

Digital Communications

Chapter 2: Deterministic and Random Signal Analysis

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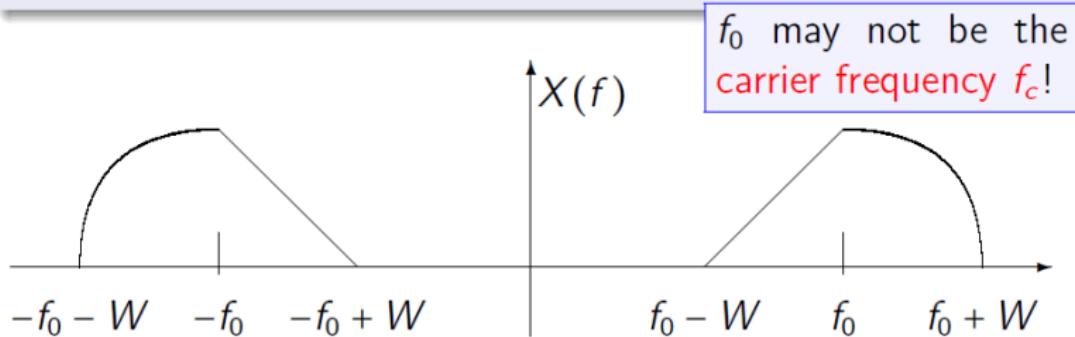
2.1 Bandpass and lowpass signal representation

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Definition (Bandpass signal)

A *bandpass signal* $x(t)$ is a *real* signal whose frequency content is located around *central frequency* f_0 , i.e.

$$X(f) = 0 \quad \text{for all } |f \pm f_0| > W$$



- The spectrum of a bandpass signal is *Hermitian symmetric*, i.e., $X(-f) = X^*(f)$. (Why? Hint: Fourier transform.)

2.1 Bandpass and lowpass signal representation

- Since the spectrum is Hermitian symmetric, we only need to retain half of the spectrum $X_+(f) = X(f)u_{-1}(f)$ (named analytic signal or pre-envelope) in order to analyze it,

$$\text{where } u_{-1}(f) = \begin{cases} 1 & f > 0 \\ \frac{1}{2} & f = 0 \\ 0 & \text{else} \end{cases}$$

Note: $X(f) = X_+(f) + X_+^*(-f)$

- A bandpass signal is very “real”, but may contain “unnecessary” content such as the carrier frequency f_c that is nothing to do with the “digital information” transmitted.
- So, it is more convenient to remove this carrier frequency and transform $x(t)$ into its lowpass equivalent signal $x_l(t)$ before “analyzing” the digital content.

2.1 Bandpass and lowpass signal representation -

Baseband and bandpass signals

Definition (Baseband signal)

- A **lowpass or baseband** (equivalent) signal $x_l(t)$ is a complex signal (because it is not necessarily Hermitian symmetric!) whose spectrum is located around zero frequency, i.e.

$$X_l(f) = 0 \text{ for all } |f| > W$$

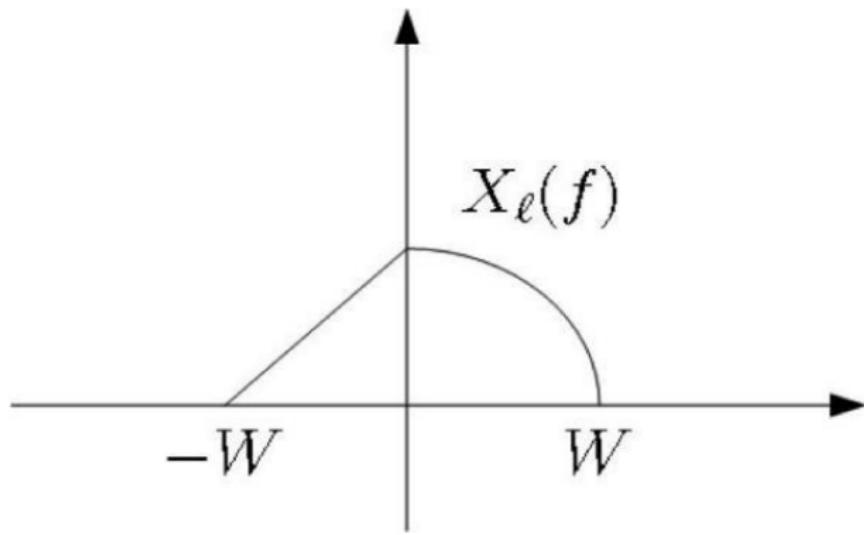
- It is generally written as

$$x_l(t) = x_i(t) + jx_q(t)$$

Where $x_i(t)$ is called the in-phase signal; $x_q(t)$ is called the quadrature signal

Baseband signal

Our goal is to relate $x_l(t)$ to $x(t)$ and vice versa



From lowpass equivalent $x_l(t)$ to $x(t)$

Definition of bandwidth. The bandwidth of a signal is one half of the entire range of frequencies over which the spectrum is (essentially) nonzero. Hence, W is the bandwidth in the lowpass signal we just defined, while $2W$ is the bandwidth of the bandpass signal by our definition

Analytic signal

- Let's start from the analytic signal $x_+(t)$.

$$\begin{aligned}x_+(t) &= \int_{-\infty}^{\infty} X_+(f) e^{j2\pi ft} df \\&= \int_{-\infty}^{\infty} X(f) u_{-1}(f) e^{j2\pi ft} df \\&= \mathcal{F}^{-1} \{X(f) u_{-1}(f)\} \quad \boxed{\mathcal{F}^{-1} \text{ Inverse Fourier transform}} \\&= \mathcal{F}^{-1} \{X(f)\} * \mathcal{F}^{-1} \{u_{-1}(f)\} \\&= x(t) * \left(\frac{1}{2} \delta(t) + j \frac{1}{2\pi t} \right) \\x_+(t) &= \frac{1}{2} x(t) + j \frac{1}{2} \hat{x}(t),\end{aligned}$$

where $\hat{x}(t) = x(t) * \frac{1}{\pi t} = \int_{-\infty}^{\infty} \frac{x(\tau)}{\pi(t-\tau)} d\tau$ is a **real-valued** signal.

Appendix: Extended Fourier transform

$$\begin{aligned}\mathcal{F}^{-1}\{2u_{-1}(f)\} &= \mathcal{F}^{-1}\{1 + \operatorname{sgn}(f)\} \\ &= \mathcal{F}^{-1}\{1\} + \mathcal{F}^{-1}\{\operatorname{sgn}(f)\} = \delta(t) + i \frac{1}{\pi t}\end{aligned}$$

Since $\int_{-\infty}^{\infty} |\operatorname{sgn}(f)| = \infty$, the inverse Fourier transform of $\operatorname{sgn}(f)$ does not exist in the **standard** sense! We therefore have to derive its inverse Fourier transform in the **extended** sense!

$$\begin{aligned}(\forall f) S(f) &= \lim_{n \rightarrow \infty} S_n(f) \text{ and } (\forall n) \int_{-\infty}^{\infty} |S_n(f)| df < \infty \\ \Rightarrow \mathcal{F}^{-1}\{S(f)\} &= \lim_{n \rightarrow \infty} \mathcal{F}^{-1}\{S_n(f)\}.\end{aligned}$$

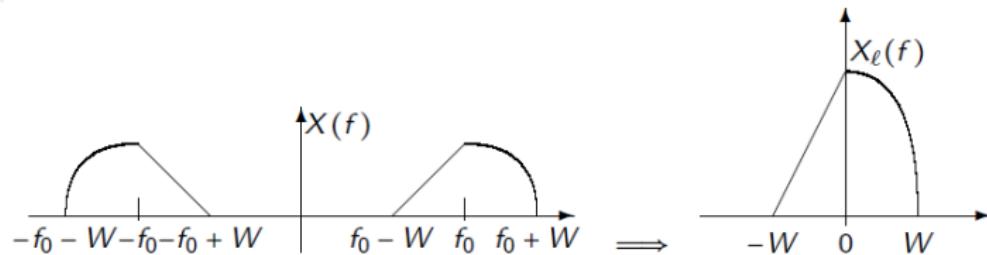
Appendix: Extended Fourier transform

Since $\lim_{a \downarrow 0} e^{-a|f|} \operatorname{sgn}(f) = \operatorname{sgn}(f)$,

$$\begin{aligned}& \lim_{a \downarrow 0} \int_{-\infty}^{\infty} e^{-a|f|} \operatorname{sgn}(f) e^{i 2\pi f t} df \\&= \lim_{a \downarrow 0} \left[- \int_{-\infty}^0 e^{f(a + i 2\pi t)} df + \int_0^{\infty} e^{f(-a + i 2\pi t)} df \right] \\&= \lim_{a \downarrow 0} \left[-\frac{1}{a + i 2\pi t} + \frac{1}{a - i 2\pi t} \right] \\&= \lim_{a \downarrow 0} \left[\frac{i 4\pi t}{a^2 + 4\pi^2 t^2} \right] = \begin{cases} 0 & t = 0 \\ i \frac{1}{\pi t} & t \neq 0 \end{cases}\end{aligned}$$

Hence, $\mathcal{F}^{-1} \{2u_{-1}(f)\} = \mathcal{F}^{-1} \{1\} + \mathcal{F}^{-1} \{\operatorname{sgn}(f)\} = \delta(t) + i \frac{1}{\pi t}$.

From $x_+(t)$ to $x_l(t)$



- We then observe

$$X_l(f) = 2X_+(f + f_0).$$

- This implies

$$\begin{aligned}x_l(t) &= \mathcal{F}^{-1}\{X_l(f)\} \\&= \mathcal{F}^{-1}\{2X_+(f + f_0)\} \\&= 2x_+(t)e^{-\imath 2\pi f_0 t} \\&= (x(t) + \imath \hat{x}(t))e^{-\imath 2\pi f_0 t}\end{aligned}$$

As a result,

$$x(t) + \imath \hat{x}(t) = x_\ell(t) e^{\imath 2\pi f_0 t}$$

which gives:

$$\color{red} x(t) \quad \left(= \mathbf{Re} \{ x(t) + \imath \hat{x}(t) \} \right) \quad = \mathbf{Re} \{ x_\ell(t) e^{\imath 2\pi f_0 t} \}$$

By $x_\ell(t) = x_i(t) + \imath x_q(t)$,

$$\begin{aligned} \color{red} x(t) &= \left(= \mathbf{Re} \{ (x_i(t) + \imath x_q(t)) e^{\imath 2\pi f_0 t} \} \right) \\ &= x_i(t) \cos(2\pi f_0 t) - x_q(t) \sin(2\pi f_0 t) \end{aligned}$$

From $X_\ell(f)$ to $X(f)$

$$\begin{aligned}
 &= \int_{-\infty}^{\infty} \frac{1}{2} \left[x_\ell(t) e^{i2\pi f_0 t} + (x_\ell(t) e^{i2\pi f_0 t})^* \right] e^{-i2\pi ft} dt \\
 &= \frac{1}{2} \int_{-\infty}^{\infty} x_\ell(t) e^{-i2\pi(f-f_0)t} dt \\
 &\quad + \frac{1}{2} \int_{-\infty}^{\infty} x_\ell^*(t) e^{-i2\pi(f+f_0)t} dt \\
 &= \frac{1}{2} [X_\ell(f - f_0) + X_\ell^*(-f - f_0)]
 \end{aligned}$$

$$X_\ell^*(-f) = \int_{-\infty}^{\infty} (x_\ell(t) e^{-i2\pi(-f)t})^* dt = \int_{-\infty}^{\infty} x_\ell^*(f) e^{-i2\pi ft} dt$$

Terminologies & relations

- Bandpass signal

$$\begin{cases} x(t) = \operatorname{Re} \{x_\ell(t)e^{\imath 2\pi f_0 t}\} \\ X(f) = \frac{1}{2} [X_\ell(f - f_0) + X_\ell^*(-f - f_0)] \end{cases}$$

