Department of Mathematics

MTL107: Numerical Methods and Computations

Exercise Set 9: Numerical Differentiation-three point formulas, five point formulas.

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1. Use the forward-difference formulas and backward-difference formulas to determine each missing entry in the following tables:

a.

x	f(x)	$f'(x)$
0.5	0.4794	
0.6	0.5646	
0.7	0.6442	

b.

x	f(x)	$f'(x)$
0.0	0.00000	
0.2	0.74140	
0.4	1.3718	

2. The data in Exercise 1 were taken from the following functions. Compute the actual errors in Exercise 1, and find error bounds using the error formulas.
a. $f(x) = \sin x$ b. $f(x) = e^x - 2x^2 + 3x - 1$.

3. Use the most accurate 3-point formula to determine each missing entry in the following tables:

a.

x	f(x)	$f'(x)$
1.1	9.025013	
1.2	11.02318	
1.3	13.46374	
1.4	16.44465	

b.

x	f(x)	$f'(x)$
8.1	16.94410	
8.3	17.56492	
8.5	18.19056	
8.7	18.82091	

c.

x	f(x)	$f'(x)$
2.9	-4.827866	
3.0	-4.240058	
3.1	-3.496909	
3.2	-2.596792	

d.

x	f(x)	$f'(x)$
2.0	3.6887983	
2.1	3.6905701	
2.2	3.6688192	
2.3	3.6245909	

4. The data in Exercise 3 were taken from the following functions. Compute the actual errors in Exercise 3, and find error bounds using the error formulas.
a. $f(x) = e^{2x}$ b. $f(x) = x \ln x$.
c. $f(x) = x \cos x - x^2 \sin x$ d. $f(x) = 2(\ln x)^2 + 3 \sin x$.

a.

x	f(x)	$f'(x)$
2.1	-1.709847	
2.2	-1.373823	
2.3	-1.119214	
2.4	-0.9160143	
2.5	-0.7470223	
2.6	-0.6015966	

b.

x	f(x)	$f'(x)$
-3.0	9.367879	
-2.8	8.233241	
-2.6	7.180350	
-2.4	6.209329	
-2.2	5.320305	
-2.0	4.513417	

5. Use the formulas given in this section to determine, as accurately as possible, approximations for each missing entry in the above tables:

6. The data in Exercise 5 were taken from the following functions. Compute the actual errors in Exercise 5, and find error bounds using the error formulas.
   a. $f(x) = \tan x$ b. $f(x) = e^{x/3} + x^2$.

7. Use the following data and the knowledge that the first five derivatives of f are bounded on $[1, 5]$ by 2,3,6, and 23, respectively, to approximate $f'(3)$ as accurately as possible. Find a bound for the error.

x	1	2	3	4	5
f(x)	2.4142	2.6734	2.8974	3.0976	3.2804

8. Repeat Exercise 1 using 4-digit rounding arithmetic, and compare the errors to those in Exercise 2.

9. Repeat Exercise 5 using 4-digit rounding arithmetic, and compare the errors to those in Exercise 6.

10. Let $f(x) = \cos \pi x$. Approximate $f''(0.5)$ using the values of $f(x)$ at $x = 0.24, 0.5,$ and 0.75 and the second derivative midpoint formula

$$f''(x_0) = \frac{1}{h^2}[f(x_0 - h) - 2f(x_0) + f(x_0 + h)] - \frac{h^2}{12} f^{(4)}(\xi)$$

for some ξ, where $x_0 - h < \xi < x_0 + h$. Compare this result to the exact value. Find a bound for the error.

x	0.2	0.4	0.6	0.8	1.0
f(x)	0.9798652	0.9177710	0.8080348	0.6386093	0.3843735

11. Consider the above table of data:
a. Approximate $f'(0.2)$ and $f'(1.0)$ using the five-point endpoint formula. For $x_0 < \xi < x_0 + 4h$.

$$f'(x_0) = \frac{1}{12h}[-25f(x_0)+48f(x_0+h)-36f(x_0+2h)+16f(x_0+3h)-3f(x_0+4h)]+\frac{h^4}{5}f^{(5)}(\xi)$$

b. Approximate $f'(0.6)$ using the five-point midpoint formula

$$f'(x_0) = \frac{1}{12h}[f(x_0-2h) - 8f(x_0-h) + 8f(x_0+h) - f(x_0+2h)] + \frac{h^4}{30}f^{(5)}(\xi)$$

for some ξ, where $x_0 - 2h < \xi < x_0 + 2h$.

ANSWERS

1. From the forward and backward difference formula, we have the following approximations:
1a. $f'(0.5) \approx 0.8520, f'(0.6) \approx 0.8520, f'(0.7) \approx 0.7960$

1b. $f'(0.0) \approx 3.7070, f'(0.2) \approx 3.1520, f'(0.4) \approx 3.1520$

2a.	x	Actual Error	Error Bound
	0.5	0.0255	0.0282
	0.6	0.0267	0.0282
	0.7	0.0312	0.0322

2b.	x	Actual Error	Error Bound
	0.0	0.2930	0.3000
	0.2	0.2694	0.2779
	0.4	0.2602	0.2779

2.

