Introduction to Mathematical Logic

Chapter 1 Genesis

By *Johannes Yuan Hong*
Or just *John*

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Logic

- Is anyone going to give us some examples or a definition of Logic?

Logic

- **If it was so, it might be, and if it were so, it would be, but since it isn't, it ain't. That's logic.**
  – Lewis Carroll, *Through the Looking Glass*
- **Since it is possible to touch the clock without stopping it, it follows that one can start a clock without touching it. That is logic, as I understand it.**
  – James Thurber, *The 13 Clocks*

Logic

- **Logic: the art of thinking and reasoning that is in strict accordance with the limitations and incapacities of human misunderstanding. The basis of logic is syllogism, consisting of a major and minor premise and a conclusion.**
  – Ambrose Bierce, *The Devil's Dictionary*

Logic

- A sample of syllogism
  - **Major Premise**
    Sixty men can do sixty times as quick as one man.
  - **Minor Premise**
    One man can dig a post-hole in 60 seconds. Therefore,
  - **Conclusion**
    Sixty men can dig a post-hole in 1 second.

Logic

- (of a syllogism) Valid vs. Sound
  - **A valid syllogism**
    a syllogism of which the conclusion is a logical consequence of the premises, regardless of whether the premises are true or not.
  - **A sound syllogism**
    a syllogism that is not only valid but also all its premises are true and therefore, so is its conclusion.

Logic

- Examples of a valid and a sound syllogism
  - **<u>of sound syllogism</u>**
    - **All men are mortal.**
    - **Socrates is a man. Therefore**
    - **Socrates is mortal.**
  - **<u>of valid syllogism</u>**
    - **All bats can fly.**
    - **Socrates is a bat. Therefore**
    - **Socrates can fly.**

Logic

- Are these syllogisms sound or just valid, or neither?
  - **Syllogism 1**
    - **All big winners were once losers.**
    - **Elon Musk is a big winner. Therefore**
    - **Elon Musk was once a loser.**
  - **Syllogism 2**
    - **All bands need a vocalist.**
    - **The Doors is a band of rock music. Therefore**
    - **The Doors needs a vocalist.**

Logic

- Are these syllogisms sound or just valid, or neither?
  - **Syllogism 3**
    - **Everyone loves my baby.**
    - **My baby loves only me. Therefore**
    - **I am my own baby.**
  - **Syllogism 4**
    - **Everybody loves a lover (or a lover is loved by everybody).**
    - **Romeo loves Juliet. Therefore**
    - **Lupin loves Hagrid.**

Logic

- What are the purposes of logic?
  - **To prove and many others**
    - **What is a proof?**
    - **Are the following two statements saying the same thing or different ones? Try proving your conclusion, please.**
      - **Good food is not cheap.**
      - **Cheap food is not good.**
    - **I'd like to hear your opinions about the following story, and their PROOFs, please.**

Logic

- **A boy was hungry and stole a bread to eat, for which he was caught. You passed by and told him: “You should have bought the bread.” “But I have no money”, he replied. “You can go working and get paid”, you said, and he nodded: “Yes, sir, you are right.”**
- **“Would you pay me back if I paid for you and get you released?” You asked. “Sure. Of course, sir”, replied he.**
- **You paid for him. He got released and paid you back. But you finally found that the money he gave you was from stealing, too.**
- **What would be the right thing you have to do then? Why?**

Mathematical Logic

- **Gottfried Wilhelm (von) Leibniz**
  - **<u>Preface to the General Science (1677):</u> It is obvious that if we could find characters or signs suited for expressing all our thoughts as clearly and as exactly as arithmetic expresses numbers or geometry expresses lines, we could do in all matters insofar as they are subject to reasoning all that we can do in arithmetic and geometry. For all investigations which depend on reasoning would be carried out by transposing these characters and by a species of calculus.**

Mathematical Logic

- **Gottfried Wilhelm (von) Leibniz**
  - **<u>The Art of Discovery (1685):</u> The only way to rectify our reasonings is to make them as tangible as those of the Mathematicians, so that we can find our error at a glance, and when there are disputes among persons, we can simply say: Let us calculate [calculemus], without further ado, to see who is right.**

Mathematical Logic

- **George Boole**
  - **Published a small book in 1847, *The Mathematical Analysis of Logic*, which was expanded seven years later into *An Investigation of the Laws of Thought***

Mathematical Logic

- **George Boole**
  - **Boole's aim in his *An Investigation of Laws of Thought* (1854) was to "investigate the fundamental laws of those operations of the mind by which the reasoning is performed; to give expression to them in the symbolical language of a calculus, and upon this foundation to establish the science of Logic and construct its method."**

