Chapter 12：STATIC GAMES OF INCOMPLETE INFORMATION: Bayesian Games

2

作为一个不吃辣的四川人
是的，我就是我！是不一样的烟火！
哪个看到我都要冒火！
因为吃不得辣从小被屋头人赏到大
我妈不承认我是她亲生的
我老汉儿喊我各人滚去广东
就连读书了也被同学们当成外星人
你们能体会到20多年每次出门吃火锅
大家都说不能因为你一个人搞特殊
自己默默准备五碗开水来涮的痛吗
真是闻者伤心，听者落泪

6

不吃辣在四川真的很难拥有爱情
我恋爱道路上最大的障碍就是
如何解决人家海椒当顿吃
而我不沾海椒的重大分歧
每盘耍朋友都是因为点菜拉豁
如果哪天让我找到不吃辣的同类
我一定要握着他的手，感动地说：
“同志，终于找到你了！”


BID
BID
BID
BID
BID
BID
LOT
On Line
AUCTION

In all the examples and tools for analysis that we have encountered thus far, we have made an important assumption that the game played is common knowledge.

In particular we have assumed that the players are aware of *who is playing*, *what the possible actions of each player are*, *and how outcomes translate into payoffs*. Furthermore we have assumed that this knowledge of the game is itself common knowledge.

Little effort is needed to convince anyone that these idealized situations are rarely encountered in reality.

For example, consider one of our early examples, the duopoly market game. We have analyzed both the Cournot and Bertrand models of duopolistic competition, and for each we have a clear and precise, easily understood outcome. One assumption of the model was that the payoffs of the firms, like their action spaces, are common knowledge.

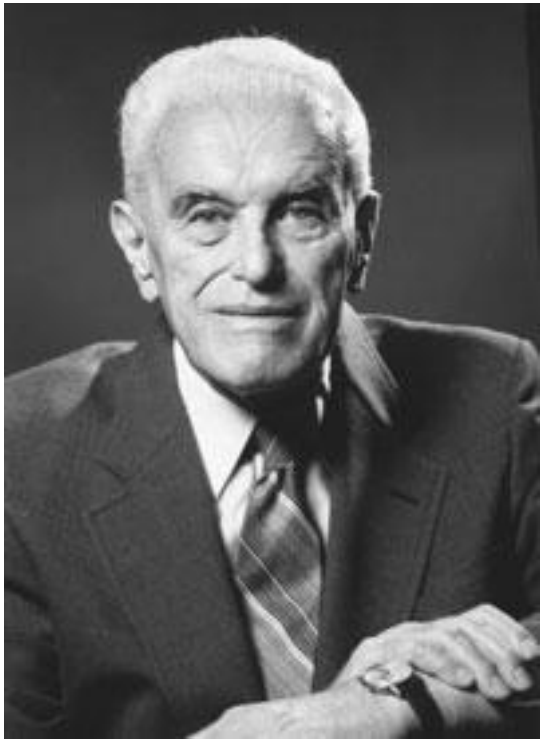
However, is it reasonable to assume that the production technologies are indeed common knowledge? And if they are, should we believe that the productivity of workers in each firm is known to the other firm? More generally, is it reasonable to assume that the cost function of each firm is precisely known to its opponent?

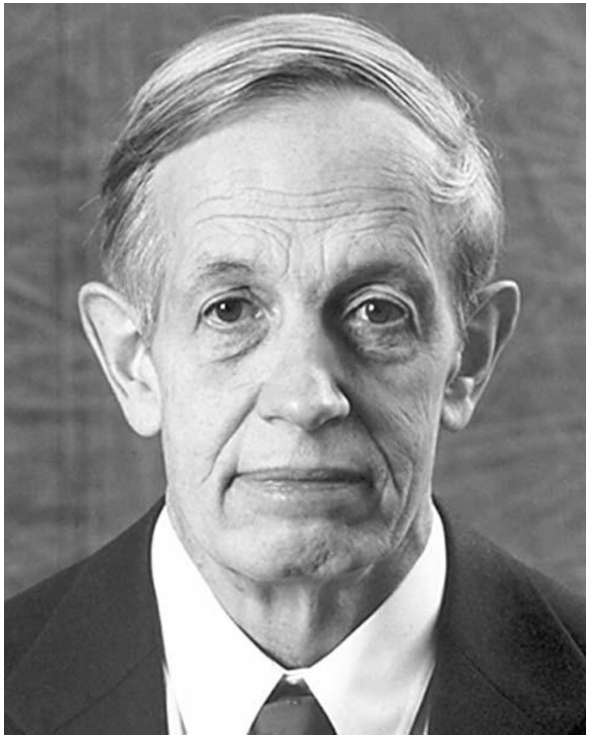
Perhaps it is more convincing to believe that firms have a good idea about their opponents' costs but do not know exactly what they are. The analysis toolbox we have developed so far is not adequate to address such situations.

How do we think of situations in which players have some idea about their opponents' characteristics but don't know for sure what these characteristics are? At some level this is not so different from the situation in a simultaneous-move game in which a player does not know what actions his opponents are taking, but instead knows what the set of actions can be. As we have seen earlier, a player must form *a conjecture about the behavior of his opponents in order to choose his best response*, and we identified this idea as the player's belief over the actions that his opponents will choose.

In the mid-1960s John Harsanyi realized the similarity between beliefs over a player's actions and beliefs over his other characteristics, such as costs and preferences.

Harsanyi proceeded to develop an elegant and extremely operational way to capture the idea that beliefs over the characteristics of other players—their *types*—can be embedded naturally into the framework of game theory that we have already developed. This advancement sets Harsanyi up to be the third Nobel Laureate to share the prestigious prize with John Nash and Reinhard Selten in 1994.





Reinhard Selten

We call games that incorporate the possibility that players could be of different types (a concept soon to be well defined) games of incomplete information.

As with games of complete information, we will develop a theory of equilibrium behavior that requires players to have beliefs about their opponents' characteristics and their actions, and furthermore requires that these beliefs be consistent or correct. It should be no surprise that this will require very strong assumptions about the cognition of the players: we assume that common knowledge reigns over the possible characteristics of players and over the likelihood that each type of player is indeed part of the game.

To capture the idea that a player's characteristics may be unknown to other players, we introduce uncertainty over the preferences of the players.

That is, instead of having a unique payoff function for each player that maps profiles of actions into payoffs, games of incomplete information allow players to have one of possibly many payoff functions. We associate each of a player's possible payoff functions with the player's type, which captures the idea that a player's preferences, or type, may not be common knowledge.

Harsanyi (1967–68) suggested the following framework.

Imagine that before the game is played Nature chooses the preferences, or type, of each player from his possible set of types.

Another way to think about this approach is that Nature is choosing a game from among a large set of games, in which each game has the same players with the same action sets, but with different payoff functions. If Nature is randomly choosing among many possible games, then there must be a well-defined probability distribution over the different games.

It is this observation, together with the requirement that everything about a game must be common knowledge, that will make this setting amenable to equilibrium analysis.

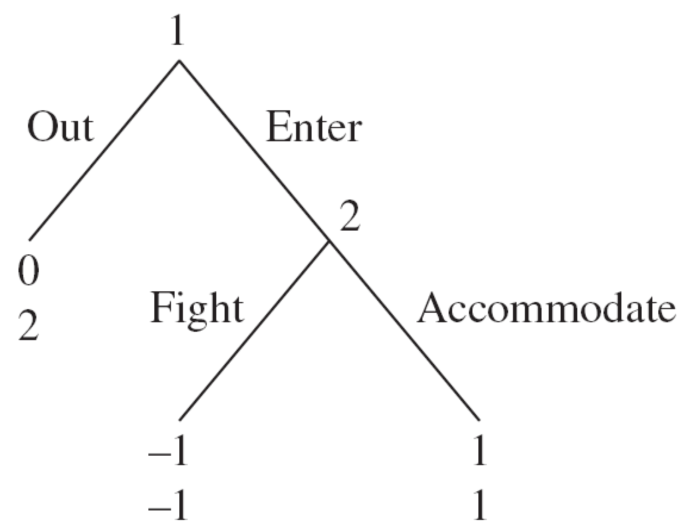


FIGURE 12.1 A simple entry game.

Nash equilibria？

Subgame-perfect equilibrium ？

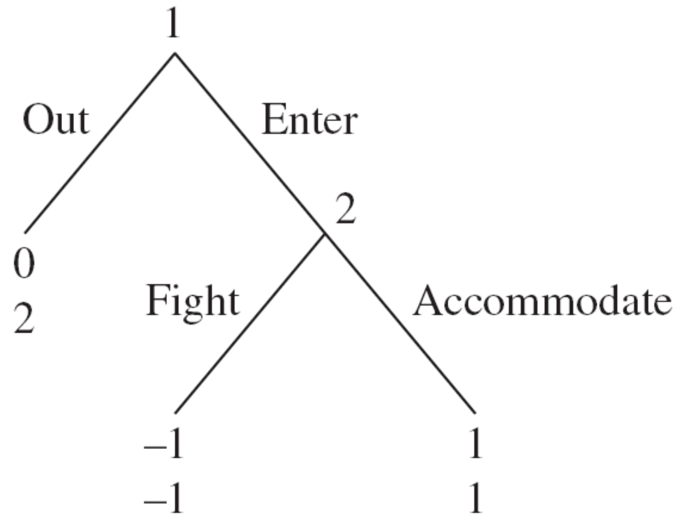


FIGURE 12.1 A simple entry game.

