Solutions Manual

Introduction to Data Compression

Third Edition

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Chapter 2

Problem 1.

$$H(X) = \sum_{i=1}^{M} P(x_i)\log_2 \frac{1}{P(x_i)} \geq \sum_{i=1}^{M} P(x_i)\log_2 \frac{1}{1} = 0$$

$$\begin{aligned}
H(X) - \log_2 M &= \sum_{i=1}^{M} P(x_i)\log_2 \frac{1}{P(x_i)} - \sum_{i=1}^{M} P(x_i)\log_2 M \\
&= \sum_{i=1}^{M} P(x_i)(\log_2 \frac{1}{P(x_i)} - \log_2 M) \\
&= \sum_{i=1}^{M} P(x_i)\log_2 \frac{1}{MP(x_i)} \\
&= E[\log_2 \frac{1}{MP(X)}]
\end{aligned}$$

Now we use Jensen's inequality which states that if $g(X)$ is a convex $\bigcap$ function $E[g(X)] \leq g(E[X])$.

$$\begin{aligned}
E[\log_2 \frac{1}{MP(x_i)}] &\leq \log_2 E[\frac{1}{MP(X)}] \\
&= \log_2(\sum_{i=1}^{M} P(x_i)\frac{1}{MP(x_i)}) \\
&= 0
\end{aligned}$$

or

$$H(X) \leq \log_2 M$$

Problem 2.

Without loss of generality we can assume that the source alphabet is $\{1, 2, \cdots, M\}$. Let H_1 represent the first order entropy and let H be the entropy of the source. Define

$$G_K = -\sum_{i_1=1}^{M} \cdots \sum_{i_K=1}^{M} P(X_1 = i_1, X_2 = i_2 \cdots X_K = i_K)\log_2 P(X_1 = i_1, X_2 = i_2 \cdots X_K = i_K)$$

then by definition

$$H = \lim_{K\to\infty} \frac{1}{K} G_K$$

If the X_i are independent we can write

$$P(X_1 = i_1, X_2 = i_2 \cdots X_K = i_K) = P(X_1 = i_1)P(X_2 = i_2) \cdots P(X_K = i_K)$$

and

$$
\begin{aligned}
G_K &= -\sum_{i_1=1}^{M} \cdots \sum_{i_K=1}^{M} P(X_1=i_1)P(X_2=i_2)\cdots P(X_K=i_K)[\log_2 P(X_1=i_1) \\
&\quad + \log_2 P(X_2=i_2) + \cdots \log_2 P(X_K=i_K)] \\
&= -\sum_{i_2=1}^{M} P(X_2=i_2)\cdots \sum_{i_K=1}^{M} P(X_K=i_K) \sum_{i_1=1}^{M} P(X_1=i_1)\log_2 P(X_1=i_1) \\
&- \sum_{i_1=1}^{M} P(X_1=i_1) \sum_{i_3=1}^{M} P(X_3=i_3)\cdots \sum_{i_K=1}^{M} P(X_K=i_K) \sum_{i_2=1}^{M} P(X_2=i_2)\log_2 P(X_2=i_2) \\
&\vdots \\
&- \sum_{i_1=1}^{M} P(X_1=i_1)\cdots \sum_{i_{K-1}=1}^{M} P(X_{K-1}=i_{K-1}) \sum_{i_K=1}^{M} P(X_K=i_K)\log_2 P(X_K=i_K)
\end{aligned}
$$

The first summations in each term sum to 1 and the last summation (assuming identical distributions) are equal to H_1, and

$$G_K = KH_1$$

Therefore,

$$H = \lim_{K\to\infty} \frac{1}{K} KH_1 = H_1$$

Problem 3.

a) 2 bits.

b) 1.75 bits.

c) 1.739818 bits.

Problem 4.

$$
\begin{aligned}
H_Q - H_P &= -\sum_{i=1}^{m} q_i \log_2 q_i + \sum_{i=1}^{m} p_i \log_2 p_i \\
&= -q_{j-1}\log_2 q_{j-1} - q_j \log_2 q_j + p_{j-1}\log_2 p_{j-1} + p_j \log_2 p_j
\end{aligned}
$$

Given a function

$$f_a(x) = -x\log x - (a-x)\log(a-x)$$

we can easily show that $f_a(x)$ is maximum for $x = \frac{a}{2}$ Let

$$p_{j-1} + p_j = c$$

then

$$q_{j-1} = q_j = \frac{c}{2}$$

Then

$$\begin{aligned} H_Q - H_P &= -\frac{c}{2}\log_2\frac{c}{2} - \frac{c}{2}\log_2\frac{c}{2} + p_j\log_2 p_j + (c-p_j)\log_2(c-p_j) \qquad (1) \\ &= f_c(\frac{c}{2}) - f_c(p_j) \\ &\geq 0 \end{aligned}$$

Therefore $H_Q \geq H_P$.

Problem 7.

a) Start with the list of codewords

$$\{0, 01, 11, 111\}$$

0 is a prefix for 01 generating a dangling suffix of 1. 11 is a prefix to 111 also generating a dangling suffix of 1. Augment the codeword list with the dangling suffix.

$$\{0, 01, 11, 111, 1\}$$

Now 1 is a prefix to 111 generating a dangling suffix of 11. As 11 is a codeword the code is not uniquely decodeable.

b) Start with the list of codewords

$$\{0, 01, 110, 111\}$$

0 is a prefix for 01 generating a dangling suffix of 1. Augment the codeword list with the dangling suffix.

$$\{0, 01, 110, 111, 1\}$$

1 is a prefix of 110 and 111 generating the dangling suffixes 10 and 11. As neither is a codeword we augment the list and continue.

$$\{0, 01, 110, 111, 1, 10, 11\}$$

At this point we can get no new dangling suffixes therefore this code is uniquely decodeable.

c) Start with the list of codewords

$$\{0, 10, 110, 111\}$$

No codeword is a prefix of any other codeword. Therefore, this code is uniquely decodeable.

d) Start with the list of codewords

$$\{1, 10, 110, 111\}$$

1 is a prefix of 110 which generates a dangling suffix of 10 which is a codeword. Therefore this code is not uniquely decodeable.

Chapter 3

Problem 4.

a)

$$H = \sum_{i=1}^{5} P(a_i) \log P(a_i) = 1.817684\text{bits}$$

b) If we sort the probabilities in descending order we can see that the two letters with the lowest probabilities are a_2 and a_4. These will become the leaves on the lowest level of the binary tree. The parent node of these leaves will have a probability of 0.9. If we consider parent node as a letter in a reduced alphabet the it will be one of the two letters with the lowest probability: the other one being a_1. Continuing in this manner we get the binary tree shown in Figure 1. and the code is

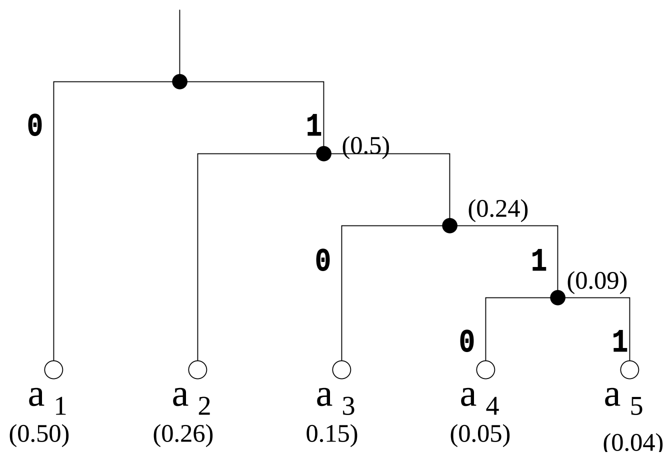

Figure 1: Huffman code for the five letter alphabet.

a_1	110
a_2	1111
a_3	10
a_4	1110
a_5	0

c)

$$\bar{l} = 0.15 \times 3 + 0.04 \times 4 + 0.26 \times 2 + 0.05 \times 4 + 0.5 \times 1 = 1.83\text{bits/symbol}$$

Problem 5.

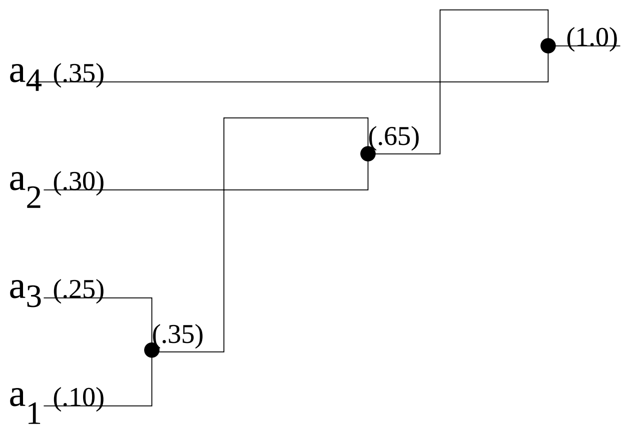

Figure 2: Huffman code for the four letter alphabet in Problem 5.

a) The Huffman code tree is shown in Figure 2. The code is

a_1	011
a_2	01
a_3	010
a_4	1

The average length of the code is $0.1 \times 3 + 0.3 \times 2 + 0.25 \times 3 + 0.35 \times 1 = 2$ bits/symbol.

b) The Huffman code tree is shown in Figure 3. The code is

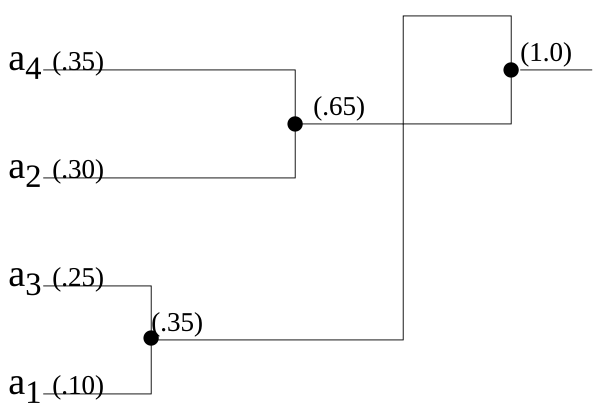

Figure 3: Minimum variance Huffman code for the four letter alphabet in Problem 5.

a_1	01
a_2	11
a_3	00
a_4	10

The average length of the code is obviously 2 bits/symbol.

While the average length of the codeword is the same for both codes, that is they are both equally efficient in terms of rate. However, the second code has a variance of zero for the codelengths. This means that we would not have any problems with buffer control if we were using this code in a communication system. We cannot make the same assertion about the first code.

Problem 6. Examining the Huffman code generated in Problem 4 (not 3!) along with the associated probabilities, we have

a_1	110	0.15
a_2	1111	0.04
a_3	10	0.26
a_4	1110	0.05
a_5	0	0.50

The proportion of zeros in a given sequence can be obtained by first computing the probability of observing a zero in a codeword $\sum_{k=1}^{5} P(0|a_k)P(a_k)$ and then dividing that by the average length of a codeword. The probability of observing a zero in a codeword is

$$1 \times 0.15 + 0 \times 0.04 + 1 \times 0.26 + 1 \times 0.05 + 1 \times 0.50 = 0.96$$

$0.96/1.83 = 0.52$. Thus the proportion of zeros is close to a half. If we examine Huffman codes for sources with dyadic probabilities we would find that the proportion is exactly a half. Thus the use of a Huffman code will not lead to inefficient channel usage.

Problem 8.

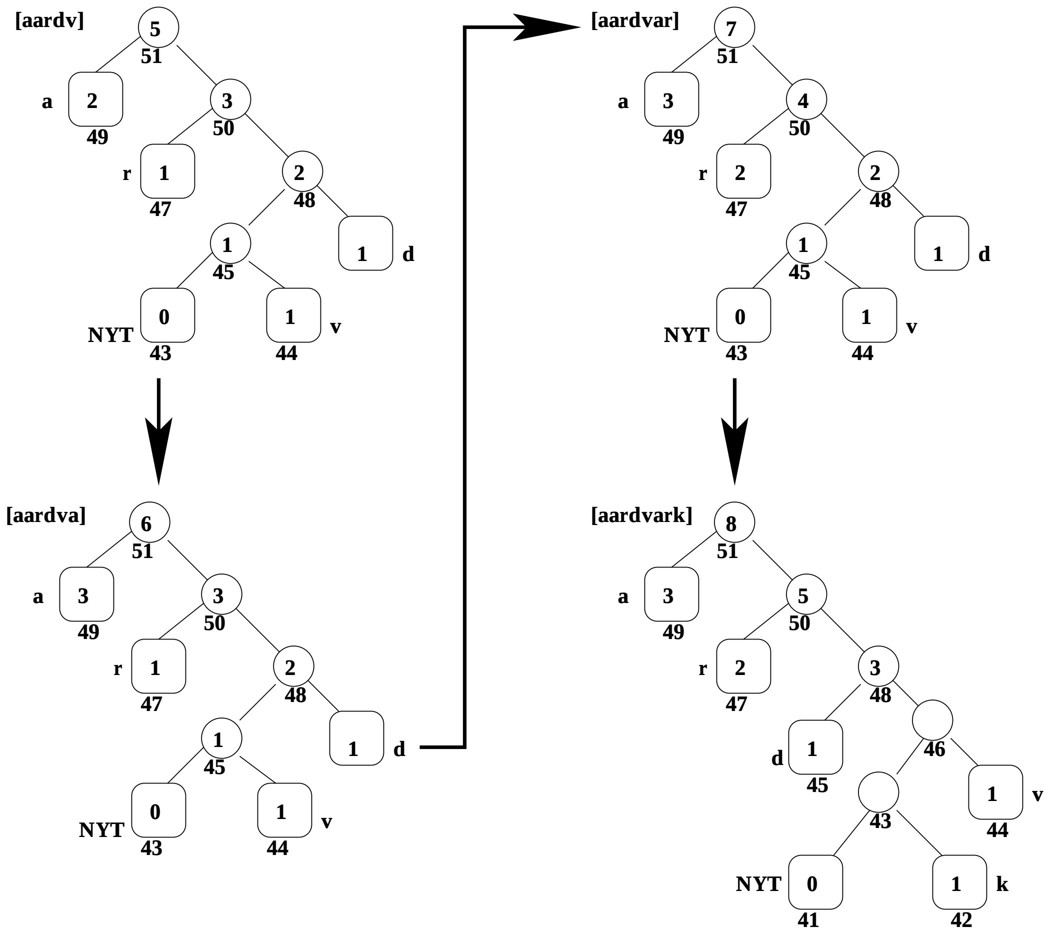

Figure 4: Figure for Problem 8 in Chapter 3.

