Problem Set 1

Suggested Solutions

1. Define three random variables as follows: $X = \{-1, 1\}$, $Y = \{-1, 1\}$, $Z = XY$. The probability distribution of X and Y is same as $p_X = p_Y = \{1/2, 1/2\}$, and X is independent of Y.

a) Calculate the probability distribution of Z;

b) Are X, Y, Z mutually independent? Are they pairwise-independent？

Solution.

a) $Z = \{-1, 1\}$

$$p\{Z = -1\} = p\{X = -1, Y = 1\} + p\{X = 1, Y = -1\}$$
$$= p\{X = -1\}p\{Y = 1\} + p\{X = 1\}p\{Y = -1\} \text{ // as } X\perp Y$$
$$= 1/2,$$
$$p\{Z = 1\} = 1 - p\{Z = -1\} = 1/2.$$

Obviously X, Y, Z are obviously not mutually independent as

$$p\{X = -1, Y = -1, Z = -1\} = 0 \neq 1/8 = p\{X = -1\}p\{Y = -1\}p\{Z = -1\}$$

Next, calculate the joint probability mass function p{X, Z}, p{Y, Z}:

$$p\{X = -1, Z = -1\} = p\{X = -1, Z = -1, Y = -1\} + p\{X = -1, Z = -1, Y = 1\}$$
$$= 0 + p\{Z = -1 \mid X = -1, Y = 1\}p\{ X = -1, Y = 1\} = 1\times 1/4 = 1/4$$
$$p\{X = -1, Z = 1\} = p\{X = -1, Z = 1, Y = -1\} + p\{X = -1, Z = 1, Y = 1\}$$
$$= p\{Z = 1 \mid X = -1, Y = -1\}p\{X = -1, Y = -1\} + 0 = 1\times 1/4 = 1/4$$
$$p\{X = 1, Z = -1\} = p\{X = 1, Z = -1, Y = -1\} + p\{X = 1, Z = -1, Y = 1\}$$
$$= p\{Z = -1 \mid X = 1, Y = -1\}p\{X = 1, Y = -1\} + 0 = 1\times 1/4 = 1/4$$
$$p\{X = 1, Z = 1\} = p\{X = 1, Z = 1, Y = -1\} + p\{X = 1, Z = 1, Y = 1\}$$
$$= 0 + p\{Z = 1 \mid X = 1, Y = 1\}p\{X = 1, Y = 1\} = 1\times 1/4 = 1/4$$

Hence, $X\perp Z$. Symmetrically, it can be readily calculated that $p\{Y = -1, Z = -1\} = p\{Y = -1, Z = 1\} = p\{Y = 1, Z = -1\} = p\{Y = 1, Z = 1\} = 1/4$, and hence $Y\perp Z$ too. In conclusion, X, Y, Z are pairwise independent.

2. Let $X = \{-1, 0, 1\}$ be a random variable with probability mass function $P(-1) = 0.2$, $p(0) = 0.5$, $p(1) = 0.3$. Compute $E[X]^2$ and $E[X^2]$.

Solution. $E[X]^2 = (-1\times 0.2 + 0\times 0.5 + 1\times 0.3)^2 = 0.01$, $E[X^2] = (1\times 0.2 + 0\times 0.5 + 1\times 0.3) = 0.5$

3. Let $X = \{1, 2\}$, $Y = \{1, 2\}$, $p\{X = 1, Y = 1\} = 0$, $P\{X = 1, Y = 2\} = P\{X = 2, Y = 1\} = P\{X = 2, Y = 2\} = 1/3$. Is X independent of Y? Compute $E[X^2]$, $E[Y^2]$, $E[X^2+Y^2]$, $E[X^2Y^2]$.

Solution. $p\{X = 1\} = p\{X = 1, Y = 1\} + P\{X = 1, Y = 2\} = 1/3$， $p\{X = 2\} = 1 – p\{X = 1\} = 2/3$；

$p\{Y = 1\} = p\{X = 1, Y = 1\} + P\{X = 2, Y = 1\} = 1/3$， $p\{Y = 2\} = 1 – p\{Y = 1\} = 2/3$；

Since $p\{X = 1, Y = 1\} = 0 \neq 1/9 = p\{X = 1\}p\{Y = 1\}$, X and Y are NOT independent.

