Lecture 3
Point to Point Communications

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Wire vs. Wireless Communication

Wired Channel

$$y[m] = \sum_l h_l x[m-l] + w[m]$$

- Deterministic channel gains
- Main issue: combat noise
- Key technique: coding to exploit degrees of freedom and increase data rate

(coding gain)

Wireless Channel

$$y[m] = \sum_l h_l[m] x[m-l] + w[m]$$

- Random channel gains
- Main issue: combat fading
- Key technique: coding to exploit diversity and increase reliability

(diversity gain)

- **Remark**: In wireless channel, there is still additive noise, and hence the techniques developed in wire communication are still useful.

Plot

- Study detection in flat fading channel to learn
  - Communication over flat fading channel has poor performance due to significant probability that the channel is in deep fade
  - How the performance scale with SNR
- Investigate various techniques to provide diversity across
  - Time
  - Frequency
  - Space
- Key: how to exploit additional diversity efficiently

Outline

- Detection in Rayleigh fading channel vs. static AWGN channel
- Code design and degrees of freedom
- Time diversity
- Antenna (space) diversity
- Frequency diversity

Detection In
Rayleigh Fading Channel

Baseline: AWGN Channel

$$y = x + w, \quad w \sim \mathcal{CN}\left(0, \sigma^2\right)$$

BPSK: $x = \pm a$

Transmitted constellation is real, it suffices to consider the real part:

$$\Re\{y\} = x + \Re\{w\}, \quad \Re\{w\} \sim \mathcal{N}\left(0, \sigma^2/2\right)$$

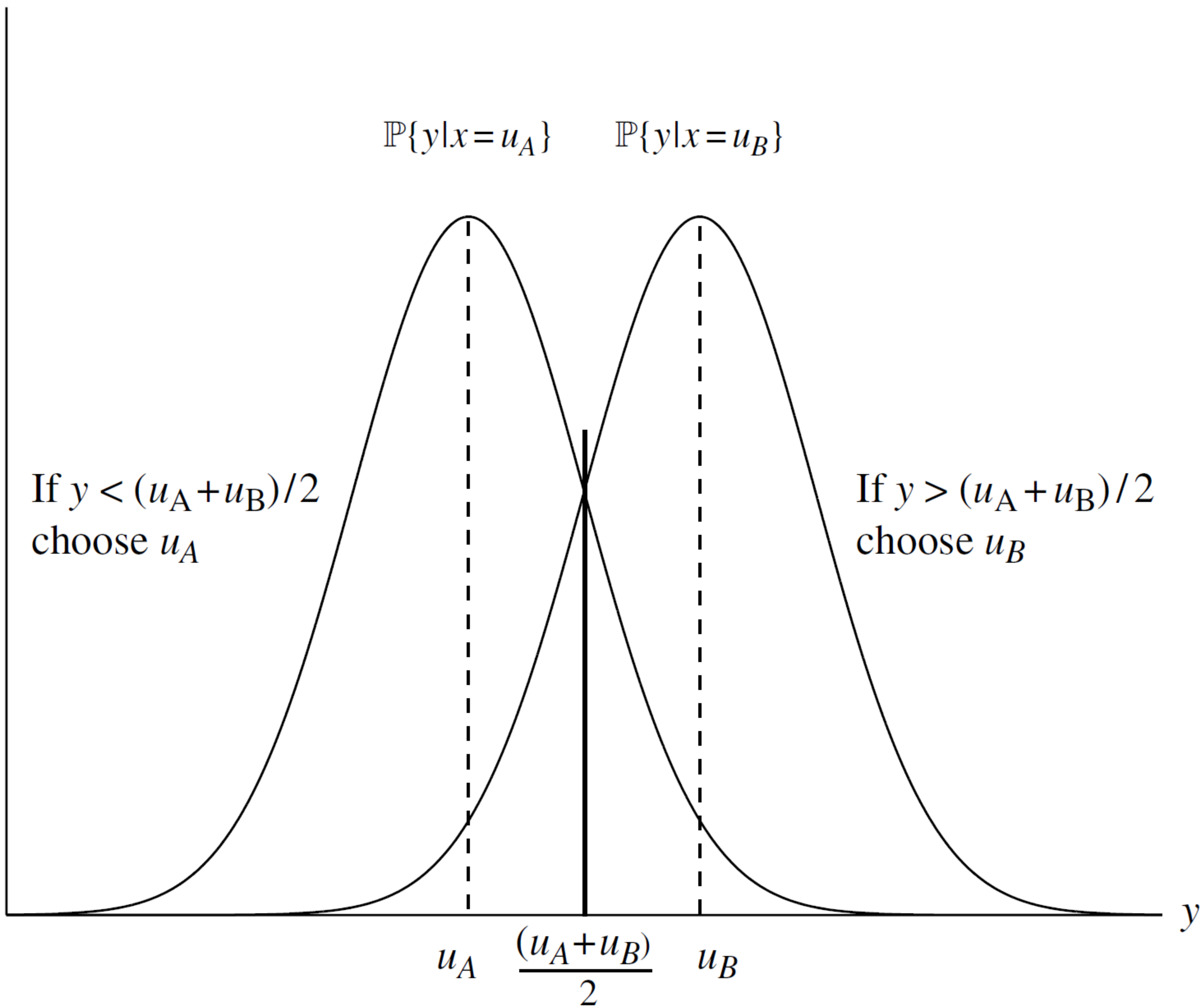
ML rule: $\widehat{x} = \begin{cases} a, & \text{if } |\Re\{y\} - a| < |\Re\{y\} - (-a)| \\ -a, & \text{otherwise} \end{cases}$

Probability of error:

$$\Pr\{\mathcal{E}\} = \Pr\left\{\Re\{w\} > \frac{a - (-a)}{2}\right\} = Q\left(\frac{a}{\sqrt{\sigma^2/2}}\right) = \boxed{Q\left(\sqrt{2\mathsf{SNR}}\right)}$$

$$\boxed{\mathsf{SNR} := \frac{\text{average received signal energy per (complex) symbol time}}{\text{noise energy per (complex) symbol time}}} \begin{matrix} a^2 \\ \sigma^2 \end{matrix}$$

Gaussian Scalar Detection



Gaussian Vector Detection

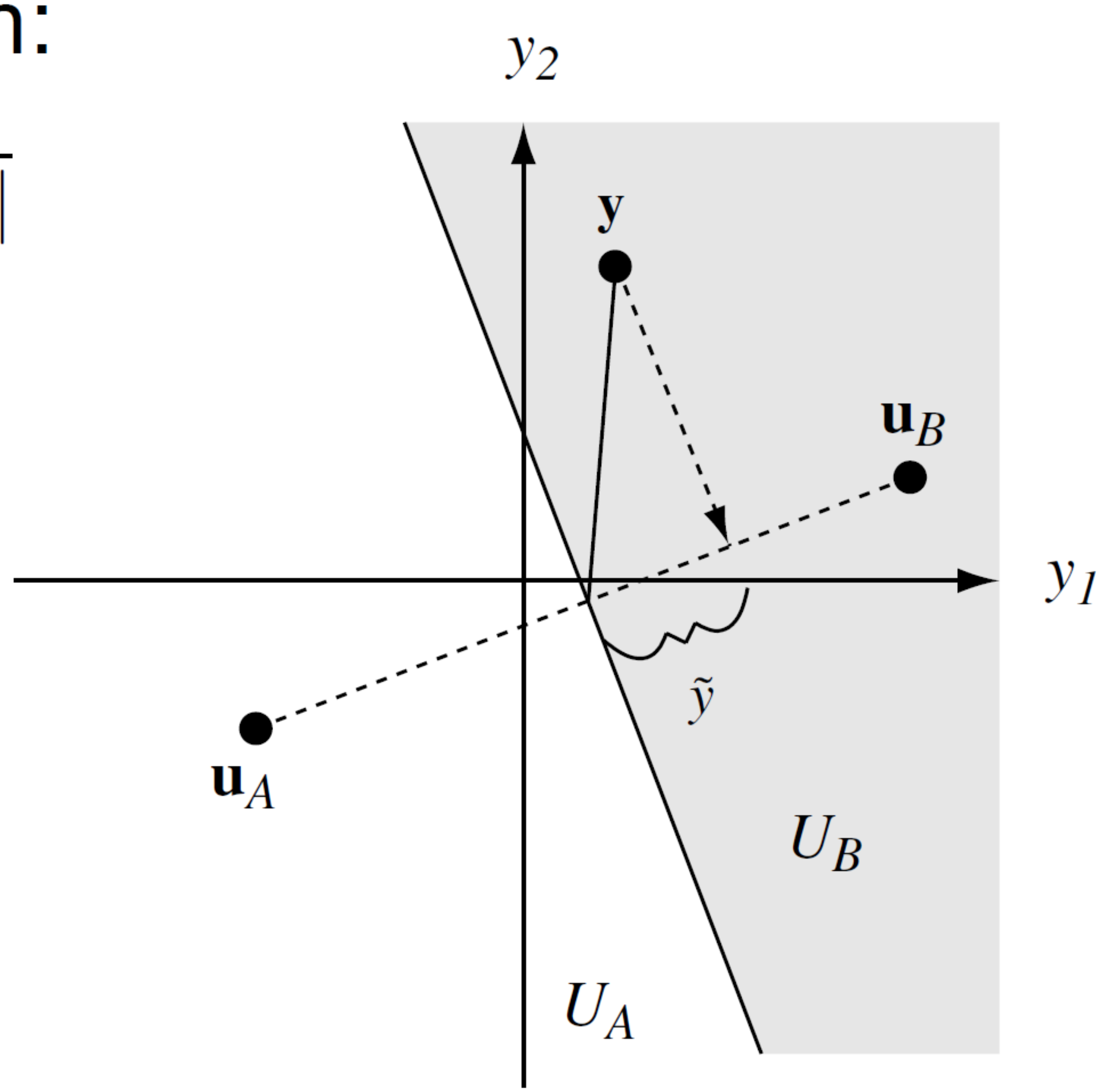
$$\mathbf{y} = \mathbf{x} + \mathbf{w}, \quad \mathbf{w} \sim \mathcal{CN}\left(0, \sigma^2 \mathbf{I}\right)$$

- Sufficient statistic for detection: projection on to $\mathbf{v} := \dfrac{\mathbf{u}_A - \mathbf{u}_B}{||\mathbf{u}_A - \mathbf{u}_B||}$

$$\widetilde{y} := \mathbf{v}^* \left(\mathbf{y} - \frac{\mathbf{u}_A + \mathbf{u}_B}{2}\right) = \widetilde{x} + \widetilde{w}$$

$$\widetilde{x} := \mathbf{v}^* \left(\mathbf{x} - \frac{\mathbf{u}_A + \mathbf{u}_B}{2}\right)$$

$$= \begin{cases} \frac{||\mathbf{u}_A - \mathbf{u}_B||}{2}, & \mathbf{x} = \mathbf{u}_A \\ -\frac{||\mathbf{u}_A - \mathbf{u}_B||}{2}, & \mathbf{x} = \mathbf{u}_B \end{cases}$$



- Since $\mathbf{w}$ is circular symmetric,

$$\implies \widetilde{w} \sim \mathcal{CN}\left(0, \sigma^2\right)$$

Binary Detection in Gaussian Noise

$$\mathbf{y} = \mathbf{x} + \mathbf{w}, \quad \mathbf{w} \sim \mathcal{CN}\left(0, \sigma^2 \mathbf{I}\right)$$

Binary signaling: $\mathbf{x} = \mathbf{u}_A, \mathbf{u}_B$

It suffices to consider the projection onto $(\mathbf{u}_A - \mathbf{u}_B)$

$$\widetilde{y} = x||\mathbf{u}_A - \mathbf{u}_B|| + \widetilde{w}, \quad x = \pm\frac{1}{2}, \quad \widetilde{w} \sim \mathcal{CN}\left(0, \sigma^2\right)$$

Probability of error:

$$\Pr\left\{\Re\{w\} > \frac{||\mathbf{u}_A - \mathbf{u}_B||}{2}\right\} = \boxed{Q\left(\frac{||\mathbf{u}_A - \mathbf{u}_B||}{2\sqrt{\sigma^2/2}}\right)}$$

Rayleigh Fading Channel

$$y = hx + w, \quad h \sim \mathcal{CN}(0,1), \quad w \sim \mathcal{CN}(0,\sigma^2)$$

BPSK: $x = \pm a \qquad \text{SNR} = \frac{\mathbb{E}\left[|h|^2\right] a^2}{\sigma^2} = \frac{a^2}{\sigma^2}$

- Note: $|h|$ is an exponential random variable with mean 1
  - Fair comparison with the AWGN case (same avg. signal power)
- Coherent detection:
  - The receiver knows h perfectly (channel estimation through pilots)
  - For a given realization of h, the error probability is

$$\Pr\{\mathcal{E} \mid h\} = Q\left(\frac{a|h|}{\sqrt{\sigma^2/2}}\right) = Q\left(\sqrt{2|h|^2\text{SNR}}\right)$$

- Probability of error:

$$\Pr\{\mathcal{E}\} = \mathbb{E}\left[Q\left(\sqrt{2|h|^2\text{SNR}}\right)\right] = \boxed{\frac{1}{2}\left(1 - \sqrt{\frac{\text{SNR}}{1+\text{SNR}}}\right)}$$

Check!
Hint: exchange the order in the double integral

Non-coherent Detection

$$y = hx + w, \quad h \sim \mathcal{CN}(0,1), \quad w \sim \mathcal{CN}(0,\sigma^2)$$

