

UNIT 2. SETS AND RELATIONS

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1

SETS

Section 2.1

2

SECTION SUMMARY

- Definition of sets
- Describing Sets
 - Roster Method
 - Set-Builder Notation
- Some Important Sets in Mathematics
- Empty Set and Universal Set
- Subsets and Set Equality
- Cardinality of Sets
- Tuples
- Cartesian Product

INTRODUCTION

- Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.
 - Important for counting.
 - Programming languages have set operations.
- Set theory is an important branch of mathematics.

SETS

- A *set* is an unordered collection of objects.
 - the students in this class
 - the chairs in this room
- The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.
- The notation $a \in A$ denotes that a is an element of the set A .
- If a is not a member of A , write $a \notin A$

DESCRIBING A SET: ROSTER METHOD

- $S = \{a, b, c, d\}$
- Order not important

$$S = \{a, b, c, d\} = \{b, c, a, d\}$$

- Each distinct object is either a member or not; listing more than once does not change the set.

$$S = \{a, b, c, d\} = \{a, b, c, b, c, d\}$$

- Ellipses (...) may be used to describe a set without listing all of the members when the pattern is clear.

$$S = \{a, b, c, d, \dots, z\}$$

ROSTER METHOD (枚举法)

- Set of all vowels in the English alphabet:

$$V = \{a, e, i, o, u\}$$

- Set of all odd positive integers less than 10:

$$O = \{1, 3, 5, 7, 9\}$$

- Set of all positive integers less than 100:

$$S = \{1, 2, 3, \dots, 99\}$$

- Set of all integers less than 0:

$$S = \{\dots, -3, -2, -1\}$$

SOME IMPORTANT SETS

\mathbf{N} = *natural numbers* = $\{0,1,2,3,\dots\}$

\mathbf{Z} = *integers* = $\{\dots,-3,-2,-1,0,1,2,3,\dots\}$

\mathbf{Z}^+ = *positive integers* = $\{1,2,3,\dots\}$

\mathbf{R} = *set of real numbers*

\mathbf{R}^+ = *set of positive real numbers*

\mathbf{C} = *set of complex numbers.*

\mathbf{Q} = *set of rational numbers*

SET-BUILDER NOTATION (构造符号)

- Specify the property or properties that all members must satisfy:

$$S = \{x \mid x \text{ is a positive integer less than } 100\}$$

$$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$$

$$O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}$$

- A predicate may be used:

$$S = \{x \mid P(x)\}$$

- Example: $S = \{x \mid \text{Prime}(x)\}$

- Positive rational numbers:

$$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p, q\}$$

INTERVAL NOTATION

$$[a,b] = \{x \mid a \leq x \leq b\}$$

$$[a,b) = \{x \mid a \leq x < b\}$$

$$(a,b] = \{x \mid a < x \leq b\}$$

$$(a,b) = \{x \mid a < x < b\}$$

closed interval $[a,b]$

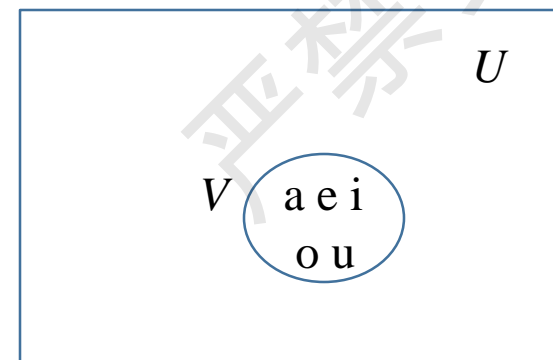
open interval (a,b)

UNIVERSAL SET AND EMPTY SET

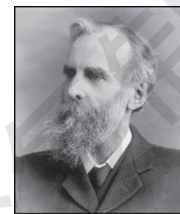
- The *universal set* U is the set containing everything currently under consideration.
 - Sometimes implicit
 - Sometimes explicitly stated.
 - Contents depend on the context.
- The empty set is the set with no elements.

Symbolized \emptyset , but $\{ \}$ also used.

Venn Diagram
文氏图



John Venn (1834-1923)
Cambridge, UK



SOME THINGS TO REMEMBER

- Sets can be elements of sets.

$$\{\{1,2,3\}, a, \{b,c\}\}$$

$$\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$$

- The empty set is different from a set containing the empty set.

$$\emptyset \neq \{ \emptyset \}$$

RUSSELL'S PARADOX

- Let S be the set of all sets which are not members of themselves. A paradox results from trying to answer the question “Is S a member of itself?”
- Related Paradox:
 - Henry is a barber who shaves all people who do not shave themselves. A paradox results from trying to answer the question “Does Henry shave himself?”

Bertrand Russell (1872-1970)
Cambridge, UK
Nobel Prize Winner



SET EQUALITY

Definition: Two sets are *equal* if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if
• $\forall x(x \in A \leftrightarrow x \in B)$
- We write $A = B$ if A and B are equal sets.

$$\{1, 3, 5\} = \{3, 5, 1\}$$

$$\{1, 5, 5, 5, 3, 3, 1\} = \{1, 3, 5\}$$

SUBSETS

Definition: The set A is a *subset* of B , if and only if every element of A is also an element of B .

