REPRESENTING RELATIONS

Section 9.3

SECTION SUMMARY

- Representing Relations using Matrices
- Representing Relations using Digraphs

REPRESENTING RELATIONS USING MATRICES

- A relation between finite sets can be represented using a zero-one matrix.
- Suppose R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$.
  - The elements of the two sets can be listed in any particular arbitrary order. When $A = B$, we use the same ordering.
- The relation R is represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

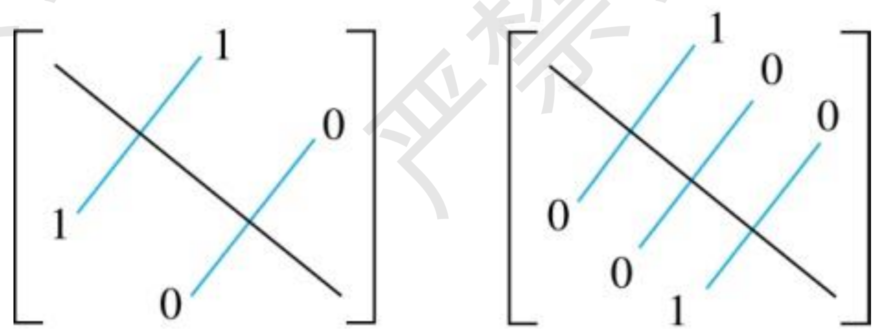
- The matrix representing R has a 1 as its (i,j) entry when a_i is related to b_j and a 0 if a_i is not related to b_j.

MATRICES OF RELATIONS ON SETS

- If R is a reflexive relation, all the elements on the main diagonal of M_R are equal to 1.

$$\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix}$$

- R is a symmetric relation, if and only if $m_{ij} = 1$ whenever $m_{ji} = 1$. R is an antisymmetric relation, iff $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.



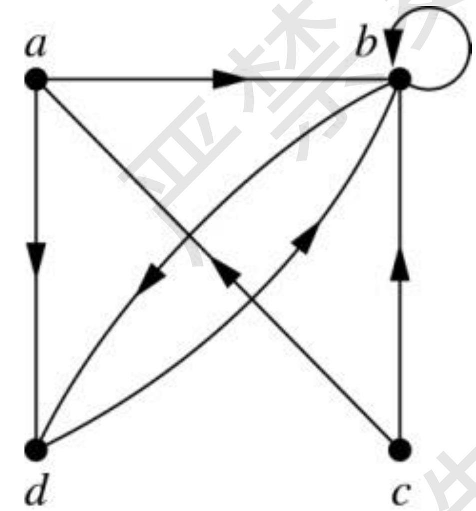
(a) Symmetric (b) Antisymmetric

REPRESENTING RELATIONS USING DIGRAPHS

Definition: A *directed graph*, or *digraph*, consists of

- **a set *V* of *vertices* (or *nodes*) together with**
- **a set *E* of ordered pairs of elements of *V* called *edges* (or *arcs*).**

- The vertex *a* is called the *initial vertex* of the edge (*a*,*b*), and the vertex *b* is called the *terminal vertex* of this edge.
- An edge of the form (*a*,*a*) is called a *loop*.



Example: A drawing of the directed graph with vertices *a*, *b*, *c*, and *d*, and edges (*a*, *b*), (*a*, *d*), (*b*, *b*), (*b*, *d*), (*c*, a), (*c*, *b*), and (*d*, *b*) is shown here.

DETERMINING WHICH PROPERTIES A RELATION HAS FROM ITS DIGRAPH

- *Reflexivity*: A loop must be present at all vertices in the graph.
- *Symmetry*: If (x,y) is an edge, then so is (y,x).
- *Antisymmetry*: If (x,y) with $x \neq y$ is an edge, then (y,x) is not an edge.
- *Transitivity*: If (x,y) and (y,z) are edges, then so is (x,z).

