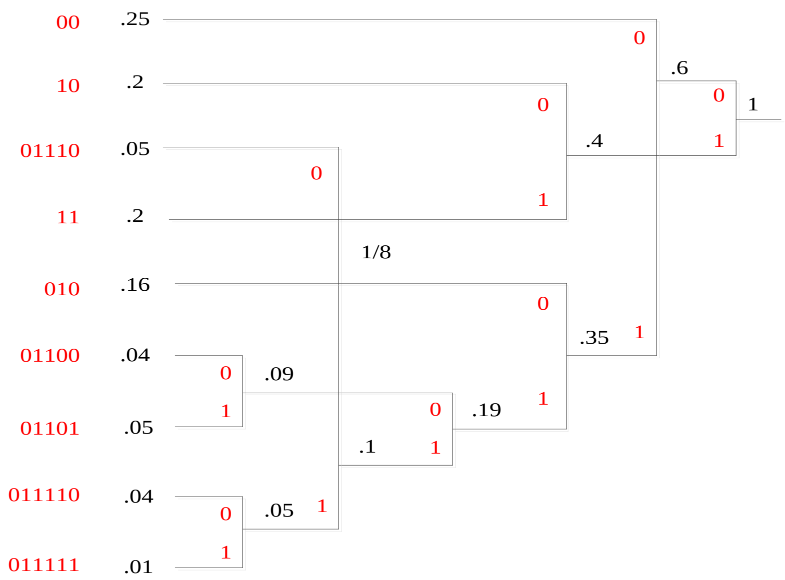
Problem Set 4

Suggested Solutions

1. Let X = {x_1, x_2, x_3} be a ternary random variable with probability distribution {0.5, 0.4, 0.1}.
   a) Construct a binary Huffman code for X. Calculate the code's expected length $\bar{L}_1$ and code efficiency $\eta_1 = H(X)/\bar{L}_1$.
   b) Construct a binary Huffman code for two i.i.d. copies X^2 of X, calculate the code's expected length $\bar{L}_2$ and code efficiency $\eta_2 = H(X^2)/\bar{L}_2 = 2H(X)/\bar{L}_2$.
   c) Make a comparison between η_1 and η_2. If a binary Huffman code is used for n i.i.d. copies X^n, what is the asymptotic value of $\eta_n = H(X^n)/\bar{L}_n = nH(X)/\bar{L}_n$ when $n \to \infty$?

Sol. a) Binary Huffman code: {0, 10, 11}, expected length $\bar{L}_1$ = 1*.5+2*.4+2*.1 = 1.5, code efficiency η_1 = H(0.5, 0.4, 0.1)/1.5 = 1.361/1.5 = .9073.

b) The probability distribution of X^2 is (.25, .2, .05, .2, .16, .04, .05, .04, .01). The construction of a binary Huffman code (not unique) is



$\bar{L}_2$ = .5+.4+.25+.4+.48+.2+.25+.24+.06 = 2.78, code efficiency η_2 = 2*H(0.5, 0.4, 0.1)/2.78 = 2.722/2.78 = .9791.

c) η_n tends to 1 with $n \to \infty$.

2. Consider 4 different codes:

{000, 10, 00, 11};
{100, 101, 0, 11};
{01, 100, 011, 00, 111, 1010, 1011, 1101};
{01, 111, 011, 00, 010, 110}

a) Which ones do not satisfy Kraft inequality?
b) Is each of these codes qualified as a prefix code? If not, please explain.

Sol. a) 3rd one does not satisfy Kraft inequality.

b) 1st (including 000 and 00), 3rd (including 01 and 011), 4th (including 01 and 011) are not prefix codes.

3. For the following three codes, which ones cannot be constructed by Huffman coding?
   a) {0, 10, 11};
   b) {00, 01, 10, 110};
   c) {01, 10};

Sol. b) and c) cannot be constructed by Huffman coding, because code b) can be reduced to another uniquely decodable code {00,01,10,11}, which has a shorter expected length, and c) can be reduced to another uniquely decodable code {0, 1}.

4. Consider a binary symmetric erasure channel (BSEC) with X = {0, 1}, Y = {0, e, 1}, and $p_{Y|X} = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$. Assume $P(X = 0) = p_0$, $P(X = 1) = 1 - p_0$.
   a) Calculate H(Y|X);
   b) Find the distribution of Y;
   c) What is the value of p_0 that maximizes H(Y)?
   d) What is the capacity C of this channel?

Sol. a) $H(Y|X) = p_0H(Y \mid X = 0) + (1 - p_0)H(Y \mid X = 1)$

$= p_0H(1/2, 1/4, 1/4) + (1 - p_0)H(1/4, 1/4, 1/2) = H(1/2, 1/4, 1/4) = 3/2.$

b) $p(Y = 0) = (1/2)p_0 + (1/4)(1 - p_0) = (1/4)(p_0+1)$,

$p(Y = 1) = (1/4)p_0 + (1/4)(1 - p_0) = 1/4$, $p(Y = 2) = (1/4)((1-p_0) +1)$.

c) When $p(Y = 0) = p(Y = 2)$, i.e., when $p_0 = 1/2$, H(Y) is maximized.

d) $C = \max_{p0} I(X;Y) = \max_{p0}H(Y) - H(Y \mid X)$

$= H(3/8, 1/4, 3/8) - 3/2 = 2*(3/8)\log(8/3)+(1/4)\log4 - 3/2$

$= 5/4 - (3/4)\log3 = 0.0613.$