Problem 7. If we combine three letters in the first and second steps we get the following code:

a_1	01
a_2	12
a_3	02
a_4	10
a_5	00
a_6	11

Thus the average length of the code is two ternary digits per symbol. If we calculate the average length of the code in Example 3.3.1 we obtain an average length of 1.7 ternary digits per symbol.

Problem 10

a)

Message	$a_2a_1a_3a_2a_1a_2$
Transmitted binary sequence	1010001011
Received binary sequence	0010001011
Decoded sequence	$a_4a_4a_2a_2$

Depending on how you count the errors five characters are received in error before the first correctly decoded character.

b)

Message	$a_2a_1a_3a_2a_1a_2$
Transmitted binary sequence	001011001000
Received binary sequence	101011001000
Decoded sequence	$a_1a_1a_3a_2a_1a_2$

Only a single character is received in error.

c)

Message	$a_2a_1a_3a_2a_1a_2$
Transmitted binary sequence	1010001011
Received binary sequence	1000001011
Decoded sequence	$a_2a_3a_4a_2a_2$

four characters are received in error before the first correct character.

For the minimum variance code the situation is different

Message	$a_2a_1a_3a_2a_1a_2$
Transmitted binary sequence	001011001000
Received binary sequence	000011001000
Decoded sequence	$a_2a_2a_3a_2a_1a_2$

Again, only a single character is received in error.

Problem 12.

Let's first do the preprocessing: Note that $y_{max} = 41$ and because we are going to assume a zero for the prediction of the first element of the sequence $y_{min} = 0$.

y_i	32	33	35	39	37	38	39	40	40	40	40	39	40	40	41	40
$\hat{y}_i$	0	32	33	35	39	37	38	39	40	40	40	40	39	40	40	41
d_i	32	1	2	4	-2	1	1	1	0	0	0	-1	1	0	1	-1
T_i	0	9	8	6	2	4	3	2	1	1	1	1	2	1	1	0
x_i	32	2	4	8	3	2	2	2	0	0	0	1	2	0	2	1

As $J = 8$ we divide the sequence of sixteen into two packets. The first packet consists of the numbers:

32	2	4	8	3	2	2	2

There is significant variation in the values in this packet so we would be best served with one of the split sample options. The split sample option with $m = 1$ would require 43 bits, $m = 2$ would require 35 bits, $m = 3$ would require 39 bits and for $m > 3$ would require more than 40 bits so we pick $m = 2$. In the table we show the translation for each number in the packet

Number	binary representation	code
32	100000	00000000001
2	000010	101
4	000100	0001
8	001000	00001
3	000011	111
2	000010	101
2	000010	101
2	000010	101

Stringing the codes together with an identifier for the code option gives us the code for the first packet.

The second packet looks very different.

0	0	0	1	2	0	2	1

It looks like a candidate for the low entropy option. However, as we do not have the lookup tables available we can just use the fundamental sequence option which gives us the following code for the packet.

11101001100101

Problem 13.

First iteration:

Letter	Probability
a_1	0.7
a_2	0.2
a_3	0.1

Second iteration:

Letter	Probability
a_2	0.2
a_3	0.1
a_1a_1	0.49
a_1a_2	0.14
a_1a_3	0.07

Final iteration:

Letter	Code
a_2	000
a_3	001
a_1a_2	010
a_1a_3	011
$a_1a_1a_1$	100
$a_1a_1a_2$	101
$a_1a_1a_3$	110

Chapter 4

Problem 1. Given a number $a \in [0, 1)$ with binary representation $[b_1 b_2 \dots b_n]$

$$a = b_1 2^{-1} + b_2 2^{-2} + \cdots + b_n 2^{-n}$$

If b has a binary representation with $[b_1 b_2 \dots b_n]$ as prefix, then

$$b = b_1 2^{-1} + b_2 2^{-2} + \cdots + b_n 2^{-n} + b_{n+1} 2^{-(n+1)} + \cdots$$

Therefore,

$$b - a = b_{n+1} 2^{-(n+1)} + \cdots$$

Obviously $b - a \geq 0$ and $b \geq a$. To show $b < a + \frac{1}{2^n}$ we note that

$$\begin{aligned} b - a &= b_{n+1} 2^{-(n+1)} + b_{n+2} 2^{-(n+2)} + \cdots \\ &\leq 2^{-(n+1)} + 2^{-(n+2)} + \cdots \\ &< \frac{1}{2^n} \end{aligned}$$

Problem 4.

$Count(1) = 40$	$Cum\_Count(0) = 0$	Scale3 $= 0$
$Count(2) = 1$	$Cum\_Count(1) = 40$	
$Count(3) = 9$	$Cum\_Count(2) = 41$	
$Total\_Count = 50$	$Cum\_Count(3) = 50$	

If $m = 6$

$$l^{(0)} = 000000; \qquad u^{(0)} = 111111 = 63$$

The first element of the sequence to be encoded is **1**.

$$\begin{aligned} l^{(1)} &= 0 + \lfloor \frac{64 \times Cum\_Count(0)}{50} \rfloor = 0 = (00000000)_2 \\ u^{(1)} &= 0 + \lfloor \frac{64 \times Cum\_Count(1)}{50} \rfloor - 1 = 50 = (110010)_2 \end{aligned}$$

The next element of the sequence is **3**.

$$\begin{aligned} l^{(2)} &= 0 + \lfloor \frac{51 \times Cum\_Count(2)}{50} \rfloor = 41 = (101001)_2 \\ u^{(2)} &= 0 + \lfloor \frac{51 \times Cum\_Count(3)}{50} \rfloor - 1 = 50 = (110010)_2 \end{aligned}$$

The MSBs of $l^{(2)}$ and $u^{(2)}$ are both 1. Therefore, we shift this value out and send it to the decoder. All other bits are shifted left by one bit, giving

$$\begin{aligned} l^{(2)} &= (010010)_2 = 18 \\ u^{(2)} &= (100101)_2 = 37. \end{aligned}$$

Notice that while the MSBs of the limits are different, the second MSB of the upper limit is zero, while the second MSB of the lower limit is one. This is the condition for the E_3 mapping. We complement the second MSB of both limits and shift one bit to the left, shifting in a zero as the LSB of $l^{(2)}$ and a one as the LSB of $u^{(2)}$. This gives us

$$\begin{aligned} l^{(2)} &= (000100)_2 = 4 \\ u^{(2)} &= (101011)_2 = 43. \end{aligned}$$

We also increment Scale3 to a value of one.

The next element in the sequence is **2**. Updating the limits, we have

$$\begin{aligned} l^{(3)} &= 4 + \lfloor \frac{40 \times Cum\_Count(1)}{50} \rfloor = 36 = (100100)_2 \\ u^{(3)} &= 4 + \lfloor \frac{40 \times Cum\_Count(2)}{50} \rfloor - 1 = 35 = (100011)_2 \end{aligned}$$

and the lower limit is greater than the upper limit. Clearly an unacceptable situation. Therefore we cannot use $m = 6$.

Problem 5

Letter	Probability	cdf
a_1	.2	$F_X(1) = 0.2$
a_2	.3	$F_X(2) = 0.5$
a_3	.5	$F_X(3) = 1.0$

$$\begin{aligned} l^{(0)} &= 0 \\ u^{(0)} &= 1 \end{aligned}$$

First letter is a_1

$$\begin{aligned} l^{(1)} &= 0 + (1 - 0) \times 0 = 0 \\ u^{(1)} &= 0 + (1 - 0) \times .2 = .2 \end{aligned}$$

Second letter is a_1

$$\begin{aligned} l^{(2)} &= 0 + (.2 - 0) \times 0 = 0 \\ u^{(2)} &= 0 + (.2 - 0) \times .2 = .04 \end{aligned}$$

Third letter is a_3

$$\begin{aligned} l^{(3)} &= 0 + (.04 - 0) \times 0.5 = 0.02 \\ u^{(3)} &= 0 + (.04 - 0) \times 1.0 = 0.04 \end{aligned}$$

Fourth letter is a_2.

$$\begin{aligned} l^{(4)} &= 0.02 + (.04 - 0.02) \times 0.2 = 0.024 \\ u^{(4)} &= 0.02 + (.04 - 0.02) \times 0.5 = 0.03 \end{aligned}$$

Fifth letter is a_3

$$\begin{aligned} l^{(5)} &= 0.024 + (.03 - 0.024) \times 0.5 = 0.027 \\ u^{(5)} &= 0.024 + (.03 - 0.024) \times 1.0 = 0.03 \end{aligned}$$

Sixth letter is a_1

$$\begin{aligned} l^{(6)} &= 0.027 + (.03 - 0.027) \times 0.0 = 0.027 \\ u^{(6)} &= 0.027 + (.03 - 0.027) \times 0.2 = 0.0276 \end{aligned}$$

Therefore a possible tag value is 0.0273.

Problem 6. The tag decodes to the following sequence:

$$a_3 a_2 a_2 a_1 a_2 a_1 a_3 a_2 a_2 a_3$$

Problem 7

Letter	Count	Cum_Count
		Cum_Count[0] = 0
$a(=1)$	37	Cum_Count[1] = 37
$b(=2)$	38	Cum_Count[2] = 75
$c(=3)$	25	Cum_Count[3] = 100

a) Total Count = 100

$$m = \lceil \log_2(Total\_Count) \rceil + 2 = 7 + 2 = 9\text{bits}$$

b)

$$\begin{aligned} l^{(0)} &= (000000000)_2 = (0)_{10} \\ u^{(0)} &= (111111111)_2 = (511)_{10} \end{aligned}$$

First letter is $a(1)$

$$\begin{aligned} l^{(1)} &= 0 + \lceil \frac{512 \times 0}{100} \rceil = (0)_{10} = (000000000)_2 \\ u^{(1)} &= 0 + \lceil \frac{512 \times 37}{100} \rceil - 1 = (188)_{10} = (010111100)_2 \end{aligned}$$

<u>Transmit 0</u>.

$$\begin{aligned} l^{(1)} &= (000000000)_2 = (0)_{10} \\ u^{(1)} &= (101111001)_2 = (377)_{10} \end{aligned}$$

Input $= b$ (2)

$$\begin{aligned} l^{(2)} &= 0 + \lceil \frac{378 \times 37}{100} \rceil = (139)_{10} = (010001011)_2 \\ u^{(2)} &= 0 + \lceil \frac{378 \times 75}{100} \rceil - 1 = (282)_{10} = (100011010)_2 \end{aligned} \tag{2}$$

This is an $E3$ condition $\Rightarrow$ scale3=1

$$\begin{aligned} l^{(2)} &= (000010110)_2 = (22)_{10} \\ u^{(2)} &= (100110101)_2 = (309)_{10} \end{aligned} \tag{3}$$

Input=a (1)

$$\begin{aligned} l^{(3)} &= 22 + \lceil \frac{288 \times 0}{100} \rceil = (22)_{10} = (000010110)_2 \\ u^{(3)} &= 22 + \lceil \frac{288 \times 37}{100} \rceil - 1 = (105)_{10} = (001101001)_2 \end{aligned} \tag{4}$$

<u>Transmit 0</u>
<u>Transmit 1</u>. Set scale3=0
<u>Transmit 0</u>

$$\begin{aligned} l^{(3)} &= (001011000)_2 = (88)_{10} \\ u^{(3)} &= (110100111)_2 = (423)_{10} \end{aligned}$$

input $= c$ (3)

$$\begin{aligned} l^{(4)} &= 88 + \lceil \frac{336 \times 75}{100} \rceil = (340)_{10} = (101010100)_2 \\ u^{(4)} &= 88 + \lceil \frac{336 \times 100}{100} \rceil - 1 = (423)_{10} = (110100111)_2 \end{aligned}$$

<u>Transmit 1</u>

$$\begin{aligned} l^{(4)} &= (010101000)_2 \\ u^{(4)} &= (101001111)_2 \end{aligned}$$

E3 condition $\Rightarrow$ Set scale3=1.

$$\begin{aligned} l^{(4)} &= (001010000)_2 = (80)_{10} \\ u^{(4)} &= (110011111)_2 = (415)_{10} \end{aligned}$$

Input $= a$ (1)

$$\begin{aligned} l^{(5)} &= 80 + \lceil \frac{336 \times 0}{100} \rceil = (80)_{10} = (001010000)_2 \\ u^{(5)} &= 80 + \lceil \frac{336 \times 37}{100} \rceil - 1 = (203)_{10} = (011001011)_2 \end{aligned}$$

<u>Transmit 0</u>
<u>Transmit 1</u>. Set scale3 = 0.

$$\begin{aligned} l^{(5)} &= (010100000)_2 = (160)_{10} \\ u^{(5)} &= (110010111)_2 = (407)_{10} \end{aligned}$$

Input $= b$ (2)

$$\begin{aligned} l^{(6)} &= 160 + \lceil \frac{248 \times 37}{100} \rceil = (251)_{10} = (011111011)_2 \\ u^{(6)} &= 160 + \lceil \frac{248 \times 75}{100} \rceil - 1 = (345)_{10} = (101011001)_2 \end{aligned}$$

E3 condition $\Rightarrow$ Set scale3=1.

$$\begin{aligned} l^{(6)} &= (011110110)_2 = (246)_{10} \\ u^{(6)} &= (110110011)_2 = (435_{10} \end{aligned}$$

Input=b (2)

$$\begin{aligned} l^{(7)} &= 246 + \lceil \frac{190 \times 37}{100} \rceil = (316)_{10} = (100111100)_2 \\ u^{(7)} &= 246 + \lceil \frac{190 \times 75}{100} \rceil - 1 = (388)_{10} = (110000100)_2 \end{aligned}$$

<u>Transmit 1</u>
<u>Transmit 0</u>. Scale3 = 0

$$\begin{aligned} l^{(7)} &= (001111000)_2 \\ u^{(7)} &= (100001001)_2 \end{aligned}$$

To terminate the encoding send the bits in the lower limit.

<u>Transmit 00111100</u>.

c) Received sequence: 001010110001111000.

$$\begin{aligned} l^{(0)} &= (000000000)_2 = (0)_{10} \\ u^{(0)} &= (111111111)_2 = (511)_{10} \\ t &= (001010010)_2 = (86)_{10} \end{aligned}$$

$$\frac{86 \times 100 - 1}{512} = 16$$

Compare this against the Cum_Counts

$$\begin{aligned} k = 0 \quad & 16 \geq Cum\_Count[0] = 0 \\ k = 1 \quad & 16 < Cum\_Count[1] = 37 \end{aligned}$$

Decode k=1 $\Rightarrow a$.