Consider the following simple "entry game", in which an entrant firm, player 1, decides whether or not to enter a market. The incumbent firm in that market, player 2, decides how to respond to an entry decision of player 1 by either fighting or accommodating entry. The payoffs given in Figure 12.1 show that if player 1 enters, player 2's best response is to accommodate, which in turn implies that the unique subgame-perfect equilibrium is for player 1 to enter and for player 2 to accommodate entry.

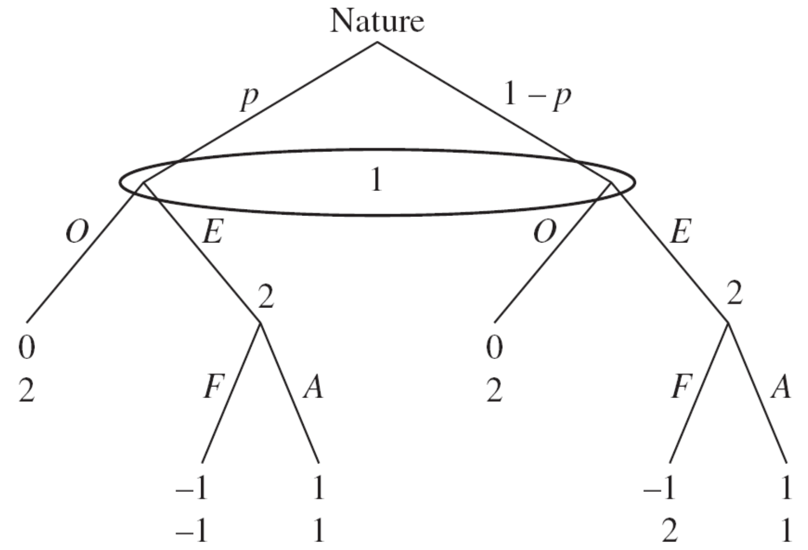


FIGURE 12.2 An incomplete-information entry game.

Imagine that there is one type of player 1, and there are two types of player 2. The 1st type, called "rational," has payoffs as shown in Figure 12.1. The 2nd type, called "crazy," enjoys fighting and the payoff he gets from (Enter, Fight) is 2 instead of −1. The structure of the game is fixed by the set of players N and the action spaces A_i for each player $i \in N$, yet Nature chooses which type of player 2 is playing the game with player 1. We need to state the likelihood or probability of each type being selected by Nature. Let p denote the probability that Nature chooses the rational type.

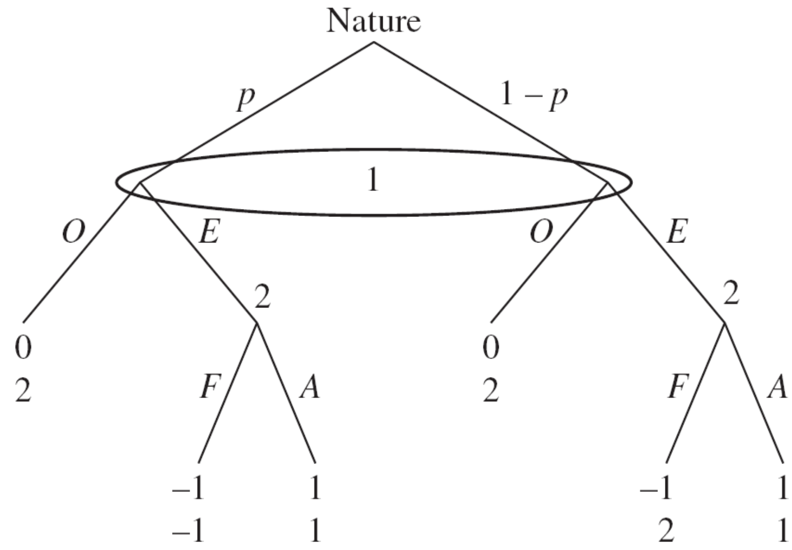


FIGURE 12.2 An incomplete-information entry game.

We assume that players know their own preferences, which in turn will allow us to analyze a player's best response given his assumptions about the behavior of his opponents. This is the reason for the single information set in Figure 12.2, which shows that player 1 is uncertain about the preferences of player 2, but player 2 knows what his preferences are when he needs to make a decision.

A final issue still needs to be addressed. If players know their own preferences, but they do not know the preferences or types of their opponents, then what must players know in order for them to have a well-defined best response and in turn let us perform equilibrium analysis?

The natural step, as Harsanyi realized, is to require players to form correct beliefs about the preferences and types of their opponents. For this reason we assume that, despite each player not necessarily knowing the actual preferences of his opponents, he does know the precise way in which Nature chooses these preferences. That is, each player knows the probability distribution over types, and this itself is common knowledge among the players of the game.

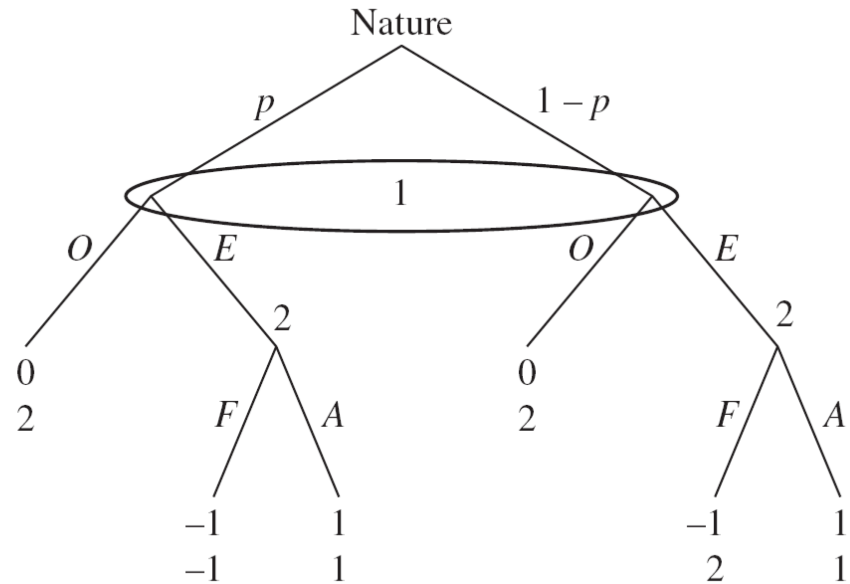


FIGURE 12.2 An incomplete-information entry game.

In the entry game of Figure 12.2 this is given by specifying that p is common knowledge, and hence player 1 knows that the probability that he is at the left node in his information set is p. This is often called the common prior assumption, and it means that all the players agree on the way the world works, as described by the probabilities according to which Nature chooses the different types of players.

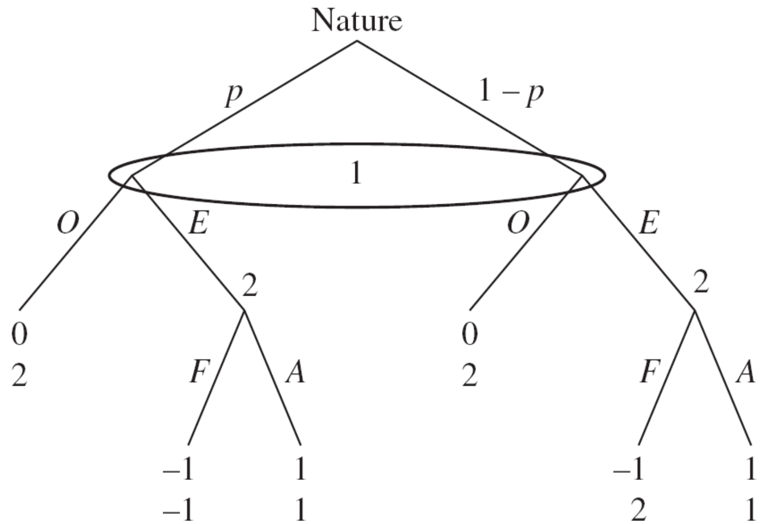


FIGURE 12.2 An incomplete-information entry game.

Now that we have completed the extensive-form representation of this game, we can turn to its normal form. Notice that player 2 must have four pure strategies: in each of his information sets he has two actions from which to choose. Let's define a strategy of player 2 as $xy \in \{AA, AF, FA, FF\}$, where x describes what a rational player 2 does and y what a crazy one does. When we introduce incomplete information, a strategy of a player is now a prescription that tells each type of a player what he should do if this is the type that Nature chose for the game. The strategy set for player 1 is simply $\{E, O\}$.