- Analytic signal or pre-envelope $x_+(t)$ and $X_+(f)$
- Lowpass equivalent signal or complex envelope

$$\begin{cases} x_\ell(t) = (x(t) + \imath \hat{x}(t))e^{-\imath 2\pi f_0 t} \\ X_\ell(f) = 2X(f + f_0)u_{-1}(f + f_0) \end{cases}$$

Useful to know

Terminologies & relations

- From $x_\ell(t) = x_i(t) + \imath x_q(t) = (x(t) + \imath \hat{x}(t)) e^{-\imath 2\pi f_0 t}$,

$$\begin{cases} x_i(t) = \mathbf{Re} \left\{ (x(t) + \imath \hat{x}(t)) e^{-\imath 2\pi f_0 t} \right\} \\ x_q(t) = \mathbf{Im} \left\{ (x(t) + \imath \hat{x}(t)) e^{-\imath 2\pi f_0 t} \right\} \end{cases}$$

- Also from $x_\ell(t) = (x(t) + \imath \hat{x}(t)) e^{-\imath 2\pi f_0 t}$,

$$\begin{cases} x(t) = \mathbf{Re} \left\{ x_\ell(t) e^{\imath 2\pi f_0 t} \right\} \\ \hat{x}(t) = \mathbf{Im} \left\{ x_\ell(t) e^{\imath 2\pi f_0 t} \right\} \end{cases}$$

Useful to know

Terminologies & relations

- From $x_\ell(t) = x_i(t) + \imath x_q(t) = (x(t) + \imath \hat{x}(t)) e^{-\imath 2\pi f_0 t}$,

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- Also from $x_\ell(t) = (x(t) + \imath \hat{x}(t)) e^{-\imath 2\pi f_0 t}$,

$$\begin{cases} x(t) = \mathbf{Re} \left\{ (x_i(t) + \imath x_q(t)) e^{\imath 2\pi f_0 t} \right\} \\ \hat{x}(t) = \mathbf{Im} \left\{ (x_i(t) + \imath x_q(t)) e^{\imath 2\pi f_0 t} \right\} \end{cases}$$

Terminologies & relations

- pre-envelope $x_+(t)$
- complex envelope $x_\ell(t)$
- envelope $|x_\ell(t)| = \sqrt{x_i^2(t) + x_q^2(t)} = r_\ell(t)$

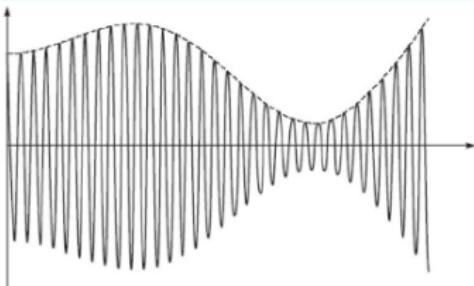


FIGURE 2.1-4

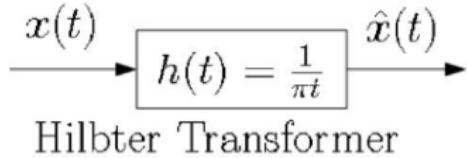
A bandpass signal. The dashed curve denotes the envelope.

- phase $\theta_\ell(t) = \arctan[x_q(t)/x_i(t)]$

Modulator/demodulator and Hilbert transformer

Usually, we will modulate and demodulate with respect to carrier frequency f_c , which may not be equal to the center frequency f_0 .

- $x_\ell(t) \rightarrow x(t) = \operatorname{Re} \{x_\ell(t)e^{i2\pi f_c t}\} = \text{modulation}$
- $x(t) \rightarrow x_\ell(t) = (x(t) + i\hat{x}(t))e^{-i2\pi f_c t} = \text{demodulation}$
- The modulation requires to generate $\hat{x}(t)$, a Hilbert transform of $x(t)$



Hilbert transform is basically a 90-degree phase shifter.

$$H(f) = \mathcal{F}\left\{\frac{1}{\pi t}\right\} = -i \operatorname{sgn}(f) = \begin{cases} -i, & f > 0 \\ 0, & f = 0 \\ i, & f < 0 \end{cases}$$

Recall that on page 10, we have shown

$$\mathcal{F}^{-1}\{\operatorname{sgn}(f)\} = i \frac{1}{\pi t} \mathbf{1}\{t \neq 0\};$$

hence

$$\mathcal{F}\left\{\frac{1}{\pi t}\right\} = \frac{1}{i} \operatorname{sgn}(f) = -i \operatorname{sgn}(f).$$

Tip: $x_+(t) = \frac{1}{2}[x(t) + i\hat{x}(t)] \Rightarrow X_+(f) = \frac{1}{2}[X(f) + i\hat{X}(f)] = \begin{cases} X(f) & f > 0 \\ 0 & f < 0 \end{cases} \Rightarrow$

$$i\hat{X}(f) = iX(f)H(f) = \begin{cases} X(f) & f > 0 \\ -X(f) & f < 0 \end{cases} \Rightarrow iH(f) = \begin{cases} 1 & f > 0 \\ -1 & f < 0 \end{cases}$$

Example: $\sin(2\pi f_c t) = \cos(2\pi f_c t) * h(t) = \cos(2\pi f_c t - \pi/2)$

Energy considerations

Definition (Energy of a signal)

- The energy \mathcal{E}_s of a (complex) signal $s(t)$

$$\mathcal{E}_s = \int_{-\infty}^{\infty} |s(t)|^2 dt$$

Hence,

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$\mathcal{E}_{x_+} = \int_{-\infty}^{\infty} |x_+(t)|^2 dt$$

$$\mathcal{E}_{x_\ell} = \int_{-\infty}^{\infty} |x_\ell(t)|^2 dt$$

We are interested in the connections among \mathcal{E}_x , \mathcal{E}_{x_+} , and \mathcal{E}_{x_ℓ}

- From Parseval's Theorem we see

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

- Secondly

- Parsevals theorem: $\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$
 - Rayleigh's theorem: $\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$

$$X(f) = \underbrace{\frac{1}{2}X_\ell(f - f_c)}_{=X_+(f)} + \underbrace{\frac{1}{2}X_\ell^*(-f - f_c)}_{=X_+^*(-f)}$$

- Thirdly, $f_c \gg W$ and

$$X_\ell(f - f_c)X_\ell^*(-f - f_c) = 4X_+(f)X_+^*(-f) = 0 \text{ for all } f$$

It then shows

$$\begin{aligned}\mathcal{E}_x &= \int_{-\infty}^{\infty} \left| \frac{1}{2} X_\ell(f - f_c) + \frac{1}{2} X_\ell^*(-f - f_c) \right|^2 df \\ &= \frac{1}{4} \mathcal{E}_{x_\ell} + \frac{1}{4} \mathcal{E}_{x_\ell}^* = \frac{1}{2} \mathcal{E}_{x_\ell}\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}_x &= \int_{-\infty}^{\infty} |X_+(f) + X_+^*(-f)|^2 df \\ &= \mathcal{E}_{x_+} + \mathcal{E}_{x_+}^* = 2\mathcal{E}_{x_+}\end{aligned}$$

Theorem (Energy considerations)

$$\mathcal{E}_{x_l} = 2\mathcal{E}_x = 4\mathcal{E}_{x_+}$$

Extension of energy considerations

Definition (Inner product)

We define the inner product of two (complex) signals $x(t)$ and $y(t)$ as

$$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x(t)y^*(t)dt.$$

- Parseval's relation immediately gives

$$\langle x(t), y(t) \rangle = \langle X(f), Y(f) \rangle.$$

- $\mathcal{E}_x = \langle x(t), x(t) \rangle = \langle X(f), X(f) \rangle$
- $\mathcal{E}_{x_\ell} = \langle x_\ell(t), x_\ell(t) \rangle = \langle X_\ell(f), X_\ell(f) \rangle$

We can similarly prove that

$$\begin{aligned} & \langle x(t), y(t) \rangle \\ &= \langle X(f), Y(f) \rangle \\ &= \left\langle \frac{1}{2} X_\ell(f - f_c) + \frac{1}{2} X_\ell^*(-f - f_c), \frac{1}{2} Y_\ell(f - f_c) + \frac{1}{2} Y_\ell^*(-f - f_c) \right\rangle \\ &= \frac{1}{4} \langle X_\ell(f - f_c), Y_\ell(f - f_c) \rangle + \underbrace{\frac{1}{4} \langle X_\ell(f - f_c), Y_\ell^*(-f - f_c) \rangle}_{=0} \\ &\quad + \underbrace{\frac{1}{4} \langle X_\ell^*(-f - f_c), Y_\ell(f - f_c) \rangle}_{=0} + \frac{1}{4} \langle X_\ell^*(-f - f_c), Y_\ell^*(-f - f_c) \rangle \\ &= \frac{1}{4} \langle x_\ell(t), y_\ell(t) \rangle + \frac{1}{4} (\langle x_\ell(t), y_\ell(t) \rangle)^* = \left[\frac{1}{2} \mathbf{Re} \{ \langle x_\ell(t), y_\ell(t) \rangle \} \right]. \end{aligned}$$

Corss-correlation of two signals

Definition (Cross-correlation)

The *cross-correlation* of two signals $x(t)$ and $y(t)$ is defined as

$$\rho_{x,y} = \frac{\langle x(t), y(t) \rangle}{\sqrt{\langle x(t), x(t) \rangle} \sqrt{\langle y(t), y(t) \rangle}} = \frac{\langle x(t), y(t) \rangle}{\sqrt{\mathcal{E}_x \mathcal{E}_y}}.$$

Definition (Orthogonality)

Two signals $x(t)$ and $y(t)$ are said to be *orthogonal* if $\rho_{x,y} = 0$.

- The previous slide then shows $\rho_{x,y} = \mathbf{Re}\{\rho_{x_\ell, y_\ell}\}$.
- $\rho_{x_\ell, y_\ell} = 0 \Rightarrow \rho_{x,y} = 0$ but $\rho_{x,y} = 0 \not\Rightarrow \rho_{x_\ell, y_\ell} = 0$

2.1-4 Lowpass equivalence of a bandpass system

Definition (Bandpass system)

A **bandpass system** is an LTI system with real impulse response $h(t)$ whose transfer function is located around a frequency f_c

- Using a similar concept, we set the lowpass equivalent impulse response as

$$h(t) = \operatorname{Re} \{ h_\ell(t) e^{j2\pi f_c t} \}$$

and

$$H(f) = \frac{1}{2} [H_\ell(f - f_c) + H_\ell^*(-f - f_c)]$$

Baseband input-output relation

- Let $x(t)$ be a bandpass input signal and let

$$y(t) = h(t) * x(t) \quad \text{or equivalently} \quad Y(f) = H(f)X(f)$$

- Then, we know

$$x(t) = \operatorname{Re} \{x_\ell(t)e^{j2\pi f_c t}\}$$

$$h(t) = \operatorname{Re} \{h_\ell(t)e^{j2\pi f_c t}\}$$

$$y(t) = \operatorname{Re} \{y_\ell(t)e^{j2\pi f_c t}\}$$

and

Theorem (Baseband input-output relation)

$$y(t) = h(t) * x(t) \iff y_\ell(t) = \frac{1}{2}h_\ell(t) * x_\ell(t)$$

Proof:

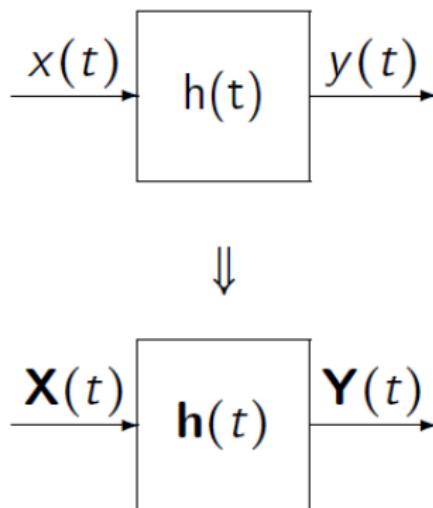
For $f \neq -f_c$ (or specifically, for $u_{-1}(f + f_c) = u_{-1}^2(f + f_c)$),

Note $\frac{1}{2} = u_{-1}(0) \neq u_{-1}^2(0) = \frac{1}{4}$.

$$\begin{aligned} Y_\ell(f) &= 2Y(f + f_c)u_{-1}(f + f_c) \\ &= 2H(f + f_c)X(f + f_c)u_{-1}(f + f_c) \\ &= \frac{1}{2} [2H(f + f_c)u_{-1}(f + f_c)] \cdot [2X(f + f_c)u_{-1}(f + f_c)] \\ &= \frac{1}{2} H_\ell(f) \cdot X_\ell(f) \end{aligned}$$

The case for $f = -f_c$ is valid since $Y_\ell(-f_c) = X_\ell(-f_c) = 0$. □

- The above applies to deterministic system. How about stochastic system?



