Responses of Digital Filters

Chapter Intended Learning Outcomes:

(i) Understanding the relationships between impulse response, frequency response, difference equation and transfer function in characterizing a LTI system

(ii) Ability to identify infinite impulse response (IIR) and finite impulse response (FIR) filters. Note that a digital filter is system which processes discrete-time signals

(iii) Ability to compute system frequency response

LTI System Characterization

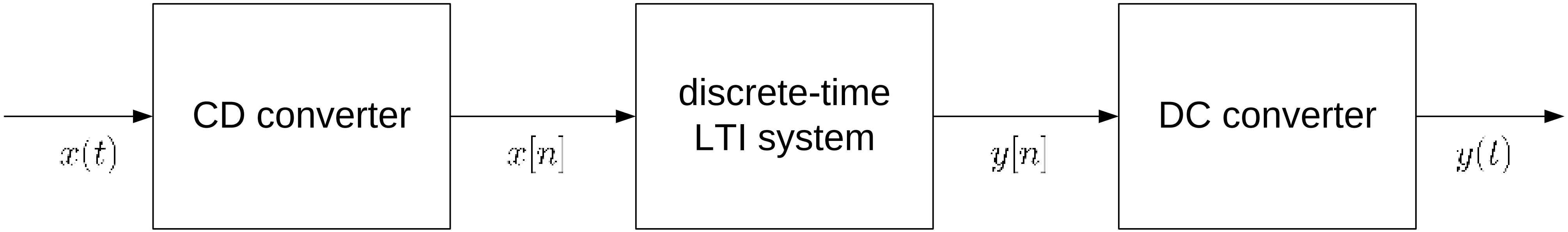

Fig.8.1: Processing of analog signal with LTI filter

The LTI system can be characterized as

- Impulse Response

Let $h[n]$ be the impulse response of the LTI filter. Recall from (3.17), it characterizes the system via the convolution:

$$y[n] = x[n] \otimes h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \qquad (8.1)$$

$h[n]$ is the time-domain response of the LTI filter

- Frequency Response

We have from (6.17), which is the discrete-time Fourier transform (DTFT) of (8.1):

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) \tag{8.2}$$

$H(e^{j\omega})$ is the frequency-domain response of the LTI filter

- Difference Equation

A LTI system satisfies a difference equation of the form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \tag{8.3}$$

- Transfer Function

Taking the z transform on both sides of (8.3):

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \tag{8.4}$$

Which of them can uniquely characterize a system? Which of them cannot?

Example 8.1

Given the difference equation with input $x[n]$ and output $y[n]$:

$$y[n] = ay[n-1] + x[n]$$

Find all possible ways to compute $y[n]$ given $x[n]$.

In fact, there are two ways to compute $y[n]$.

A straightforward and practical way is to implement a causal system by using the difference equation recursively with a given initial condition of $y[-1]$:

$$\begin{aligned} y[0] &= ay[-1] + x[0] \\ y[1] &= ay[0] + x[1] \\ y[2] &= ay[1] + x[2] \\ \cdots & \quad \cdots \quad \cdots \end{aligned}$$

On the other hand, it is possible to implement a noncausal system via reorganizing the difference equation as:

$$y[n] = ay[n-1] + x[n] \Rightarrow y[n-1] = \frac{1}{a}\left(y[n] - x[n]\right)$$

In doing so, we need an initial future value of $y[-1]$ and future inputs, and the recursive implementation is:

$$\begin{aligned} y[-2] &= \frac{1}{a}\left(y[-1] - x[-1]\right) \\ y[-3] &= \frac{1}{a}\left(y[-2] - x[-2]\right) \\ y[-4] &= \frac{1}{a}\left(y[-3] - x[-3]\right) \\ \cdots & \quad \cdots \quad \cdots \end{aligned}$$

As a result, generally the difference equation cannot uniquely characterize the system.

Nevertheless, if causality is assumed, then the difference equation corresponds to a unique LTI system.

Alternatively, we can also study the computation of $y[n]$ using system transfer function $H(z)$:

$$Y(z) = az^{-1}Y(z) + X(z) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - az^{-1}}$$

Since the ROC is not specified, there are two possible cases, namely, $|z| > |a|$ and $|z| < |a|$.

