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Numerical Analysis
Assignment 5

亚历克上 M202161029 Chapter 4 : Ex1b, 2, 3a

1) Using forward-difference and backward-difference formulas to determine each missing entry.

b)

x	$f(x)$	$f'(x)$
0,0	0,0000	
0,2	0,74140	
0,4	1,3718	

$h = n - 1 = 0{,}3 - 0{,}1 = 0{,}2$

$\therefore f'(0{,}0),\ f'(0{,}2),\ f'(0{,}4)$

By Forward difference

Formula $f'(x_0) = \dfrac{f(x_0+h) - f(x_0)}{h}$

$$f'(0{,}0) = \frac{f(0{,}2) - f(0{,}4)}{0{,}2} \Rightarrow \frac{0{,}7414 - 0}{0{,}2} = 3{,}707$$

$$f'(0{,}2) = \frac{f(0{,}4) - f(0{,}2)}{0{,}2} = \frac{1{,}3718 - 0{,}7414}{0{,}2} = 3{,}152$$

By backward difference $f'(0{,}4)$

Formula $f'(x_0) = \dfrac{f(x_0) - f(x_0-h)}{h}$

$$f'(0{,}4) = \frac{f(0{,}4) - f(0{,}2)}{0{,}2} = \frac{1{,}3718 - 0{,}7414}{0{,}2} = 3{,}152$$

$\therefore f'(0{,}0) \approx 3{,}7070\ ;\ f'(0{,}2) \approx 3{,}1520\ ;\ f'(0{,}4) \approx 3{,}1520$

2) Compute the actual errors and find error bounds using the error formulas.

a) $f(x) = \sin x$

function

x	$f(x)$	
0,5	0,4794	
0,6	0,5646	
0,7	0,6442	

∴ for small values of $h = 0,2$;

$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h}$ error bound $\frac{Mh}{2}$

where M is a bound of $|f^2(x)|$ for $x \in [a,b]$.

∴ For forward-difference $h > 0$

∴ For backward-difference $h < 0$

The forward difference at $x_0 = 0,5$ for $h = 0,1$ of accuracy $O(h)$

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}$$

$$f'(0,5) = \frac{f(0,5+0,1) - f(0,5)}{0,1} = \frac{0,5646 - 0,4794}{0,1} = \underline{0,852}$$

$f'(x) = f(x) = \sin x \Rightarrow \cos x = \cos(0,5) = \underline{0,877582}$

Actual error $\Rightarrow 0,877582 - 0,852 = \underline{0,02558}$

Error bound $\Rightarrow |E_p| \leq \frac{k(b-a)^3}{12} \Rightarrow 0,255 \leq \frac{0,1(0,1-0)^3}{12}$

$\Rightarrow \underline{0,0282}$

$$f'(x_1) \approx \frac{f(0,6+0,1) - f(0,6)}{0,1} = \frac{0,6442 - 0,5646}{0,1} = \underline{0,796}$$

$f'(\sin(x)) = \frac{\cos x}{\cancel{\sin x}} \Rightarrow \frac{\cos x}{\cancel{\sin}}(0,6) \Rightarrow \cancel{0,843}\ \underline{0,82533}$

Actual error $\Rightarrow 0,82533 - 0,796 \Rightarrow \underline{0,0293}$

Error bound $\Rightarrow 0,0293 < \frac{0,1(0,1-0)^3}{12} = \underline{0,0282}$

$$f'(0,7) = \frac{f(x_n+h) - f(x_n)}{h}$$

$$= \frac{f(0,6) - f(0,5)}{0,2} = \frac{0,5646 - 0,4744}{0,2} = \underline{0,426}$$

$$f'(\sin x) = \cos x = \cos(0,7) = \underline{0,76484218728}$$

Actual error $\Rightarrow 0,7648 - 0,426 \Rightarrow \underline{0,3388}$

Error bound $\Rightarrow 0,3388 \leq \frac{0,2(0,2-0,1)^3}{12} \Rightarrow \underline{0,03645}$

$\therefore$ for $x = \sin x$

x	$f(x)$	$f'(x)$	$\cos x$	Actual Error	Error Bound
0,5	0,4744	0,852	0,8875	0,02558	0,0282
0,6	0,5646	0,746	0,68624	0,0243	0,0282
0,7	0,6442	0,426	0,7648	0,3388	0,03645

2b) $f(x) = e^x - 2x^3 + 3x - 1$

$h = 0,2$ for $O(h)$

$f'(x) = \frac{f(x_0+h) - f(x_0)}{h}$

$f'(0,0) = \frac{f(0,2) - f(0,0)}{0,2} = \frac{0,74140 - 0,000}{0,2} = 3,7070$

$f(x) = e^x - 2x^3 + 3x - 1$
$f'(x) = e^x - 4x + 3$
$f''(x) = e^x - 4$

Using the bound at

$\left|\frac{h}{2} f^{(2)}(\xi_l)\right| \le \frac{M}{2}\left(\max_{\epsilon_p \in [a,\ a+h]} \left|f^{(2)}(\epsilon_p)\right|\right) <$