First letter is a. Update upper and lower limits.

$$\begin{aligned} l^{(1)} &= 0 + \lceil \frac{512 \times 0}{100} \rceil = (0)_{10} = (000000000)_2 \\ u^{(1)} &= 0 + \lceil \frac{512 \times 37}{100} \rceil - 1 = (188)_{10} = (010111100)_2 \end{aligned} \tag{5}$$

Shift out 0 from upper and lower limit and the tag and read in one more bit for the tag

$$\begin{aligned} l^{(1)} &= (000000000)_2 = (0)_{10} \\ u^{(1)} &= (101111001)_2 = (377)_{10} \\ t &= (010101100)_2 = (172)_{10} \end{aligned}$$

$$\frac{172 \times 100 - 1}{378} = 45$$

$$\begin{aligned} k = 0 \quad & 45 \geq Cum\_Count[0] = 0 \\ k = 1 \quad & 45 \geq Cum\_Count[1] = 37 \\ k = 2 \quad & 45 < Cum\_Count[2] = 75 \end{aligned}$$

Decode $k = 2 \Rightarrow b$

Update upper and lower limits

$$\begin{aligned} l^{(2)} &= 0 + \lceil \frac{378 \times 37}{100} \rceil = (139)_{10} = (010001011)_2 \\ u^{(2)} &= 0 + \lceil \frac{378 \times 75}{100} \rceil - 1 = (282)_{10} = (100011010)_2 \end{aligned} \tag{6}$$

This is an $E3$ condition. Complement second msb on upper and lower limits and on tag and shift left. Read in a 0 into lsb of the lower limit, a 1 into lsb of the upper limit, and a bit from the bitstream for the tag.

$$\begin{aligned} l^{(2)} &= (000010110)_2 = (22)_{10} \\ u^{(2)} &= (100110101)_2 = (309)_{10} \\ t &= (001011000)_2 = (88)_{10} \end{aligned}$$

$$\frac{(88 - 22 + 1) \times 100 - 1}{288} = 23$$

$$\begin{aligned} k = 0 \quad & 23 \geq Cum\_Count[0] = 0 \\ k = 1 \quad & 23 < Cum\_Count[1] = 37 \end{aligned}$$

Decode $k = 1 \Rightarrow a$

Update upper and lower limits:

$$\begin{aligned} l^{(3)} &= 22 + \lceil \frac{288 \times 0}{100} \rceil = (22)_{10} = (000010110)_2 \\ u^{(3)} &= 22 + \lceil \frac{288 \times 37}{100} \rceil - 1 = (105)_{10} = (001101001)_2 \end{aligned} \tag{7}$$

Shift out two 0's from the upper and lower limits and the tag. Shift in 0's into the lsbs of the lowerlimit, 1's into the lsb's of the upper limit and read two bits from the received bitstream into the lsb's of the tag.

$$\begin{aligned} l^{(3)} &= (001011000)_2 = (88)_{10} \\ u^{(3)} &= (110100111)_2 = (423)_{10} \\ t &= (101100011)_2 = (355)_{10} \end{aligned}$$

$$\frac{(355-88+1)\times 100-1}{336}=79$$

$$\begin{array}{ll} k=0 & 79 \geq Cum\_Count[0]=0 \\ k=1 & 79 \geq Cum\_Count[1]=37 \\ k=2 & 79 \geq Cum\_Count[2]=75 \\ k=3 & 79 < Cum\_Count[3]=100 \end{array}$$

Decode $k=3 \Rightarrow c$

Update upper and lower limits

$$\begin{aligned} l^{(4)} &= 88+\lceil\frac{336\times 75}{100}\rceil=(340)_{10}=(101010100)_2 \\ u^{(4)} &= 88+\lceil\frac{336\times 100}{100}\rceil-1=(423)_{10}=(110100111)_2 \end{aligned}$$

Shift out 1 from upper and lower limits and tag.

$$\begin{aligned} l^{(4)} &= (010101000)_2 \\ u^{(4)} &= (101001111)_2 \\ t &= (011000111)_2 \end{aligned}$$

E3 condition. Complement second msb of upper and lower limits and tag and shift left.

$$\begin{aligned} l^{(4)} &= (001010000)_2=(80)_{10} \\ u^{(4)} &= (110011111)_2=(415)_{10} \\ t &= (010001111)_2=(143)_{10} \end{aligned}$$

$$\frac{(143-80+1)\times 100-1}{336}=19$$

$$\begin{array}{ll} k=0 & 19 \geq 0 \\ k=1 & 19 < 37 \end{array}$$

Decode $k = 1 \Rightarrow a$.

Update upper and lower limits:

$$\begin{aligned} l^{(5)} &= 80 + \lceil \frac{336 \times 0}{100} \rceil = (80)_{10} = (001010000)_2 \\ u^{(5)} &= 80 + \lceil \frac{336 \times 37}{100} \rceil - 1 = (203)_{10} = (011001011)_2 \end{aligned}$$

Shift 0 out of upper and lower limits and tag.

$$\begin{aligned} l^{(5)} &= (010100000)_2 = (160)_{10} \\ u^{(5)} &= (110010111)_2 = (407)_{10} \\ t &= (100011110)_2 = (286)_{10} \end{aligned}$$

$$\frac{(286 - 160 + 1) \times 100 - 1}{248} = 51$$

$$\begin{array}{ll} k = 0 & 51 \geq 0 \\ k = 1 & 51 \geq 37 \\ k = 2 & 51 < 75 \end{array}$$

Decode $k = 2 \Rightarrow b$.

Update upper and lower limits:

$$\begin{aligned} l^{(6)} &= 160 + \lceil \frac{248 \times 37}{100} \rceil = (251)_{10} = (011111011)_2 \\ u^{(6)} &= 160 + \lceil \frac{248 \times 75}{100} \rceil - 1 = (345)_{10} = (101011001)_2 \end{aligned}$$

E3 condition.

$$\begin{aligned} l^{(6)} &= (011110110)_2 = (246)_{10} \\ u^{(6)} &= (110110011)_2 = (435_{10} \\ t &= (100111100)_2 = (316)_{10} \end{aligned}$$

$$\frac{(316 - 246 + 1) \times 100 - 1}{190} = 37$$

$$\begin{array}{ll} k=0 & 37 \geq 0 \\ k=1 & 37 \geq 37 \\ k=2 & 37 < 75 \end{array}$$

Decode $k = 2 \Rightarrow b$.

Assuming we know that only seven symbols were transmitted we can stop decoding.

Chapter 5

Problem 3.

Index	Codebook entry
1	a
2	b
3	r
4	y
5	$\not b$
6	$a\not b$
7	$\not b b$
8	ba
9	ar
10	$r\not b$
11	$\not b a$
12	arr
13	ra
14	ay
15	$y\not b$
16	$\not b by$
17	$y\not b b$
18	bar
19	rr
20	ray
21	ya
22	$ar\not b$
23	$\not b a_{-}$

Transmitted sequence is

$$1, 5, 2, 1, 3, 5, 9, 3, 1, 4, 7, 15, 8, 3, 13, 4, 9, 7, 14$$

Problem 4.

Index	Codebook entry
1	a
2	$\not b$
3	h
4	i
5	s
6	t
7	th
8	hi
9	is
10	$s\not b$
11	$\not b h$
12	ha
13	at
14	$t\not b$
15	$\not b i$
16	$is\not b$
17	$\not b hi$
18	$is\not b h$
19	hat
20	$t\not b i$
21	it
22	$t\not b is$
23	$s\not b h$
24	his
25	$s\not b ha$
26	at_{-}

The decoded message is *this hat is his hat it is his hat*

Problem 5.

Index	Codebook entry
1	a
2	$\not b$
3	r
4	t
5	ra
6	at
7	ta
8	ata
9	$atat$
10	$t\not b$
11	$\not b a$
12	$a\not b$
13	$\not b r$
14	rat
15	$t\not b a$
16	$at\not b$
17	$\not b ab$
18	$\not b ra$
19	at_{-}

The decoded message is *ratatatat a rat at a rat.*

Problem 6.

$$< 0, 0, C(b) >< 0, 0, C(a) >< 0, 0, C(r) >< 1, 1, C(a) >< 0, 0, C(y) >< 5, 2, C(\not b) >$$

$$< 9, 3, C(\not b) >< 4, 1, C(y) >< 7, 4, C(r) >< 3, 1, C(y) >< 12, 4, C(a) >$$

Problem 7

The decoded sequence is *ratatatat a rat at a rat.*

Problem 8.

Initial dictionary:

Index	Entry
1	S
2	$\not b$
3	I
4	T
5	H

Received sequence: 4, 5, 3, 1, 2, 8, 2, 7, 9, 7, 4

Received	Decode	Update Dictionary	
4	T	Entry 6: $T\ldots$	
5	H	Entry 6: TH	Entry 7: $H\ldots$
3	I	Entry 7: HI	Entry 8: $I\ldots$
1	S	Entry 8: IS	Entry 9: $S\ldots$
2	$\not b$	Entry 9: $S\not b$	Entry 10: $\not b\ldots$
8	IS	Entry 10: $\not bI$	Entry 11: $IS\ldots$
2	$\not b$	Entry 11: $IS\not b$	Entry 12: $\not b\ldots$
7	HI	Entry 12: $\not bH$	Entry 13: $HI\ldots$
9	$S\not b$	Entry 13: HIS	Entry 14: $S\not b$
7	HI	Entry 14: $S\not bH$	Entry 15: $HI\ldots$
4	T	Entry 15: HIT	Entry 16: $T\ldots$

Final Dictionary:

Index	Entry
1	S
2	$\not b$
3	I
4	T
5	H
6	TH
7	HI
8	IS
9	$S\not b$
10	$\not bI$
11	$IS\not b$
12	$\not bH$
13	HIS
14	$S\not bH$
15	HIT
16	$T\ldots$

Chapter 6

Problem 3(a)

Lets first obtain all cyclical shifts of the message to be encoded

e	t	a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c	e	t	a
a	e	t	a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c	e	t
t	a	e	t	a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c	e
e	t	a	e	t	a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c
c	e	t	a	e	t	a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$
$\not b$	c	e	t	a	e	t	a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b	e	t	a
a	$\not b$	c	e	t	a	e	t	a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b	e	t
t	a	$\not b$	c	e	t	a	e	t	a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b	e
e	t	a	$\not b$	c	e	t	a	e	t	a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b
b	e	t	a	$\not b$	c	e	t	a	e	t	a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$
$\not b$	b	e	t	a	$\not b$	c	e	t	a	e	t	a	$\not b$	c	e	t	a	$\not b$	a	n	d
d	$\not b$	b	e	t	a	$\not b$	c	e	t	a	e	t	a	$\not b$	c	e	t	a	$\not b$	a	n
n	d	$\not b$	b	e	t	a	$\not b$	c	e	t	a	e	t	a	$\not b$	c	e	t	a	$\not b$	a
a	n	d	$\not b$	b	e	t	a	$\not b$	c	e	t	a	e	t	a	$\not b$	c	e	t	a	$\not b$
$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c	e	t	a	e	t	a	$\not b$	c	e	t	a
a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c	e	t	a	e	t	a	$\not b$	c	e	t
t	a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c	e	t	a	e	t	a	$\not b$	c	e
e	t	a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c	e	t	a	e	t	a	$\not b$	c
c	e	t	a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c	e	t	a	e	t	a	$\not b$
$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c	e	t	a	e	t	a
a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c	e	t	a	e	t
t	a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c	e	t	a	e

Now let's sort the cyclically shifted sequences:

$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c	e	t	a	e	t	a	$\not b$	c	e	t	a
$\not b$	b	e	t	a	$\not b$	c	e	t	a	e	t	a	$\not b$	c	e	t	a	$\not b$	a	n	d
$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c	e	t	a	e	t	a
$\not b$	c	e	t	a	e	t	a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b	e	t	a
a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c	e	t	a	e	t	a	$\not b$	c	e	t
a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c	e	t	a	e	t
a	$\not b$	c	e	t	a	e	t	a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b	e	t
a	e	t	a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c	e	t
a	n	d	$\not b$	b	e	t	a	$\not b$	c	e	t	a	e	t	a	$\not b$	c	e	t	a	$\not b$
b	e	t	a	$\not b$	c	e	t	a	e	t	a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$
c	e	t	a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c	e	t	a	e	t	a	$\not b$
c	e	t	a	e	t	a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$
d	$\not b$	b	e	t	a	$\not b$	c	e	t	a	e	t	a	$\not b$	c	e	t	a	$\not b$	a	n
e	t	a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c	e	t	a	e	t	a	$\not b$	c
e	t	a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c	e	t	a
e	t	a	$\not b$	c	e	t	a	e	t	a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b
e	t	a	e	t	a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c
n	d	$\not b$	b	e	t	a	$\not b$	c	e	t	a	e	t	a	$\not b$	c	e	t	a	$\not b$	a
t	a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c	e	t	a	e	t	a	$\not b$	c	e
t	a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c	e	t	a	e
t	a	$\not b$	c	e	t	a	e	t	a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b	e
t	a	e	t	a	$\not b$	c	e	t	a	$\not b$	a	n	d	$\not b$	b	e	t	a	$\not b$	c	e

The sequence consisting of the last letters of the cyclically sorted set is

a	d	a	a	t	t	t	t	$\not b$	$\not b$	$\not b$	$\not b$	n	c	a	b	c	a	e	e	e	e

If we now begin with the index assignments

$\not b$	0
a	1
b	2
c	3
d	4
e	5
n	6
t	7

The move to front indices are

1	4	1	0	7	0	0	0	3	0	0	0	7	6	4	6	2	2	7	0	0	0

and the original sequence has an index of 14 (the first sequence in the list has an index of 0).

Problem 3(b)

In order to decode we first need to obtain the sequence of last letters from the move-to-front indices. Using the initial index assignment the first letter (with index of 1) is decoded to be a. The index assignment changes to

a	0	d	4
$\not b$	1	e	5
b	2	n	6
d	3	t	7

The next index is 4 and the element with an index of 4 is d. The index assignment now becomes

d	0	d	4
a	1	e	5
$\not b$	2	n	6
b	3	t	7

nd the next decoded letter is a. Continuing in this fashion the sequence of index assignments and decoded letters are:

a	0	d	4
d	1	e	5
$\not b$	2	n	6
b	3	t	7

Decode 0 as a.

Decode 7 as t and change index assignment to:

t	0	b	4
a	1	c	5
d	2	e	6
$\not b$	3	n	7

Decode 0 as t.

Decode 0 as t.

Decode 0 as t.

Decode 3 as $\not b$ and change the index assignment to:

$\not b$	0	b	4
t	1	c	5
a	2	e	6
d	3	n	7

Decode 0 as $\not b$.

Decode 0 as $\not b$.

Decode 0 as $\not b$.