The text abuses the notation by using $X(f)$ as the spectrum of $x(t)$ but using $X(t)$ as the stochastic counterpart of $x(t)$.

2.7 Random processes

Random Process

Definition

A random process is a set of indexed random variables $\{X(t), t \in T\}$, where T is often called the index set.

Classification

- ① If T is a finite set \Rightarrow Random Vector
- ② If $T = \mathbb{Z}$ or \mathbb{Z}^+ \Rightarrow Discrete Random Process
- ③ If $T = \mathbb{R}$ or \mathbb{R}^+ \Rightarrow Continuous Random Process
- ④ If $T = \mathbb{R}^2, \mathbb{Z}^2, \dots, \mathbb{R}^n, \mathbb{Z}^n$ \Rightarrow Random Field

Examples of random process

Example

Let \mathbf{U} be a random variable uniformly distributed over $[-\pi, \pi]$. Then

$$\mathbf{X}(t) = \cos(2\pi f_c t + \mathbf{U})$$

is a (continuous) random process.

Example

Let \mathbf{B} be a random variable taking values in $\{-1, 1\}$. Then

$$\mathbf{X}(t) = \begin{cases} \cos(2\pi f_c t) & \text{if } \mathbf{B} = -1 \\ \sin(2\pi f_c t) & \text{if } \mathbf{B} = +1 \end{cases} = \cos\left(2\pi f_c t - \frac{\pi}{4}(\mathbf{B} + 1)\right)$$

is a (continuous) random process.

Statistical properties of random process

For any integer $k > 0$ and any $t_1, t_2, \dots, t_k \in \mathcal{T}$, the finite-dimensional cumulative distribution function (cdf) for $\mathbf{X}(t)$ is given by:

$$F_{\mathbf{X}}(t_1, \dots, t_k; x_1, \dots, x_k) = \Pr \{ \mathbf{X}(t_1) \leq x_1, \dots, \mathbf{X}(t_k) \leq x_k \}$$

As event $[\mathbf{X}(t) \leq \infty]$ (resp. $[\mathbf{X}(t) \leq -\infty]$) is always regarded as **true** (resp. **false**),

$$\begin{aligned} & \lim_{x_s \rightarrow \infty} F_{\mathbf{X}}(t_1, \dots, t_k; x_1, \dots, x_k) \\ &= F_{\mathbf{X}}(t_1, \dots, \textcolor{blue}{t}_{s-1}, \textcolor{blue}{t}_{s+1}, t_k; x_1, \dots, \textcolor{blue}{x}_{s-1}, \textcolor{blue}{x}_{s+1}, \dots, x_k) \end{aligned}$$

and

$$\lim_{x_s \rightarrow -\infty} F_{\mathbf{X}}(t_1, \dots, t_k; x_1, \dots, x_k) = 0$$

Definition

Let $\mathbf{X}(t)$ be a random process; then the *mean function* is

$$m_{\mathbf{X}}(t) = \mathbb{E}[\mathbf{X}(t)],$$

the *(auto)correlation function* is

$$R_{\mathbf{X}}(t_1, t_2) = \mathbb{E}[\mathbf{X}(t_1)\mathbf{X}^*(t_2)],$$

and the *(auto)covariance function* is

$$K_{\mathbf{X}}(t_1, t_2) = \mathbb{E}\left[(\mathbf{X}(t_1) - m_{\mathbf{X}}(t_1)) (\mathbf{X}(t_2) - m_{\mathbf{X}}(t_2))^* \right]$$

Definition

Let $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ be two random processes; then the cross-correlation function is

$$R_{\mathbf{X}, \mathbf{Y}}(t_1, t_2) = \mathbb{E}[\mathbf{X}(t_1)\mathbf{Y}^*(t_2)],$$

and cross-covariance function is

$$K_{\mathbf{X}, \mathbf{Y}}(t_1, t_2) = \mathbb{E}[(\mathbf{X}(t_1) - m_{\mathbf{X}}(t_1))(\mathbf{Y}(t_2) - m_{\mathbf{Y}}(t_2))^*]$$

Proposition

$$R_{\mathbf{X}, \mathbf{Y}}(t_1, t_2) = K_{\mathbf{X}, \mathbf{Y}}(t_1, t_2) + m_{\mathbf{X}}(t_1)m_{\mathbf{Y}}^*(t_2)$$

$$R_{\mathbf{Y}, \mathbf{X}}(t_2, t_1) = R_{\mathbf{X}, \mathbf{Y}}^*(t_1, t_2) \quad R_{\mathbf{X}}(t_2, t_1) = R_{\mathbf{X}}^*(t_1, t_2)$$

$$K_{\mathbf{Y}, \mathbf{X}}(t_2, t_1) = K_{\mathbf{X}, \mathbf{Y}}^*(t_1, t_2) \quad K_{\mathbf{X}}(t_2, t_1) = K_{\mathbf{X}}^*(t_1, t_2)$$

Stationary random processes

Definition

A random process $\mathbf{X}(t)$ is said to be **strictly** or **strict-sense stationary (SSS)** if its finite-dimensional joint distribution function is shift-invariant, i.e. for any integer $k > 0$, any $t_1, \dots, t_k \in \mathcal{T}$ and any τ ,

$$F_{\mathbf{X}}(t_1 - \tau, \dots, t_k - \tau; x_1, \dots, x_k) = F_{\mathbf{X}}(t_1, \dots, t_k; x_1, \dots, x_k)$$

Definition

A random process $\mathbf{X}(t)$ is said to be **weakly** or **wide-sense stationary (WSS)** if its mean function and (auto)correlation function are shift-invariant, i.e. for any $t_1, t_2 \in \mathcal{T}$ and any τ ,

$$m_{\mathbf{X}}(t - \tau) = m_{\mathbf{X}}(t) \text{ and } R_{\mathbf{X}}(t_1 - \tau, t_2 - \tau) = R_{\mathbf{X}}(t_1, t_2).$$

The above condition is equivalent to

$$m_{\mathbf{X}}(t) = \text{constant} \text{ and } R_{\mathbf{X}}(t_1, t_2) = R_{\mathbf{X}}(t_1 - t_2).$$

Wide-sense stationary random processes

Definition

Two random processes $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ are said to be *jointly wide-sense stationary* if

- Both $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ are WSS;
- $R_{\mathbf{X},\mathbf{Y}}(t_1, t_2) = R_{\mathbf{X},\mathbf{Y}}(t_1 - t_2)$.

Proposition

For jointly WSS $\mathbf{X}(t)$ and $\mathbf{Y}(t)$,

$$R_{\mathbf{Y},\mathbf{X}}(t_2, t_1) = R_{\mathbf{X},\mathbf{Y}}^*(t_1, t_2) \implies R_{\mathbf{X},\mathbf{Y}}(\tau) = R_{\mathbf{Y},\mathbf{X}}^*(-\tau)$$

$$K_{\mathbf{Y},\mathbf{X}}(t_2, t_1) = K_{\mathbf{X},\mathbf{Y}}^*(t_1, t_2) \implies K_{\mathbf{X},\mathbf{Y}}(\tau) = K_{\mathbf{Y},\mathbf{X}}^*(-\tau)$$

Gaussian random process

Definition

A random process $\{\mathbf{X}(t), t \in \mathcal{T}\}$ is said to be Gaussian if for any integer $k > 0$ and for any $t_1, \dots, t_k \in \mathcal{T}$, the finite-dimensional joint cdf

$$F_{\mathbf{X}}(t_1, \dots, t_k; x_1, \dots, x_k) = \Pr [\mathbf{X}(t_1) \leq x_1, \dots, \mathbf{X}(t_k) \leq x_k]$$

is Gaussian.

Remark

The joint cdf of a Gaussian process is fully determined by its mean function and its (auto)covariance function.

Gaussian random process

Definition

Two real random processes $\{\mathbf{X}(t), t \in \mathcal{T}_X\}$ and $\{\mathbf{Y}(t), t \in \mathcal{T}_Y\}$ are said to be jointly Gaussian if for any integers $j, k > 0$ and for any $s_1, \dots, s_j \in \mathcal{T}_X$ and $t_1, \dots, t_k \in \mathcal{T}_Y$, the finite-dimensional joint cdf

$$\Pr [\mathbf{X}(s_1) \leq x_1, \dots, \mathbf{X}(s_j) \leq x_j, \mathbf{Y}(t_1) \leq y_1, \dots, \mathbf{Y}(t_k) \leq y_k]$$

is Gaussian.

Definition

A complex process is Gaussian if the real and imaginary processes are jointly Gaussian.

Gaussian random process

Remark

For joint (in general complex) Gaussian processes,
“uncorrelatedness”, defined as

$$\begin{aligned} R_{\mathbf{X}, \mathbf{Y}}(t_1, t_2) &= \mathbb{E}[\mathbf{X}(t_1)\mathbf{Y}^*(t_2)] \\ &= \mathbb{E}[\mathbf{X}(t_1)]\mathbb{E}[\mathbf{Y}^*(t_2)] = m_{\mathbf{X}}(t_1)m_{\mathbf{Y}}^*(t_2), \end{aligned}$$

implies “independence”, i.e.,

$$\begin{aligned} &\Pr[\mathbf{X}(s_1) \leq x_1, \dots, \mathbf{X}(s_j) \leq x_j, \mathbf{Y}(t_1) \leq y_1, \dots, \mathbf{Y}(t_k) \leq y_k] \\ &= \Pr[\mathbf{X}(s_1) \leq x_1, \dots, \mathbf{X}(s_k) \leq x_k] \cdot \Pr[\mathbf{Y}(t_1) \leq y_1, \dots, \mathbf{Y}(t_k) \leq y_k] \end{aligned}$$

Theorem

If a Gaussian random process $\mathbf{X}(t)$ is WSS, then it is SSS.