CLOSURES OF RELATIONS

Section 9.4

INTRODUCTION – WHY NEED CLOSURES

- Let R be the relation containing (a, b) if there is a telephone line from the data center in a to that in b.
  - How can we determine if there is some (possibly indirect) link composed of one or more telephone lines from one center to another?
- Similarly, how to determine the links between cities through high-speed trains?

Solution:

- By constructing a transitive relation S containing R such that S is a subset of every transitive relation containing R. Here, S is the smallest transitive relation that contains R.

CLOSURES

- **Definition:** Let R be a relation on a set A.
  - R may or may not have some property P, such as reflexivity, symmetry, or transitivity.
  - If there is a relation S with property P containing R such that S is a subset of every relation with property P containing R, then
  - S is called the **closure of *R* with respect to *P*.**
    - Reflexive closures
    - Symmetric closures, and
    - Transitive closures

REFLEXIVE CLOSURES

- Construction method
  - Given a relation R on a set A, the reflexive closure of R can be formed by
    - **adding to R all pairs of the form (a, a) with $a \in A$, not already in R.**
  - The addition of these pairs produces a new relation that is reflexive, contains R, and is contained within any reflexive relation containing R.
- The reflexive closure of R,
  - $r(R) = R \cup \Delta$, where

    $\Delta = \{(a, a) \mid a \in A\}$ is the **diagonal relation** on A.

SYMMETRIC CLOSURES

- Construction method
  - The symmetric closure of a relation R can be constructed by
    - **adding all ordered pairs of the form (b, a), where (a, b) is in the relation, that are not already present in R.**
  - Adding these pairs produces a relation that is symmetric, that contains R, and that is contained in any symmetric relation that contains R.
- The symmetric closure of a relation can be constructed by taking the union of a relation with its **inverse**; that is,
  - $s(R) = R \cup R^{-1}$, where
    $R^{-1} = \{(b, a) \mid (a, b) \in R\}$.

TRANSITIVE CLOSURES

- Suppose that a relation R is not transitive. How can we produce a ***transitive relation*** that contains R such that this new relation is contained within any transitive relation that contains R?
- It is more complicated than reflexive closures and symmetric closures!
  - Representing relations by directed graphs helps in the construction of transitive closures.

PATHS IN DIRECTED GRAPHS

Definition: A *path* from a to b in the directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$ in G, where

- n is a nonnegative integer, and
- $x_0 = a$ and $x_n = b$.

- That is, it is a sequence of edges where the terminal vertex of an edge is the same as the initial vertex in the next edge in the path.
- This path is denoted by $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ and has *length* n.
- We view the empty set of edges as a path of length zero from a to a.
- A path of length $n \geq 1$ that begins and ends at the same vertex is called a *circuit* or *cycle*.

TRANSITIVE CLOSURES

Definition: Let R be a relation on a set A. The *connectivity relation* R^* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R.

$$R^* = \bigcup_{n=1}^{\infty} R^n.$$

TRANSITIVE CLOSURES

- **Theorem**: The transitive closure of a relation R equals the connectivity relation R^*
- **Lemma**: Let A be a set with ***n* elements**, and let R be a relation on A. If there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n. Moreover, when $a = b$, if there is a path of length at least one in R from a to b, then there is such a path with length not exceeding $n - 1$.
- **Theorem**: Let $\mathbf{M}_R$ be the zero–one matrix of the relation R on a set with n elements. Then the zero–one matrix of the transitive closure R^* is

$$\mathbf{M}_{R*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdot \cdot \cdot \vee \mathbf{M}_R^{[n]}$$

TRANSITIVE CLOSURES

Examples: Find the zero–one matrix of the transitive closure of the relation R where

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution: By the Theorem, it follows that the zero–one matrix of R^* is

$$\mathbf{M}_{R*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]}.$$

$$\mathbf{M}_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

WARSHALL'S ALGORITHM

- Warshall's algorithm is an efficient method for computing the transitive closure of a relation.