Decode 7 as n and change index assignment to:

n	0	d	4
$\not b$	1	b	5
t	2	c	6
a	3	e	7

Decode 6 as c and change index assignment to

c	0	a	4
n	1	d	5
$\not b$	2	b	6
t	3	e	7

Decode 4 as a and change the index assignment to

a	0	t	4
c	1	d	5
n	2	b	6
$\not b$	3	e	7

Decode 6 as b and change the index assignment to:

b	0	$\not b$	4
a	1	t	5
c	2	d	6
n	3	e	7

Decode 2 as c and change the index assignment to:

c	0	$\not b$	4
b	1	t	5
a	2	d	6
n	3	e	7

Decode 2 as a and change the index assignment to:

a	0	$\not b$	4
c	1	t	5
b	2	d	6
n	3	e	7

Decode 7 as e and change the index assignment to:

e	0	n	4
a	1	$\not b$	5
c	2	t	6
b	3	d	7

Decode 0 as e.

Decode 0 as e.

Decode 0 as e.

Thus the decoded sequence of last letters becomes:

a	d	a	a	t	t	t	t	$\not b$	$\not b$	$\not b$	$\not b$	n	c	a	b	c	a	e	e	e	e

In order to get the transformation array we need to get the sequence of first letters. This is simply the sorted version of our decoded sequence.

L:	a	d	a	a	t	t	t	t	$\not b$	$\not b$	$\not b$	$\not b$	n	c	a	b	c	a	e	e	e	e
F:	$\not b$	$\not b$	$\not b$	$\not b$	a	a	a	a	a	b	c	c	d	e	e	e	e	n	t	t	t	t

From this we can construct T as

4	12	5	6	18	19	20	21	0	1	2	3	17	10	7	9	11	8	13	14	15	16

Now we begin decoding the original sequence. The last letter of the original sequence is in location 14 (starting the count with 0) which is an a. $T[14] = 7$ therefore the preceding letter is t. $T[7] = 21$, therefore the letter preceding t is e. $T[21] = 16$, therefore the letter preceding e is c. $T[16] = 11$ therefore the letter preceding c is $\not b$. Continuing in this fashion

$$\begin{aligned} T[11] = 3 \quad &\Rightarrow \quad a \\ T[3] = 6 \quad &\Rightarrow \quad t \\ T[6] = 20 \quad &\Rightarrow \quad e \\ T[20] = 15 \quad &\Rightarrow \quad b \end{aligned} \tag{8}$$

$$\begin{aligned}
T[15] = 9 &\quad\Rightarrow\quad \not b \\
T[9] = 1 &\quad\Rightarrow\quad d \\
T[1] = 12 &\quad\Rightarrow\quad n \\
T[12] = 17 &\quad\Rightarrow\quad a \\
T[17] = 8 &\quad\Rightarrow\quad \not b \\
T[8] = 0 &\quad\Rightarrow\quad a \\
T[0] = 4 &\quad\Rightarrow\quad t \\
T[4] = 18 &\quad\Rightarrow\quad e \\
T[18] = 13 &\quad\Rightarrow\quad c \\
T[13] = 10 &\quad\Rightarrow\quad \not b \\
T[10] = 2 &\quad\Rightarrow\quad a \\
T[2] = 5 &\quad\Rightarrow\quad t \\
T[5] = 19 &\quad\Rightarrow\quad e
\end{aligned}$$

and we obtain the decoded sequence.

Chapter 7

Problem 1

Let $w(n)$ and $b(n)$ represent white and black runs of length n respectively.

The lines of the image are $w(16)$
$w(4)$, $b(1)$, $w(2)$, $b(1)$, $w(1)$, $b(1)$, $w(6)$
$w(4)$, $b(1)$, $w(2)$, $b(1)$, $w(8)$
$w(4)$, $b(4)$, $w(1)$, $b(1)$, $w(6)$
$w(4)$, $b(1)$, $w(2)$, $b(1)$, $w(1)$, $b(1)$, $w(6)$
$w(4)$, $b(1)$, $w(2)$, $b(1)$, $w(1)$, $b(1)$, $w(6)$
$w(16)$
$w(16)$

Replacing the runlengths with the corresponding modified Huffman codes:

101010
1011, 010, 0111, 010, 000111, 010, 1110
1011, 010, 0111, 010, 10011
1011, 011, 000111, 010, 1110
1011, 010, 0111, 010, 000111, 010, 1110
1011, 010, 0111, 010, 000111, 010, 1110
101010
101010

Problem 4 a

Number of bits in 15 seconds: $9600 \times 15 = 144,000$.

This corresponds to $\frac{144,000}{8} = 18,000$ pixels.

At 512 pixels per line this is a little more than 35 lines which corresponds to about 6.9% of the image.

Problem 4 b

There are $\frac{512 \times 512}{8 \times 8} = 4,096$ 8×8 blocks in the image. If each block is represented by a single pixel, this corresponds to 32,768 bits which would be transmitted in 3.4133 seconds over a 9600 bps line. The second approximation would require three times as many pixel or 10.24 seconds. Together the first two approximations would require about 13.65 seconds.

Chapter 8

Problem 2.

a)

$$M = 1: \qquad R = 1 \quad D = 0;$$

$$M = 2: \qquad R = \frac{1}{2}$$

We have four possibilities with the following probabilities and distortions:

Message	Probability	Distortion
00	$P(0)^2$	0
01	$P(0)(1 - P(0))$	$\frac{1}{2}$
10	$P(0)(1 - P(0))$	$\frac{1}{2}$
11	$(1 - P(0))^2$	0

Total distortion $D = 0.16$

Similarly (with more calculations) we obtain

$$M = 4: \qquad R = 0.25 \quad D = 0.1856$$

$$M = 8: \qquad R = 0.125 \quad D = 0.1970$$

$$M = 16: \qquad R = 0.0625 \quad D = 0.199780$$

For a binary source

$$R = \begin{cases} H_b(p) - H_b(D) & D < \min p, 1 - p \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

Comparing with the rate distortion function

Distortion	Rate from scheme	Optimal rate
0.0	1.0	0.721928
0.16	0.5	0.087619
0.1856	0.25	0.029752
0.1970	0.125	0.006041
0.199780	0.0625	0.000440

b) If we encoded the output of the encoder using an entropy code which achieves a performance equal to the entropy the comparison is not as lopsided

Distortion	Rate from entropy coded scheme	Optimal rate
0.0	0.721928	0.721928
0.16	0.12	0.087619
0.1856	0.045	0.029752
0.1970	0.01043	0.006041
0.199780	0.0010	0.000440

Problem 4.

$$
\begin{aligned}
H(X_d|Y_d) &= -\sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} P(x_i|y_j)P(y_j)\log P(x_i|y_j) \\
&= -\sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} f_{X|Y}(x_i|y_j)\Delta f_Y(y_j)\Delta \log f_{X|Y}(x_i|y_j)\Delta \\
&= -\sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} f_{X|Y}(x_i|y_j) f_Y(y_j)\Delta\Delta \log f_{X|Y}(x_i|y_j)\Delta \\
&- \sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} f_{X|Y}(x_i|y_j) f_Y(y_j)\Delta\Delta \log \Delta \\
&= -\sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} f_{X|Y}(x_i|y_j) f_Y(y_j)\Delta\Delta \log f_{X|Y}(x_i|y_j) - \log\Delta
\end{aligned}
\tag{10}
$$

Problem 5.

$$
\begin{aligned}
H(X|Y) - H(X) &= -\sum_{i=0}^{N-1}\sum_{j=0}^{M-1} P(x_i,y_j)\log P(x_i|y_j) + \sum_{i=0}^{N-1}\sum_{j=0}^{M-1} P(x_i,y_j)\log P(x_i) \\
&= \sum_{i=0}^{N-1}\sum_{j=0}^{M-1} P(x_i,y_j)\log\frac{P(x_i)}{P(x_i|y_j)} \\
&= E\left[\log\frac{P(x_i)}{P(x_i|y_j)}\right]
\end{aligned}
$$

Now we use Jensen's inequality

$$
\begin{aligned}
H(X|Y) - H(X) &= E\left[\log\frac{P(X)}{P(X|Y)}\right] \\
&\leq \log\left[E\left[\frac{P(X)}{P(X|Y)}\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \log\left[\sum_{i=0}^{N-1}\sum_{j=0}^{M-1} P(x_i, y_j)\frac{P(x_i)}{P(x_i|y_j)}\right] \\
&= \log\left[\sum_{i=0}^{N-1}\sum_{j=0}^{M-1} P(y_j)P(x_i)\right] \\
&= 0
\end{aligned}
$$

where equality is achieved when $P(x_i) = P(x_i|y_j)$ or in other word the random variables X and Y are independent.

Problem 6

$$
\begin{aligned}
I(X;Y) &= \sum_{i=0}^{N-1}\sum_{j=0}^{M-1} P(x_i, y_j)\log\frac{P(x_i|y_j)}{P(x_i)} \\
&= \sum_{i=0}^{N-1}\sum_{j=0}^{M-1} P(x_i, y_j)\log\frac{P(x_i, y_j)}{P(x_i)P(y_j)} \\
&= \sum_{i=0}^{N-1}\sum_{j=0}^{M-1} P(x_i, y_j)\log\frac{P(y_j|x_i)}{P(y_j)} \\
&= I(Y;X)
\end{aligned}
$$

Problem 7

$$I(X;Y) = H(X) - H(X|Y)$$

$$H(X) = -p\log p - (1-p)\log(1-p) = H_b(p)$$

$$
\begin{aligned}
H(X|Y) &= -P(0|0)P(Y=0)\log P(0|0) - P(0|1)P(Y=1)\log P(0|1) - P(1|0)P(Y=0)\log P(1|0) - P(\\
&= -(1-D)P(Y=0)\log(1-D) - DP(Y=1)\log D - DP(Y=0)\log D - (1-D)P(Y=1)\,\mathrm{lo} \\
&= -[(1-D)\log(1-D)][P(Y=0)+P(Y=1)] - [D\log D][P(Y=0)+P(Y=1)] \\
&= -(1-D)\log(1-D) - D\log D \\
&= H_b(D)
\end{aligned}
$$

Therefore

$$I(X;Y) = H_b(p) - H_b(D)$$

Chapter 9

Problem 1. Leibnitz rule is given by

$$\frac{\delta}{\delta t}\int_{a(t)}^{b(t)} f(x,t)dx = \int_{a(t)}^{b(t)} \frac{\delta}{\delta t} f(x,t)dx + f(b(t),t)\frac{\delta}{\delta t}b(t) - f(a(t),t)\frac{\delta}{\delta t}a(t)$$

where $a(t)$ and $b(t)$ are monotonic functions of t.

The quantization noise power is given by:

$$\sigma_q^2 = 2\sum_{i=1}^{\frac{M}{2}-1}\int_{(i-1)\Delta}^{i\Delta}\left(x-\frac{2i-1}{2}\Delta\right)^2 f_X(x)dx + 2\int_{(\frac{M}{2}-1)\Delta}^{\infty}\left(x-\frac{M-1}{2}\Delta\right)^2 f_X(x)dx$$

Taking a derivative with respect to Δ and using Leibnitz rule we obtain:

$$\begin{aligned}\frac{\delta}{\delta t}\sigma_q^2 &= 2\sum_{i=1}^{\frac{M}{2}-1}\left[2(-\frac{(2i-1)}{2})\int_{(i-1)\Delta}^{i\Delta}\left(x-\frac{2i-1}{2}\Delta\right)f_X(x)dx + \frac{i\Delta^2}{4}f_X(i\Delta) - \frac{(i-1)\Delta^2}{4}f_X((i-1)\Delta)\right] \\ &+ \left[2(-\frac{(M-1)}{2})\int_{(\frac{M}{2}-1)\Delta}^{\infty}\left(x-\frac{M-1}{2}\Delta\right)f_X(x)dx - \frac{(\frac{M}{2}-1)\Delta^2}{4}f_X((\frac{M}{2}-1)\Delta)\right]\end{aligned}$$

Setting this to zero and noticing that

$$\sum_{i=1}^{\frac{M}{2}-1}\left[\frac{i\Delta^2}{4}f_X(i\Delta) - \frac{(i-1)\Delta^2}{4}f_X((i-1)\Delta)\right] = \frac{(\frac{M}{2}-1)\Delta^2}{4}f_X((\frac{M}{2}-1)\Delta)$$

we obtain the desired result.

Problem 6.

Input	$c(x)$	Quantize	$c^{-1}(x)$	Uniformly quantized
-0.8	-1.6	-1.5	-0.75	-0.5
1.2	2.13	2.5	1.75	1.5
0.5	1.0	0.5	0.25	0.5
0.6	1.2	1.5	0.75	0.5
3.2	3.47	3.5	3.25	3.5
-0.3	-0.6	-0.5	-0.25	-0.5

Chapter 10

Problem 1

Decrease in mse due to the inclusion of an output point at $(\Delta, 0)$ is 0.0038. By symmetry, the same is true for the inclusion of outputs at the other three points. The increased mse because of the removal of one of the corner points is 0.00026

$$\sigma^2_{q,original} = 10^{-\frac{SNR}{10}} = 0.07178$$

$$\sigma^2_{q,new} = 0.05768$$

$$SNR_{new} = 12.39dB$$

Therefore the increase in SNR is 0.95 dB

Problem 5

a) Referring to Figure 5:

$$P_A = \int_{2\Delta}^{3\Delta} e^{-\sqrt{2}x} dx \int_{3\Delta}^{4\Delta} e^{-\sqrt{2}y} dy = 0.00059$$

$$P_B = \left(\int_{3\Delta}^{4\Delta} e^{-\sqrt{2}x} dx \right)^2 = 0.00021$$

Total increase in overload probability is $8P_A + 4P_B = 0.0056$.

b)

$$P_C = \int_0^{\Delta} \frac{1}{\sqrt{2}} e^{-\sqrt{2}x} dx \int_{4\Delta}^{5\Delta} \frac{1}{\sqrt{2}} e^{-\sqrt{2}y} dy = 0.00166$$

Decrease of probability of overload is $8P_C = 0.01329$.

