5/30/2021

# Mathematical Methods in Philosophy

Subject: Mathematical logic

School of Computer and Communication engineering

University of Science and Technology Beijing(USTB), Beijing 10083, China.

Husnain Ahmed

STUDENT ID: S20201481

**Abstract:**

If mathematics is considered a science, then the philosophy of mathematics, like the philosophy of physics and the philosophy of biology, might be considered a branch of scientific philosophy. The philosophy of mathematics, on the other hand, holds a unique place in the philosophy of science due to its subject matter. While the natural sciences study entities that are spatially and temporally situated, it is far from evident that the same is true of mathematical objects. Furthermore, the methods of investigation used in mathematics differ significantly from those used in the scientific sciences. Unlike general knowledge, which is gained through inductive processes, mathematical information appears to be learned through deduction from basic concepts. Mathematical knowledge appears to have a different status than knowledge in the scientific sciences. Natural science theories appear to be less definite and more subject to modification than mathematical theories. As a result, mathematics presents philosophy with a unique set of issues. As a result, philosophers have paid close attention to the ontological and epistemological issues surrounding mathematics.

## 1. Introduction:

In this paper discussion will take place about different philosophical aspects. Many Philosophers and Scholars depicts their ideas regarding mathematical Methods which will be discussed in detail.

### 1.1. Philosophy of Mathematics, Logics and foundation of Mathematics:

On the one hand, mathematical philosophy is concerned with issues that are intimately related to core metaphysical and epistemological issues. At first glance, mathematics appears to be concerned with abstract concepts. This makes one wonder what mathematical entities are made of and how we might know about them. If these difficulties are deemed insurmountable, one can consider whether mathematical objects may, in fact, belong in the real world.

On the other hand, it has been discovered that mathematical approaches may be applied to philosophical problems about mathematics to some extent. This has been done in the context of mathematical logic, which is roughly defined as including the subfields of proof theory, model theory, set theory, and computability theory. As a result, the mathematical exploration of the repercussions of what are essentially philosophical views about the nature of mathematics has dominated the twentieth century.

Professional mathematicians are said to be involved in fundamental research when they are concerned with the subject's underpinnings. Professional philosophers are said to contribute to the philosophy of mathematics when they examine philosophical concerns about mathematics. Of course, the line between mathematics philosophy and mathematics foundations is blurry, and the more contact there is between philosophers and mathematicians working on concerns about mathematics' nature, the better.

## 2. Four School of Thoughts:

In the nineteenth century, the overall philosophical and scientific attitude leaned toward the empirical: platonistic parts of rationalistic theories of mathematics were rapidly losing favour. The once highly acclaimed faculty of rational perception of ideas, in particular, was viewed with mistrust. As a result, formulating a philosophical theory of mathematics free of platonistic features became a task. Three non-platonistic views of mathematics emerged in the first decades of the twentieth century: logicism, formalism, and intuitionism. Predictiveism was a fourth programme that appeared at the turn of the twentieth century. Its true potential was not realized until the 1960s, due to unforeseen historical events. It does, however, merit a position alongside the three main schools of mathematical philosophy, which are mentioned in most standard current introductions to the subject.[1]:[2]

### 2.1 Logicism:

The goal of the logicist endeavour is to reduce mathematics to logic. Because logic is supposed to be unbiased when it comes to ontological issues, this endeavour looked to fit in with the anti-platonistic climate of the moment.

The assumption that mathematics is just another form of logic dates back to Leibniz. However, it was only in the nineteenth century that the basic principles of core mathematical theories were stated (by Dedekind and Peano) and the principles of logic were uncovered that a serious attempt to carry out the logicist programme in detail could be made (by Frege).

Frege spent most of his life attempting to demonstrate that mathematics may be reduced to logic [3]. From the basic laws of a system of second-order logic, he was able to extract the principles of (second-order) Peano arithmetic. His derivation was error-free. He did, however, rely on one concept that turned out not to be logical after all. Even worse, it is untenable. The principle in question is Frege’s Basic Law V: $\{x|Fx\} = \{x|Gx\}$ if and only if $\forall x(Fx\equiv Gx)$, In words: the set of the Fs is identical with the set of the Gs if the Fs are precisely the Gs.

In a famous letter to Frege, Russell showed that Frege’s Basic Law V entails a contradiction[4]. This argument has come to be known as Russell’s paradox.

Russell then attempted to reduce mathematics to logic in a different manner. According to Frege's Basic Law V, every property of mathematical entities corresponds to a class of mathematical beings with that property. This was obviously too powerful, because Russell's dilemma was caused by this very outcome. As a result, Russell proposed that classes are determined solely by qualities of mathematical objects that have already been demonstrated to exist. Predicates that implicitly refer to the class that they were supposed to determine if one existed don't do so. As a result, a typed structure of attributes emerges: ground object properties, ground object properties and classes of ground objects, and so on. This typed structure of properties determines a layered universe of mathematical objects, starting from ground objects, proceeding to classes of ground objects, then to classes of ground objects and classes of ground objects, and so on.

Unfortunately, Russell found that the principles of his typed logic did not suffice for deducing even the basic laws of arithmetic. He needed, among other things, to lay down as a basic principle that there exists an infinite collection of ground objects. This could hardly be regarded as a logical principle. Thus the second attempt to reduce mathematics to logic also faltered.

And things stayed that way for more than fifty years. Crispin Wright's book on Frege's theory of natural numbers was published in 1983 [5]. Wright gives the logicist project a new lease on life in it. Frege's derivation of second-order Peano Arithmetic, he remarks, can be split down into two steps.

The number of the Fs = the number of the Gs if and only if F≈G where FG denotes a one-to-one correspondence between the Fs and the Gs. (Second-order logic can be used to express this one-to-one correlation.) The principles of second-order Peano Arithmetic are then derived from Hume's Principle and acknowledged second-order logic concepts in a second stage. The second component of the derivation, in particular, does not require Basic Law V. Furthermore, Wright hypothesized that, unlike Frege's Basic Law V, Hume's Principle is consistent. Hume's Principle is consistent, according to George Boolos and others [6]. Wright went on to say that Hume's Principle can be considered a logical fact. If that is so, then at least second-order Peano arithmetic is reducible to logic alone. Thus a new form of logicism was born; today this view is known as neo-logicism [7].

Most philosophers of mathematics today doubt that Hume's Principle is a principle of logic. Indeed, even Wright has in recent years sought to qualify this claim: he now argues that Hume's Principle is analytic of our concept of number, and therefore at least a law of reason.

