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DIGITAL COMMUNICATION SUMMARY

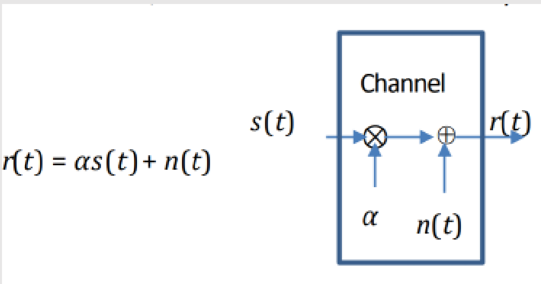
1.3 Mathematical model

For analysis and design purpose of communication system, we need a mathematical model or equations. It Is convenient to construct mathematical model models that reflects the most important characteristics of the transmission medium. **These mathematical models for the channels are then used in the design of channel encoder and modulator at the transmitter and demodulator and channel decoder at the receiver.**

The channel models used frequently to characterize many of the physical channels are discussed below:

1. Additive noise channel

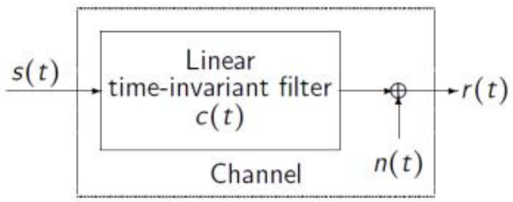
It is the simplest math model. Within the channel there is a noise n(t) which add up to the transmitted signal and send to the receiver as a sum of transmitted signal and the noise. If the signal undergoes attenuation in the transmitter, then the received signal can described as:



- It is statically characterized as gaussian noise channel. This channel model applies to a broad class of communication channels and hence mostly used in communication system design.

2. Linear filter channel with additive noise

In wire line channel the signal does not exceed specified bandwidth, so to meet this limitation we use this mathematical model. It’s a channel characterized mathematically as linear filter (for limit the bandwidth) with additive noise.



$$\begin{aligned} r(t) &= s(t) \star c(t) + n(t) \\ &= \int_{-\infty}^{\infty} c(\tau) s(t-\tau)\, d\tau + n(t) \end{aligned}$$

Here, c(t) is the impulse response of the system. The operation s(t) and c(t) is convolutional.

3. LTV filter channel with additive noise

In wireless communication specially in cellular networks there are multipaths which is from the transmitter to the receiver. So, in this situation we need the channel model to sum up all the paths together and also the noise. Mathematically described as:

$c(\tau;t)$ usually has the form

$$c(\tau;t) = \sum_{k=1}^{L} a_k(t) \delta(\tau - \tau_k)$$

where

- $\{a_k(t)\}_{k=1}^{L}$ represent the possibly time-varying attenuation factor for the L multipath propagation paths
- $\{\tau_k\}_{k=1}^{L}$ are the corresponding time delays.

Hence

$$r(t) = \sum_{k=1}^{L} a_k(t) s(t - \tau_k) + n(t)$$

Three models

- ✓ **Linear Time-Invariant Filter Channel**

$$r(t) = s(t) * c(t) + n(t)$$

- ✓ **Linear Time-Variant Filter Channel**

$$r(t) = s(t) * c(t;\tau) + n(t)$$

- ✓ **Multipath propagation**

$$r(t) = \sum_{k=1}^{L} \alpha(t) s(t - \tau_k) + n(t)$$

Historical perspective in the development of digital communication

- Morse code (1837, Baudot code (1875, Shannon (1948), Hartley (1928), Kolmogorov (1939) and Wiener (1942), Kotenikov (1947), Wozencraft and Jacobs (1965), Hamming(1950), Muller (1954), Reed(1954), Reed and Solomon (1960), Bose and Ray-Chaudhuri (1960), and Goppa (1970,1971), Forney (1966), Chien (1964), Berlekamp (1968), Wozencraft and Reiffen (1961), Fano (1963), Zigangirov (1966), Jelinek, (1969), Forney (1970, 1972, 1974) and Viterbi (1967, 1971), Ungerboeck (1982), Forney et al. (1984), Wei (1987), Ziv and Lempel (1977, 1978) and Linde et al. (1980), Berrou et al. (1993), Gallager (1963), Davey and Mackay (1998).