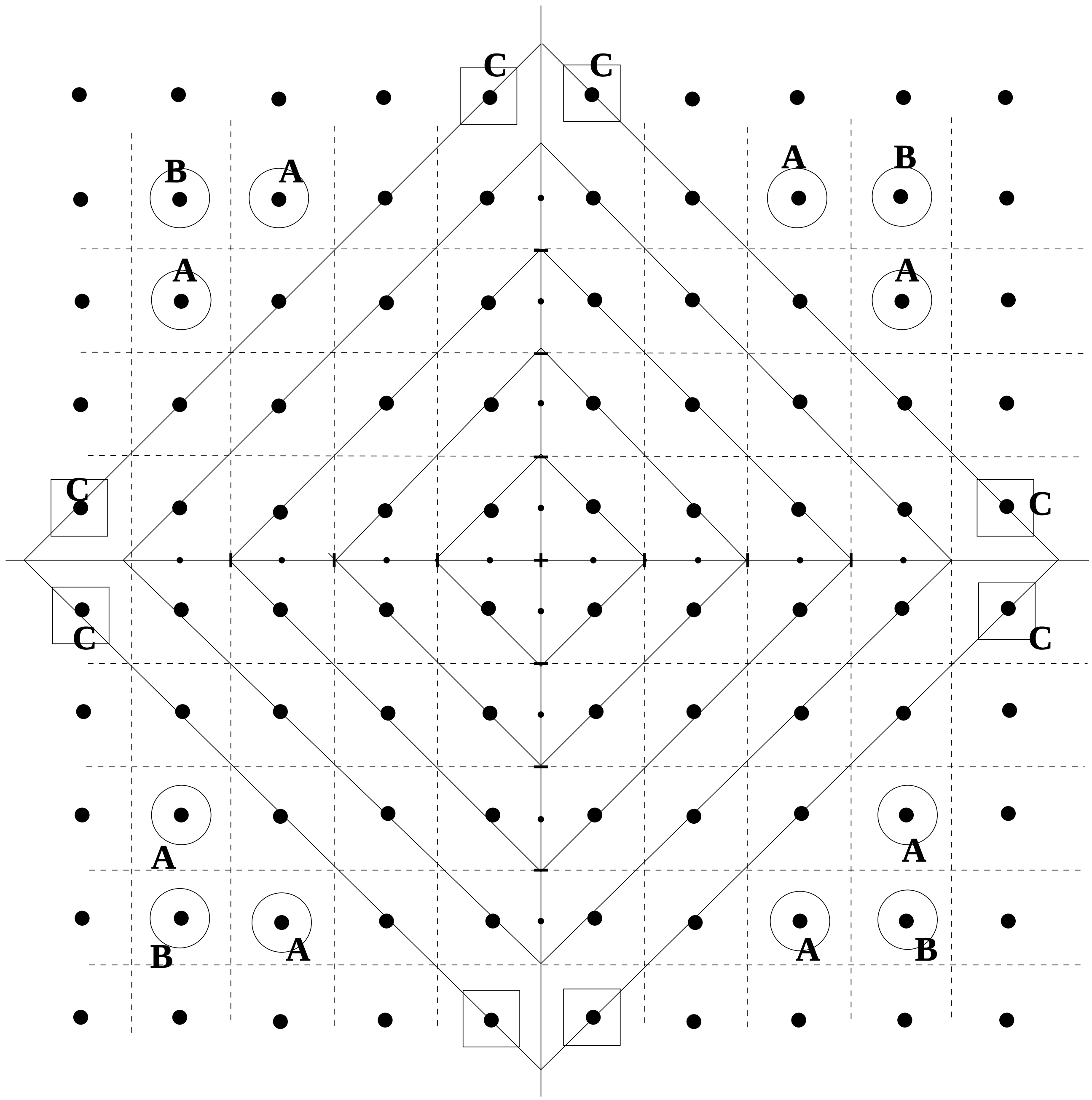

Figure 5: Figure for Problem 5 in Chapter 10.

Chapter 12

Problem 1. Let $X = \{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_N\}$ then

$$V = \{\mathbf{v} : \mathbf{v} = \sum_{i=1}^{N} c_i \mathbf{a}_i, \qquad c_i \in \mathcal{R}\}$$

where $\mathcal{R}$ denotes the set of real numbers.

1a. Let $\mathbf{v}_1, \mathbf{v}_2 \in V$. Then by definition

$$\mathbf{v}_1 = \sum_{i=1}^{N} \alpha_i \mathbf{a}_i, \quad \text{and} \quad \mathbf{v}_2 = \sum_{i=1}^{N} \beta_i \mathbf{a}_i$$

for $\{\alpha_i\},\ \{\beta_i\} \in \mathcal{R}$, and

$$\mathbf{v}_1 + \mathbf{v}_2 = \sum_{i=1}^{N} \alpha_i \mathbf{a}_i + \sum_{i=1}^{N} \beta_i \mathbf{a}_i = \sum_{i=1}^{N} (\alpha_i + \beta_i) \mathbf{a}_i$$

Define $\gamma_i = \alpha_i + \beta_i$, then $\gamma_i \in \mathcal{R}$ and

$$\mathbf{v}_1 + \mathbf{v}_2 = \sum_{i=1}^{N} \gamma_i \mathbf{a}_i$$

and $\mathbf{v}_1 + \mathbf{v}_2 \in V$

1b. By picking c_i to be zero for all i, we obtain a vector θ which by definition is in V and for which $\mathbf{v}_i + \theta = \mathbf{v}_i$ for all $\mathbf{v}_i \in V$. Thus θ is the additive identity and it is contained in V.

1c. Let $\mathbf{x} = \sum_{i=1}^{N} \alpha_i \mathbf{a}_i$. Define

$$-\mathbf{x} = \sum_{i=1}^{N} (-\alpha_i) \mathbf{a}_i$$

If $\alpha_i \in \mathcal{R}$ then $(-\alpha_i) \in \mathcal{R}$ and $-\mathbf{x} \in V$

Problem 2. Let $F(\omega) = \mathcal{F}[f(t)]$ and assume all integrals exist and we can switch the order of integration.

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(t)|^2 dt &= \int_{-\infty}^{\infty} f(t) f^*(t) dt \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \right] f^*(t) dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \left[\int_{-\infty}^{\infty} f^*(t) e^{j\omega t} dt \right] d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \left[\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right]^* d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) F^*(\omega) d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega
\end{aligned}$$

Problem 3. Let $F(\omega) = \mathcal{F}[f(t)]$.

$$\begin{aligned}
\mathcal{F}\left[f(t)e^{j\omega_o t}\right] &= \int_{-\infty}^{\infty} f(t)e^{j\omega_o t}e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_o)t}dt \\
&= F(\omega - \omega_o)
\end{aligned}$$

Problem 4. Let $F(\omega) = \mathcal{F}[f(t)]$, $F_1(\omega) = \mathcal{F}[f_1(t)]$, and $F_2(\omega) = \mathcal{F}[f_2(t)]$, and assume that all the integrals exist and we can change the order of integration.

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)e^{-j\omega t}d\tau dt
\end{aligned}$$

Let $u = t - \tau$, then $t = u + \tau$ and $dt = du$.

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f_1(\tau)\int_{-\infty}^{\infty} f_2(u)e^{-j\omega(u+\tau)}dud\tau \\
&= \int_{-\infty}^{\infty} f_1(\tau)e^{-j\omega\tau}\left[\int_{-\infty}^{\infty} f_2(u)e^{-j\omega u}du\right]d\tau \\
&= \int_{-\infty}^{\infty} f_1(\tau)e^{-j\omega\tau}F_2(\omega)d\tau \\
&= F_1(\omega)F_2(\omega)
\end{aligned}$$

Problem 5. We will make use of the fact that as the inverse Fourier transform of the delta function is given by

$$\mathcal{F}^{-1}[\delta(\omega)] = \frac{1}{2\pi}\delta(\omega)e^{-j\omega t}d\omega = \frac{1}{2\pi}$$

Therefore, by uniqueness of the Fourier transform

$$\mathcal{F}\left[\frac{1}{2\pi}\right] = \delta(\omega)$$

We will also make use of the modulation property of the Fourier transform.

First note that $f(t) = \sum_{n=-\infty}^{\infty}\delta(t - nT)$ is a periodic function with period T and therefore has a Fourier series representation

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t}, \qquad \omega_o = \frac{2\pi}{T}$$

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{n=-\infty}^{\infty} \delta(t - nT)dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)dt = \frac{1}{T}$$

Therefore,

$$\begin{aligned} f(t) &= \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_o t} \\ &= = \frac{\omega_o}{2\pi}\sum_{n=-\infty}^{\infty} e^{jn\omega_o t} \\ &= \omega_o \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} e^{jn\omega_o t} \end{aligned}$$

Now using the fact that $\mathcal{F}\left[\frac{1}{2\pi}\right] = \delta(\omega)$ and the modulation property

$$\mathcal{F}\left[\frac{1}{2\pi}e^{jn\omega_o t}\right] = \delta(\omega - n\omega_o)$$

$$\begin{aligned} F(\omega) &= \mathcal{F}[f(t)] \\ &= \mathcal{F}\left[\omega_o \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} e^{jn\omega_o t}\right] \\ &= \omega_o \sum_{n=-\infty}^{\infty} \mathcal{F}\left[\frac{1}{2\pi}e^{jn\omega_o t}\right] \\ &= \omega_o \sum_{n=-\infty}^{\infty} \delta(\omega - \omega_o) \end{aligned}$$

Problem 6.

(a).

$$\begin{aligned} H(z) &= \sum_{n=0}^{\infty} 2^{-n} z^{-n} \\ &= \sum_{n=0}^{\infty} (2^{-1} z^{-1})^n \\ &= \frac{1}{1 - 2^{-1}z^{-1}} \qquad |2^{-1}z^{-1}| < 1 \\ &= \frac{2z}{2z - 1} \qquad |z| > \frac{1}{2} \end{aligned}$$

(b).

$$H(z) = \sum_{n=0}^{\infty} (n^2 - n)3^{-n} z^{-n}$$

$$\begin{aligned} &= \sum_{n=0}^{\infty} n(n-1)3^{-n}z^{-n} \\ &= 3^{-2}\sum_{n=0}^{\infty} n(n-1)(3^{-1})^{n-2}z^{-n} \end{aligned}$$

Setting $m = 2$ in Example 11.9.2

$$H(z) = \frac{1}{9}\frac{2z}{(z-\frac{1}{3})^2}$$

(c).

$$\begin{aligned} H(z) &= \sum_{n=0}^{\infty}(n2^{-n} + 0.6^n)z^{-n} \\ &= \sum_{n=0}^{\infty} n2^{-n}z^{-n} + \sum_{n=0}^{\infty} 0.6^n z^{-n} \\ &= \frac{1}{2}\sum_{n=0}^{\infty} n(2^{-1})^{n-1}z^{-n} + \sum_{n=0}^{\infty} 0.6^n z^{-n} \\ &= \frac{1}{2}\frac{z}{(z-\frac{1}{2})^2} + \frac{z}{z-0.6} \qquad |z| > 0.6 \end{aligned}$$

Problem 7.

(a).

$$\begin{aligned} y_n &= 0.6y_{n-1} + 0.5x_n + 0.2x_{n-1} \\ Y(z) &= 0.6z^{-1}Y(z) + 0.5X(z) + 0.2z^{-1}X(z) \end{aligned}$$

Therefore,

$$Y(z)(1-0.6z^{-1}) = (0.5+0.2z^{-1})X(z)$$

and

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} = \frac{0.5+0.2z^{-1}}{1-0.6z^{-1}} \\ &= \frac{5z+2}{10z-6} \end{aligned}$$

(b).

$$H(z) = \frac{5z+2}{10z-6}$$

Performing long division we obtain the series

$$0.5 + 0.5z^{-1} + 0.3z^{-2} + 0.18z^{-3} + 0.108z^{-4} + 0.0648z^{-5} + \cdots$$

At this time we can see the pattern. From the third term on the nth element of the series is of the form $0.3(0.6)^{n-2}$. Therefore,

$$h_n = \begin{cases} .5 & n = 0, 1 \\ 0.3(0.6)^{n-2} & n > 1 \end{cases}$$

Problem 8.

(a)

$$H(z) = \frac{5}{z-2}$$

We have a pole outside the unit circle therefore this is the Z-transform of a left handed sequence. Therefore, $\{h_n\}$ is of the form

$$h_0 + h_{-1}z + h_{-2}z^2 + \cdots$$

Using the long division method we get the series

$$-\frac{5}{2} - \frac{5}{4}z - \frac{5}{8}z^2 - \frac{5}{16}z^3 - \cdots$$

Therefore,

$$h_n = \begin{cases} -5(2)^{-(n+1)} & n = 0, -1, -2, \cdots \\ 0 & n = 1, 2, 3, \cdots \end{cases}$$

(b)

$$H(z) = \frac{z}{z^2 - 0.25} = \frac{z}{(z-0.5)(z+0.5)}$$

Let's expand this using partial fraction expansion:

$$\frac{H(z)}{z} = \frac{\cancel{z}}{\cancel{z}(z-0.5)(z+0.5)} = \frac{A_1}{z-0.5} + \frac{A_2}{z+0.5}$$

To find the coefficients A_1 and A_2

$$A_1 = \frac{1}{z+0.5}|_{z=0.5} = 1$$

$$A_2 = \frac{1}{z-0.5}|_{z=-0.5} = -1$$

Therefore,

$$H(z) = \frac{z}{z-0.5} - \frac{z}{z+0.5}$$

and

$$h_n = 0.5^n u[n] - (-0.5)^n u[n]$$

(c)

$$H(z) = \frac{z}{z - 0.5}$$

By inspection,

$$h_n = 0.5^n u[n]$$

Chapter 13

Problem 1

$$\begin{aligned} |X_1 - X_2|^2 &= (X_1 - X_2)^T(X_1 - X_2) \\ &= (A^T\Theta_1 - A^T\Theta_2)^T(A^T\Theta_1 - A^T\Theta_2) \\ &= \Theta_1^T AA^T\Theta_1 - \Theta_1^T AA^T\Theta_2 - \Theta_2^T AA^T\Theta_1 + \Theta_2^T AA^T\Theta_2 \\ &= \Theta_1^T\Theta_1 - \Theta_1^T\Theta_2 - \Theta_2^T\Theta_1 + \Theta_2^T\Theta_2 \\ &= |\Theta_1 - \Theta_2|^2 \end{aligned}$$

Problem 2a.

If we transform

10	11	12	11	12	13	12	11

we obtain the coefficients

32.527	-1.281	-1.307	0.450	-1.414	-0.301	0.541	0.255

Notice that the DC coefficient has a rather high value while the other coefficients have relatively low values.

When we transform the vector

10	-10	8	-7	8	-8	7	-7

we obtain the coefficients

0.354	4.251	0.350	5.046	2.475	8.384	1.769	20.388

This time the higher frequency coefficients have larger magnitudes than the low frequency coefficients.

Concatenating the coefficients we obtain the plot shown in Figure 6.

Problem 2b. If we concatenate the two vectors and take a 16 point DCT the results are

23.25	21.78	-3.92	-6.96	-0.68	6.04	-3.25	-2.57	0.75	3.16	-6.14	1.61	1.63	5.50	-14.24	13.56

The results are plotted in Figure 7

Problem 2c. Comparing the plots shown in Figures 6 and 7 we can see that we obtain a larger number of coefficients with relatively large magnitudes when we use the 16 point DCT. This is due to the differing natures of the two vectors. As more large magnitude coefficients mean more bits required, it would be better from a compression perspective to use two 8 points DCT rather than one 16 point DCT in this example.

