Introduction to Mathematical Logic

Chapter 5 Beginning Propositional Logic

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Propositions

- All men are mortal.
- Romeo loves Juliet.
- There is no largest natural number.
- He speaks French or German.
- Albert Camus was a great novelist and philosopher.
- If he is a spy, he has to die.
- …

Logical Connectives

- Negation / not / ~
  - p: There is the largest natural number.
  - ~p: There is no largest natural number.

p	~p
T	F
F	T

Logical Connectives

- Conjunction / and / $\wedge$
  - p: Albert Camus was a novelist.
  - q: Albert Camus was a philosopher.
  - $p \wedge q$: Albert Camus was a novelist and philosopher.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Logical Connectives

- Disjunction / or / $\vee$
  - p: He speaks French.
  - q: He speaks German.
  - $p \vee q$: He speaks French or German.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Logical Connectives

- Disjunction / or / ∨
  - p: He'll finish the job today.
  - q: He'll finish the job tomorrow.
  - $p \vee q$: He'll finish the job today or tomorrow.

p	q	$p \vee q$
T	T	
T	F	T
F	T	T
F	F	F

Logical Connectives

- Disjunction / or / $\vee$
  - p: You come with me.
  - q: You stay here.
  - $p \vee q$: You come with me or stay here.

p	q	$p \vee q$
T	T	
T	F	T
F	T	T
F	F	F

Logical Connectives

- Conditional / imply / if...then... / ⊃
  - p: He is a spy.
  - q: He has to die.
  - p⊃q: If he is a spy, he has to die.

p	q	p⊃q
T	T	T
T	F	F
F	T	T
F	F	T

Logical Connectives

- Biconditional / if and only if / equivalent / ≡
  - p: He is a spy.
  - q: He has to die.
  - p≡q: He has to die if and only if he is a spy.

p	q	p≡q
T	T	T
T	F	F
F	T	F
F	F	T

Formulae

- Letters like *p*, *q*, *r*, with or without subscripts, are *propositional variables*.
- By *a formula* is meant any expression that is constructed according to the following rules.
  - (1) A propositional variable is a formula.
  - (2) Given formulae X and Y already constructed, $\sim X$, $(X \wedge Y)$, $(X \vee Y)$, $(X \supset Y)$ and $(X \equiv Y)$ are all formulae.
  - No expression is a formula unless its being so is the consequence of rule (1) and (2).

Formulae

- Which, if any, are formulae among the follow expressions?
  - (1) p
  - (2) $\sim q$
  - (3) $((p \supset q) \wedge r)$
  - (4) $\sim(p_1 \wedge (p_2 \supset p_3) \wedge (p_1 \supset p_3))$
  - (5) $\sim(p_1 \wedge (p_2 \supset p_3) \vee (p_1 \supset p_3))$
  - (6) $((p_1 \supset (p_2 \supset p_3)) \vee (p_1 \supset (\sim p_3)))$
  - (7) $((p_1 \supset (p_2 \supset p_3)) \vee \sim(p_1 \supset p_3))$

Formulae

- Matching parentheses
  - $((p_1 \supset (p_2 \supset p_3)) \vee \sim(p_1 \supset p_3))$
  - Define *a pair of matching parentheses*.
- The *level* of a matching parentheses pair
  - A pair of matching parentheses between that there is no matching parenthesis is said to be parentheses of *level 0*.
  - A pair of matching parentheses is said to be of *level n+1* given that the top pair between the pair of parentheses is of level *n*.

Formulae

- Simplifications
  - Since $((X \vee Y) \vee Z)$ has the same truth table as $(X \vee (Y \vee Z))$, they both can be put as $(X \vee Y \vee Z)$.
  - Therefore, all formulae consist of X_1, X_2, …, X_n and disjunctions only for connectives like $((\ldots(X_1 \vee X_2) \vee \ldots) \vee X_n)$ can be put as $(X_1 \vee X_2 \vee \ldots \vee X_n)$. (*prove it by yourselves*)
  - Similarly, all those of X_1, X_2, …, X_n and only conjunctions, can be put as $(X_1 \wedge X_2 \wedge \ldots \wedge X_n)$.