Proof:

For any $k > 0$, consider the sampled random vector

$$\vec{\mathbf{X}}_k = \begin{bmatrix} \mathbf{X}(t_1) \\ \mathbf{X}(t_2) \\ \vdots \\ \mathbf{X}(t_k) \end{bmatrix}$$

The mean vector and covariance matrix of $\vec{\mathbf{X}}_k$ are respectively

$$m_{\vec{\mathbf{X}}_k} = \mathbb{E}[\vec{\mathbf{X}}_k] = \begin{bmatrix} \mathbb{E}[\mathbf{X}(t_1)] \\ \mathbb{E}[\mathbf{X}(t_2)] \\ \vdots \\ \mathbb{E}[\mathbf{X}(t_k)] \end{bmatrix} = m_{\mathbf{X}}(0) \cdot \vec{\mathbf{1}}$$

and

$$K_{\bar{\mathbf{X}}} = \mathbb{E}[\vec{\mathbf{X}}_k \vec{\mathbf{X}}_k^H] = \begin{bmatrix} K_{\mathbf{X}}(0) & K_{\mathbf{X}}(t_1 - t_2) & \dots \\ K_{\mathbf{X}}(t_2 - t_1) & K_{\mathbf{X}}(0) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

It can be shown that for a new sampled random vector

$$\begin{bmatrix} \mathbf{X}(t_1 + \tau) \\ \mathbf{X}(t_2 + \tau) \\ \vdots \\ \mathbf{X}(t_k + \tau) \end{bmatrix}$$

the mean vector and covariance matrix remain the same.
Hence, $\mathbf{X}(t)$ is SSS. □

Power spectral density

Definition

Let $R_{\mathbf{X}}(\tau)$ be the correlation function of a WSS random process $\mathbf{X}(t)$. The power spectral density (PSD) or power spectrum of $\mathbf{X}(t)$ is defined as

$$S_{\mathbf{X}}(f) = \int_{-\infty}^{\infty} R_{\mathbf{X}}(\tau) e^{-i2\pi f \tau} d\tau$$

Let $R_{\mathbf{X},\mathbf{Y}}(\tau)$ be the cross-correlation function of two jointly WSS random processes $\mathbf{X}(t)$ and $\mathbf{Y}(t)$; then the cross spectral density (CSD) is

$$S_{\mathbf{X},\mathbf{Y}}(f) = \int_{-\infty}^{\infty} R_{\mathbf{X},\mathbf{Y}}(\tau) e^{-i2\pi f \tau} d\tau$$

Properties of PSD

- PSD (in units of watts per Hz) describes the distribution/density of power as a function of frequency.
 - Analogously, probability density function (pdf) describes the distribution/density of probability as a function of outcome.
 - The integration of PSD gives power of the random process over the considered range of frequency.
 - Analogously, the integration of pdf gives probability over the considered range of outcome.

Theorem

$S_X(f)$ is non-negative and real (which matches that the power of a signal cannot be negative).

Proof: $S_X(f)$ is real because

$$\begin{aligned} S_X(f) &= \int_{-\infty}^{\infty} R_X(\tau) e^{-\imath 2\pi f \tau} d\tau \\ &= \int_{-\infty}^{\infty} R_X(-s) e^{\imath 2\pi fs} ds \quad (s = -\tau) \\ &= \int_{-\infty}^{\infty} R_X^*(s) e^{\imath 2\pi fs} ds \\ &= \left(\int_{-\infty}^{\infty} R_X(s) e^{-\imath 2\pi fs} ds \right)^* \\ &= S_X^*(f) \end{aligned}$$

$S_{\mathbf{X}}(f)$ is non-negative because of the following (we only prove this based on that $\mathcal{T} \subset \mathbb{R}$ and $\mathbf{X}(t) = 0$ outside $[-T, T]$).

$$\begin{aligned}
 S_{\mathbf{X}}(f) &= \int_{-\infty}^{\infty} \mathbb{E}[\mathbf{X}(t+\tau)\mathbf{X}^*(t)]e^{-\imath 2\pi f\tau} d\tau \\
 &= \mathbb{E}\left[\mathbf{X}^*(t) \int_{-\infty}^{\infty} \mathbf{X}(t+\tau)e^{-\imath 2\pi f\tau} d\tau\right] \quad (s = t + \tau) \\
 &= \mathbb{E}\left[\mathbf{X}^*(t) \int_{-\infty}^{\infty} \mathbf{X}(s)e^{-\imath 2\pi f(s-t)} ds\right] \\
 &= \mathbb{E}\left[\mathbf{X}^*(t)\tilde{\mathbf{X}}(f)e^{\imath 2\pi ft}\right]
 \end{aligned}$$

In notation, $\tilde{\mathbf{X}}(f) = \mathcal{F}\{\mathbf{X}(t)\}$

Since the above is a constant with respect to t (by WSS),

$$\begin{aligned}
 S_{\mathbf{X}}(f) &= \frac{1}{2T} \int_{-T}^T \mathbb{E}\left[\mathbf{X}^*(t)\tilde{\mathbf{X}}(f)e^{\imath 2\pi ft}\right] dt \\
 &= \frac{1}{2T} \mathbb{E}\left[\tilde{\mathbf{X}}(f) \int_{-T}^T \mathbf{X}^*(t)e^{\imath 2\pi ft} dt\right] \\
 &= \frac{1}{2T} \mathbb{E}\left[\tilde{\mathbf{X}}(f)\tilde{\mathbf{X}}^*(f)\right] = \frac{1}{2T} \mathbb{E}[|\tilde{\mathbf{X}}(f)|^2] \geq 0.
 \end{aligned}$$



Wiener-Khintchine theorem

Theorem (Wiener-Khintchine)

Let $\{\mathbf{X}(t), t \in \mathbb{R}\}$ be a WSS random process. Define

$$\mathbf{X}_T(t) = \begin{cases} \mathbf{X}(t) & \text{if } t \in [-T, T] \\ 0, & \text{otherwise.} \end{cases}$$

and set

$$\tilde{\mathbf{X}}_T(f) = \int_{-\infty}^{\infty} \mathbf{X}_T(t) e^{-\imath 2\pi f t} dt = \int_{-T}^{T} \mathbf{X}(t) e^{-\imath 2\pi f t} dt.$$

If $S_{\mathbf{X}}(f)$ exists (i.e., $R_{\mathbf{X}}(\tau)$ has a Fourier transform), then

$$S_{\mathbf{X}}(f) = \lim_{T \rightarrow \infty} \frac{1}{2T} \mathbb{E} \left\{ |\tilde{\mathbf{X}}_T(f)|^2 \right\}$$

Variations of PSD definitions

- Power density spectrum : Alternative definition
 - Fourier transform of auto-covariance function (e.g., Robert M. Gray and Lee D. Davisson, Random Processes: A Mathematical Approach for Engineers, p. 193)
- I remark that from the viewpoint of digital communications, the text's definition is more appropriate since
 - the auto-covariance function is independent of a mean-shift; however, random signals with different means consume different "powers."

- What can we say about, e.g., the PSD of stochastic system input and output?

$$\begin{array}{ccc} x(t) & \xrightarrow{\quad h(t) \quad} & y(t) \\ \xrightarrow{x_\ell(t)} & \boxed{\frac{1}{2}h_\ell(t)} & \xrightarrow{y_\ell(t)} \end{array} \quad \begin{cases} \square(t) = \mathbf{Re}\{\square_\ell(t)e^{i2\pi f_c t}\} \\ \square_\ell(t) = (\square(t) + i\hat{\square}(t))e^{-i2\pi f_c t} \end{cases}$$

where “ \square ” can be x, y or h .



$$\begin{array}{ccc} \mathbf{X}(t) & \xrightarrow{\quad \mathbf{h}(t) \quad} & \mathbf{Y}(t) \\ \mathbf{X}_\ell(t) & \xrightarrow{\boxed{\frac{1}{2}\mathbf{h}_\ell(t)}} & \mathbf{Y}_\ell(t) \end{array} \quad \begin{cases} \square(t) = \mathbf{Re}\{\square_\ell(t)e^{i2\pi f_c t}\} \\ \square_\ell(t) = (\square(t) + i\hat{\square}(t))e^{-i2\pi f_c t} \end{cases}$$

where “ \square ” can be \mathbf{X}, \mathbf{Y} or \mathbf{h} .

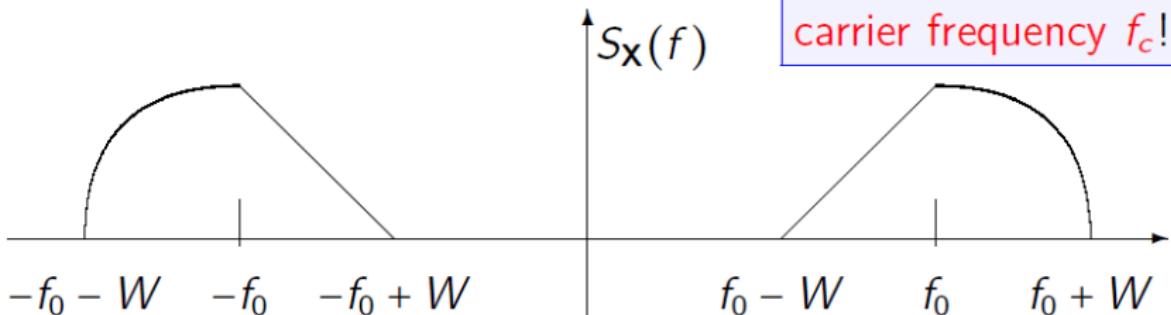
2.9 Bandpass and lowpass random processes

Definition (Bandpass random signal)

A *bandpass* (WSS) stochastic signal $\mathbf{X}(t)$ is a *real* random process whose *PSD* is located around *central frequency* f_0 , i.e.

$$S_{\mathbf{X}}(f) = 0 \quad \text{for all } |f \pm f_0| > W$$

f_0 may not be the carrier frequency f_c !



- We know $\begin{cases} \mathbf{X}(t) = \mathbf{Re}\{\mathbf{X}_\ell(t)e^{i2\pi ft}\} \\ \mathbf{X}_\ell(t) = (\mathbf{X}(t) + i\hat{\mathbf{X}}(t))e^{-i2\pi f_0 t} \end{cases}$

Assumption

The bandpass signal $\mathbf{X}(t)$ is WSS.

In addition, its complex lowpass equivalent process $\mathbf{X}_\ell(t)$ is WSS. In other words,

- $\mathbf{X}_i(t)$ and $\mathbf{X}_q(t)$ are WSS.
- $\mathbf{X}_i(t)$ and $\mathbf{X}_q(t)$ are jointly WSS.

Under this **fundamental assumption**, we obtain the following properties:

P1) If $\mathbf{X}(t)$ zero-mean, both $\mathbf{X}_i(t)$ and $\mathbf{X}_q(t)$ zero-mean because $m_{\mathbf{X}} = m_{\mathbf{X}_i} \cos(2\pi f_c t) - m_{\mathbf{X}_q} \sin(2\pi f_c t)$.

P2)
$$\begin{cases} R_{\mathbf{X}_i}(\tau) = R_{\mathbf{X}_q}(\tau) \\ R_{\mathbf{X}_i, \mathbf{X}_q}(\tau) = -R_{\mathbf{X}_q, \mathbf{X}_i}(\tau) \end{cases}$$

Proof of P2):

$$R_{\mathbf{X}}(\tau)$$

$$= \mathbb{E}[\mathbf{X}(t + \tau)\mathbf{X}(t)]$$

$$= \mathbb{E}\left[\mathbf{Re}\{\mathbf{X}_l(t + \tau)e^{\imath 2\pi f_c(t + \tau)}\} \mathbf{Re}\{\mathbf{X}_l(t)e^{\imath 2\pi f_c t}\}\right]$$

$$= \mathbb{E}\left[(\mathbf{X}_i(t + \tau) \cos(2\pi f_c(t + \tau)) - \mathbf{X}_q(t + \tau) \sin(2\pi f_c(t + \tau))) (\mathbf{X}_i(t) \cos(2\pi f_c t) - \mathbf{X}_q(t) \sin(2\pi f_c t))\right]$$

$$= \frac{R_{\mathbf{X}_i}(\tau) + R_{\mathbf{X}_q}(\tau)}{2} \cos(2\pi f_c \tau)$$

$$+ \frac{R_{\mathbf{X}_i, \mathbf{X}_q}(\tau) - R_{\mathbf{X}_q, \mathbf{X}_i}(\tau)}{2} \sin(2\pi f_c \tau)$$

$$+ \frac{R_{\mathbf{X}_i}(\tau) - R_{\mathbf{X}_q}(\tau)}{2} \cos(2\pi f_c(2t + \tau)) \quad (= 0)$$

$$- \frac{R_{\mathbf{X}_i, \mathbf{X}_q}(\tau) + R_{\mathbf{X}_q, \mathbf{X}_i}(\tau)}{2} \sin(2\pi f_c(2t + \tau)) \quad (= 0)$$



$$P3) \quad R_{\mathbf{X}}(\tau) = \mathbf{Re} \left\{ \frac{1}{2} R_{\mathbf{X}_{\ell}}(\tau) e^{\imath 2\pi f_c \tau} \right\}.$$

