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Numerical Analysis
Assignment 3

亚历克上 M202161029

Homework 3

1. Construction of interpolation polynomials

a) $f(x) = \cos x$

$x_0 = 0$

$x_1 = 0,6$

$x_2 = 0,9$

$y_0 = f(x_0) = \cos(0) = 1$

$y_1 = f(x_1) = \cos(0,6) = 0,82533531$

$y_2 = f(x_2) = \cos(0,9) = 0,62160996$

$\therefore$ The lagrange interpolation of degree one is:

$$L_1(x) = \begin{cases} \dfrac{x_1 - x}{x_1 - x_0} y_0 + \dfrac{x - x_0}{x_1 - x_0} y_1, \text{ for } x \in [x_0, x_1] \\ \dfrac{x_2 - x}{x_2 - x_1} y_1 + \dfrac{x - x_1}{x_2 - x_1} y_2, \text{ for } x \in [x_1, x_2] \end{cases}$$

$\therefore$

$$L_1(x) = \left.\begin{cases} 1 - 0,2912x, \text{ for } x \in [0, 0,6] \\ 1,2327 - 0,679x, \text{ for } \in [0,6, 0,9] \end{cases}\right\} \Rightarrow 0,8690$$

The lagrange polynomial of $f(0,45) \Rightarrow \cos(0,45) = 0,900447$

$\therefore f(0,45) = 0,9004 \approx 08690$

The actual error for degree at most one $\Rightarrow$

$0,9004 - 0,8690 \Rightarrow \underline{0,0314}$

* For at most two degree $\Rightarrow$

The lagrange interpolation polynomial of degree at most two :

$$L_2(x) = l_0(x) y_0 + l_1(x) y_1 + l_2(x) y_2$$

for

$$l_0(x) = \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right) \Rightarrow \left(\frac{x - 0{,}8253}{1 - 0{,}8252}\right)\left(\frac{x - 0{,}6216}{1 - 0{,}6216}\right)$$

$$l_1(x) = \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right) \Rightarrow \left(\frac{x - 1}{0{,}8253 - 1}\right)\left(\frac{x - 0{,}6216}{0{,}8253 - 0{,}6216}\right)$$

$$l_2(x) = \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right) \Rightarrow \left(\frac{x - 1}{0{,}6216 - 0{,}8253}\right)\left(\frac{x - 0{,}8253}{0{,}6216 - 0{,}8253}\right)$$

$$\Rightarrow 0{,}8980$$

$\therefore$ $0{,}9004 - 0{,}8980$

The error at degree two is $\underline{\underline{0{,}0024}}$

1b) $f(x) = \sqrt{1+x}$

$$x_0 = 0$$
$$x_1 = 0{,}6$$
$$x_2 = 0{,}9$$

$$y_0 = f(x_0) = \sqrt{1+0} = 1$$
$$y_1 = f(x_1) = \sqrt{1+0{,}6} = 1{,}26491106$$
$$y_2 = f(x_2) = \sqrt{1+0{,}9} = 1{,}37840487$$
$$f(0{,}45) = \sqrt{1+0{,}45} = 1{,}20415945$$

∴ For the lagrange interpolation of degree one is :-

$$L_1(x) = \begin{cases} \dfrac{x_1 - x}{x_1 - x_0} y_0 + \dfrac{x - x_0}{x_1 - x_0} y_1, & \text{for } x \in [x_0, x_1] \\ \dfrac{x_2 - x}{x_2 - x_1} y_1 + \dfrac{x - x_1}{x_2 - x_1} y_2, & \text{for } x \in [x_1, x_2] \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{1{,}2649 - x}{1{,}2649 - 1} + \dfrac{x - 1}{1{,}2649 - 1} y_1, \\ \dfrac{1{,}3784 - x}{1{,}3784 - 1{,}2649} y_1 + \dfrac{x - 1{,}26491}{1{,}3784 - 1{,}26491} y_2 \end{cases}$$

$$L_1(x) = \begin{cases} x, & \text{for } x \in [0, 0{,}6] \\ x + 0{,}02413, & \text{for } x \in [0{,}6, 0{,}9] \end{cases} \Rightarrow 1{,}1\cancel{98321}\ 997345611$$

∴ The exact error is :-

$$1{,}20415945 - 1{,}198734$$

$$\cancel{\Rightarrow 0{,}0107} \Rightarrow 0{,}0055$$

* For at most two degree :-

$$L_2(x) = l_0(x)\, y_0 + l_1(x)\, y_1 + l_2(x)\, y_2$$

for $l_0(x) = \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right) \Rightarrow \left(\frac{x - 1,2649}{1 - 1,2644}\right)\left(\frac{x - 1,3784}{1 - 1,3784}\right)$

