MSBD 5004 Mathematical Methods for Data Analysis
Homework 5

Due date: Nov 28, Thursday

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation. If T satisfies

$$T \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \text{and} \quad T \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

then find

$$T \begin{bmatrix} 8 \\ 3 \\ 7 \end{bmatrix}.$$

2. Find the Jacobian matrix of the following vector-valued multi-variable functions.

   (a) $f : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}$, where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^n$.

   (b) $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $f(\boldsymbol{x}) = \boldsymbol{x}\boldsymbol{x}^T\boldsymbol{a}$, where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{a} \in \mathbb{R}^n$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (x^2y, x - y)$ and $h = f \circ g = f(g(x, y))$. Find $\frac{\partial h}{\partial x}|_{x=1,y=2}$ if $\frac{\partial f}{\partial x}|_{x=2,y=-1} = 3$ and $\frac{\partial f}{\partial y}|_{x=2,y=-1} = -2$. (Hint: use the chain rule)

4. Let $f(t) = f_1(t) * f_2(t)$ be the convolution of two functions $f_1(t)$ and $f_2(t)$ on $\mathbb{R}$, i.e.,

$$f(t) = \int_{-\infty}^{+\infty} f_1(t - s) f_2(s) ds$$

   Let a, a_1, a_2 be real numbers.

   (i) Prove the following identity:

$$f_1(t - a) * f_2(t) = f_1(t) * f_2(t - a) = f(t - a).$$

   (ii) Prove the following identity:

$$f_1(t - a_1) * f_2(t - a_2) = f(t - a_1 - a_2).$$

5. Let V_1 and V_2 be two Hilbert spaces with the inner products $\langle \cdot, \cdot \rangle_{V_1}$ and $\langle \cdot, \cdot \rangle_{V_2}$, respectively. Let $T \in \mathcal{L}(V_1, V_2)$, i.e., $T : V_1 \to V_2$ be a bounded linear operator.

   (a) Let $S : V_2 \to V_1$ be an operator satisfying $\langle T\boldsymbol{x}, \boldsymbol{y} \rangle_{V_2} = \langle \boldsymbol{x}, S\boldsymbol{y} \rangle_{V_1}$ for any $\boldsymbol{x} \in V_1$ and $\boldsymbol{y} \in V_2$. Prove that S is a bounded linear operator. (Consequently, S is the adjoint of T, i.e., $S = T^*$)

(b) Prove that $(T^*)^* = T$.

(c) Prove that $\|T\| = \|T^*\|$.

6. Consider the vector space ℓ_∞ equipped with the norm $||\cdot||_\infty$. Define the operator $T : \ell_\infty \to \ell_\infty$ by $T(\{x_n\}_{n\in\mathbb{N}}) = \{y_n\}_{n\in\mathbb{N}}$ where $y_n = x_{n+1}$.

   (a) Prove that T is a linear operator.

   (b) Prove that T is a bounded operator.

   (c) Prove that $\|T\| = 1$.