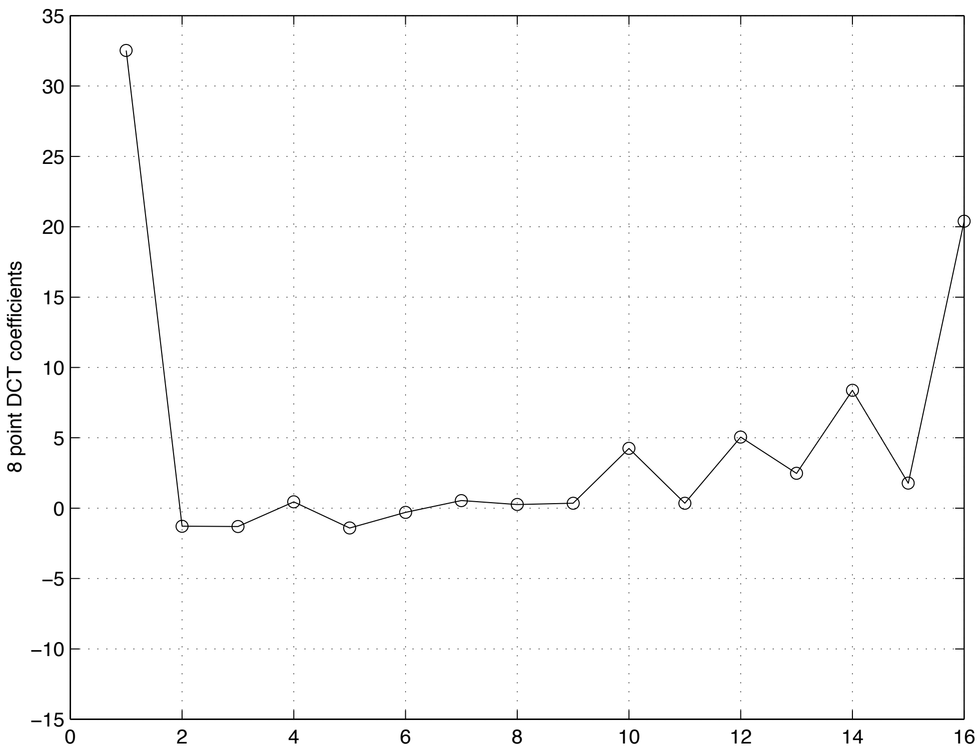

Figure 6: Plot of the 8 point coefficients of the two vectors. The first eight coefficients belong to the first vector while the second eight coefficients are the result of an 8 point DCT of the second vector.

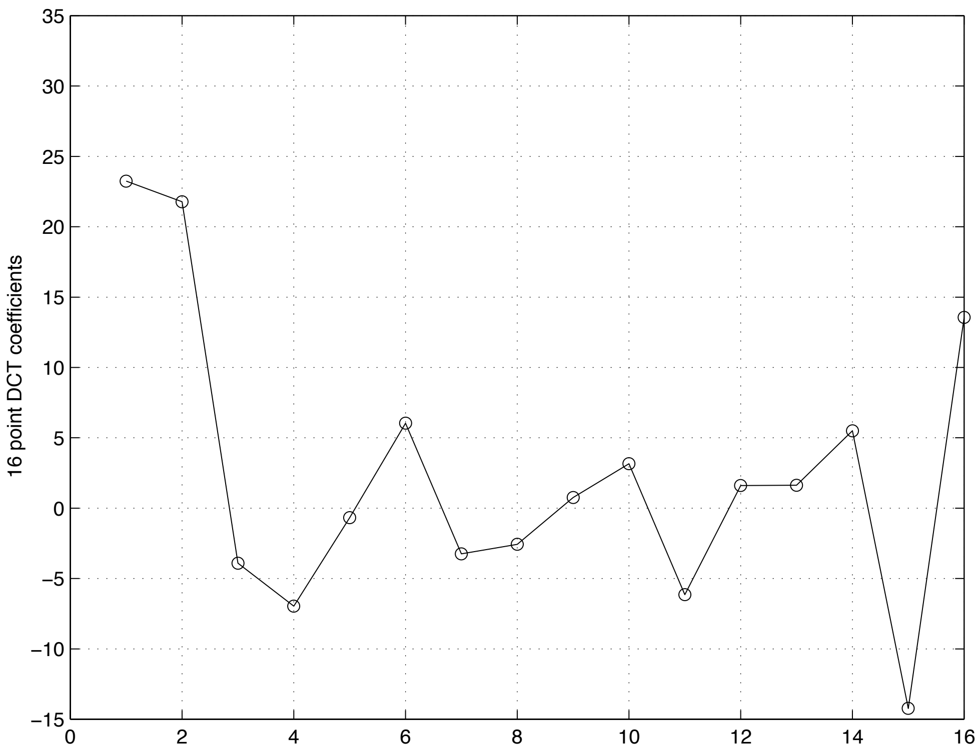

Figure 7: Plot of the 16 point DCT coefficients obtained by taking the 16 point DCT of the concatenation of the two vectors.

Problem 3

a) By rows:

$$\begin{bmatrix} 10 & 2 & 4 & 0 \\ 7 & 1 & 3 & 1 \\ 5 & 1 & 1 & 1 \\ 4 & 0 & 0 & 0 \end{bmatrix}$$

Then by columns:

$$\begin{bmatrix} 26 & 4 & 8 & 2 \\ 4 & 2 & 2 & 0 \\ 8 & 2 & 6 & 0 \\ 2 & 0 & 0 & -2 \end{bmatrix}$$

b) By columns first:

$$\begin{bmatrix} 10 & 7 & 5 & 4 \\ 2 & 1 & 1 & 0 \\ 4 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Then by rows:

$$\begin{bmatrix} 26 & 4 & 8 & 2 \\ 4 & 2 & 2 & 0 \\ 8 & 2 & 6 & 0 \\ 2 & 0 & 0 & -2 \end{bmatrix}$$

c) Results of a) and b) are identical.

Chapter 14

Problem 1 The impulse response $\{h_n\}$ of a system is the output when the input is δ_n where

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

By the second property if the input to the system is $a\delta_n$ then the output is $\{y_n\} = \{\alpha h_n\}$. Let's write this as

$$\alpha\delta_n \Rightarrow \{\alpha h_n\}$$

We can write the input sequence $\{x_n\}$ as

$$\{x_n\} = x_0\delta_n + x_1\delta_{n-1} + x_2\delta_{n-2} + \cdots$$

Using both properties we can see that

$$\begin{array}{rcl} x_0\delta_n & \Rightarrow & \{x_0 h_n\} \\ x_1\delta_{n-1} & \Rightarrow & \{x_1 h_{n-1}\} \\ x_2\delta_{n-2} & \Rightarrow & \{x_2 h_{n-2}\} \\ \vdots & & \vdots \\ x_M\delta_{n-M} & \Rightarrow & \{x_M h_{n-M}\} \\ \vdots & & \vdots \end{array}$$

Adding the left side we obtain the input sequence. By the second property the corresponding output is the sum of the right side or

$$y_n = \sum x_k h_{n-k}$$

Problem 2.

(a) Let's use a very simple impulse response:

$$h_{1,0} = \frac{1}{2}, h_{1,1} = \frac{1}{2}, h_{1,2} = \frac{1}{2}, h_{1,3} = -\frac{1}{2}$$

This satisfies the requirement that $|h_{1,j}| = |h_{1,k}|$. Furthermore,

$$\sum_{i=0}^{3} h_{1,i}^2 = 4 \times \frac{1}{4} = 1$$

and

$$\sum_{i=0}^{3} h_{1,i}h_{1,i+2} = h_{1,0}h_{1,2} + h_{1,1}h_{1,3} = \frac{1}{4} - \frac{1}{4} = 0$$

and the orthonormality requirement is satisfied.

(b) The magnitude and phase response of this filter are plotted in Figure 8. It is not a very good filter.

(c) Using Equation (13.23) we obtain the following coefficients for the high pass filter:

$$h_{2,0} = -\frac{1}{2}, h_{2,1} = -\frac{1}{2}, h_{1,2} = \frac{1}{2}, h_{1,3} = -\frac{1}{2}$$

(d) The magnitude and phase response of th high pass filter are plotted in Figure 9.

Problem 3.

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	$\cdots$
1	-1	1	-1	1	-1	1	-1	1	-1	1	$\cdots$

(a)

$$\begin{aligned} y_n &= h_0 x_n + h_1 x_{n-1} \\ &= \frac{1}{\sqrt{2}} x_n + \frac{1}{\sqrt{2}} x_{n-1} \end{aligned}$$

$$\begin{aligned} y_0 &= \frac{1}{\sqrt{2}} \times 1 + \frac{1}{\sqrt{2}} \times 0 = \frac{1}{\sqrt{2}} \\ y_1 &= \frac{1}{\sqrt{2}} \times (-1) + \frac{1}{\sqrt{2}} \times 1 = 0 \\ y_2 &= \frac{1}{\sqrt{2}} \times 1 + \frac{1}{\sqrt{2}} \times (-1) = 0 \\ y_3 &= \frac{1}{\sqrt{2}} \times (-1) + \frac{1}{\sqrt{2}} \times 1 = 0 \\ &\vdots \quad \vdots \end{aligned}$$

Or

$$y_n = \begin{cases} \frac{1}{\sqrt{2}} & n = 0 \\ 0 & n > 0 \end{cases}$$

(b)

$$\begin{aligned} w_n &= h_0 x_n + h_1 x_{n-1} \\ &= \frac{1}{\sqrt{2}} x_n - \frac{1}{\sqrt{2}} x_{n-1} \end{aligned}$$

$$w_0 = \frac{1}{\sqrt{2}} \times 1 - \frac{1}{\sqrt{2}} \times 0 = \frac{1}{\sqrt{2}}$$

$$
\begin{aligned}
w_1 &= \frac{1}{\sqrt{2}} \times (-1) - \frac{1}{\sqrt{2}} \times 1 = -\sqrt{2} \\
w_2 &= \frac{1}{\sqrt{2}} \times 1 - \frac{1}{\sqrt{2}} \times (-1) = \sqrt{2} \\
w_3 &= \frac{1}{\sqrt{2}} \times (-1) - \frac{1}{\sqrt{2}} \times 1 = -\sqrt{2} \\
&\vdots \quad \vdots
\end{aligned}
$$

Or

$$
w_n = \begin{cases} \frac{1}{\sqrt{2}} & n = 0 \\ \sqrt{2}(-1)^n & n > 0 \end{cases}
$$

(c) The filter response in part (a) corresponds to a low pass filter and the filter response in part (b) corresponds to a high pass filter. After the initial transient the output of the low pass filter is zero while the output of the high pass filter is an amplified version of the input. Thus the input is a high pass signal.

Problem 4.

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	$\cdots$
1	1	1	1	1	1	1	1	1	1	1	$\cdots$

(a)

$$
\begin{aligned}
y_n &= h_0 x_n + h_1 x_{n-1} \\
&= \frac{1}{\sqrt{2}} x_n + \frac{1}{\sqrt{2}} x_{n-1}
\end{aligned}
$$

$$
\begin{aligned}
y_0 &= \frac{1}{\sqrt{2}} \times 1 + \frac{1}{\sqrt{2}} \times 0 = \frac{1}{\sqrt{2}} \\
y_1 &= \frac{1}{\sqrt{2}} \times 1 + \frac{1}{\sqrt{2}} \times 1 = \sqrt{2} \\
y_2 &= \frac{1}{\sqrt{2}} \times 1 + \frac{1}{\sqrt{2}} \times 1 = \sqrt{2} \\
y_3 &= \frac{1}{\sqrt{2}} \times 1 + \frac{1}{\sqrt{2}} \times 1 = \sqrt{2} \\
&\vdots \quad \vdots
\end{aligned}
$$

Or

$$
y_n = \begin{cases} \frac{1}{\sqrt{2}} & n = 0 \\ \sqrt{2} & n > 0 \end{cases}
$$

(b)

$$\begin{aligned} w_n &= h_0 x_n + h_1 x_{n-1} \\ &= \frac{1}{\sqrt{2}} x_n - \frac{1}{\sqrt{2}} x_{n-1} \end{aligned}$$

$$\begin{aligned} w_0 &= \frac{1}{\sqrt{2}} \times 1 - \frac{1}{\sqrt{2}} \times 0 = \frac{1}{\sqrt{2}} \\ w_1 &= \frac{1}{\sqrt{2}} \times 1 - \frac{1}{\sqrt{2}} \times 1 = 0 \\ w_2 &= \frac{1}{\sqrt{2}} \times 1 - \frac{1}{\sqrt{2}} \times 1 = 0 \\ w_3 &= \frac{1}{\sqrt{2}} \times 1 - \frac{1}{\sqrt{2}} \times 1 = 0 \\ &\vdots \quad \vdots \end{aligned}$$

Or

$$w_n = \begin{cases} \frac{1}{\sqrt{2}} & n = 0 \\ 0 & n > 0 \end{cases}$$

(c) After the initial transient the output of the high pass filter is zero while the output of the low pass filter is an amplified version of the input. Thus the input is a low pass signal.

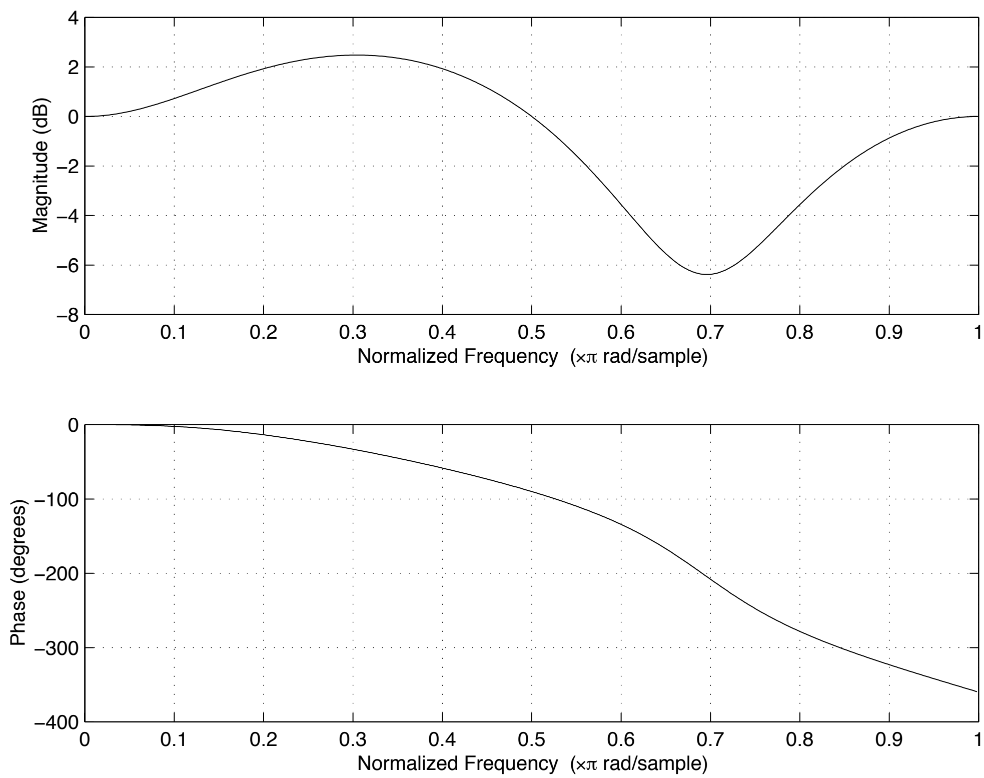

Figure 8: Magnitude and phase response of a filter with impulse response {0.5,0.5,0.5,-0.5}.

