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Assignment 2

Number1)

Two binary random variables

$P(X=0, Y=0) = 1/3$
$P(X=0, Y=1) = 1/3$
$P(X=1, Y=1) = 1/3$
$P(X=1, Y=0) = 0$

$H(X) = \Sigma\, p(x) \log p(x)$

a) $H(X, Y)$

$P(X=0) = 2/3$
$P(X=1) = 1/3$
$P(Y=0) = 1/3$
$P(Y=1) = 2/3$

$P_r(2/3, 1/3) = (1/3, 2/3)$

$H(X) = 2/3 \log(3/2) \cdot 1/3 \log(3)$
$0{,}35 + 0{,}52 = \underline{0{,}91}$

b) $H(X,Y) = 1/3 \log(3) + 1/3 \log 3 + 1/3 \log 3 + 0$
$= \underline{21{,}5}$

c) $H(Y/X) = P(Y=0/X=0) \cdot \frac{P(Y=0, X=0)}{P(X=0)}$

$= \frac{1/3}{2/3} \Rightarrow \underline{1/2}$

$P(Y=1/X=0) = \underline{1/2}$
$P(Y=0 \mid X=1) = \underline{0}$
$P(Y=1 \mid X=1) = \underline{1}$

(d) $H(Y/X) = P(X=0, Y=0) \log \frac{1}{P(X=0),(Y=0)} +$
$+ P(X=0),(Y=1) \log \frac{1}{P(Y=1)(X=0)} +$
$P(X=1, Y=0) \log \frac{1}{P(Y=0 \mid X=1)} +$
$P(X=1, Y=1) \log \frac{1}{P(Y=1 \mid X=1)}$

$\Rightarrow 1/3 \log 3 + \frac{1}{3} \log(2) + 0 + 1/3 \log$

$\Rightarrow \underline{5/3}$

$I(X,Y) = 1/3 \cdot \log \frac{1/3}{(2/3)(1/3)} + 1/3 \log \cdot \frac{1/3}{(2/3)(2/3)}$

$1/3 \log \left[\frac{1/3}{(1/3)(2/3)}\right]$

$\Rightarrow \underline{1/4}$

Number 2)

$X=(0,1)$, $y=(0,1)$, $z=(0,1)$

$P(z=0)\, P(X=0, Y=0) + P(X=1, Y=1)$

$P(z=0) = \frac{1}{2}$ $\quad P(X) = P(Y = (\frac{1}{2}, \frac{1}{2})$

$P(z=1) = \frac{1}{2}$

$H(z) = \frac{1}{2}\log_2 2 + \frac{1}{2}\log_2 2$

a) $H(z) = 1 = H(X) = H(Y)$

$\cancel{I(X;Y)=0}$

$H(z) = 1$

b) $H(z|X,Y) = 0$

$H(X,Y,z) = 2$

c) $H(Y/X,z) = 0$

$H(X|Y,z) = 0$

$H(Y,z) = \frac{1}{2}\log_2 2 + \frac{1}{2}\log_2 2 = 1$

d) $H(Y|z) = H(Y) = 1$

$H(X|z) = H(X) = 1$

$H(X,Y|z) = H(X|z) + H(Y|X,z) = 1$

e) $I(X;z) = 0$

$I(X;Y|z) = 1$

Number 3)

Assume that $X \to Y \to Z$ forms a Markov Chain.

a) $I(X;Z|Y) = 0$

b) $H(X|Y,Z) - H(X/Y)$

$H(X|Y,X) = H(X/Y) = X \perp Z|Y$

$H(X,Z/Y) = H(X/Y) + H(Z/Y)$

$I(X;Z|Y) = 0$

$H(Z|Y,X) = H(Z/Y)$ $\therefore$ we conclude that $X \to Y \to Z$ forms a Markov Chain.