Wright's work has drawn the attention of philosophers of mathematics to the kind of principles of which Basic Law V and Hume's Principle are examples. These principles are called abstraction principles. At present, philosophers of mathematics attempt to construct general theories of abstraction principles that explain which abstraction principles are acceptable and which are not, and why [8];[9]. Also, it has emerged that in the context of weakened versions of second-order logic, Frege's Basic Law V is consistent. But these weak background theories only allow very weak arithmetical theories to be derived from Basic Law V [10].

### 2.2 Intutionism:

Intuitionism is based on the mathematician L.E.J. Brouwer's work [11], and it is motivated by Kantian notions of what objects are [12]. Mathematics, according to intuitionism, is primarily a construction activity. Proofs and theorems are mental constructions, as are natural numbers, real numbers, and proofs and theorems mathematical meaning is a mental construction. Mathematical constructions are produced by the ideal mathematician, i.e., abstraction is made from contingent, physical limitations of the real life mathematician. But even the ideal mathematician remains a finite being. She can never complete an infinite construction, even though she can complete arbitrarily large finite initial parts of it. This entails that intuitionism resolutely rejects the existence of the actual (or completed) infinite; only potentially infinite collections are given in the activity of construction. A basic example is the successive construction in time of the individual natural numbers.

Intuitionists deduce a revisionist attitude in logic and mathematics from these broad questions about the nature of mathematics based on the state of the human mind[13]. Non-constructive existence proofs are unacceptable to them. Intuitionism rejects non-constructive existence proofs as 'theological' and 'metaphysical'. The characteristic feature of non-constructive existence proofs is that they make essential use of the principle of excluded third

$\varphi \vee \neg\varphi$,

or one of its equivalents, such as the principle of double negation

$\neg\neg\varphi \rightarrow \varphi$.

These ideas are valid in classical logic. The logic of intuitionistic mathematics is created by removing from classical logic the notion of excluded third (and its counterparts). Of course, this necessitates a review of mathematical understanding. For example, Peano Arithmetic, a classical theory of elementary arithmetic, can no longer be accepted. Instead, an intuitionistic arithmetic theory (dubbed Heyting Arithmetic) is offered, which does not include the excluded third premise. Although intuitionistic elementary arithmetic is weaker than classical elementary arithmetic, the difference is not all that great. There exists a simple syntactical translation which translates all classical theorems of arithmetic into theorems which are intuitionistically provable.

Parts of the mathematical community were sympathetic to the intuitionistic critique of classical mathematics and the alternative it presented in the early twentieth century. This changed when it became evident that the intuitionistic alternative departs significantly from the classical theory in higher mathematics. Intuitionistic mathematical analysis, for example, is a complex theory that differs significantly from traditional mathematical analysis. The mathematics community's excitement for the intuitionistic effort was dimmed as a result. However, Brouwer's disciples have continued to develop intuitionistic mathematics to this day[14].

### 2.3 Formalism:

David Hilbert agreed with the intuitionists that there is a sense in which the natural numbers are basic in mathematics. But unlike the intuitionists, Hilbert did not take the natural numbers to be mental constructions. Instead, he argued that the natural numbers can be taken to be symbols. Symbols are strictly speaking abstract objects Nonetheless, it is essential to symbols that they can be embodied by concrete objects, so we may call them quasi-concrete objects [12]. Perhaps physical entities could play the role of the natural numbers. For example, we could assign the number 0 to a concrete ink trace of the type |, the number 1 to a concretely realised ink trace ||, and so on. Higher mathematics, Hilbert believed, could not be directly described in a same plain and perhaps even concrete manner.

Hilbert, unlike the intuitionists, was not willing to take a revisionist approach to the existing body of mathematical knowledge. Instead, when it came to advanced mathematics, he took an instrumentalist attitude. Higher mathematics, he believed, was nothing more than a formal game. Higher-order mathematics assertions are uninterrupted strings of symbols. Proving such claims is like to playing a game in which symbols are modified according to predetermined rules. According to Hilbert, the goal of the "game of higher mathematics" is to prove propositions of simple arithmetic that have a direct interpretation[15].

There can be no reasonable dispute regarding the soundness of classical Peano Arithmetic — or at least about the soundness of a subsystem of it known as Primitive Recursive Arithmetic, according to Hilbert[16]. And he believed that any arithmetical proposition that could be proven indirectly through higher mathematics could be demonstrated directly in Peano Arithmetic. And he thought that every arithmetical statement that can be proved by making a detour through higher mathematics, can also be proved directly in Peano Arithmetic. In fact, he strongly suspected that every problem of elementary arithmetic can be decided from the axioms of Peano Arithmetic. Of course solving arithmetical problems in arithmetic is in some cases practically impossible. The history of mathematics has shown that making a "detour" through higher mathematics can sometimes lead to a proof of an arithmetical statement that is much shorter and that provides more insight than any purely arithmetical proof of the same statement.

Hilbert realized, albeit somewhat dimly, that some of his convictions can actually be considered to be mathematical conjectures. For a proof in a formal system of higher mathematics or of elementary arithmetic is a finite combinatorial object which can, modulo coding, be considered to be a natural number. But in the 1920s the details of coding proofs as natural numbers were not yet completely understood.

A minimum condition of formal systems of higher mathematics, according to formalists, is that they are at least consistent. Otherwise, they can be used to establish any elementary arithmetic proposition. Hilbert also realised (though dimly) that the coherence of a higher mathematics system implies that it is at least somewhat arithmetically sound. As a result, Hilbert and his pupils set out to verify statements like the consistency of mathematical analysis' common postulates. Of course, such assertions would have to be proven in a "safe" area of mathematics, such as elementary arithmetic. Otherwise, the demonstration does not strengthen our belief in mathematical analysis' consistency. And, fortunately, it seemed possible in principle to do this, for in the final analysis consistency statements are, again modulo coding, arithmetical statements. So, to be precise, Hilbert and his students set out to prove the consistency of, e.g., the axioms of mathematical analysis in classical Peano arithmetic. This project was known as Hilbert's program[17]. It turned out to be more difficult than they had expected. In fact, they did not even succeed in proving the consistency of the axioms of Peano Arithmetic in Peano Arithmetic.