Proof: Observe from P2),

$$\begin{aligned} R_{\mathbf{X}_{\ell}}(\tau) &= \mathbb{E} [\mathbf{X}_{\ell}(t + \tau) \mathbf{X}_{\ell}^*(t)] \\ &= \mathbb{E} [(\mathbf{X}_i(t + \tau) + \imath \mathbf{X}_q(t + \tau)) (\mathbf{X}_i(t) - \imath \mathbf{X}_q(t))] \\ &= R_{\mathbf{X}_i}(\tau) + R_{\mathbf{X}_q}(\tau) - \imath R_{\mathbf{X}_i, \mathbf{X}_q}(\tau) + \imath R_{\mathbf{X}_q, \mathbf{X}_i}(\tau) \\ &= 2R_{\mathbf{X}_i}(\tau) + \imath 2R_{\mathbf{X}_q, \mathbf{X}_i}(\tau). \end{aligned}$$

Hence, also from P2),

$$\begin{aligned} R_{\mathbf{X}}(\tau) &= R_{\mathbf{X}_i}(\tau) \cos(2\pi f_c t) - R_{\mathbf{X}_q, \mathbf{X}_i}(\tau) \sin(2\pi f_c t) \\ &= \mathbf{Re} \left\{ \frac{1}{2} R_{\mathbf{X}_{\ell}}(\tau) e^{\imath 2\pi f_c \tau} \right\} \end{aligned}$$

$$P4) \quad S_{\mathbf{X}}(f) = \frac{1}{4} [S_{\mathbf{X}_\ell}(f - f_c) + S_{\mathbf{X}_\ell}^*(-f - f_c)].$$

Proof. A direct consequence of P3). □

Note:

- Amplitude $\tilde{\mathbf{X}}(f) = \frac{1}{2} [\tilde{\mathbf{X}}_\ell(f - f_c) + \tilde{\mathbf{X}}_\ell^*(-f - f_c)]$
- Amplitude square

$$\begin{aligned} |\tilde{\mathbf{X}}(f)|^2 &= \frac{1}{4} \left| \tilde{\mathbf{X}}_\ell(f - f_c) + \tilde{\mathbf{X}}_\ell^*(-f - f_c) \right|^2 \\ &= \frac{1}{4} \left(\left| \tilde{\mathbf{X}}_\ell(f - f_c) \right|^2 + \left| \tilde{\mathbf{X}}_\ell^*(-f - f_c) \right|^2 \right) \end{aligned}$$

- Wiener-Khintchine: $S_{\mathbf{X}}(f) \equiv |\tilde{\mathbf{X}}(f)|^2$.

P5) $\mathbf{X}_i(t)$ and $\mathbf{X}_q(t)$ uncorrelated if one of them has zero-mean.

Proof. From P2),

$$R_{\mathbf{X}_i, \mathbf{X}_q}(\tau) = -R_{\mathbf{X}_q, \mathbf{X}_i}(\tau) = -R_{\mathbf{X}_i, \mathbf{X}_q}(-\tau).$$

Hence, $R_{\mathbf{X}_i, \mathbf{X}_q}(0) = 0$ (i.e.,

$$\mathbb{E}[\mathbf{X}_i(t)\mathbf{X}_q(t)] = 0 = \mathbb{E}[\mathbf{X}_i(t)]\mathbb{E}[\mathbf{X}_q(t)].$$

□

P6) If $S_{\mathbf{X}_\ell}(-f) = S_{\mathbf{X}_\ell}^*(f)$, $\mathbf{X}_i(t + \tau)$ and $\mathbf{X}_q(t)$ uncorrelated for any τ if one of them has zero-mean.

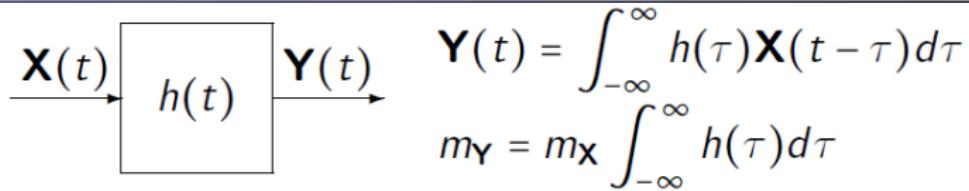
Proof. From P3),

$$R_{\mathbf{X}_\ell}(\tau) = 2R_{\mathbf{X}_i}(\tau) - \imath 2R_{\mathbf{X}_q, \mathbf{X}_i}(\tau).$$

$S_{\mathbf{X}_\ell}(-f) = S_{\mathbf{X}_\ell}^*(f)$ implies $R_{\mathbf{X}_\ell}(\tau)$ is real;

hence, $R_{\mathbf{X}_q, \mathbf{X}_i}(\tau) = 0$ for any τ . □

We next discuss the PSD of a system.



$$\mathbf{Y}(t) = \int_{-\infty}^{\infty} h(\tau) \mathbf{X}(t - \tau) d\tau$$

$$m_{\mathbf{Y}} = m_{\mathbf{X}} \int_{-\infty}^{\infty} h(\tau) d\tau$$

$$\begin{aligned}
 R_{\mathbf{X},\mathbf{Y}}(\tau) &= \mathbb{E} \left[\mathbf{X}(t + \tau) \left(\int_{-\infty}^{\infty} h(u) \mathbf{X}(t - u) du \right)^* \right] \\
 &= \int_{-\infty}^{\infty} h^*(u) R_{\mathbf{X}}(\tau + u) du = \int_{-\infty}^{\infty} h^*(-v) R_{\mathbf{X}}(\tau - v) dv \\
 &= R_{\mathbf{X}}(\tau) * h^*(-\tau)
 \end{aligned}$$

$$\begin{aligned}
 R_{\mathbf{Y}}(\tau) &= \mathbb{E} \left[\left(\int_{-\infty}^{\infty} h(u) \mathbf{X}(t + \tau - u) du \right) \left(\int_{-\infty}^{\infty} h(v) \mathbf{X}(t - v) dv \right)^* \right] \\
 &= \int_{-\infty}^{\infty} h(u) \left(\int_{-\infty}^{\infty} h^*(v) R_{\mathbf{X}}((\tau - u) + v) dv \right) du \\
 &= \int_{-\infty}^{\infty} h(u) R_{\mathbf{X},\mathbf{Y}}(\tau - u) du \\
 &= R_{\mathbf{X},\mathbf{Y}}(\tau) * h(\tau) = R_{\mathbf{X}}(\tau) * h^*(-\tau) * h(\tau).
 \end{aligned}$$

Thus,

$$S_{X,Y}(f) = S_X(f)H^*(f) \text{ since } \int_{-\infty}^{\infty} h^*(-\tau) e^{-i2\pi f\tau} d\tau = H^*(f)$$

and

$$S_Y(f) = S_{X,Y}(f)H(f) = S_X(f)|H(f)|^2.$$

White process

Definition (White process)

A (WSS) process $\mathbf{W}(t)$ is called a white process if its PSD is **constant** for all frequencies:

$$S_W(f) = \frac{N_0}{2}$$

- This constant is usually denoted by $\frac{N_0}{2}$ because the PSD is **two-sided**. So, the power spectral **density** is actually N_0 per Hz ($N_0/2$ at $f = -f_0$ and $N_0/2$ at $f = f_0$).
- The autocorrelation function $R_W(\tau) = \frac{N_0}{2}\delta(\cdot)$, where $\delta(\cdot)$ is the Dirac delta function.

Why negative frequency?

A “sample answer”:

- It is just an imaginary convenient way created by Human to correspond to the “imaginary” domain of a complex signal (that is why we call it “imaginary part”).
- By giving respectively the spectrum for f_0 and $-f_0$ (which may not be symmetric), we can tell the amount of **real part** and **imaginary part** in time domain corresponding to this frequency.
- For example, if the spectrum is conjugate symmetric, we know **imaginary part (in time domain) = 0**.
- Notably, in communications, **imaginary part** is the part that will be modulated by (or transmitted with carrier) $\sin(2\pi f_c t)$; on the contrary, **real part** is the part that will be modulated by (or transmitted with carrier) $\cos(2\pi f_c t)$.

Why $\delta(\cdot)$ function?

Definition (Dirac delta function)

Define the Dirac delta function $\delta(t)$ as

$$\delta(t) = \begin{cases} \infty, & t = 0; \\ 0, & t \neq 0 \end{cases},$$

which satisfies the *replication property*, i.e., for every **continuous** point of $g(t)$,

$$g(t) = \int_{-\infty}^{\infty} g(\tau) \delta(t - \tau) d\tau.$$

Hence, by replication property,

$$\int_{-\infty}^{\infty} \delta(u) du = \int_{-\infty}^{\infty} \delta(t - \tau) d\tau = \int_{-\infty}^{\infty} 1 \cdot \delta(t - \tau) d\tau = 1.$$

- Note that it seems $\delta(t) = 2\delta(t) = \begin{cases} \infty, & t = 0; \\ 0, & t \neq 0 \end{cases}$; but with $g_1(t) = 1$ and $g_2(t) = 2$ continuous at all points,

$$1 = \int_{-\infty}^{\infty} g_1(t)\delta(t)dt \neq \int_{-\infty}^{\infty} g_2(t)\delta(t)dt = 2.$$

- So, mathematician does not “like” this function as it contradicts intuition:

$f(t) = g(t)$ for $t \in \mathbb{R}$ except for countably many points

$$\Rightarrow \int_{-\infty}^{\infty} f(t)dt = \int_{-\infty}^{\infty} g(t)dt \quad \left(\text{if } \int_{-\infty}^{\infty} f(t)dt \text{ is finite} \right).$$

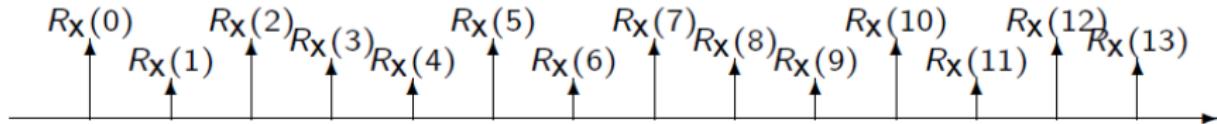
- Hence, $\delta(t)$ and $2\delta(t)$ are two “different” Dirac delta functions by definitions. (The multiplicative constant cannot be omitted!) Very “artificial” indeed.

- **Comment:** $x + a = y + a \Rightarrow x = y$ is incorrect if $a = \infty$.
As a result, saying $\infty = \infty$ (or $\delta(t) = 2\delta(t)$) is not a “rigorously defined” statement.
- **Summary:** The Dirac delta function, like “ ∞ ”, is simply a concept *defined* only through its *replication property*.
- Hence, a white process $\mathbf{W}(t)$ that has autocorrelation function $R_{\mathbf{W}}(\tau) = \frac{N_0}{2}\delta(\tau)$ is just a convenient and simplified notion for theoretical research about real world phenomenon. Usually, $N_0 = KT$, where T is the ambient temperature in kelvins and k is Boltzman's constant.