Formulae

- Simplifications
  - The most outer pair of parentheses can be lifted without misunderstanding.
  - So $(X \supset Y)$ can be put as $X \supset Y$, and $(X_1 \wedge X_2 \wedge ... \wedge X_n)$ as $X_1 \wedge X_2 \wedge ... \wedge X_n$.

Formulae

- Compound truth tables
  - $((p_1 \supset (p_2 \supset p_1)) \vee \sim(p_1 \supset p_2))$

p_1	p_2	$(p_2 \supset p_1)$	$(p_1 \supset p_2)$	$(p_1 \supset (p_2 \supset p_1))$	$\sim(p_1 \supset p_2)$	$((p_1 \supset (p_2 \supset p_1)) \vee \sim(p_1 \supset p_2))$
T	T	T	T	T	F	T
T	F	T	F	T	T	T
F	T	F	T	T	F	T
F	F	T	T	T	F	T

Formulae

- Find, if any, the wrong calculated row of the truth table.
  - $((p_1 \supset (p_2 \supset p_3)) \vee \sim(p_1 \supset p_3))$

p_1	p_2	p_3	$(p_2 \supset p_3)$	$(p_1 \supset p_3)$	$(p_1 \supset (p_2 \supset p_3))$	$\sim(p_1 \supset p_3)$	$((p_1 \supset (p_2 \supset p_3)) \vee \sim(p_1 \supset p_3))$
T	T	T	T	T	T	F	T
T	T	F	F	F	F	T	T
T	F	T	T	T	T	F	T
T	F	F	T	F	F	T	T
F	T	T	T	T	T	F	T
F	T	F	F	T	T	F	T
F	F	T	T	T	F	F	F
F	F	F	T	T	T	F	T

Tautologies

- Consider the following formula and its truth table
  - $((p_1 \supset (p_2 \supset p_3)) \vee \sim(p_1 \supset p_3))$

p_1	p_2	p_3	$(p_2 \supset p_3)$	$(p_1 \supset p_3)$	$(p_1 \supset (p_2 \supset p_3))$	$\sim(p_1 \supset p_3)$	$((p_1 \supset (p_2 \supset p_3)) \vee \sim(p_1 \supset p_3))$
T	T	T	T	T	T	F	T
T	T	F	F	F	F	T	T
T	F	T	T	T	T	F	T
T	F	F	T	F	T	T	T
F	T	T	T	T	T	F	T
F	T	F	F	T	T	F	T
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

Tautologies

- State which of the following are *tautologies*, which are *contradictions*, and which are *contingent*.
  - (1) $((p_1 \supset (p_2 \supset p_3)) \vee \sim(p_1 \supset p_3))$
  - (2) $((p \supset q) \supset (q \supset p))$
  - (3) $((p \supset q) \supset (\sim p \supset \sim q))$
  - (4) $((p \supset q) \supset (\sim q \supset \sim p))$
  - (5) $(p \supset \sim p)$
  - (6) $(p \equiv \sim p)$

Tautologies

- State which of the following are *tautologies*, which are *contradictions*, and which are *contingent*.
  - (7) $((p \equiv q) \equiv (\sim p \equiv \sim q))$
  - (8) $(\sim(p \wedge q) \equiv (\sim p \wedge \sim q))$
  - (9) $(\sim(p \wedge q) \equiv (\sim p \vee \sim q))$
  - (10) $((\sim p \vee \sim q) \equiv \sim(p \vee q))$
  - (11) $(\sim(p \vee q) \equiv (\sim p \wedge \sim q))$
  - (12) $((p \equiv (p \wedge q)) \equiv (q \equiv (p \vee q)))$

Logical Implication and Equivalence

- Let X, Y be formulae and S be a set of formulae
  - By X (logically) *implies* Y is meant in all cases where X is true, Y is also true. Or what is the same thing, $X \supset Y$ is a tautology.
  - By S (logically) *implies* X is meant in all cases where all elements of S are true, so is X.
  - By X is (logically) *equivalent* to Y is meant both X and Y are true in **just** the same cases. Or what is the same thing, $X \equiv Y$ is a tautology.