$l_1(x) \Rightarrow \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right) \Rightarrow \left(\frac{x - 1}{1,2649 - 1}\right)\left(\frac{x - 1,3784}{1,2649 - 1,3784}\right)$

$l_2(x) \Rightarrow \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right) \Rightarrow \left(\frac{x - 1}{1,3784 - 1}\right)\left(\frac{x - 1,2649}{1,3784 - 1,2649}\right)$

$$\Rightarrow \frac{1}{0,54} \cdot (x - 0,6)(x - 0,9) - \frac{\sqrt{1.6}}{0,18} \cdot x(x - 0,9) + \frac{\sqrt{1,9}}{0,27} \cdot x(x - 0,6).$$

$$\Rightarrow 1,2034013$$

$\therefore$ The actual error is : $f(0,45) - P_2(0,45)$

$$\Rightarrow 1,2042 - 1,2034$$

$$\approx \underline{\underline{0,0007}}$$

3) Using appropriate Lagrange interpolating polynomials degree one, two and three to approximate each :-

a) $f(8.4)$ if $f(8.1) = 16.94410$, $f(8,3) = 17,56492$, $f(8,6) = 18,50515$,
~~$f(8,7) = 0,33493750$, $f(\cdot) = 1,10100000$~~ $f(8,7) = 18.82091$

$\therefore$ The linear Lagrange requires only **two nodes** :-
→ we know that → $x_0 = 8,1$ where $f(x_0) = f(8,1) = 16.94410$
$x_1 = 8,3$ $\quad f(x_1) = f(8,3) = 17,56492$
$x_2 = 8,6$ $\quad f(x_2) = f(8,6) = 18,50515$

$$f(8,4) \Rightarrow \begin{array}{l} x_0 = 8,1 \\ x_1 = 8,3 \\ x_2 = 8,6 \end{array} \rightarrow \begin{array}{l} (x_1, f(x_1)) \\ (x_2, f(x_2)) \end{array}$$

$$P_1(x) = \frac{x - x_2}{x_1 - x_2} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2)$$

$$P_1(x) = \frac{x - 8,6}{8,3 - 8,6} f(8,3) + \frac{x - 8,3}{8,6 - 8,3} f(8,6)$$

$$\Rightarrow 1/_{0,3} (x - 8,6)(17,56492) + 1/_{0,3} (x - 8,3)(18,50515)$$

$$\Rightarrow 3,1341x - 8.44811$$

$\therefore$ The approximation of the functional value $f(8,4)$ would be

$$P_1(8,4) = 3,1341(8,4) - 8.44811$$

$$\Rightarrow \underline{\underline{17.87833}}$$

For a quadratic interpolating polynomial we need **3** nodes. therefore we add $x_3 = 8,7$

The values will be $\rightarrow x_0 = 8,1 = f(8,1) = 16.94410$

$x_1 = 8,3 = f(8,3) = 17,56492$

$x_2 = 8,6 = f(8,6) = 18,50515$

$x_3 = 8,7 = f(8,7) = 18,82091$

Formular $\Rightarrow$ The polynomial is $P_2(x) = L_0(x) f(x_1) + L_1(x) f(x_2) + L_2(x) f(x_3)$

where

$$L_0(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} = \frac{(x-8,6)(x-8,7)}{(8,3-8,6)(8,3-8,7)} \Rightarrow 8,3333x^2 - 144,16667x + 623,5$$

$$L_1(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} \Rightarrow \frac{(x-8,3)(x-8,7)}{(8,6-8,3)(8,6-8,7)} \Rightarrow -33,3333x^2 + 586,6667x - 2407$$

$$L_2(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} = \frac{(x-8,3)(x-8,6)}{(8,7-8,3)(8,7-8,6)} \Rightarrow 25x^2 - 422,5x + 1784,5$$

From the formular $P_2(x) = L_0(x) f(x_1) + L_1(x) f(x_2) + L_2(x) f(x_3)$

$$P_2(x) \Rightarrow (8,3333x^2 - 144,16667x + 623,5)(f(8,3)) + (-33,3333x^2 + 566,6667x - 2407)(f(8,6))$$
$$+ (25x^2 - 422,5x + 1784,5)(f(8,7))$$
$$\Rightarrow 0,05875x^2 + 2,14123x - 4,254535$$

$\therefore$ The approximate value of $f(8,4)$ is:-

$$P_2(8,4) = 0,05875(8,4)^2 + 2,14123(8,4) - 4,254535$$

$$\Rightarrow \underline{\underline{17,8772}}$$

5a) For a cubic interpolating polynomial we will need all **4 nodes** therefore :-