For $|z| > |a|$ or $|az^{-1}| < 1$, using inverse z transform:

$$
\begin{aligned}
\frac{Y(z)}{X(z)} &= \frac{1}{1-az^{-1}} = 1 + az^{-1} + a^2 z^{-2} + \cdots \\
&\Rightarrow Y(z) = X(z)\left(1 + az^{-1} + a^2 z^{-2} + \cdots\right) \\
&\Rightarrow y[n] = x[n] + ax[n-1] + a^2 x[n-2] + \cdots = \sum_{k=0}^{\infty} a^k x[n-k]
\end{aligned}
$$

which corresponds to a causal system.

The same result can also be produced by first finding the system impulse response $h[n]$ via inverse z transform of $H(z)$:

$$
h[n] = a^n u[n] \leftrightarrow H(z) = \frac{1}{1-az^{-1}}, \quad |z| > |a|
$$

and then computing $x[n] \otimes h[n]$.

For $|z| < |a|$ or $|a^{-1}z| < 1$, using inverse z transform:

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - az^{-1}} = \frac{-a^{-1}z}{1 - a^{-1}z} = -a^{-1}z\left(1 + a^{-1}z + a^{-2}z^2 + \cdots\right)$$
$$\Rightarrow Y(z) = -X(z)\left(a^{-1}z + a^{-2}z^2 + a^{-3}z^3 + \cdots\right)$$
$$\Rightarrow y[n] = -a^{-1}x[n+1] - a^{-2}x[n+2] - a^{-3}x[n+3] + \cdots$$
$$= \sum_{k=-\infty}^{-1} -a^k x[n-k]$$

which corresponds to a noncausal system.

<u>Example 8.2</u>

Discuss all possibilities for the LTI system whose input $x[n]$ and output $y[n]$ are related by:

$$y[n] - 5.4y[n-1] + 2y[n-2] = x[n]$$

Taking z transform on the difference equation yields:

$$H(z) = \frac{1}{1 - 5.4z^{-1} + 2z^{-2}} = \frac{1}{(1 - 0.4z^{-1})(1 - 5z^{-1})} = \frac{z^2}{(z - 0.4)(z - 5)}$$

There are 3 possible choices:

$|z| > 5$: The ROC is outside the circle with radius characterized by the largest-magnitude pole. The system can be causal but is not stable since the ROC does not include the unit circle

$0.4 < |z| < 5$: The system is stable because the ROC includes the unit circle but not causal

$|z| < 0.4$: The system is unstable and noncausal

Impulse Response of Digital Filters

When $H(z)$ is a rational function of z^{-1} with only first-order poles, we have from (5.26):

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \sum_{l=0}^{M-N} B_l z^{-l} + \sum_{k=1}^{N} \frac{A_k}{1 - c_k z^{-1}} \tag{8.5}$$

where the first component is present only if $M \geq N$.

If the system is causal, then the ROC must be of the form $|z| > |p_{\max}|$ where $p_{\max}$ is the largest-magnitude pole.

According to this ROC, the impulse response $h[n]$ is:

$$h[n] = \sum_{l=0}^{M-N} B_l \delta[n-l] + \sum_{k=1}^{N} A_k (c_k)^n u[n] \qquad (8.6)$$

There are two possible cases for (8.6) which correspond to IIR and FIR filters:

- IIR Filter

If $N \geq 1$ or there is at least one pole, the system is referred to as an IIR filter because $h[n]$ is of infinite duration.

- FIR Filter

If $N = 0$ or there is no pole, the system is referred to as a FIR filter because $h[n]$ is of finite duration.

Notice that the definitions of IIR and FIR systems also hold for noncausal systems.

Example 8.3

Determine if the following difference equations correspond to IIR or FIR systems. All systems are assumed causal.

(a) $y[n] = 0.1y[n-1] + x[n]$

(b) $y[n] = x[n] + 2x[n-1] + 3x[n-2]$

Taking the z transform on (a) yields

$$Y(z) = az^{-1}Y(z) + X(z) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-0.1z^{-1}}$$

which has one pole. For causal system:

$$h[n] = (0.1)^n u[n] \leftrightarrow H(z) = \frac{1}{1 - 0.1z^{-1}}, \quad |z| > 0.1$$

which is a right-sided sequence and corresponds to an IIR system as $h[n]$ is of infinite duration.

Similarly, we have for (b):

$$Y(z) = X(z) + 2z^{-1}X(z) + 3z^{-2}X(z) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = 1 + 2z^{-1} + 3z^{-2}$$

which does not have any nonzero pole. Hence

$$h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]$$

which corresponds to a FIR system as $h[n]$ is of finite duration.