ALGORITHM 2 Warshall Algorithm.

procedure *Warshall* ($\mathbf{M}_R$: $n \times n$ zero–one matrix)
$\mathbf{W} := \mathbf{M}_R$
for $k := 1$ **to** n
  for $i := 1$ **to** n
    for $j := 1$ **to** n
      $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$
return $\mathbf{W}\{\mathbf{W} = [w_{ij}]$ is $\mathbf{M}_{R^*}\}$

EQUIVALENCE RELATIONS

Section 9.5

SECTION SUMMARY

- Equivalence Relations
- Equivalence Classes
- Equivalence Classes and Partitions

EQUIVALENCE RELATIONS

Definition 1: A relation on a set A is called an *equivalence relation* if it is **reflexive, symmetric, and transitive**.

Definition 2: Two elements a, and b that are related by an equivalence relation are called *equivalent*.

- Denoted as $a \sim b$ with respect to a particular equivalence relation.

EQUIVALENCE CLASSES

Definition 3: Let R be an equivalence relation on a set A. **The set of all elements that are related to an element *a* of *A*** is called the ***equivalence class* of *a***.

- The equivalence class of a with respect to R is denoted by $[a]_R$.
- When only one relation is under consideration, we can write $[a]$, without the subscript R, for this equivalence class.
- $[a]_R = \{s|(a,s) \in R\}$.
- If $b \in [a]_R$, then b is called a representative of this equivalence class. Any element of a class can be used as a representative of the class.

EQUIVALENCE CLASSES AND PARTITIONS

Theorem 1: let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

(i) aRb

(ii) $[a] = [b]$

(iii) $[a] \cap [b] \neq \emptyset$

Proof: We show that (i) implies (ii). Assume that aRb. Now suppose that $c \in [a]$. Then aRc. Because aRb and R is symmetric, bRa. Because R is transitive and bRa and aRc, it follows that bRc. Hence, $c \in [b]$. Therefore, $[a] \subseteq [b]$. A similar argument shows that $[b] \subseteq [a]$. Since $[a] \subseteq [b]$ and $[b] \subseteq [a]$, we have shown that $[a] = [b]$.

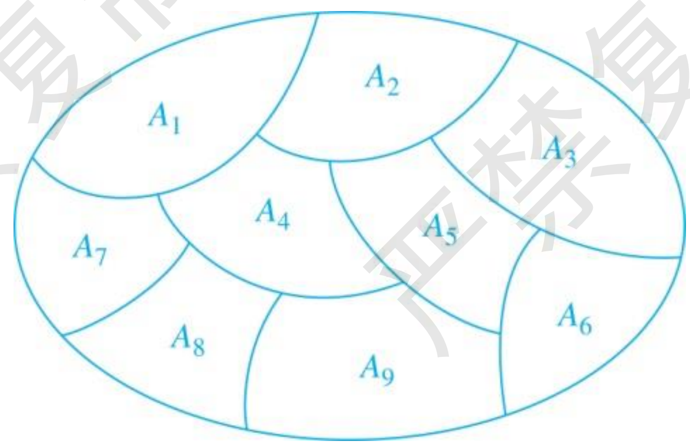
(*see text for proof that* (ii) *implies* (iii) *and* (iii) *implies* (i))

PARTITION OF A SET

Definition: A ***partition*** of a set S is a collection of disjoint nonempty subsets of S that have S as their union.

In other words, the collection of subsets A_i, where $i \in I$ (where I is an index set), forms a partition of S if and only if

- $A_i \neq \emptyset$ for $i \in I$,
- $A_i \cap A_j = \emptyset$ when $i \neq j$,
- and $\bigcup_{i \in I} A_i = S.$



A Partition of a Set

AN EQUIVALENCE RELATION PARTITIONS A SET

- Let R be an equivalence relation on a set A. **The union of all the equivalence classes of *R*** is all of A, since an element a of A is in its own equivalence class $[a]_R$.
- In other words,

$$\bigcup_{a \in A} [a]_R = A.$$

  - From Theorem 1, it follows that these equivalence classes are either equal or disjoint, so $[a]_R \cap [b]_R = \emptyset$ when $[a]_R \neq [b]_R$.
  - Therefore, the equivalence classes form a partition of A, because they split A into disjoint subsets.

