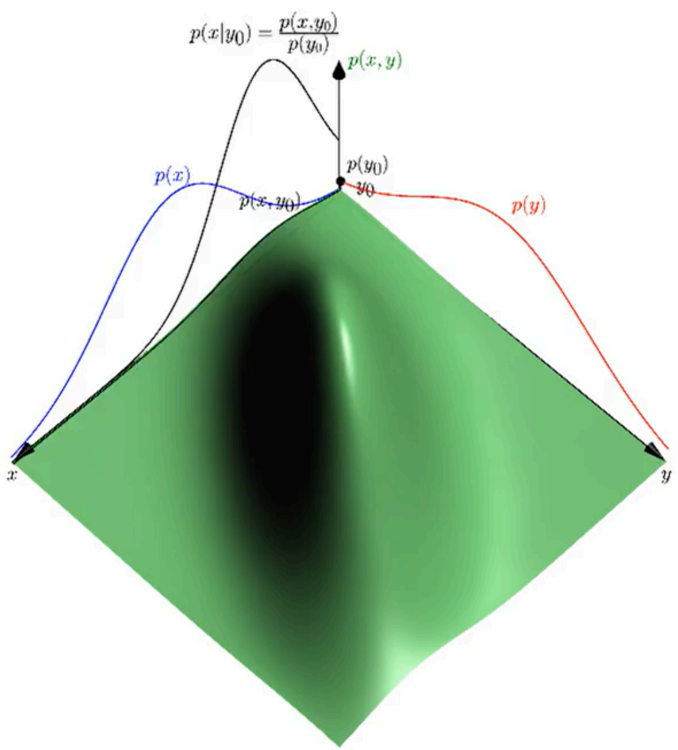
MCHA6100
Advanced Estimation



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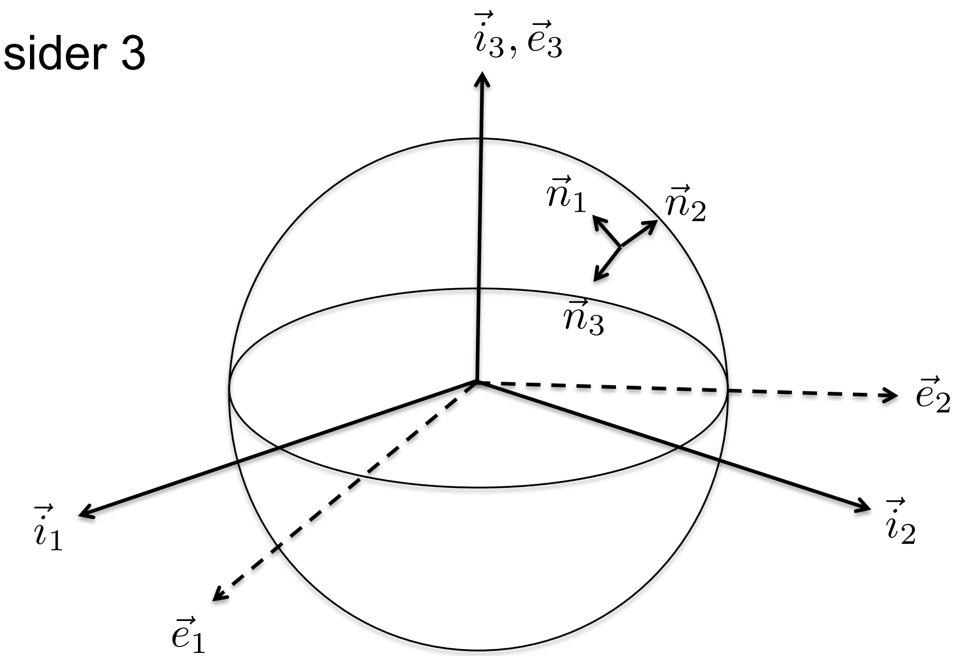
Navigation & Vehicle Kinematics

For general vehicle navigation, we consider 3 reference frames:

1. Stars
2. Earth
3. Body



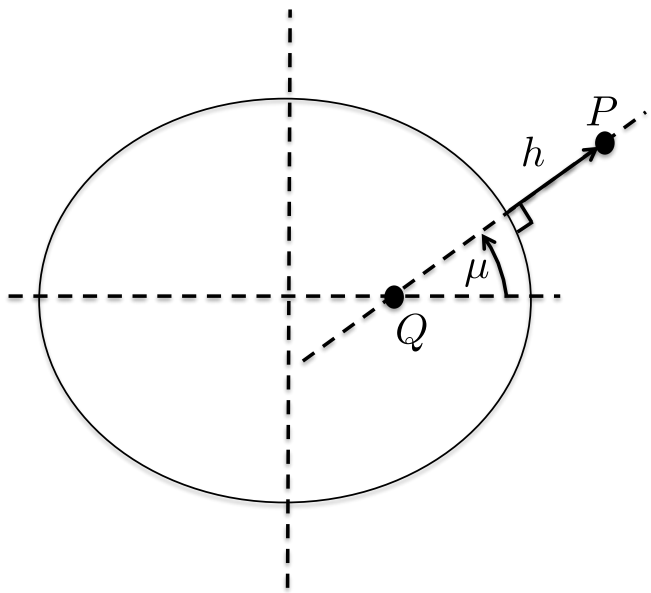
And 4 coordinate systems:

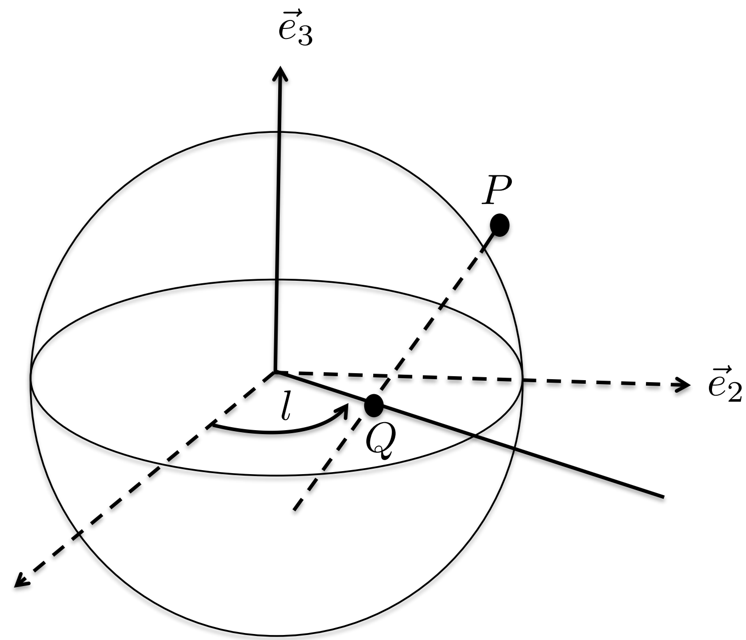
1. Earth Centered Inertial (ECI) $\{i\} \equiv \vec{i}_1, \vec{i}_2, \vec{i}_3$
2. Earth Centered Earth Fixed (ECEF) $\{e\} \equiv \vec{e}_1, \vec{e}_2, \vec{e}_3$
3. North East Down (NED) $\{n\} \equiv \vec{n}_1, \vec{n}_2, \vec{n}_3$
4. Body $\{b\} \equiv \vec{b}_1, \vec{b}_2, \vec{b}_3$

Earth models

The shape of the Earth can be modelled using a ellipsoid of revolution (spheroid) centred at the Earth's centre of mass and rotating about an axis through the north and south poles. The parameters of this spheroid have been documented by the US department of defence World Geodesic System in 1984 (WGS-84).

Latitude and longitude:



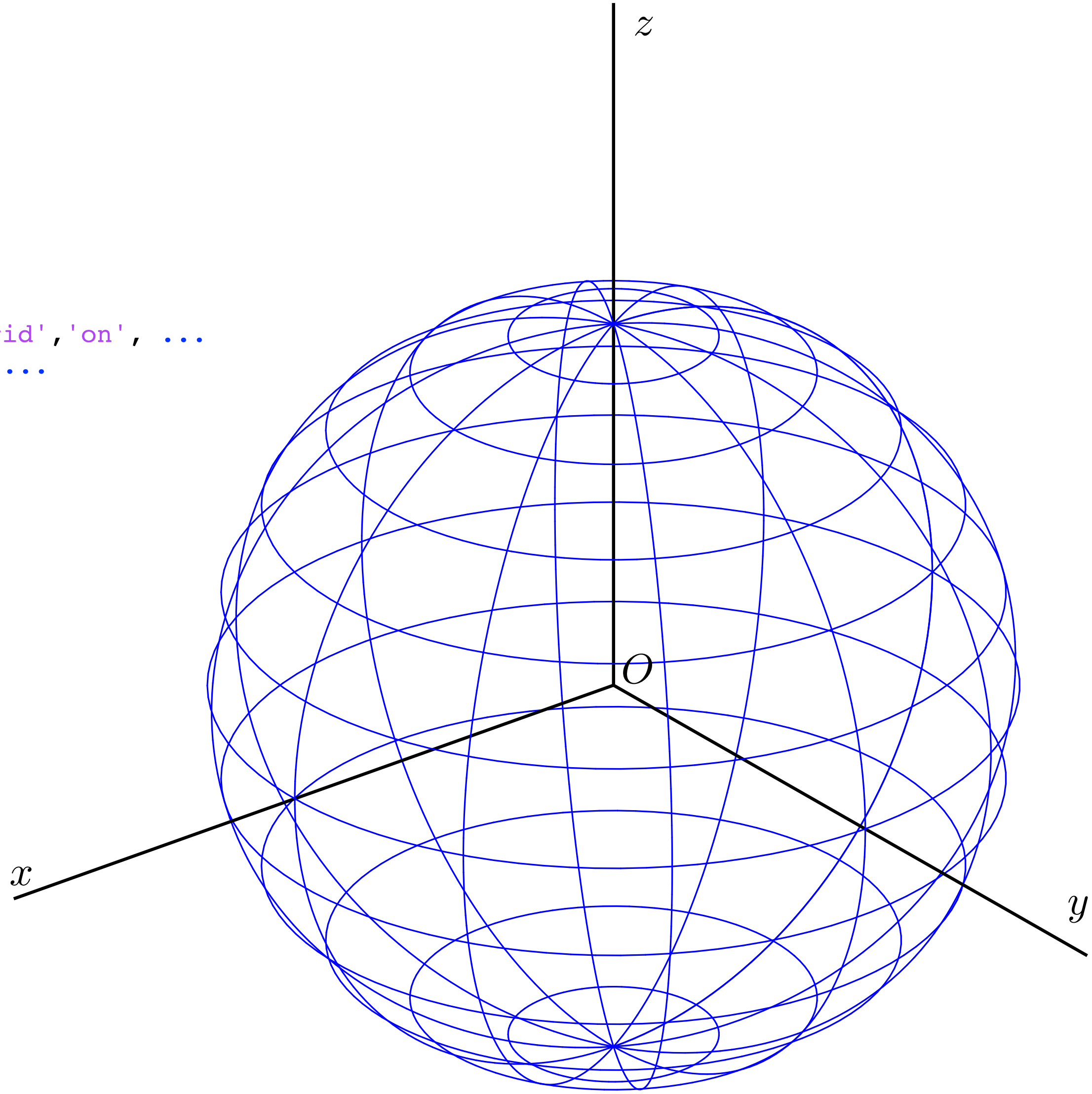


WGS-84 Ellipsoid Model


```
%Get the WGS-84 ellipsoid model
wgs84 = wgs84Ellipsoid('meters');

%Plot the ellisoid
figure('Renderer','opengl')
ax = axesm('globe','Geoid',wgs84,'Grid','on', ...
     'GLineWidth',1,'GLineStyle','-',...
     'Gcolor','b','Galtitude',100);
ax.Position = [0 0 1 1];
axis equal off;
view(3);

%Plot the ECEF principle axes
l1=line([0 12000000],[0 0],[0 0]);
l2=line([0 0],[0 12000000],[0 0]);
l3=line([0 0],[0 0],[0 12000000]);
set(l1,'color','k','linewidth',2);
set(l2,'color','k','linewidth',2);
set(l3,'color','k','linewidth',2);
```



Points on the Earths Surface

The WGS-84 model can be expressed in different ways. The following uses the equitorial semi-axis and the so-called eccentricity.