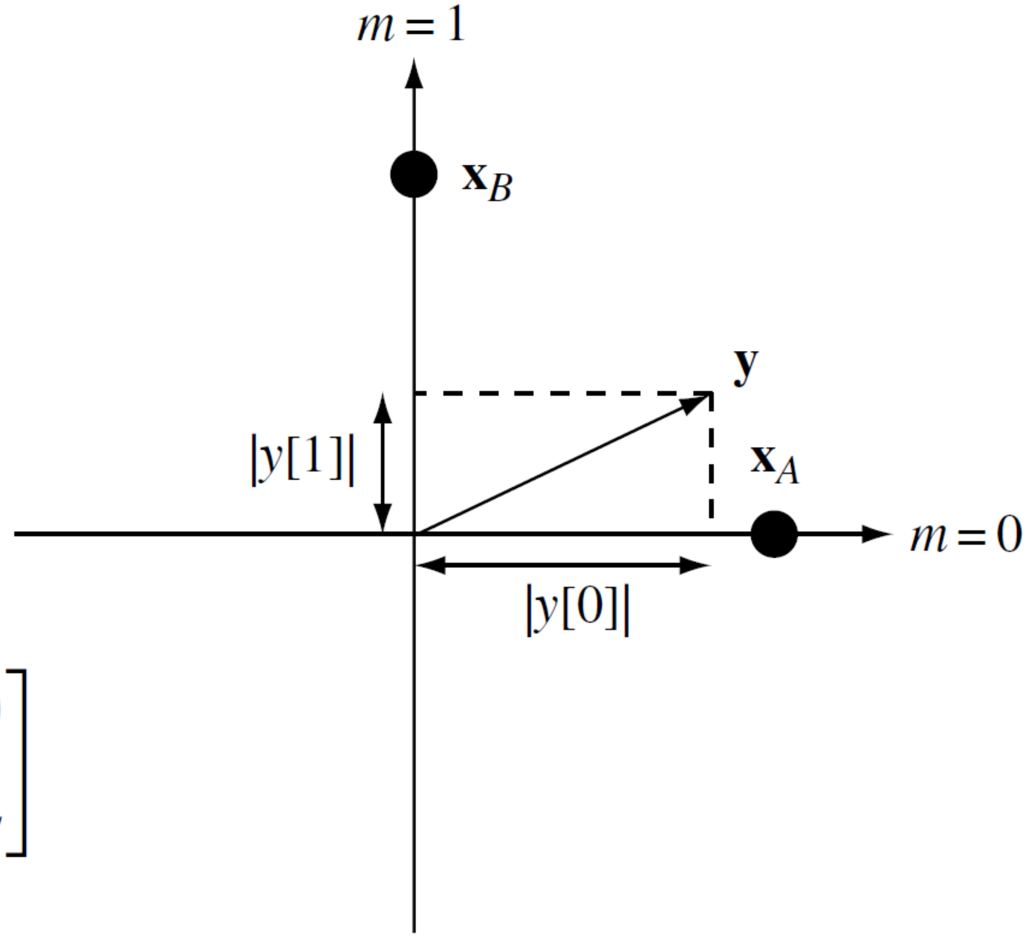
- If Rx does not know the realization of h:
  - Scalar BPSK ($x = \pm a$) completely fails
  - Because the phase of h is uniform over $[0, 2\pi]$
- Orthogonal modulation:
  - Use two time slots $m = 0{,}1$

$$\Longrightarrow \mathbf{y} := \begin{bmatrix} y[0] \\ y[1] \end{bmatrix} = h \begin{bmatrix} x[0] \\ x[1] \end{bmatrix} + \begin{bmatrix} w[0] \\ w[1] \end{bmatrix}$$

$$:= h\mathbf{x} + \mathbf{w}$$

  - Modulation:

$$\mathbf{x}_A = \begin{bmatrix} a \\ 0 \end{bmatrix} \quad \text{or} \quad \mathbf{x}_B = \begin{bmatrix} 0 \\ a \end{bmatrix}$$



Non-coherent Detection

$$\mathbf{y} = h\mathbf{x} + \mathbf{w}, \quad h \sim \mathcal{CN}\left(0, 1\right), \quad \mathbf{w} \sim \mathcal{CN}\left(0, \sigma^2 \mathbf{I}_2\right)$$

Orthogonal modulation: $\mathbf{x}_A = \begin{bmatrix} a \\ 0 \end{bmatrix}$ or $\mathbf{x}_B = \begin{bmatrix} 0 \\ a \end{bmatrix}$ $\quad \text{SNR} = \dfrac{a^2}{2\sigma^2}$

- ML rule:
  - Given $\mathbf{x} = \mathbf{x}_A \implies \mathbf{y} \sim \mathcal{CN}\left(0, \begin{bmatrix} a^2 + \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}\right)$

    $\mathbf{x} = \mathbf{x}_B \implies \mathbf{y} \sim \mathcal{CN}\left(0, \begin{bmatrix} \sigma^2 & 0 \\ 0 & a^2 + \sigma^2 \end{bmatrix}\right)$
  - LLR:

$$\Lambda(\mathbf{y}) := \ln \frac{f(\mathbf{y} \mid \mathbf{x}_A)}{f(\mathbf{y} \mid \mathbf{x}_B)} = \frac{a^2}{(a^2 + \sigma^2)\sigma^2} \left(|y[0]|^2 - |y[1]|^2\right)$$

$$\frac{\left\{\left(\sigma^2 + a^2\right)|y[0]|^2 + \sigma^2 |y(1)|^2\right\} - \left\{\sigma^2 |y(0)|^2 + \left(\sigma^2 + a^2\right)|y[0]|^2\right\}}{(a^2 + \sigma^2)\sigma^2}$$

- Energy detector: $\boxed{\begin{array}{l} \hat{\mathbf{x}} = \mathbf{x}_A \iff |y[0]| > |y[1]| \\ \hat{\mathbf{x}} = \mathbf{x}_B \iff |y[0]| < |y[1]| \end{array}}$

Non-Coherent Detection

$$\mathbf{y} = h\mathbf{x} + \mathbf{w}, \quad h \sim \mathcal{CN}\left(0,1\right), \quad \mathbf{w} \sim \mathcal{CN}\left(0, \sigma^2 \mathbf{I}_2\right)$$

Orthogonal modulation: $\mathbf{x}_A = \begin{bmatrix} a \\ 0 \end{bmatrix}$ or $\mathbf{x}_B = \begin{bmatrix} 0 \\ a \end{bmatrix}$ $\quad \mathsf{SNR} = \frac{a^2}{2\sigma^2}$

- Probability of error:
  - Given $\mathbf{x} = \mathbf{x}_A \implies \mathbf{y} \sim \mathcal{CN}\left(0, \begin{bmatrix} a^2+\sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}\right)$

$$\implies |y[0]|^2 \sim \mathrm{Exp}\left((a^2+\sigma^2)^{-1}\right), \ |y[1]|^2 \sim \mathrm{Exp}\left((\sigma^2)^{-1}\right)$$

$$|y[0]|^2 \text{ and } |y[1]|^2 \text{ are independent}$$

  - Hence $\Pr\left\{\mathcal{E}\right\} = \Pr\left\{\mathrm{Exp}\left((\sigma^2)^{-1}\right) > \mathrm{Exp}\left((a^2+\sigma^2)^{-1}\right)\right\}$

Check!

$$= \frac{(a^2+\sigma^2)^{-1}}{(\sigma^2)^{-1} + (a^2+\sigma^2)^{-1}} = \frac{1}{2 + a^2/\sigma^2}$$

$$= \boxed{\frac{1}{2(1+\mathsf{SNR})}}$$

Comparison: AWGN vs. Rayleigh

- AWGN: Error probability decays faster than $e^{\text{-SNR}}$

$$\Pr\{\mathcal{E}\} = Q\left(\sqrt{2\mathsf{SNR}}\right) \approx \frac{1}{\sqrt{2\mathsf{SNR}}\sqrt{2\pi}} e^{-\mathsf{SNR}} \quad \text{at high SNR}$$

$$Q(x) := \Pr\{\mathcal{N}(0,1) > a\} \approx \frac{1}{x\sqrt{2\pi}} e^{-x^2/2} \quad \text{when } x \gg 1$$

- Rayleigh fading: Error probability decays as SNR^{-1}
  - Coherent detection:

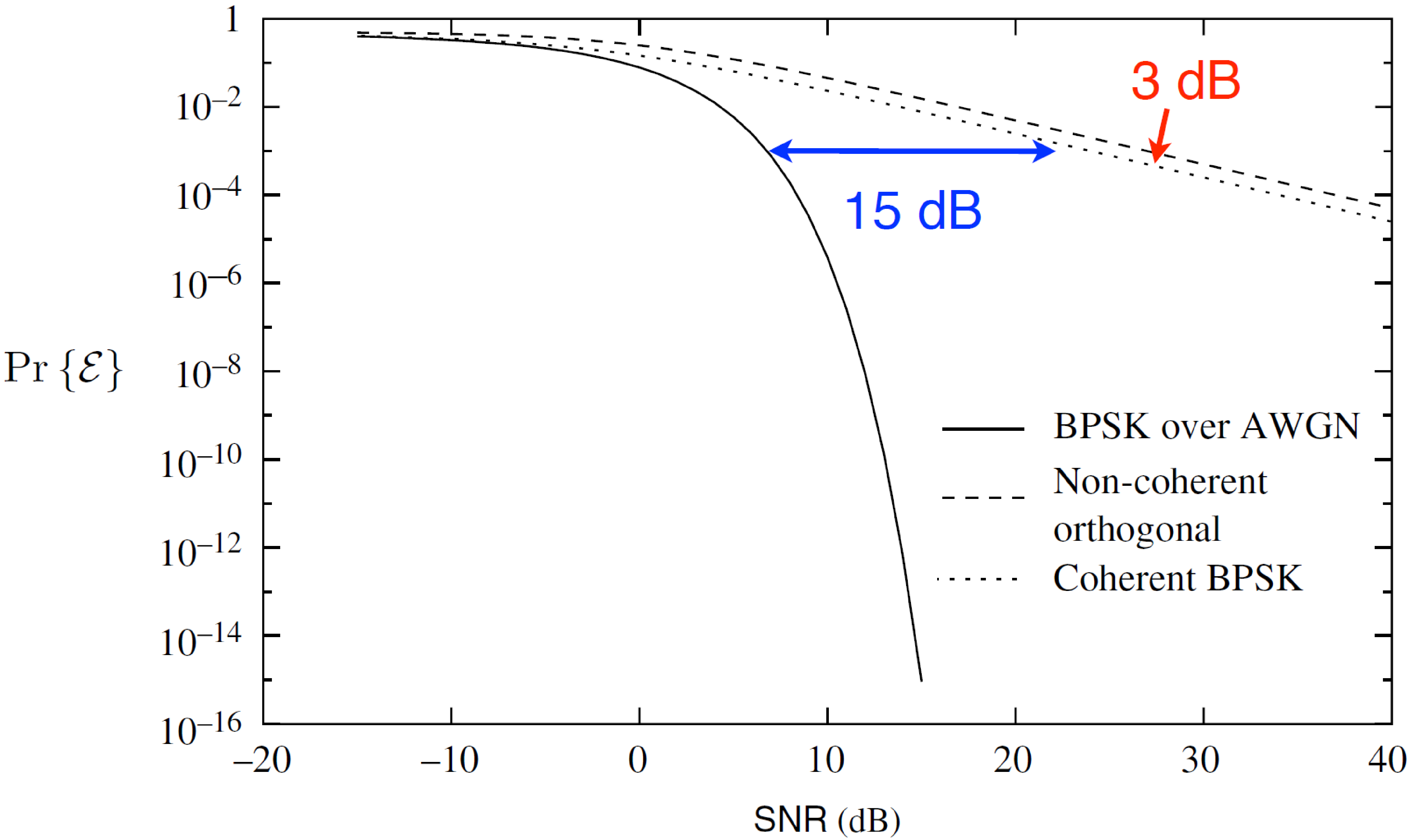
$$\Pr\{\mathcal{E}\} = \frac{1}{2}\left(1 - \sqrt{\frac{\mathsf{SNR}}{1+\mathsf{SNR}}}\right) \approx (4\mathsf{SNR})^{-1} \quad \text{at high SNR}$$

$$\sqrt{\frac{x}{1+x}} = \left(1 - \frac{1}{1+x}\right)^{1/2} \approx 1 - \frac{1}{2(1+x)} \approx 1 - \frac{1}{2x} \quad \text{when } x \gg 1$$

  - Non-coherent detection:

$$\Pr\{\mathcal{E}\} = \frac{1}{2(1+\mathsf{SNR})} \approx (2\mathsf{SNR})^{-1} \quad \text{at high SNR}$$

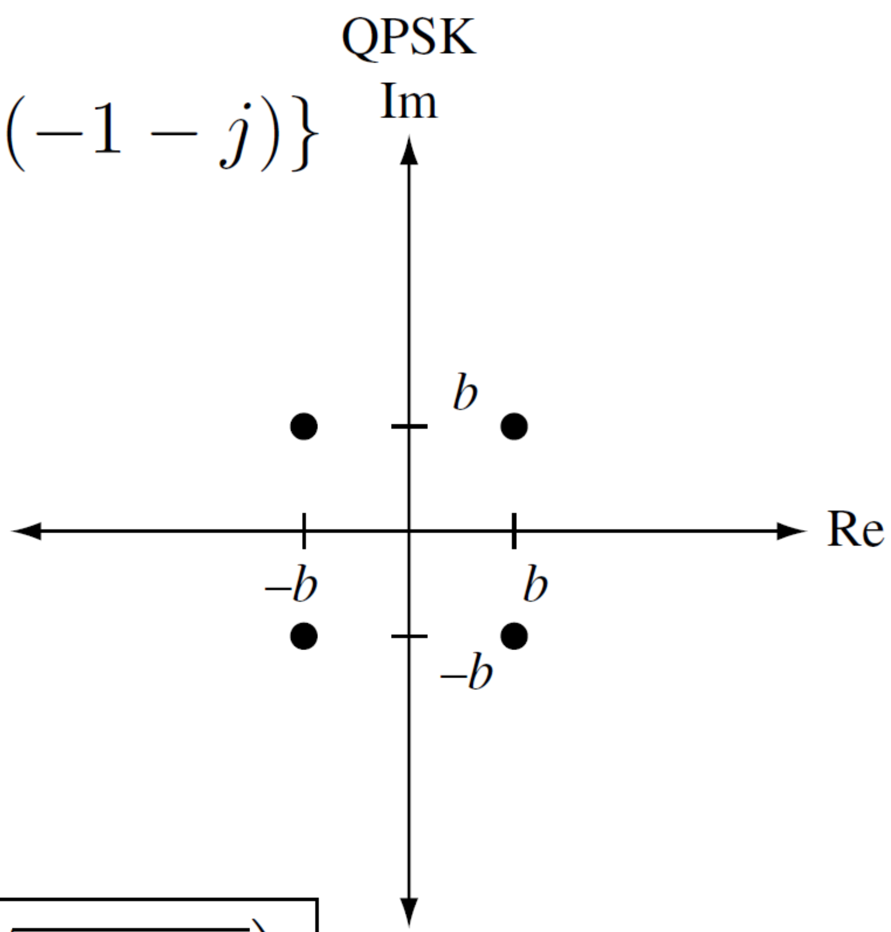
Comparison: AWGN vs. Rayleigh



Coherent Detection under QPSK

- BPSK only makes use of the real dimension (I channel)
- Rate can be increased if an additional bit is sent on the imaginary dimension (Q channel)
- QPSK: $x \in \{b(1+j), b(1-j), b(-1+j), b(-1-j)\}$

$$\text{SNR} = \frac{2b^2}{\sigma^2}$$ Double of BPSK!

