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Assignment No. "3rd"

{ Fourier Series and Fourier Transform }

Date: 3rd oct, 2022

⇒ Fourier Series:

· It is the method of representing a periodic function $F(x)$ as infinite sum of sines and cosines.

Periodic Function:

A function is said to be periodic with period "p", if

$$g(x) = g(x+np) \quad , \text{ where } n = 1, 2, 3, \ldots$$

⇒ For expansion of a function, the Fourier Series is making use of orthogonality relation between the sine and cosine functions.

Orthogonal Functions:

Two functions $g(t)$ and $x(t)$ are said to be orthogonal, if:

$$\langle g(t) | x(t) \rangle = \int_a^b g(t)\, x(t)\, w(t)\, dt = 0$$

where; $w(t)$ = weighting function

and $a \leq t \leq b$

⇒ Fourier Series is a very convenient way of dividing a periodic function into simple terms, and then after solving them separately, the final output can be recombined to present the solution to a given problem.

Date: 3rd oct, 2022

⇒ Dirichlet's Conditions:

For the implementation of Fourier Series to represent a periodic function, the following two conditions should be satisfied:

a: For a single-valued periodic signal, there should be a finite number of maxima and minima, and at most finite number of discontinues points in an interval of one period.

b: The signal is completely integrable over the interval of one period:

let, we have an arbitrary periodic signal $g(t)$, then;

$$g(t) = \int_{-T_0/2}^{T_0/2} |g(t)|\, dt < \infty.$$

⇒ Fourier Series Representation of a signal:

⇒ Consider a continuous periodic signal;

$$g(t) = e^{j\omega_0 t}$$

Due to its continuity property, there will be a set of complex exponentials associated with, ranging from $-\infty$ to $+\infty$.

Date: 3rd oct, 2022

So, we can write;

$$\phi_k = \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$$

Now, by using vector anology;

$$\cancel{x}g(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad (i)$$

where a_k = Fourier Series Co-efficient

eq. (i) shows the Fourier Series for representing any signal in space.

Derivation of Fourier Series Co-efficient

As we know that;

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

multiplying both sides by $e^{-jn\omega_0 t}$

$$\Rightarrow x(t).\ e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k.\ e^{jk\omega_0 t}.\ e^{-jn\omega_0 t}$$

Taking $j\omega_0 t$ as common,

$$\Rightarrow x(t).\ e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k.\ e^{j(k-n)\omega_0 t}$$

Date: 3rd oct, 2022

Taking $\int_0^T$ on both sides:

$$\Rightarrow \int_0^T x(t).e^{-jn\omega_0 t}\,dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}\,dt$$

Taking constants out of integral;

$$\Rightarrow \int_0^T x(t).e^{-jn\omega_0 t}\,dt = \sum_{k=-\infty}^{\infty} a_k.\left[\int_0^T e^{j(k-n)\omega_0 t}\,dt\right] \quad \text{(i)}$$

Using Euler's Formula on $\int_0^T e^{j(k-n)\omega_0 t}\,dt$

$$\Rightarrow \int_0^T e^{j(k-n)\omega_0 t}\,dt = \int_0^T \cos(k-n)\omega_0 t\,dt + j\int_0^T \sin(k-n)\omega_0 t\,dt$$

There are two possibilities;

$$\Rightarrow \int_0^T e^{j(k-n)\omega_0 t}\,dt = \begin{cases} T, & k=n \\ 0, & k\neq n \end{cases}$$

Hence, eq.(i) can be represented as;

$$a_k = \frac{1}{T}\int_0^T x(t).e^{-jn\omega_0 t}\,dt.$$

which is the required Fourier series co-efficient.

Date: 3rd oct, 2022

⇒ Fourier Transform from Fourier series:

⇒ Fourier Transform is a mathematical process which can be used to transform a signal from time domain to frequency domain and vice versa.

⇒ It can be applied in many fields of engineering like signal processing and RADAR.

Derivation of Fourier Transform:

A signal $x(t)$ in terms of Fourier series can be written as;

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]\, e^{jk\omega t}$$

$$\Rightarrow x(t) = \sum_{k=-\infty}^{\infty} X[k]\, e^{jk\frac{2\pi}{T}t} \quad (i) \qquad \because \omega = 2\pi/T$$

$\because$ change in frequency $= \Delta f = \dfrac{1}{T}$

$$\text{eq. (i)} \Rightarrow x(t) = \sum_{k=-\infty}^{\infty} X[k]\, e^{jk2\pi\Delta f t}$$

$$\because X[k] = \frac{1}{T}\int_{t_0}^{t_0+T} x(t) \cdot e^{-jk\omega t}\, dt$$

Date: 3rd oct. 2022

$$\Rightarrow X[K] = \Delta f \int_{t_0}^{t_0+T} x(t).\, e^{-jK2\pi \Delta f t} \, dt \quad \text{(ii)} \qquad \because \Delta f = \frac{1}{T}$$

put eq. (ii) in eq. (i)

$$\text{eq. (i)} \Rightarrow x(t) = \sum_{K=-\infty}^{\infty} \left[\Delta f \int_{t_0}^{t_0+T} x(t).\, e^{-jK2\pi \Delta f t} \, dt \right].\, e^{jK2\pi \Delta f t}$$

Let; $t_0 = -T/2$

$$\Rightarrow x(t) = \sum_{K=-\infty}^{\infty} \left[\Delta f \int_{-T/2}^{T/2} x(t).\, e^{-j2\pi K \Delta f t} . dt \right].\, e^{j2\pi K \Delta f t} \quad \text{(iii)}$$

By taking limit: $-\infty$ to ∞ i.e $T \to \infty$.
we will make the following changes:

"Σ" becomes "$\int$".
"Δf" becomes "df"
"$K\Delta f$" becomes "f"

$$\text{eq (iii)} \Rightarrow x(t) = \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} x(t).\, e^{-j2\pi f t} . dt .\, e^{j2\pi f t}$$

$$\Rightarrow x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X[\omega].\, e^{j\omega t} . d\omega \quad \text{(iv)}$$

from eq. (iv): we can write:

$$X[\omega] = \int_{-\infty}^{\infty} x(t).\, e^{-j\omega t} \, dt \quad \text{(V)}$$

Date: 3rd oct. 2022

Result:

eq. (iv) is The inverse Fourier Transform while eq. (v) is The Fourier Transform of any signal.