The Geocentric Coordinates are equivalently given in ECEF

$$\begin{bmatrix} x_e \\ y_e \\ z_e \end{bmatrix} = \begin{bmatrix} (R_N + h)\cos\mu \, \cos l \\ (R_N + h)\cos\mu \, \sin l \\ (R_N(1 - e^2) + h)\sin\mu \end{bmatrix}$$

where e is the eccentricity e=0.08181919 and the equatorial semi-axis is r_e = 6378137m, and the eccentricity is with the normal radius defined as

$$R_N = \frac{r_e}{\sqrt{1 - e^2 \sin^2 \mu}}$$

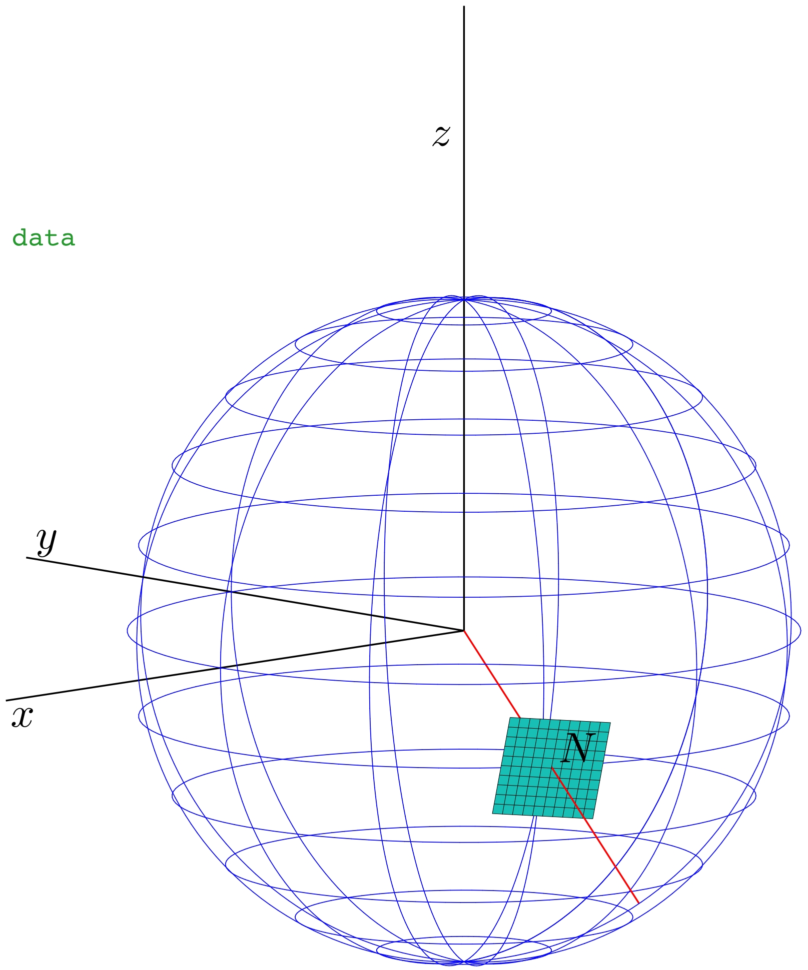
ECEF to NED

How can we define a local NED coordinate system at some reference longitude and latitude?

```
%Continued…

%Draw the NE plane centred at UoN ES Building
[x y] = meshgrid(-1e6:2e5:1e6); % Generate y and z data
z     = 0*y;

%rotate to new lat,long (this is an approximation)
R=rotatezyx([0;-90+32.891019;151.701059]);
[sx,sy]=size(x);
xyz = [x(:) y(:) z(:)]*R.';
x=rNOe(1) + reshape(xyz(:,1),sx,sy);
y=rNOe(2) + reshape(xyz(:,2),sx,sy);
z=rNOe(3) + reshape(xyz(:,3),sx,sy);
surf(x,y,z,0.9*ones(size(z))) %Plot the surface
```



ECEF to NED

The rotation matrix that takes {e} into {*n*} is

$$\mathbf{R}_n^e(l,\mu) = \mathbf{R}_{z,l}\,\mathbf{R}_{y,-\mu-\pi/2} = \begin{bmatrix} -\cos l\ \sin\mu & -\sin l & -\cos l \cos\mu \\ -\sin l\ \sin\mu & \cos l & -\sin l\ \cos\mu \\ 0 & 0 & -\sin\mu \end{bmatrix}$$

In applications of mobile robot and vehicle local positioning relative to a reference point *N*, we can find the coordinates of a point *B* on the robot from measurements of lat, long and height as follows:

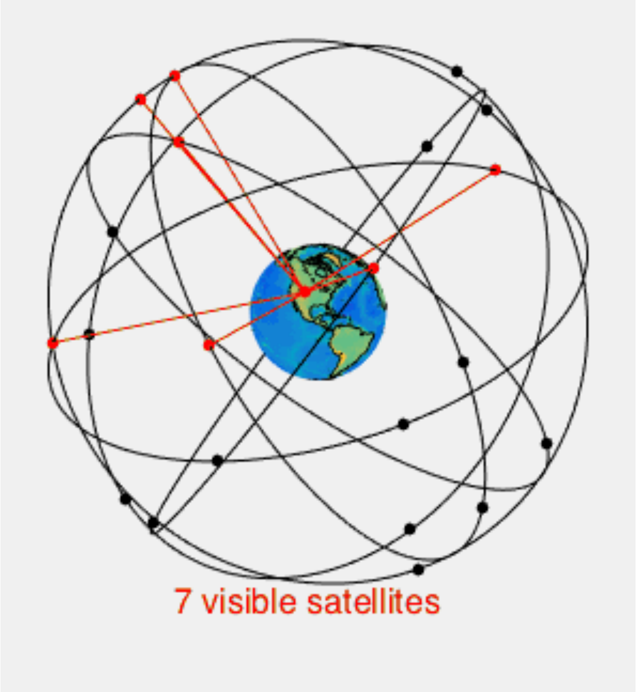
North, East, Down: $$\mathbf{r}^n_{B/N} = \mathbf{R}^e_n(l_N,\mu_N)^{\mathsf{T}}(\mathbf{r}^e_{B/O} - \mathbf{r}^e_{N/O})$$

GPS

The Global Positioning System is a satellite based radio navigation system. It was started by the US Government in the 1970's for defence navigation purposes.

Principles of operation:

- Satellites transmit information about their position and the time of transmission (50 bits/sec, 750 sec for a full message)
- The receiver matches a copy of all the CDMA codes to detect which satellite transmitted the data and extracts timing and position data
- The time of flight can be calculated based on the time of transmission and time of arrival (the receiver clock is typically not accurate)
- A range estimate can be obtained from the time of flight and position can be estimated using four or more satellites (position plus time bias estimation)



Pseudo-range Measurements

Some GPS units can output ECEF Euclidean coordinates, in which case we need to transform from the local coordinate system {n} (where the states are defined) to {e} (where the measurements are defined)

$$\mathbf{r}^e_{B/O} = \mathbf{R}^e_n(l_N, \mu_N)\mathbf{r}^n_{B/N} + \mathbf{r}^e_{N/O}$$

$$\begin{aligned}
\mathbf{r}^e_{B/O} &= \text{position of vehicle in ECEF coordinates} \\
\mathbf{R}^e_n(l_N, \mu_N) &= \text{rotation matrix defined at the longitude } l_N \text{ and latitude } \mu_N \\
&\quad\ \text{of the reference point} \\
\mathbf{r}^n_{B/N} &= \text{position of vehicle relative to local reference point} \\
\mathbf{r}^e_{N/O} &= \text{reference point relative to ECEF centre}
\end{aligned}$$

Pseudo-range Measurements

Some GPS units can output ECEF Euclidean coordinates, in which case we need to transform from the local coordinate system {n} (where the states are defined) to {e} (where the measurements are defined)

$$\mathbf{r}^e_{B/O} = \mathbf{R}^e_n(l_N, \mu_N)\mathbf{r}^n_{B/N} + \mathbf{r}^e_{N/O}$$

Therefore, the measurement equation is

$$\mathbf{y}_{GPS} = \mathbf{R}^e_n(l_N, \mu_N)\mathbf{r}^n_{B/N} + \mathbf{r}^e_{N/O} + \mathbf{e}_{GPS}$$

with

$$\mathbf{e}_{GPS} \sim \mathcal{N}(0, R_{GPS})$$

Pseudo-range Measurements

Other GPS units only output position in geodetic ECEF curvilinear coordinates, in which case we need to transform from the local coordinate system {n} (where the states are defined) to {e} (where the measurements are defined), to the curvilinear coordinates {g}

$$\mathbf{r}^g_{B/O} = \mathbf{\Phi}^g_e \left(\mathbf{R}^e_n(l_N, \mu_N)\mathbf{r}^n_{B/N} + \mathbf{r}^e_{N/O} \right)$$

$$\begin{aligned}
\mathbf{r}^g_{B/O} &= \text{position of vehicle in ECEF curvilinear coordinates} \\
\mathbf{\Phi}^g_e &= \text{transformation from euclidean to curvilinear coordinates (to be defined)} \\
\mathbf{R}^e_n(l_N, \mu_N) &= \text{rotation matrix defined at the longitude } l_N \text{ and latitude } \mu_N \\
& \quad \text{of the reference point} \\
\mathbf{r}^n_{B/N} &= \text{position of vehicle relative to local reference point} \\
\mathbf{r}^e_{N/O} &= \text{reference point relative to ECEF centre}
\end{aligned}$$

Pseudo-range Measurements

The map from ECEF Euclidean coordinates to curvilinear coordinates is given by

$$\mathbf{r}^g_{B/O} = \begin{bmatrix} \phi^g \\ \lambda^g \\ h^g \end{bmatrix}$$

$$\phi^g = \arctan \frac{z^e + f^2\, b \sin^3 \theta}{p - d^2\, a \cos^3 \theta},$$

$$\lambda^g = 2 \arctan \frac{y^e}{x^e + p},$$

$$h^g = \frac{p}{\cos \phi^g}$$

where

$$f = \sqrt{\frac{a^2 - b^2}{b^2}}$$

$$p = \sqrt{(x^e)^2 + (y^e)^2}$$

$$\theta = \arctan \frac{z^e\, a}{p\, b}$$

$$N = \frac{a^2}{\sqrt{a^2 \cos^2 \phi^g + b^2 \sin^2 \phi^g}}$$

$$a = 6378137.0$$

$$b = 6356752.3142$$

$$d = 0.08181919$$

Pseudo-range Measurements

Therefore, if the GPS output is longitude, latitude and height, then the measurement equations becomes

$$\mathbf{y}_{GPS} = \boldsymbol{\Phi}_e^g \left(\mathbf{R}_n^e(l_N, \mu_N)\mathbf{r}_{B/N}^n + \mathbf{r}_{N/O}^e \right) + \mathbf{e}_{GPS}$$

With

$$\mathbf{e}_{GPS} \sim \mathcal{N}(0, R_{GPS})$$

Therefore, where possible it is easier to work in Euclidean ECEF coordinates!