Logical Implication and Equivalence

- Which, if any, of the following statements are true?
  - (1) $(p \wedge q)$ implies p.
  - (2) $(p \vee q)$ implies p.
  - (3) p implies $(p \wedge q)$.
  - (4) p implies $(p \vee q)$.
  - (5) $\{p, p \supset q\}$ implies q.
  - (6) $\{q, p \supset q\}$ implies p.
  - (7) $\{q, p \supset q\}$ implies q.

Logical Implication and Equivalence

- Which, if any, of the following statements are true?
    - (8) $(p \supset q)$ is equivalent to $(\sim p \supset \sim q)$.
    - (9) $(p \supset q)$ is equivalent to $(\sim q \supset \sim p)$.

Formulae Involving t and f

- Let *t* stands for truth and *f* stands for falsity, we define *t* and *f* as *propositional constants*.
- We thus have a new version of definition of *formulae* as following.
  - (1) A propositional variable is a formula.
  - (2) A propositional constant is a formula.
  - (3) Given formulae X and Y already constructed, $\sim X$, $(X \wedge Y)$, $(X \vee Y)$, $(X \supset Y)$ and $(X \equiv Y)$ are all formulae.
  - An expression is a formula unless its being so is the consequence of rules (1) through (3).

Formulae Involving t and f

- Every formula involving *t*'s or *f*'s or both is equivalent to another that involves neither, or to *t* alone, or to *f* alone. Related facts are following with *equ* standing for *is equivalent to*.
  - $X \wedge t$ equ X, $t \wedge X$ equ X
  - $X \wedge f$ equ f, $f \wedge X$ equ f
  - $X \vee t$ equ t, $t \vee X$ equ t
  - $X \vee f$ equ X, $f \vee X$ equ X

Formulae Involving t and f

- Every formula involving *t*'s or *f*'s or both is equivalent to another that involves neither, or to *t* alone, or to *f* alone. Related facts are following with *equ* standing for *is equivalent to*.
  - $X \supset t$ equ t, $t \supset X$ equ X
  - $X \supset f$ equ $\sim X$, $f \supset X$ equ t
  - $X \equiv t$ equ X, $t \equiv X$ equ X
  - $X \equiv f$ equ $\sim X$, $f \equiv X$ equ $\sim X$
  - t equ $\sim f$, f equ $\sim t$

Formulae Involving t and f

- Reduce following formulae to anothers that involve neither, or to *t* alone, or to *f* alone.
  - $(((t \supset p) \wedge (q \vee f)) \supset ((q \supset f) \vee (r \wedge t)))$
  - $((p \vee t) \supset q)$
  - $(\sim(p \vee t) \equiv (f \supset q))$

Liars and Truth Tellers

- There is a land where inhabitants are of one of two types only – truth tellers who tell only truths and liars who tell only lies. Now let's consider two inhabitants A and B from the land when A is asked to say something about they two.
    - Suppose A says 'We are both liars.' What can be determined about the types of A and B?

Liars and Truth Tellers

- There is a land where inhabitants are of one of two types only – truth tellers who tell only truths and liars who tell only lies. Now let's consider two inhabitants A and B from the land when A is asked to say something about they two.
  - If, instead, A says 'At lease one of us is a liar.' then what can be determined?

Liars and Truth Tellers

- There is a land where inhabitants are of one of two types only – truth tellers who tell only truths and liars who tell only lies. Now let's consider two inhabitants A and B from the land when A is asked to say something about they two.
  - If, instead, A says 'We are of the same type. B and I are both truth tellers or liars.' then what can be determined about them?

The End

- **Chapter 5: Beginning Propositional Logic**