Then Kurt Gödel proved that in Peano Arithmetic, there are undecidable arithmetical assertions [18] His first incompleteness theorem is known as Gödel's first incompleteness theorem. This did not bode well for Hilbert's proposal, but it did leave open the possibility that higher mathematics' consistency is not one of these indefensible claims. Unfortunately, Gödel immediately discovered that the consistency of Peano Arithmetic is independent of Peano Arithmetic, unless (God forbid!) Peano Arithmetic is inconsistent. The second incompleteness theorem by Gödel. The incompleteness theorems of Gödel are found to be generalizable to all sufficiently strong but consistent recursively axiomatizable systems. They imply that Hilbert's programme is doomed. Higher mathematics, it turns out, cannot be interpreted solely instrumentally. Higher mathematics can prove arithmetical sentences that Peano Arithmetic cannot, such as consistency statements.

All this does not spell the end of formalism. Even in the face of the incompleteness theorems, it is coherent to maintain that mathematics is the science of formal systems.

One version of this view was proposed by Curry[19]. On this view, mathematics consists of a collection of formal systems which have no interpretation or subject matter. (Curry here makes an exception for metamathematics.) Relative to a formal system, one can say that a statement is true if and only if it is derivable in the system. But on a fundamental level, all mathematical systems are on a par. There can be at most pragmatical reasons for preferring one system over another. Inconsistent systems can prove all statements and therefore are pretty useless. So when a system is found to be inconsistent, it must be modified. It is simply a lesson from Gödel's incompleteness theorems that a sufficiently strong consistent system cannot prove its own consistency.

There is a canonical objection against Curry's formalist position. Mathematicians do not in fact treat all apparently consistent formal systems as being on a par. Most of them are unwilling to admit that the preference of arithmetical systems in which the arithmetical sentence expressing the consistency of Peano Arithmetic are derivable over those in which its negation is derivable, for instance, can ultimately be explained in purely pragmatical terms. Many mathematicians want to maintain that the perceived correctness (incorrectness) of certain formal systems must ultimately be explained by the fact that they correctly (incorrectly) describe certain subject matters.

Detlefsen has underlined that the incompleteness theorems do not impede the arithmetical establishment of the consistency of sections of higher mathematics that are in practise employed for solving arithmetical problems of importance to mathematicians[20]. In this sense, something can perhaps be rescued from the flames even if Hilbert's instrumentalist stance towards all of higher mathematics is ultimately untenable.

Isaacson made another attempt to save a piece of Hilbert's programme [21]. He argues that Peano Arithmetic may be comprehensive in certain ways after all[21] True sentences undecidable in Peano Arithmetic, he claims, can only be proved using higher-order notions. For instance, the consistency of Peano Arithmetic can be proved by induction up to a transfinite ordinal number[22]. But the notion of an ordinal number is a set-theoretic, and hence non-arithmetical, concept. If the only ways of proving the consistency of arithmetic make essential use of notions which arguably belong to higher-order mathematics, then the consistency of arithmetic, even though it can be expressed in the language of Peano Arithmetic, is a non-arithmetical problem. And generalizing from this, one can wonder whether Hilbert's conjecture that every problem of arithmetic can be decided from the axioms of Peano Arithmetic might not still be true.

### 2.4 Predicativism:

As mentioned earlier, predicativism is not commonly referred to as one of the schools. But it is only for circumstantial reasons that predicativism did not grow to the level of prominence of the other schools prior to the outbreak of World War II.

The origin of predicativism lies in the work of Russell. On a cue of Poincaré, he arrived at the following diagnosis of the Russell paradox. The argument of the Russell paradox defines the collection C of all mathematical entities that satisfy $\neg x \in x$. The argument then proceeds by asking whether C itself meets this condition, and derives a contradiction.

This definition does not pick out a collection at all, according to the Poincaré-Russell diagnosis: it is impossible to define a collection S by a condition that indirectly refers to S itself. The vicious circle principle is what it's called. Impredicative definitions are those that violate the vicious circle principle. A proper definition of a collection is one that only includes entities that exist independently from the defined collection. Predictive definitions are what they're termed. A platonist, as Gödel later pointed out, would find this line of reasoning unpersuasive. If mathematical collections exist irrespective of the act of definition, it is unclear why collections that can only be defined impredicatively cannot exist[23].

All of this prompted Russell to devise the basic and ramified theory of types, which included syntactical constraints that rendered impredicative definitions ill-formed. The free variables in defining formulas in simple type theory range over entities to which the collection to be defined does not belong. Furthermore, in ramified type theory, the bound variables' range in defining formulas must not include the collection to be defined. Russell's type theory, as stated in section 2.1, cannot be viewed as a reduction of mathematics to logic. But even aside from that, it was observed early on that especially in ramified type theory it is too cumbersome to formalize ordinary mathematical arguments.

When Russell turned to other areas of analytical philosophy, Hermann Weyl took up the predicativist cause [24]. Like Poincaré, Weyl did not share Russell's desire to reduce mathematics to logic. And right from the start he saw that it would be in practice impossible to work in a ramified type theory. Weyl developed a philosophical stance that is in a sense intermediate between intuitionism and platonism. He took the collection of natural numbers as unproblematically given. But the concept of an arbitrary subset of the natural numbers was not taken to be immediately given in mathematical intuition. Only those subsets which are determined by arithmetical (i.e., first-order) predicates are taken to be predicatively acceptable.

On the one hand, many of the traditional definitions in mathematical analysis have proven to be impredicative. The intersection of all sets that are closed under applications of the operation, for example, is usually described as the minimal closure of an operation on a set. However, the minimal closure is one of the sets that is closed when the operation is used. Thus, the definition is impredicative. In this way, attention gradually shifted away from concern about the set-theoretical paradoxes to the role of impredicativity in mainstream mathematics. On the other hand, Weyl showed that it is often possible to bypass impredicative notions. It even emerged that most of mainstream nineteenth century mathematical analysis can be vindicated on a predicative basis[25].

History intervened in the 1920s. Brouwer's more radical intuitionistic project drew Weyl over. Meanwhile, mathematicians began to believe that Russell's paradox posed less of a danger to Cantor and Zermelo's extremely impredicative transfinite set theory than previously thought. Predictivism lapsed into a latent condition for several decades as a result of these circumstances.