Inertial Measurement Unit

Magnetometer, Gyroscopes, Accelerometers

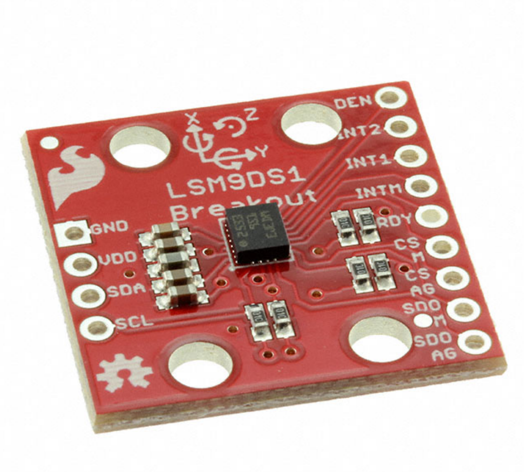
Calibration

Inertial Measurement Unit

An Inertial Measurement Unit is an electronic device that detects local accelerations, rotations, and possibly the magnetic field vector.

Principles of operation:

- Typically includes 3 accelerometers, 3 gyroscopes and 3 magnetometers in 3D orthogonal arrangement
- Does not depend on external signals (unlike GPS)
- Can apply the dead-reckoning (DR) principle
  - DR is the determination of pose using prior pose information (integration or mechanization)
- Sensors always exhibit a bias, which when integrated to give velocity/position, can result in large errors
- Therefore, it is vital to calibrate IMU's before deploying on a vehicle.



Inertial Measurement Unit

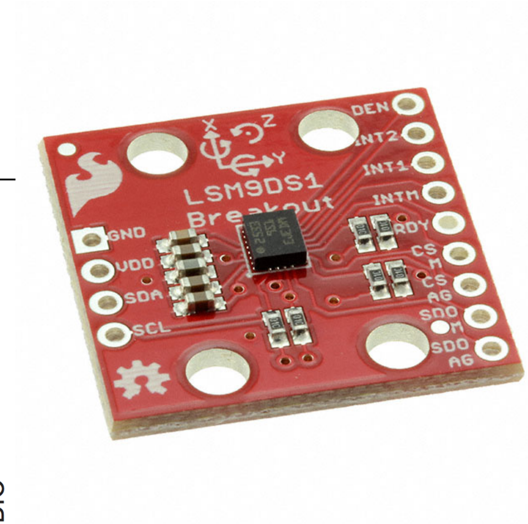
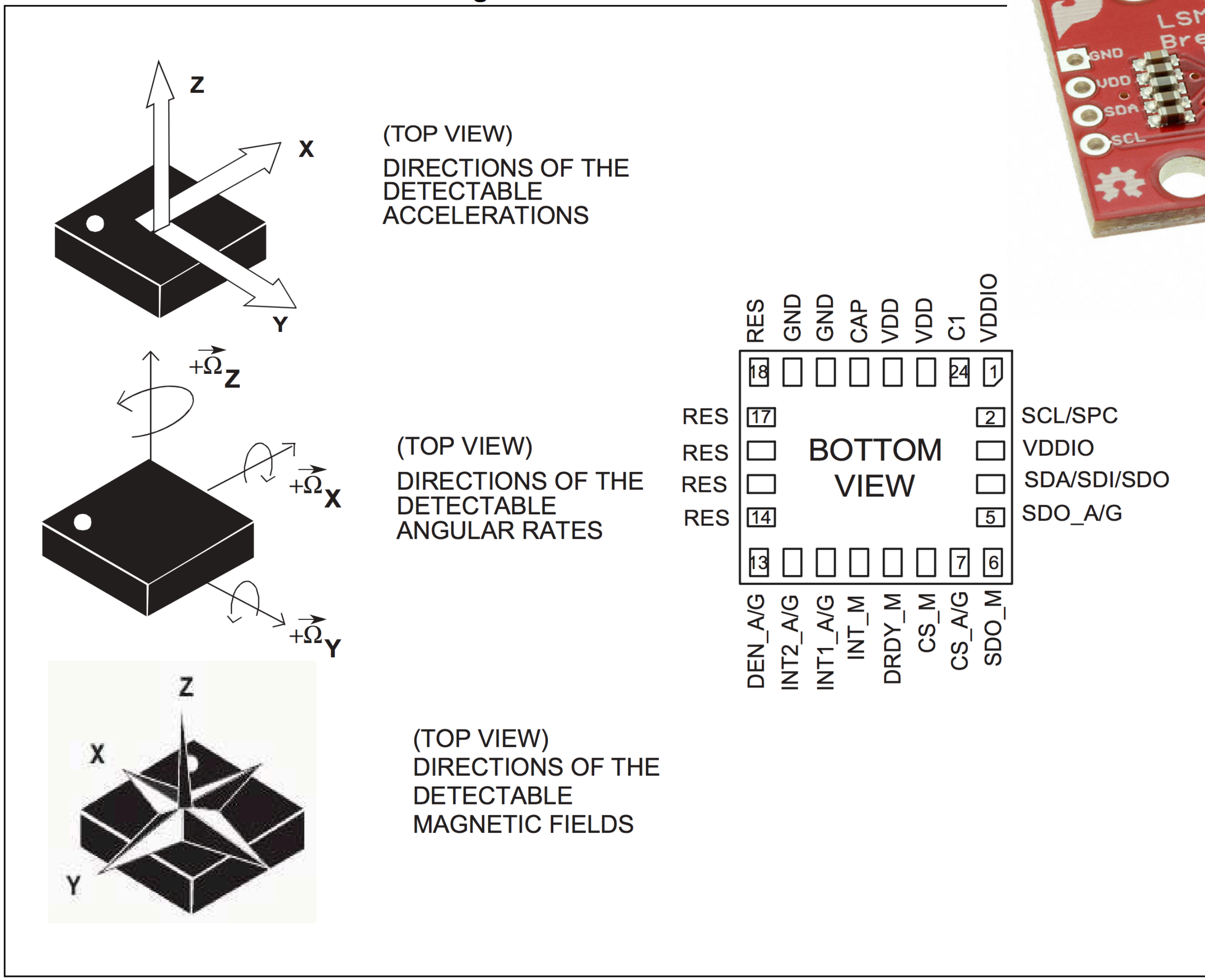


Figure 1. Pin connections



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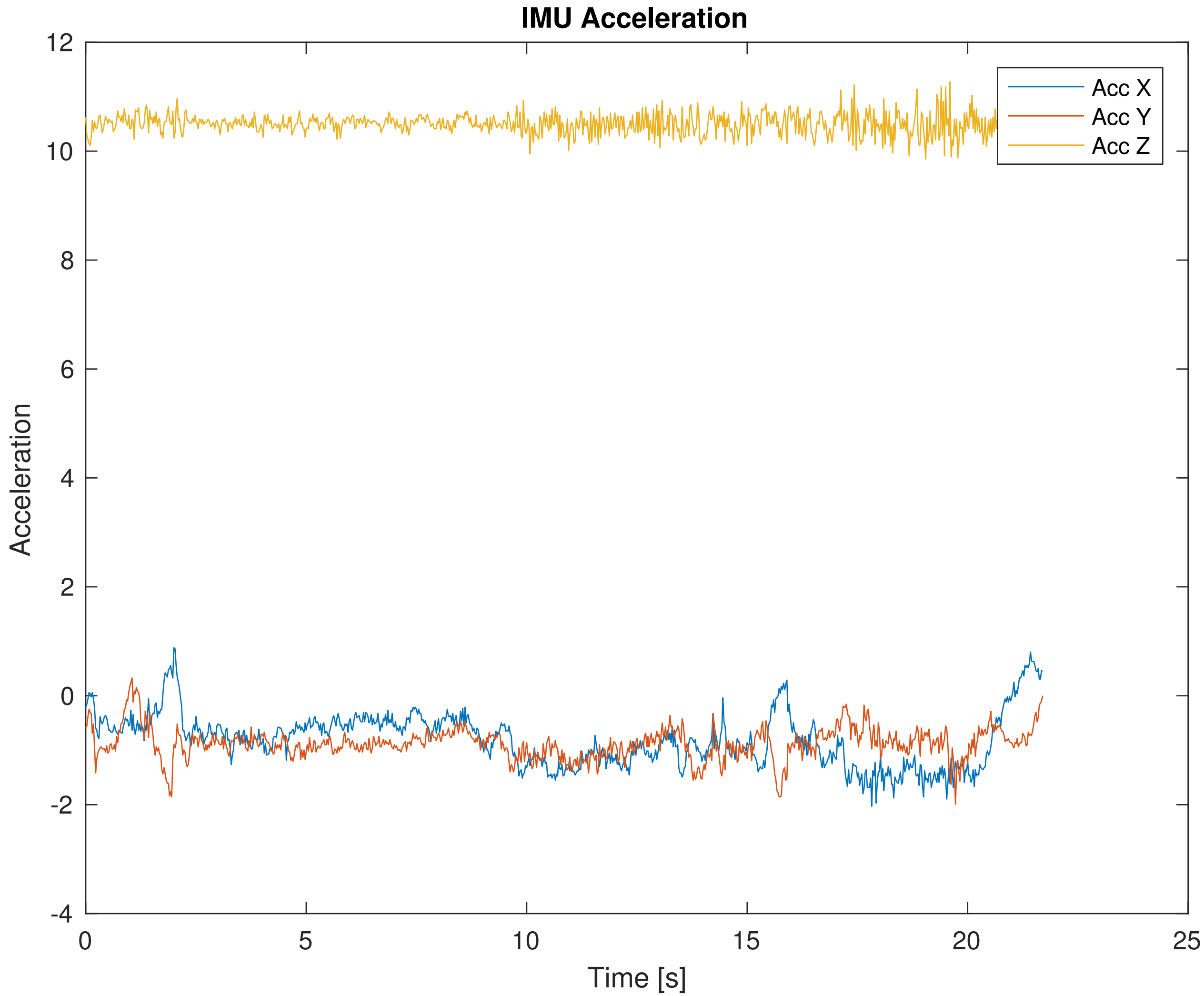
Inertial Measurement Unit – Example Data