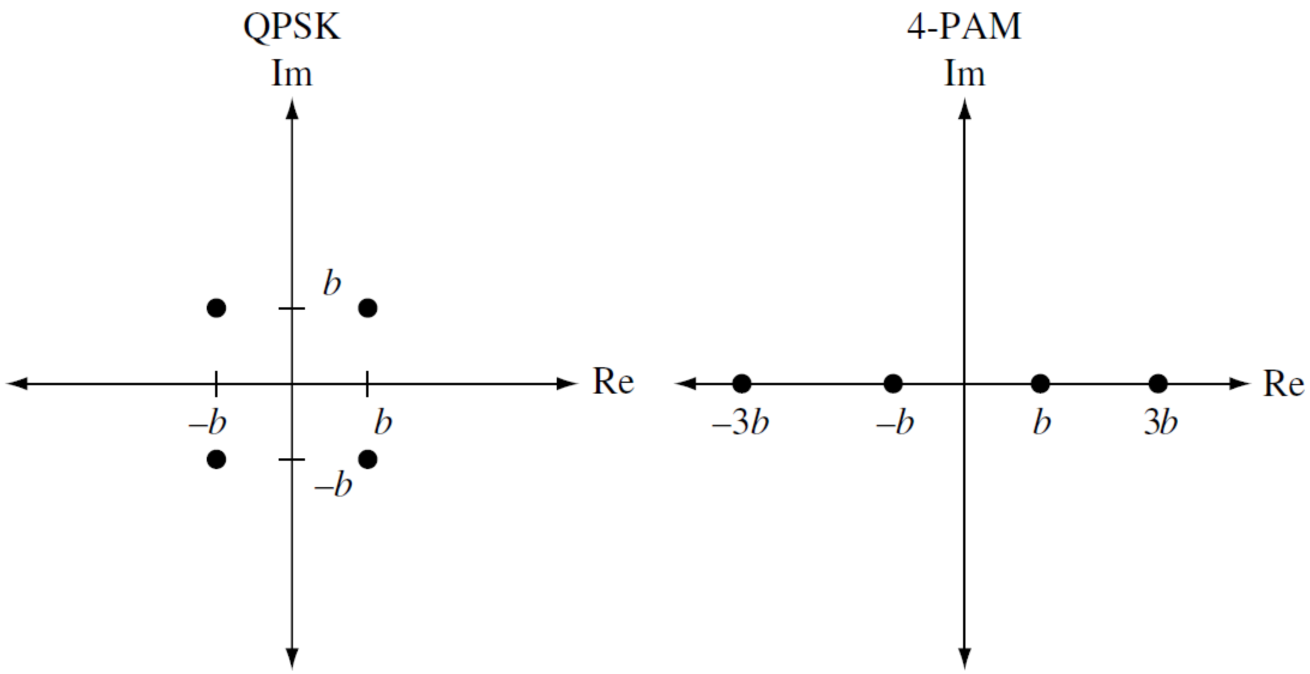


- (Bit) Probability of error
  - Simply a product of two BPSK
  - Analysis is the same
  - Simply replace SNR by SNR/2

$$\Pr\{\mathcal{E}\}_{\text{AWGN}} = \boxed{Q\left(\sqrt{\text{SNR}}\right)} \qquad \Pr\{\mathcal{E}\}_{\text{Rayleigh}} = \boxed{\frac{1}{2}\left(1 - \sqrt{\frac{\text{SNR}}{2+\text{SNR}}}\right)}$$

$\approx (2\text{SNR})^{-1}$ at high SNR

Degree of Freedom



QPSK

$$\text{symbol error probability} \approx 2Q\left(\sqrt{\frac{2b^2}{\sigma^2}}\right)$$

4-PAM

$$\text{symbol error probability} = \frac{3}{2}Q\left(\sqrt{\frac{2b^2}{\sigma^2}}\right)$$

- A complex scalar channel has 2 degrees of freedom
  - BPSK only uses 1 but QPSK uses 2 $\Longrightarrow$ QPSK rate is doubled
- QPSK is 2.5× more energy efficient than 4-PAM
  - QPSK avg. Tx energy = $2b^2$
  - 4-PAM avg. Tx energy = $5b^2$

Typical Error Event: Deep Fade

- In Rayleigh fading channel, regardless of constellation size and detection method (coherent/non-coherent),

$$\Pr\{\mathcal{E}\} \sim \frac{1}{\mathsf{SNR}}$$

- For BPSK, $\Pr\{\mathcal{E} \mid h\} = Q\left(\sqrt{2|h|^2\mathsf{SNR}}\right)$
  - If $|h|^2 \gg \mathsf{SNR}^{-1} \implies$ the conditional probability is very small
  - If $|h|^2 < \mathsf{SNR}^{-1} \implies$ the conditional probability is very large
  - Hence, $\Pr\{\mathcal{E}\} = \Pr\{|h|^2 > \mathsf{SNR}^{-1}\}\Pr\{\mathcal{E} \mid |h|^2 > \mathsf{SNR}^{-1}\}$

probability of deep fade

$$\begin{aligned} &+ \Pr\{|h|^2 < \mathsf{SNR}^{-1}\}\Pr\{\mathcal{E} \mid |h|^2 < \mathsf{SNR}^{-1}\} \\ &\lessapprox \Pr\{|h|^2 < \mathsf{SNR}^{-1}\} = 1 - e^{\mathsf{SNR}^{-1}} \\ &\approx \boxed{\mathsf{SNR}^{-1}} \end{aligned}$$

Diversity

$$y = hx + w, \quad h \sim \mathcal{CN}(0,1), \quad w \sim \mathcal{CN}(0,\sigma^2)$$

$$\boxed{\text{Deep Fade Event:} \quad \left\{ |h|^2 < \mathsf{SNR}^{-1} \right\}}$$

- Reception only relies on a single “path” h
- If h is in deep fade $\Longrightarrow$ trouble (low reliability)
- Increase the number of “paths” $\Longleftrightarrow$ Increase diversity
  - If one path is in deep fade, other paths can compensate!
- Diversity over time, space, and frequency

Outlook

- Time diversity
  - Coding + Interleaving: obtain diversity over time
  - Repetition coding
  - Rotation coding: utilize degrees of freedom better
- Space (Antenna) diversity
  - Receive diversity: multiple Rx antennas
  - Transmit diversity: multiple Tx antennas
  - Space-time codes
- Frequency diversity
  - ISI mitigation
  - Time-domain equalization
  - Direct-sequence spread spectrum
  - OFDM

Time Diversity

Repetition Coding + Interleaving

- A simple idea: Repetition Coding
  - Repeat the symbol over L time slots (note: L is NOT the # of taps)

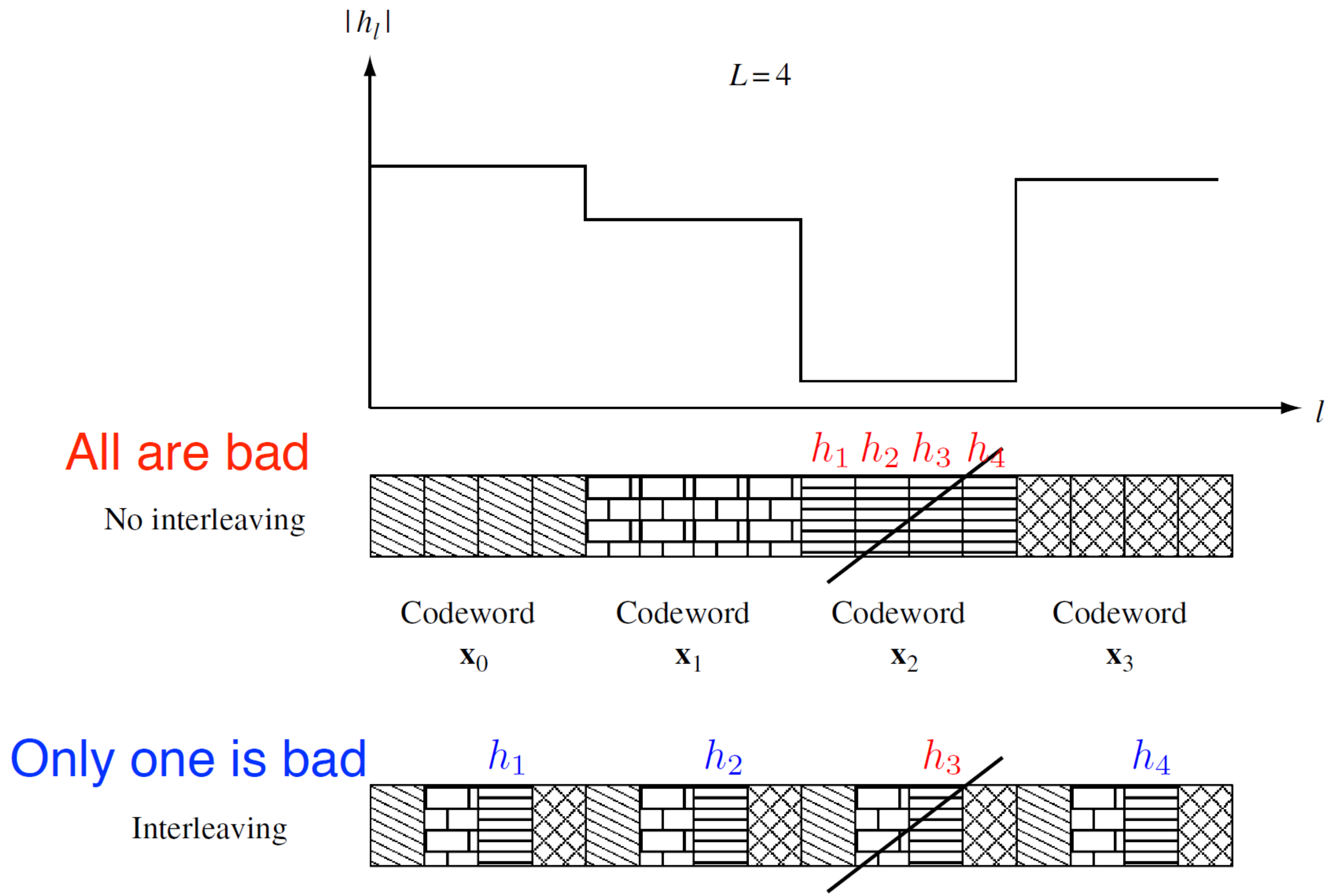
$$\text{Info. Symbol } b \rightarrow \boxed{\text{ENC}} \rightarrow \text{Codeword } \mathbf{x} := \begin{bmatrix} b & b & \cdots & b \end{bmatrix}$$

$$y_l = h_l x_l + w_l, \quad l = 1, 2, \ldots, L$$

  - As long as the channels $\{h_l \mid l = 1, 2, \ldots L\}$ are not ALL in deep fade, there is a good probability that we can decode the symbol

- Interleaving:
  - Channels within coherence time are highly correlated
  - Diversity is obtained if we interleave the codeword across multiple coherence time periods

Coding + Interleaving Increases Diversity



Analysis of Repetition Coding

- Equivalent vector channel
  - Original channel: $y_l = h_l x_l + w_l, \quad l = 1, 2, \ldots, L, \ w_l \sim \mathcal{CN}\left(0, \sigma^2\right)$
  - Sufficient interleaving $\Longrightarrow \{h_l \mid l = 1, 2, \ldots, L\}$: i.i.d. $\mathcal{CN}(0, 1)$
  - Repetition coding $\Longrightarrow x_l = x, \quad l = 1, 2, \ldots, L$
  - Vector channel: $\boxed{\mathbf{y} = \mathbf{h}x + \mathbf{w}}$

$$\mathbf{y} := \begin{bmatrix} y_1 & y_2 & \cdots & y_L \end{bmatrix}^T \ \mathbf{h} := \begin{bmatrix} h_1 & h_2 & \cdots & h_L \end{bmatrix}^T \ \mathbf{w} := \begin{bmatrix} w_1 & w_2 & \cdots & w_L \end{bmatrix}^T$$

- Analysis of error probability: under BPSK $x = \pm a$,
  - After projection we get a scalar equivalent channel

$$\boxed{\widetilde{y} = ||\mathbf{h}||x + \widetilde{w}, \quad x = \pm a, \quad \widetilde{w} \sim \mathcal{CN}\left(0, \sigma^2\right)}$$