Discrete-time random processes

- The property of a time-discrete process $\{\mathbf{X}[n], n \in \mathbb{Z}^+\}$ can be “obtained” using sampling notion via the Dirac delta function.
- $\mathbf{X}[n] = \mathbf{X}(nT)$, a sample at $t = nT$ from a time-continuous process $\mathbf{X}(t)$, where we assume $T = 1$ for convenience.
- The autocorrelation function of a time-discrete process is given by:

$$\begin{aligned} R_{\mathbf{X}}[m] &= \mathbb{E}\{\mathbf{X}[n+m]\mathbf{X}[n]\} \\ &= \mathbb{E}\{\mathbf{X}(n+m)\mathbf{X}(n)\} \\ &= R_{\mathbf{X}}(m), \text{ a sample from } R_{\mathbf{X}}(t). \end{aligned}$$



$$\begin{aligned}
 S_X[f] &= \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} R_X(t) \delta(t-n) \right) e^{-\imath 2\pi f t} dt \\
 &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} R_X(t) e^{-\imath 2\pi f t} \delta(t-n) dt \\
 &= \sum_{n=-\infty}^{\infty} R_X(n) e^{-\imath 2\pi f n} \text{ (Replication Property)} \\
 &= \sum_{n=-\infty}^{\infty} R_X[n] e^{-\imath 2\pi f n} \text{ (Fourier Series)}
 \end{aligned}$$

Hence, by Fourier series,

$$R_X[n] = \int_{-1/2}^{1/2} S_X[f] e^{\imath 2\pi f m} df \quad (= R_X(n) = \int_{-\infty}^{\infty} S_X(f) e^{\imath 2\pi f m} df).$$

2.8 Series expansion of random processes

2.8-1 Sampling bandLimited random process

Deterministic case

- A deterministic signal $x(t)$ is called **band-limited** if $X(f) = 0$ for all $|f| > W$
- Shannon-Nyquist theorem: $x(t)$ can be perfectly reconstructed if the sampling rate $f_s > 2W$ and

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \text{sinc}\left[2W\left(t - \frac{n}{2W}\right)\right]$$

- Note that the above is only sufficient, not necessary.

Stochastic case

- A WSS stochastic process $\mathbf{X}(t)$ is said to be **band-limited** if its PSD $S_{\mathbf{X}}(f) = 0$ for all $|f| > W$
- It follows that

$$R_{\mathbf{X}}(\tau) = \sum_{n=-\infty}^{\infty} R_{\mathbf{X}}\left(\frac{n}{2W}\right) \text{sinc}\left[2W\left(\tau - \frac{n}{2W}\right)\right]$$

- In fact, the random process $\mathbf{X}(t)$ can be reconstructed by its (random) samples in the sense of mean square.

Theorem

$$\mathbb{E} \left| \mathbf{X}(t) - \sum_{n=-\infty}^{\infty} \mathbf{X}\left(\frac{n}{2W}\right) \text{sinc}\left[2W\left(t - \frac{n}{2W}\right)\right] \right|^2 = 0$$

The random samples

- Problems of using these random samples.
 - These random samples $\left\{ \mathbf{X}\left(\frac{n}{2W}\right) \right\}_{n=-\infty}^{\infty}$ are in general **correlated** unless $\mathbf{X}(t)$ is **zero-mean white**.

$$\begin{aligned}\mathbb{E}\left\{\mathbf{X}\left(\frac{n}{2W}\right)\mathbf{X}^*\left(\frac{m}{2W}\right)\right\} &= R_{\mathbf{X}}\left(\frac{n-m}{2W}\right) \\ &\neq \mathbb{E}\left\{\mathbf{X}\left(\frac{n}{2W}\right)\right\}\mathbb{E}\left\{\mathbf{X}^*\left(\frac{m}{2W}\right)\right\} = |m_{\mathbf{X}}|^2.\end{aligned}$$

- If $\mathbf{X}(t)$ is zero-mean white,

$$\begin{aligned}\mathbb{E}\left\{\mathbf{X}\left(\frac{n}{2W}\right)\mathbf{X}^*\left(\frac{m}{2W}\right)\right\} &= R_{\mathbf{X}}\left(\frac{n-m}{2W}\right) = \frac{N_0}{2}\delta\left(\frac{n-m}{2W}\right) \\ &= \mathbb{E}\left\{\mathbf{X}\left(\frac{n}{2W}\right)\right\}\mathbb{E}\left\{\mathbf{X}^*\left(\frac{m}{2W}\right)\right\} = |m_{\mathbf{X}}|^2 = 0 \text{ except } n = m.\end{aligned}$$

- Thus, we will introduce the **uncorrelated KL expansions** in slide 87.

2.9 Bandpass and lowpass random processes

Definition (Filtered white noise)

Definition (Filtered white noise)

A process $\mathbf{N}(t)$ is called a *filtered white noise* if its PSD equals

$$S_{\mathbf{N}}(f) = \begin{cases} \frac{N_0}{2}, & |f \pm f_c| < W \\ 0, & \text{otherwise} \end{cases}$$

- Applying P4) $S_{\mathbf{X}}(f) = \frac{1}{4} [S_{\mathbf{X}_\ell}(f - f_c) + S_{\mathbf{X}_\ell}^*(-f - f_c)]$, we learn the PSD of the lowpass equivalent process $\mathbf{N}_\ell(t)$ of $\mathbf{N}(t)$ is

$$S_{\mathbf{N}_\ell}(f) = \begin{cases} 2N_0, & |f| < W \\ 0, & \text{otherwise} \end{cases}$$

- From P6), $S_{\mathbf{N}_\ell}(-f) = S_{\mathbf{N}_\ell}^*(f)$ implies $\mathbf{N}_i(t + \tau)$ and $\mathbf{N}_q(t)$ are uncorrelated for any τ if one of them has zero mean.

Now we explore more properties for PSD of bandlimited $\mathbf{X}(t)$ and complex $\mathbf{X}_\ell(t)$.

P0-1) By **fundamental assumption** on slide 52, we obtain that $\mathbf{X}(t)$ and $\hat{\mathbf{X}}(t)$ are jointly WSS.

$R_{\mathbf{X}, \hat{\mathbf{X}}}(\tau)$ and $R_{\hat{\mathbf{X}}}(\tau)$ are only functions of τ because $\hat{\mathbf{X}}(t)$ is the Hilbert transform of $\mathbf{X}(t)$, i.e., $R_{\mathbf{X}, \hat{\mathbf{X}}}(\tau) = R_{\mathbf{X}}(\tau) * h^*(-\tau) = -R_{\mathbf{X}}(\tau) * h(\tau)$ (since $h^*(-\tau) = -h(\tau)$) and $R_{\hat{\mathbf{X}}}(\tau) = R_{\mathbf{X}, \hat{\mathbf{X}}}(\tau) * h(\tau)$.

P0-2) $\mathbf{X}_i(t) = \operatorname{Re} \{ (\mathbf{X}(t) + i\hat{\mathbf{X}}(t)) e^{-i2\pi f_c t} \}$ is WSS by **fundamental assumption**.

$$P2') \begin{cases} R_{\mathbf{X}}(\tau) = R_{\hat{\mathbf{X}}}(\tau) & (\mathbf{X}(t) + i\hat{\mathbf{X}}(t) \text{ is the "lowpass equivalent" signal of } \mathbf{X}_i(t)!) \\ R_{\mathbf{X}, \hat{\mathbf{X}}}(\tau) = -R_{\hat{\mathbf{X}}, \mathbf{X}}(\tau) & (\mathbf{X}_i(t) + i\mathbf{X}_q(t) \text{ is the lowpass equivalent signal of } \mathbf{X}(t)!) \end{cases}$$

Also, $R_{\hat{\mathbf{X}}, \mathbf{X}}(\tau) = \hat{R}_{\mathbf{X}}(\tau)$, where $\hat{R}_{\mathbf{X}}(\tau)$ is the Hilbert transform output due to input $R_{\mathbf{X}}(\tau)$.

$$P3') \quad R_{\mathbf{X}_i}(\tau) = \mathbf{Re} \left\{ \frac{1}{2} R_{(\mathbf{X} + \imath \hat{\mathbf{X}})}(\tau) e^{-\imath 2\pi f_c \tau} \right\}$$

$$\begin{aligned} R_{\mathbf{X}_i}(\tau) &= \mathbf{Re} \left\{ \frac{1}{2} R_{(\mathbf{X} + \imath \hat{\mathbf{X}})}(\tau) e^{-\imath 2\pi f_c \tau} \right\} \\ &= \mathbf{Re} \left\{ (R_{\mathbf{X}}(\tau) + \imath R_{\hat{\mathbf{X}}, \mathbf{X}}(\tau)) e^{-\imath 2\pi f_c \tau} \right\} \\ &= R_{\mathbf{X}}(\tau) \cos(2\pi f_c \tau) + \hat{R}_{\mathbf{X}}(\tau) \sin(2\pi f_c \tau) \end{aligned}$$

Note that $\hat{S}_{\mathbf{X}}(f) = S_{\mathbf{X}}(f) H_{\text{Hilbert}}(f) = S_{\mathbf{X}}(f)(-\imath \text{sgn}(f))$.

$$P4') \quad S_{\mathbf{X}_i}(f) = S_{\mathbf{X}}(f - f_c) + S_{\mathbf{X}}(f + f_c) \quad (= S_{\mathbf{X}_q}(f)) \text{ for } |f| < f_c$$

$$\begin{aligned} S_{\mathbf{X}_i}(f) &= \frac{1}{2} (S_{\mathbf{X}}(f - f_c) + S_{\mathbf{X}}(f + f_c)) \\ &+ \frac{1}{2\imath} (-\imath \text{sgn}(f - f_c) S_{\mathbf{X}}(f - f_c) + \imath \text{sgn}(f + f_c) S_{\mathbf{X}}(f + f_c)) \\ &= S_{\mathbf{X}}(f - f_c) + S_{\mathbf{X}}(f + f_c) \quad \text{for } |f| < f_c \end{aligned}$$

$$P4'') \quad S_{\mathbf{X}_q, \mathbf{X}_i}(f) = \imath [S_{\mathbf{X}}(f - f_c) - S_{\mathbf{X}}(f + f_c)] \text{ for } |f| < f_c$$

Terminologies & relations

- $$\bullet \begin{cases} R_{\mathbf{X}}(\tau) = \mathbf{Re} \left\{ \frac{1}{2} R_{\mathbf{X}_\ell}(\tau) e^{\imath 2\pi f_c \tau} \right\} & (P3) \\ \underbrace{R_{\hat{\mathbf{X}}, \mathbf{X}}(\tau) = R_{\mathbf{X}}(\tau) \star h_{\text{Hilbert}}(\tau) = \mathbf{Im} \left\{ \frac{1}{2} R_{\mathbf{X}_\ell}(\tau) e^{\imath 2\pi f_c \tau} \right\}}_{\text{P0-1}} \end{cases}$$
- $$\bullet \text{ Then: } \underbrace{\frac{1}{2} R_{\mathbf{X}_\ell}(\tau)}_{\text{Proof of P3}} = R_{\mathbf{X}_i}(\tau) + \imath R_{\mathbf{X}_q, \mathbf{X}_i}(\tau) = (R_{\mathbf{X}}(\tau) + \imath R_{\hat{\mathbf{X}}, \mathbf{X}}(\tau)) e^{-\imath 2\pi f_c \tau}$$
- $$\bullet \begin{cases} R_{\mathbf{X}_i}(\tau) = \mathbf{Re} \left\{ (R_{\mathbf{X}}(\tau) + \imath R_{\hat{\mathbf{X}}, \mathbf{X}}(\tau)) e^{-\imath 2\pi f_c \tau} \right\} & (P3') \\ R_{\mathbf{X}_q, \mathbf{X}_i}(\tau) = \mathbf{Im} \left\{ (R_{\mathbf{X}}(\tau) + \imath R_{\hat{\mathbf{X}}, \mathbf{X}}(\tau)) e^{-\imath 2\pi f_c \tau} \right\} = R_{\mathbf{X}_i}(\tau) \star h_{\text{Hilbert}}(\tau) \end{cases}$$

Proof: Hence,

$$\begin{aligned} R_{\mathbf{x}_q, \mathbf{x}_i}(\tau) &= \mathbf{Im} \left\{ (R_{\mathbf{x}}(\tau) + i R_{\hat{\mathbf{x}}, \mathbf{x}}(\tau)) e^{-i 2\pi f_c \tau} \right\} \\ &= -R_{\mathbf{x}}(\tau) \sin(2\pi f_c \tau) + R_{\hat{\mathbf{x}}, \mathbf{x}}(\tau) \cos(2\pi f_c \tau) \\ &= -R_{\mathbf{x}}(\tau) \sin(2\pi f_c \tau) + \hat{R}_{\mathbf{x}}(\tau) \cos(2\pi f_c \tau). \end{aligned}$$

The property can be proved similarly to P4'.