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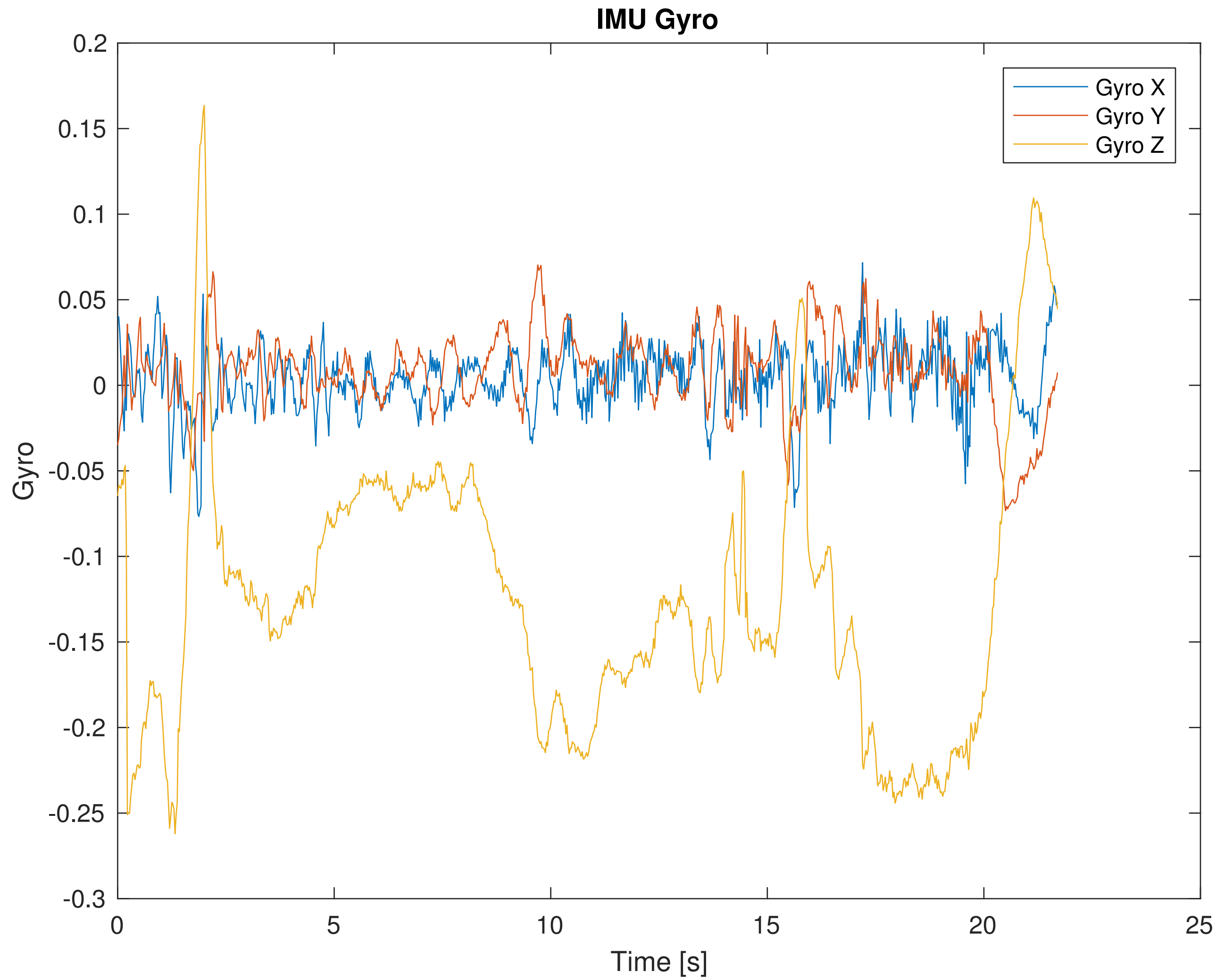
Inertial Measurement Unit – Example Data



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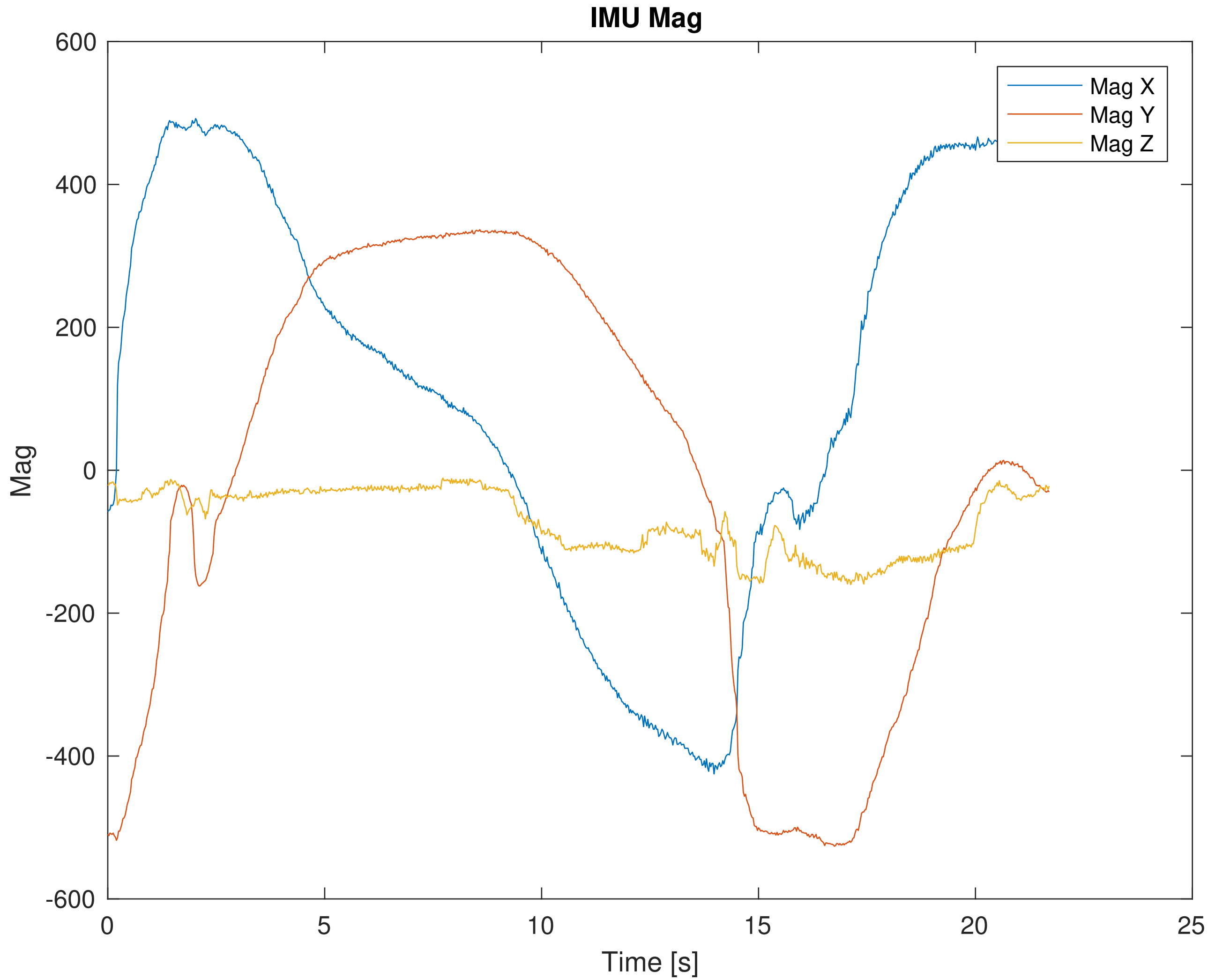
Inertial Measurement Unit – Example Data



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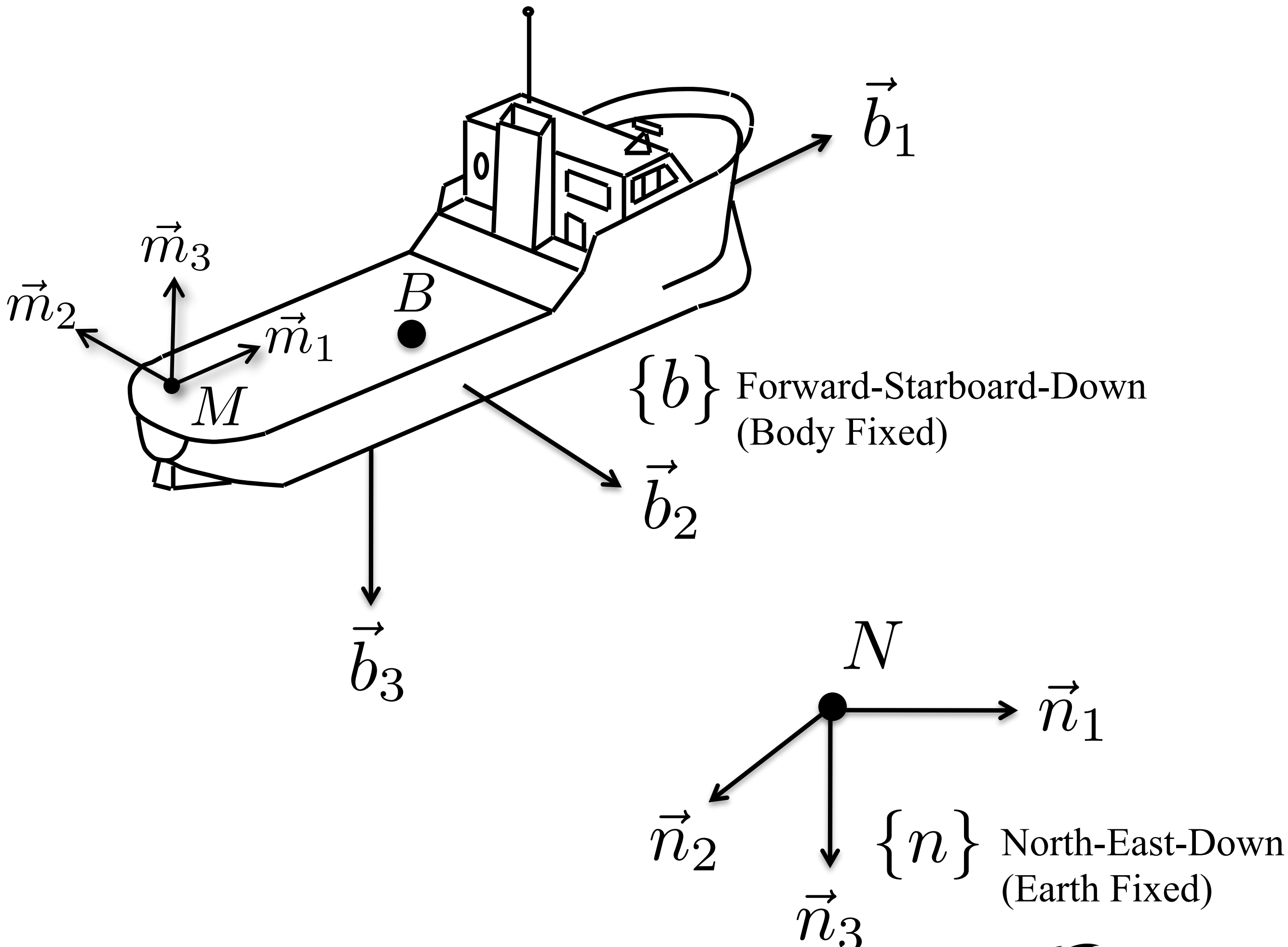
IMU, Body and Inertial Frames

The IMU is typically attached to some fixed point M, relative to the body frame with origin B, that is moving relative to some inertial frame with origin N.



Reference Frame & Coordinate Systems

A **reference frame** is perspective from which the motion of a body or an object is described by an observer. A reference frame can be defined by a set of at least 3 non-co-linear points in space that are rigidly connected.

A **coordinate system** is a mathematical entity that allows us to establish a one-to-one correspondence between vector magnitudes and scalars called coordinates. A basis defines a coordinate system.

Therefore, a reference frame is not the same as a coordinate system, one is physical and the other is mathematical.

We will denote reference frames by $\mathcal{A}, \mathcal{B}$
and bases associated with reference frames by $\{a\}, \{b\}$

Time-derivative of a vector

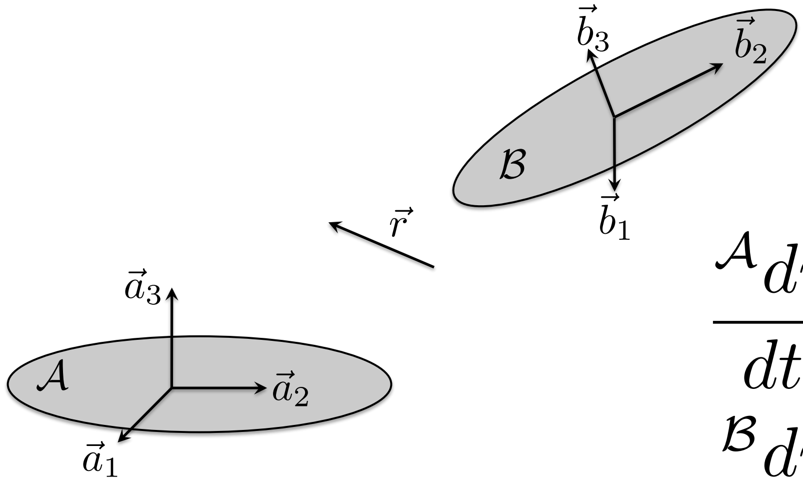
The time derivative of a scalar magnitude is independent of the reference frame in which the magnitude is observed.