In this example, if, say, player 2 plays *AF* (*A* if he is rational and *F* if he is crazy), and if player 1 plays E, then with probability p the outcome will yield payoffs of (1, 1), and with probability $(1-p)$ the payoffs will be (−1, 2). Thus in expectations the pair of payoffs from the pair of strategies (AF, E) is

$$v_1 = p \times 1 + (1-p) \times (-1) = 2p - 1$$
$$v_2 = p \times 1 + (1-p) \times 2 = 2 - p.$$

In this way we can compute the expected payoffs of both players from each pair of strategies, which results in the following normal-form matrix game:

		Player 2			
		AA	*AF*	*FA*	*FF*
Player 1	*O*	0, 2	0, 2	0, 2	0, 2
	E	1, 1	$2p$-1, 2-p	1-2p, 1-2p	-1, 2-3p

For concreteness, set $p = 2/3$, which results in the following normal-form matrix game:

Player 2

Player 1		*AA*	*AF*	*FA*	*FF*
	O	0, 2	0, 2	0, 2	0, 2
	E	1, 1	1/3, 4/3	-1/3, -1/3	-1, 0

This game has three pure strategy Nash equilibria: (*O*, *FA*), (*O*, *FF*), and (*E*, *AF*). Also notice that of these three, only (*E*, *AF*) is a subgame-perfect equilibrium.

We can see how each type of player i can calculate his expected payoffs given a belief over the actions of each type of his opponents. But by writing a single game of imperfect information as we did, we are averaging the payoffs of all the types for each player using the likelihood of each type as the weight.

That is, we have produced a “meta-player” (e.g., player 2 in the example) who cares about this average payoff across his different types rather than looking at the game one type at a time.

12.1 Strategic Representation of Bayesian Games

12.1.1 Players, Actions, Information, and Preferences

To model situations in which players know their own payoffs from outcomes (different profiles of actions), but do not know the payoffs of the other players, we introduce the concept of **incomplete information**, which is composed of three new components.

First, a player's preferences are associated with his type. If a player can have several different preferences over outcomes, each of these will be associated with a different type.

More generally information that the player has about his own payoffs, or information he might have about other relevant attributes of the game, is also part of what defines a player's type.

Second, uncertainty over types is described by Nature choosing types for the different players. Thus we introduce type spaces for each player, which represent the sets from which Nature chooses the players' types.

Last but not least, there is common knowledge about the way in which Nature chooses between profiles of types of players. This is represented by a common prior, which is the probability distribution over types that is common knowledge among the players. Because every agent learns his own type, he can use the common prior to form posterior beliefs over the types of other agents.

Definition 12.1 The normal-form representation of an n-player **static Bayesian game of incomplete information** is

$$\left\langle N, \{A_i\}_{i=1}^n, \{\Theta_i\}_{i=1}^n, \{v_i(\cdot; \theta_i), \theta_i \in \Theta_i\}_{i=1}^n, \{\phi_i\}_{i=1}^n \right\rangle,$$

where $N = \{1, 2, \ldots, n\}$ is the set of players; A_i is the *action set* of player i; $\Theta_i = \{\theta_{i1}, \theta_{i2}, \ldots, \theta_{ik_i}\}$ is the *type space* of player i; $v_i : A \times \Theta_i \to \mathbb{R}$ is the type-dependent payoff function of player i, where $A \equiv A_1 \times A_2 \times \cdots \times A_n$; and ϕ_i describes the *belief* of player i with respect to the uncertainty over the other players' types, that is, $\phi_i(\theta_{-i}|\theta_i)$ is the (posterior) conditional distribution on θ_{-i} (all other types but i) given that i knows his type is θ_i.

Aside from the three basic components of players, actions, and preferences, *the addition of types, type-dependent preferences, and beliefs about the types of other players* captures the ideas illustrated earlier.

It is convenient to think about a static Bayesian game as one that proceeds through the following steps:

1. Nature chooses a profile of types $(\theta_1, \theta_2, . . . , \theta_n)$.

2. Each player i learns his own type, θ_i, which is his private information, and then uses his prior φ_i to form posterior beliefs over the other types of players.

3. Players simultaneously (hence this is a static game) choose actions $a_i \in A_i, i \in N$.

4. Given the players' choices $a = (a_1, a_2, . . . , a_n)$, the payoffs $v_i(a; \theta_i)$ are realized for each player $i \in N$.

In the above definition a player's payoff $v_i\,(a;\,\theta_i)$ depends on the actions of all players and only on i's type, but it does not depend on the types of the other players θ_{-i}.

This particular assumption is known as the **private values case** because each type's payoff depends only on his private information.

This case is not rich enough to capture all the interesting examples that we will analyze, and for this reason we will later introduce the case of **common values**, in which $v_i(a_1, a_2, \ldots, a_n\,;\theta_1, \theta_2, \ldots, \theta_n)$ is possible.

12.1.2 Deriving Posteriors from a Common Prior: A Player's Beliefs

In this section we explore the meaning of each player *i* using the common prior to derive a posterior belief about the distribution of the other types of players.

Conditional probabilities follow a mathematical rule that derives the way in which a player or decision maker should change a **prior (initial) belief** in the light of new evidence, resulting in a **posterior (updated) belief.**

The idea can be described as follows.

First, before Nature chooses the actual type of each player, imagine that every player does not yet know what his type will be; but he does know the probability distribution that Nature uses to choose the types for all the players.

Later, after Nature has chosen a type for each player, they all independently and privately learn their types. This new piece of information for each player, his type, may provide some new piece of evidence about how the other players' types may have been chosen. It is in this respect that a player may derive new beliefs about the other players once he learns his type.

Consider the following example. Imagine that there are two players, 1and 2, each having two possible types, $\theta_1 \in \{a, b\}$ and $\theta_2 \in \{c, d\}$. Nature chooses these types according to a prior over the four possible type combinations, where the following joint distribution matrix describes Nature's prior:

Player 2's type

Player 1's type		c	d
	a	1/6	1/3
	b	1/3	1/6

Now imagine that player 1 learns that his type is a. What must be his belief about player 2's type?

$$\phi_1(\theta_2 = c|\theta_1 = a) = \frac{\Pr\{\theta_1 = a \cap \theta_2 = c\}}{\Pr\{\theta_1 = a\}} = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{3}} = \frac{1}{3},$$

and similarly

$$\phi_1(\theta_2 = d|\theta_1 = a) = \frac{\Pr\{\theta_1 = a \cap \theta_2 = d|\}}{\Pr\{\theta_1 = a\}} = \frac{\frac{1}{3}}{\frac{1}{6} + \frac{1}{3}} = \frac{2}{3}.$$

12.1.3 Strategies and Bayesian Nash Equilibrium

The representation of a Bayesian game has action sets, A_i , for each player $i \in$ N. However, each player i can be one of several types $\theta_i \in \Theta_i$, and each type θ_i may choose a different action from the set A_i . To define a strategy for player i we need to specify what each type $\theta_i \in \Theta_i$ of player i will choose when Nature calls upon this type to play the game.

Definition 12.3 Consider a static Bayesian game

$$\left\langle N, \{A_i\}_{i=1}^n, \{\Theta_i\}_{i=1}^n, \{v_i(\cdot;\theta_i), \theta_i \in \Theta_i\}_{i=1}^n, \{\phi_i\}_{i=1}^n \right\rangle .$$

A **pure strategy** for player i is a function $s_i : \Theta_i \to A_i$ that specifies a pure action $s_i(\theta_i)$ that player i will choose when his type is θ_i. A **mixed strategy** is a probability distribution over a player's pure strategies.

This should remind you of strategies for extensive-form games that are defined as mappings from information sets to actions, where a pure strategy is a rulebook of what to choose in each information set.

In Bayesian games *we can think of the types of players as being their information sets: when player i learns his type then it is as if he is in a unique information set that is identified with this type.*

Second, we are effectively specifying player *i*'s strategy for all his information sets—one for each type—just as in any extensive form game.

Given a certain realization of Nature, we are specifying what an player i will do even for those types that have not been realized. *The reason we must specify i's strategy completely is so that player i's opponents can form well defined beliefs over i's behavior.*

Players $j \neq i$ *need to combine their beliefs from their posterior over i's types together with their beliefs over what each type* θ_i *of player i plans to do.*

Without this complete specification, players $j \neq i$ cannot calculate their expected payoffs from their own actions.

We have completely defined what a static Bayesian game is, and what the strategies for each player are, it is easy to define a solution concept that is derived from Nash equilibrium as follows:

Definition 12.4 In the Bayesian game

$$\left\langle N, \{A_i\}_{i=1}^n, \{\Theta_i\}_{i=1}^n, \{v_i(\cdot; \theta_i), \theta_i \in \Theta_i\}_{i=1}^n, \{\phi_i\}_{i=1}^n \right\rangle,$$

a strategy profile $s^* = (s_1^*(\cdot), s_2^*(\cdot), \ldots, s_n^*(\cdot))$ is a ***pure-strategy Bayesian Nash equilibrium*** if, for every player i, for each of player i's types $\theta_i \in \Theta_i$, and for every $a_i \in A$, $s_i^*(\cdot)$ solves

$$\sum_{\theta_{-i} \in \Theta_{-i}} \phi_i(\theta_{-i}|\theta_i) v_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}); \theta_i) \geq \sum_{\theta_{-i} \in \Theta_{-i}} \phi_i(\theta_{-i}|\theta_i) v_i(a_i, s_{-i}^*(\theta_{-i}); \theta_i). \tag{12.2}$$

A Bayesian Nash equilibrium has each player choose a type-contingent strategy s^*_i (.) so that given any one of his types $\theta_i \in \Theta_i$, and his beliefs about the strategies of his opponents s^*_{-i} (.), his expected payoff from $s^*_i(\theta_i)$ is at least as large as that from any one of his actions $a_i \in A_i$.