Solomon Feferman extended the predicativist project in the 1960s, building on work in expanded recursion theory [26]. Weyl's technique may be iterated into the transfinite, he discovered. Also counted as predicatively acceptable are those sets of numbers that can be specified using quantification over the sets that Weyl viewed as predicatively justified, and so on. This ordinal path stretches as far into the transfinite as the predicative ordinals reach, where an ordinal is predicative if it measures the length of a provable well-ordering of the natural numbers. This calibration of the strength of predicative mathematics, which is due to Feferman and (independently) Schütte, is nowadays fairly generally accepted. Feferman then investigated how much of standard mathematical analysis can be carried out within a predicativist framework. The research of Feferman and others (most notably Harvey Friedman) shows that most of twentieth century analysis is acceptable from a predicativist point of view. But it is also clear that not all of contemporary mathematics that is generally accepted by the mathematical community is acceptable from a predicativist standpoint: transfinite set theory is a case in point.

## 3. Platonism:

In the years before the second world war it became clear that weighty objections had been raised against each of the three anti-platonist programs in the philosophy of mathematics. Predicativism was perhaps an exception, but it was at the time a program without defenders. Thus room was created for a renewed interest in the prospects of platonistic views about the nature of mathematics. On the platonistic conception, the subject matter of mathematics consists of abstract entities.

### 3.1 Gödel’s Platonism:

Gödel was a platonist with respect to mathematical objects and with respect to mathematical concepts [27]. But his platonistic view was more sophisticated than that of the mathematician in the street.

On the one hand, Gödel believed that viable theories of mathematical objects and concepts and feasible theories of physical things and qualities have a significant parallelism. Mathematical objects and concepts, like physical objects and qualities, are not created by humans. Mathematical objects and notions, like physical things and qualities, cannot be reduced to mental beings. Physical things and attributes are as objective as mathematical objects and notions. Mathematical objects and concepts, like physical objects and qualities, are hypothesized in order to arrive at a good and adequate theory of our experience. Our perception of physical objects and concepts is fallible and can be corrected. In the same way, mathematical intuition is not fool-proof — as the history of Frege's Basic Law V shows— but it can be trained and improved. Unlike physical objects and properties, mathematical objects do not exist in space and time, and mathematical concepts are not instantiated in space or time

### 3.2 Naturalism and Indispensability:

Quine formulated a methodological critique of traditional philosophy. He suggested a different philosophical methodology instead, which has become known as naturalism [28]. According to naturalism, our best theories are our best scientific theories. If we want to obtain the best available answer to philosophical questions such as What do we know? and Which kinds of entities exist? we should not appeal to traditional epistemological and metaphysical theories. We should also refrain from embarking on a fundamental epistemological or metaphysical inquiry starting from first principles. Rather, we should consult and analyze our best scientific theories. They contain, albeit often implicitly, our currently best account of what exists, what we know, and how we know it that Putnam applied Quine's naturalistic stance to mathematical ontology [29]. At least since Galilei, our best theories from the natural sciences are mathematically expressed. Newton's theory of gravitation, for instance, relies heavily on the classical theory of the real numbers. Thus an ontological commitment to mathematical entities seems inherent to our best scientific theories. This line of reasoning can be strengthened by appealing to the Quinean thesis of confirmational holism. Empirical evidence does not bestow its confirmatory power on any one individual hypothesis. Rather, experience globally confirms the theory in which the individual hypothesis is embedded. Since mathematical theories are part and parcel of scientific theories, they too are confirmed by experience. Thus, we have empirical confirmation for mathematical theories. Even more appears true. It seems that mathematics is indispensable to our best scientific theories: it is not at all obvious how we could express them without using mathematical vocabulary. Hence the naturalist stance commands us to accept mathematical entities as part of our philosophical ontology. This line of argumentation is called an indispensability argument[30].

### 3.3 Deflating Platonism:

When a mathematician is at work, Bernays observes that she "naively" considers the subjects she is dealing with in a platonistic manner. He claims that every working mathematician is a platonist [31].However, if a philosopher questions the mathematician about her ontological commitments while she is off duty, she is likely to shuffle her feet and retreat to a vaguely non-platonistic position.

Carnap introduced a distinction between questions that are internal to a framework and questions that are external to a framework [32]. Tait has worked out in detail how something like this distinction can be applied to mathematics [33]. This has resulted in what might be regarded as a deflationary version of platonism.

According to Tait, questions of existence of mathematical entities can only be sensibly asked and reasonably answered from within (axiomatic) mathematical frameworks. If one is working in number theory, for instance, then one can ask whether there are prime numbers that have a given property. Such questions are then to be decided on purely mathematical grounds.

Philosophers have a tendency to step outside the framework of mathematics and ask “from the outside” whether mathematical objects *really* exist and whether mathematical propositions are *really* true. In this question they are asking for supra-mathematical or metaphysical grounds for mathematical truth and existence claims. Tait argues that it is hard to see how any sense can be made of such external questions. He attempts to deflate them, and bring them back to where they belong: to mathematical practice itself. Of course not everyone agrees with Tait on this point. Linsky and Zalta have developed a systematic way of answering precisely the sort of external questions that Tait approaches with disdain [34].

### 3.4 Benacerraf’s Epistemological Problem:

In the philosophy of science, Benacerraf established an epistemological issue for a range of platonistic perspectives [35]. The argument is specifically addressed against Gödel's concept of mathematical intuition. The premise of Benacerraf's argument is that our best theory of knowing is the causal theory of knowing. It is then pointed out that, according to platonism, abstract objects are neither physically nor temporally localized, although flesh and blood mathematicians are. Knowledge of mathematical entities should, according to our best epistemological theory, emerge through causal contact with these entities. However, it's impossible to believe that this is the case.

Hodes has formulated a semantical variant of Benacerraf’s epistemological problem [36]. According to our currently best semantic theory, causal-historical connections between humans and the world of concreta enable our words to refer to physical entities and properties. According to platonism, mathematics refers to abstract entities. The platonist therefore owes us a plausible account of how we (physically embodied humans) are able to refer to them. On the face of it, it appears that the causal theory of reference will be unable to supply us with the required account of the ‘microstructure of reference’ of mathematical discourse.

### 3.5 Plenitudinous Platonism:

A version of platonism has been developed which is intended to provide a solution to Benacerraf’s epistemological problem [34];[37]. This position is known as plenitudinous platonism. The central thesis of this theory is that every logically consistent mathematical theory necessarily refers to an abstract entity. Whether the mathematician who formulated the theory knows that it refers or does not know this, is largely immaterial. By entertaining a consistent mathematical theory, a mathematician automatically acquires knowledge about the subject matter of the theory. So, on this view, there is no epistemological problem to solve anymore.