2.1 Bandpass and lowpass signal representation

A lowpass or baseband (equivalent) signal $x_l(t)$ is a complex signal (because it is not necessarily Hermitian symmetric) whose spectrum is located around zero frequency. It's the real signal. Baseband signal is normally referred to as the original message signal which is intended to be transmitted whereas passband signal refers to modulated or filtered signal which ultimately gets converted back to baseband signal. The band is very low with frequency 0. They are vulnerable to noise and interference so we need to move them to higher band which is stronger to combat the noise then we transmit them. First, we should know the mathematical model to represent lowpass or bandpath.

$$\begin{cases} x(t) = \mathbf{Re}\left\{x_\ell(t)e^{\imath 2\pi f_0 t}\right\} \\ X(f) = \frac{1}{2}\left[X_\ell(f - f_0) + X_\ell^*(-f - f_0)\right] \end{cases}$$

Analytical signal or pre-envelope: $x_+(t)$ and $X_+(f)$

Lowpass equivalent signal or complex envelope:

$$\begin{cases} x_\ell(t) = (x(t) + \imath\,\hat{x}(t))e^{-\imath 2\pi f_0 t} \\ X_\ell(f) = 2X(f + f_0)u_{-1}(f + f_0) \end{cases}$$

Terminologies & relations

- From $x_\ell(t) = x_i(t) + \imath x_q(t) = (x(t) + \imath \hat{x}(t))e^{-\imath 2\pi f_0 t}$,

$$\begin{cases} x_i(t) = \mathbf{Re}\left\{ \boxed{(x(t) + \imath \hat{x}(t))}\, e^{-\imath 2\pi f_0 t} \right\} \\ x_q(t) = \mathbf{Im}\left\{ \boxed{(x(t) + \imath \hat{x}(t))}\, e^{-\imath 2\pi f_0 t} \right\} \end{cases}$$

- Also from $x_\ell(t) = (x(t) + \imath \hat{x}(t))e^{-\imath 2\pi f_0 t}$,

$$\begin{cases} x(t) = \mathbf{Re}\left\{ \boxed{x_\ell(t)}\, e^{\imath 2\pi f_0 t} \right\} \\ \hat{x}(t) = \mathbf{Im}\left\{ \boxed{x_\ell(t)}\, e^{\imath 2\pi f_0 t} \right\} \end{cases}$$

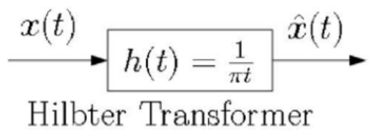
- ***The bandwidth of a signal*** is one half of the entire range of frequencies over which the spectrum is (essentially) nonzero. Hence, W is the bandwidth in the lowpass signal we just defined, while 2W is the bandwidth of the bandpass signal by our definition.

Modulator/demodulator and Hilbert transform

$x_\ell(t) \to x(t) = \mathbf{Re}\{x_\ell(t)e^{\imath 2\pi f_c t}\}$, mathematical representation of **modulation process.**

$x(t) \to x_\ell(t) = (x(t) + \imath \hat{x}(t))e^{-\imath 2\pi f_c t}$, mathematical representation of **demodulation process.**

The modulation requires to generate $\hat{x}(t)$, a Hilbert transform of $x(t)$



- **Hilbert transform is basically a 90-degree phase shifter, which means we need to have sign, cosine.**

$$H(f) = \mathcal{F}\left\{\frac{1}{\pi t}\right\} = -\imath\,\mathrm{sgn}(f) = \begin{cases} -\imath, & f > 0 \\ 0, & f = 0 \\ \imath, & f < 0 \end{cases}$$

Recall that on page 10, we have shown

$$\mathcal{F}^{-1}\{\mathrm{sgn}(f)\} = \imath\frac{1}{\pi t}\mathbf{1}\{t \neq 0\};$$

hence

$$\mathcal{F}\left\{\frac{1}{\pi t}\right\} = \frac{1}{\imath}\mathrm{sgn}(f) = -\imath\,\mathrm{sgn}(f).$$

Tip: $x_+(t) = \frac{1}{2}[x(t) + \imath\hat{x}(t)] \Rightarrow X_+(f) = \frac{1}{2}\left[X(f) + \imath\hat{X}(f)\right] = \begin{cases} X(f) & f > 0 \\ 0 & f < 0 \end{cases} \Rightarrow$

$\imath\hat{X}(f) = \imath X(f)H(f) = \begin{cases} X(f) & f > 0 \\ -X(f) & f < 0 \end{cases} \Rightarrow \imath H(f) = \begin{cases} 1 & f > 0 \\ -1 & f < 0 \end{cases}$

Example: $\sin(2\pi f_c t) = \cos(2\pi f_c t) \star h(t) = \cos(2\pi f_c t - \pi/2)$

Energy relationships between these 3 types of signals:

Definition (Energy of a signal)

- The energy $\mathcal{E}_s$ of a (complex) signal s(t)

$$\mathcal{E}_s = \int_{-\infty}^{\infty} |s(t)|^2\,dt$$

Hence,

$$\begin{aligned} \mathcal{E}_x &= \int_{-\infty}^{\infty} |x(t)|^2\,dt \\ \mathcal{E}_{x_+} &= \int_{-\infty}^{\infty} |x_+(t)|^2\,dt \\ \mathcal{E}_{x_\ell} &= \int_{-\infty}^{\infty} |x_\ell(t)|^2\,dt \end{aligned}$$