The time derivative of a vector magnitude depends, in general, on the reference frame in which the magnitude is observed.

We will use a notation that indicates this explicitly by using a left superscript:

$$\frac{{}^{\mathcal{A}}d\vec{r}}{dt} = \overset{\mathcal{A}}{\lim_{\Delta t \to 0}} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

Time-derivative of a vector



$$\frac{{}^{\mathcal{A}}d\vec{r}}{dt} = \frac{dr_1^a}{dt}\vec{a}_1 + \frac{dr_2^a}{dt}\vec{a}_2 + \frac{dr_3^a}{dt}\vec{a}_3$$

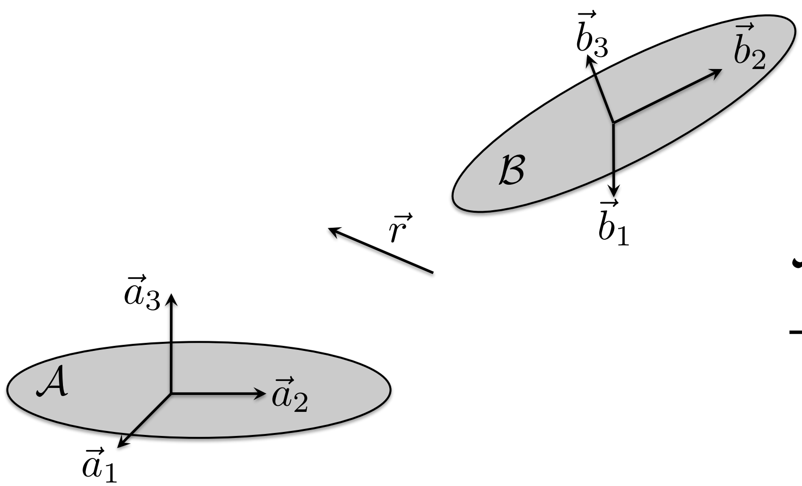
$$\frac{{}^{\mathcal{B}}d\vec{r}}{dt} = \frac{dr_1^b}{dt}\vec{b}_1 + \frac{dr_2^b}{dt}\vec{b}_2 + \frac{dr_3^b}{dt}\vec{b}_3$$

In general,

$$\frac{{}^{\mathcal{A}}d\vec{r}}{dt} \neq \frac{{}^{\mathcal{B}}d\vec{r}}{dt}$$

(Rate of) Transport Theorem (TT)

Theorem 1 (Rate of Change Transport Theorem) *Consider the scenario depicted in the figure, and assume that $\mathcal{B}$ only rotates with respect of $\mathcal{A}$. Then there exist a unique vector $\vec{\omega}_{\mathcal{B}/\mathcal{A}}$ called the* **angular velocity** *of $\mathcal{B}$ with respect of $\mathcal{A}$ such that*

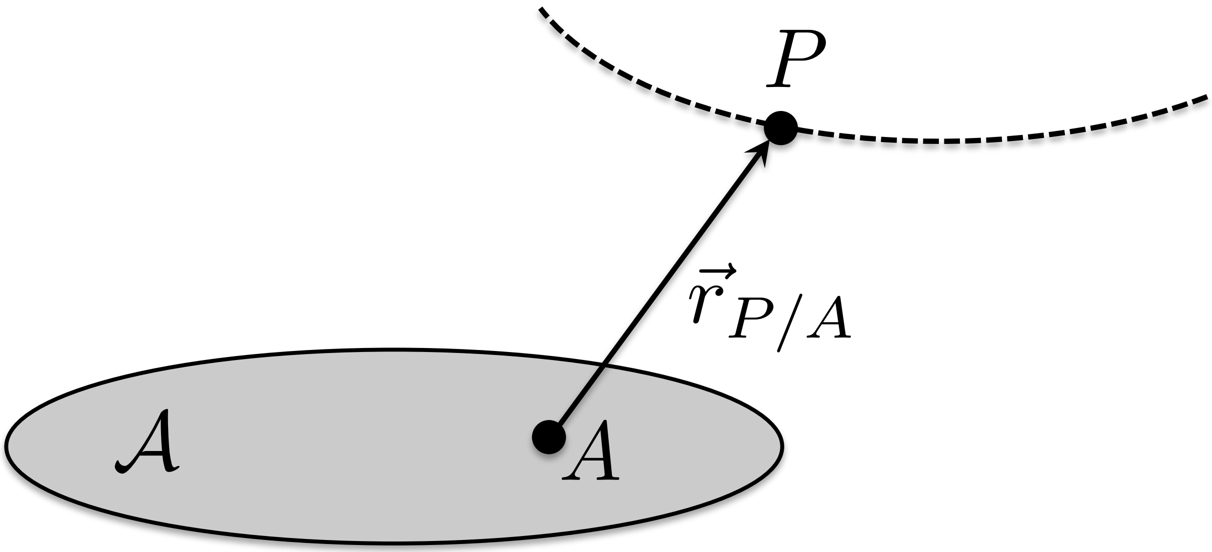


$$\frac{{}^{\mathcal{A}}d\vec{r}}{dt} = \frac{{}^{\mathcal{B}}d\vec{r}}{dt} + \vec{\omega}_{\mathcal{B}/\mathcal{A}} \times \vec{r}$$

This formula is key to derive kinematic models. We will be using it a lot!

Position

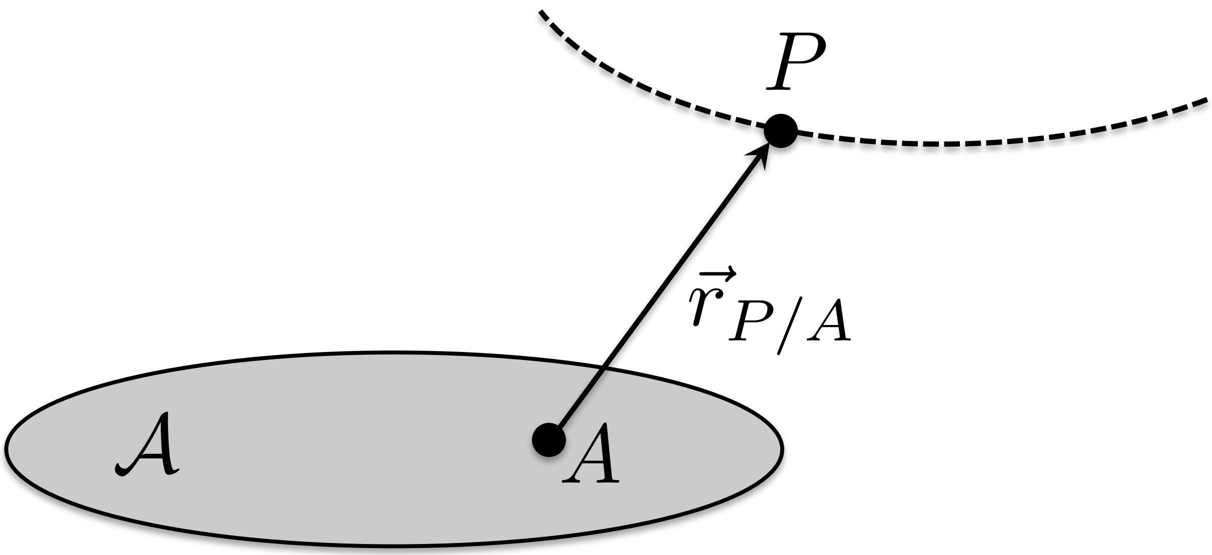
The **position** vector denoted by $\vec{r}_{P/A}$ indicates "*the position of the point P with respect to the point A.*"



Velocity

If $\mathcal{A}$ is considered an inertial frame, then the rate of change of position of a point P with respect to a point A fixed in $\mathcal{A}$ is called **velocity**,

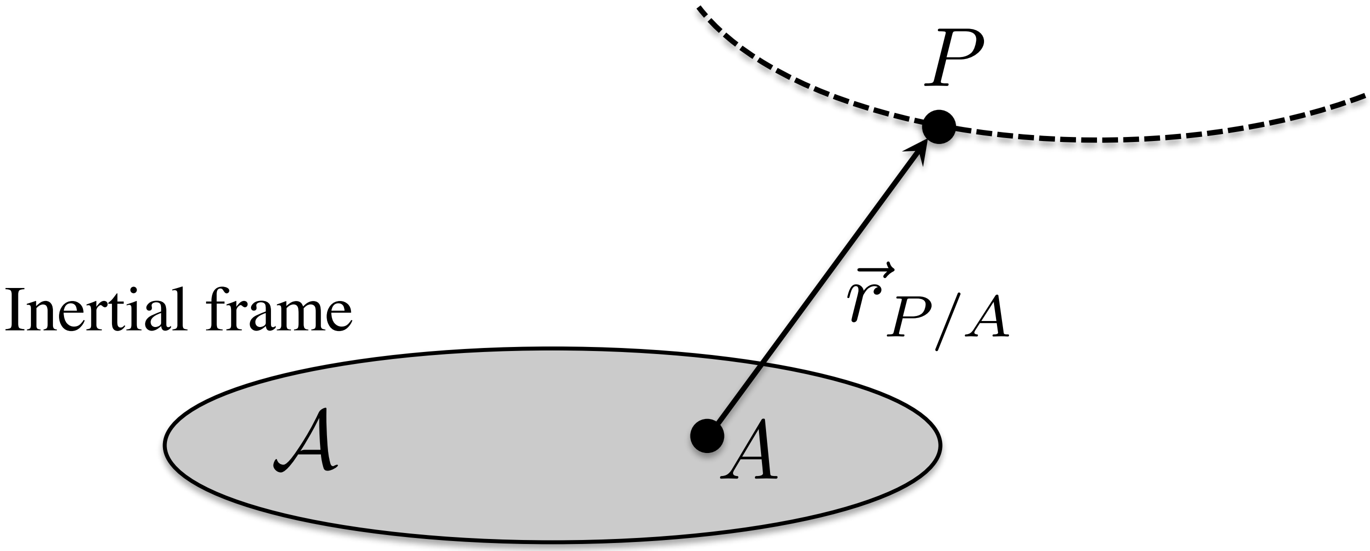
$$\vec{v}_{P/\mathcal{A}} \triangleq \frac{{}^{\mathcal{A}}d\vec{r}_{P/A}}{dt}.$$



Acceleration

The **rate of change of velocity** of a point P with respect to a point A fixed in $\mathcal{A}$ is called **acceleration** and will be denoted by

$$\vec{a}_{P/\mathcal{A}} = \frac{{}^{\mathcal{A}}d\vec{v}_{P/\mathcal{A}}}{dt} = \frac{{}^{\mathcal{A}}d^2\vec{r}_{P/A}}{dt^2},$$