His expectations are derived from the strategies played by other players and the mixing that occurs owing to the randomization of nature that each player faces through his beliefs $\varphi_i(\theta_{-i}| \theta_i)$.

On the right side of the inequality (12.2) we can replace a_i with $s'_i(\theta_i)$ so that it reads

$$\sum_{\theta_{-i}\in\Theta_{-i}} \phi_i(\theta_{-i}|\theta_i)v_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}); \theta_i) \geq \sum_{\theta_{-i}\in\Theta_{-i}} \phi_i(\theta_{-i}|\theta_i)v_i(s_i'(\theta_i), s_{-i}^*(\theta_{-i}); \theta_i),$$

which means that player i of type θ_i does not want to replace the strategy s^*_i (.) with any other s'_i (.) $\in S_i$ because this inequality holds for every type $\theta_i \in \Theta_i$.

It suffices to consider any deviation from s^*_i (.) to an action, a_i, instead of the more complex notion of deviating to a type-dependent strategy, s'_i (.).

How to define the strategic form game associated with the incomplete information game?

Two firms are engaged in Bertrand price competition. Firm 1's marginal cost of production is known, and firm 2's is either high or low, with each possibility being equally likely. There are no fixed costs. Thus, firm 1 has but one type, and firm 2 has two types – high cost and low cost. The two firms each have the same strategy set, namely the set of non-negative prices. Firm 2's payoff depends on his type, but firm 1's payoff is independent of firm 2's type; it depends only on the chosen prices.

To derive from this game of incomplete information a strategic form game, imagine that there are actually *three firms rather than two*, *namely*, *firm 1*, *firm 2 with high cost*, *and* firm 2 with low cost.

Imagine also that *each of the three firms must simultaneously choose* a price and that firm 1 believes that each of the firm 2's is equally likely to be his only competitor. Some thought will convince you that this way of looking at things beautifully captures all the relevant strategic features of the original situation.

In particular, firm 1 must choose its price without knowing whether its competitor has high or low costs. Moreover, firm 1 understands that the competitor's price may differ according to its costs.

12.2 Examples

12.2.1 Teenagers and the Game of Chicken

It is often the case that a player who finds himself in some game of conflict will hold out and suffer in order to appear strong, instead of letting his rival get the better of him.

Be it firms in the marketplace, politicians in government, countries at war, or even kids on the playground, the optimal behavior will depend on some combination of each player’s tendency to be aggressive and his belief about his opponent’s tendency to be aggressive.

To illustrate this idea, consider a simple game of aggression that is known to many teenagers: the game of “chicken.” The 1955 film Rebel Without a Cause（无因的背叛）featured James Dean as a juvenile delinquent and introduced to the silver screen one rather dangerous variant of the game. In the movie two teenagers simultaneously drive their cars toward the edge of a cliff. The first one to jump out is considered chicken and loses the contest.

Two teenagers, players 1 and 2, have borrowed their parents' cars and decided to play the game of chicken as follows.

They drive toward each other in the middle of a street, and just before impact they must simultaneously choose whether to be chicken and swerve to the right, or continue driving head on.

If both are chicken then both gain no respect from their friends but suffer no losses; thus both get a payoff of 0.

If i continues to drive while $j \neq i$ plays chicken then i gains all the respect, which is a payoff of R, and j gets no respect, which is worth 0. In this case both players suffer no additional losses.

If both continue to drive head on then they split the respect，but an accident is bound to happen and they will at least be reprimanded by their parents, if not seriously injured. An accident imposes a personal loss of k on each player, so the payoff to each one is $R/2 - k$.

However, a potential difference exists between these two youngsters. The punishment, k, depends on the type of parents they have.

For each player i, his parents are harsh (H) with probability 1/2 and is independent of the type of parents that another player j has. If player i's parents are harsh then they will beat the living daylights out of their child, which imposes a high cost of an accident, denoted by $k = H$.

If instead the parents are lenient (L) then they will give their child a long lecture on why his behavior is unacceptable, which imposes a lower cost of an accident, denoted by $k = L < H$.

Each kid knows the type of his parents but does not know the type of his opponent's parents. The distribution of types is common knowledge (this is the common prior assumption).

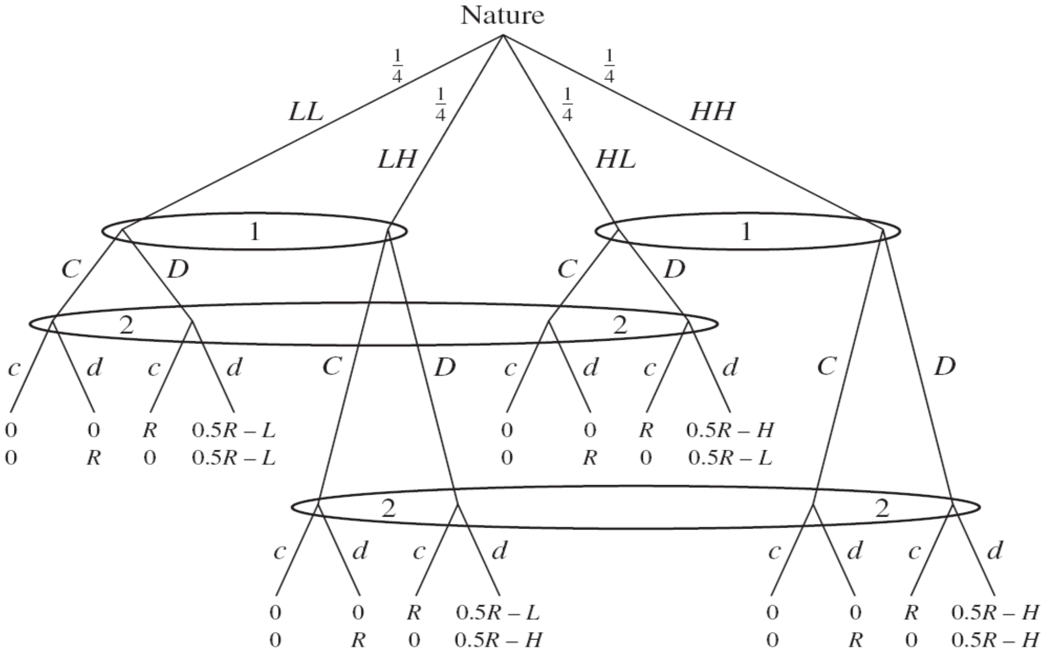


FIGURE 12.3 The game of chicken with incomplete information.

There are four states of Nature, $\theta_1\ \theta_2 \in \{LL, LH, HL, HH\}$ that denote the types of player 1 and player 2, respectively, and each of these four states occurs with equal probability.

The structure of the information sets follows from the knowledge of the players when they make their moves. Player 1, for example, cannot distinguish between the states of Nature *LL* and *LH* (his type is *L*), nor can he distinguish between *HL* and *HH* (his type is *H*). However, he knows what his own type is, and hence he can distinguish between these two pairs of states. The action sets of each player are $A_1 = \{C, D\}$ and $A_2 = \{c, d\}$, where *C* (or *c*) stands for “chicken” and *D* (or *d*) stands for “drive.”

A strategy for player 1 is denoted by $xy \in S_1 = \{CC, CD, DC, DD\}$, where x is what he does if he is an L type and y is what he does if he is an H type. Similarly, $S_2 = \{cc, cd, dc, dd\}$. The payoffs are calculated using the probabilities over the states of Nature together with the strategies of the players. For example, if player 1 chooses CD and player 2 chooses dd then the expected payoffs for player 1 are

$$\begin{aligned} Ev_1(CD, dd) &= \frac{1}{4} \times v_1(C, d; L) + \frac{1}{4} \times v_1(C, d; L) + \frac{1}{4} \times v_1(D, d; H) + \frac{1}{4} \times v_1(D, d; H) \\ &= \frac{1}{4} \times 0 + \frac{1}{4} \times 0 + \frac{1}{4} \times \left(\frac{R}{2} - H\right) + \frac{1}{4} \times \left(\frac{R}{2} - H\right) = \frac{R}{4} - \frac{H}{2} \end{aligned}$$

and the expected payoffs for player 2 are

$$\begin{aligned} Ev_2(CD, dd) &= \frac{1}{4} \times v_2(C, d; L) + \frac{1}{4} \times v_2(C, d; H) + \frac{1}{4} \times v_2(D, d; L) + \frac{1}{4} \times v_2(D, d; H) \\ &= \frac{1}{4} \times R + \frac{1}{4} \times R + \frac{1}{4} \times \left(\frac{R}{2} - L\right) + \frac{1}{4} \times \left(\frac{R}{2} - H\right) = \frac{3R}{4} - \frac{L}{4} - \frac{H}{4}. \end{aligned}$$