2.2 Signal space representation

Key idea & motivation

- The low-pass equivalent representation removes the dependence of system performance analysis on carrier frequency.
- Equivalent vectorization of the (discrete or continuous) signals further removes the “waveform” redundancy in the analysis of system performance.

Vector space concepts

- Inner product: $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \sum_{i=1}^n v_{1,i} v_{2,i}^* = \mathbf{v}_2^H \mathbf{v}_1$
("H" denotes **Hermitian transpose**)
- Orthogonal if $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$
- Norm: $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$
- Orthonormal: $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$ and $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = 1$
- Linearly independent:

$$\sum_{i=1}^k a_i \mathbf{v}_i = \mathbf{0} \text{ iff } a_i = 0 \text{ for all } i$$

- Triangle inequality

$$\|\mathbf{v}_1 + \mathbf{v}_2\| \leq \|\mathbf{v}_1\| + \|\mathbf{v}_2\|$$

- Cauchy-Schwartz inequality

$$|\langle \mathbf{v}_1, \mathbf{v}_2 \rangle| \leq \|\mathbf{v}_1\| \|\mathbf{v}_2\|.$$

Equality holds iff $\mathbf{v}_1 = a\mathbf{v}_2$ for some a

- Norm square of sum:

$$\|\mathbf{v}_1 + \mathbf{v}_2\|^2 = \|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 + \langle \mathbf{v}_1, \mathbf{v}_2 \rangle + \langle \mathbf{v}_2, \mathbf{v}_1 \rangle$$

- Pythagorean: if $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$, then

$$\|\mathbf{v}_1 + \mathbf{v}_2\|^2 = \|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2$$

Eigen-decomposition

- ① Matrix transformation w.r.t. matrix A

$$\hat{\mathbf{v}} = A\mathbf{v}$$

- ② Eigenvalues of square matrix A are solutions $\{\lambda\}$ of characteristic polynomial

$$\det(A - \lambda I) = 0$$

- ③ Eigenvectors for eigenvalue λ is solution \mathbf{v} of

$$A\mathbf{v} = \lambda\mathbf{v}$$

Signal space concept

How to extend the signal space concept to a (complex) function/signal $z(t)$ defined over $[0, T]$?

Answer: We can start by defining the **inner product** for complex functions.

- Inner product: $\langle z_1(t), z_2(t) \rangle = \int_0^T z_1(t)z_2^*(t) dt$
- Orthogonal if $\langle z_1(t), z_2(t) \rangle = 0$.
- Norm: $\|z(t)\| = \sqrt{\langle z(t), z(t) \rangle}$
- Orthonormal: $\langle z_1(t), z_2(t) \rangle = 0$ and $\|z_1(t)\| = \|z_2(t)\| = 1$.
- Linearly independent: $\sum_{i=1}^k a_i z_i(t) = 0$ iff $a_i = 0$ for all $a_i \in \mathbb{C}$

- Triangle Inequality

$$\|z_1(t) + z_2(t)\| \leq \|z_1(t)\| + \|z_2(t)\|$$

- Cauchy Schwartz inequality

$$|\langle z_1(t), z_2(t) \rangle| \leq \|z_1(t)\| \cdot \|z_2(t)\|$$

Equality holds iff $z_1(t) = a \cdot z_2(t)$ for some $a \in \mathbb{C}$

- Norm square of sum:

$$\begin{aligned}\|z_1(t) + z_2(t)\|^2 &= \|z_1(t)\|^2 + \|z_2(t)\|^2 \\ &\quad + \langle z_1(t), z_2(t) \rangle + \langle z_2(t), z_1(t) \rangle\end{aligned}$$

- Pythagorean property: if $\langle z_1(t), z_2(t) \rangle = 0$,

$$\|z_1(t) + z_2(t)\|^2 = \|z_1(t)\|^2 + \|z_2(t)\|^2$$

- Transformation w.r.t. a function $C(t, s)$

$$\hat{z}(t) = \int_0^T C(t, s) z(s) ds$$

This is in parallel to

$$\hat{\mathbf{v}} \quad \left(\hat{v}_t = \sum_{s=1}^n A_{t,s} v_s \right) = A\mathbf{v}.$$

Eigenvalues and eigenfunctions

Given a complex continuous function $C(t, s)$ over $[0, T]^2$, the eigenvalues and eigenfunctions are $\{\lambda_k\}$ and $\{\varphi_k(t)\}$ such that

$$\int_0^T C(t, s) \varphi_k(s) ds = \lambda_k \varphi_k(t) \quad (\text{In parallel to } A\mathbf{v} = \lambda \mathbf{v})$$

Mercer's theorem

Theorem (Mercer's theorem)

Give a complex continuous function $C(t, s)$ over $[0, T]^2$ that is Hermitian symmetric (i.e., $C(t, s) = C^*(s, t)$) and nonnegative definite (i.e., $\sum_i \sum_j a_i C(t_i, t_j) a_j^* \geq 0$ for any $\{a_i\}$ and $\{t_i\}$). Then the eigenvalues $\{\lambda_k\}$ are reals, and $C(t, s)$ has the following eigen-decomposition

$$C(t, s) = \sum_{k=1}^{\infty} \lambda_k \varphi_k(t) \varphi_k^*(s).$$

Karhunen-Loeve theorem

Theorem (Karhunen-Loève theorem)

Let $\{\mathbf{Z}(t), t \in [0, T]\}$ be a zero-mean random process with a continuous autocorrelation function $R_{\mathbf{Z}}(t, s) = \mathbb{E}[\mathbf{Z}(t)\mathbf{Z}^*(s)]$. Then $\mathbf{Z}(t)$ can be written as

$$\mathbf{Z}(t) \stackrel{\mathcal{M}_2}{=} \sum_{k=1}^{\infty} \mathbf{Z}_k \cdot \varphi_k(t) \quad 0 \leq t < T$$

where “ $=$ ” is in the sense of mean-square,

$$\mathbf{Z}_k = \langle \mathbf{Z}(t), \varphi_k(t) \rangle = \int_0^T \mathbf{Z}(t) \varphi_k^*(t) dt$$

and $\{\varphi_k(t)\}$ are orthonormal eigenfunctions of $R_{\mathbf{Z}}(t, s)$.

- Merit of KL expansion: $\{\mathbf{Z}_k\}$ are uncorrelated. (But samples $\{\mathbf{Z}(k/(2W))\}$ are not uncorrelated even if $\mathbf{Z}(t)$ is bandlimited!)

Proof.

$$\begin{aligned}\mathbb{E}[\mathbf{Z}_i \mathbf{Z}_j^*] &= \mathbb{E}\left[\left(\int_0^T \mathbf{Z}(t) \varphi_i^*(t) dt\right)\left(\int_0^T \mathbf{Z}(s) \varphi_j^*(s) ds\right)^*\right] \\ &= \int_0^T \left(\int_0^T R_{\mathbf{Z}}(t, s) \varphi_j(s) ds\right) \varphi_i^*(t) dt \\ &= \int_0^T \lambda_j \varphi_j(t) \varphi_i^*(t) dt \\ &= \begin{cases} \lambda_j & \text{if } i = j \\ 0 & \left(= \mathbb{E}[\mathbf{Z}_i] E[\mathbf{Z}_j^*] \right) \quad \text{if } i \neq j \end{cases}\end{aligned}$$

□

Lemma

From a given orthonormal set $\{\phi_k(t)\}$, how to minimize the energy of error signal $e(t) = s(t) - \hat{s}(t)$ for $\hat{s}(t)$ spanned by (i.e., expressed as a linear combination of) $\{\phi_k(t)\}$?

Assume $\hat{s}(t) = \sum_k a_k \phi_k(t)$; then

$$\begin{aligned}\|e(t)\|^2 &= \|s(t) - \hat{s}(t)\|^2 \\&= \|s(t) - \sum_{k:k \neq i} a_k \phi_k(t) - a_i \phi_i(t)\|^2 \\&= \|s(t) - \sum_{k:k \neq i} a_k \phi_k(t)\|^2 + \|a_i \phi_i(t)\|^2 \\&\quad - \langle s(t) - \sum_{k:k \neq i} a_k \phi_k(t), a_i \phi_i(t) \rangle \\&\quad - \langle a_i \phi_i(t), s(t) - \sum_{k:k \neq i} a_k \phi_k(t) \rangle \\&= \|s(t) - \sum_{k:k \neq i} a_k \phi_k(t)\|^2 + |a_i|^2 \\&\quad - a_i^* \langle s(t), \phi_i(t) \rangle - a_i \langle \phi_i(t), s(t) \rangle\end{aligned}$$

By taking derivative w.r.t. $\text{Re}\{a_i\}$ and $\text{Im}\{a_i\}$, we obtain
 $a_i = \langle s(t), \phi_i(t) \rangle$. □

Definition

If every finite energy signal $s(t)$ (i.e., $\|s(t)\|^2 < \infty$) satisfies

$$\|e(t)\|^2 = \left\| s(t) - \sum_k \langle s(t), \phi_k(t) \rangle \phi_k(t) \right\|^2 = 0$$

(equivalently,

$$s(t) \stackrel{\mathcal{L}_2}{=} \sum_k \langle s(t), \phi_k(t) \rangle \phi_k(t) = \sum_k a_k \cdot \phi_k(t)$$

(in the sense that the norm of the difference between left-hand-side and right-hand-side is zero), then the set of orthonormal functions $\{\phi_k(t)\}$ is said to be **complete**.

Example (Fourier series)

$$\left\{ \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi kt}{T}\right), \sqrt{\frac{2}{T}} \sin\left(\frac{2\pi kt}{T}\right) : 0 \leq k \in \mathbb{Z} \right\}$$

is a complete orthonormal set for signals defined over $[0, T]$ with finite number of discontinuities. □

- For a complete orthonormal basis, the energy of $s(t)$ is equal to

$$\begin{aligned}\|s(t)\|^2 &= \left\langle \sum_j a_j \phi_j(t), \sum_k a_k \phi_k(t) \right\rangle \\ &= \sum_j \sum_k a_j a_k^* \langle \phi_j(t), \phi_k(t) \rangle \\ &= \sum_j a_j a_j^* \\ &= \sum_j |a_j|^2\end{aligned}$$

- Given a deterministic function $s(t)$, and a set of **complete orthonormal basis** $\{\phi_k(t)\}$ (possibly countably infinite), $s(t)$ can be written as

$$s(t) \stackrel{\mathcal{L}_2}{=} \sum_{k=0}^{\infty} a_k \phi_k(t), \quad 0 \leq t \leq T$$

where

$$a_k = \langle s(t), \phi_k(t) \rangle = \int_0^T s(t) \phi_k^*(t) dt.$$

In addition,

$$\|s(t)\|^2 = \sum_k |a_k|^2.$$

Remark

In terms of energy (and error rate):

- A bandpass signal $s(t)$ can be equivalently “analyzed” through lowpass equivalent signal $s_\ell(t)$ without the burden of carrier freq f_c ;
- A lowpass equivalent signal $s_\ell(t)$ can be equivalently “analyzed through (countably many) $\{a_k = \langle s_\ell(t), \phi_k(t) \rangle\}$ without the burden of continuous waveforms.