3. For the endpoints of the tables, we use 3-Point Endpoint Formula. The other approximations come from 3-Point Midpoint formula:

3a. $f'(1.1) \approx 17.769705, f'(1.2) \approx 22.193635, f'(1.3) \approx 27.107350, f'(1.4) \approx 32.150850$
3b. $f'(8.1) \approx 3.092050, f'(8.3) \approx 3.116150, f'(8.5) \approx 3.139975, f'(8.7) \approx 3.163525$
3c. $f'(2.9) \approx 5.101375, f'(3.0) \approx 6.654785, f'(3.1) \approx 8.216330, f'(3.2) \approx 9.786010$
3d. $f'(2.0) \approx 0.13533150, f'(2.1) \approx -0.09989550, f'(2.2) \approx -0.3298960, f'(2.3) \approx -0.5546700.$

4.

4a.	x	Actual Error	Error Bound
	1.1	0.280322	0.359033
	1.2	0.147282	0.179517
	1.3	0.179874	0.219262
	1.4	0.378444	0.438524

4b.	x	Actual Error	Error Bound
	8.1	1.8594×10^{-4}	2.0322×10^{-5}
	8.3	1.0551×10^{-4}	1.0161×10^{-5}
	8.5	9.116×10^{-5}	9.677×10^{-6}
	8.5	2.0197×10^{-4}	1.9355×10^{-5}

5. The approximations and the formulas: 5PEP=5 Point Endpoint formula and 5PMP=5 Point Mid Point formula used are:

   5a. $f^{'}(2.1) \approx 3.899344(5PEP), f^{'}(2.2) \approx 2.876876(5PEP), f^{'}(2.3) \approx 2.249704(5PMP),$ $f^{'}(2.4) \approx 1.837756(5PMP), f^{'}(2.5) \approx 1.544210(5PEP), f^{'}(2.6) \approx 1.355496(5PEP).$

   5b. $f^{'}(-3.0) \approx -5.877358(5PEP), f^{'}(-2.8) \approx -5.468933(5PEP), f^{'}(-2.6) \approx -5.059884(5PMP),$ $f^{'}(-2.4) \approx -4.650223(5PMP), f^{'}(-2.2) \approx -4.239911(5PEP), f^{'}(-2.0) \approx -3.828853(5PEP).$

6.

7. $f^{'}(3) \approx \frac{1}{12}[f(1) - 8f(2) + 8f(4) - f(5)] = 0.21062$, with an error bound given by

$$\max_{1 \le x \le 5} \frac{|f^{(}5)(x)|h^4}{30} \le \frac{23}{30} = 0.7\bar{6}.$$

8. From the forward-backward difference formula, we have the following approximations:
   8a. $f^{'}(0.5) \approx 0.852, f^{'}(0.6) \approx 0.852, f^{'}(0.7) \approx 0.7960$

   8b. $f^{'}(0.0) \approx 3.707, f^{'}(0.2) \approx 3.153, f^{'}(0.4) \approx 3.153$

9. For the endpoints of the tables, we use Formula 5PEP. The other approximations come from Formula 5PMP. 9a. $f^{'}(2.1) \approx 3.884, f^{'}(2.2) \approx 2.896, f^{'}(2.3) \approx 2.249,$ $f^{'}(2.4) \approx 1.836, f^{'}(2.5) \approx 1.550, f^{'}(2.6) \approx 1.348.$

   9b. $f^{'}(-3.0) \approx -5.883, f^{'}(-2.8) \approx -5.467, f^{'}(-2.6) \approx -5.059,$ $f^{'}(-2.4) \approx -4.650, f^{'}(-2.2) \approx -4.208, f^{'}(-2.0) \approx -3.875.$

10. The approximation is -4.8×10^{-9}. $f^{''}(0.5) = 0$. The error bound is 0.35874. The method is very accurate since the function is symmetric about $x = 0.5$.

4c.	x	Actual Error	Error Bound
	2.9	0.011956	0.0180988
	3.0	0.0049251	0.00904938
	3.1	0.0004765	0.00493920
	3.2	0.0013745	0.00987840

4d.	x	Actual Error	Error Bound
	2.0	0.00252235	0.00410304
	2.1	0.00142882	0.00205152
	2.2	0.00204851	0.00260034
	2.3	0.00437954	0.00520068

11. 11a. $f^{'}(0.2) \approx -0.1951027$
    11b. $f^{'}(1.0) \approx -1.541415$
    11c. $f^{'}(0.6) \approx -0.6824175$.

6a.	x	Actual Error	Error Bound
	2.1	0.0242312	0.109271
	2.2	0.0105138	0.0386885
	2.3	0.0029352	0.0182120
	2.4	0.0013262	0.00644808
	2.5	0.0138323	0.109271
	2.6	0.0064225	0.0386885

6b.	x	Actual Error	Error Bound
	-3.0	1.55×10^{-5}	6.33×10^{-7}
	-2.8	1.32×10^{-5}	6.76×10^{-7}
	-2.6	7.95×10^{-7}	1.05×10^{-7}
	-2.4	6.79×10^{-7}	1.13×10^{-7}
	-2.2	1.28×10^{-5}	6.76×10^{-7}
	-2.0	7.96×10^{-6}	6.76×10^{-7}