AN EQUIVALENCE RELATION PARTITIONS A SET

Theorem 2: Let R be an equivalence relation on a set S.

- The equivalence classes of R form a partition of S.
- Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Proof: We have already shown the first part of the theorem.

For the second part, assume that $\{A_i \mid i \in I\}$ is a partition of S. Let R be the relation on S consisting of the pairs (x, y) where x and y belong to the same subset A_i in the partition. We must show that R satisfies the properties of an equivalence relation.

- *Reflexivity*: For every $a \in S$, $(a,a) \in R$, because a is in the same subset as itself.
- *Symmetry*: If $(a,b) \in R$, then b and a are in the same subset of the partition, so $(b,a) \in R$.
- *Transitivity*: If $(a,b) \in R$ and $(b,c) \in R$, then a and b are in the same subset of the partition, as are b and c. Since the subsets are disjoint and b belongs to both, the two subsets of the partition must be identical. Therefore, $(a,c) \in R$ since a and c belong to the same subset of the partition.

PARTIAL ORDERINGS

Section 9.6

SECTION SUMMARY

- Partial Orderings and Partially-ordered Sets
- Lexicographic Orderings
- Hasse Diagrams

PARTIAL ORDERINGS

Definition 1: A relation R on a set S is called a *partial ordering*, or *partial order*, if it is **reflexive**, **antisymmetric**, and **transitive**.

A set together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R). Members of S are called *elements* of the poset.

COMPARABILITY

Definition 2:

- The elements a and b of a poset $(S,\preccurlyeq)$ are ***comparable*** if either $a \preccurlyeq b$ or $b \preccurlyeq a$.
- When a and b are elements of S so that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, then a and b are called ***incomparable***.

The symbol $\preccurlyeq$ is used to denote the relation in any poset.

Definition 3: If $(S,\preccurlyeq)$ is a poset and every two elements of S are comparable,

- S is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order.*
- A totally ordered set is also called a *chain.*

Definition 4: $(S,\preccurlyeq)$ is a well-ordered set if

- it is a poset such that $\preccurlyeq$ is **a total ordering** and every nonempty subset of S has a least element.

LEXICOGRAPHIC ORDER

Definition: Given two posets $(A_1,\preccurlyeq_1)$ and $(A_2,\preccurlyeq_2)$, the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1,b_2), that is,

$$(a_1, a_2) \prec (b_1,b_2), \quad \text{either if } a_1 \prec_1 b_1 \text{ or if } a_1 = b_1 \text{ and } a_2 \prec_2 b_2.$$

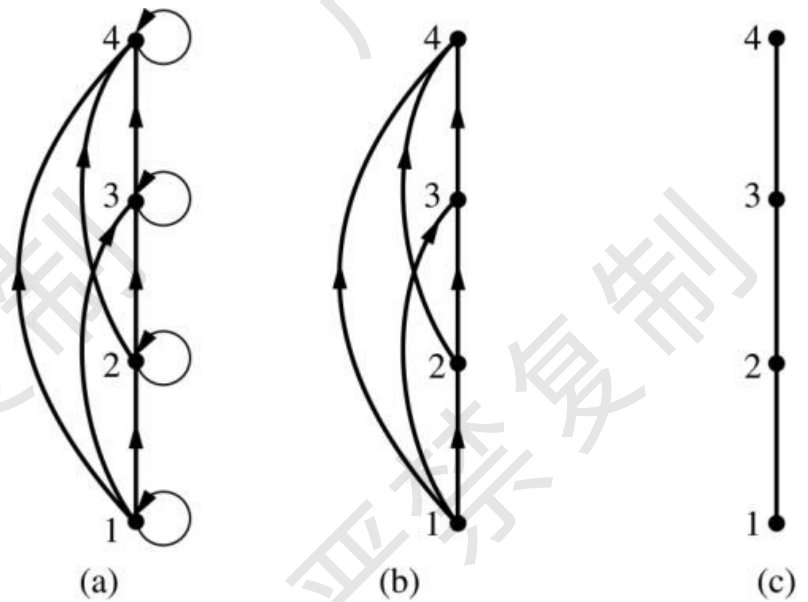
This definition can be easily extended to a lexicographic ordering on strings (*see text*).

