Workbook 3 Hand-in

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Contents

1 Ex 11: The stability of spin motion for an irregular asteroid **1**

1.1 Spin kinetic energy 5

2 Ex 12: The Spin Ellipsoid **5**

2.1 Period of motion 9

2.2 Euler's equations period 12

2.2.1 Axis X 13

2.2.2 Axis Z 13

3 Ex 13: Kuiper belt object interception **14**

3.1 Long duration burn simulation 14

3.2 Burn aftermath 17

3.3 Fuel calculation 21

4 Ex 14: Fast track to the Moon **24**

4.1 Comparing to Hohmann transfer 29

1 Ex 11: The stability of spin motion for an irregular asteroid

To study the stability of spin motion for an irregular shape we need a mass model to represent it. In the case of this asteroid we are going to use the following:

z = 1		z = 0		z = -1	
x	y	x	y	x	y
-1	4	-3	3	-2	2
0	3	-3	1	-1	5
0	-5	-2	4	-1	3
1	4	-2	2	-1	1
1	-2	-2	0	-1	-5
		-2	-4	0	6
		-1	5	0	4
		-1	3	0	2
		-1	1	0	0
		-1	-1	0	2
		-1	-3	0	4
		-1	-5	0	6

z = 1	z = 0		z = -1	
	-1	-7	1	5
	0	6	1	3
	0	4	1	1
	0	2	1	-3
	0	0	1	-5
	0	-2	2	4
	0	-4		
	0	-6		
	1	5		
	1	3		
	1	1		
	1	-1		
	1	-3		
	1	-5		
	2	4		
	2	2		
	2	0		
	2	-2		
	3	1		

[1]:
```
import numpy as np

SCALE = 300
masslumps=np.array([[-1,4,1],
[0,3,1],
[0,-5,1],
[1,4,1],
[1,-2,1],
[-3,3,0],
[-3,1,0],
[-2,4,0],
[-2,2,0],
[-2,0,0],
[-2,-4,0],
[-1,5,0],
[-1,3,0],
[-1,1,0],
[-1,-1,0],
[-1,-3,0],
[-1,-5,0],
[-1,-7,0],
[0,6,0],
[0,4,0],
[0,2,0],
[0,0,0],
```

```
[0,-2,0],
[0,-4,0],
[0,-6,0],
[1,5,0],
[1,3,0],
[1,1,0],
[1,-1,0],
[1,-3,0],
[1,-5,0],
[2,4,0],
[2,2,0],
[2,0,0],
[2,-2,0],
[3,1,0],
[-2,2,-1],
[-1,5,-1],
[-1,3,-1],
[-1,1,-1],
[-1,-5,-1],
[0,6,-1],
[0,4,-1],
[0,2,-1],
[0,0,-1],
[0,2,-1],
[0,4,-1],
[0,6,-1],
[1,5,-1],
[1,3,-1],
[1,1,-1],
[1,-3,-1],
[1,-5,-1],
[2,4,-1]])

unit_com = np.mean(masslumps, axis=0)
center_of_mass = unit_com*SCALE
points_remap = masslumps*SCALE - center_of_mass
list(unit_com) # Center of mass coordinates in the form [x, y, z] in the unit␣
 ↪cell
```

[1]: [-0.037037037037037035, 0.7962962962962963, -0.24074074074074073]

[2]:
```
list(center_of_mass) # Center of mass coordinates in the form [x, y, z] in␣
 ↪physical space
```

[2]: [-11.11111111111111, 238.88888888888889, -72.22222222222221]

To be sure that the new coordinate system is centered around the center of mass, we can calculate its center of mass and expect a point centered at the origin. The result will not be perfect due to computer float number precision.

```
[3]: list(np.mean(points_remap, axis=0)) # Should be [0, 0, 0]
```

```
[3]: [1.0947621410229691e-13, 6.736997790910579e-14, -3.0527021240063567e-14]
```

Now, knowing that the total mass of the asteroid is $5.10^{13} kg$, it is possible to deduce the mass of each lumps.

```
[4]: total_mass = 5e13 # Total mass of the system (in kg)
     mass_per_point = total_mass / len(masslumps)
     mass_per_point # Mass of each point (in kg)
```

```
[4]: 925925925925.9259
```

Now, it is possible to calculate the inertia matrix as well as the three principal moments of inertia with the formula:

$$I = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{pmatrix} = \begin{pmatrix} \sum m(y^2+z^2) & -\sum mxy & -\sum mxz \\ -\sum mxy & \sum m(x^2+z^2) & -\sum myz \\ -\sum mxz & -\sum myz & \sum m(x^2+y^2) \end{pmatrix}$$