$E[X^2] = E[Y^2] = (1/3)\times 1^2 + (2/3)\times 2^2 = 3$；

$E[X^2 + Y^2] = E[X^2] + E[Y^2] = 6$;

$E[X^2Y^2] = P\{X = 1, Y = 2\}\times(1^2\times 2^2) + P\{X = 2, Y = 1\}\times(2^2\times 1^2) + P\{X = 2, Y = 2\}\times(2^2\times 2^2) = 8$

(Note that $E[X^2Y^2] \neq E[X^2]E[Y^2] = 9$ here).

4. For three binary random variables X, Y, Z, their joint probability mass function is given as $P(X = 0, Y = 0, Z = 0) = P(X = 1, Y = 1, Z = 0) = P(X = 1, Y = 0, Z = 1) = P(X = 0, Y = 1, Z = 1) = 0.2$, $P(X = 0, Y = 0, Z = 1) = P(X = 1, Y = 1, Z = 1) = P(X = 1, Y = 0, Z = 0) = P(X = 0, Y = 1, Z = 0) = 0.05$. Calculate:

a) The distribution of Z, i.e., $p(Z = 0)$ and $p(Z = 1)$;

b) $p(X = 0, Y = 0 \mid Z = 1)$, $p(X = 1, Y = 0 \mid Z = 1)$.

Solution.

a) $p(Z = 0) = P(X = 0, Y = 0, Z = 0) + P(X = 1, Y = 1, Z = 0) + P(X = 1, Y = 0, Z = 0) = P(X = 0, Y = 1, Z = 0) = 0.5$； $P(Z = 1) = 1 – P(Z = 0) = 0.5$

b) $p(X = 0, Y = 0 \mid Z = 1) = p(X = 0, Y = 0, Z = 1)/p(Z = 1) = 0.1$

$p(X = 1, Y = 0 \mid Z = 1) = p(X = 1, Y = 0, Z = 1)/p(Z = 1) = 0.4$

5. Define three random variables as follows. $X = \{0, 1\}$, $p(X = 0) = 1/3$, $p(X = 1) = 2/3$; $Y = \{0, 1\}$, $p(Y = 0) = 1/4$, $p(Y = 1) = 3/4$, and Y is independent of X; $Z = XY$.

a) Find $p(Z = 0 \mid X = 0, Y = 0)$, $p(Z = 1 \mid X = 0, Y = 1)$;

b) Find $p(X = Z)$, $p(Y = Z)$;

c) Find the joint probability mass function $p(X, Y, Z)$;

d) Find the probability distribution of Z?

e) Find $E[Z]$.

Solution.

a) $p(Z = 0 \mid X = 0, Y = 0) = 1$, $p(Z = 1 \mid X = 0, Y = 1) = 0$

b) $p(X = Z) = p(X = 0, Y = 0) + p(X = 0, Y = 1) + p(X = 1, Y = 1) = 1 – p(X = 1, Y = 0) = 5/6$

Similarly, $p(Y = Z) = 1 – p(X = 0, Y = 1) = 3/4$

c) $p(X = 0, Y = 0, Z = 0) = p(Z = 0 \mid X = 0, Y = 0)p(X = 0, Y = 0) = 1\times 1/3\times 1/4 = 1/12$

$p(X = 0, Y = 0, Z = 1) = p(Z = 1 \mid X = 0, Y = 0)p(X = 0, Y = 0) = 0$

$p(X = 0, Y = 1, Z = 0) = p(Z = 0 \mid X = 0, Y = 1)p(X = 0, Y = 1) = 1\times 1/3\times 3/4 = 1/4$