Number 4)

$X = Y = \{0,1\}$ $Z \oplus Y$

$P(x) = P(y) = (1/2, 1/2)$

$P(z) = (1/2, 1/2)$

$I(X,Y) = 0+6$

$I(X, Y = 1 + (-1) = 0$

$I(X; Y/Z) = 1$

$X = Y = (0,1)$

$P(X) = (1/2, 1/2)$

$P(Y=1|X=0, Z = 1/4$

$Z = Y \Rightarrow H(Y/X) = 0$

$I(X,Y) = 0$

$P(X=0, Y=1) = 1/4$ $\Rightarrow P(X) = (1/2, 1/2)$

$P(X=1, Y=0) = 1/4$ $\Rightarrow P(Y) = (1/2, 1/2)$

$P(X=1, Y=1) = 1/4$ $\Rightarrow P(Z) = (1/2, 1/2)$

$P(X=0, Y=0) = 1/4$

$I(X,Y) = 1/4 \log \frac{1/4}{\cancel{X} \cdot 1/4} + 3/4 \log \frac{1/4}{(1/2)(1/4)})$

$\Rightarrow 1$

$I(X;\cancel{Y}/Z) = 0$

Number 5)

$H(x) \leq H(Y)$

Expansion of the chain rule for two random variables.

$$H(x) + \underbrace{H(Y/X)}_{0} = H(Y) + H(X/Y)$$

$$\cancel{H(x) + H(Y)}$$

$$H(x) = H(Y) + H(X/Y) \geq 0$$

Number 6)

a) $H(X,Y|Z) \geq H(Y|X,Z)$

$H(X,Y|Z) = H(X|Z) + H(Y|X,Z) \geq H(X|Z)$

→ Entropy is non-negative.

⇒ This inequality is tight when $H(Y|X,Z)$ is zero.

⇒ Conditionally on both X,Z the value of Y is deterministic

b) $I(X,Y;Z) \geq I(X;Z)$

$\therefore I(X,Y;Z) = H(X,Y) - H(X,Y|Z) = H(X) + H(Y|X) - H(X|Z) - H(Y|X,Z) = I(X,Z) + H(Y|X) - H(Y|X,Z) \geq I(X;Z)$

⇒ The condition of last ~~entropy~~ inequality cannot reduce entropy.

⇒ So this inequality is tight when $Y \perp Z|X$.

c) $H(X,Y,Z) - H(X,Y) \leq H(X,Z) - H(X)$

Using chain rule :- left ~~hand~~ hand is $H(Z|X,Y)$
right hand is $H(Z|X)$

⇒ The condition of this inequality cannot reduce entropy.

⇒ So this inequality is tight when $Z \perp Y/X$.

d) $I(X;Z/Y) > I(Z;Y/X) - I(Z;Y) + I(X;Z)$

$I(X;Z/Y) - I(Y;Z/X) = H(Z/Y) - H(Z/X,Y) - H(Z/X) + H(Z/X,Y)$
$= H(Z/Y) - H(Z/X)$

while:

$I(X;Z) - I(Y;Z) = H(Z) - H(Z/X) - H(Z) + H(Z/Y)$
$= H(Z/Y) - H(Z/X)$

From this we can see that this inequality is always an equality

Number 7)

a) $I(X;Y_1) = H(Y_1) - H(Y_1|X)$

$H(Y_1|X) = H(Y_1|X=0)\,P(\overset{1/4}{X=0}) + H(Y_1|X=1)\,P(\underset{1/4}{X=1}) + H(Y_1|X=2)\,P(\underset{1/2}{X=2})$

$\Rightarrow 1/4 + 1/4 + 1/8$

$\Rightarrow 5/8$

$H(Y_1) = 2 \cdot 1/2 \log 2 = 1$

$\therefore 1 - 5/8 \Rightarrow 3/8$

$I(X, Y_2) = H(Y_2) - H(Y_2|X)$

$\Rightarrow 0 - \cancel{1/4}\,0$

$= \cancel{3/8}\,0$

$H(Y_2|X) = H(Y_2|X=0)\,P(\overset{1/4}{X=0}) + H(Y_2|X=1)\,P(\underset{1/4}{X=1}) + H(Y_2|X=2)\,P(\underset{1/2}{X=2})$

$= (0 \cdot 1/4) + 0 + (0 \cdot 1/2)$

$\Rightarrow 0$

Experiment Y_1 is better than Y_2

b) $I(X; Y_1, Y_2) = H(X) + \underbrace{H(Y_1, Y_2)}_{2} - H(X, Y_1, Y_2)$

$\Rightarrow X=0 \Rightarrow Y_1 = Y=0;\ X=1, \Rightarrow Y_2 = 0, Y=1, X=2 = Y_2 = 1$