$f(x_0) = 16.9441$, $f(x_1) = 17,56492$, $f(x_2) = 18,50515$, $f(x_3) = 18,182091$

The polynomial is $P_3(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3)$
where $L_k(x)$ for $k = 0,1,2,3$. is determined by theorm (3,2)

The values will be $x_0 - x_3$

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

$$\Rightarrow \frac{(x-8,3)(x-8,6)(x-8,7)}{(8,1-8,3)(8,1-8,6)(8,1-8,7)}$$

$$\Rightarrow -16.66667x^3 + 426.6667x^2 - 3640,1667x + 10350,1$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$\Rightarrow \frac{(x-8,1)(x-8,6)(x-8,7)}{(8,3-8,1)(8,3-8,6)(8,3-8,7)}$$

$$\Rightarrow 41,66667x^3 - 1058.33333x^2 + 8956.25x - 25251,75$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$\Rightarrow \frac{(x-8,1)(x-8,3)(x-8,7)}{(8,6-8,1)(8,6-8,3)(8,6-8,7)}$$

$$\Rightarrow -66,66667x^3 + 1673,3333x^2 - 13994x + 38993,4$$

$$L_3(x) = \frac{(x-x_0)(x-x_1(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$\Rightarrow \frac{(x-8,1)(x-8,3)(x-8,6)}{(8,7-8,1)(8,7-8,3)(8,7-8,6)}$$

$$\Rightarrow 41,66667x^3 - 1041,66667x^2 + 8677,91667x - 24090,75$$

$\therefore$ The Third degree lagrange interpolating polynomial is :-

$$P_3(x) = (-16,66667x^3 + 426.66667x^2 - 3640.1667x + 10350.1)f(8,1)$$
$$+(41,66667x^3 - 1058,33333x^2 + 8956,25x - 25251,75)f(8,3)$$
$$+(-66.66667x^3 + 1673,33333x^2 - 13994x + 38993,4)f(8,6)$$
$$+(41,66667x^3 - 1041,66667x^2 + 8677,91667x - 24090,75)f(8,7)$$
$$\Rightarrow -0,00208x^3 + 0,11207x^2 + 1,6862x - 2.96077$$

$\therefore$ The approximate value of $f(8,4)$ is

$$P_3(8,4) = -0,00208(8,4)^3 + 0,11207(8,4)^2 + 1,686(8,4) - 2,96077$$

$$\Rightarrow \underline{\underline{17,8773}}$$

3) b) $f(-\frac{1}{3})$ is $f(-0{,}75) = -0{,}07181250$, $f(-0{,}5) = -0{,}0247500$,
$f(-0{,}25) = 0{,}33493750$, $f(0) = 1{,}10100000$

$\therefore$ The linear Lagrange requires only **two nodes**:
As $x = -\frac{1}{3}$ is between the nodes $x_1 = -0{,}5$ and $x_2 = -0{,}25$, we are looking for the Lagrange linear polynomial through $(x_1, f(x_1))$ and $(x_2, f(x_2))$:

$$P_1(x) = \frac{x - x_2}{x_1 - x_2} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2)$$

The value of this polynomial at $x = -\frac{1}{3}$ is

$$P_1(-\tfrac{1}{3}) = \frac{(-\frac{1}{3}) - x^2}{x_1 - x_2} f(x_1) + \frac{(-\frac{1}{3}) - x_1}{x_2 - x_1} f(x_2)$$

$$\Rightarrow \frac{(-\frac{1}{3}) - (-0{,}\cancel{0}25)}{-0{,}5 - (-0{,}25)} f(0{,}5) + \frac{(-\frac{1}{3}) - (-0{,}5)}{\cancel{0{,}25}\ -0{,}25 - (-0{,}5)} f(-0{,}25)$$

$$\Rightarrow (\tfrac{1}{3}) \cdot (-0{,}02475) + (\tfrac{2}{3}) \cdot 0{,}3349375$$

$$\Rightarrow \underline{\underline{0{,}21504167}}$$

3b) For a quadratic interpolating polynomial we need **3 nodes** - we add node $x_3 = 0$, but node $x_0 = -0,75$ could be added instead.