$\frac{0,2}{2}\left(\max_{\epsilon_p \in [a,\ a+h]} \left|f^{(2)}(\epsilon_p)\right|\right) \le 0,1\left(\max_{\epsilon_p \in [a,\ a+h]} \left|f^{(2)}(\epsilon_p)\right|\right)$

$\therefore \left|\frac{h}{2} f^{(2)}(\epsilon_p)\right| \le 0{\cdot}1\left(\max_{\epsilon_p \in [a,\ a+h]} \left|e^{\epsilon_p} - 4\right|\right)$

e^x increases exponentially in $[0,0 \quad 0,4]$

$\max_{\epsilon_p \in [a,\ a+h]} \left|e^0 - 4\right| = 3$

Actual Error $\left|3,707 - (e^0 - 4 \times 0 + 3)\right| = \underline{0,293}$

with forward difference at $x_0 = 0,2$

$f'(0,2) = \frac{-f(0,2) + f(0,2+0,2)}{0,2} = \frac{1,3718 - 0,74140}{0,2} = \underline{3,152}$

$\therefore \max_{\epsilon_p \in [0,2 \quad 0,4]} \left|e^{\epsilon_p} - 4\right| = 4 - e^{0,2} = \underline{2,7786}$

$\therefore$ Actual error at 0,2 $\Rightarrow$

$$|3,152 - (e^{0,2} - 4 \times 0,2 + 3)| = |3,152 - 3,4214| = 0,2464$$

For backward difference at 0,2

$$f(x_0) = \cancel{f(x_0+h)} \cancel{\quad} \frac{f(x_0) - f(x_0+h)}{h}$$

$$f(0,2) = \frac{f(0,2) - f(0,2-0,2)}{0,2} = \frac{0,74140 - 0,0}{0,2} = 3,707$$

we know that e^x increases exponentially at [0,0 0,4]

$$\max_{E_P \in [0,0 \; 0,2]} |e^{E_P} - 4| = 4 - e^{0,0} = \underline{3}$$

$$\text{Actual error} = |3,707 - (e^{0,2} - 4 \times 0\text{-}2 + 3)| = |3,707 - e^{0,2} - 3,8| = \underline{0,2856}$$

$\Rightarrow$ Backward difference is:-

$$f'(0,2) = \frac{f(0,4) - f(0,2}{0,2} = 3,1520$$

we know that e^x increases exponentially at [0,0 0,4]

$$\max_{E_P \in [0,2 \; 0,4]} |e^{E_P} - 4| = 4 - e^{0,2} = \underline{2,7786}$$

$\therefore$ the error $\Rightarrow$ $0,1 \underline{(\max)} E_P [0,2 \; 0,4] \; |e^{E_P} - 4| = \underline{0,27780}$

Actual error $\Rightarrow |3,152 - (e^{0,4} - 4\,0,4 + 3) = \underline{0,2602}$

2b

x	$f(x)$	$f'(x)$	Actual Error	Error bound
0,0	0,00000	0,030703	0,2430	0,3000
0,2	0,74140	3,1520545	0,2694	0,2779
0,4	1,3718	3,1520545	0,2602	0,2779

3a) Using the most accurate three-point formula to determine each missing entry in the following table :-

	x	$f(x)$	$f'(x)$
x_0	1,1	9,025013	
x_1	1,2	11,02318	
x_2	1,3	13,46374	
x_3	1,4	16,44465	

using 3 points formula :-

$\Rightarrow f'(x_0) \times f'(x_1)$

$\Rightarrow f'(x_0) = \frac{1}{2h}\left[-3f(x_0) + 4f(x_1) - f(x_2)\right] = 17,7641$

$f'(x_1) = \frac{1}{2h}\left[-3f(x_1) + 4f(x_2) - f(x_3)\right] = 21,70385$

for $f(x_2) \times f'(x_3)$

$f'(x_2) = \frac{1}{2h}\left[f(x_0) - 4f(x_1) + 3f(x_2)\right] = 26,6173458$

$f'(x_3) = \frac{1}{2h}\left[f(x_1) - 4f(x_2) + 3f(x_3)\right] = 32,511$

$Error = \frac{h^2}{3} \cdot f^{(3)}(\varepsilon_p)$

Since $f(x) = e^{2x}$

$f'''(x) = 8e^{2x}$

$\left|f^{(III)}(\varepsilon_p)\right| = 8e^{2\varepsilon_p} \leq \frac{h^2}{3} 8e^{2 \times 1,4} = 0,43 \quad \varepsilon_p [1,1 \quad 1,4]$

$\therefore f'(x) = 2e^{2x}$

$f'(1,1) = 2e^{2x} = 18,050$

$f'(1,2) = 22,046$

$f'(1,3) = 26,927$

$f'(1,4) = 32,889$

x	$f(x)$	$f'(x)$	actual $f'(x)$
1,1	9,025013	17,7691	18,050
1,2	11,02310	21,70385	22,046
1,3	13,39674	26,617	26,927
1,4	16,44465	32,571	32,889