UNIT 2. SETS AND RELATIONS

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RELATIONS

Chapter 9

CHAPTER SUMMARY

- Relations and Their Properties
- *n*-ary Relations and Their Applications
- Representing Relations
- Closures of Relations
- Equivalence Relations
- Partial Orderings

RELATIONS AND THEIR PROPERTIES

Section 9.1

SECTION SUMMARY

- Relations and Functions
- Properties of Relations
  - Reflexive Relations
  - Symmetric and Antisymmetric Relations
  - Transitive Relations
- Combining Relations

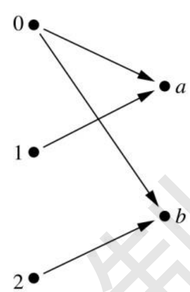
BINARY RELATIONS

Definition: A *binary relation* R from a set A to a set B is a subset

$$R \subseteq A \times B.$$

Example:

- Let $A = \{0,1,2\}$ and $B = \{a,b\}$
- $\{(0, a), (0, b), (1,a) , (2, b)\}$ is a relation from A to B.
- We can represent relations from a set A to a set B graphically or using a table:



R	a	b
0	×	×
1	×	
2		×

Relations are more general than functions. A function is a relation where exactly one element of B is related to each element of A.

BINARY RELATIONS

- $a \; R \; b$ denotes that $(a,b) \in R$, a is related to b by R.
- $a \not R \; b$ denotes that $(a,b) \notin R$
- e.g., $1 \; R \; a$ *and* $1 \not R \; b$

- Relationships between the elements of two sets are binary, in contrast with n-ary relations, which express relationships among elements of more than two sets.

RELATIONS ON A SET

Definition: A binary relation R *on a set* A is a subset of $A \times A$ or a relation from A to A.

Example:

- Suppose that $A = \{a,b,c\}$. Then $R = \{(a,a),(a,b), (a,c)\}$ is a relation on A.
- Let $A = \{1, 2, 3, 4\}$. The ordered pairs in the relation $R = \{(a,b) \mid a \text{ divides } b\}$ are

  $(1,1), (1, 2), (1,3), (1, 4), (2, 2), (2, 4), (3, 3)$, and $(4, 4)$.

PROPERTIES OF RELATIONS

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REFLEXIVE RELATIONS

Definition: R is *reflexive* iff $(a,a) \in R$ for every element $a \in A$. Written symbolically, R is reflexive if and only if

$$\forall x[x \in A \longrightarrow (x, x) \in R]$$

Example: The following relations on the integers are reflexive:

$R_1 = \{(a,b) \mid a \leq b\},$

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$

$R_4 = \{(a,b) \mid a = b\}.$

The following relations are not reflexive:

$R_2 = \{(a,b) \mid a > b\}$ (note that $3 \ngtr 3$),

$R_5 = \{(a,b) \mid a = b + 1\}$ (note that $3 \neq 3 + 1$),

$R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that $4 + 4 \nleq 3$).

SYMMETRIC RELATIONS

Definition: R is *symmetric* iff $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$.
Written symbolically, R is symmetric if and only if

$$\forall x \forall y\, [(x,y) \in R \longrightarrow (y,x) \in R]$$

Example: The following relations on the integers are symmetric:

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$,

$R_6 = \{(a,b) \mid a + b \leq 3\}$.

The following are not symmetric:

$R_1 = \{(a,b) \mid a \leq b\}$ (note that $3 \leq 4$, but $4 \nleq 3$),

$R_2 = \{(a,b) \mid a > b\}$ (note that $4 > 3$, but $3 \ngtr 4$),

$R_5 = \{(a,b) \mid a = b + 1\}$ (note that $4 = 3 + 1$, but $3 \neq 4 + 1$).

ANTISYMMETRIC RELATIONS

Definition:A relation R on a set A such that for all $a, b \in A$ if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.
Written symbolically, R is antisymmetric if and only if

$$\forall x \forall y\, [((x,y) \in R \wedge (y, x) \in R) \longrightarrow x = y]$$

- **Example**: The following relations on the integers are antisymmetric:

  $R_1 = \{(a,b) \mid a \leq b\}$,
  $R_2 = \{(a,b) \mid a > b\}$,
  $R_4 = \{(a,b) \mid a = b\}$,
  $R_5 = \{(a,b) \mid a = b + 1\}$.

  For any integer, if $a \leq b$ and $b \leq a$, then $a = b$.

  The following relations are not antisymmetric:

  $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$
  (note that both $(1,-1)$ and $(-1,1)$ belong to R_3),

  $R_6 = \{(a,b) \mid a + b \leq 3\}$
  (note that both $(1,2)$ and $(2,1)$ belong to R_6).

TRANSITIVE RELATIONS

Definition: A relation R on a set A is called *transitive* if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$. Written symbolically, R is transitive if and only if

$$\forall x \forall y \forall z[(x,y) \in R \wedge (y,z) \in R \longrightarrow (x,z) \in R]$$

- **Example**: The following relations on the integers are transitive:

  For every integer, $a \leq b$ and $b \leq c$, then $a \leq c$.

  $R_1 = \{(a,b) \mid a \leq b\}$,
  $R_2 = \{(a,b) \mid a > b\}$,
  $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
  $R_4 = \{(a,b) \mid a = b\}$.

