亚历克上 - M202161029

Numerical Exam

PART A

Final exam

亚历克上 – M202161029

2021 – 2022 – Exam

PART A

1)

Bisection

Advantages	disadvantages
– Brackets the root much quicker since it discharges 50% of the current interval. – Example: if the interval is between 0–1 it takes 6–7 functions evaluations to estimate the root to within 0,1 accuracy. – The root evaluation with 6–7 functions estimation is within 0,625 to 0,031 accuracy	– Difficult to find complex roots of polynomials. – Can be ~~easi~~ easily fooled by singularities in the functions. – Doesn't ~~wo~~ work when the function has a tangent to the x-axis.

Fixed Point Iteration

→ Global optimum → Pareto ranking → Converges → Formula: $X_{x+n} = F(x_n)$ Examples → Jacobi iterations and Gauss-Seidel.	→ not enough numbers covered. ~~Formu~~

Newton's Methods

Advantages	disadvantages
- Can identify repeated roots	- Can delay to find the root.
- It converges quicker	- Only works if we have a function representation of $f(x)$.
- Can find complex roots assuming we start at x_1	- Root is ~~gee~~ granted.
- Doesn't get fooled by singularity	- Hard to find the root for example $f(x) = \tan^{-1}(x)$
	- Can oscillate around the point due to change of sign.

2) Interpolation refers to estimation of unknown values.

~~Ag~~. Polynomial Approximation refers to an approximation of a curve with a polynomial.

Examples of Interpolation and polynomial approximation

(i) Taylors polynomial which illustrates that for a given function $f(x)$ and a point x_0, we approximate $f(x)$ by the Taylor's polynomial $P_n(x)$.

(ii) Neville's method which illustrates that we don't know which choice is better since true $f(x)$ is unknown, but we can compute all elements and see the trend.

→ the data is arranged closer to the interpolated point.

-The formular is:- $P(x) = (x-x_i) \dfrac{P_{0,i,\ldots,j+1,\ldots k}(x) - P_{0,1,i-1,i+1\ldots k}(x)}{x_i - x_j}$

Lagrange interpolating polynomial

Advantages	Disadvantages
→ Its more efficient when interpolating several functions on the same set of points	- The formular length of interpolation is large. - Its too tedious as the approximation increases.

Newtonian interpolating polynomial.

Advantages	disadvantages.
- Fast convergence - The length of interpolation formular is small. - No need to recompute as more interpolation points are added.	- convergence is not guaranteed - division by zero can occur - root jumping might take place - Inflection point issue might occur

3)

Gauss Elimination method

Advantages

- An augumented matrix is form from the linear systems.
- Operation are (i) $\lambda E_i \to E_i$
  (ii) $E_i + \lambda E_j \to E_i$
- It can solve more than 2 linear equations simultaneously.
- The procedure is $O(n^3/3)$.
- Useful for solving large problems

disadvantages

- The method require more operations
- Prone to false results.
-

Pivot strategies

- Its Gaussian with partial pivoting.

The formular is

$$a_{RK}^{(k)} = \max_{k \le i \le n} \left| a_{ik}^{(k)} \right|$$

- only converge for gradually dominant matrix.

The formular is

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

4) <u>Numerical differention</u> refers to the process of finding the numerical value of a derivative of a given function at a given point.

- <u>good example is Taylor series or lagrange interpolations</u>

Taylor's series expansion is defined as:-

$$f(x_i) = f(x_0) + \Delta x \frac{df}{dx}\bigg|_{x=x_0} + \frac{(\Delta x)^2}{2!}\frac{d^2f}{dx^2}\bigg|_{x=x_0} + \dots \qquad \Delta x = x_i - x_0$$

$$\Rightarrow f(x_1) = f(x_0) + (x_1 - x_0) f'(x_0) + \frac{(x_1 - x_0)^2}{2!} f''(x_0) + \frac{(x_i - x_0)^3}{3!} f'''(x_0) + \dots$$

Numerical Intergration refer to the approximation of a definite integration by a weighted sum of function values at discretized points within the interval of integration.