Gram-Schmidt procedure

Given a set of functions $v_1(t), v_2(t), \dots, v_k(t)$

① $\phi_1(t) = \frac{v_1(t)}{\|v_1(t)\|}$

② Compute for $i = 2, 3, \dots, k$ (or until $\|\phi_i(t)\| = 0$),

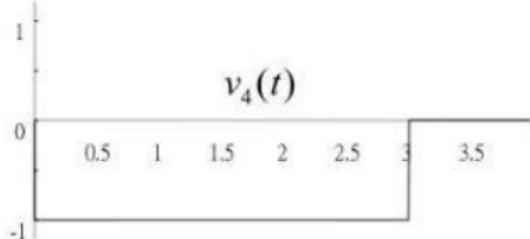
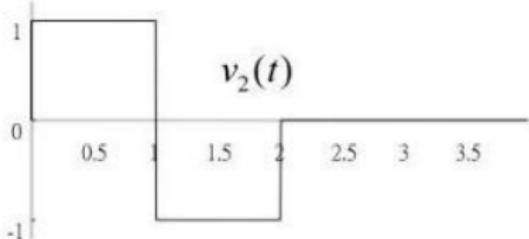
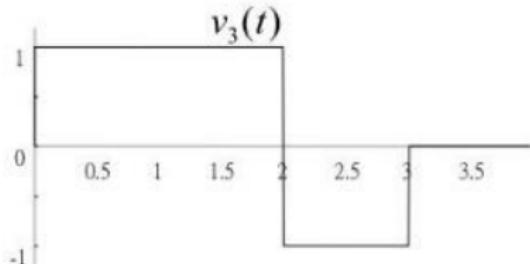
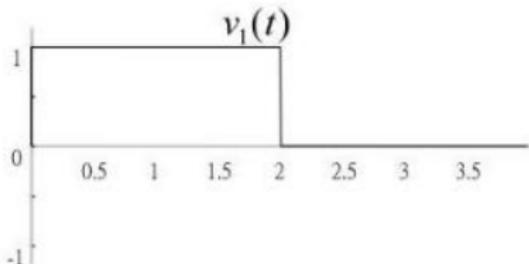
$$\gamma_i(t) = v_i(t) - \sum_{j=1}^{i-1} \langle v_i(t), \phi_j(t) \rangle \phi_j(t)$$

and set $\phi_i(t) = \frac{\gamma_i(t)}{\|\gamma_i(t)\|}$.

This gives an orthonormal basis $\phi_1(t), \phi_2(t), \dots, \phi_{k'}(t)$, where $k' \leq k$.

Example

Find a Gram-Schmidt orthonormal basis of the following signals.



Sol.

- $\phi_1(t) = \frac{v_1(t)}{\|v_1(t)\|} = \frac{v_1(t)}{\sqrt{2}}$

-

$$\begin{aligned}\gamma_2(t) &= v_2(t) - \langle v_2(t), \phi_1(t) \rangle \phi_1(t) \\ &= v_2(t) - \left(\int_0^3 v_2(t) \phi_1^*(t) dt \right) \phi_1(t) = v_2(t)\end{aligned}$$

Hence $\phi_2(t) = \frac{\gamma_2(t)}{\|\gamma_2(t)\|} = \frac{v_2(t)}{\sqrt{2}}.$

-

$$\begin{aligned}\gamma_3(t) &= v_3(t) - \langle v_3(t), \phi_1(t) \rangle \phi_1(t) - \langle v_3(t), \phi_2(t) \rangle \phi_2(t) \\ &= v_3(t) - \sqrt{2} \phi_1(t) - 0 \cdot \phi_2(t) = \begin{cases} -1, & 2 \leq t < 3 \\ 0, & \text{otherwise} \end{cases}\end{aligned}$$

Hence $\phi_3(t) = \frac{\gamma_3(t)}{\|\gamma_3(t)\|}.$

$$\begin{aligned}\gamma_4(t) &= v_4(t) - \langle v_4(t), \phi_1(t) \rangle \phi_1(t) - \langle v_4(t), \phi_2(t) \rangle \phi_2(t) \\ &\quad - \langle v_4(t), \phi_3(t) \rangle \phi_3(t) \\ &= v_4(t) - (-\sqrt{2})\phi_1(t) - (0)\phi_2(t) - \phi_3(t) = 0\end{aligned}$$

- Orthonormal basis = $\{\phi_1(t), \phi_2(t), \phi_3(t)\}$, where $3 \leq 4$.

Example

Represent the signals in slide 95 in terms of the orthonormal basis obtained in the same example.

Sol.

$$v_1(t) = \sqrt{2}\phi_1(t) + 0 \cdot \phi_2(t) + 0 \cdot \phi_3(t) \implies [\sqrt{2}, 0, 0]$$

$$v_2(t) = 0 \cdot \phi_1(t) + \sqrt{2} \cdot \phi_2(t) + 0 \cdot \phi_3(t) \implies [0, \sqrt{2}, 0]$$

$$v_3(t) = \sqrt{2}\phi_1(t) + 0 \cdot \phi_2(t) + 1 \cdot \phi_3(t) \implies [\sqrt{2}, 0, 1]$$

$$v_4(t) = -\sqrt{2}\phi_1(t) + 0 \cdot \phi_2(t) + 1 \cdot \phi_3(t) \implies [-\sqrt{2}, 0, 1]$$



The vectors are named **signal space representations** or **constellations** of the signals.

Remark

The orthonormal basis is **not unique**.

For example, for $k = 1, 2, 3$, re-define

$$\phi_k(t) = \begin{cases} 1, & k-1 \leq t < k \\ 0, & \text{otherwise} \end{cases}$$

Then

$$v_1(t) \xrightarrow{\Phi} (+1, +1, 0)$$

$$v_2(t) \xrightarrow{\Phi} (+1, -1, 0)$$

$$v_3(t) \xrightarrow{\Phi} (+1, +1, -1)$$

$$v_4(t) \xrightarrow{\Phi} (-1, -1, -1)$$

Euclidean distance

$s_1(t) \implies (a_1, a_2, \dots, a_n)$ for some complete basis

$s_2(t) \implies (b_1, b_2, \dots, b_n)$ for the same complete basis

$d_{12} =$ Euclidean distance between $s_1(t)$ and $s_2(t)$

$$= \sqrt{\sum_{i=1}^n (a_i - b_i)^2}$$

$$= \|s_1(t) - s_2(t)\| \quad \left(= \sqrt{\int_0^T |s_1(t) - s_2(t)|^2 dt} \right)$$

Bandpass and lowpass orthonormal basis

- Now let's change our focus from $[0, T)$ to $(-\infty, \infty)$

- A time-limited signal cannot be bandlimited to W .
- A bandlimited signal cannot be time-limited to T .

Hence, in order to talk about the **ideal** bandlimited signal, we have to deal with signals with unlimited time span.

- Re-define the inner product as:

$$\langle f(t), g(t) \rangle = \int_{-\infty}^{\infty} f(t)g^*(t) dt$$

- Let $s_{1,\ell}(t)$ and $s_{2,\ell}(t)$ be lowpass equivalent signals of the bandpass $s_1(t)$ and $s_2(t)$

$$S_{1,\ell}(f) = S_{2,\ell}(f) = 0 \text{ for } |f| > f_B$$

$$s_i(t) = \mathbf{Re} \left\{ s_{i,\ell}(t) e^{i 2\pi f_c t} \right\} \text{ where } f_c \gg f_B$$

Then, as we have proved in slide 24,

$$\langle s_1(t), s_2(t) \rangle = \frac{1}{2} \mathbf{Re} \left\{ \langle s_{1,\ell}(t), s_{2,\ell}(t) \rangle \right\}.$$

Proposition

If $\langle s_{1,\ell}(t), s_{2,\ell}(t) \rangle = 0$, then $\langle s_1(t), s_2(t) \rangle = 0$.

Proposition

If $\{\phi_{n,\ell}(t)\}$ is a complete basis for the set of lowpass signals, then

$$\begin{cases} \phi_n(t) = \mathbf{Re} \left\{ (\sqrt{2}\phi_{n,\ell}(t)) e^{i2\pi f_c t} \right\} \\ \tilde{\phi}_n(t) = -\mathbf{Im} \left\{ (\sqrt{2}\phi_{n,\ell}(t)) e^{i2\pi f_c t} \right\} \\ \quad = \mathbf{Re} \left\{ (i\sqrt{2}\phi_{n,\ell}(t)) e^{i2\pi f_c t} \right\} \end{cases}$$

is a *complete orthonormal set* for the set of bandpass signals.

Proof: First, orthonormality can be proved by

$$\langle \phi_n(t), \phi_m(t) \rangle = \frac{1}{2} \mathbf{Re} \left\{ \left\langle \sqrt{2}\phi_{n,\ell}(t), \sqrt{2}\phi_{m,\ell}(t) \right\rangle \right\} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

$$\langle \tilde{\phi}_n(t), \tilde{\phi}_m(t) \rangle = \frac{1}{2} \mathbf{Re} \left\{ \left\langle i\sqrt{2}\phi_{n,\ell}(t), i\sqrt{2}\phi_{m,\ell}(t) \right\rangle \right\} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

and

$$\begin{aligned}\langle \tilde{\phi}_n(t), \phi_m(t) \rangle &= \frac{1}{2} \mathbf{Re} \left\{ \left\langle i\sqrt{2}\phi_{n,\ell}(t), \sqrt{2}\phi_{m,\ell}(t) \right\rangle \right\} \\ &= \mathbf{Re} \{ i \langle \phi_{n,\ell}(t), \phi_{m,\ell}(t) \rangle \} \\ &= \begin{cases} \mathbf{Re} \{ i \} = 0 & n = m \\ 0 & n \neq m \end{cases}\end{aligned}$$

Now, with

$$\begin{cases} s(t) = \mathbf{Re} \{ s_\ell(t) e^{i2\pi f_c t} \} \\ \hat{s}(t) = \mathbf{Re} \{ \hat{s}_\ell(t) e^{i2\pi f_c t} \} \\ \hat{s}_\ell(t) \stackrel{\mathcal{L}_2}{=} \sum_n a_{n,\ell} \phi_{n,\ell}(t) \text{ with } a_{n,\ell} = \langle s_\ell(t), \phi_{n,\ell}(t) \rangle \\ \|s_\ell(t) - \hat{s}_\ell(t)\|^2 = 0 \end{cases}$$

we have

$$\|s(t) - \hat{s}(t)\|^2 = \frac{1}{2} \|s_\ell(t) - \hat{s}_\ell(t)\|^2 = 0$$

and

$$\begin{aligned}\hat{s}(t) &= \mathbf{Re} \left\{ \hat{s}_\ell(t) e^{\imath 2\pi f_c t} \right\} \\ &= \mathbf{Re} \left\{ \sum_n a_{n,\ell} \phi_{n,\ell}(t) e^{\imath 2\pi f_c t} \right\} \\ &= \sum_n \left(\mathbf{Re} \left\{ \frac{a_{n,\ell}}{\sqrt{2}} \right\} \mathbf{Re} \left\{ \sqrt{2} \phi_{n,\ell}(t) e^{\imath 2\pi f_c t} \right\} \right. \\ &\quad \left. + \mathbf{Im} \left\{ \frac{a_{n,\ell}}{\sqrt{2}} \right\} \mathbf{Im} \left\{ \left(-\sqrt{2} \phi_{n,\ell}(t) \right) e^{\imath 2\pi f_c t} \right\} \right) \\ &= \sum_n \left(\mathbf{Re} \left\{ \frac{a_{n,\ell}}{\sqrt{2}} \right\} \phi_n(t) + \mathbf{Im} \left\{ \frac{a_{n,\ell}}{\sqrt{2}} \right\} \tilde{\phi}_n(t) \right)\end{aligned}$$

□

What you learn from Chapter 2

- Random process
 - WSS
 - autocorrelation and crosscorrelation functions
 - PSD and CSD
 - White and filtered white
- Relation between (bandlimited) bandpass and lowpass equivalent deterministic signals
- Relation between (bandlimited) bandpass and lowpass equivalent random signals
 - Properties of autocorrelation and power spectrum density
- Role of Hilbert transform Signal space concept