$p(X = 0, Y = 1, Z = 1) = p(Z = 1 \mid X = 0, Y = 1)p(X = 0, Y = 1) = 0$

$p(X = 1, Y = 0, Z = 0) = p(Z = 0 \mid X = 1, Y = 0)p(X = 1, Y = 0) = 1\times 2/3\times 1/4 = 1/6$

$p(X = 1, Y = 0, Z = 1) = p(Z = 1 \mid X = 1, Y = 0)p(X = 1, Y = 0) = 0$

$p(X = 1, Y = 1, Z = 0) = p(Z = 0 \mid X = 1, Y = 1)p(X = 1, Y = 1) = 0$

$p(X = 1, Y = 1, Z = 1) = p(Z = 1 \mid X = 1, Y = 1)p(X = 1, Y = 1) = 1\times 2/3\times 3/4 = 1/2$

d) $p(Z = 0) = p(X = 0, Y = 0, Z = 0) + p(X = 0, Y = 1, Z = 0) + p(X = 1, Y = 0, Z = 0) + p(X = 1, Y = 1, Z = 0)$

$= 1/12 + 1/4 + 1/6 + 0 = 1/2$;

$p(Z = 1) = p(X = 0, Y = 0, Z = 1) + p(X = 0, Y = 1, Z = 1) + p(X = 1, Y = 0, Z = 1) + p(X = 1, Y = 1, Z = 1)$

$= 0 + 0 + 0 + 1/2 = 1/2$.

e) $E[Z] = 1/2$.

6. (Binary Symmetric Erasure Channel) Let $X = \{0, 1\}$ and $Y = \{0, 1, e\}$. It is known that

$p(Y = 0 \mid X = 0) = p(Y = 1 \mid X = 1) = 0.7$,

$p(Y = 1 \mid X = 0) = p(Y = 0 \mid X = 1) = 0.1$

$p(Y = e \mid X = 0) = p(Y = e \mid X = 1) = 0.2$

When $p(X = 0) = p(X = 1) = 1/2$, compute $p(X = 1 \mid Y = 1)$ and $p(X \neq Y)$.

Solution.

$p(X = 1 \mid Y = 1) = p(X = 1, Y = 1)/p(Y = 1)$

$= p(Y = 1 \mid X = 1)p(X = 1) / [p(X = 0, Y = 1) + p(X = 1, Y = 1)]$

$= p(Y=1 \mid X=1)p(X = 1) / [p(Y=1 \mid X=0)p(X=0) + p(Y=1 \mid X=1)p(X=1)]$

$= 0.7\times 0.5/[0.1\times 0.5 + 0.7\times 0.5] = 7/8$

$p(X = Y) = p(X = 0, Y = 0) + p(X = 1, Y = 1)$

$= p(Y = 0 \mid X = 0)p(X = 0) + p(Y = 1 \mid X = 1)p(X = 1) = 0.7$,

so $p(X \neq Y) = 0.3$.

7. Suppose that we have 3 cards that are identical in form, except that both sides of the first card are colored red, both sides of the second card are colored black, and one side of the third card is colored red and the other side black. The 3 cards are mixed up in a hat, and 1 card is randomly selected and put down on the ground. If the upper side of the chosen card is colored red, what is the probability that the other side is colored black?

(Hint: let $X = \{1, 2, 3\}$ denote the card that is selected, and $Y = \{R, B\}$ denote the color of the upper side of the selected card. What the question asks is to essentially compute $P(X = 3 \mid Y = R)$.)

Solution.

$P(X = 3 \mid Y = R)$

$= p(X = 3, Y = R)/p(Y = R)$

$= p(Y = R \mid X = 3)p(X = 3)/[p(X = 1, Y = R) + p(X = 2, Y = R) + p(X = 3, Y = R)]$

$= p(Y=R \mid X=3)p(X=3)/[p(Y=R \mid X=1)p(X=1)+p(Y=R \mid X=2)p(X=2)+p(Y=R \mid X=3)p(X=3)]$

$= 1/2\times1/3/[1\times1/3+0\times1/3+1/2\times1/3] = 1/3$