The formular is:-

$$\int_a^b f(x)\, dx \approx \sum_{i=0}^{N} w_i f(x_i)$$

Composite Trapezoidal rule	Composite Simpson's rule
→ Evaluates the area under the curves by dividing the total area into smaller trapezoids rather than using rectangles.	→ uses fundamental theorems of Calculus.
→ Works by approximating the region under the graph of a function as a trapezoid.	→ Used to evaluate definite ~~integ~~ integral.
→ takes the average of the left and the right sum.	→ Uses the quadratic approximation ~~insted~~ instead of linear approximation.
→ No accurate value when underlying function is smooth.	→ Provides accurate values when the underlying function is smooth.
	→
→ gives approximation value by. $\Delta x = \frac{(b-a)}{n}$ where ~~a=~~ $a = x_0 < x_1 < x_2 \dots < x_n = b$	→ gives approximation value by:- $\Delta x = \frac{b-a}{n}$ where $x_i = a + i\Delta x$

PART B

PART B

1a) Bisection method → P_3 → $f(x) = \sqrt{x} - \cos x = 0$

∴ We know that there is atleast one root.

$f(x) = \sqrt{x} - \cos x$

$f(0) = \sqrt{0} - \cos(0)$; $f(1) = \sqrt{1} - \cos(1)$

$f(0) = -1$; $f(1) = 0,459697$

$$P_1 = \frac{x_1 - x}{2} = \frac{0+1}{2} = \frac{1}{2}$$

$f(x) = \sqrt{x} - \cos x$

$f(0,5) = \sqrt{0,5} - \cos(0,5)$

$f(0,5) \Rightarrow \underline{\underline{-0,17}}$

Since $f(0)$ and $f(0,5) < 0$ new root is $[P_1, x_1] \Rightarrow [0,5 ; 1]$

$$P_2 = \frac{1 + 0,5}{2} = \frac{1,5}{2} = 0,75$$

$f(0,75) = \sqrt{0,75} - \cos(0,75)$

$\Rightarrow \underline{\underline{0,13}}$

Since P_1 and $P_2 < 0$ new root is $[P_1, x_1] \Rightarrow [0,5 ; 0,75]$

$$P_3 = \frac{0,5 + 0,75}{2} = \underline{\underline{0,625}}$$

b) $f(x) = x^2 - 6 = 0$ Find P_2 with $p_0 = -1$

$f(x) = x^2 - 6$ $\therefore p_0 = -1$

$f'(x) = 2x$

$f'(p_0) = 2(-1)$
$= -2$

$-2 \neq 0$
applicable.

$$P_n = P_{n-1} - \frac{f(P_{n-1})}{f'(P_{n-1})}$$

$$P_1 = -1 - \frac{(-1)^2 - 6}{2(-1)} \Rightarrow -1 - \frac{-5}{-2} = -3,5$$

$$P_2 = (-3,5) - \frac{(-3,5)^2 - 6}{2(-3,5)} \Rightarrow -4,392857143$$

$P_2 = -4,39286$

2)

~~x~~	~~x_0~~	~~1,3~~	1,9	1,4
~~$f(x)$~~	[illegible]	[illegible]	0,4555	[illegible]

$X_2 = 1,3$, $x_3 = 1,6$, $x_4 = 1,9$

degree I $= P_1(x) = L_0(x) f(x_0) + L_1(x) f(x_1) + \dots$

$$P_1(x) = \frac{x - x_3}{x_2 - x_3} f(x_2) + \frac{x - x_2}{x_3 - x_2} f(x_3)$$

$$\Rightarrow \frac{x - 1,6}{1,3 - 1,6} (0,65234) + x - 1,3 (0,4554)$$

$$\Rightarrow -0,548946x + 1,33376$$

Since $x = 1,5$ $P_1(1,5) \Rightarrow -0,549\ (1,5) + 1,3376$

$$\Rightarrow 0,5103$$

degree II $\quad X_2 = 1,3,\ X_3 = 1,6,\ x_4 = 1,9$

$$P_2(x) \Rightarrow \frac{(x - x_3)(x - x_4)}{(x_2 - x_3)(x_2 - x_4)} f(x_2) + \frac{(x - x_2)(x - x_4)}{x_3 - x_2\ (x_3 - x_4)} f(x_3) + \frac{(x - x_2)(x - x_3)}{(x_4 - x_2)(x_4 - x_3)} f(x_4)$$

$$\Rightarrow \frac{(x - 1,6)\ (x - 1,9)}{(1,3 - 1,6)\ (1,3 - 1,9)} (0,6200860) + \frac{(x - 1,3)\ (x - 1,6)}{(1,9 - 1,3)} (0,2818186)$$