  - Probability of error: $\mathbb{E}\left[Q\left(\sqrt{2||\mathbf{h}||^2 \mathsf{SNR}}\right)\right] \approx \binom{2L-1}{L} \frac{1}{(4\mathsf{SNR})^L}$

Probability of Deep Fade

- Deep fade event: $\{||\mathbf{h}||^2 < \mathsf{SNR}^{-1}\}$

$$||\mathbf{h}||^2 = \sum_{l=1}^{L} |h_l|^2 : \text{ sum of i.i.d. Exp(1) RV's}$$

$$\Longrightarrow \text{density of } ||\mathbf{h}||^2 : \; f(x) = \frac{1}{(L-1)!} x^{L-1} e^{-x}, \; x \geq 0$$

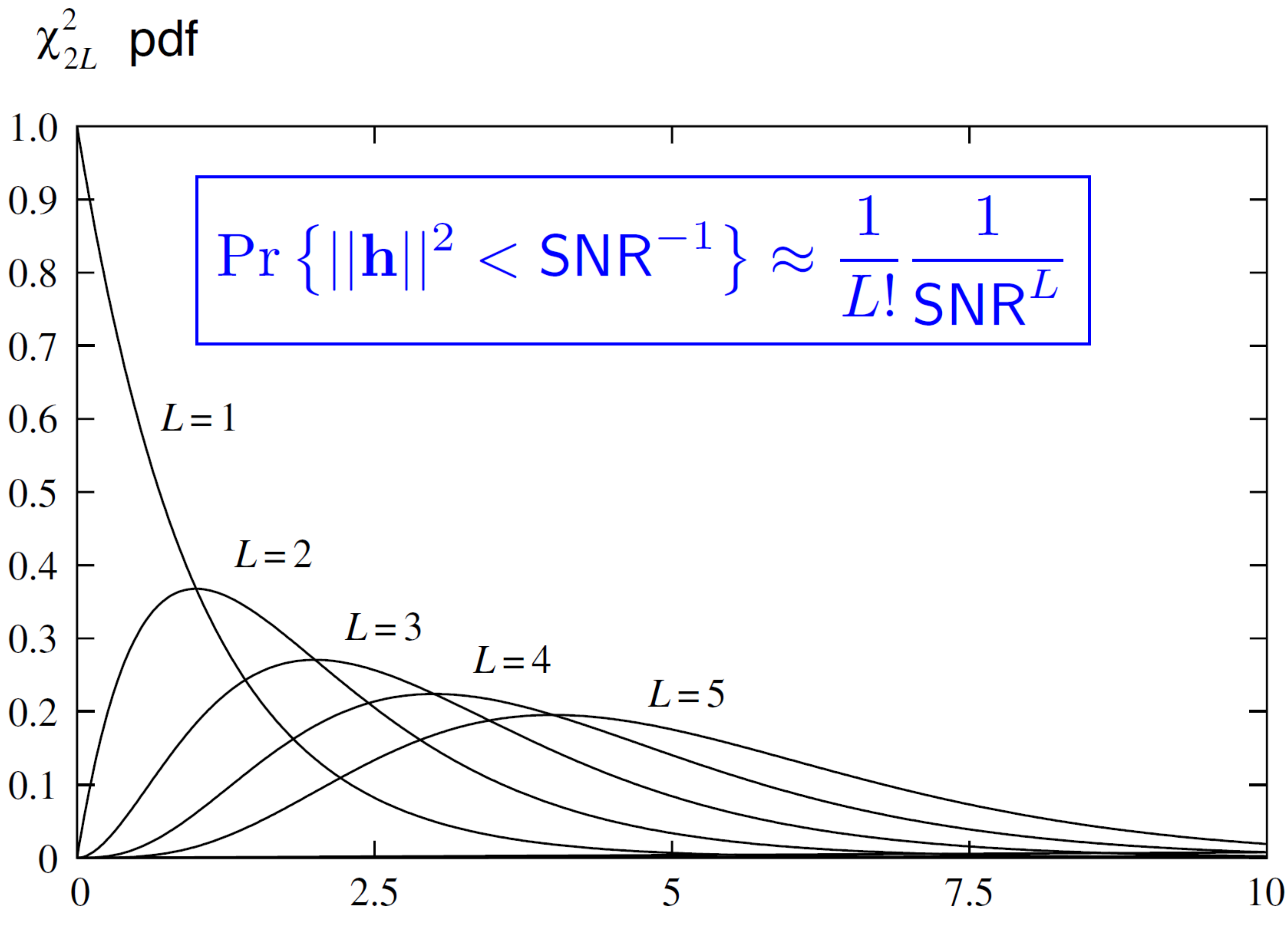
  - Chi-squared distribution with $2L$ degrees of freedom

- Probability of deep fade:
  - Approximation:

$$f(x) = \frac{1}{(L-1)!} x^{L-1} e^{-x} \approx \frac{1}{(L-1)!} x^{L-1}, \; 0 \leq x \ll 1$$

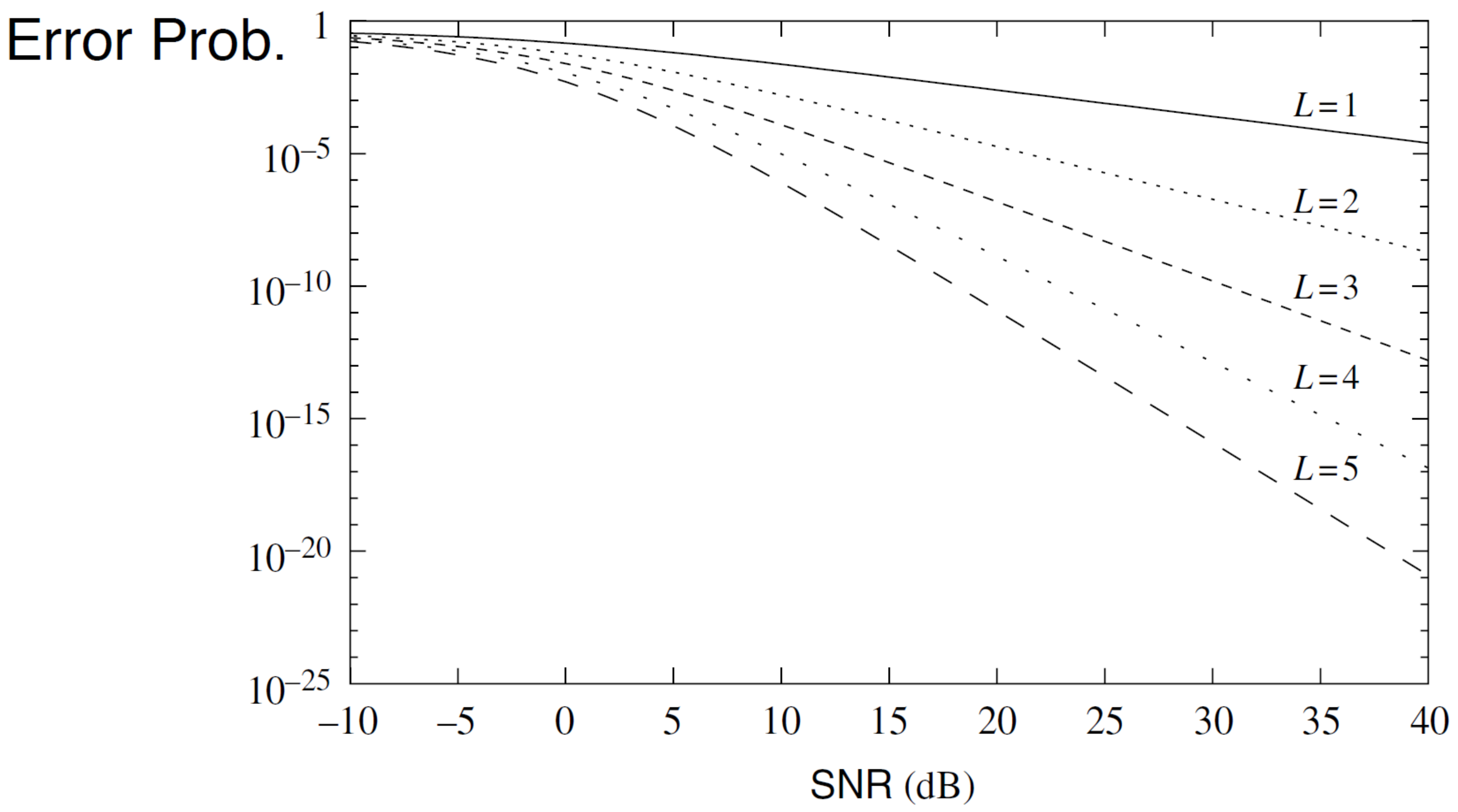
$$\Longrightarrow \Pr\{||\mathbf{h}||^2 < \mathsf{SNR}^{-1}\} \approx \int_0^{\mathsf{SNR}^{-1}} \frac{1}{(L-1)!} x^{L-1}\, dx = \boxed{\frac{1}{L!} \frac{1}{\mathsf{SNR}^L}}$$

Deep Fades Become Rarer



Diversity Gain: $1 \rightarrow L$

- Comparison of probabilities of deep fade
  - Without coding and interleaving: $\sim \text{SNR}^{-1}$
  - With coding and interleaving: $\sim \text{SNR}^{-L}$
  - Diversity: increase from 1 to L



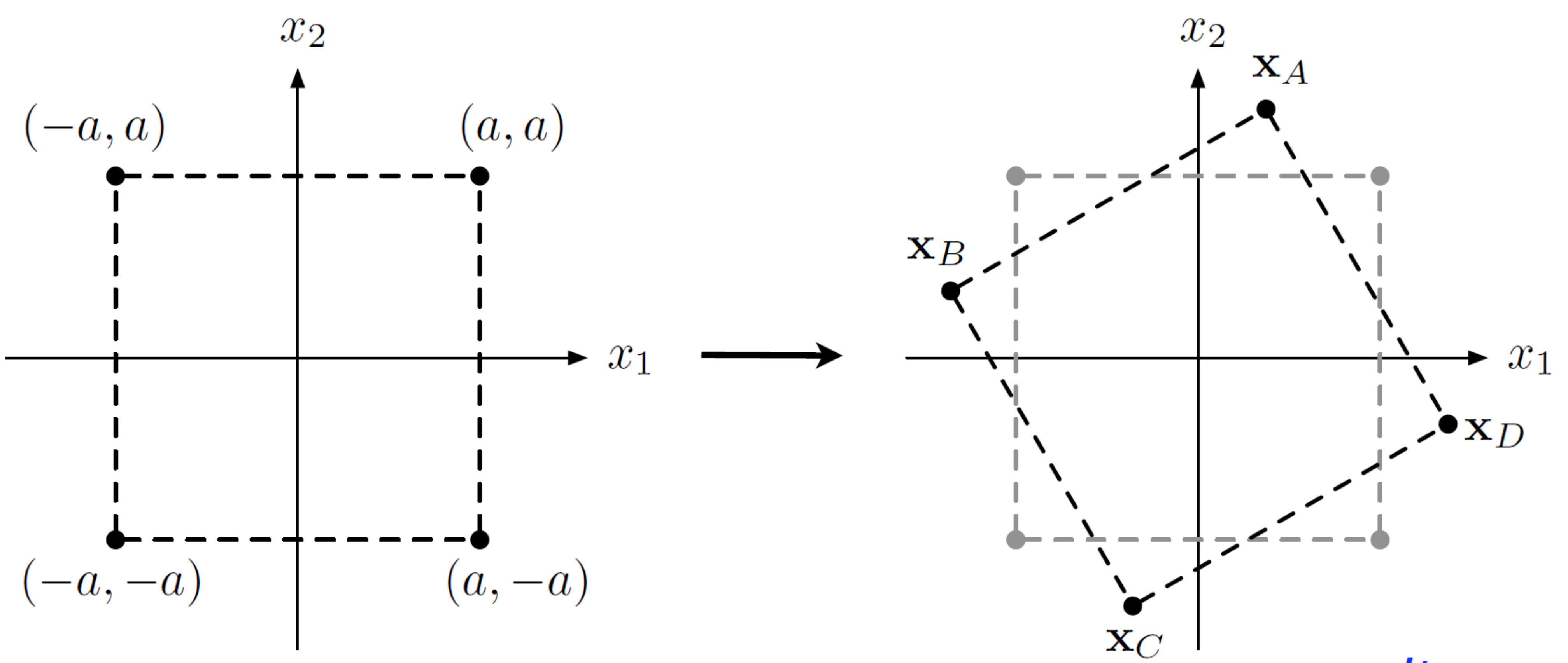
Beyond Repetition Coding

- Repetition coding:
  - Achieves full diversity gain L
  - Only one symbol per L symbol times
  - Does not fully exploit the degrees of freedom

- How to do better?