This results in the following matrix-form Bayesian game:

Player 2

Player 1		cc	cd	dc	dd
	CC	$0, 0$	$0, \frac{R}{2}$	$0, \frac{R}{2}$	$0, R$
	CD	$\frac{R}{2}, 0$	$\frac{3R}{8}-\frac{H}{4}, \frac{3R}{8}-\frac{H}{4}$	$\frac{3R}{8}-\frac{H}{4}, \frac{3R}{8}-\frac{L}{4}$	$\frac{R}{4}-\frac{H}{2}, \frac{3R}{4}-\frac{L}{4}-\frac{H}{4}$
	DC	$\frac{R}{2}, 0$	$\frac{3R}{8}-\frac{L}{4}, \frac{3R}{8}-\frac{H}{4}$	$\frac{3R}{8}-\frac{L}{4}, \frac{3R}{8}-\frac{L}{4}$	$\frac{R}{4}-\frac{L}{2}, \frac{3R}{4}-\frac{L}{4}-\frac{H}{4}$
	DD	$R, 0$	$\frac{3R}{4}-\frac{L}{4}-\frac{H}{4}, \frac{R}{4}-\frac{H}{2}$	$\frac{3R}{4}-\frac{L}{4}-\frac{H}{4}, \frac{R}{4}-\frac{L}{2}$	$\frac{R}{2}-\frac{L}{2}-\frac{H}{2}, \frac{R}{2}-\frac{L}{2}-\frac{H}{2}$

To solve for the Bayesian Nash equilibria we need to have more information about the parameters R, H, and L. Assume that R = 8, H = 16, and L = 0, which results in the following matrix-form game:

		Player 2			
		cc	*cd*	*dc*	*dd*
Player 1	*CC*	0, 0	0, 4	0, 4	$\overline{0,\ 8}$
	CD	4, 0	−1, −1	$\overline{-1,\ 3}$	−6, 2
	DC	4, 0	$\underline{3,\ -1}$	$\overline{\underline{3,\ 3}}$	$\underline{2,\ 2}$
	DD	$\underline{8,\ 0}$	2, −6	$\overline{2,\ 2}$	−4, −4

The game has a unique pure-strategy Bayesian Nash equilibrium: (*DC*, *dc*): children of lenient parents will continue driving head on, while those of harsh parents will swerve to avoid the costly consequences. If the payoffs we assumed are indeed representative of this situation then the view that harsh upbringing yields better outcomes can be supported.

12.2.2 Study Groups

Some if not most of you have participated in study groups, in which each student takes upon himself some effort, the fruits of which are shared with the other group members.

Let's consider such a situation with two students, players 1 and 2, who have to hand in a joint lab assignment. Each student *i* can either put in the effort ($e_i = 1$) or shirk ($e_i = 0$), where the cost of putting in the effort is the same for each student and is given by some $0<c<1$, while shirking involves no cost.

If either one or both of the students put in the effort then the lab assignment is a success, while if both shirk then it is a failure.

Students vary in how much they care about their educational success, which is described by the following specification.

Each student has a type $\theta_i \in [0, 1]$, which is independently and uniformly distributed over this interval.

Student *i*'s personal value from a successful assignment is given by the square of his type, θ_i^2.

If student i chooses to put in the effort, his payoff is $\theta_i^2 - c$.

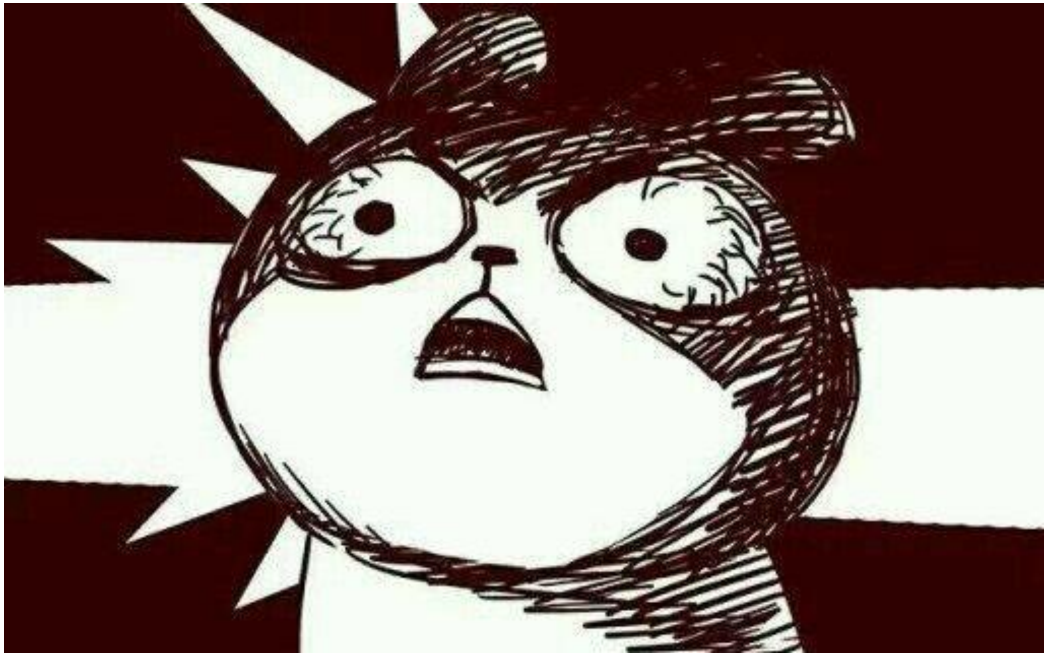
If he chooses to shirk, his payoff depends on what his partner does. If his partner j puts in the effort, student i's payoff is θ_i^2. If his partner j shirks, student i's payoff is 0.

Each student knows only his own type before choosing his effort e.

It is common knowledge that the types are distributed independently and uniformly on [0, 1] and that the cost of effort is c.

This is an example of a Bayesian game with continuous type spaces and discrete sets of actions. Because of the continuous types, it is not useful to draw the game, and we can not derive a matrix.

Despite the two possible actions, $e \in \{0, 1\}$, the continuous types imply that there is a continuum of strategies: a pure strategy for player i is a function $s_i : [0, 1] \rightarrow \{0, 1\}$ that identifies a choice of $e_i \in \{0, 1\}$ for every type $\theta_i \in [0, 1]$.



Given a belief of player i about the strategy of player j, the only factor that affects i's payoff is the probability that player j chooses $e_j = 1$.

If we consider two strategies of player j, $s_j(\theta_j)$ and $s'_j(\theta_j)$—which result in the same probability that he chooses $e_j = 1$, so that $\Pr\{s_j(\theta_j) = 1\} = \Pr\{s'_j(\theta_j) = 1\}$—player i gets the same expected payoff.

The best response of player i will be to choose e = 1if this is (weakly) better than choosing e = 0, which will be true if and only if

$$\theta_i^2 - c \geq \theta_i^2 \Pr\{s_j(\theta_j) = 1\},$$

which can be rewritten as

$$\theta_i \geq \sqrt{\frac{c}{1 - \Pr\{s_j(\theta_j) = 1\}}}. \qquad (12.3)$$

Claim 12.1 *The best response of player i to any strategy $s_j(\theta_j)$ is a **threshold rule:** there exists some $\widehat{\theta}_i$ such that i's best response is to choose $e = 1$ if $\theta_i \geq \widehat{\theta}_i$ and to choose $e = 0$ if $\theta_i \leq \widehat{\theta}_i$.*

This means that we are looking for a Bayesian Nash equilibrium in which each student has a threshold type $\widehat{\theta}_i \in [0, 1]$ such that[4]

$$s_i(\theta_i) = \begin{cases} 0 & \text{if } \theta_i < \widehat{\theta}_i \\ 1 & \text{if } \theta_i \geq \widehat{\theta}_i. \end{cases}$$

This observation lets us calculate what a player's best-response function is given his belief about the threshold strategy of his opponent. If player j is using a threshold $\widehat{\theta}_j$ so that $e = 1$ if and only if $\theta_j \geq \widehat{\theta}_j$ then it follows that $\Pr\{s_j(\theta_j) = 1\} = 1 - \widehat{\theta}_j$. This in turn means that player i will choose $e = 1$ if and only if

$$\theta_i \geq \sqrt{\frac{c}{\widehat{\theta}_j}}. \tag{12.4}$$

This in turn results in the best response of player i. If $\widehat{\theta}_j > c$ then the right side of (12.4) is less than 1, implying that player i's threshold is less than 1. If $\widehat{\theta}_j < c$ then the right side of (12.4) is greater than 1, implying that player i's threshold is equal to 1 (it cannot of course be greater than 1). Define the best response of player i, $BR_i(\widehat{\theta}_j)$, as his best-response threshold strategy $\widehat{\theta}_i$ given that player j is using the threshold $\widehat{\theta}_j$. We therefore have

$$BR_i(\widehat{\theta}_j) = \begin{cases} \sqrt{\frac{c}{\widehat{\theta}_j}} & \text{if } \widehat{\theta}_j \geq c \\ 1 & \text{if } \widehat{\theta}_j < c. \end{cases} \quad (12.5)$$

Claim 12.2 *In the unique Bayesian Nash equilibrium each player chooses the same threshold level $\theta_i = \theta^*$, where $0 < \theta^* < 1$.*

This claim follows directly from the best-response function given in (12.5). First, consider the possibility of a Bayesian Nash equilibrium in which each player i is choosing his threshold on the part of the best-response function where $\widehat{\theta}_j \geq c$ (the top part of (12.5)). We then have two best-response equations with two unknowns,

$$\widehat{\theta}_1 = \sqrt{\frac{c}{\widehat{\theta}_2}} \quad \text{and} \quad \widehat{\theta}_2 = \sqrt{\frac{c}{\widehat{\theta}_1}}, \quad (12.6)$$

which together imply that $\widehat{\theta}_1^2\widehat{\theta}_2 = c$ and $\widehat{\theta}_2^2\widehat{\theta}_1 = c$. Because we assumed that $0 < c < 1$ the equality can be satisfied only if $\widehat{\theta}_1 = \widehat{\theta}_2 = \theta^*$, where $0 < \theta^* < 1$. Substituting $\widehat{\theta}_1 = \widehat{\theta}_2 = \theta^*$ into any of the two best-response functions in (12.6) implies that $\theta^* = c^{\frac{1}{3}} < 1$ (because $c < 1$ and $c^{\frac{1}{3}} > c$).