Example: Consider strings of lowercase English letters. A lexicographic ordering can be defined using the ordering of the letters in the alphabet. This is the same ordering as that used in dictionaries.

- *discreet* $\prec$ *discrete*, because these strings differ in the seventh position and $e \prec t$.
- *discreet* $\prec$ *discreetness*, because the first eight letters agree, but the second string is longer.

HASSE DIAGRAMS

Definition: A *Hasse diagram* is a visual representation of a partial ordering that leaves out edges that must be present because of the reflexive and transitive properties.



Poset ({1,2,3,4}, ≤)

A partial ordering is shown in (a) of the figure above. The loops due to the reflexive property are deleted in (b). The edges that must be present due to the transitive property are deleted in (c). The Hasse diagram for the partial ordering (a), is depicted in (c).

PROCEDURE FOR CONSTRUCTING A HASSE DIAGRAM

- To represent a finite poset $(S, \preccurlyeq)$ using a Hasse diagram, start with the directed graph of the relation:
  - Remove the loops (a, a) present at every vertex due to the reflexive property.
  - Remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec y$. These are the edges that must be present due to the transitive property.
  - Arrange each edge so that its initial vertex is *below* the terminal vertex. Remove all the arrows, because all edges point *upwards* toward their terminal vertex.

MAXIMAL AND MINIMAL ELEMENTS

Definition: An element of a poset is called maximal if it is not less than any element of the poset. That is, a is **maximal** in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$.

Definition: An element of a poset is called minimal if it is not greater than any element of the poset. That is, a is **minimal** if there is no element $b \in S$ such that $b \prec a$.

GREATEST AND LEAST ELEMENTS

Definition: An element in a poset that is greater than every other element. Such an element is called the greatest element. That is, a is the **greatest element** of the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ for all $b \in S$.

Definition: an element is called the least element if it is less than all the other elements in the poset. That is, a is the **least element** of $(S, \preccurlyeq)$ if $a \preccurlyeq b$ for all $b \in S$.

- The greatest element is unique when it exists.
- The least element is unique when it exists.

UPPER BOUND AND LOWER BOUND

It is possible to find an element that is greater than or equal to all the elements in a **subset** ***A*** of a poset $(S, \preccurlyeq)$.

Definition: If u is an element of S such that $a \preccurlyeq u$ for all elements $a \in A$, then u is called an **upper bound** of A. Likewise, there may be an element less than or equal to all the elements in A. If l is an element of S such that $l \preccurlyeq a$ for all elements $a \in A$, then l is called a **lower bound** of A.

UPPER BOUND AND LOWER BOUND

Definition: The element x is called the **least upper bound (lub)** of the subset A if x is an upper bound that is less than every other upper bound of A. That is, x is the lub of A if $a \preccurlyeq x$ whenever $a \in A$, and $x \preccurlyeq z$ whenever z is an upper bound of A.

Definition: The element y is called the **greatest lower bound (glb)** of A if y is a lower bound of A and $z \preccurlyeq y$ whenever z is a lower bound of A.

- The lub of a subset A is unique when it exists.
- The glb of a subset A is unique when it exists.

SECTION SUMMARY

- Representing Relations using Matrices
- Representing Relations using Digraphs
- Closures of Relations
- Equivalence Relations
- Partial Orderings