  The following are not transitive:

  $R_5 = \{(a,b) \mid a = b + 1\}$
  (note that both (4,3) and (3,2) belong to R_5, but not (4,2)),

  $R_6 = \{(a,b) \mid a + b \leq 3\}$
  (note that both (2,1) and (1,2) belong to R_6, but not (2,2)).

COMBINING RELATIONS

COMBINING RELATIONS

- Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.
- **Example**: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined using basic set operations to form new relations:

$$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\} \qquad R_1 - R_2 = \{(2,2),(3,3)\}$$

$$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$$

COMPOSITION

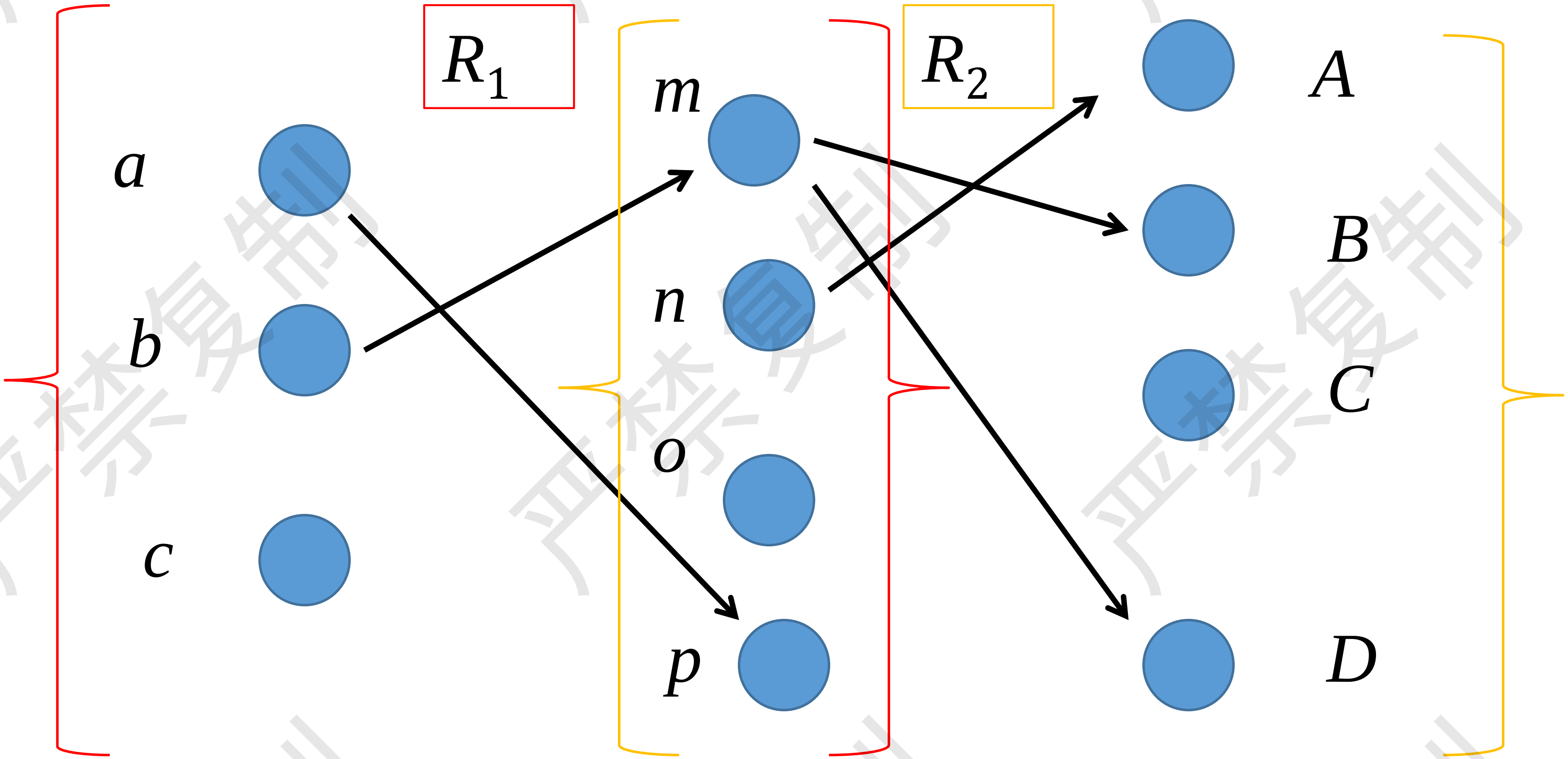
Definition: Suppose

- R_1 is a relation from a set A to a set B.
- R_2 is a relation from a set B to a set C.

Then the *composition* (or *composite*) of R_2 with R_1, is a relation from A to C where

- if (x,y) is a member of R_1 and (y,z) is a member of R_2**,** then (x, z) is a member of $R_2 \circ R_1$.

REPRESENTING THE COMPOSITION OF A RELATION



$$R_2 \circ R_1 = \{(b, D), (b, B)\}$$

POWERS OF A RELATION

Definition: Let R be a binary relation on A. Then the powers R^n of the relation R can be defined inductively by:

- Basis Step: $R^1 = R$
- Inductive Step: $R^{n+1} = R^n \circ R$

Example: Let $R = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$. Find the powers R^n, $n = 2, 3, 4, \ldots$.