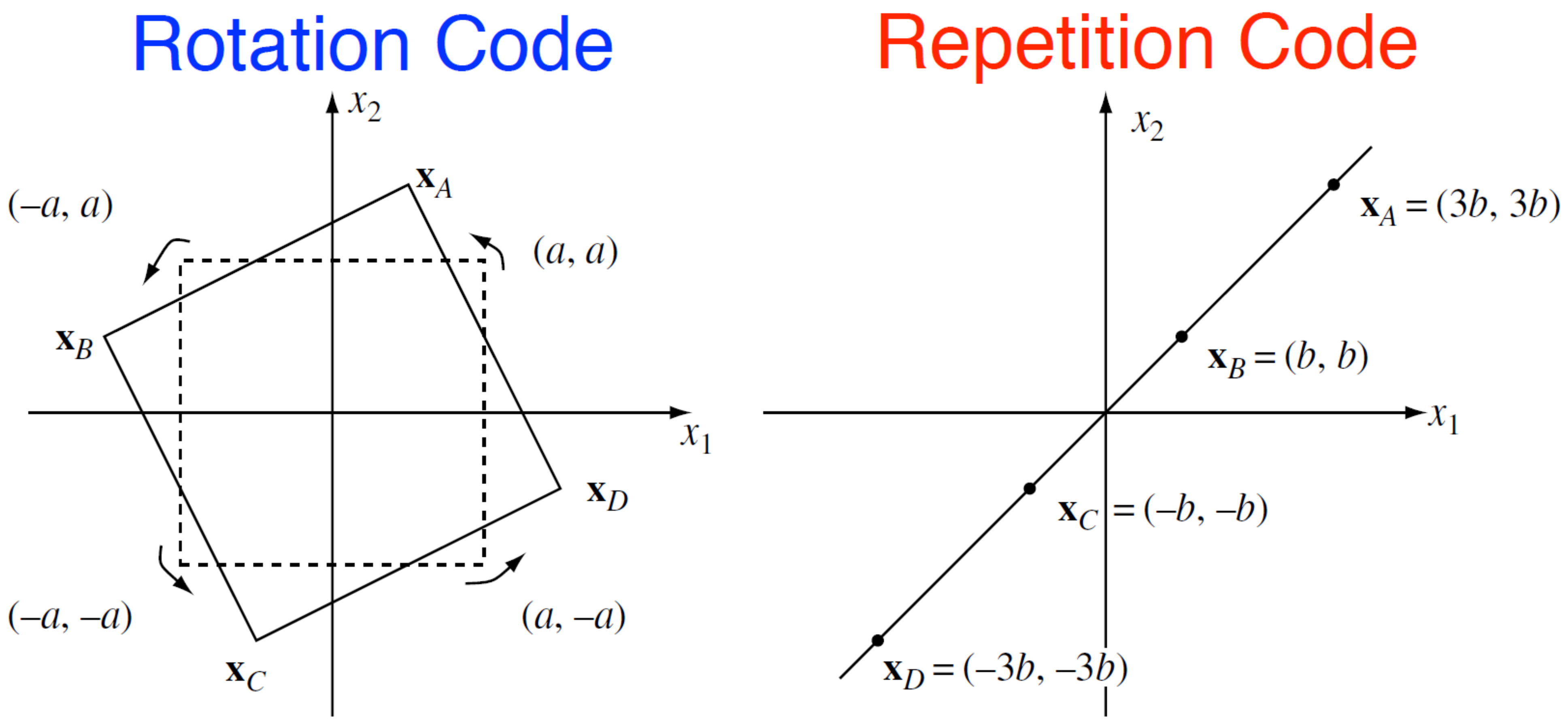
Rotation Code (L=2)

- 2 BPSK symbols ($x_1, x_2 = \pm a$) over two time slots ($L = 2$)
  - No diversity, as each BPSK symbol experiences only one “path”



- Rotation: $\boxed{\mathbf{x} = \mathbf{R}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{R} := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$ *unitary matrix*
  - 4 codewords: $\mathbf{x}_A = \mathbf{R}\begin{bmatrix} a \\ a \end{bmatrix}, \quad \mathbf{x}_B = \mathbf{R}\begin{bmatrix} -a \\ a \end{bmatrix}, \quad \mathbf{x}_C = \mathbf{R}\begin{bmatrix} -a \\ -a \end{bmatrix}, \quad \mathbf{x}_D = \mathbf{R}\begin{bmatrix} a \\ -a \end{bmatrix}$

Rotation vs Repetition Coding



- Again, like QPSK vs 4-PAM, rotation code uses the available DoF better.
- Coding Gain: saving power by 3.5 dB ($\sqrt{5}$)

Vector Channel

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w} = \mathbf{u} + \mathbf{w}$$

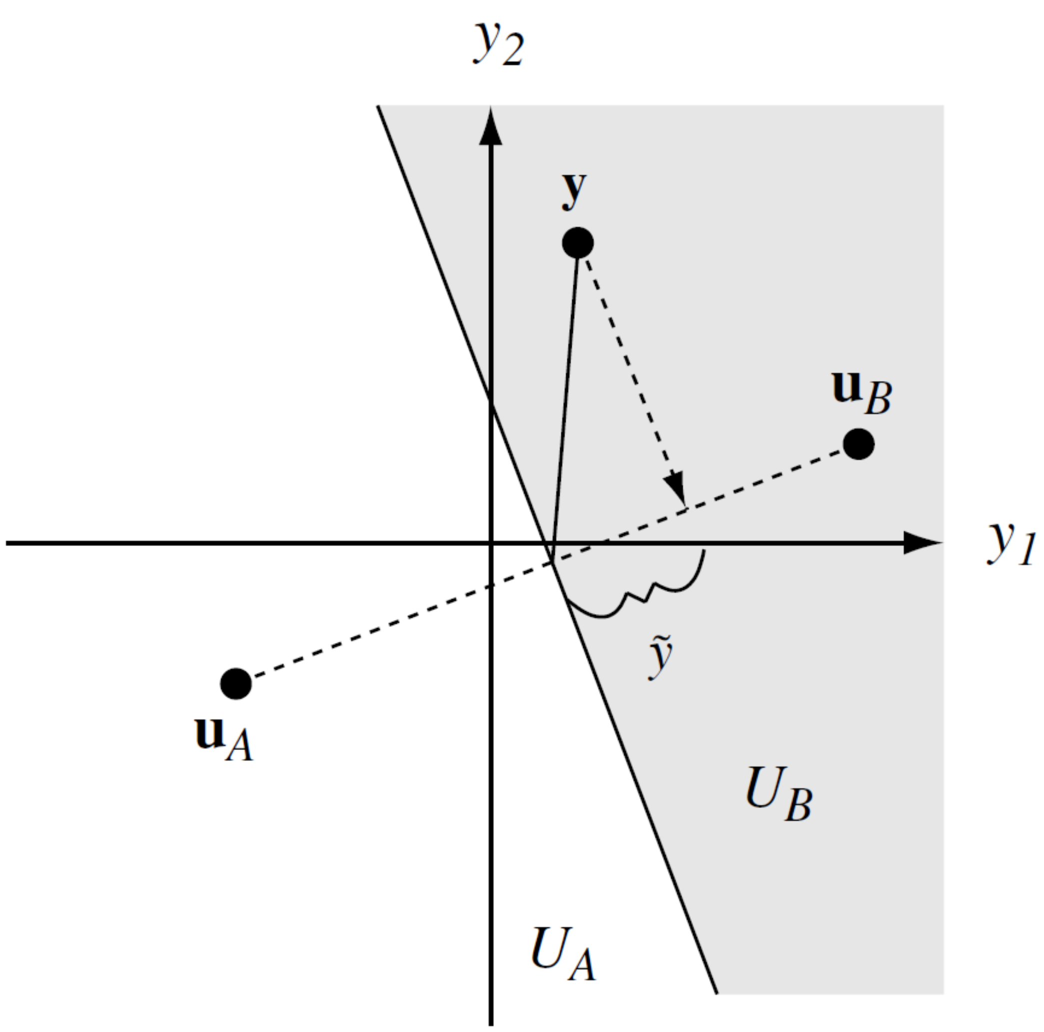
$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Probability of error (given channel):

$$\Pr\{\mathbf{x}_A \to \mathbf{x}_B \mid \mathbf{H}, \mathbf{x}_A \text{ is sent}\}$$

$$= \Pr\left\{\Re\{w\} > \frac{||\mathbf{u}_A - \mathbf{u}_B||}{2}\right\}$$

$$= \boxed{Q\left(\frac{||\mathbf{u}_A - \mathbf{u}_B||}{2\sqrt{\sigma^2/2}}\right)}$$



Pairwise Error Probability

- Hard to compute the exact error probability
- Union bound: WLOG assume $\mathbf{x}_A$ is sent.

conditional probability

$$\Pr\{\mathcal{E}\} \le \Pr\{\mathbf{x}_A \to \mathbf{x}_B\} + \Pr\{\mathbf{x}_A \to \mathbf{x}_C\} + \Pr\{\mathbf{x}_A \to \mathbf{x}_D\}$$

- Pairwise error probability:

$$\mathbf{u}_A = \begin{bmatrix} h_1 x_{A,1} \\ h_2 x_{A,2} \end{bmatrix} \quad \mathbf{u}_B = \begin{bmatrix} h_1 x_{B,1} \\ h_2 x_{B,2} \end{bmatrix}$$

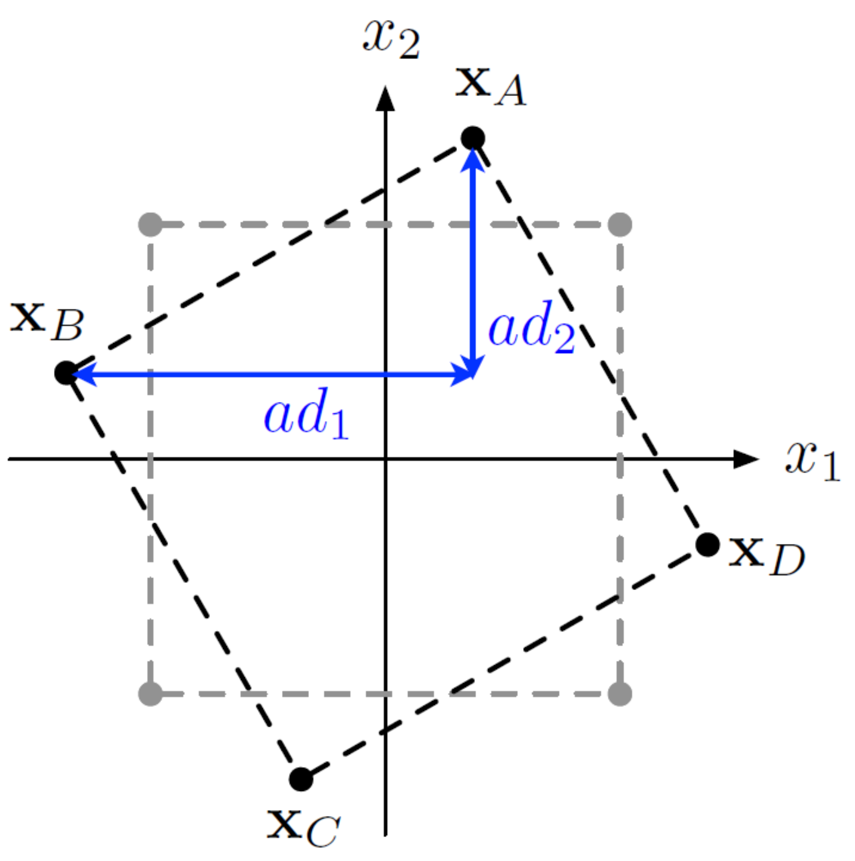
$$\Pr\{\mathbf{x}_A \to \mathbf{x}_B \mid h_1, h_2\} = Q\left(\frac{||\mathbf{u}_A - \mathbf{u}_B||}{2\sqrt{\sigma^2/2}}\right)$$

$$= Q\left(\sqrt{\frac{|h_1|^2(ad_1)^2 + |h_2|^2(ad_2)^2}{2\sigma^2}}\right)$$

$$= Q\left(\sqrt{\frac{\text{SNR}\left(|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2\right)}{2}}\right)$$

$$\overset{Q(x) \le e^{-x^2/2}}{\le} \exp\left(-\frac{\text{SNR}\left(|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2\right)}{4}\right)$$

$$\Longrightarrow \Pr\{\mathbf{x}_A \to \mathbf{x}_B\} \le \left(\frac{1}{1 + \frac{\text{SNR}|d_1|^2}{4}}\right)\left(\frac{1}{1 + \frac{\text{SNR}|d_2|^2}{4}}\right) \approx \boxed{\frac{16}{|d_1|^2|d_2|^2}\frac{1}{\text{SNR}^2}}$$



Product Distance

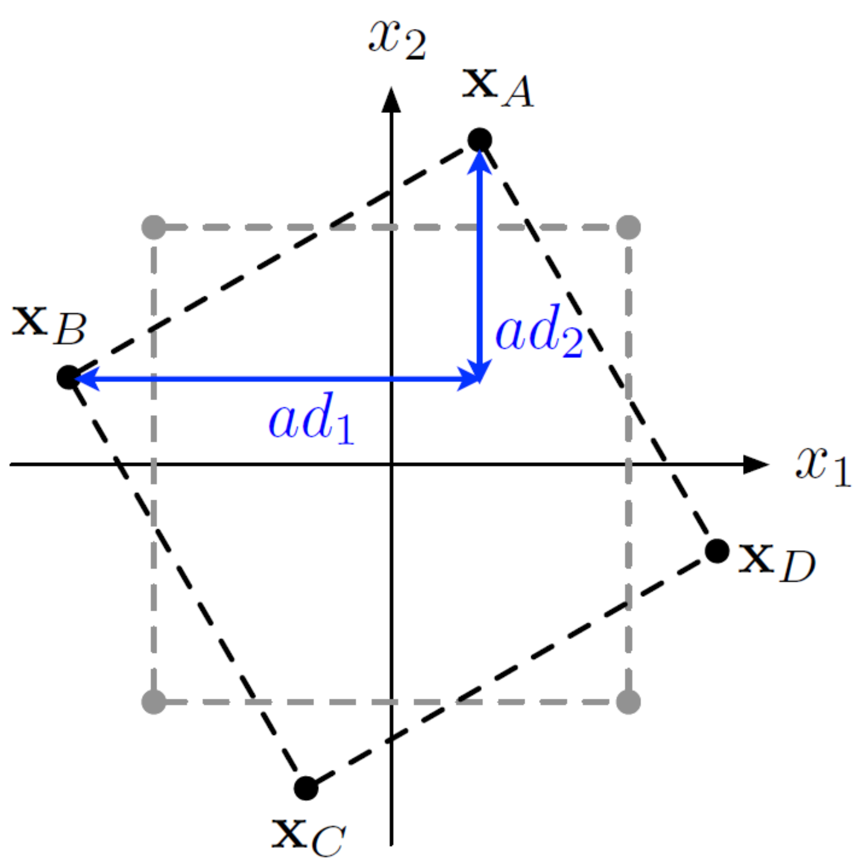
- Diversity Order = 2
  - Intuition:

$$\Pr\{\mathbf{x}_A \to \mathbf{x}_B \mid h_1, h_2\} = Q\left(\sqrt{\frac{\mathsf{SNR}\left(|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2\right)}{2}}\right)$$

$$\implies \Pr\{\text{Deep Fade}\}$$

$$\approx \Pr\left\{|h_1|^2 < \frac{1}{|d_1|^2\mathsf{SNR}},\ |h_2|^2 < \frac{1}{|d_2|^2\mathsf{SNR}}\right\}$$

$$\approx \frac{1}{|d_1|^2|d_2|^2}\frac{1}{\mathsf{SNR}^2}$$



- Squared product distance:

$$\boxed{\delta_{AB} := |d_1 d_2|^2}$$

$$\implies \Pr\{\mathbf{x}_A \to \mathbf{x}_B\} \lessapprox \boxed{\frac{16}{\delta_{AB}}\frac{1}{\mathsf{SNR}^2}}$$

$$\mathbf{d}_{AB} := \frac{\mathbf{x}_A - \mathbf{x}_B}{a} = \begin{bmatrix} 2\cos\theta \\ 2\sin\theta \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