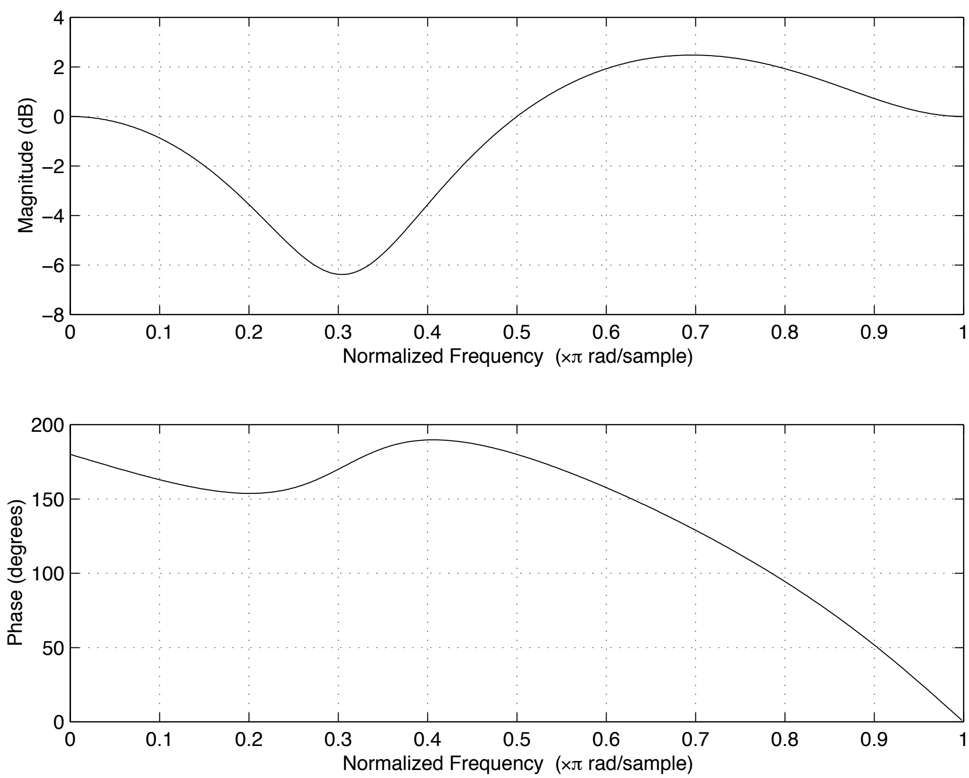

Figure 9: Magnitude and phase response of a filter with impulse response {-0.5,-0.5,0.5,-0.5}.

Chapter 15

Problem 1

(a)

$$f(t) = \sin(2t) = \frac{e^{j2t} - e^{-j2t}}{2j}$$

The Fourier transform is given by

$$\begin{aligned}
F(\omega) &= \mathcal{F}\left[\frac{e^{j2t} - e^{-j2t}}{2j}\right] \\
&= \frac{1}{2j}\mathcal{F}\left[e^{j2t}\right] - \frac{1}{2j}\mathcal{F}\left[e^{-j2t}\right] \qquad \text{by linearity} \\
&= \frac{1}{2j}\left[\delta(\omega-2) - \delta(\omega+2)\right]
\end{aligned}$$

(b)

$$f_1(t) = \begin{cases} \sin(2t) & -2 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
F_1(\omega) &= \int_{-2}^{2} \sin(2t) e^{-j\omega t} dt \\
&= \int_{-2}^{2} \left[\frac{e^{j2t} - e^{-j2t}}{2j}\right] e^{-j\omega t} dt \\
&= \frac{1}{2j}\left[\int_{-2}^{2} e^{-j(w-2)t} dt - \int_{-2}^{2} e^{-j(w+2)t} dt\right] \\
&= \frac{2}{j}\left[\frac{\sin(2(\omega-2))}{2(\omega-2)} - \frac{\sin(2(\omega+2))}{2(\omega+2)}\right]
\end{aligned}$$

(c)

$$f_2(t) = \begin{cases} (1+\cos(\frac{\pi}{2}t))\sin(2t) & -2 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
F_2(\omega) &= \int_{-2}^{2} (1+\cos(\frac{\pi}{2}t))\sin(2t) e^{-j\omega t} dt \\
&= \int_{-2}^{2} \sin(2t) e^{-j\omega t} dt + \int_{-2}^{2} \cos(\frac{\pi}{2}t)\sin(2t) e^{-j\omega t} dt
\end{aligned}$$

The first integral is identical to part (b). Evaluating the second integral we obtain

$$\frac{1}{j}\left[\frac{\sin(2(w-\frac{\pi}{2}-2))}{2(w-\frac{\pi}{2}-2)} - \frac{\sin(2(w-\frac{\pi}{2}+2))}{2(w-\frac{\pi}{2}+2)} + \frac{\sin(2(w+\frac{\pi}{2}-2))}{2(w+\frac{\pi}{2}-2)} - \frac{\sin(2(w+\frac{\pi}{2}+2))}{2(w+\frac{\pi}{2}+2)}\right]$$

(d) The magnitudes of $F(\omega)$, $F_1(\omega)$, and $F_2(\omega)$ are plotted in Figure 10, Figure 11 and Figure 12.

While both short term Fourier transforms broaden out the peak, the use of the rectangular window introduces a number of extraneous peaks of significant size. The use of the raised cosine window considerably reduces the size of these extraneous peaks.

Problem 2.

$$f(t) = \begin{cases} 1+\sin(2t) & 0 \le t \le 1 \\ \sin(2t) & \text{otherwise} \end{cases}$$

$$\psi(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2} \\ -1 & \frac{1}{2} \le t < 1 \end{cases}$$

$$c_{j,k} = \int f(t)\psi_{j,k}(t)dt$$

and

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$$

$$\begin{aligned} c_{0,0} &= \int_0^1 f(t)\psi(t)dt \\ &= \int_0^{\frac{1}{2}} (1+\sin(2t) - \int_{\frac{1}{2}}^1 (1+\sin(2t))dt \\ &= \frac{1}{2}[-\cos(0) + 2\cos(1) - \cos(2)] \end{aligned}$$

$$\begin{aligned} c_{0,1} &= \int_1^2 f(t)\psi(t-1)dt \\ &= \int_1^{\frac{3}{2}} \sin(2t)dt - \int_{\frac{3}{2}}^2 \sin(2t))dt \\ &= \frac{1}{2}[-\cos(2) + 2\cos(3) - \cos(4)] \end{aligned}$$

$$\begin{aligned} c_{0,2} &= \int_2^3 f(t)\psi(t-2)dt \\ &= \int_2^{\frac{5}{2}} \sin(2t)dt - \int_{\frac{5}{2}}^3 \sin(2t))dt \\ &= \frac{1}{2}[-\cos(4) + 2\cos(5) - \cos(6)] \end{aligned}$$

We can see a pattern here:

$$c_{0,k} = \frac{1}{2}\left[-\cos(2k) + 2\cos(2k+1) - \cos(2k+2)\right]$$

To find $c_{1,k}$ we use

$$c_{1,k} = \sqrt{2}\int \psi(2t-k)f(t)dt$$

So

$$\begin{aligned} c_{1,0} &= \sqrt{2}\int_0^{\frac{1}{4}}(1+\sin(2t))dt - \sqrt{2}\int_{\frac{1}{4}}^{\frac{1}{2}}(1+\sin(2t))dt \\ &= \frac{\sqrt{2}}{2}\left[-\cos(0) + 2\cos(\frac{1}{2} - \cos(1)\right] \end{aligned}$$

$$\begin{aligned} c_{1,1} &= \sqrt{2}\int_{\frac{1}{2}}^{\frac{3}{4}}(1+\sin(2t))dt - \sqrt{2}\int_{\frac{3}{4}}^{1}(1+\sin(2t))dt \\ &= \frac{\sqrt{2}}{2}\left[-\cos(1) + 2\cos(\frac{3}{2} - \cos(2)\right] \end{aligned}$$

$$\begin{aligned} c_{1,2} &= \sqrt{2}\int_{1}^{\frac{5}{4}}\sin(2t)dt - \sqrt{2}\int_{\frac{5}{4}}^{\frac{3}{2}}\sin(2t)dt \\ &= \frac{\sqrt{2}}{2}\left[-\cos(2) + 2\cos(\frac{5}{2} - \cos(3)\right] \end{aligned}$$

Again, we can see the pattern:

$$c_{1,k} = \frac{\sqrt{2}}{2}\left[-\cos(k) + \cos(\frac{2k+1}{2}) - \cos(k+1)\right]$$

Continuing for $c_{2,k}$ we find that

$$c_{2,k} = \frac{2}{2}\left[-\cos(\frac{k}{2}) + \cos(\frac{2k+1}{4}) - \cos(\frac{k+1}{2})\right]$$

Looking at the expressions for $c_{0,k}$, $c_{1,k}$ and $c_{2,k}$ we can see a common pattern:

$$c_{j,k} = \frac{2^{\frac{j}{2}}}{2}\left[-\cos(\frac{2k}{2^j}) + 2\cos(\frac{2k+1}{2^j}) - \cos(\frac{2k+2}{2^j})\right]$$

Problem 3. We assume the following:

- We have a bit budget of 30 bits.
- We can send the initial threshold value T_o separately without effecting the bit budget.
- We use the following codes

Zerotree root	zr	00
Significant positive	sp	11
Significant negative	sn	01
Isolated zero	iz	10

21	6	15	12
-6	3	6	3
3	-3	0	-3
3	0	0	0

$$\begin{array}{rcll} T_0 = 16 & & & \\ \text{Dominant Pass:} & & & \\ 21 & \Rightarrow & sp & [11] \\ 6 & \Rightarrow & zr & [00] \\ -6 & \Rightarrow & zr & [00] \\ 3 & \Rightarrow & zr & [00] \\ L_s = \{21\} & & & \\ \text{Refinement Pass:} & & & \\ 21 - 24 = 3 & \Rightarrow & \text{Correction term} = -4 & [0] \end{array}$$

Second Pass:

$$\begin{array}{rcll} T_0 = 8 & & & \\ \text{Dominant Pass:} & & & \\ 6 & \Rightarrow & iz & [10] \\ -6 & \Rightarrow & zr & [00] \\ 3 & \Rightarrow & zr & [00] \\ 15 & \Rightarrow & sp & [11] \\ 12 & \Rightarrow & sp & [11] \\ 6 & \Rightarrow & iz & [10] \\ 3 & \Rightarrow & iz & [10] \\ L_s = \{21, 15, 12\} & & & \end{array}$$

$$\begin{array}{llll} \text{Refinement Pass:} & & & \\ & 21 - 20 = 1 & \Rightarrow & \text{Correction term} = 2 \quad [1] \\ & 15 - 12 = 3 & \Rightarrow & \text{Correction term} = 2 \quad [1] \\ & 12 - 12 = 0 & \Rightarrow & \text{Correction term} = -2 \quad [0] \end{array}$$

Third Pass:

Pass:

$$\begin{array}{rlll} T_0 = 4 & & & \\ \text{Dominant Pass:} & & & \\ 6 & \Rightarrow & sp & [11] \\ -6 & \Rightarrow & sn & [01] \end{array}$$

And we have used up thirty bits.

The transmitted bitstream is

$$110000000100000111110101101101$$

(b)

The received bitstream is

$$110000000100000111110101101101$$

and we begin with knowledge of the initial threshold and size of the image. For sake reference lets represent the "image" as

a	b	e	f
c	d	g	h
i	j	m	n
k	l	o	p

$T_o = 16$ and the first codeword is 11 or sp. Therefore

$$a = 24$$

The next three codewords are 00 corresponding to a zerotree root. Therefore,

$$b = c = d = e = f = g = h = i = j = k = l = m = n = o = p = 0$$

There are no tree roots that are significant so this is the end of the first dominant pass. At the end of this pass $L_s = \{a\}$. In the refinement pass we get a 0 corresponding to a negative correction for a. Therefore,

$$a = 24 - 4 = 20$$

The next bits are from the second dominant pass.

$T_o = 8$ and 10 means an isolated zero. Thus $b < T_o$ but some of the descendants of b are greater than T_o.

The next codeword is 00 therefore, c is a zerotree root.

The next codeword is 00 therefore, d is a zerotree root.

As b was an isolated zero the next bits correspond to its descendants.

11 means e is significant, therefore, $e = 12$

11 means f is significant, therefore, $f = 12$

10 means g is insignificant, therefore, $g = 0$

10 means h is insignificant, therefore, $h = 0$

The significance list now becomes $L_s = \{a, e, f\}$

1 means a positive correction, therefore, $a = 20 + 2 = 22$.

1 means a positive correction, therefore, $e = 12 + 2 = 14$.

0 means a negative correction, therefore, $f = 12 - 2 = 10$.

Now we start the third dominant pass with $T_o = 4$.

11 means b is significant, therefore, $b = 6$

01 means c is significant negative, therefore, $c = -6$.

and we have reached the end of the received bitstream.

The reconstructed image is as follows

22	6	14	10
-6	0	0	0
0	0	0	0
0	0	0	0

Problem 4. We assume the following:

- We have a bit budget of 30 bits.
- We can send the initial value of n separately without effecting the bit budget.

