2021-2022 Final Exam

Part A

① Explain the advantages and disadvantages of Bisection, Fixed point and Newton's method for finding the roots of equation $f(x)=0$

a) Bisection method

Advantages		Disadvantages
The method always converges and for that reason it is often use as a starter for more efficient methods.	1	It is slow to converge (that is, N may become quite large before $\|P-P_n\|$ is sufficiently small.
	2	A good ~~approximation~~ intermediate approximation can be inadvertently discarded
	3	Complex roots of polynomial is ~~hard to~~ cannot be found.

b) Fixed Point method (if $f(P)=0$ with fixed point P $g(x)=x-f(x)$ or $f(x)=x-g(x)$.)

Advantages		Disadvantages
* It is the simplest method * Converges * formula : $x_{n+1}=f(x_n)$ ex: Jacobi iteration and Gauss-Seidel	1	If convergence exists, it can be slow

(c)

Newton's method

$$P_n = P_{n-1} - \frac{f(P_{n-1})}{f'(P_{n-1})}, \text{ for } n \geq 1$$

Advantages	Disadvantage
① Converges to the root quickly	① Cannot be continued if $f'(P_{n-1}) = 0$ for some n.
② Easy to convert to multiple ~~dimention~~ dimension..	② Derivative must be derived first before using this method
③ The number of significant digits doubles with each step as we go near the root.	③ Depeddent on initial guess ~~④ may loop indefinitely~~
④ It converges on the root quadratically	④ The local root may be far far.

② Explain the idea of Interpolation and polynomial Approximation, the advantages and disadvantages of Lagrange Interpolating polynomial and Newtonian Interpolating polynomial.

~~Ans: Interpol~~

Ans: Interpolation is a method of precisely finding some quantity by known individual values of it or other quantities related to it. On the basic of Interpolation a whole series of approximate method for solving problem has been developed.

and Polynomial approximation:

refers to, the estimation of unknown value to be found.

Examples of Interpolation and Polynomial Approximation are as follow: ⓐ Taylor's polynomial ⓑ Neville's method and etc.

(2 continue)

(9) Lagrange Interpolating polynomial is given by

$\Rightarrow f(x_k) = P(x_k)$ for each $k = 0, 1, 2 \cdots n$

~~Advantages~~ $P(x) = f(x_0) L_{n,0}(x) + \cdots f(x_n) L_{n,n}(x)$

$= \sum_{k=0}^{n} f(x_k) L_{n,k}(x)$ where $k = 0, 1, \cdots n$

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1) \cdots \cdots (x-x_{k-1})(x-x_{k+1}) \cdots (x-x_n)}{(x_k-x_0)(x_k-x_1) \cdots (x_k-x_{k-1})(x_k-x_{k+1}) \cdots (x_k-x_n)}$$

$L_{n,k}(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x-x_i}{x_k-x_i}$ the ~~P~~

Advantage	Disadvantage
Efficient when interpolation many functions on the same set of points	The length of interpolation formula is large
	As approximation increases, it becomes tedious
Suitable for arithmetic operation of same data points but different value	Error term is difficult to apply and until all component are determined, the degree of polynomial needed for desired accuracy is not known

(9) Newtonian Interpolating ~~method~~ Polynomial:

In general, an $(n-1)^{th}$ Newton interpolating polynomial has all the terms of the $(n-2)^{th}$ polynomial one extra. The general formula is: $f_{n-1}(x) = b_1 + b_2(x-x_1) + \cdots b_n(x-x_1)(x-x_2) \cdots (x-x_{n-1})$

where $b_1 = f(x_1)$, $b_2 = f[x_2, x_1]$, $b_3 = f[x_3 \ldots x_2, x_1]$

$b_n = f[x_n, x_{n-1}, \ldots x_2, x_1]$ and $f[\cdot\cdot]$ represent divided difference.

Newtonian Interpolating Polynomial

Advantage	Disadvantages
(1) If more interpolation points are added. then there is no need to recompute the previously. Computed interpolation	Convergence to not generated
(2) The length of interpolation formal is small	Root might be jumped
(3) Convergence is fast	Inflection point may happened
	Decision by zero may occure.

③ Part A

Gaussian Elimination method is a step process where an argumented matrix is form from linear system of equation

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & & a_{2n} & \vdots & b_2 \\ \vdots & & & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} & \vdots & b_n \end{bmatrix}$$

Advantages

* It can solve more than 2 linear equations simultaneously
* Requires less computation for longer problems.