$\therefore H(Y_2|X) = 0$

$H(X, Y_1, Y_2) = H(X|Y_1, Y_2)$

$Y_1 = 0, Y_2 = 0 \Rightarrow X=0;\ Y_1 = 0, Y_2 = 1 \Rightarrow \boxed{\cancel{X}};$

$Y_1 = 1, Y_2 = 0, \Rightarrow X=1;\ Y_1 = 1, Y_2 = 1, \Rightarrow X=2$

$\therefore \Rightarrow 0$ due to lack of uncertainty.

$I(X; Y_1, Y_2) = H(X)$

$\Rightarrow 1/4 \log 4 + 1/2 \log 2$

$\Rightarrow 1 + 1/2$

$\Rightarrow 3/2$

c) $I(X; Y_1|Y_2) \Rightarrow H(Y_1|X) - H(Y_1|X, Y_2)$

$\Rightarrow 0$

$I(X; Y_2|Y_1) = H(Y_2|X) - H(Y_2|X, Y_1)$

$= 0$

Number 8)

$$I(X_1, X_2; Y) = H(Y) \Rightarrow \frac{1}{2}\log 2 \Rightarrow \frac{1}{2}$$

$$I(Y,Z) = H(Z) - H(Z/Y)$$
$$= H(Z)$$
$$\Rightarrow 2 \cdot \frac{1}{4}\log 4$$
$$\Rightarrow 2$$

$$I(X_1, X_2; Z) \Rightarrow H(Z) \Rightarrow 2 \cdot \frac{1}{4}\log 4 + \frac{1}{2} + 0$$
$$\Rightarrow 2{,}5$$

$\Rightarrow H(X) \Rightarrow$ Formular 1/2

Number 9)

$H(X_j) = H(1/3, 1/4, 1/4, 1/6) = 1/3 \log 3 + 2 \cdot 1/4 \log 4 + 1/6 \log 6 \Rightarrow \underline{1,959148}$

$H(X_j, X_{j+1})/2 = \left(H(X_j) + H(X_{j+1} / X_j)\right)/2$

$\Rightarrow H(X_j) = 1,959$

$\Rightarrow H(X_{j+1} / X_j) \Rightarrow \{ P(X_j = a)\, \underline{H(X_{j+1} | X_j = a)} + P(X_j = b)\, \underline{H(X_{j+1} | X_j = b)}$
$+ P(X_j = c)\, \underline{H(X_{j+1} | X_j = c)} +$
$P(X_j = d)\, \underline{(X_{j+1} | X_j = d)}$

$H(X_{j+1} | X_j = a) = H(1/4, 1/4, 1/4, 1/4) = 4 \cdot 1/4 \log 4 = 2$
$H(X_{j+1} / X_j = b) = H(1/3, 1/3, 1/3, 0) = 3 \cdot 1/3 \log 3 = 1,584963$
$H(X_{j+1} / X_j = c) = H(1/3, 1/3, 0, 1/3) = 3 \cdot 1/3 \log 3 = 1,584963$
$H(X_{j+1} / X_j = d) = H(1/2, 0, 1/2, 0) = 2 \cdot 1/2 \log 2 = 1$

$\Rightarrow (1,959 + 2 + 1,5849 + 1,5849 + 1)/2$

$\Rightarrow \underline{4,064537}$

$H_x = H(X_{j+1}, X_j) = H(1/4, 1/3, 1/3, 1/2) = 1/4 \log 4 + 2 \cdot 1/3 \log 3 + 1/2 \log 2 = 2,0566$
$H(1/4, 1/3, 1/3, 0) = 1/4 \log 4 + 2 \cdot 1/3 \log 3 + 0 = 1,556642$
$H(1/4, 1/3, 0, 1/2) = 1/4 \log 4 + 1/3 \log 3 + 1/2 \log 2 = 1,528324$
$H(1/4, 0, 1/3, 0) = 1/4 \log 4 + 1/3 \log 3 = 1,028321$

$\Rightarrow \underline{6,19437}$

$\Rightarrow$ Comparing the values 4,065 and 6,1944.
The condition does increase entropy.