# University of Science and Technology, Beijing

# Digital Communication

## Introduction to Fourier Transform

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## What is Fourier Transform?

The Fourier transform is a mathematical function that converts a waveform, from the time domain, f(t), into a complex valued function of frequency, f(w). The absolute value of the Fourier transform denotes the frequency value in the original function and its complex argument equates to the phase offset of the basic sinusoidal in that frequency.

The Fourier transform helps in extending the Fourier series to non-periodic functions, which allows viewing any function as a sum of simple cosine and sine waveforms.

The Fourier transform of a function f(x) is given by:

$$f(x) = \int_{-\infty}^{\infty} F(k)\, e^{2\pi i k x}\, dk$$

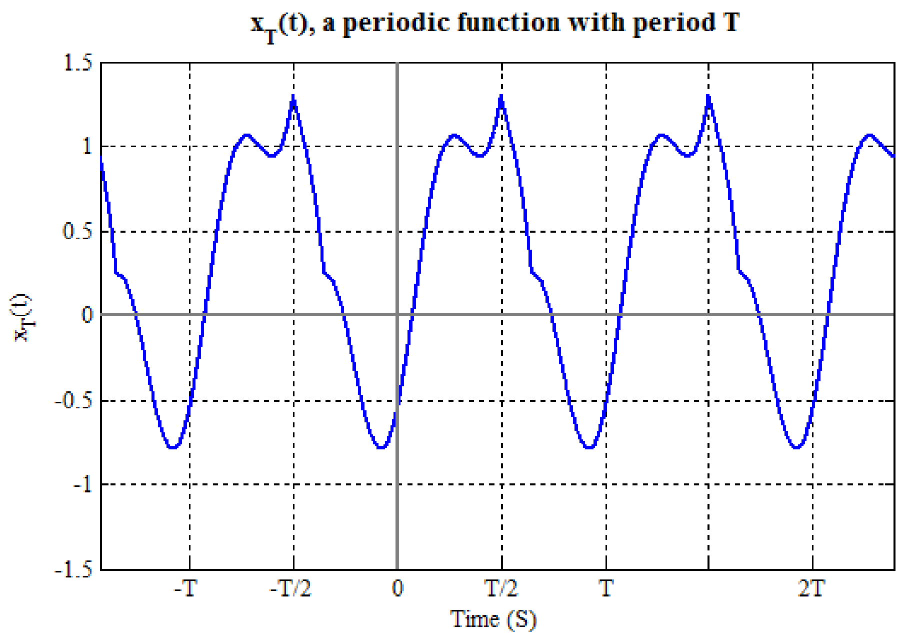
$$F(k) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i k x}\, dx$$

Where F(k) can be obtained using inverse Fourier transform.

Some of the properties of Fourier transform include:

- It is a linear transform – If f(t) and g(t) are two Fourier transforms resulting in F(f) and G(f) respectively, then the Fourier transform of the linear product of f and g can be easily calculated.
- Time-shift characteristic – The Fourier transform of g(t–a) where a is a real number that shifts the original function by the same amount of shift in the magnitude of the waveform.
- Modulation property – A function is modulated by another function when it is multiplied in time.
- Parseval’s theorem – Fourier transform is unitary, i.e., the sum of square of a function g(t) equals the sum of the square of its Fourier transform, G(f).
- Duality – If g(t) has the Fourier transform G(f), then the Fourier transform of G(t) is g(-f).

- Consider a periodic signal $x_T(t)$ with period $T$. Since the period is $T$, we take the fundamental frequency to be $\omega_0=2\pi/T$. We can represent any such function (with some very minor restrictions) using Fourier Series.




- In the early 1800's Joseph Fourier determined that such a function can be represented as a series of sines and cosines. In other words he showed that a function such as the one above can be represented as a sum of sines and cosines of different frequencies, called a Fourier Series. There are two common forms of the Fourier Series, "Trigonometric" and "Exponential" synthesis equations.

$$x\left(t\right) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right) \qquad Trigonometric\ Form$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \qquad Exponential\ Form$$

**How do we find $a_n$?**

$$a_n = \frac{2}{T} \int_T x_e(t) \cos(n\omega_0 t) dt$$

**How do we find $b_n$?**

$$b_n = \frac{2}{T} \int_T x_o(t) \sin(n\omega_0 t) dt$$

**How do we find $a_0$?**

$$a_0 = \frac{1}{T} \int_T x_T(t)\, dt \; = \; average$$

A more compact version of the Fourier Series uses complex exponentials. This is described by the following synthesis and analysis equations:

$$x_T(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t} \qquad Synthesis$$

$$c_n = \frac{1}{T} \int_T x(t) e^{-jn\omega_0 t} dt \qquad Analysis$$

Though the exponential form is more compact than the trigonometric form, the benefits include: a single analysis equation (versus three equations for the trigonometric form); notation is similar to that of the Fourier Transform and it is often easier to mathematically manipulate exponentials rather sines and cosines.