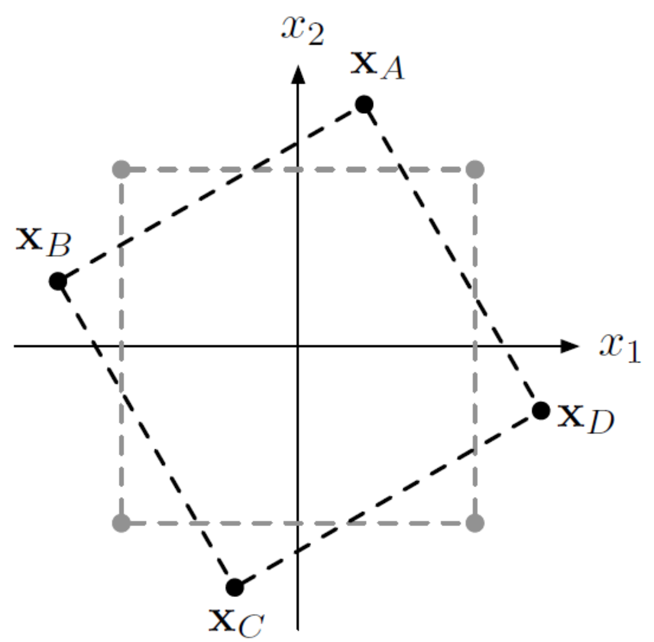
Rotation code achieve full diversity

- Total probability of error: upper bounded by

$$\Pr\{\mathcal{E}\} \le \Pr\{\mathbf{x}_A \to \mathbf{x}_B\} + \Pr\{\mathbf{x}_A \to \mathbf{x}_C\} + \Pr\{\mathbf{x}_A \to \mathbf{x}_D\}$$

$$\lessapprox 16\left(\frac{1}{\delta_{AB}} + \frac{1}{\delta_{AC}} + \frac{1}{\delta_{AD}}\right)\mathsf{SNR}^{-2}$$

$$\le \boxed{\frac{48}{\min\{\delta\}}\mathsf{SNR}^{-2}}$$



- Diversity Order = 2
- Coding Gain: maximize the minimum product distance

$$\delta_{AB} = \delta_{AD} = 4\sin^2 2\theta, \quad \delta_{AC} = 16\cos^2 2\theta$$

  - max-min is achieved when

$$4\sin^2 2\theta = 16\cos^2 2\theta \implies \theta = \frac{1}{2}\tan^{-1} 2$$

Summary: Time---Diversity Code

- Code: $\mathbf{x} \in \{\mathbf{x}_1, \ \mathbf{x}_2, \cdots, \mathbf{x}_M\}, \quad \mathbf{x}_i \in \mathbb{C}^L$

If full diversity L is obtained

- Union bound on error probability:

$$\Pr\{\mathcal{E}\} \leq \frac{1}{M} \sum_{i \neq j} \Pr\{\mathbf{x}_i \to \mathbf{x}_j\} \lessapprox \frac{4^L}{M} \sum_{i \neq j} \left(\frac{1}{\delta_{ij}}\right) \mathsf{SNR}^{-L}$$

- Pairwise error probability:

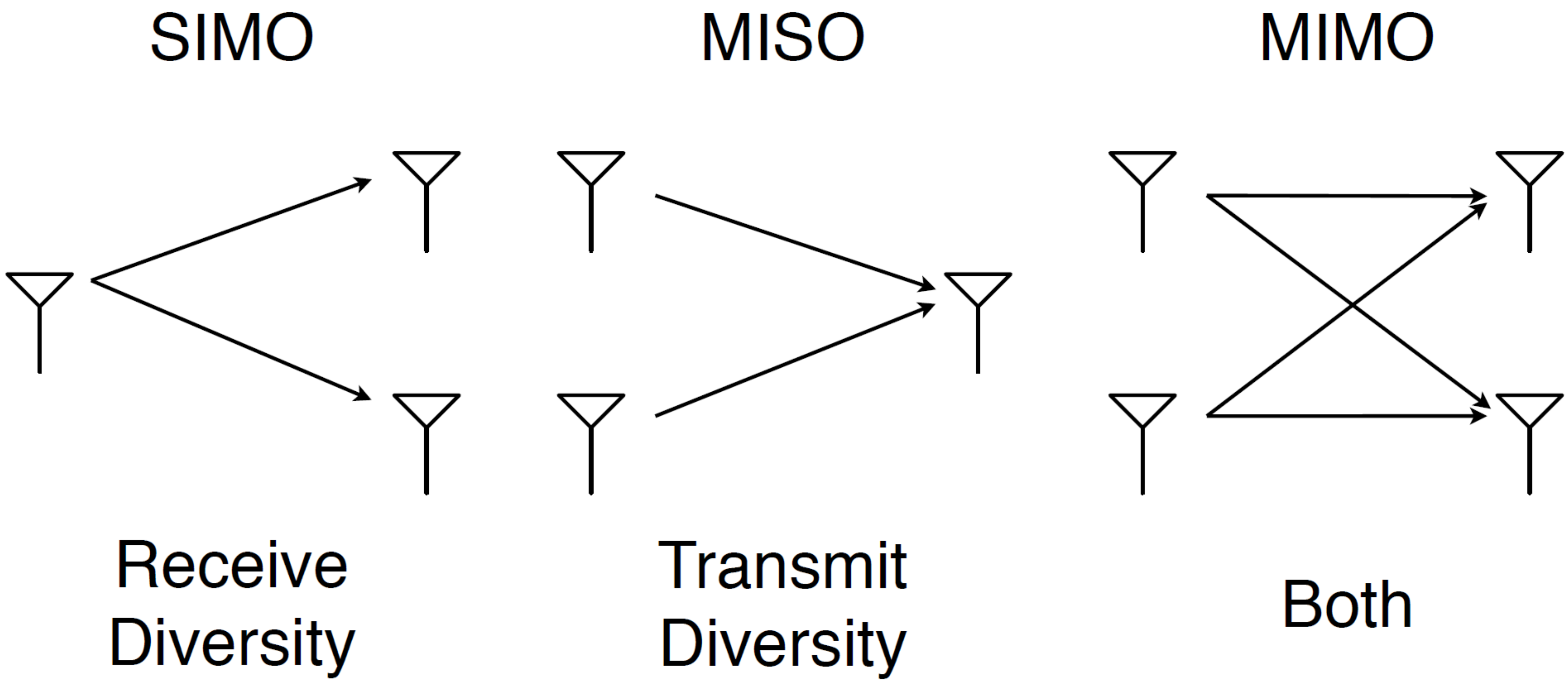
$$\Pr\{\mathbf{x}_i \to \mathbf{x}_j\} \leq \prod_{l=1}^{L} \frac{1}{1 + \mathsf{SNR}|x_{i,l} - x_{j,l}|^2/4}$$

- Diversity order: $\min_{i \neq j} \{L_{ij}\}, \quad L_{ij} = \sum_{l=1}^{L} \mathbb{I}\{x_{i,l} \neq x_{j,l}\}$

- Squared product distance $\delta_{ij} = \prod_{l=1}^{L} |x_{i,l} - x_{j,l}|^2$

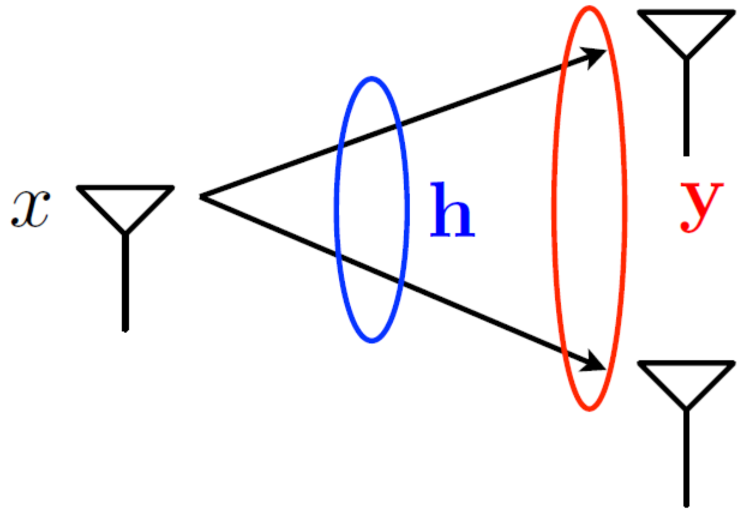
Antenna Diversity

Multiple Antennas



Typical antenna separation for space diversity ~ λ_c

Receive Diversity



$$\mathbf{y} = \mathbf{h}x + \mathbf{w} \in \mathbb{C}^L, \quad x = \pm a$$

$$\downarrow$$

$$\boxed{\text{MRC}, \ \frac{\mathbf{h}^*}{||\mathbf{h}||}}$$

$$\downarrow$$

$$\widetilde{y} = ||\mathbf{h}||x + \widetilde{w}, \quad \widetilde{w} \sim \mathcal{CN}\left(0, \sigma^2\right)$$

Diversity Order = L

L-fold Power Gain

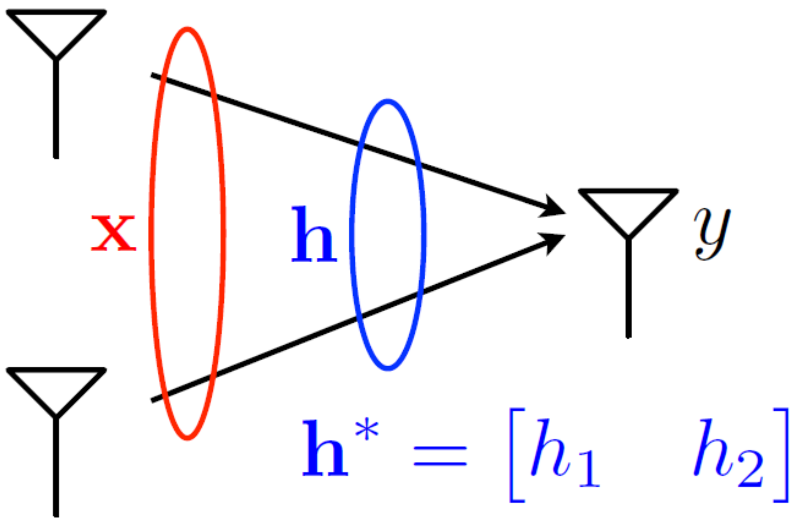
- Same as repetition coding in time diversity
- Except that there is a further power gain
  - Receive SNR in repetition coding = $\dfrac{a^2}{\sigma^2}$
  - Receive SNR in SIMO = $L\dfrac{a^2}{\sigma^2}$

tends to 1 as L tends to ∞: Diversity Gain

- Probability of Error: $\mathbb{E}\left[Q\left(\sqrt{2\left(\frac{1}{L}||\mathbf{h}||^2\right)\frac{La^2}{\sigma^2}}\right)\right]$

L-fold Power Gain

Transmit Diversity



$$y = \mathbf{h}^*\mathbf{x} + w \in \mathbb{C}, \quad \mathbf{x}, \mathbf{h} \in \mathbb{C}^L$$

$$\downarrow$$

$$\boxed{\text{Tx Beamform, } \mathbf{x} = x\mathbf{h}/||\mathbf{h}||}$$

$$\downarrow$$

$$y = x||\mathbf{h}|| + w$$

- SIMO: Rx beamforming
- MISO: if Tx knows the channel, it can send

$$\mathbf{x} = x\frac{\mathbf{h}}{||\mathbf{h}||} \quad \text{Tx Beamforming}$$

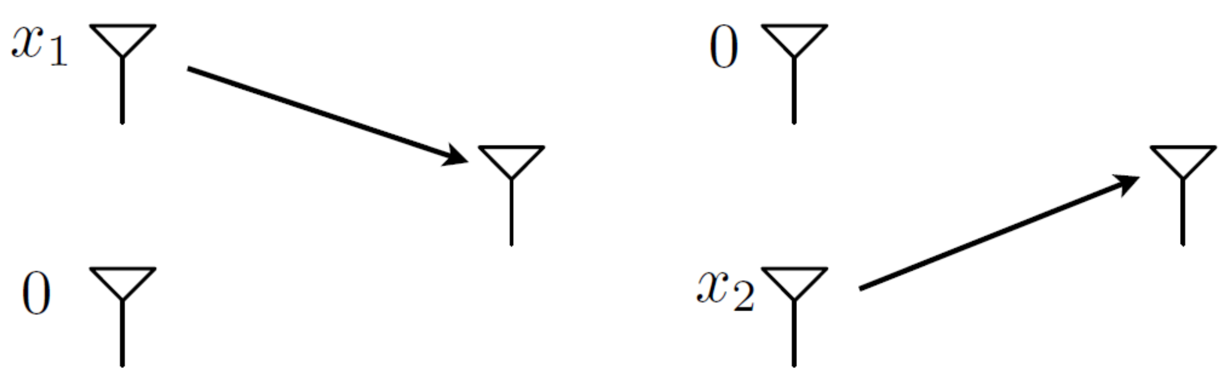
- Same as SIMO: diversity order = L; L-fold power gain
- What if Tx does not know the channel?

