Assignment 1

CSE471-Statistical Methods in Artificial Intelligence (SMAI)
Spring 2017

January 11, 2017

Instructions

- All questions are compulsory to solve.
- Total marks are 25.

Problems

Problem 1 (2 Marks). Let $\mathbf{w}, \mathbf{v} \in \Re^d$. Using Pythagorean theorem, show that $\mathbf{w}.\mathbf{v} = 0$ if $\mathbf{w}$ and $\mathbf{v}$ are perpendicular to each other.

Problem 2 (1 Mark). Let $\mathbf{x}, \mathbf{y} \in \Re^d$. Using Cauchy-Shwarz Inequality show that

$$||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$$

This is the famous triangle inequality.

Problem 3 (3 Marks). Let $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $W = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ both form two different bases for the same vector space, then show that $m = n$.

Problem 4 (2 Marks). Let A be an $n \times n$ matrix such that it has $\lambda_1, \ldots, \lambda_n$ are its distinct eigenvalues. Shown that corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

Problem 5 (2 Marks). Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show that the sum of the eigenvalues of A is same as the sum of the diagonal entries of A.

Problem 6 (1 Mark). Show that the eigenvalues of A are same as the eigenvalues of A^T.

Problem 7 (1 Mark). Let A be an $n \times n$ matrix whose columns $(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ are orthonormal to each other. Orthonomal means that $||\mathbf{a}_i|| = 1, \ \forall i = 1 \ldots n$ and $\mathbf{a}_i.\mathbf{a}_j = 0, \ \forall i \neq j$. Such matrices are called **Unitary Matrices**.

Show that $A^{-1} = A^T$.

Problem 8 (1 Mark). Let S be a linearly independent subset of vector space $\mathcal{V}$. Let $\mathbf{v} \in \mathcal{V}$ which is not in the subspace spanned by S. Prove that the set obtained by adjoining $\mathbf{v}$ to S is linearly independent.

Problem 9 (2 Mark). Let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ forms an orthogonal basis for a subspace S of the vector space $\Re^n$. Orthogonal basis means that $\mathbf{v}_i.\mathbf{v}_j = 0,\ \forall i \neq j$. Show that, for each $\mathbf{x} \in S$, the weights of the linear combination

$$\mathbf{x} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$$

are given by

$$c_j = \frac{\mathbf{x}.\mathbf{v}_j}{\mathbf{v}_j.\mathbf{v}_j}$$

Problem 10 (2 Marks). Show that the null space of A and A^TA are same.

Problem 11 (4 Marks). Let U be an $m \times n$ matrix with orthonormal columns. Let $\mathbf{x}, \mathbf{y} \in \Re^n$, then show that

1. (1 Mark) $||U\mathbf{x}|| = ||\mathbf{x}||$
2. (1 Mark) $(U\mathbf{x}).(U\mathbf{y}) = \mathbf{x}.\mathbf{y}$
3. (2 Marks) $(U\mathbf{x}).(U\mathbf{y}) = 0$ if and only if $\mathbf{x}.\mathbf{y} = 0$

Problem 12 (2 Marks). Let A be an $m \times n$ matrix with a singular value decomposition $A = U\Sigma V^T$ where U is $m \times m$ orthogonal matrix, V is $n \times n$ orthogonal matrix and Σ is $m \times n$ diagonal matrix with r positive entries and no negative entry.

Show that the columns of U are eigenvectors of AA^T and columns of V are eigenvectors of A^TA and the diagonal entries of Σ are the singular values of A.

Problem 13 (2 Marks). Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthonormal basis of $\Re^n$ consisting eigenvectors of A^TA arranged so that the corresponding eigenvalues of A^TA satisfy $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and suppose A has r nonzero singular values. Then $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_n\}$ is an orthogonal basis for column space of A.