Next we need to make sure that there is no Bayesian Nash equilibrium in which for some player the threshold is $\widehat{\theta}_j < c$, implying that player i will choose the threshold $\widehat{\theta}_i = BR_i(\widehat{\theta}_j) = 1$ (never choose $e = 1$). But notice that if $\widehat{\theta}_i = 1$ then player j's best response is to choose a threshold equal to $BR_j(1) = \sqrt{c}$, which contradicts the premise that $\widehat{\theta}_j < c$ because $\sqrt{c} > c$ for $c < 1$. Hence the unique Bayesian Nash equilibrium is the symmetric threshold choices $\theta^* = c^{\frac{1}{3}}$, which are implemented by the following strategies for each player $i \in \{1, 2\}$:

$$s_i^*(\theta_i) = \begin{cases} 0 & \text{if } \theta_i < c^{\frac{1}{3}} \\ 1 & \text{if } \theta_i \geq c^{\frac{1}{3}}. \end{cases}$$

12.3 Inefficient Trade and Adverse Selection

One of the main conclusions of competitive market analysis in economics is that markets allocate goods to the people who value them the most.

The simple intuition behind this conclusion works as follows: If a good is misallocated so that some people who have it value it less than people who do not, then so-called market pressures will cause the price of that good to increase to a level at which the current owners will prefer to sell it rather than hold on to it, and the people who value it more will be willing to pay that price.

This powerful argument is based on some assumptions, one of which is that the value of the good is easily understood by all market participants, or in our terminology, *there is perfect information about the value of the good.*

It is important to understand the extent to which these arguments stand or fall in the face of incomplete information, when some people are better informed about the value of goods than others.

To address this question we will develop a simple example that follows in the spirit of the important contribution made by George Akerlof (1970), a contribution that introduced the idea of *adverse selection* into economics and earned its author a Nobel Prize.

Imagine a scenario in which player 1 owns an orange grove. The yield of fruit depends on the quality of the soil and other local conditions, and we assume that through his experience only player 1 knows the quality of the land.

Local geological surveys conclude that the quality of the land may be low, mediocre, or high, each with equal probability 1/3. Thus we can think of the knowledge of player 1 about the quality of the land as his type, $\Theta_1 = \{L, M, H\}$. Assume that the value for player 1 of owning land of type θ_1 is given by the following monetary-equivalent values:

$$v_1(\theta_1) = \begin{cases} 10 & \text{if } \theta_1 = L \\ 20 & \text{if } \theta_1 = M \\ 30 & \text{if } \theta_1 = H. \end{cases}$$

Imagine that player 2 is a soybean grower who is considering the purchase of this land for production. Player 2's family expertise in growing soybeans has been very profitable, but that pursuit also depends on the quality of the land. In particular player 2's monetary-equivalent values are given by

$$v_2(\theta_1) = \begin{cases} 14 & \text{if } \theta_1 = L \\ 24 & \text{if } \theta_1 = M \\ 34 & \text{if } \theta_1 = H. \end{cases}$$

The problem, however, is that player 2 knows only that the quality is distributed equally among the three options (the geological survey's results); he does not know which of the three it is.

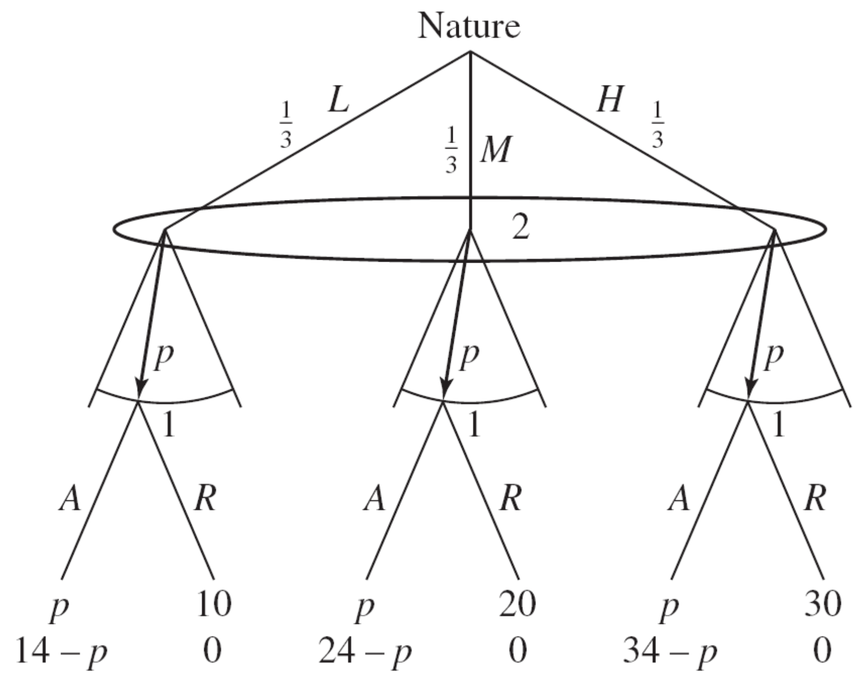


FIGURE 12.5 A trading game with incomplete information.

Consider the following game: player 2 makes a take-it-or-leave-it price offer to player 1, after which player 1 can accept (A) or reject (R) the offer, and the game ends with either a transfer of land for the suggested price or no transfer. A strategy for player 2 is therefore a single price offer, $p \geq 0$, and a pure strategy for player 1 is a mapping from the offered price and his type space Θ_1 to a response, $s_1 : [0,\infty) \times \Theta_1 \rightarrow \{A, R\}$.

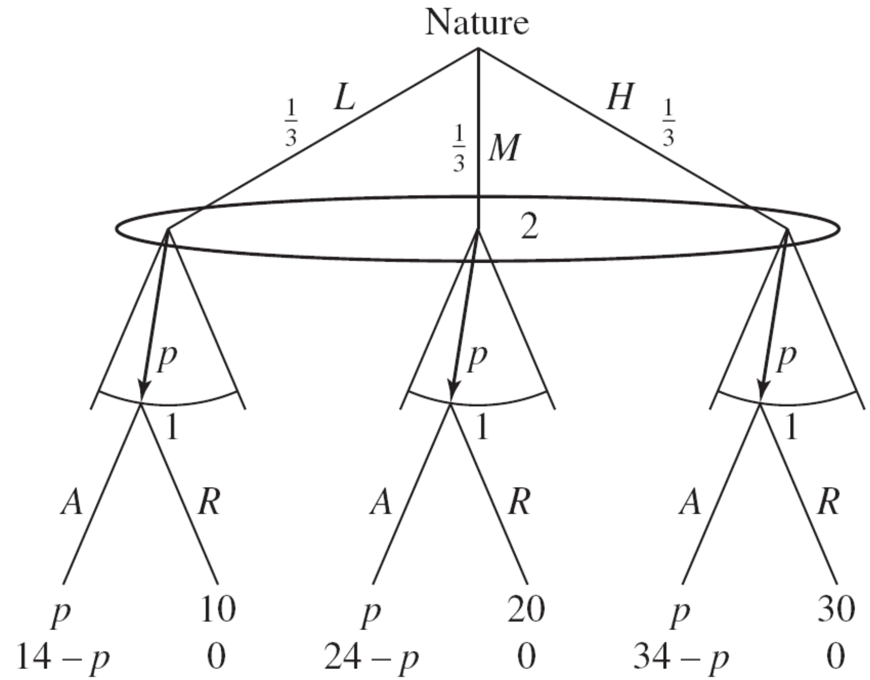


FIGURE 12.5 A trading game with incomplete information.

The assumptions on the payoffs of the two players imply that from an efficiency point of view player 2 should own the land. Indeed *if the quality of the land were common knowledge then there would be many prices at which both player 1 and player 2 would be happy to trade the property*. For example, if the quality were known to be low then, the unique subgame-perfect equilibrium would have player 2 offering player 1 a price of 10 and player 1 accepting.