We are interested in the connections among $\mathcal{E}_x$, $\mathcal{E}_{x_+}$, and $\mathcal{E}_{x_l}$

- From Parseval's Theorem we see

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

- Parsevals theorem: $\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$
- Rayleigh's theorem: $\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$

- Secondly

$$X(f) = \underbrace{\frac{1}{2}X_\ell(f-f_c)}_{=X_+(f)} + \underbrace{\frac{1}{2}X_\ell^*(-f-f_c)}_{=X_+^*(-f)}$$

- Thirdly, $f_c \gg W$ and

$$X_\ell(f-f_c)X_\ell^*(-f-f_c) = 4X_+(f)X_+^*(-f) = 0 \text{ for all } f$$

It then shows

$$\begin{aligned}\mathcal{E}_x &= \int_{-\infty}^{\infty} \left|\frac{1}{2}X_\ell(f-f_c) + \frac{1}{2}X_\ell^*(-f-f_c)\right|^2 df \\ &= \frac{1}{4}\mathcal{E}_{x_\ell} + \frac{1}{4}\mathcal{E}_{x_\ell} = \frac{1}{2}\mathcal{E}_{x_\ell}\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}_x &= \int_{-\infty}^{\infty} |X_+(f) + X_+^*(-f)|^2 \, df \\ &= \mathcal{E}_{x_+} + \mathcal{E}_{x_+} = 2\mathcal{E}_{x_+}\end{aligned}$$

Theorem (Energy considerations)

$$\mathcal{E}_{x_l} = 2\mathcal{E}_x = 4\mathcal{E}_{x_+}$$

- Energy of xt is equal to half of xl, means xlt take two-time power of xt because ext is complex signal.
- Ex is two times of analytical signal.
- The final result is, xlt has the largest power which is 2 times of Ex and 4 times of analytical signal (Ex+).

Inner product: It is the inner product of two complex signals, x(t) and y(t). It is a general way to define power or energy. Its integration over t, with two signals multiplied together.

$$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x(t)y^*(t)dt.$$

- Parseval's relation immediately gives

$$\langle x(t), y(t)\rangle = \langle X(f), Y(f)\rangle.$$

- $\mathcal{E}_x = \langle x(t), x(t)\rangle = \langle X(f), X(f)\rangle$
- $\mathcal{E}_{x_\ell} = \langle x_\ell(t), x_\ell(t)\rangle = \langle X_\ell(f), X_\ell(f)\rangle$

Corss-correlation of two signals: Its defined as, the inner product of two signals divided by their square of the energy.

$$\rho_{x,y} = \frac{\langle x(t), y(t)\rangle}{\sqrt{\langle x(t), x(t)\rangle}\sqrt{\langle y(t), y(t)\rangle}} = \frac{\langle x(t), y(t)\rangle}{\sqrt{\mathcal{E}_x\mathcal{E}_y}}.$$

Orthogonality: We call two signals are orthogonal if they don't interfere with each other. Because the inner product is 0, they will not overlap.

$$\boxed{\rho_{x_\ell,y_\ell} = 0 \Rightarrow \rho_{x,y} = 0 \text{ but } \rho_{x,y} = 0 \not\Rightarrow \rho_{x_\ell,y_\ell} = 0}$$

2.1-4 Lowpass equivalence of a bandpass system

A bandpass system is an LTI system with real impulse response h(t) whose transfer function is located around a frequency fc which means already appear on fc or its modulated. It's a time invariant system.

Using a similar concept, we set the lowpass equivalent impulse response as

$$h(t) = \mathbf{Re}\left\{h_\ell(t)e^{\imath 2\pi f_c t}\right\}$$

and

$$H(f) = \frac{1}{2}\left[H_\ell(f - f_c) + H_\ell^*(-f - f_c)\right]$$

- Let $x(t)$ be a bandpass input signal and let

$$y(t) = h(t) \star x(t) \quad \text{or equivalently} \quad Y(f) = H(f)X(f)$$

- Then, we know

$$\begin{aligned} x(t) &= \mathbf{Re}\left\{x_\ell(t)e^{\imath 2\pi f_c t}\right\} \\ h(t) &= \mathbf{Re}\left\{h_\ell(t)e^{\imath 2\pi f_c t}\right\} \\ y(t) &= \mathbf{Re}\left\{y_\ell(t)e^{\imath 2\pi f_c t}\right\} \end{aligned}$$

and

Theorem (Baseband input-output relation)

$$y(t) = h(t) \star x(t) \quad \Longleftrightarrow \quad y_\ell(t) = \frac{1}{2}h_\ell(t) \star x_\ell(t)$$