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B .
- $A \subseteq B$ holds if and only if $\forall x(x \in A \rightarrow x \in B)$ is true.
 1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$, for every set S .
 2. Because $a \in S \rightarrow a \in S$, $S \subseteq S$, for every set S .

SHOWING A SET IS OR IS NOT A SUBSET OF ANOTHER SET

- **Showing that A is a Subset of B:** To show that $A \subseteq B$, show that if x belongs to A , then x also belongs to B .
- **Showing that A is not a Subset of B:** To show that A is not a subset of B , $A \not\subseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

Examples:

1. The set of all computer science majors at your school is a subset of all students at your school.
2. The set of integers with squares less than 100 is not a subset of the set of nonnegative integers.

ANOTHER LOOK AT EQUALITY OF SETS

- Recall that two sets A and B are *equal*, denoted by $A = B$, iff $\forall x(x \in A \leftrightarrow x \in B)$

- Using logical equivalences we have that $A = B$ iff

$$\forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

- This is equivalent to

$$A \subseteq B \quad \text{and} \quad B \subseteq A$$

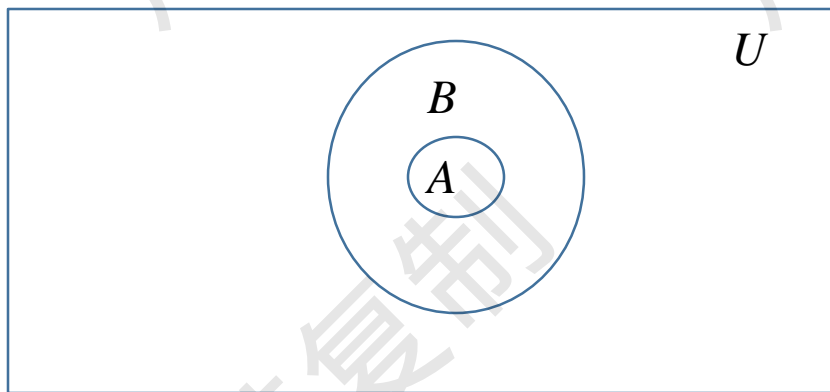
PROPER SUBSETS (真子集)

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B , denoted by $A \subset B$. If $A \subset B$, then

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true.

Venn Diagram



SET CARDINALITY (基数, 势)

Definition: If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is *finite*. Otherwise it is *infinite*.

Definition: The *cardinality* of a finite set A , denoted by $|A|$, is the number of (distinct) elements of A .

Examples:

1. $|\emptyset| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1, 2, 3\}| = 3$
4. $|\{\emptyset\}| = 1$
5. The set of integers is infinite.

POWER SETS (幂集)

Definition: The set of all subsets of a set A , denoted $P(A)$, is called the *power set* of A .

Example: If $A = \{a, b\}$ then

$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

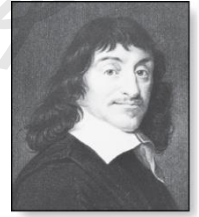
If a set has n elements, then the cardinality of the power set is 2^n .

TUPLES (元组)

- The *ordered n-tuple* (a_1, a_2, \dots, a_n) is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.
- Two n-tuples are equal if and only if their corresponding elements are equal.
- 2-tuples are called *ordered pairs* (序偶) .
- The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

CARTESIAN PRODUCT

René Descartes
(1596-1650)



Definition: The *Cartesian Product* of two sets A and B , denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a,b) | a \in A \wedge b \in B\}$$

Example:

$$A = \{a,b\} \quad B = \{1,2,3\}$$

$$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$$

- **Definition:** A subset R of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B .

CARTESIAN PRODUCT

Definition: The Cartesian products of the sets A_1, A_2, \dots, A_n , denoted by $A_1 \times A_2 \times \dots \times A_n$, is the set of ordered n -tuples (a_1, a_2, \dots, a_n) where a_i belongs to A_i for $i = 1, \dots, n$.

$$A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \dots, n\}$$

Example: What is $A \times B \times C$ where $A = \{0,1\}$, $B = \{1,2\}$ and $C = \{0,1,2\}$

Solution: $A \times B \times C = \{(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)\}$

TRUTH SETS OF QUANTIFIERS (真值集合)

- Given a predicate P and a domain D , we define the *truth set* of P to be the set of elements in D for which $P(x)$ is true. The truth set of $P(x)$ is denoted by $\{x \in D \mid P(x)\}$
- **Example:** What are the truth sets of $P(x)$, $Q(x)$ and $R(x)$ where the domain is the integers and $P(x)$ is “ $|x| = 1$ ”, $Q(x)$ is “ $x^2 = 2$ ”, and $R(x)$ is “ $|x| = x$ ”
 - The truth set of $P(x)$ is the set $\{-1, 1\}$;
 - The truth set of $Q(x)$ is the set \emptyset ;
 - The truth set of $R(x)$ is the set \mathbf{N} ;