For principal moments of inertia, it is the eigen values of this matrix.

```
[5]: def inertia_moment(mass, x, y):
         return mass * (x**2 + y**2)
     def inertia_product(mass, x, y):
         return mass * x * y

     inertia = np.array([[0, 0, 0],
                         [0, 0, 0],
                         [0, 0, 0]], dtype=float)

     m = mass_per_point
     for [x, y, z] in points_remap:
         inertia += np.array([[inertia_moment(m, y, z), -inertia_product(m, x, y),␣
      ↪-inertia_product(m, x, z)],
                              [-inertia_product(m, x, y), inertia_moment(m, x, z),␣
      ↪-inertia_product(m, y, z)],
                              [-inertia_product(m, x, z), -inertia_product(m, y, z),␣
      ↪inertia_moment(m, x, y)]])

     inertia
```

```
[5]: array([[ 5.73858025e+19, -2.16049383e+17,  4.01234568e+16],
            [-2.16049383e+17,  9.31635802e+18,  1.72067901e+18],
            [ 4.01234568e+16,  1.72067901e+18,  6.33904321e+19]])
```

```
[6]: w,_=np.linalg.eig(inertia)
print('E-value:', w) # Eigenvalues of the inertia tensor (in kg m^2) in the
 ↪form [Ixx, Iyy, Izz]
```

```
E-value: [9.26067960e+18 5.73866005e+19 6.34453125e+19]
```

1.1 Spin kinetic energy

Knowing that the spin kinetic energy is given by:

$$E = \sum_{k=1}^{3} \frac{1}{2} I_k \Omega_k^2$$

By differentiating the expression with respect to time:

$$\frac{dE}{dt} = \sum_{k=1}^{3} I_k \Omega_k \frac{d\Omega_k}{dt}$$

Or, without external torque $T = 0 = I_k \frac{d\Omega_k}{dt}$. This means that $\frac{dE}{dt} = 0$. Therefore, E is constant if any external torque is apply on the system.

2 Ex 12: The Spin Ellipsoid

Knowing that the spin kinetic energy is:

$$E = \sum_{k=1}^{3} \frac{1}{2} I_k \Omega_k^2$$

It can be written as:

$$1 = \sum_{k=1}^{3} \frac{1}{2E} \frac{1}{I_k^{-1}} \Omega_k^2$$

$$1 = \sum_{k=1}^{3} \frac{\Omega_k^2}{2\frac{E}{I_k}}$$

This, is an ellipsoid equation as $(\frac{x}{a})^2 + (\frac{y}{b})^2 + (\frac{z}{c})^2 = 1$. If $a = b = c = 1$ the ellipsoid become a sphere of radius 1. In our case it would be: $2\frac{E}{I_x} = 2\frac{E}{I_y} = 2\frac{E}{I_z} = 1$

```
[7]: import matplotlib.pyplot as plt
import numpy as np
from scipy.integrate import odeint

SCALE = 300
masslumps=np.array([[-1,4,1],
[0,3,1],
[0,-5,1],
[1,4,1],
[1,-2,1],
[-3,3,0],
```

```
[-3,1,0],
[-2,4,0],
[-2,2,0],
[-2,0,0],
[-2,-4,0],
[-1,5,0],
[-1,3,0],
[-1,1,0],
[-1,-1,0],
[-1,-3,0],
[-1,-5,0],
[-1,-7,0],
[0,6,0],
[0,4,0],
[0,2,0],
[0,0,0],
[0,-2,0],
[0,-4,0],
[0,-6,0],
[1,5,0],
[1,3,0],
[1,1,0],
[1,-1,0],
[1,-3,0],
[1,-5,0],
[2,4,0],
[2,2,0],
[2,0,0],
[2,-2,0],
[3,1,0],
[-2,2,-1],
[-1,5,-1],
[-1,3,-1],
[-1,1,-1],
[-1,-5,-1],
[0,6,-1],
[0,4,-1],
[0,2,-1],
[0,0,-1],
[0,2,-1],
[0,4,-1],
[0,6,-1],
[1,5,-1],
[1,3,-1],
[1,1,-1],
[1,-3,-1],
[1,-5,-1],
```

```
[2,4,-1]])

center_of_mass = np.mean(masslumps, axis=0)*SCALE
points_remap = masslumps*SCALE - center_of_mass
total_mass = 5e13 # Total mass of the system (in kg)
mass_per_point = total_mass / len(masslumps)

def inertia_moment(mass, x, y):
    return mass * (x**2 + y**2)
def inertia_product(mass, x, y):
    return mass * x * y

inertia = np.array([[0, 0, 0],
                    [0, 0, 0],
                    [0, 0, 0]], dtype=float)

m = mass_per_point
for [x, y, z] in points_remap:
    inertia += np.array([[inertia_moment(m, y, z), -inertia_product(m, x, y),␣
↪-inertia_product(m, x, z)],
                         [-inertia_product(m, x, y), inertia_moment(m, x, z),␣
↪-inertia_product(m, y, z)],
                         [-inertia_product(m, x, z), -inertia_product(m, y, z),␣
↪inertia_moment(m, x, y)]])

w,_=np.linalg.eig(inertia)
w
```