Disadvantages

* Solution of one set of linear equation at a time. The method
* calls for more operation by performing backward substitution of each augumented matrices
*

③ Explain the idea of direct method. (Gauss elimination method, pivoting strategies) and Iterative Technique for solving linear system of equations:

ⓐ Gaussian Elimination with partial pivoting where

$$a_{pk}^{(k)} = \max_{k \le i \le n} |a_{ik}^{(k)}| \text{ and perform } E(k) \leftrightarrow E(P)$$

(Interchange is done when there is no Column)

ⓑ Scale pivoting techniques:

① $S_i = \max_{i \le j \le n} |a_{ij}|$ ③ $\frac{|a_{pi}|}{s_p} = \max_{i \le k \le} \frac{|a_{ki}|}{s_k}$ and performing $(E_i) \leftrightarrow E_P$

③ Select the smallest integer $P \ge i$ with $\frac{|a_{pi}|}{s_p} = \max_{i \le k \le n} \frac{|a_{ki}|}{s_k}$

④ $i \ne P$

(4) Explain the Idea of numerical differentiation and integration, the of Composite Trapezoidal rule and Composite Simpson's rule :

Numerical differentiation is the process of the numerical value of a derivative of a given function at some point.

Example : Lagrange Interpolation and Taylor Series

The Taylor's series can be expanded as written below :

$$f(x) = f(x_0) + \Delta x \left.\frac{dy}{dx}\right|_{x=x_0} + \frac{(\Delta x)^2}{2!} \left.\frac{d^2 f}{dx^2}\right|_{x=0} + \dots \quad \Delta x = x_i - x_0$$

and it can be expand below as :

$$f(x_i) = f(x_0) + (x_i - x_0) f'(x_0) + \frac{(x_i - x_0)^2}{2!} + f''(x_0) + \frac{(x_i - x_0)^3}{3!} f'''(x_0) + \dots$$

And numerical integration is the approximation of a definite integration by weighted sum of functions. Numerical integration uses the same information to compute numerical approximation of derivatives and integral for a function that is only known at isolated points.

$$\int_a^b f(x)\,dx = \sum_{k=0}^{N} w_i f(x_i)$$

Composite Trapezoidal Rule	Composite Simpson's Rule
* Operates by approximating the area under the curve of the function as trapezoid	* It gives the approximation value by the below formula $\Delta x = \frac{b-a}{n}$ where $x_1 = a + \Delta x$
* It takes the left and right averages	* It is used for definite integral
* When underlying function is smooth, no accurate value	* When the underlying function is smooth, it provides accurate value.
* The approximation value is given by: $\Delta x = \frac{(b-a)}{n}$ and $a = x_0 < x_1 < x_2 \dots < x_n = b$	* It uses the fundamental of Calculus.
* It works by evaluating the area under the curve by dividing the total area into trapezoids.	

Part A

5) Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & -3 \\ -2 & 4 & 5 \end{bmatrix}$, to Compute $\|A\|_1 = .\ \|A\|_\infty$

$$\|A\|_1 \Rightarrow \max_{1 \le j \le n} \sum_{K=1}^{n} |a_{Kj}| \Rightarrow \sum_{K=1}^{3} |a_{1j}| = |1| + |1| + |-2| = -4$$

$$\Rightarrow \sum_{K=1}^{3} |a_{1j_2}| = |1| + |-2| + |4| = 7$$

$$\Rightarrow \sum_{K=1}^{3} |a_{j_3}| = |2| + |-3| + |5| = 10$$

$$\therefore \|A\|_1 = \max\{4, 7, 10\} = \underline{10}$$

$$\|A\|_\infty \Rightarrow A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & -3 \\ -2 & 4 & 5 \end{bmatrix}$$

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1} |a_{ij}|$$

$$= \sum_{j=1}^{3} |a_{1j}| = |1| + |-1| + |2| = 4$$

$$= \sum_{j=1}^{3} |a_{2j}| = |1| + |-2| + |-3| = 6$$

$$= \sum_{j=1}^{3} |a_{3j}| = |-2| + |4| + |5| = 11$$

$$\therefore \|A\|_\infty = \max\{4, 6, 11\} = \underline{11}$$

Part B

(1) **Bisection method:**

$P_3 \Rightarrow f(x) = \sqrt{x} - \cos x = 0$

Soln:

$f(x) = \sqrt{x} - \cos x$ $\quad f(1) = \sqrt{1} - \cos(1)$

$f(0) = \sqrt{0} - \cos(0)$ $\quad f(1) = 0.459697$

$f(0) = -1$

$P_1 = \frac{x_1 + x}{2} = \frac{0+1}{2} = 0.5$

$f(x) = \sqrt{x} - \cos x$

$f(0.5) = \sqrt{0.5} - \cos(0.5)$

$f(0.5) = -0.17$

Because $f(0)$ and $f(0.5) < 0$ a new root is

$[P_1, x_1] \Rightarrow [0.5, 1]$

$P_2 = \frac{1+0.5}{2} = 0.75$

$f(0.75) = \sqrt{0.75} - \cos(0.75) = 0.13$

Since P_1 and $P_2 < 0$ New root $= [P_1, x_1] \Rightarrow [0.5, 0.75]$

$\therefore P_3 = \frac{0.5 + 0.75}{2} = 0.625$

(b) **Using Newton's method**

$f(x) = x^2 - 6 = 0$ Find P_2 with $P_0 = -1$

$f(x) = x^2 - 6$, ~~$f'(x) = 2x$~~ $f'(x) = 2x$

$f'(P_0) = 2(-1) = -2$ $\quad -2 \neq 0$

Thus is applicable

$P_n = P_{n-1} - \frac{f(P_{n-1})}{f'(P_{n-1})} = -1 - \frac{(-1)^2 - 6}{2(-1)} = -3.5 \Rightarrow P_1$

$P_2 \Rightarrow (-3.5) - \frac{(-3.5)^2 - 6}{2(-3.5)} = -2.607142857$

Part B

(2) Langrange Interpolating polynomial of degree 1 and of degree 2 by $P_{2,3,4}(1.5)$

Table 1

X	1.0	1.3	1.6	1.9
$f(X)$	0.7651977	0.6200860	0.4554022	0.2818186

Considering: $X_2 = 1.3$
$X_3 = 1.6$
$X_4 = 1.9$

1^{st} Degree Lagrang Interpolating:

$$P_2(x) = L_0(x) f(x_0) + L_1(x) f(x_1) + \dots$$

$$P_2(x) = \frac{x - x_3}{x_2 - x_3} f(x_2) + \frac{x - x_2}{x_3 - x_2} f(x_3)$$

$$P_2(x) = \frac{x - 1.6}{1.3 - 1.6}(0.6200860) + \frac{x - 1.3}{1.6 - 1.3}(0.4554022)$$

$$P_2(x) = -0.548946x + 1.3337158$$

$$P_2(1.5) = -0.548946(1.5) + 1.3337158$$

$$\boxed{P_2(1.5) = 0.5102968}$$

For Two degree! we Consider $x_2 = 1.3$
$x_3 = 1.6$
$x_4 = 1.9$

$$P_2(x) = L_0(x) f(x_2) + L_1(x) f(x_3) + L_2(x) f(x_4)$$

$$P_2(x) = \frac{(x - x_3)(x - x_4)}{(x_2 - x_3)(x_2 - x_4)} f(x_2) + \frac{(x - x_2)(x - x_4)}{(x_3 - x_2)(x_3 - x_4)} f(x_3) + \frac{(x - x_2)(x - x_3)}{(x_4 - x_2)(x_4 - x_3)} f(x_4)$$

$$P_2(x) = \frac{(x - 1.6)(x - 1.9)}{(1.3 - 1.6)(1.3 - 1.9)}(0.6200860) + \frac{(x - 1.3)(x - 1.9)}{(1.6 - 1.3)(1.6 - 1.9)}(0.4554022) + \frac{(x - 1.3)(x - 1.6)}{(1.9 - 1.3)(1.9 - 1.6)}(0.2818186)$$

$$P_2(1.5) = \frac{(1.5 - 1.6)(1.5 - 1.9)}{(1.3 - 1.6)(1.3 - 1.9)}(0.6200860) + \frac{(1.5 - 1.3)(1.5 - 1.9)}{(1.6 - 1.3)(1.6 - 1.9)}(0.4554022) + \frac{(1.5 - 1.3)(1.5 - 1.6)}{(1.9 - 1.3)(1.9 - 1.6)}(0.2818186)$$

$$P_2(1.5) = 0.1377968889 + 0.4048019556 + (-0.0312424288 9)$$

$$\therefore P_2(1.5) = 0.511356415 6$$

Part B

(3) Use the trapezoidal rule with value of 4 to approximate the integral $\int_{-2}^{2} e^{x} x^{2}\, dx$, $n=4$