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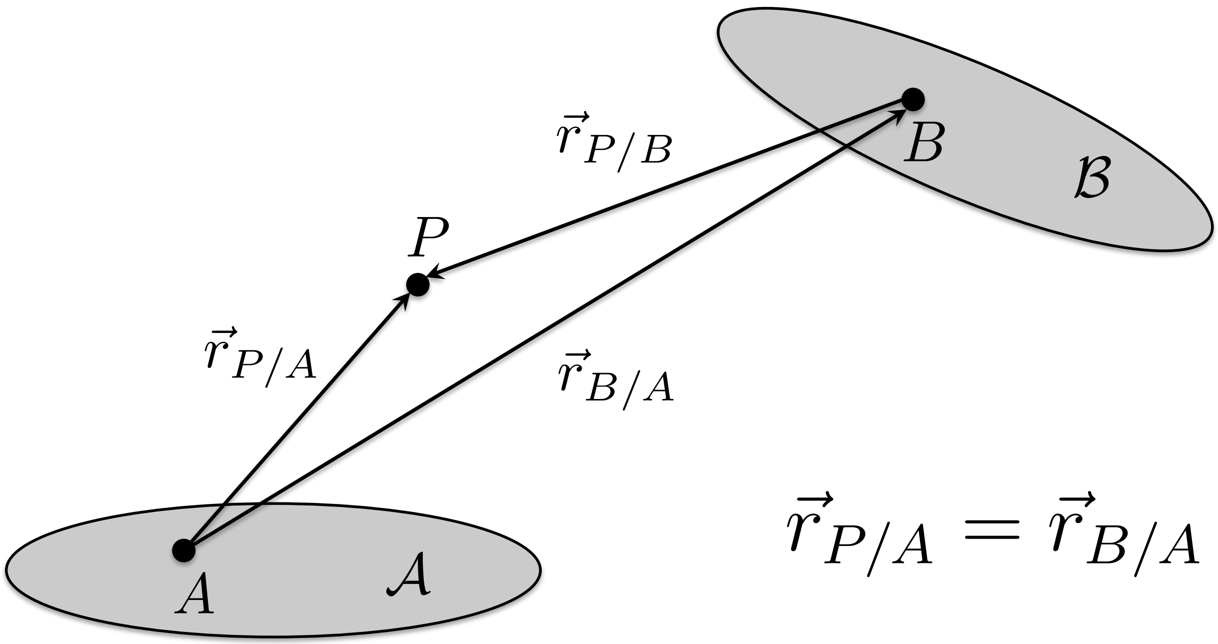
Motion in Different Frames

A practical way to relate motion in different frames is to start from the position vectors and the apply the transport theorem.



$$\vec{r}_{P/A} = \vec{r}_{B/A} + \vec{r}_{P/B}$$

Motion in Different Frames

We can take the derivative of $\vec{r}_{P/A} = \vec{r}_{B/A} + \vec{r}_{P/B}$

$$\frac{{}^{\mathcal{A}}d\vec{r}_{P/A}}{dt} = \frac{{}^{\mathcal{A}}d\vec{r}_{B/A}}{dt} + \frac{{}^{\mathcal{A}}d\vec{r}_{P/B}}{dt}$$

If we assume *A* inertial:

$$\vec{v}_{P/\mathcal{A}} = \vec{v}_{B/\mathcal{A}} + \frac{{}^{\mathcal{A}}d\vec{r}_{P/B}}{dt}$$

The transport theorem leads to the **velocity formula**:

$$\vec{v}_{P/\mathcal{A}} = \vec{v}_{B/\mathcal{A}} + \frac{{}^{\mathcal{B}}d\vec{r}_{P/B}}{dt} + \vec{\omega}_{\mathcal{B}/\mathcal{A}} \times \vec{r}_{P/B}$$

Vehicle kinematics

To compute the accelerations

$$\frac{{}^{\mathcal{A}}d^2\vec{r}_{P/A}}{dt^2} = \frac{{}^{\mathcal{A}}d^2\vec{r}_{B/A}}{dt^2} + \frac{{}^{\mathcal{A}}d^2\vec{r}_{P/B}}{dt^2}$$

$$\vec{a}_{P/\mathcal{A}} = \vec{a}_{B/\mathcal{A}} + \frac{{}^{\mathcal{A}}d}{dt}\left(\frac{{}^{\mathcal{A}}d\vec{r}_{P/B}}{dt}\right)$$

$$\vec{a}_{P/\mathcal{A}} = \vec{a}_{B/\mathcal{A}} + \frac{{}^{\mathcal{A}}d}{dt}\left(\frac{{}^{\mathcal{B}}d\vec{r}_{P/B}}{dt} + \vec{\omega}_{\mathcal{B}/\mathcal{A}} \times \vec{r}_{P/B}\right)$$

Vehicle kinematics

$$\vec{a}_{P/\mathcal{A}} = \vec{a}_{B/\mathcal{A}} + \frac{{}^{\mathcal{A}}d}{dt}\left(\frac{{}^{\mathcal{B}}d\vec{r}_{P/B}}{dt} + \vec{\omega}_{\mathcal{B}/\mathcal{A}} \times \vec{r}_{P/B}\right)$$

$$\vec{a}_{P/\mathcal{A}} = \vec{a}_{B/\mathcal{A}} + \frac{{}^{\mathcal{B}}d}{dt}\left(\frac{{}^{\mathcal{B}}d\vec{r}_{P/B}}{dt} + \vec{\omega}_{\mathcal{B}/\mathcal{A}} \times \vec{r}_{P/B}\right) + \vec{\omega}_{\mathcal{B}/\mathcal{A}} \times \left(\frac{{}^{\mathcal{B}}d\vec{r}_{P/B}}{dt} + \vec{\omega}_{\mathcal{B}/\mathcal{A}} \times \vec{r}_{P/B}\right)$$

$$\vec{\alpha}_{\mathcal{B}/\mathcal{A}} = \frac{{}^{\mathcal{A}}d}{dt}\vec{\omega}_{\mathcal{B}/\mathcal{A}} = \frac{{}^{\mathcal{B}}d}{dt}\vec{\omega}_{\mathcal{B}/\mathcal{A}} + \underbrace{\vec{\omega}_{\mathcal{B}/\mathcal{A}} \times \vec{\omega}_{\mathcal{B}/\mathcal{A}}}_{=\vec{0}}$$

Vehicle kinematics

$$\vec{a}_{P/\mathcal{A}} = \vec{a}_{B/\mathcal{A}} + \frac{{}^{\mathcal{B}}d}{dt}\left(\frac{{}^{\mathcal{B}}d\vec{r}_{P/B}}{dt} + \vec{\omega}_{\mathcal{B}/\mathcal{A}} \times \vec{r}_{P/B}\right) + \vec{\omega}_{\mathcal{B}/\mathcal{A}} \times \left(\frac{{}^{\mathcal{B}}d\vec{r}_{P/B}}{dt} + \vec{\omega}_{\mathcal{B}/\mathcal{A}} \times \vec{r}_{P/B}\right)$$

Distributing terms leads to the **acceleration formula**:

$$\vec{a}_{P/\mathcal{A}} = \vec{a}_{B/\mathcal{A}} + \frac{{}^{\mathcal{B}}d^2\vec{r}_{P/B}}{dt^2} + \vec{\alpha}_{\mathcal{B}/\mathcal{A}} \times \vec{r}_{P/B} + 2\vec{\omega}_{\mathcal{B}/\mathcal{A}} \times \frac{{}^{\mathcal{B}}d\vec{r}_{P/B}}{dt} + \vec{\omega}_{\mathcal{B}/\mathcal{A}} \times \vec{\omega}_{\mathcal{B}/\mathcal{A}} \times \vec{r}_{P/B}$$

$$\vec{\alpha}_{\mathcal{B}/\mathcal{A}} \times \vec{r}_{P/B} \qquad \text{(Transverse or Euler acceleration)}$$

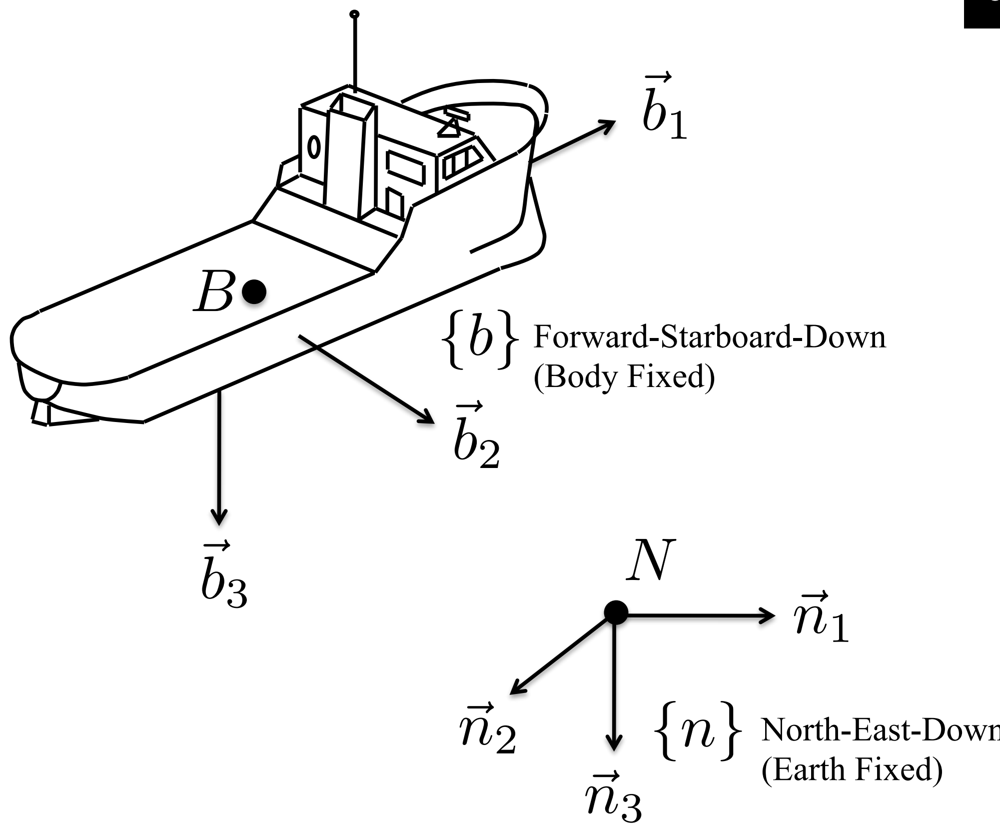
$$2\vec{\omega}_{\mathcal{B}/\mathcal{A}} \times \frac{{}^{\mathcal{B}}d\vec{r}_{P/B}}{dt} \qquad \text{(Coriolis acceleration)}$$

$$\vec{\omega}_{\mathcal{B}/\mathcal{A}} \times \vec{\omega}_{\mathcal{B}/\mathcal{A}} \times \vec{r}_{P/B} \qquad \text{(Centripetal acceleration)}$$