In Balaguer's version, plenitudinous platonism postulates a multiplicity of mathematical universes, each corresponding to a consistent mathematical theory. Thus, in particular a question such as the continuum problem (cf. section 4.1) does not receive a unique answer: in some set-theoretical universes the continuum hypothesis holds, in others it fails to hold. However, not everyone agrees that this picture can be maintained. Martin has developed an argument to show that multiple universes can always to a large extent be "accumulated" into a single universe [38].

In Linsky and Zalta's version of plenitudinous platonism, the mathematical entity that is postulated by a consistent mathematical theory has exactly the mathematical properties which are attributed to it by the theory. The abstract entity corresponding to ZFC, for instance, is partial in the sense that it neither makes the continuum hypothesis true nor false. The reason is that ZFC neither entails the continuum hypothesis nor its negation. This does not entail that all ways of consistently extending ZFC are on a par. Some ways may be fruitful and powerful, others less so. But the view does deny that certain consistent ways of extending ZFC are preferable because they consist of true principles, whereas others contain false principles.

## 4. Important Topics:

Sub disciplines of mathematics philosophy have begun to emerge in recent years. They develop in a way that isn't entirely governed by the "grand arguments" concerning mathematics' essence. We'll look at a few of these disciplines in this section.

### 4.1 Foundations and Set theory:

Many people consider set theory to be the cornerstone of mathematics. Even if it is sometimes a difficult environment for doing so, it appears that almost any piece of mathematics may be carried out in set theory. In recent years, set theory philosophy has emerged as a distinct philosophical discipline. This is not to say that in specific debates in the philosophy of set theory it cannot make an enormous difference whether one approaches it from a formalistic point of view or from a platonistic point of view, for instance.

The claim that set theory is best suited to serve as the foundations of mathematics is far from settled. In recent decades, category theory has emerged as a contender for this position. The mathematical theory of category theory was established in the middle of the twentieth century. Unlike set theory, mathematical objects in category theory are only specified up to isomorphism. As a result, Benacerraf's identification problem does not apply to category theoretical notions and "objects." Simultaneously, (nearly) everything that can be done in set theory can also be done in category theory (albeit not necessarily naturally), and vice versa (again not always in a natural manner). As a result, from a structuralist standpoint, category theory is an appealing candidate for providing the foundations of mathematics[39].

The distinction between sets and appropriate classes has been a topic of discussion in set theory since its inception. (There is a natural analogue to this dilemma in category theory: the distinction between small and large categories.) Cantor's diagonal argument forces us to acknowledge that the entire set-theoretical cosmos cannot be considered a set. The power set (i.e., the set of all subsets) of each given set has a bigger cardinality than the given set itself, according to Cantor's Theorem. Assume that the set-theoretical world is made up of a single set: the set of all sets. The set of all sets' power set would then have to be a subset of the set of all sets. This would contradict the fact that the power set of the set of all sets would have a larger cardinality than the set of all sets. So we must conclude that the set-theoretical universe cannot form a set.

Multiplicities that are too vast to be considered a set are referred to as inconsistent multiplicities by Cantor [40]. Cantor's inconsistent multiplicities are now referred to as appropriate classes. Proper classes, according to certain math philosophers, nonetheless constitute unities and hence can be considered a collection. In a Cantorian sense, they're just collections that are too big to be sets. However, there are flaws with this viewpoint. There can't be a valid class of all proper classes for diagonalization reasons, just as there can't be a set of all sets. As a result, the proper class approach appears driven to include a realm of super-proper classes, and so on. As a result, Zermelo asserted that suitable classes do not exist. This situation isn't as unusual as it appears. On close inspection, one sees that in ZFC one never needs to quantify over entities that are too large to be sets (although there exist systems of set theory that do quantify over proper classes). On this view, the set-theoretical universe is potentially infinite in an absolute sense of the word. It never exists as a completed whole, but is forever growing, and hence forever unfinished [27]. This way of speaking indicates that in our attempts to understand this notion of potential infinity, we are drawn to temporal metaphors. It is not surprising that these temporal metaphors cause some philosophers of mathematics acute discomfort.

The justification of recognized basic principles of mathematics, i.e. the ZFC axioms, is a second topic in set theory philosophy. The process through which the Axiom of Choice was adopted by the mathematical community in the early decades of the twentieth century is an interesting historical case study [41]. The value of this case study stems partly from the fact that the mathematical community held an open and explicit discussion about its acceptability. General arguments for adopting or refusing to recognize a proposition as a basic axiom surfaced during this conversation. On the systematic side, two notions of set have been developed with the goal of justifying all ZFC axioms. On the one hand, there's the iterative idea of sets, which explains how the set-theoretical world is formed from an empty set using the power set operation [42]:[43] On the other side, there is the size notion of sets limitation, which holds that any collection that is not too large to be a set is a set [44]. Some ZFC axioms (for example, the power set axiom) are well motivated by the iterative idea, whereas others, such as the replacement axiom, are less so [45]The limitation of size conception motivates other axioms better (such as the restricted comprehension axiom). It seems fair to say that there is no uniform conception that clearly justifies all axioms of ZFC.

A third focus of set theory philosophy is the motivation of putative axioms that go beyond ZFC [46, 47];[48] The large cardinal axioms are an example of this type of principle. Large cardinal hypotheses are now commonly understood to refer to some form of embedding features between the set theoretic cosmos and set theory's inner models.

The weaker of the large cardinal principles are supported by intrinsic evidence (see section 3.1). They follow from what are called reflection principles. These are principles that state that the set theoretic universe as a whole is so rich that it is very similar to some set-sized initial segment of it. The stronger of the large cardinal principles hitherto only enjoy extrinsic support. Many researchers are skeptical about the possibility that reflection principles, for instance, can be found that support them [49]others, however, disagree [50].

### 4.2 Categoricity:

In the second half of the nineteenth century Dedekind proved that the basic axioms of arithmetic have, up to isomorphism, exactly one model, and that the same holds for the basic axioms of Real Analysis. If a theory has, up to isomorphism, exactly one model, then it is said to be categorical. So modulo isomorphisms, arithmetic and analysis each have exactly one intended model. Half a century later Zermelo proved that the principles of set theory are "almost" categorical or quasi-categorical: for any two models M1 and M2 of the principles of set theory, either M1 is isomorphic to M2, or M1 is isomorphic to a strongly inaccessible rank of M2, or M2 is isomorphic to a strongly inaccessible rank of M1[27]. In recent years, attempts have been made to develop arguments to the effect that Zermelo's conclusion can be strengthened to a full categoricity assertion [51];[38], but we will not discuss these arguments here.