Let's consider the value that player 2 places on the land. He knows that with equal probability it is worth one of the values $v_2 \in \{14, 24, 34\}$, so on average it is worth 24. He also knows that on average it is worth 20 to player 1. It would seem that the natural equilibrium candidate would be to offer the lowest price at which player 2 thinks that player 1 will accept, and such an offer would be $p = 20$.

What then would be player 1's best response? Player 1 knows the quality of the land, and he would accept the offer only if his type is L or M.

Then, player 2 would get a parcel of land that is of low or mediocre quality, each with equal probability, and receive an expected value of $1/2 \times 14 + 1/2 \times 24 = 19$. Player 2's expected payoff from trading is $Ev_2 = 19 - 20 < 0$, implying that he would be better off not buying the land.

For any offer $p \in [20, 30)$ player 1 would accept the offer only if his type is L or M, and player 2's expected value is still 19, making such a trade impossible in equilibrium. If player 2 is to take into account the best response of player 1, he knows that he will get all types of player 1 to agree to sell only if player 2 offers 30, but in this case he would get only an expected value of 24, which is not profitable. No trade can occur at a price greater than 20.

If trade is to occur it will occur at a price less than 20, which in turn implies that player 1 will agree to such a trade only if his type is L. Taking this into account, player 2 should offer a price no greater than p =14, commensurate with player 1 trading if his type is indeed L.

Proposition 12.1 *Trade can occur in a Bayesian Nash equilibrium only if it involves the lowest type of player 1 trading. Furthermore any price* $p^* \in [10, 14]$ *can be supported as a Bayesian Nash equilibrium.*

To see that any price $p^* \in [10, 14]$ can be supported as a Bayesian Nash equilibrium, consider the following strategies: player 2 offers a price p^*, and the strategy for player 1 is

$$s_1(\theta_1) = \begin{cases} A \text{ if and only if } p \geq p^* \text{ and } R \text{ otherwise} & \text{when } \theta_1 = L \\ A \text{ if and only if } p \geq 20 \text{ and } R \text{ otherwise} & \text{when } \theta_1 = M \\ A \text{ if and only if } p \geq 30 \text{ and } R \text{ otherwise} & \text{when } \theta_1 = H. \end{cases}$$

In this case the strategies of the two players are mutual best responses, and therefore they constitute a Bayesian Nash equilibrium.

The conclusion is that trade will occur only if the quality of the land is the lowest. This happens because of what is called **adverse selection**. When the buyer is willing to pay a price equal to his average value, then the type of seller who is willing to sell at this price is below average, because the best types select not to sell for an average price, hence the adverse selection of lower-than-average sellers. This unraveling causes traded quality to drop to its lowest level, preventing the market from implementing efficient trade outcomes.

It is also worth mentioning that this scenario falls into the category of games with ***common values.*** In this category of games the type of one player affects the payoffs of another player. In this example the type of player 1 affects the payoffs of both player 1 and player 2. It is precisely this feature of the game that causes the adverse effects of equilibrium.

人人车围绕车辆检测建立起来了包含30个大项的249项精细化、体系化检测程序，可以覆盖车况的每个细节。这249个检测项目又分为上门检测和台架检测两个步骤来进行，每一种检测都达到了国家级的标准。人人车专业评估师不仅持有国家《二手车鉴定评估师资格证书》，而且有着丰富车辆检测经验锻炼出来的“火眼金睛”，可以排除掉9成以上的坏车。而后续的台架检测可以进一步排除特情，不放过任何一辆坏车。人人车的车辆检测体系得到了国家二手车检测标准、中国汽车流通协会“行认证”的充分认可。


adverse selection
EXAMPLE :
SMOKERS TEND TO BUY HEALTH INSURANCE MORE THAN NON-SMOKERS.
Buzzle.com

12.4 Committee Voting

Many decisions are made by committees. Examples include legislatures, firms, membership clubs, and juries.

Each member of the committee will have different information, or different ways of interpreting information, and the goals of the committee members may be congruent or diverse. Because each committee member has private information about his own preference or about the value of different decisions to other players, committee votes can be modeled as Bayesian games.

As an example, consider a jury made up of two players (jurors) who must collectively decide whether to acquit (*A*) or to convict (*C*) a defendant. The process calls for each player to cast a sealed vote, and *the defendant is convicted only if both vote C*.

There is uncertainty about whether the defendant is guilty (*G*) or innocent (*I*). The prior probability that the defendant is guilty is $q > 1/2$ and is common knowledge.

Assume that *each player cares about making the right decision*. If the defendant is guilty, each player receives a payoff of 1 from a conviction and 0 from an acquittal. If the defendant is innocent, each player receives a payoff of 0 from a conviction and 1 from an acquittal.

If the only information available to the players is the probability q then the game can be described by:

		Player 2	
		A	*C*
Player 1	*A*	1-q, 1-q	1-q, 1-q
	C	1-q, 1-q	q, q

If either of the players chooses *A*, the defendant is acquitted and each player receives an expected payoff of $1-q$. If both choose *C* the defendant will be convicted and each will receive a payoff of q.

Because $q > 1/2$ each player has a weakly dominant strategy, which is to vote *C* and convict the defendant.

Imagine that things are a bit more complex. Each player i has a different expertise and, when observing the evidence, gets a private signal $\theta_i \in \{\theta_G, \theta_I\}$ that contains valuable information.

When the defendant is guilty player i is more likely to receive the signal $\theta_i = \theta_G$ than when the defendant is innocent, and vice versa for $\theta_i = \theta_I$.

The signals are independent. The probability of receiving the signal θ_G when the defendant is guilty is equal to the probability of receiving the signal θ_I when the defendant is innocent, so that

$$\Pr\{\theta_i = \theta_G | G\} = \Pr\{\theta_i = \theta_I | I\} = p > \tfrac{1}{2} \quad \text{for } i \in \{1, 2\}$$

and

$$\Pr\{\theta_i = \theta_G | I\} = \Pr\{\theta_i = \theta_I | G\} = 1 - p < \tfrac{1}{2} \quad \text{for } i \in \{1, 2\}.$$

With this information structure, we have a Bayesian game of incomplete information in which each player i will choose a signal-dependent action to maximize the probability that a guilty defendant is convicted while an innocent one is acquitted.

Each player has two types, given by the signal he observes. Each player will have four pure strategies, $s_i \in \{AA, AC, CA, CC\}$, where xy means that he chooses x when his signal is θ_G and he chooses y when his signal is θ_I.

A natural first question is as follows: if each player would be able to convict or acquit the defendant by himself, how would his signal determine his choice?

Consider the decision problem with just one player. Without receiving the signal, the player knows that the defendant is guilty with probability $q > 1/2$, so that he would choose to convict the defendant. After receiving the signal, however, the player will update his beliefs about the defendant. If the signal was θ_G then the updated belief is

$$\Pr\{G|\theta_i = \theta_G\} = \frac{\Pr\{G \text{ and } \theta_i = \theta_G\}}{\Pr\{\theta_i = \theta_G\}} = \frac{qp}{qp + (1-q)(1-p)} > q,$$

which means that the player is more convinced that the defendant is guilty, and will choose to convict him.

If the signal was θ_I then the updated belief is

$$\Pr\{G|\theta_i=\theta_I\}=\frac{\Pr\{G \text{ and } \theta_I\}}{\Pr\{\theta_I\}}=\frac{q(1-p)}{q(1-p)+(1-q)p}<q.$$

The player is less sure of the defendant's guilt than he was before the signal, and whether this is enough to persuade him to acquit the defendant depends on the value of p. He will choose to acquit the defendant if and only if

$$\frac{q(1-p)}{q(1-p)+(1-q)p}<\frac{1}{2},$$

which reduces to $p>q$. When p is high enough， signal is informative enough about the defendant's actual condition.

The observation that if $p > q$ then each player would choose to vote according to his signal in the one-person decision problem leads to the next question: *is the pair of strategies in the game,* $s_i = CA$*, where each votes according to his signal, a Bayesian Nash equilibrium*?

Intuitively it seems like it should be because the signal that each player receives is informative about the actual condition of the defendant.

To see whether this is indeed the case we need to verify whether *CA* is a best response to *CA* for each player.

Before proceeding to answer this question, it will be useful to calculate the probabilities of the different informational states of the two signals that the players can receive. There are four relevant informational states: either both receive the signal θ_G, both receive the signal θ_I, or each player receives a different signal (two states).

	$\theta_2=\theta_G$	$\theta_2=\theta_I$
$\theta_1=\theta_G$	$qp^2+(1-q)(1-p)^2$	$p(1-p)$
$\theta_1=\theta_I$	$p(1-p)$	$q(1-p)^2+(1-q)p^2$

For example, the probability that player 1 receives a signal θ_G and that player 2 receives a signal θ_I is equal to $qp(1-p)+(1-q)(1-p)p = p(1-p)$.

Unanimity is needed to convict the defendant. A player is decisive, or “pivotal,” only if the other player chooses C. If player j chooses A then the defendant will be acquitted for sure, regardless of what player i choose. If player j chooses C then the decision of whether the defendant will be convicted or not depends on the choice of player i.