$\Delta x = h = \frac{b-a}{n} \Rightarrow \frac{2+2}{4} = 1$

$I_4 = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$

$f(x_0) = f(-2) = e^{(-2)}(-2) = 0.54134113$

$f(x_1) = f(-1) = e^{-1}(-1)^2 = 0.36787944$

$f(x_2) = f(0) = e^{0}(0)^2 = 0$

$f(x_3) = f(1) = e^{1}(1)^2 = 2.71828182$

$f(x_4) = f(2) = 29.55622439$

$\therefore I_4 = \frac{1}{2}\left[0.54134113 + 2(0.36787944) + 0 + 2(2.71828182) + 29.55622439\right]$

$I_4 = 18.13494402$

(4)(a) Jacobi iteration

$$\begin{bmatrix} 10 & -1 & -1 \\ -1 & 10 & -2 \\ -2 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6.2 \\ 8.5 \\ 3.2 \end{bmatrix}$$

$$\Rightarrow \quad 10x_1 - x_2 - x_3 = 6.2$$
$$-x_1 + 10x_2 - 2x_3 = 8.5$$
$$-2x_1 - x_2 + 5x_3 = 3.2$$

$$x_1 = \frac{6.2 + x_2 + x_3}{10} \qquad x_2 = \frac{8.5 + 2x_3 + x_1}{10} \qquad x_3 = \frac{3.2 + 2x_1 + x_2}{5}$$

Initially we guess $(x_1, x_2, x_3) \Rightarrow (0, 0, 0)$

1st iteration $= x_1^{(1)} = \frac{6.2}{10}$, $x_2^{(1)} = \frac{8.5}{10}$, $x_3^{(1)} = \frac{3.2}{10}$

II iteration $= x_1^{(2)} = 6.2$, $x_2^{(2)} = 0.2$, $x_3^{(2)} = 0.13$

$$\therefore \; x_1 = \frac{1}{10}[6.2 + 6.2 + 0.13]$$

$$\boxed{x_1 = 0.653}$$

Part B

5) $\begin{cases} y' = -5y + 5t^2 + 2t \\ y(0) = \frac{1}{3} \end{cases}$ $\quad 0 \le t \le 1$

$y(0) = \frac{1}{3}$, $h = 0.1$

$y' = -5y + 5t^2 + 2t$

Using $w_{i+1} = w_i + hf(t_i, w_i)$

$y = f(t, y)$.

$y(a) = \alpha$

$h = \frac{b-a}{N}$, $N = \frac{b-a}{h} = \frac{1}{0.1} = 10$

$y_1 = y_0 + hf(t_0, y_0)$

$y_1 = \frac{1}{3} + 0.1[-5(1/3) + 5(0)^2 + 2(0)] = 1/6$

$y_2 = \frac{1}{6} + 0.1[-5(1/6) + 5(1/10)^2 + 2(1/10)] = 13/120$

$y_3 = y_2 + 0.1[-5(y_2) + 5(0.2) + 2(0.2)] = 0.114167$

$y_4 = y_3 + 0.1[5(y_3) + 5(1/3)^2 + 2t_3] = 0.1620835$

$y_5 = y_4 + 0.1[-5(y_5) + 5(t_4)^2 + 2t_4] = 0.24104175$

$y_6 = y_5 + 0.1[-5(y_5) + 5t_4^2 + 2t_5] = 0.345520875$

$y_7 = y_6 + 0.1[-5(y_6) + 5t_6^2 + 2t_6] = 0.4727604375$

$y_8 = y_7 + 0.1[-5y_7 + 5t_7^2 + 2t_7] = 0.621382188$

$y_9 = y_8 + 0.1[-5y_8 + 5t_8^2 + 2t_8] = 0.7906961094$

$y_{10} = y_9 + 0.1[-5y_9 + 5t_9^2 + 2t_9] = 0.9803450547$

N	t_i	y_i
0	0	1/3
1	0.1	1/6
2	0.2	13/120
3	0.3	0.114167
4	0.4	0.1620835
5	0.5	0.24104175
6	0.6	0.345520875
7	0.7	0.4727604375
8	0.8	0.621382188
9	0.9	0.7906901094
10	1	0.9803450547