}, \boldsymbol{z})$. Therefore, we can finally prove f defines an inner product. This question showed that the parallelogram identity is also a sufficient condition for a norm to be induced by an inner product. Combined with the parallelogram law on inner product spaces, we see that the parallelogram identity is a necessary and sufficient condition for a norm to be an induced by an inner product.

4. Consider the kernel $K(\boldsymbol{x}, \boldsymbol{y}) = e^{\boldsymbol{x}^T\boldsymbol{y}}$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$. Find an explicit feature space H (a Hilbert space) and the feature map $\phi : \mathbb{R}^2 \to H$ satisfying $\langle \phi(\boldsymbol{x}), \phi(\boldsymbol{y})\rangle = K(\boldsymbol{x}, \boldsymbol{y})$. What is the inner product and the induced norm on H? (*H might be infinite dimensional, and consider the Taylor's expansion* $e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$.)

5. Let $X \subset \mathbb{R}^2$ be a two-dimensional input space, and consider the feature map $\phi : X \to \mathbb{R}^3$ defined by

$$\phi(\boldsymbol{x}) = (x_1^2, x_2^2, \sqrt{2}x_1x_2),$$

where $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$. We are given the function $K : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$K(\boldsymbol{x}, \boldsymbol{y}) = \langle \phi(\boldsymbol{x}), \phi(\boldsymbol{y}) \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^3$. Prove that K is a kernel function.