Frequency Response of Digital Filters

The frequency response of a LTI system whose impulse response $h[n]$ is obtained by taking the DTFT of $h[n]$:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \tag{8.7}$$

According to (5.8), $H(e^{j\omega})$ is also obtained if $H(z)$ is available:

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-\omega k}} \tag{8.8}$$

assuming that the ROC of $H(z)$ includes the unit circle.

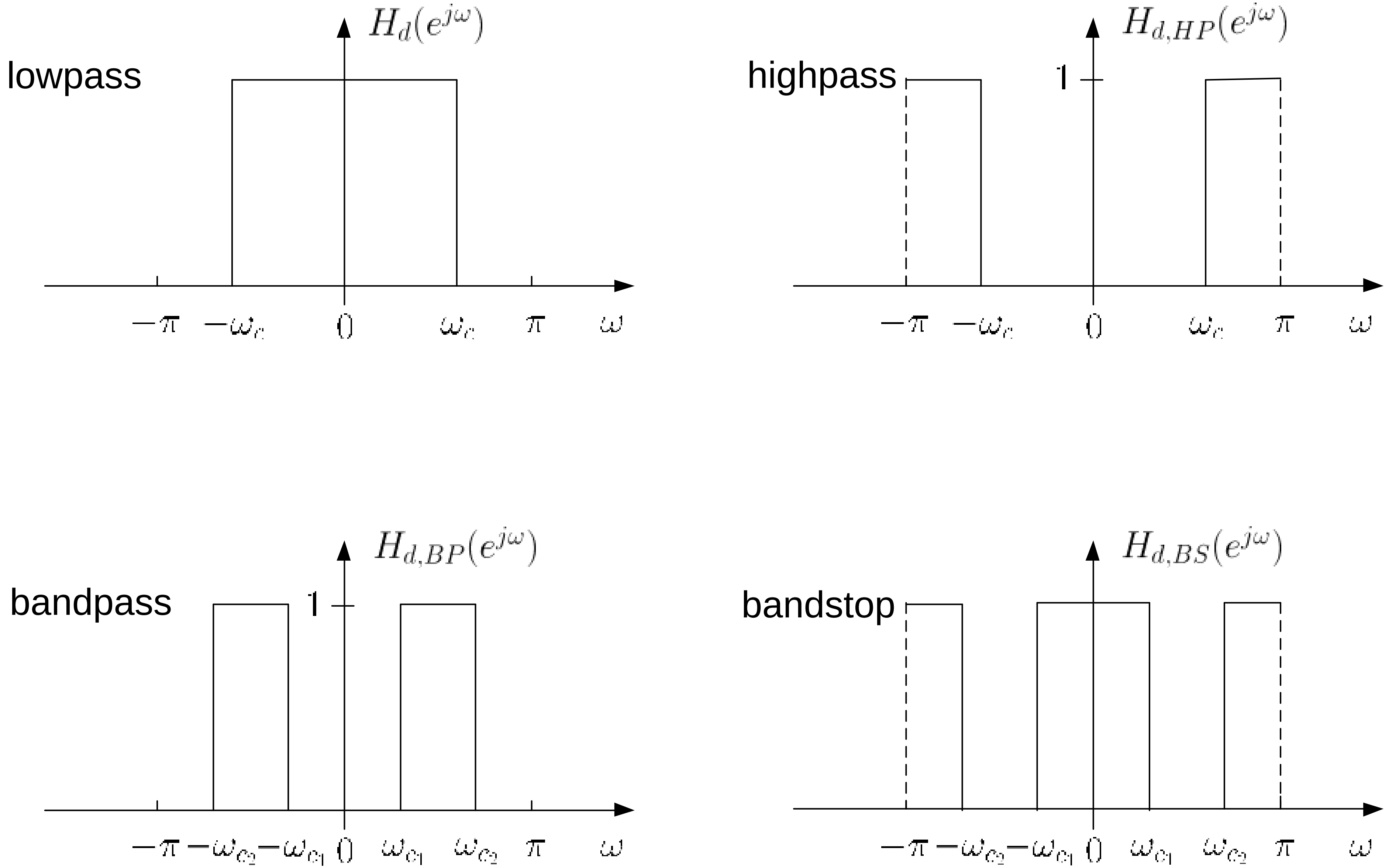

Fig.8.2: Typical filters described in frequency domain

Example 8.4

Plot the frequency response of the system with impulse response $h[n]$ of the form:

$$h[n] = \begin{cases} \mathrm{sinc}(0.1(n-50)), & 0 \le n \le 100 \\ 0, & \text{otherwise} \end{cases}$$

where

$$\mathrm{sinc}(u) = \frac{\sin(\pi u)}{\pi u}$$

Following Example 7.6, we append a large number of zeros at the end of $h[n]$ prior to performing discrete Fourier transform (DFT) to produce more DTFT samples.

