Introduction to Mathematical Logic

Chapter 2 Infinite Sets

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1-1 Correspondence

- For two sets A and B, a *1-1 correspondence* between them means an assignment where
  - Each and every element of A is paired up with one and only one element of B, and
  - No two elements of A is paired up with a same element of B, and
  - No element of B is left unpaired-up with.

1-1 Correspondence

- For example
  - If a cinema having 500 seats for audience sells 500 tickets for a movie on show, it'll have to guarantee that there is a *1-1 correspondence* between the set of all seats and that of all tickets applied.
  - You can certainly give more examples by your own.

Sizes of Infinite Sets

- What do you believe is the rank of sizes of the following three sets?
  - The set of natural numbers, even natural numbers, and perfect squares.
- Try problem on the next slide, please.

Sizes of Infinite Sets

- Try this problem, please
  - A hotel has infinite number of rooms of which
  - Each has a traveler living in it, and
  - Now come another infinite number of travelers.
  - The question is, can this infinite number of coming travelers be also accommodated with each room for one traveler only being assured?

Denumerable Sets

- A game
  - I put a natural number down and you guess what the number is
  - You are allowed infinitely many times at each of which
  - You are allowed only one try
- Are you sure you will certainly win? How?

Denumerable Sets

- Another game
  - I put a pair of natural numbers down and you guess what they both are
  - You are allowed infinitely many times at each of which
  - You are allowed only one try
- Are you sure you will certainly win? How?

Denumerable Sets

- One more game
  - I put a finite set of arbitrarily many natural numbers down and you guess what they all are
  - You are allowed infinitely many times at each of which
  - You are allowed only one try
- Are you sure you will certainly win? How?

Denumerable Sets

- An infinite set that has a 1-1 correspondence with the set of natural numbers is called *denumerable*.

Denumerable Sets

- Is the union of two denumerable sets necessarily denumerable?
- Let D be a denumerable set of D_1, D_2, ..., D_n, ... of which each is in turn a denumerable set, and S be the union of all elements of D, i.e. D_1, D_2, ..., D_n, ..., then is S denumerable?
- Given a denumerable set D, is the set of all finite sequence of D denumerable or not?

Cantor's Theorem

- The power set of a given set A is
  - The set of all subsets of A including $\emptyset$ and A
  - Which is denoted as P(A).
- What is P(A) if A is $\{a, b\}$?
- How many elements does P(A) have if A has 3 elements? How many if A has 5? How many if *n*?

Cantor's Theorem

- One more game*
  - I put a set of natural numbers down, finite or infinite, and you guess what they all are
  - You are allowed infinitely many times at each of which
  - You are allowed only one try
- Are you sure you will certainly win? How?

Cantor's Theorem

- Cantor's Theorem tells us that
  - For any set A, finite or infinite, denumerable or not, it is always true that
  - **The size of P(A) is greater than that of A.**

Cantor's Theorem

- Prove that the set of all infinite sequences of 0s and 1s is of the same size as *P(N)* with *N* stands for the set of natural numbers.
- Prove that all infinite subsets of a denumerable set are denumerable.
- Prove that every infinite set has a denumerable subset.
- Prove that every infinite set, denumerable or not, have a 1-1 correspondence with a proper subset of itself.

The End

- **Chapter 2: Infinite Sets**