Since $x = 1,5$ $P_2(1,5) \Rightarrow \frac{(1,5 - 1.6)\ (1,5 - 1,6)}{(1,3 - 1,6)(1,3 - 1,9)} \dots \dots 0,2818186)$

$$\Rightarrow \underline{\underline{0,51135}}$$

3)

$$\int_{-2}^{2} e^{x} x^{2}\, dx \quad , n = 4$$

The width of each subinterval is

$$\Delta x = h = \frac{b-a}{n} \Rightarrow \frac{2-(-2)}{4} = \frac{4}{4} = 1$$

$$T_4 = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$f(x_0) = f(-2) = e^{(-2)}(-2)^2 \Rightarrow 0{,}54134113$$

$$f(x_1) = f(-1) = e^{-1}(-1)^2 \Rightarrow 0{,}36787944$$

$$f(x_2) = f(0) \quad e^{0} 0^{2} \Rightarrow 0$$

$$f(x_3) = f(1) \quad e^{1} 1^{2} \Rightarrow 2{,}71828182$$

$$f(x_4) = f(2) \quad e^{2} 2^{2} \Rightarrow 29{,}55622439$$

$$T_4 \Rightarrow \frac{1}{2}\left[0{,}5413 + 2(0{,}36788) + 0 + 2(2{,}71828) + 29{,}556\right]$$

$$\Rightarrow 18{,}13494402$$

4) Jacobi iteration ~~&~~ Gauss - Seidel

$$\begin{bmatrix} 10 & -1 & -1 \\ -1 & 10 & -2 \\ -2 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6,2 \\ 8,5 \\ 3,2 \end{bmatrix}$$

$$10x_1 - x_2 - x_3 = 6,2$$
$$-x_1 + 10x_2 - 2x_3 = 8,5$$
$$-2x_1 - x_2 + 5x_3 = 3,2$$

$$x_1 = \frac{6,2 + x_2 + x_3}{10}$$

$$x_2 = \frac{8,5 + 2x_3 + x_1}{10}$$

$$x_3 = \frac{3,2 + 2x_1 + x_2}{5}$$

initial guess $(x_1, x_2, x_3) = (0, 0, 0)$

1st iteration $x_1^{(1)} = \frac{6,2}{10}$, $x_2^{(1)} = \frac{8,5}{10}$, $x_3^{(1)} = \frac{3,2}{5}$

IInd iteration $x_1^{(2)} = 6,2$, $x_2^{(2)} = 0,2$, $x_3^{(2)} = 0,13$

$$\therefore x_1 = \frac{1}{10}\left[6,2 + 0,2 + 0,13\right]$$

$$\Rightarrow 0,653$$

5.

$$\begin{cases} y' = -5y + 5t^2 + 2t \\ y(0) = \frac{1}{3} \end{cases}$$

$0 \le t \le 1$

$h = 0,1$

$n = 1 - 0 = 1$

$y' = -5y + 5t^2 + 2t$

t	y	$y' = -5y + 5t^2 + 2t$	new y
0,0	~~0,00~~ 1	1,00	1,00
0,1	1,10	1,20	1,22
0,2	1,22	1,42	1,36
0,3	1,36	1,66	1,53
0,4	1,53	1,93	0,3456
0,5	1,72	2,22	0,6654
0,6	1,44	2,54	0,47275
0,7	2,19	2,89	0,6321
0,8	2,48	3,29	0,783455
0,9	2,81	3,71	0,98039999
1,0	3,18		

$\Rightarrow 0,9803$