Is it a low-pass filter? Why?

The MATLAB program is provided as `ex8_4.m`.

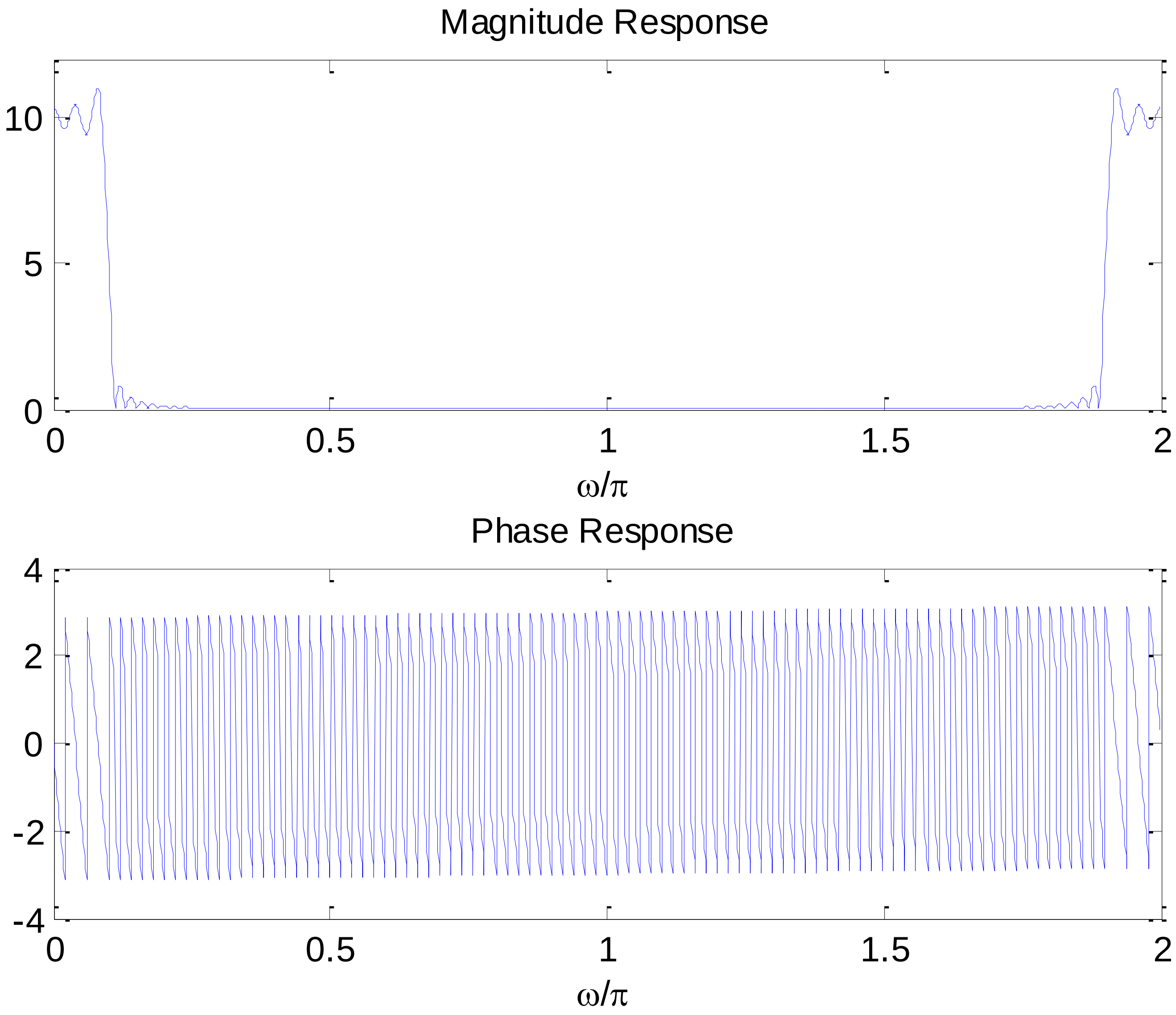

Fig.8.3: Frequency response for sinc $h[n]$

Example 8.5

Plot the frequency response of the system with transfer function $H(z)$ of the form:

$$H(z)=\frac{1+2z^{-1}+3z^{-2}}{2+3z^{-1}+4z^{-2}}$$

It is assumed that the ROC of $H(z)$ includes the unit circle.