Simultaneously, the Löwenheim-Skolem theorem states that every first-order formal theory with at least one infinite domain must also have models with domains of all infinite cardinalities. The Löwenheim-Skolem theorem appears to apply to arithmetic, analysis, and set theory because they must all have at least one infinite model. Isn't this in conflict with Dedekind's categoricity theorem?

The solution of this conundrum lies in the fact that Dedekind did not even implicitly work with first-order formalizations of the basic principles of arithmetic and analysis. Instead, he informally worked with second-order formalizations.

Let us focus on arithmetic to see what this amounts to. The basic postulates of arithmetic contain the induction axiom. In first-order formalizations of arithmetic, this is formulated as a scheme: for each first-order arithmetical formula of the language of arithmetic with one free variable, one instance of the induction principle is included in the formalization of arithmetic. Elementary cardinality considerations reveal that there are infinitely many properties of natural numbers that are not expressed by a first-order formula. But intuitively, it seems that the induction principle holds for all properties of natural numbers. So in a first-order language, the full force of the principle of mathematical induction cannot be expressed. For this reason, a number of philosophers of mathematics insist that the postulates of arithmetic should be formulated in a second-order language [52]. Second-order languages contain not just first-order quantifiers that range over elements of the domain, but also second-order quantifiers that range over properties (or subsets) of the domain. In full second-order logic, it is insisted that these second-order quantifiers range over all subsets of the domain. If the principles of arithmetic are formulated in a second-order language, then Dedekind's argument goes through and we have a categorical theory. For similar reasons, we also obtain a categorical theory if we formulate the basic principles of real analysis in a second-order language, and the second-order formulation of set theory turns out to be quasi-categorical.

A second objection against second-order logic can be traced back to Quine [53]This objection states that the interpretation of full second-order logic is connected with set-theoretical questions. This is already indicated by the fact that most regimentations of second-order logic adopt a version of the axiom of choice as one of its axioms. But more worrisome is the fact that second-order logic is inextricably intertwined with deep problems in set theory, such as the continuum hypothesis. For theories such as arithmetic that intend to describe an infinite collection of objects, even a matter as elementary as the question of the cardinality of the range of the second-order quantifiers, is equivalent to the continuum problem. Also, it turns out that there exists a sentence which is a second-order logical truth if and only if the continuum hypothesis holds [54]. We have seen that the continuum problem is independent of the currently accepted principles of set theory. And many researchers believe it to be absolutely truth-valueless. If this is so, then there is an inherent indeterminacy in the very notion of second-order infinite model. And many contemporary philosophers of mathematics take the latter not to have a determinate truth value. Thus, it is argued, the very notion of an (infinite) model of full second-order logic is inherently indeterminate.

If one admits that the collection of arithmetical predicates is open-ended, a categoricity theorem for arithmetic can be obtained without going beyond the constraints of first-order logic or relying on an ad hoc notion of computability. Assume A and B are two mathematicians who both state the first-order Peano-axioms in their own dialects. Assume that A and B regard the set of predicates for which mathematical induction is permitted as open-ended, and that they are both prepared to accept the truth of the other's induction scheme. Then A and B have the resources to persuade themselves that both idiolects depict the same structures [55].

### 4.3 Computation and Proof:

Until recently, the philosophy of mathematics did not pay much attention to the subject of computing. This could be owing to the fact that computation is reduced to proof in Peano Arithmetic in Hilbert-style axiomatizations of number theory. However, in recent years, the situation has altered. It appears that, as computation becomes increasingly important in mathematical practise, philosophical reflections on the concept of computing will become increasingly significant in the philosophy of mathematics in the next years.

Church’s Thesis occupies a central place in computability theory. It says that every algorithmically computable function on the natural numbers can be computed by a T Church's Thesis has a peculiar status as a principle. It appears to be a fundamental concept. On the one hand, practically everyone agrees that the principle is correct. However, it is difficult to understand how it can be mathematically proven. The reason for this is that its antecedent comprises a non-mathematical concept (algorithmic computability), but the latter has only a mathematical concept truing machine. Mathematical proofs can only connect purely mathematical notions—or so it seems. The received view was that our evidence for Church’s Thesis is quasi-empirical. Attempts to find convincing counterexamples to Church’s Thesis have come to naught. Independently, various proposals have been made to mathematically capture the algorithmically computable functions on the natural numbers. Instead of Turing machine computability, the notions of general reclusiveness, Herbrand-Gödel computability, lambda-definability… have been proposed. But these mathematical notions all turn out to be equivalent. Thus, to use Gödelian terminology, we have accumulated extrinsic evidence for the truth of Church’s Thesis.

Even though a thesis cannot be formally proved, it may still be feasible to get intrinsic evidence for it from a thorough but informal study of intuitive conceptions, as Kreisel pointed out many years ago [56]. These are what Kreisel refers to as "informal rigour exercises." Sieg's meticulous research found that Turing's foundational essay [57] is an ideal example of this type of analysis of the intuitive concept of algorithmic computability [58].

The following appear to be the most active research topics in the domain of foundations and philosophy of computation at the moment. For starters, time and effort has gone into creating theories of algorithmic computation on structures other than natural numbers. Analogues of Church's Thesis for algorithmic computation on diverse architectures have been sought in particular. In this regard, significant work has been made in recent decades in creating a theory of effective real-number computation [59]. Second, humans have attempted to explain computability concepts other than algorithmic computability. Quantum computation is one topic of great interest in this context [60].

We have a good understanding of the ideas of formal proof and formal provability, as well as their relationship to algorithmic computability and the principles that govern them. For example, we know that a formal system's proofs are computably enumerable and that provability in a sound (enough) formal system is susceptible to Gödel's incompleteness theorems. A mathematical proof in a mathematical journal, on the other hand, is not a formal proof in the sense of logicians: it is a (rigorous) informal proof [61, 62]. Informal proof and informal provability, as well as the laws that govern them, are far less well understood than formal proof and provability. In particular, despite the fact that the question has been debated since the early 1960s [63], it is still completely unclear whether the extension of informal mathematical provability coincides, for some formal theory T, with the extension of provability in T, or whether the concept of informal mathematical provability is even clear enough for this question to have a definite answer [64].