Vehicle position-orientation

$$\mathbf{r}^n_{B/N} \triangleq [N,\ E,\ D]^T$$

$$\boldsymbol{\Theta}^n_b \triangleq [\phi,\ \theta,\ \psi]^T$$



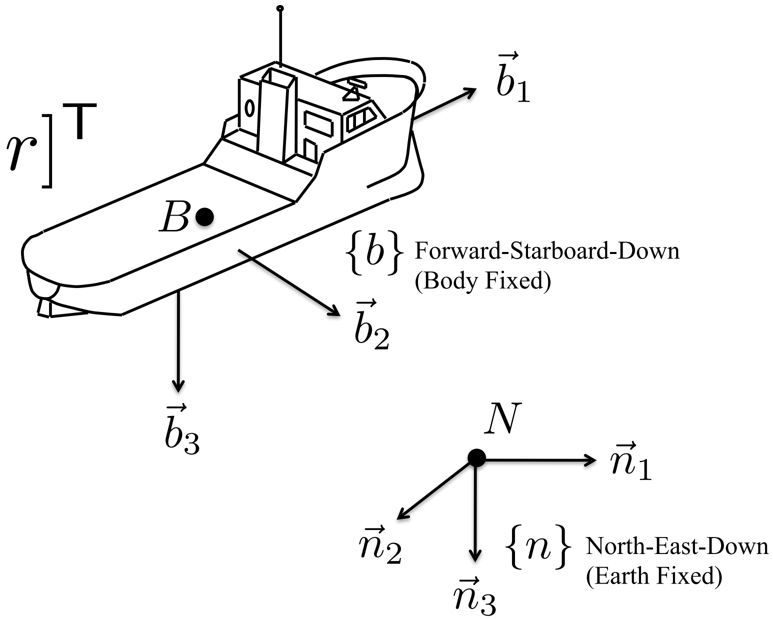
Position-orientation vector:

$$\boldsymbol{\eta} \triangleq \begin{bmatrix} \mathbf{r}^n_{B/N} \\ \boldsymbol{\Theta}^n_b \end{bmatrix} = [N,\ E,\ D,\ \phi,\ \theta,\ \psi]^T$$

Vehicle body-fixed velocity

Body-fixed velocity vector:

$$\boldsymbol{\nu} \triangleq \begin{bmatrix} \mathbf{v}^b_{B/\mathcal{N}} \\ \boldsymbol{\omega}^b_{\mathcal{B}/\mathcal{N}} \end{bmatrix} = [u,\ v,\ w,\ p,\ q,\ r]^\mathsf{T}$$



Vehicle trajectory:

$$\boldsymbol{\eta}(t) = \boldsymbol{\eta}(t_0) + \int_{t_0}^{t} \dot{\boldsymbol{\eta}}(t')\, dt'$$

$$\dot{\boldsymbol{\eta}} = \mathbf{J}(\boldsymbol{\eta})\,\boldsymbol{\nu}$$

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Vehicle KDE

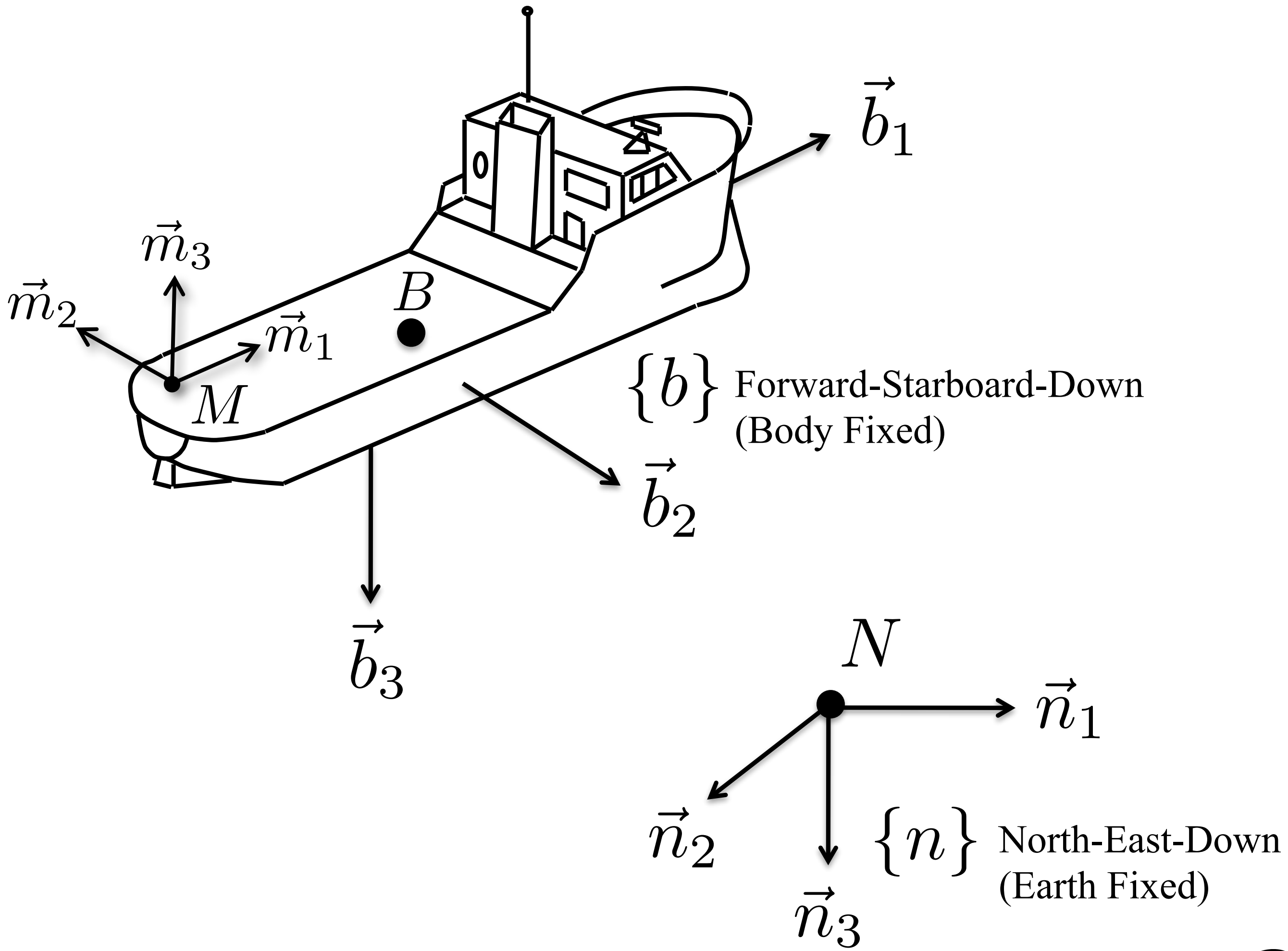
$$\dot{\boldsymbol{\eta}} = \mathbf{J}(\boldsymbol{\eta})\,\boldsymbol{\nu}, \qquad \mathbf{J}(\boldsymbol{\eta}) = \begin{bmatrix} \mathbf{R}_b^n(\boldsymbol{\Theta}) & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_K(\boldsymbol{\Theta}) \end{bmatrix}$$

$$\mathbf{R}_b^n(\boldsymbol{\Theta}) = \begin{bmatrix} c_\psi c_\theta & -s_\psi c_\phi + c_\psi s_\theta s_\phi & s_\psi s_\phi + c_\psi c_\phi s_\theta \\ s_\psi c_\theta & c_\psi c_\phi + s_\phi s_\theta s_\psi & -c_\psi s_\phi + s_\psi c_\phi s_\theta \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}$$

$$\mathbf{T}_K(\boldsymbol{\Theta}) = \begin{bmatrix} 1 & s_\phi t_\theta & c_\phi t_\theta \\ 0 & c_\phi & -s_\phi \\ 0 & \frac{s_\phi}{c_\theta} & \frac{c_\phi}{c_\theta} \end{bmatrix}, \quad \cos(\theta) \neq 0$$

IMU, Body and Inertial Frames

The IMU is typically attached to some fixed point M, relative to the body frame with origin B, that is moving relative to some inertial frame with origin N.



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States

The state are earth fixed positions and body fixed velocities

$$\boldsymbol{\eta} = \begin{bmatrix} \mathbf{r}^n_{B/N} \\ \boldsymbol{\Theta}^n_b \end{bmatrix} \qquad \boldsymbol{\nu} = \begin{bmatrix} {}^{\mathcal{N}}\dot{\mathbf{r}}^b_{B/N} \\ \boldsymbol{\omega}^b_{\mathcal{B}/\mathcal{N}} \end{bmatrix}$$

With kinematics and dynamics

$$\dot{\boldsymbol{\eta}} = \mathbf{J}(\boldsymbol{\eta})\,\boldsymbol{\nu}$$
$$\mathbf{M}\dot{\boldsymbol{\nu}} + \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{g}(\boldsymbol{\eta}) = \boldsymbol{\tau}$$

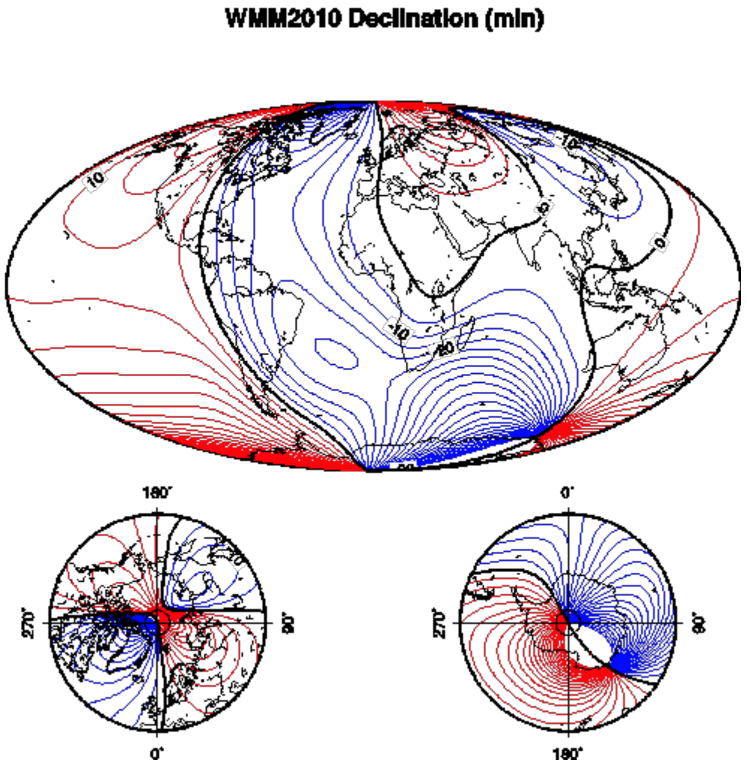
And generalised body forces

$$\boldsymbol{\tau} = \begin{bmatrix} \mathbf{F}^b \\ \mathbf{T}^b_B \end{bmatrix}$$

Magnetometer

The magnetometer measures the magnetic field vector (like a compass), which depends on location on the earth (see World Magnetic Model)

$$\mathbf{m}^n(\mathbf{r}^n_{B/N})$$



If we rotate into {m} coordinates, then

$$\mathbf{y}_{\text{mag}} = \mathbf{R}^m_b \mathbf{R}^b_n \mathbf{m}^n(\mathbf{r}^n_{B/N}) + \mathbf{b}_{\text{mag}} + \boldsymbol{\varepsilon}_{\text{mag}}$$

Where

$$\mathbf{b}_{\text{mag}} = \text{bias terms for magnetometer}$$
$$\boldsymbol{\varepsilon}_{\text{mag}} = \text{random noise term}$$

Gyroscopes

The gyroscopes measure rotation rate about each axis (according to IMU alignment).

$$\boldsymbol{\omega}^{m}_{\mathcal{B}/\mathcal{N}}$$

Therefore, in terms of relating to the states

$$\mathbf{y}_{\text{gyr}} = \mathbf{R}^{m}_{b}\boldsymbol{\omega}^{b}_{\mathcal{B}/\mathcal{N}} + \mathbf{b}_{\text{gyr}} + \boldsymbol{\varepsilon}_{\text{gyr}}$$

Where

$$\mathbf{b}_{\text{gyr}} = \text{bias terms for each gyro}$$
$$\boldsymbol{\varepsilon}_{\text{gyr}} = \text{random noise term}$$