Space-Time Codes

- Transmit the same symbol at all antennas simultaneously won’t work: (diversity order = 1)

$$\mathbf{x} = x\mathbf{1} \implies y = x\sum_{l=1}^{L} h_l + w, \quad \sum_{l=1}^{L} h_l \sim \mathcal{CN}(0, L)$$

- Time-diversity code can be used to get full Tx diversity:
  - Idea: use just one antenna at one time (let $\mathbf{x} = [x_1\ x_2]^T$ be the time-diversity codeword)



$$[y_1 \quad y_2] = \mathbf{h}^* \boxed{\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}} + [w_1 \quad w_2]$$

$\mathbf{X}$, space-time codeword

- Space-time codes

Space--Time Codes: Simple Examples

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} = \mathbf{h}^* \overset{\text{Time}}{\underset{\text{Space}}{\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}}} + \begin{bmatrix} w_1 & w_2 \end{bmatrix}$$

$$\boxed{\mathbf{y}^T = \mathbf{h}^* \mathbf{X} + \mathbf{w}^T}$$

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} : \text{ time-diversity codeword}$$

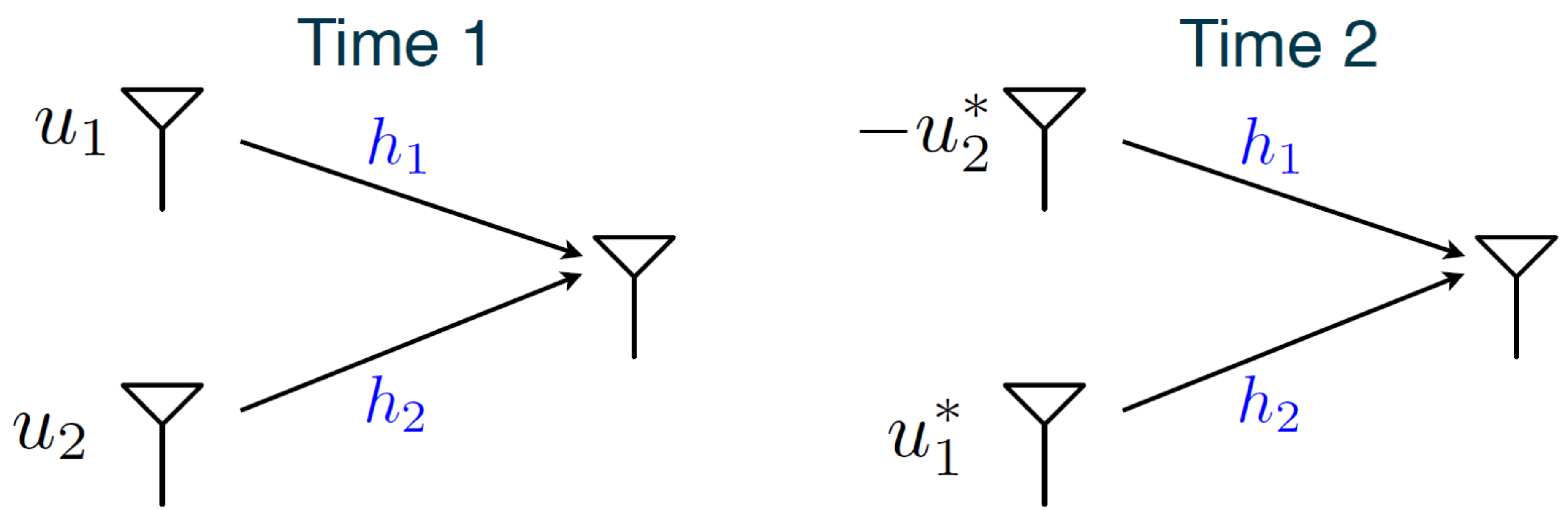
$$\updownarrow$$

$$\mathbf{X} = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} : \text{ space-time codeword}$$

- Convert a time-diversity code $\mathbf{x}$ to a space-time code $\mathbf{X}$:
  - Spatial coding: turning one antenna on per time
- Achieves full diversity; waste available DoF

- Better design is out there!

Alamouti Scheme



$u_1, u_2 \in \mathbb{C}$ $\quad \mathbf{X} = \begin{bmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{bmatrix}$ space-time codeword

Equivalent Channel:

$$\widetilde{\mathbf{h}}_1 \perp \widetilde{\mathbf{h}}_2$$

$$\begin{bmatrix} y_1 \\ y_2^* \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = u_1 \underbrace{\begin{bmatrix} h_1 \\ h_2^* \end{bmatrix}}_{\widetilde{\mathbf{h}}_1} + u_2 \underbrace{\begin{bmatrix} h_2 \\ -h_1^* \end{bmatrix}}_{\widetilde{\mathbf{h}}_2} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Projection onto the two column vectors respectively, we can get two clean channels for u_1 and u_2!

Performance of Alamouti Scheme

$$\begin{bmatrix} y_1 \\ y_2^* \end{bmatrix} = u_1 \begin{bmatrix} h_1 \\ h_2^* \end{bmatrix} + u_2 \begin{bmatrix} h_2 \\ -h_1^* \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\widetilde{\mathbf{y}} = u_1 \widetilde{\mathbf{h}}_1 + u_2 \widetilde{\mathbf{h}}_2 + \widetilde{\mathbf{w}} \qquad \widetilde{\mathbf{h}}_1 \perp \widetilde{\mathbf{h}}_2$$

- Projection onto two orthogonal directions

$$\frac{\widetilde{\mathbf{h}}_1^*}{||\widetilde{\mathbf{h}}_1||} \widetilde{\mathbf{y}} = u_1 ||\widetilde{\mathbf{h}}_1|| + \widetilde{w}_1 = u_1 ||\mathbf{h}|| + \widetilde{w}_1$$

$$\frac{\widetilde{\mathbf{h}}_2^*}{||\widetilde{\mathbf{h}}_2||} \widetilde{\mathbf{y}} = u_2 ||\widetilde{\mathbf{h}}_2|| + \widetilde{w}_2 = u_2 ||\mathbf{h}|| + \widetilde{w}_2$$

Two parallel channels, each for one symbol!

- Double the rate of repetition coding
- Diversity order = 2 Full diversity
- 3dB loss in Tx power compared to Tx beamforming

Space--Time Code Design

- In general we can extract L Tx diversity by using an $L\times L$ space-time code in an $L\times 1$ MISO channel

$$\mathbf{X} \in \{\mathbf{X}_A,\ \mathbf{X}_B, \cdots\}, \quad \mathbf{X} \in \mathbb{C}^{L\times L}$$

  - Similar to time-diversity code!
- Channel: $\mathbf{y}^T = \mathbf{h}^* \mathbf{X} + \mathbf{w}^T$
- Pairwise error probability:

$$\Pr\{\mathbf{X}_A \to \mathbf{X}_B \mid \mathbf{h}\} = Q\left(\frac{||\mathbf{h}^*(\mathbf{X}_A - \mathbf{X}_B)||}{2\sqrt{\sigma^2/2}}\right) = Q\left(\sqrt{\frac{\mathsf{SNR}}{2}\mathbf{h}^*(\mathbf{X}_A - \mathbf{X}_B)(\mathbf{X}_A - \mathbf{X}_B)^*\mathbf{h}}\right)$$

$$= Q\left(\sqrt{\frac{\mathsf{SNR}}{2}\sum_{l=1^L} |\widetilde{h}_l|^2 \lambda_l^2}\right)$$

$$(\mathbf{X}_A - \mathbf{X}_B)(\mathbf{X}_A - \mathbf{X}_B)^* = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^*$$

$$\widetilde{\mathbf{h}} := \mathbf{U}^*\mathbf{h} \quad \boldsymbol{\Lambda} = \operatorname{diag}\left(\lambda_1^2, \cdots, \lambda_L^2\right)$$

$$\Pr\{\mathbf{X}_A \to \mathbf{X}_B\} \le \prod_{l=1}^{L} \frac{1}{1 + \mathsf{SNR}|\lambda_l|^2/4} \lessapprox \frac{4^L}{\prod_{l=1}^{L}|\lambda_l|^2}\mathsf{SNR}^{-L}$$

$$\det\left\{(\mathbf{X}_A - \mathbf{X}_B)(\mathbf{X}_A - \mathbf{X}_B)^*\right\}$$

Design Criteria

- Time-diversity code:
  - Maximize the min squared product distance

$$\min_{i,j} \delta_{ij}, \quad \delta_{ij} := \prod_{l=1}^{L} |x_{i,l} - x_{j,l}|^2$$

- Space-time code
  - Maximize the min determinant

$$\min_{i,j} \det \left\{ (\mathbf{X}_i - \mathbf{X}_j)(\mathbf{X}_i - \mathbf{X}_j)^* \right\}$$

- Full diversity $\Leftrightarrow$ $(\mathbf{X}_i - \mathbf{X}_j)$ is full rank for all i, j

Frequency Diversity

Diversity in Frequency--Selective Channel

$$y[m] = \sum_l h_l x[m-l] + w[m]$$

- Resolution of multipaths provides diversity
  - A symbol can go through multiple taps
- Full diversity achieved by sending one symbol every L symbol times; inefficient (like repetition coding)
- Send more frequently $\Longrightarrow$ intersymbol interference (ISI)
- Challenge: how to mitigate the ISI while extracting the inherent diversity in the frequency-selective channel.

Approaches for mitigation ISI

- Time-domain equalization
  - GSM
- Direct-sequence spread spectrum
  - CDMA
- Orthogonal frequency-division multiplexing (OFDM)
  - 802.11a, LTE

ISI Equalization

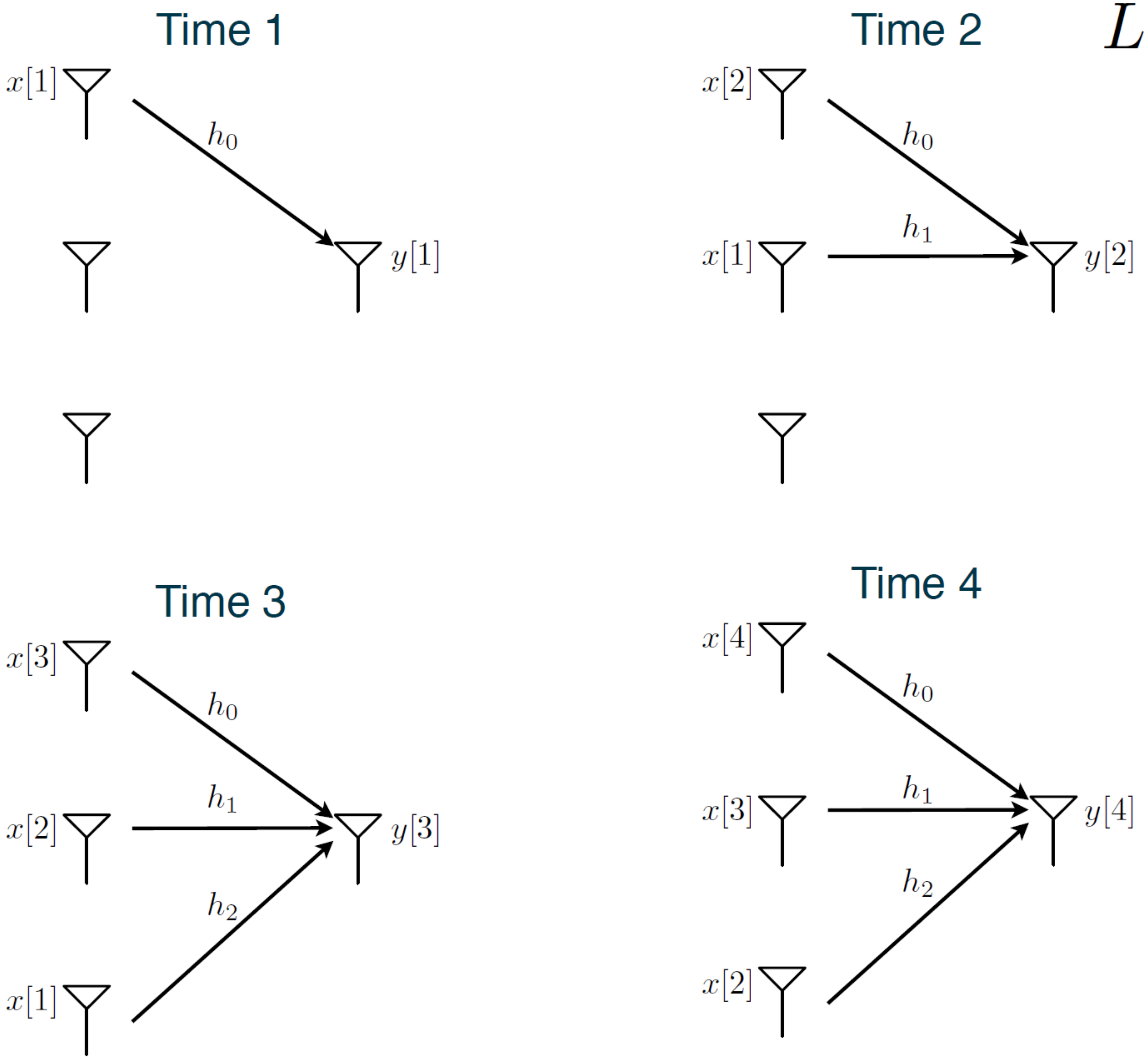
- Transmit sequence of uncoded symbols
- Maximum likelihood sequence detection (MLSD) is performed using the Viterbi algorithm