The MATLAB code is

```
b=[1,2,3];
a=[2,3,4];
freqz(b,a);
```

Is it a lowpass filter? Why?

The MATLAB program is provided as `ex8_5.m`.

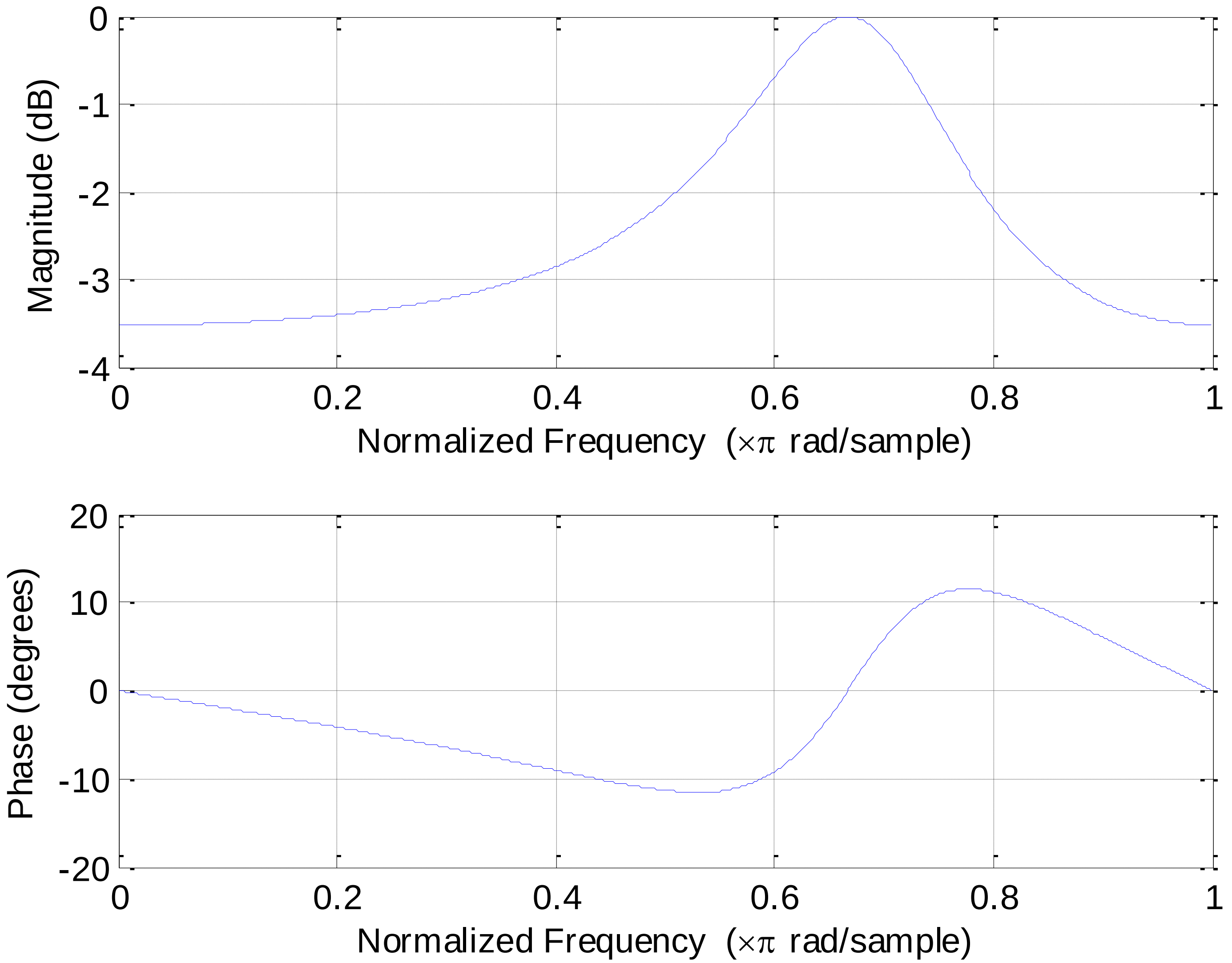

Fig.8.4: Frequency response for second-order system