Accelerometer

The accelerometer measures

$$\mathbf{a}^{m}_{M/\mathcal{N}} - \mathbf{g}^{m}$$

That is, accelerations of M relative to the inertial frame *N* and expressed in {m} coordinates, including the gravity vector. Considering the first term,

$$\begin{aligned} \vec{a}_{M/\mathcal{N}} &= \frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{dt}}\left(\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{dt}}\left(\vec{r}_{M/N}\right)\right) \\ &= \frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{dt}}\left(\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{dt}}\left(\vec{r}_{M/B} + \vec{r}_{B/N}\right)\right) \end{aligned}$$

Accelerometer

$$
\begin{aligned}
\vec{a}_{M/\mathcal{N}} &= \frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{dt}}\left(\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{dt}}\left(\vec{r}_{M/B} + \vec{r}_{B/N}\right)\right) \\
&= \frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{dt}}\left(\frac{{}^{\mathcal{N}}\mathrm{d}\vec{r}_{M/B}}{\mathrm{dt}} + \vec{v}_{B/\mathcal{N}}\right) \\
&= \frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{dt}}\left(\frac{{}^{\mathcal{B}}\mathrm{d}\vec{r}_{M/B}}{\mathrm{dt}} + \vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \vec{r}_{M/B} + \vec{v}_{B/\mathcal{N}}\right) \\
&= \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{dt}}\left(\underbrace{\frac{{}^{\mathcal{B}}\mathrm{d}\vec{r}_{M/B}}{\mathrm{dt}}}_{=0} + \vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \vec{r}_{M/B} + \vec{v}_{B/\mathcal{N}}\right) + \vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \left(\underbrace{\frac{{}^{\mathcal{B}}\mathrm{d}\vec{r}_{M/B}}{\mathrm{dt}}}_{=0} + \vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \vec{r}_{M/B} + \vec{v}_{B/\mathcal{N}}\right) \\
&= \underbrace{\frac{{}^{\mathcal{B}}\mathrm{d}\vec{\omega}_{\mathcal{B}/\mathcal{N}}}{\mathrm{dt}}}_{\vec{\alpha}_{\mathcal{B}/\mathcal{N}}} \times \vec{r}_{M/B} + \vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \underbrace{\frac{{}^{\mathcal{B}}\mathrm{d}\vec{r}_{M/B}}{\mathrm{dt}}}_{=0} + \frac{{}^{\mathcal{B}}\mathrm{d}\vec{v}_{B/\mathcal{N}}}{\mathrm{dt}} + \vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \vec{r}_{M/B} + \vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \vec{v}_{B/\mathcal{N}} \\
&= \vec{\alpha}_{\mathcal{B}/\mathcal{N}} \times \vec{r}_{M/B} + \frac{{}^{\mathcal{B}}\mathrm{d}\vec{v}_{B/\mathcal{N}}}{\mathrm{dt}} + \vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \vec{r}_{M/B} + \vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \vec{v}_{B/\mathcal{N}}
\end{aligned}
$$

Accelerometer

Re-arrange and define in terms of states

$$
\begin{aligned}
\vec{a}_{M/\mathcal{N}} &= \vec{\alpha}_{\mathcal{B}/\mathcal{N}} \times \vec{r}_{M/B} + \frac{{}^{\mathcal{B}}\mathrm{d}\vec{v}_{B/\mathcal{N}}}{\mathrm{dt}} + \vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \vec{r}_{M/B} + \vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \vec{v}_{B/\mathcal{N}} \\
&= {}^{\mathcal{B}}\,\dot{\vec{\omega}}_{\mathcal{B}/\mathcal{N}} \times \vec{r}_{M/B} + {}^{\mathcal{B}}\,\dot{\vec{v}}_{B/\mathcal{N}} + \vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \vec{r}_{M/B} + \vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \vec{v}_{B/\mathcal{N}} \\
&= -\vec{r}_{M/B} \times {}^{\mathcal{B}}\,\dot{\vec{\omega}}_{\mathcal{B}/\mathcal{N}} + {}^{\mathcal{B}}\,\dot{\vec{v}}_{B/\mathcal{N}} + \vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \vec{r}_{M/B} + \vec{\omega}_{\mathcal{B}/\mathcal{N}} \times \vec{v}_{B/\mathcal{N}}
\end{aligned}
$$

Recall the states are

$$
\boldsymbol{\eta} = \begin{bmatrix} \mathbf{r}^n_{B/N} \\ \boldsymbol{\Theta}^n_b \end{bmatrix} \qquad \boldsymbol{\nu} = \begin{bmatrix} {}^{\mathcal{N}}\dot{\mathbf{r}}^b_{B/N} \\ \boldsymbol{\omega}^b_{\mathcal{B}/\mathcal{N}} \end{bmatrix}
$$

So, expressed in {b} coordinates

$$
\begin{aligned}
\mathbf{a}^b_{M/\mathcal{N}} &= -\mathbf{S}(\mathbf{r}^b_{M/B})\,{}^{\mathcal{B}}\dot{\boldsymbol{\omega}}^b_{\mathcal{B}/\mathcal{N}} + {}^{\mathcal{B}}\,\dot{\mathbf{v}}^b_{B/\mathcal{N}} + \mathbf{S}(\boldsymbol{\omega}^b_{\mathcal{B}/\mathcal{N}})\mathbf{S}(\boldsymbol{\omega}^b_{\mathcal{B}/\mathcal{N}})\mathbf{r}^b_{M/B} + \mathbf{S}(\boldsymbol{\omega}^b_{\mathcal{B}/\mathcal{N}})\mathbf{v}^b_{B/\mathcal{N}} \\
&= \begin{bmatrix} I & -\mathbf{S}(\mathbf{r}^b_{M/B}) \end{bmatrix} \underbrace{\begin{bmatrix} {}^{\mathcal{B}}\dot{\mathbf{v}}^b_{B/\mathcal{N}} \\ {}^{\mathcal{B}}\dot{\boldsymbol{\omega}}^b_{\mathcal{B}/\mathcal{N}} \end{bmatrix}}_{\dot{\boldsymbol{\nu}}} + \mathbf{S}(\boldsymbol{\omega}^b_{\mathcal{B}/\mathcal{N}}) \left(\mathbf{S}(\boldsymbol{\omega}^b_{\mathcal{B}/\mathcal{N}})\mathbf{r}^b_{M/B} + \mathbf{v}^b_{B/\mathcal{N}} \right)
\end{aligned}
$$

Accelerometer

So we have

$$\begin{aligned}\mathbf{a}^b_{M/\mathcal{N}} &= -\mathbf{S}(\mathbf{r}^b_{M/B})^{\mathcal{B}}\dot{\boldsymbol{\omega}}^b_{\mathcal{B}/\mathcal{N}} + {}^{\mathcal{B}}\,\dot{\mathbf{v}}^b_{B/\mathcal{N}} + \mathbf{S}(\boldsymbol{\omega}^b_{\mathcal{B}/\mathcal{N}})\mathbf{S}(\boldsymbol{\omega}^b_{\mathcal{B}/\mathcal{N}})\mathbf{r}^b_{M/B} + \mathbf{S}(\boldsymbol{\omega}^b_{\mathcal{B}/\mathcal{N}})\mathbf{v}^b_{B/\mathcal{N}} \\ &= \begin{bmatrix} I & -\mathbf{S}(\mathbf{r}^b_{M/B})\end{bmatrix} \underbrace{\begin{bmatrix} {}^{\mathcal{B}}\dot{\mathbf{v}}^b_{B/\mathcal{N}} \\ {}^{\mathcal{B}}\dot{\boldsymbol{\omega}}^b_{\mathcal{B}/\mathcal{N}} \end{bmatrix}}_{\dot{\boldsymbol{\nu}}} + \mathbf{S}(\boldsymbol{\omega}^b_{\mathcal{B}/\mathcal{N}})\left(\mathbf{S}(\boldsymbol{\omega}^b_{\mathcal{B}/\mathcal{N}})\mathbf{r}^b_{M/B} + \mathbf{v}^b_{B/\mathcal{N}}\right)\end{aligned}$$

We can insert the dynamics

$$\dot{\boldsymbol{\nu}} = \mathbf{M}^{-1}\left(\boldsymbol{\tau} - \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} - \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu} - \mathbf{g}(\boldsymbol{\eta})\right)$$

to arrive at

$$\begin{aligned}\mathbf{a}^b_{M/\mathcal{N}} &= \begin{bmatrix} I & -\mathbf{S}(\mathbf{r}^b_{M/B})\end{bmatrix}\mathbf{M}^{-1}\Big(\boldsymbol{\tau} - \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} - \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu} - \mathbf{g}(\boldsymbol{\eta})\Big) + \mathbf{S}(\boldsymbol{\omega}^b_{\mathcal{B}/\mathcal{N}})\left(\mathbf{S}(\boldsymbol{\omega}^b_{\mathcal{B}/\mathcal{N}})\mathbf{r}^b_{M/B} + \mathbf{v}^b_{B/\mathcal{N}}\right) \\ &= \begin{bmatrix} \mathbf{M}^{-1} & -\mathbf{S}(\mathbf{r}^b_{M/B})\mathbf{M}^{-1}\end{bmatrix}\Big(\boldsymbol{\tau} - \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} - \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu} - \mathbf{g}(\boldsymbol{\eta})\Big) + \mathbf{S}(\boldsymbol{\omega}^b_{\mathcal{B}/\mathcal{N}})\left(\mathbf{S}(\boldsymbol{\omega}^b_{\mathcal{B}/\mathcal{N}})\mathbf{r}^b_{M/B} + \mathbf{v}^b_{B/\mathcal{N}}\right)\end{aligned}$$

Accelerometer

Recall that the accelerometer measures

$$\mathbf{a}^m_{M/\mathcal{N}} - \mathbf{g}^m$$

Therefore, the noise-free measurement model is

$$\begin{aligned} f_{\text{acc}}(\boldsymbol{\eta}, \boldsymbol{\nu}) = \mathbf{R}^m_b \Bigg(&\begin{bmatrix} \mathbf{M}^{-1} & -\mathbf{S}(\mathbf{r}^b_{M/B})\mathbf{M}^{-1} \end{bmatrix} \left(\boldsymbol{\tau} - \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} - \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu} - \mathbf{g}(\boldsymbol{\eta}) \right) \Bigg) \\ &+ \mathbf{R}^m_b \left(\mathbf{S}(\boldsymbol{\omega}^b_{\mathcal{B}/\mathcal{N}}) \left(\mathbf{S}(\boldsymbol{\omega}^b_{\mathcal{B}/\mathcal{N}})\mathbf{r}^b_{M/B} + \mathbf{v}^b_{B/\mathcal{N}} \right) - \mathbf{R}^b_n \mathbf{g}^n \right) \end{aligned}$$

Therefore, the measurement equation is

$$\mathbf{y}_{\text{acc}} = f_{\text{acc}}(\boldsymbol{\eta}, \boldsymbol{\nu}) + \mathbf{b}_{\text{acc}} + \boldsymbol{\varepsilon}_{\text{acc}}$$

where

$$\begin{aligned} \mathbf{b}_{\text{acc}} &= \text{bias term for each accelerometer} \\ \boldsymbol{\varepsilon}_{\text{acc}} &= \text{random noise term} \end{aligned}$$

Calibration of IMU

A very important aspect of IMU deployment involves careful calibration of the noise PDF and bias terms.

Calibration can be conducted in various ways:

- Use a dedicated rig to isolate particular axes
  - Accelerometer: test each axes in the up and down directions
  - Gyro: use a controlled rotating arm and isolate each axe

- Use an alternative (and high precision) position tracking system (such as the Vicon system)
  - Record position and orientation and the IMU measurements
  - Perform system identification of unknown parameter values given the data and model structure (e.g. maximum likelihood)