- Full diversity can be achieved

- Complexity grows exponentially with number of taps L

Reduction to time diversity

$L = 3$ Taps



Sequence detection

- To detect a symbol $x[N]$ and attain L-diversity, observe the received symbols up to time N+L-1
- Equivalent channel

$$\mathbf{y}^T = \mathbf{h}^* \mathbf{X} + \mathbf{w}^T$$

$$\mathbf{y}^T := y\,[1 : N + L - 1]\,, \quad \mathbf{h}^* := h_{0:L-1}, \quad \mathbf{w}^T := w\,[1 : N + L - 1]$$

  - Space-time code matrix for input sequence $\mathbf{x} := x[1 : N$+L-$1]$ is

$$\mathbf{X} = \begin{bmatrix} x[1] & x[2] & \cdot & \cdot & \cdot & x[N] & \cdot & \cdot & x[N+L-1] \\ 0 & x[1] & x[2] & \cdot & \cdot & \cdot & x[N] & \cdot & x[N+L-2] \\ 0 & 0 & x[1] & x[2] & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & x[1] & x[2] & \cdot & \cdot & x[N] \end{bmatrix}$$

Achieving Full Diversity

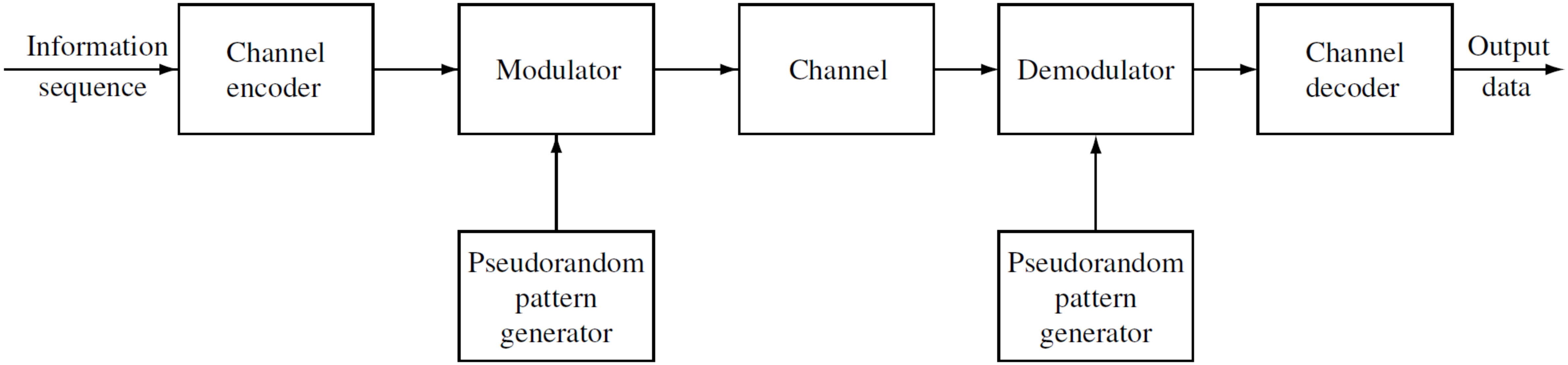
- Following the derivation in Tx diversity:
  - The difference matrix ($\mathbf{X}_A$ - $\mathbf{X}_B$) is full rank □ full diversity!

- Let $m^* \leq N$ be the first symbol time where vectors $\mathbf{x}_A$ and $\mathbf{x}_B$ differ:

$$\mathbf{X}_A - \mathbf{X}_B = \begin{bmatrix} 0 & \cdot & 0 & x_A[m^*] - x_B[m^*] & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & x_A[m^*] - x_B[m^*] & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & x_A[m^*] - x_B[m^*] & \cdot \end{bmatrix}$$

  is full rank.

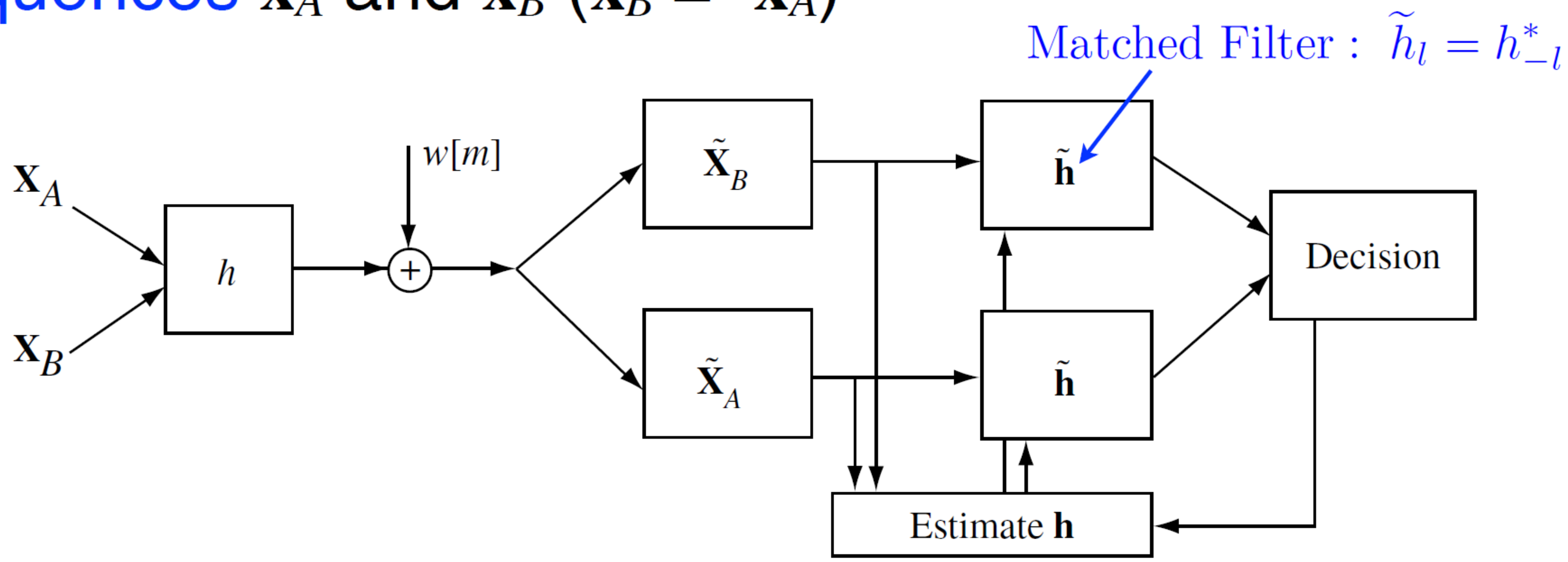
Direct Sequence Spread Spectrum



- Info. symbol rate ≪ chip rate (large processing gain)
  - Each sample period is called a *chip*
- Signal-to-noise ratio per chip is low
- ISI is not significant compared to interference from other users and match filtering (Rake) is near-optimal.

Frequency Diversity via Rake Receiver

- Considered a simplified situation (uncoded)
- Each information bit is spread into two pseudorandom sequences $\mathbf{x}_A$ and $\mathbf{x}_B$ ($\mathbf{x}_B$ = -$\mathbf{x}_A$)



- Each tap of the match filter is a finger of the Rake
- Rake is MRC □ similar to Rx diversity, full diversity can be achieved

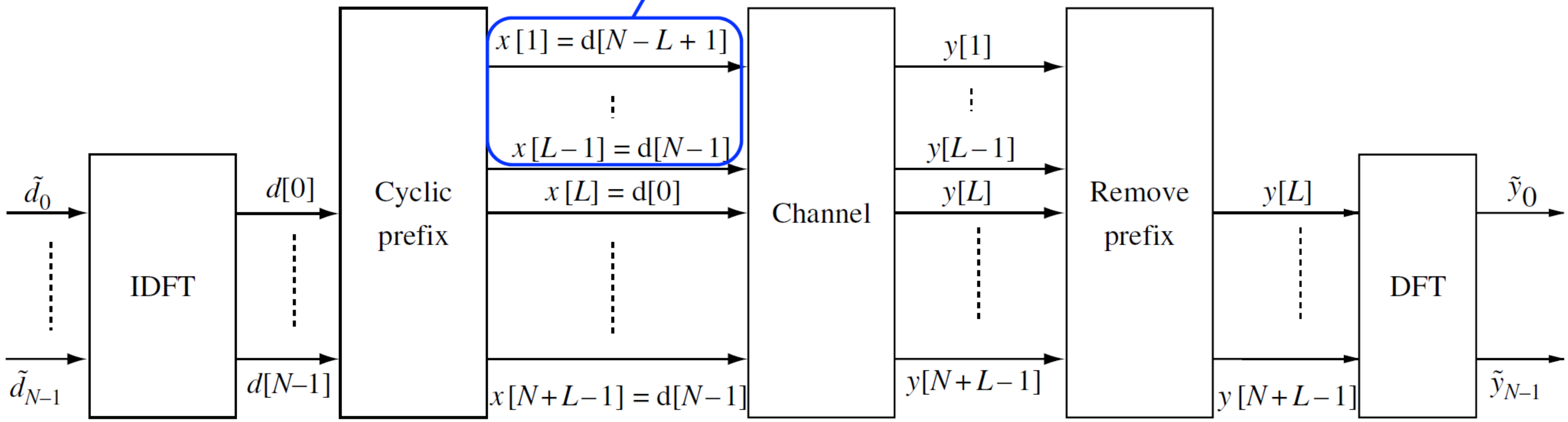
ISI vs. Frequency Diversity

- In narrowband systems, ISI is mitigated using a complex receiver.
- In asynchronous CDMA uplink, ISI is there but small compared to interference from other users.
- But ISI is not intrinsic to achieve frequency diversity.
- The transmitter needs to do some work too!

OFDM: Basic Concept

- Most wireless channels are underspread
  - Delay spread $\ll$ Coherence time.
- Can be approximated by a linear time invariant channel over a long time scale.
- Complex sinusoids are the only eigenfunctions of linear time-invariant channels.
- Should signal in the frequency domain and then transform to the time domain.

cyclic prefix: turn linear convolution to circular convolution



$$\mathbf{y} = \mathbf{h} \otimes \mathbf{d} + \mathbf{w}$$

$$\mathrm{DFT}\left(\mathbf{d}\right)_n := \frac{1}{N_c} \sum_{m=0}^{N_c-1} d[m] e^{-j2\pi \frac{nm}{N_c}}$$

(N_c-point DFT)

$$\mathbf{y} := y[L : N_c + L - 1], \quad \mathbf{w} := w[L : N_c + L - 1],$$

$$\mathbf{h} := \begin{bmatrix} h_0 & h_1 & \cdots & h_{L-1} & 0 & \cdots & 0 \end{bmatrix}^T$$

$$\boxed{\mathrm{DFT}\left(\mathbf{h} \otimes \mathbf{d}\right)_n = \sqrt{N_c}\,\mathrm{DFT}\left(\mathbf{h}\right)_n \times \mathrm{DFT}\left(\mathbf{d}\right)_n}$$

OFDM

- OFDM transforms the communication problem into the frequency domain:

$$\widetilde{y}_n = \widetilde{h}_n \widetilde{d}_n + \widetilde{w}_n, \quad n = 0, 1, \ldots, N_c - 1$$

$$\widetilde{y}_n := \mathrm{DFT}\left(\mathbf{y}\right)_n, \quad \widetilde{d}_n := \mathrm{DFT}\left(\mathbf{d}\right)_n, \quad \widetilde{w}_n := \mathrm{DFT}\left(\mathbf{w}\right)_n, \quad \widetilde{h}_n := \sqrt{N_c}\mathrm{DFT}\left(\mathbf{h}\right)_n$$

- A bunch of non-interfering sub-channels, one for each sub-carrier:

$$\widetilde{h}_n = H_b\left(\frac{nW}{N_c}\right)$$

- Can apply time-diversity techniques.

Cyclic Precix (CP)

- The N_c data symbols constitute one OFDM symbol:

$$\widetilde{d}_0, \widetilde{d}_1, \ldots, \widetilde{d}_{N_c-1}$$

- Cyclic prefix prevents inter-OFDM-symbol interference.

- It also converts linear convolution into circular convolution.

Cyclic Prefix Overhead

- OFDM overhead = length of CP / OFDM symbol time
  - Cyclic prefix dictated by delay spread
  - OFDM symbol time limited by channel coherence time

- Equivalently, the inter-carrier spacing should be much larger than the Doppler spread

- Since most channels are underspread, the overhead can be made small

Example 1: Flash OFDM

- Bandwidth = 1.25 Mz
- OFDM symbol = 128 samples = 100 μs
- Cyclic prefix = 16 samples = 11 μs delay spread
- 11 % overhead.

Example 2: Long Term Evolution

- Bandwidth = 1.25 – 20MHz
- OFDM symbol = 128 – 2048 samples (100 μs)
- Inter-carrier spacing = 15 kHz
- Cyclic prefix = 9 – 144 samples = 5 μs delay spread
- 5 % overhead.

Channel Uncertainty

- In fast varying channels, tap gain measurement errors may have an impact on diversity combining performance
- The impact is particularly significant in channel with many taps each containing a small fraction of the total received energy. (eg. Ultra-wideband channels)
- The impact depends on the modulation scheme

Summary

- Fading makes wireless channels unreliable.
- Diversity increases reliability and makes the channel more consistent.
- Smart codes yields a coding gain in addition to the diversity gain.
- This viewpoint of the adversity of fading will be challenged and enriched in later parts of the course.