Problem Set 2
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1. The joint distribution of two binary random variables X and Y is $p(X = 0, Y = 0) = p(X = 0, Y = 1) = p(X = 1, Y = 1) = 1/3$, $p(X = 1, Y = 0) = 0$. Compute
   a) $H(X, Y)$;
   b) $H(X)$, $H(Y)$;
   c) $H(Y \mid X)$, $H(Y \mid X = 0)$, $H(Y \mid X = 1)$;
   d) $H(X \mid Y)$, $H(X \mid Y = 1)$, $H(X \mid Y = 1)$;
   e) $I(X; Y)$.
2. Consider the same random variables X and Y as in Problem 1. Define a new binary random variable Z as $Z = X \oplus Y$. Compute
   a) $H(Z)$;
   b) $H(Z \mid X, Y)$, $H(X, Y, Z)$ (Hint: Since $Z = X \oplus Y$, Z is deterministic given X and Y);
   c) $H(Y \mid X, Z)$, $H(X \mid Y, Z)$, $H(Y, Z)$, $H(X, Z)$ (Hint: $Y = Z \oplus X$, $X = Y \oplus Z$, and use the results you get in a) and b));
   d) $H(Y \mid Z)$, $H(X \mid Z)$, $H(X, Y \mid Z)$ (Hint: instead of calculating by definition, make use of the results you get in a)-c));
   e) $I(X; Z)$, $I(X; Y \mid Z)$ (Hint: instead of calculating by definition, make use of the results you get in a)-d)).
3. Assume that $X \rightarrow Y \rightarrow Z$ forms a Markov Chain. What is
   a) $I(X; Z \mid Y)$?
   b) $H(X \mid Y, Z) - H(X \mid Y)$?
4. Give an example to show that $I(X; Y) = 0$ and $I(X; Y \mid Z) = 0$ does not imply each other.

   (Hint: Use the random variables which are pairwise independent but not mutually independent to show $I(X; Y) = 0$ does not imply $I(X; Y \mid Z) \neq 0$. Conversely, design three binary random variables X, Y, Z as follows: X, Y are the respective input and output of a Binary Symmetric Channel with $p(X = 0) = 1/2$, $p(Y = 1 \mid X$

$= 0) = 1/4$, $Z = Y$. Show $I(X; Y \mid Z) = 0$ but $I(X; Y) \neq 0$ for these X, Y, Z)

5. Let X be a function of Y. Prove that $H(X) \leq H(Y)$. (Hint: as X is a function of Y, X is deterministic given Y).
6. Based on the basic inequalities, prove the following inequalities on random variables X, Y, Z, and state the condition where equality holds:

   a) $H(X, Y \mid Z) \geq H(Y \mid X, Z)$

   b) $I(X, Y; Z) \geq I(X; Z)$

   c) $H(X, Y, Z) - H(X, Y) \leq H(X, Z) - H(X)$

   d) $I(X; Z \mid Y) \geq I(Z; Y \mid X) - I(Z; Y) + I(X; Z)$

   (Hint: the basic inequalities we know about information measures are: $H(X) \geq 0$, $H(X, Y) \geq 0$, $H(Y \mid X) \geq 0$, $I(X; Y) \geq 0$, $I(X; Y \mid Z) \geq 0$, $H(Y) \geq H(Y \mid X)$.)
7. A random memoryless source $X \in \{0, 1, 2\}$ with probability distribution $\{1/4, 1/4, 1/2\}$. Two experiments are designed to observe this source, with respective outcome random variables $Y_1 \in \{0, 1\}$, $Y_2 \in \{0, 1\}$. The respective conditional probability matrix of Y_1 and Y_2 given X is provided by the transition matrix (i.e. $P(Y_1 = 1 \mid X = 0) = 0$, $P(Y_1 = 0 \mid X = 1) = 1$ etc.)

$$P_{Y_1|X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}, \; P_{Y_2|X} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

   a) Compute $I(X; Y_1)$ and $I(X; Y_2)$. Which experiments is better?

   b) Compute $I(X; Y_1, Y_2)$. How much additional information on X you can obtain by doing both experiments compared with doing only experiment Y_1 or only experiment Y_2?

   c) Compute $I(X; Y_1 \mid Y_2)$ and $I(X; Y_2 \mid Y_1)$. How to interpret these two information measures?
8. Let $X_1, X_2 \in \{0, 1\}$ be two independent binary random variables with identical probability distribution $\{1/2, 1/2\}$. Define a new random variable $Y \in \{0, 1, 2\}$ to be the sum of X_1 and X_2. Define another random variable as follows. If Y is even, then $Z = 0$; if Y is odd, then $Z = 1$. Obviously, $(X_1, X_2) \to Y \to Z$ forms a Markov

chain. Respectively calculate $I(X_1, X_2; Y)$, $I(Y; Z)$ and $I(X_1, X_2; Z)$. Compare their values in terms of the data processing theorem.

9. Consider a stationary Markov source $X_1, X_2, \dots X_j, \dots$ (Markov source means $p(X_{j+1} \mid X_j, \dots, X_2, X_1) = p(X_{j+1} \mid X_j)$ for all $j \geq 1$). Every random variable X_j in the stationary source has an identical probability distribution (1/3, 1/4, 1/4, 1/6). The conditional probability distribution of X_{j+1} given X_j is known via the transition matrix

$$P_{X_{j+1}|X_j} = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}$$

Calculate $H(X_j)$, $H(X_j, X_{j+1})/2$ and entropy rate H_X of this source. Compare these values and what conclusion you can make?