Mathematical Logic

- **Bertrand Russell**
  - **He defines Mathematical Logic as the following**
    - **The subject in which nobody knows what one is saying about, or whether what one says is true.**

Set Theory Fundamentals

- **Some concepts**
  - **Set vs. element**
    - **A club and all its members**
  - **Belongs to, is element of**
    - **An owl *belongs to / is an element of* the set of all birds.**
    - **The set of all birds *contains* an owl.**
    - **o ∈ B if o stands for an owl and B stands for the set of all birds (∈, epsilon).**

Set Theory Fundamentals

- **Some concepts**
  - **Is (proper) subset of**
    - **The set of birds of prey is a *(proper) subset* of the set of all birds.**
    - **The set of all birds *includes / is a superset of* the set of birds of prey.**
    - **$B_p \subseteq B$ ($B_p \subset B$) if B_p stands for the set of birds of prey and B stands for the set of all birds ($\subseteq$, inclusion).**
  - **Equals, is the same as**
    - **The set of even numbers *equals* the set of numbers wholly divisible by 2**

Set Theory Fundamentals

- **Some concepts**
  - **Empty set**
    - **∅, or {}**
    - **Is it true that**
      - **all empty sets equal one another, or are the same set, or**
      - **empty set is subset of every set?**

Set Theory Fundamentals

- **Some concepts**
  - **Empty set**
    - **$\varnothing$, or {}**
    - **Material implication**
      - **"All Chinese students in the class wear a beret." – is this statement true or false in case that no student in this class is Chinese?**
      - **"I will marry you if I get a job next week."**
      - **For any property *P* and any element *x*, we have if $x \in \varnothing$, then *P(x)*.**

Set Theory Fundamentals

- **Some concepts**
  - **Empty set**
    - **∅, or {}**
    - **Is it true that**
      - **all empty sets equal one another, or are the same set, or**
      - **empty set is subset of every set?**

Set Theory Fundamentals

- **Boolean operations**
  - **Union, ∪**
    - **{owl, toad, cat}∪{cat, dog, wolf} results in {owl, toad, cat, dog, wolf}**
  - **Intersection, ∩**
    - **{owl, toad, cat}∩{cat, dog, wolf} results in {cat}**
  - **Complication, '**
    - **The universe of discourse *I***
    - **In case that I is {owl, toad, cat, dog, wolf}, {owl, toad, cat}' is {dog, wolf}**

Set Theory Fundamentals

- **Which, if any, of the following statements are true?**
  - **(1) If $A \cup B = A$, then $A \subseteq B$**
  - **(2) If $A \cup B = A$, then $B \subseteq A$**
  - **(3) If $A \subseteq B$, then $A \cup B = A$**
  - **(4) If $A \subseteq B$, then $A \cup B = B$**

Set Theory Fundamentals

- **Which, if any, of the following statements are true?**
  - **(1) If $A \cap B = A$, then $A \subseteq B$**
  - **(2) If $A \cap B = A$, then $B \subseteq A$**
  - **(3) If $A \subseteq B$, then $A \cap B = A$**
  - **(4) If $A \subseteq B$, then $A \cap B = B$**

Set Theory Fundamentals

- **Suppose A and B are sets such that $A \cap B = A \cup B$, does it necessarily follow that A = B?**
- **Which, if either, of the following statements is true?**
  - **(1) if $A \subseteq B$, then $A' \subseteq B'$**
  - **(2) if $A \subseteq B$, then $B' \subseteq A'$**

Set Theory Fundamentals

- **Boolean equations**
  - **A *term***
    - **A set standing alone is a *term*, and**
    - **Given two *terms t1* and *t2*, (*t1* ∪ *t2*), (*t1* ∩ *t2*) and *t1*' are all terms.**
  - **A boolean *equation***
    - **An *equation* taking the form of t1=t2 means the terms t1 and t2 equal to each other.**
    - **With capital letters, with or without subscripts, standing for sets, a boolean equation is *valid* means the equation stands with any arbitrary sets being stood for.**

Set Theory Fundamentals

- **Which, if any, of the following equations are valid?**
  - **(1) $A \cap B = A \cup B$**
  - **(2) $A' = B$**
  - **(3) $A \cup B = A \cup B$**
  - **(4) $A \cup B = A \cup B'$**
  - **(5) $A \cap B = (A \cap B) \cup C$**
  - **(6) $A \cap B = (A \cap B) \cap ((A \cap B) \cup C')$**

The End

- **Chapter 1: Genesis**