In the last few decades, we've seen the first instances of mathematical proofs in which computers appear to be crucial. One example is the four-color theorem. It claims that only four colours are required for each map to colour countries in such a way that no two countries sharing a boundary receive the same hue. In 1976, this theorem was proven [65] The proof, on the other hand, distinguishes several situations that were confirmed by a computer. Humans cannot double-check these computer verifications because they are too long. The proof of the four-color theorem sparked a dispute about whether or not computer-assisted proofs are true proofs.

According to popular belief, mathematical arguments produce a priori knowledge. When we rely on a computer to generate part of a proof, however, we appear to be relying on the accuracy of a computer programme as well as the proper operation of computer hardware. These appear to be observable variables. As a result, one would be tempted to believe that computer proofs produce quasi-empirical knowledge[66]. In other words, through the advent of computer proofs the notion of proof has lost its purely a priori character. Others hold that the empirical factors on which we rely when we accept computer proofs do not appear as premises in the argument. Hence, computer proofs can yield a priori knowledge after all [67].

## 6. Conclusion:

The nature of mathematical objects, the fundamental rules that govern them, and how we acquire mathematical knowledge about them were the focus of research in the philosophy of mathematics in the twentieth century. These are basic concerns that are inextricably linked to classical metaphysical and epistemological issues.

In the second part of the twentieth century, study in philosophy of science shifted away from foundational issues to a large extent. Rather, philosophical considerations about the evolution of scientific knowledge and understanding became more prominent. As early as the 1970s, there were voices that argued that a similar shift of attention should take place in the philosophy of mathematics [68].

For some decades, such sentiments remained restricted to a somewhat marginal school of thought in the philosophy of mathematics. However, in recent years the opposition between this new movement and mainstream philosophy of mathematics is softening. Philosophical questions relating to mathematical practice, the evolution of mathematical theories, and mathematical explanation and understanding have become more prominent, and have been related to more traditional questions from the philosophy of mathematics [69] This trend will doubtlessly continue in the years to come.

For an example, let us briefly return to the subject of computer proofs (see section 4.3). The source of the discomfort that mathematicians experience when confronted with computer proofs appears to be the following. A “good” mathematical proof should do more than to convince us that a certain statement is true. It should also explain why the statement in question holds. And this is done by referring to deep relations between deep mathematical concepts that often link different mathematical domains [70] Until now, computer proofs typically only employ fairly low level mathematical concepts. They are notoriously weak at developing deep concepts on their own, and have difficulties with linking concepts in from different mathematical fields. All this leads us to a philosophical question which is just now beginning to receive the attention that it deserves: what is mathematical understanding?

**References:**


[1] S. Shapiro, *Thinking about mathematics: The philosophy of mathematics*: OUP Oxford, 2000.

[2] Ø. Linnebo, *Philosophy of mathematics*: Princeton University Press, 2017.

[3] G. Frege, "The Foundations of Arithmetic a Logico-Mathematical Enquiry Into the Concept of Number. English Translation by JL Austin," 1953.

[4] B. Russell, "Letter to Frege, in van Heijenoort (1967)," ed: Harvard University Press, 1902.

[5] J. LUCAS, "WRIGHT, CRISPIN [1983]: Frege's Conception of Numbers as Objects. Scots Philosophical Monographs, no. 2, Aberdeen University Press. xxi+ 193 pp. Hardback£ 12.50. Paperback£ 8.50. 1 The principal claims of Frege's Grundlagen were that the natural numbers are objects, and that their-existence can be demonstrated purely."

[6] E. N. Zalta, "Frege's theorem and foundations for arithmetic," 1998.

[7] B. Hale and C. Wright, *The reason's proper study: Essays towards a neo-Fregean philosophy of mathematics*: Oxford University Press, 2001.

[8] A. Weir, "Neo-Fregeanism: An embarrassment of riches," *Notre Dame Journal of Formal Logic,* vol. 44, pp. 13-48, 2003.

[9] K. Fine, *The limits of abstraction*: Clarendon Press, 2002.

[10] J. P. Burgess, "Being explained away," *The Harvard Review of Philosophy,* vol. 13, pp. 41-56, 2005.

[11] M. Van Atten, "On Brouwer," 2004.

[12] C. Parsons, *Mathematical thought and its objects*: Cambridge University Press, 2007.

[13] A. W. Moore, *The infinite*: Routledge, 2018.

[14] A. S. Troelstra and D. Van Dalen, "Constructivism in mathematics: an introduction. II," *Studies in Logic and the Foundations of Mathematics,* vol. 123, pp. 345-879, 1988.

[15] S. J. Baek and Y. Choi, "Is the Limit Process Commutative with the Division Operation?."

[16] W. W. Tait, *Gödel on Intuition and on Hilbert’s finitism*: Cambridge Univ. Press, 2010.

[17] R. Zach, "Hilbert’s Program Then and Now. Handbook of the Philosophy of Science, Vol. 5: Philosophy of Logic, ed. Dale Jacquette," ed: Elsevier, Amsterdam, 2006.

[18] K. Gödel, *On formally undecidable propositions of Principia Mathematica and related systems*: Courier Corporation, 1992.

[19] H. B. Curry, *Outlines of a formalist philosophy of mathematics* vol. 3: Elsevier, 1951.

[20] M. Detlefsen, "Hilbert's Program: An Essay on Mathematical Instrumentalism (D. Reidel)," ed: Kluwer Academic, Dordrecht, 1986.

[21] D. Isaacson, "Arithmetical truth and hidden higher-order concepts," in *Studies in Logic and the Foundations of Mathematics*. vol. 122, ed: Elsevier, 1987, pp. 147-169.

[22] R. Péter, "Gentzen Gerhard. Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie. Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften, neue Folge, Heft 4, S. 19–44. S. Hirzel, Leipzig 1938," 1939.

[23] S. Shapiro, "Mathematics and philosophy of mathematics," *Philosophia Mathematica,* vol. 2, pp. 148-160, 1994.

[24] H. Weyl, *The Continuum: a critical examination of the foundation of analysis*: Courier Corporation, 1994.

[25] S. Feferman, *In the light of logic*: Oxford University Press on Demand, 1998.

[26] S. Shapiro and W. J. Wainwright, *The Oxford handbook of philosophy of mathematics and logic*: OUP USA, 2005.

[27] L. Horsten, "Philosophy of mathematics," 2007.

[28] W. V. Quine, *3. Epistemology Naturalized*: Columbia University Press, 1969.