$$f(x_1) = -0,02475, \quad f(x_2) = 0,3349375, \quad f(x) = f(0) = 1,101$$

The polynomial is: $P_2(x) = L_0(x) f(x_1) + L_1(x) f(x_2) + L_2(x) f(x_3)$

where L_0, L_1 are:

$$L_0(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} \Rightarrow \frac{(x+0,25)(x-0)}{(-0,5+0,25)(-0,5-0)} \Rightarrow 8(x+0,25)\cdot x$$

$$L_1(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} = \frac{(x+0,5)(x-0)}{(-0,25+0,5)(-0,25)} \Rightarrow -16(x+0,5)\cdot x$$

$$L_2(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} = \frac{(x+0,5)(x+0,25)}{(0+0,5)(0+0,25)} \Rightarrow 8(x+0,5)(x+0,25)$$

$L_k(x)$ for $k = 0,1,2$ at $x = -1/3$:

$$L_0(-1/3) = 8(-1/3 + 0,25)(-1/3) = 2/9$$

$$L_1(-1/3) = -16((-1/3)+0,5)\cdot(-1/3) = 8/9$$

$$L_2(-1/3) = 8((-1/3)+0,5)(-1/3+0,25) = -1/9$$

Now we approximate:

$$P_2(-1/3) = L_0(-1/3) f(x_1) + L_1(-1/2) f(x_2) + L_2(-1/3) f(x_3)$$

$$\Rightarrow 2/9 \cdot (-0,02475) + 8/9 \cdot 0,3349375 - 1/9 \cdot 1,101$$

$$\Rightarrow \underline{0,16988889}$$

For a Cubic interpolating we will need all **4 nodes** :-

$f(x_0) = -0{,}0718125$, $f(x_1) = -0{,}02475$, $f(x_2) = 0{,}3349375$, $f(x_3) = 1{,}101$

The polynomial is :-

$$P_3(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3)$$

Now $L_k(x)$ for $k = 0, 1, 2, 3$ as given by theorm 3,2.

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x+0{,}5)(x+0{,}25)(x-0)}{(-0{,}75+0{,}5)(-0{,}75+0{,}25)(-0{,}75-0)}$$

$$\Rightarrow -\frac{32}{3}(x+0{,}5)(x+0{,}5)x$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \Rightarrow \frac{(x+0{,}75)(x+0{,}25)(x-0)}{(-0{,}5+0{,}75)(-0{,}5+25)(-0{,}5)}$$

$$\Rightarrow 32(x+0{,}75)(x+0{,}25)x$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x+0{,}75)(x+0{,}5)(x)}{(-0{,}25+0{,}75)(-0{,}25+0{,}5)(-0{,}25-0)}$$

$$\Rightarrow -32(x+0{,}75)(x+0{,}5)x$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x+0{,}75)(x+0{,}5)(x+0{,}25)}{(0+0{,}75)(0+0{,}5)+(0+0{,}25)}$$

$$\Rightarrow \frac{32}{3}(x+0{,}75)(x+0{,}5)(x+0{,}25)$$

<u>Evaluating $f(x)$ using $P_3(x)$ at $x = -1/3$</u>

$$L_0\left(-\tfrac{1}{3}\right) = \frac{32}{3}\left(-\tfrac{1}{3}+0{,}5\right)\left(-\tfrac{1}{3}+0{,}25\right)\left(-\tfrac{1}{3}\right) \Rightarrow -0{,}04938272$$

$$L_1\left(-\tfrac{1}{3}\right) = 32\left(-\tfrac{1}{3}+0{,}75\right)\left(-\tfrac{1}{3}+0{,}25\right)\left(-\tfrac{1}{3}\right) \Rightarrow 0{,}37037037$$

$$L_2\left(-\tfrac{1}{3}\right) = -32\left(-\tfrac{1}{3}+0{,}75\right)\left(-\tfrac{1}{3}+0{,}5\right)\left(-\tfrac{1}{3}\right) \Rightarrow 0{,}7474074$$

$$L_3\left(-\tfrac{1}{3}\right) = \frac{32}{3}\left(-\tfrac{1}{3}+0{,}75\right)\left(-\tfrac{1}{3}+0{,}5\right)\left(-\tfrac{1}{3}+0{,}25\right) \Rightarrow -0{,}0617284$$

$$P_3\left(-\tfrac{1}{3}\right) = L_0\left(-\tfrac{1}{3}\right) f(-0{,}75) + L_1\left(-\tfrac{1}{3}\right) f(-0{,}5) + L_2\left(-\tfrac{1}{3}\right) f(-0{,}25) + L_3\left(-\tfrac{1}{3}\right) f(0)$$

$$\Rightarrow 0{,}04938272 \cdot (-0{,}0718125) + 0{,}37037037 \cdot (-0{,}02475) + 0{,}74074074 \cdot$$
$$0{,}3349375 - 0{,}061784 \cdot 1{,}101$$

$$\Rightarrow \underline{\underline{0{,}17451851}}$$