For this reason, if player i believes that player j is playing according to the strategy CA, then he must believe that his own vote matters only when player j observes a signal $\theta_{j=}\theta_G$.

We therefore need to calculate the posterior belief that player i has about whether the defendant is guilty conditional on his own signal and on the belief that player j's signal is θ_G.

If player *i*'s signal was $\theta_{i=}\theta_G$ then his updated belief is

$$\Pr\{G|\theta_i=\theta_G \text{ and } \theta_j=\theta_G\}=\frac{\Pr\{G \text{ and } \theta_i=\theta_G \text{ and } \theta_j=\theta_G\}}{\Pr\{\theta_i=\theta_G \text{ and } \theta_j=\theta_G\}}$$

$$=\frac{qp^2}{qp^2+(1-q)(1-p)^2}>q,$$

which means that the player is even more convinced that the defendant is guilty and hence will choose C to guarantee that the defendant is convicted when both have the signal θ_G.

If the signal was θ_I then the updated belief of player i conditional on the belief that player j's signal is θ_G is

$$\Pr\{G|\theta_i=\theta_I \text{ and } \theta_j=\theta_G\}=\frac{\Pr\{G \text{ and } \theta_i=\theta_I \text{ and } \theta_j=\theta_G\}}{\Pr\{\theta_i=\theta_I \text{ and } \theta_j=\theta_G\}}=\frac{q(1-p)p}{p(1-p)}=q.$$

When player i is conditioning the value of his signal on the event when his vote actually counts, which is when player j chooses C, then his signal $\theta_{i=}\theta_I$ becomes less convincing about the defendant's innocence.

Given the symmetric informational structure of the signals, if player i is in the situation in which he believes that he is pivotal then observing a signal of innocence is canceled out by his belief that player j saw a signal of guilt.

We conclude that playing $s_i = CA$ for both players is not a Bayesian Nash equilibrium. In fact both players choosing CC, always convict regardless of the signal, is a Bayesian Nash equilibrium (showing this is left as part of exercise 12.8).

This means that when both players receive a signal $\theta_{i=}\theta_I$, which occurs with probability $q(1\text{-}p)^2 + (1\text{-}q)\, p^2 > 0$, despite the fact that it is more likely that the defendant is innocent, he will still be convicted.

This example is a simple case of what Feddersen and Pesendorfer (1996) refer to as the “swing voter’s curse”(摇摆选民诅咒) and is also closely related to independent work by Austen-Smith and Banks (1996). When a voter believes that his vote counts, he must condition this on the situation in which his vote counts. But then it means that he will interpret his information differently and not use it in an unbiased way.

The jury game here is a simple two-player example of the more general analysis in Feddersen and Pesendorfer (1998), who show that with more than two players there will sometimes be conditions under which players will vote based on their information, but that the problem of the swing voter’s curse is generally present.

12.5 Mixed Strategies Revisited: Harsanyi's Interpretation

Consider the static game of Matching Pennies, given by the following matrix:

		Player 2	
		H	*T*
Player 1	*H*	1, -1	-1, 1
	T	-1, 1	1, -1

The unique mixed-strategy Nash equilibrium has each player playing *H* with probability 1/2. One reason this solution may be somewhat unappealing is that players are indifferent between *H* and *T*, yet they are prescribed to randomize between these strategies in a unique, particular, and precise way for this to be a Nash equilibrium. Does it make sense to expect such precision when a player is indifferent?

This question has caused some discomfort with the notion of mixed-strategy equilibria. John Harsanyi (1973) offered a twist on the basic model of behavior to resolve this problem and alleviate, to some extent, the indifference problem. His idea works as follows.

Imagine that each player may have some slight preference for choosing heads over tails or choosing tails over heads. This is done in such a way as to "break" the indifference of a player's best response if he believes that the probability of his opponent playing *H* is exactly 1/2 .

Imagine that the payoffs are given by this "perturbed" Matching Pennies game:

Player 2

Player 1		*H*	*T*
	H	$1+\varepsilon_1$, $-1+\varepsilon_2$	$-1+\varepsilon_1$, 1
	T	-1, $1+\varepsilon_2$	1, -1

where ε_1 and ε_2 are independent and uniformly distributed on the interval $[-\varepsilon, \varepsilon]$ for some small $\varepsilon > 0$.

If $\varepsilon_i > 0$ is realized, player *i* has a strict preference for choosing *H* over *T* when he believes his opponent is choosing *H* with probability ½. If $\varepsilon_i < 0$ is realized, player *i* has a strict preference for choosing *T* over *H* when he believes his opponent is choosing *H* with probability 1/2 .

Assume further that the value of ε_i is known only to player *i* but that the distribution of of ε_i is common knowledge. This perturbed Matching Pennies game is a Bayesian game of incomplete information with two actions for each player and a continuum of types. A pure strategy for each player is a mapping s_i: $[-\varepsilon, \varepsilon] \rightarrow \{H, T\}$ that assigns a choice to every type of player *i*.

Claim 12.3 *In the Bayesian perturbed Matching Pennies game, there is a unique pure-strategy Bayesian Nash equilibrium in which* $s_i(\varepsilon_i) = H$ *if and only if* $\varepsilon_i \geq 0$, *and* $s_i(\varepsilon_i) = L$ *if and only if* $\varepsilon_i < 0$. *This equilibrium converges in outcomes and payoffs to the Matching Pennies game when* $\varepsilon \rightarrow 0$.

It is easy to see that the proposed strategies are a Bayesian Nash equilibrium. Because the distribution of ε_i is uniform over the interval $[-\varepsilon, \varepsilon]$, it follows that with probability 1/2 player i is playing H, in which case the strategy of player j is a best response.

This example is a simple special case of Harsanyi's purification theorem, following the idea that we can use incomplete information to "purify" any mixed-strategy equilibrium of a game of complete information.

It implies that if people are somewhat heterogeneous in the way monetary payoffs and actions are related, then we can have uncertainty over the types of players who are playing pure strategies, but the distribution of types makes a player have beliefs as if he were facing a player who is playing a mixed strategy.

Harsanyi argues that using mixed-strategy equilibria in simple games of complete information can be thought of as a solution to the more complex games of incomplete information, in which players do not randomize but rather have strict best responses. The interested reader should refer to Govindan (2003) for a short and elegant presentation of Harsanyi’s approach.

12.2 **Cournot Revisited:** Consider the Cournot duopoly model in which two firms, 1 and 2, simultaneously choose the quantities they supply, q_1 and q_2. The price each will face is determined by the market demand function $p(q_1, q_2) = a - b(q_1 + q_2)$. Each firm has a probability μ of having a marginal unit cost of c_L and a probability $1 - \mu$ of having a marginal unit cost of c_H. These probabilities are common knowledge, but the true type is revealed only to each firm individually. Solve for the Bayesian Nash equilibrium.

12.3 **Armed Conflict:** Consider the following strategic situation: Two rival armies plan to seize a disputed territory. Each army's general can choose either to attack (A) or to not attack (N). In addition, each army is either strong (S) or weak (W) with equal probability, and the realizations for each army are *independent*. Furthermore the type of each army is known only to that army's general. An army can capture the territory if either (i) it attacks and its rival does not or (ii) it and its rival attack, but it is strong and the rival is weak. If both attack and are of equal strength then neither captures the territory. As for payoffs, the territory is worth m if captured and each army has a cost of fighting equal to s if it is strong and w if it is weak, where $s < w$. If an army attacks but its rival does not, no costs are borne by either side. Identify all the pure-strategy Bayesian Nash equilibria of this game for the following two cases, and briefly describe the intuition for your results:

a. $m = 3$, $w = 2$, $s = 1$.
b. $m = 3$, $w = 4$, $s = 2$.

$$v_i((s_i, s_j) \mid \theta) = \mu[a - b(s_i(\theta) + s_j(L)) - c_i(\theta) \times s_i(\theta)]$$

$$+ (1 - \mu)[a - b(s_i(\theta) + s_j(\mathrm{H})) - c_i(\theta) \times s_i(\theta)]$$

12.5 **Not All That Glitters:** A prospector owns a gold mine where he can dig to recover gold. His output depends on the amount of gold in the mine, denoted by x. The prospector knows the value of x, but the rest of the world knows only that the amount of gold is uniformly distributed on the interval $[0, 1]$. Before deciding to mine, the prospector can try to sell his mine to a large mining company, which is much more efficient in its extraction methods. The prospector can ask the company owner for any price $p \geq 0$, and the owner can reject (R) or accept (A) the offer. If the owner rejects the offer then the prospector is left to mine himself, and his payoff from self-mining is equal to $3x$. If the owner accepts the offer then the prospector's payoff is the price p, while the owner's payoff is given by the net value $4x - p$, and this is common knowledge.

a. Show that for a given price $p \geq 0$ there is a threshold type $x(p) \in [0, 1]$ of prospector, such that types below $x(p)$ will prefer to sell the mine, while types above $x(p)$ will prefer to self-mine.
b. Find the pure-strategy Bayesian Nash equilibrium of this game, and show that it is unique. What is the expected payoff of each type of prospector and of the company owner in the equilibrium you derived?