[29] A. Anduaga, M. Antliff, and P. Leighten, "Adams, Mary Grace Agnes. Science in the Changing World. London: George Allen & Unwin, 1933. Adrian, ED The Basis of Sensation: The Action of the Sense Organs. London: Christophers, 1928. Akin, William E. Technocracy and the American Dream: The Technocrat Movement, 1900–1941. Berkeley: University of California Press, 1977. Albright, Daniel. Quantum Poetics: Yeats, Pound Eliot, and the Science of Modernism. Cambridge," *Journalism & Mass Communication Quarterly,* vol. 83, pp. 581-595, 2006.

[30] M. Colyvan, *The indispensability of mathematics*: Oxford university press, 2001.

[31] P. Bernays, *On platonism in mathematics* vol. 34: na, 1935.

[32] R. Carnap, *Empiricism, semantics, and ontology*: Princeton University Press, 2011.

[33] W. W. Tait and W. Tait, *The provenance of pure reason: Essays in the philosophy of mathematics and its history*: Oxford University Press, USA, 2005.

[34] B. Linsky and E. N. Zalta, "Naturalized platonism versus platonized naturalism," *The Journal of Philosophy,* vol. 92, pp. 525-555, 1995.

[35] M. McEvoy, "Is reliabilism compatible with mathematical knowledge?," in *The Philosophical Forum*, 2004, pp. 423-437.

[36] H. T. Hodes, "Logicism and the ontological commitments of arithmetic," *The Journal of Philosophy,* vol. 81, pp. 123-149, 1984.

[37] M. Balaguer, *Platonism and anti-platonism in mathematics*: Oxford University Press on Demand, 1998.

[38] D. A. Martin, "Multiple universes of sets and indeterminate truth values," *Topoi,* vol. 20, pp. 5-16, 2001.

[39] C. McLarty, "Exploring categorical structuralism," *Philosophia Mathematica,* vol. 12, pp. 37-53, 2004.

[40] G. Cantor, "Gesammelte Abhandlungen, eds. A. Fraenkel and E. Zermelo," ed: Springer, Berlin, 1932.

[41] G. H. Moore, *Zermelo's axiom of choice: Its origins, development, and influence*: Courier Corporation, 2012.

[42] G. Boolos, "The iterative conception of set," *The Journal of Philosophy,* vol. 68, pp. 215-231, 1971.

[43] Ø. Linnebo, "The Potential Hierarchy of Sets," *Review of Symbolic Logic,* vol. 6, 2013.

[44] M. Hallett, *Cantorian set theory and limitation of size*: Clarendon Press Oxford, 1984.

[45] M. Potter, *Set theory and its philosophy: A critical introduction*: Clarendon Press, 2004.

[46] P. Maddy, "Believing the axioms. I," *The Journal of Symbolic Logic,* vol. 53, pp. 481-511, 1988.

[47] P. Maddy, "Believing the axioms. II," *The Journal of Symbolic Logic,* vol. 53, pp. 736-764, 1988.

[48] D. A. Martin, "Mathematical evidence," *Truth in mathematics,* pp. 215-231, 1998.

[49] P. Koellner, "On reflection principles," *Annals of Pure and Applied Logic,* vol. 157, pp. 206-219, 2009.

[50] P. Welch and L. Horsten, "Reflecting on absolute infinity," *The Journal of Philosophy,* vol. 113, pp. 89-111, 2016.

[51] V. McGee, "How we learn mathematical language," *The Philosophical Review,* vol. 106, pp. 35-68, 1997.

[52] S. Shapiro, *Foundations without foundationalism: A case for second-order logic* vol. 17: Clarendon Press, 1991.

[53] K. R. Westphal, "Conventionalism and the impoverishment of the space of reasons: Carnap, Quine and Sellars," *Journal for the History of Analytical Philosophy,* vol. 3, 2015.

[54] S. Morris, "Quine and Boolos on second-order logic: an examination of the debate," University of St Andrews, 2005.

[55] C. Parsons, "The uniqueness of the natural numbers," *Iyyun: The Jerusalem Philosophical Quarterly/עיון: רבעון פילוסופי,* pp. 13-44, 1990.

[56] G. Kreisel, "Informal rigour and completeness proofs," in *Studies in Logic and the Foundations of Mathematics*. vol. 47, ed: Elsevier, 1967, pp. 138-186.

[57] A. M. Turing, "On computable numbers, with an application to the Entscheidungsproblem," *Proceedings of the London mathematical society,* vol. 2, pp. 230-265, 1937.

[58] W. Sieg, "Mechanical procedures and mathematical experience," *Mathematics and mind,* pp. 71-117, 1994.

[59] M. B. Pour-El, "The structure of computability in analysis and physical theory: an extension of Church's thesis," *Handbook of Computability Theory,* vol. 140, pp. 449-471, 1999.

[60] D. Deutsch, A. Ekert, and R. Lupacchini, "Machines, logic and quantum physics," *The Bulletin of Symbolic Logic,* vol. 6, pp. 265-283, 2000.

[61] J. Myhill, "Some remarks on the notion of proof," *The Journal of Philosophy,* vol. 57, pp. 461-471, 1960.

[62] L. C. Pereira, "Michael Detlefsen (ed.), Proof, Logic and Formalization. Michael Detlefsen (ed.), Proof and Knowledge in Mathematics," *Erkenntnis,* vol. 47, p. 245, 1997.

[63] J. R. Lucas, "Minds, machines and Gödel," *Philosophy,* pp. 112-127, 1961.

[64] L. Horsten and P. Welch, *Gödel's Disjunction: The scope and limits of mathematical knowledge*: Oxford University Press, 2016.

[65] K. Appel, W. Haken, and J. Koch, "Every planar map is four colorable. Part II: Reducibility," *Illinois Journal of Mathematics,* vol. 21, pp. 491-567, 1977.

[66] T. Tymoczko, "The four-color problem and its philosophical significance," *The Journal of Philosophy,* vol. 76, pp. 57-83, 1979.

[67] T. Burge, "Computer proof, apriori knowledge, and other minds: The sixth philosophical perspectives lecture," *Philosophical perspectives,* vol. 12, pp. 1-37, 1998.

[68] S. Bağçe and C. Başkent, "An examination of counterexamples in proofs and refutations," *Philosophia Scientiæ. Travaux d'histoire et de philosophie des sciences,* pp. 3-20, 2009.

[69] P. Mancosu, *The philosophy of mathematical practice*: Oxford University Press on Demand, 2008.

[70] K. Manders, "Domain extension and the philosophy of mathematics," *The Journal of Philosophy,* vol. 86, pp. 553-562, 1989.