21	6	15	12
-6	3	6	3
3	-3	0	-3
3	0	0	0

$$\begin{array}{rl} \text{First Pass} & n = 4 \\ LIP: & \{(0,0),(0,1),(1,0),(1,1)\} \\ LIS: & \{(0,1)\mathcal{D},(1,0)\mathcal{D},(1,1)\mathcal{D}\} \\ LSP: & \{\} \end{array}$$

First examine the contents of LIP.

$$\begin{array}{rcl} (0,0) \rightarrow 21 & |21| \geq 16 & \text{Transmit 11 and move to } LSP \\ (0,1) \rightarrow \ \ 6 & |6| < 16 & \text{Transmit 0} \\ (1,0) \rightarrow -6 & |-6| < 16 & \text{Transmit 0} \\ (1,1) \rightarrow \ \ 3 & |3| < 16 & \text{Transmit 0} \end{array}$$

Now examine the contents of LIS

$$\begin{array}{ll} (0,1)\mathcal{D}\text{is not significant} & \text{Transmit 0} \\ (1,0)\mathcal{D}\text{is not significant} & \text{Transmit 0} \\ (1,1)\mathcal{D}\text{is not significant} & \text{Transmit 0} \end{array}$$

LSP is empty.

Transmitted bits from the first pass:

11000000

$$\begin{array}{rl} \text{Second Pass} & n = 3 \\ LIP: & \{(0,1),(1,0),(1,1)\} \\ LIS: & \{(0,1)\mathcal{D},(1,0)\mathcal{D},(1,1)\mathcal{D}\} \\ LSP: & \{(0,0)\} \end{array}$$

First examine the contents of LIP.

$$\begin{array}{rcl} (0,0) \rightarrow 21 & |21| \geq 16 & \text{Transmit 11 and move to } LSP \\ (0,1) \rightarrow \ \ 6 & |6| < 8 & \text{Transmit 0} \\ (1,0) \rightarrow -6 & |-6| < 8 & \text{Transmit 0} \\ (1,1) \rightarrow \ \ 3 & |3| < 8 & \text{Transmit 0} \end{array}$$

Now examine the contents of LIS

$$\begin{array}{rcl} (0,1)\mathcal{D}\text{is significant} & & \text{Transmit 1} \\ (0,2) \rightarrow 15 & |15| \geq 8 & \text{Transmit 11 and move to } LSP \\ (0,3) \rightarrow 12 & |12| \geq 8 & \text{Transmit 11 and move to } LSP \\ (1,2) \rightarrow 6 & |6| < 8 & \text{Transmit 0 and move to } LIP \\ (1,3) \rightarrow 3 & |3| < 8 & \text{Transmit 0 and move to } LIP \\ (1,0)\mathcal{D}\text{is not significant} & & \text{Transmit 0} \\ (1,1)\mathcal{D}\text{is not significant} & & \text{Transmit 0} \end{array}$$

Finally, for the second pass, examine the contents of LSP

$$(0,0) \rightarrow 21 \quad n = 3 \quad \text{Transmit 0}$$

Transmitted bits in the second pass.

0001111100000

$$\begin{array}{rl} \text{Third Pass} & n = 2 \\ LIP: & \{(0,1),(1,0),(1,1),(1,2),(1,3)\} \\ LIS: & \{(1,0)\mathcal{D},(1,1)\mathcal{D}\} \\ LSP: & \{(0,0),(0,2),(0,3)\} \end{array}$$

First examine the contents of LIP.

$$\begin{array}{rcl} (0,1) \rightarrow \ \ 6 & |6| \geq 4 & \text{Transmit 11 and move to } LSP \\ (1,0) \rightarrow -6 & |-6| \geq 4 & \text{Transmit 10 and move to} LSP \\ (1,1) \rightarrow \ \ 3 & |3| < 4 & \text{Transmit 0} \\ (1,2) \rightarrow \ \ 6 & |6| \geq 4 & \text{Transmit 11 and move to } LSP \\ (1,3) \rightarrow \ \ 3 & |3| < 4 & \text{Transmit 0} \end{array}$$

Now examine the contents of LIS

$$(1,0)\mathcal{D}\text{is not significant} \qquad \text{Transmit 0}$$

And we have used up the thirty bits.

The transmitted bits are

$$11000000000111110000011100 1100$$

(b)

The received bits are

$$110000000001111100000111001100$$

Begin with $n = 4$.

$$\begin{aligned} LIP: &\quad \{(0,0),(0,1),(1,0),(1,1)\} \\ LIS: &\quad \{(0,1)\mathcal{D},(1,0)\mathcal{D},(1,1)\mathcal{D}\} \\ LSP: &\quad \{\} \end{aligned}$$

First LIP

$$\begin{aligned} 11 &\;\Rightarrow\; (0,0) \rightarrow 16,\ \text{move to } LSP \\ 0 &\;\Rightarrow\; (0,1) \rightarrow 0 \\ 0 &\;\Rightarrow\; (1,0) \rightarrow 0 \\ 0 &\;\Rightarrow\; (1,1) \rightarrow 0 \end{aligned}$$

Then to LIS

$$\begin{aligned} 0 &\;\Rightarrow\; (0,1)\mathcal{D} \text{ is insignificant} \\ 0 &\;\Rightarrow\; (1,0)\mathcal{D} \text{ is insignificant} \\ 0 &\;\Rightarrow\; (1,1)\mathcal{D} \text{ is insignificant} \end{aligned}$$

LSP is empty.

Second pass, $n = 3$.

$$\begin{aligned} LIP: &\quad \{(0,1),(1,0),(1,1)\} \\ LIS: &\quad \{(0,1)\mathcal{D},(1,0)\mathcal{D},(1,1)\mathcal{D}\} \\ LSP: &\quad \{(0,0)\} \end{aligned}$$

Begin with LIP

$$\begin{aligned} 0 &\;\Rightarrow\; (0,1) \rightarrow 0 \\ 0 &\;\Rightarrow\; (1,0) \rightarrow 0 \\ 0 &\;\Rightarrow\; (1,1) \rightarrow 0 \end{aligned}$$

Then to LIS

$$\begin{aligned}
1 &\Rightarrow (0,1)\mathcal{D} \text{ is significant} \\
11 &\Rightarrow (0,2) \rightarrow 8 \text{ move to } LSP \\
11 &\Rightarrow (0,3) \rightarrow 8 \text{ move to } LSP \\
0 &\Rightarrow (0,2) \rightarrow 0 \text{ move to } LIP \\
0 &\Rightarrow (0,2) \rightarrow 0 \text{ move to } LIP \\
0 &\Rightarrow (1,0)\mathcal{D} \text{ is insignificant} \\
0 &\Rightarrow (1,1)\mathcal{D} \text{ is insignificant}
\end{aligned} \tag{11}$$

On to LSP.

$$0 \Rightarrow (0,0) \rightarrow 16$$

Third pass, $n = 2$.

$$\begin{aligned}
LIP: &\quad \{(0,1),(1,0),(1,1),(1,2),(1,3)\} \\
LIS: &\quad \{(1,0)\mathcal{D},(1,1)\mathcal{D}\} \\
LSP: &\quad \{(0,0),(0,2),(0,3)\}
\end{aligned}$$

Begin with LIP

$$\begin{aligned}
11 &\Rightarrow (0,1) \rightarrow 4 \text{ move to } LSP \\
10 &\Rightarrow (1,0) \rightarrow -4 \text{ move to } LSP \\
0 &\Rightarrow (0,1) \rightarrow 0 \\
11 &\Rightarrow (1,2) \rightarrow 4 \text{ move to } LSP \\
0 &\Rightarrow (1,3) \rightarrow 0
\end{aligned}$$

Now to LIS

$$0 \Rightarrow (1,0)\mathcal{D} \text{ is not significant}$$

We have reached the end of transmission. The reconstructed “image” is

16	4	8	8
-4	0	4	0
0	0	0	0
0	0	0	0

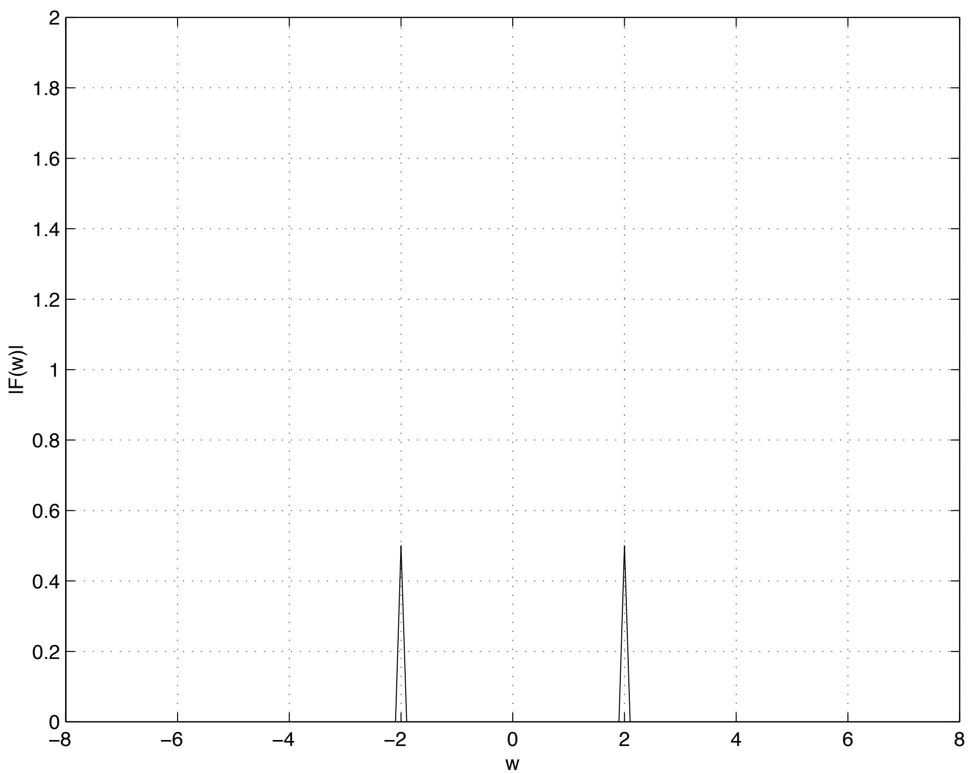

Figure 10: Magnitude of the Fourier transform of the function $\sin(2t)$.

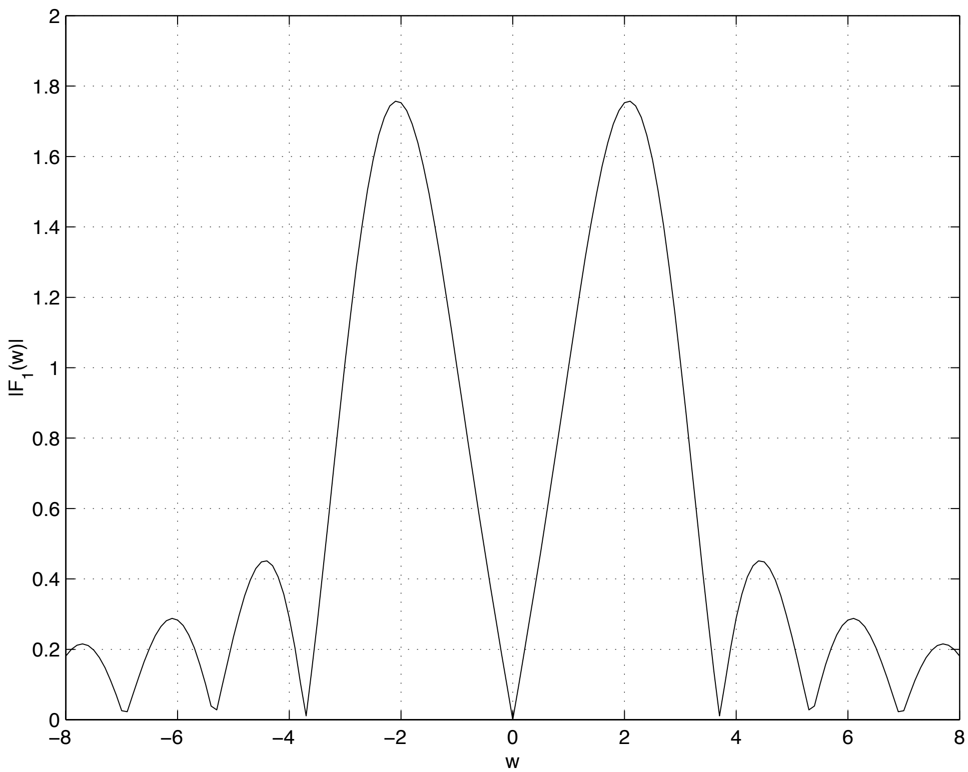

Figure 11: Magnitude of the short term Fourier transform of the function $\sin(2t)$ with a rectangular window.

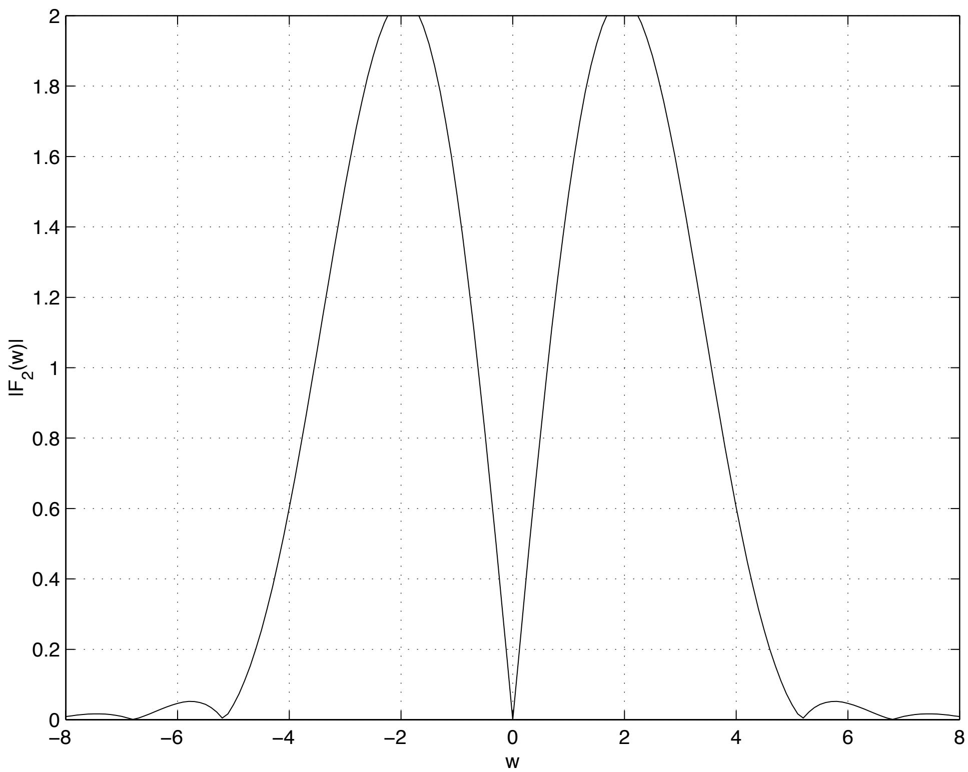

Figure 12: Magnitude of the short term Fourier transform of the function $\sin(2t)$ with a raised cosine window.