[7]: array([9.26067960e+18, 5.73866005e+19, 6.34453125e+19])

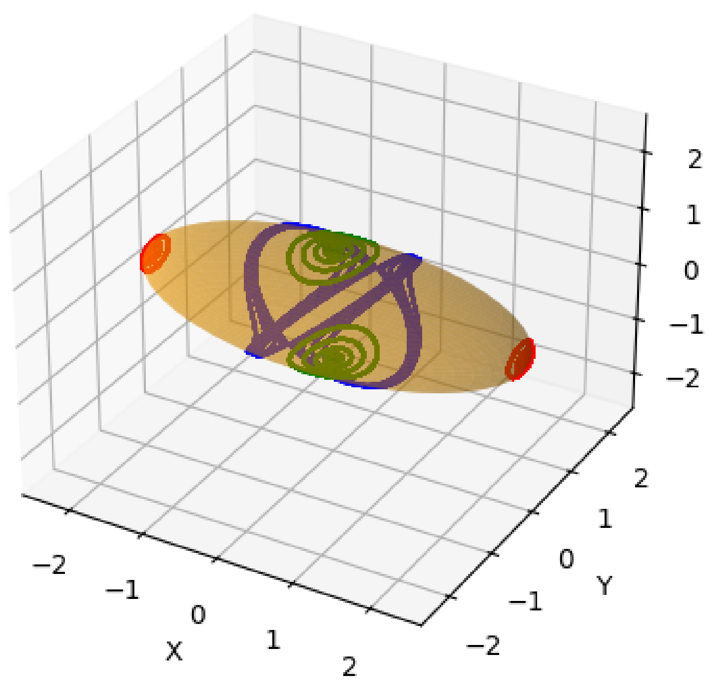
To draw the spin energy ellipsoid representing the possible values of rotation velocities for a specific energy we need to fix an energy. Then, to simulate trajectories that also lives on that shape, it will be require to evalute that ellipsis to have the starting coordinates for the solver.

[8]:
```
target_energy = 3e19
principal_intertia = np.array(w)

# Define the system of differential equations
def eulereqns(t, angvel):
    return (-np.subtract(*principal_intertia[2:0:-1]) * np.multiply(*angvel[1:
↪]) / principal_intertia[0],
            -np.subtract(*principal_intertia[::2]) * np.multiply(*angvel[::-2])␣
↪/ principal_intertia[1],
            -np.subtract(*principal_intertia[1::-1]) * np.multiply(*angvel[:2])␣
↪/ principal_intertia[2])

# Define the coefficients of the ellipsoid
```

```
rx, ry, rz =np.sqrt(2*target_energy/principal_intertia)
# Define the 3 central angles of the ellipsoid (one for each axis) [x, y, z]
uv = [(0, np.pi/2, 'red'), (np.pi/2, np.pi/2, 'blue'), (0, 0, 'green'),
      (np.pi, np.pi/2, 'red'), (3*np.pi/2, np.pi/2, 'blue'), (0, np.pi,␣
 ↪'green')]
tfinal = 100
number_of_trajectories = 6

angvels = []
for u, v, color in uv:
    # Then we add a small variation to the central angles to see multiple␣
 ↪trajectories around that central point
    for var in np.linspace(0, np.pi/24, number_of_trajectories):
        v += var/2
        u += var/2
        x = rx * np.cos(u) * np.sin(v)
        y = ry * np.sin(u) * np.sin(v)
        z = rz * np.cos(v)
        angvel0 = np.array([x, y, z])
        inenergy = 0.5 * np.sum(principal_intertia * angvel0 ** 2)
        t = np.linspace(0, tfinal, 10000)
        angvel = odeint(eulereqns, angvel0, t, tfirst=True)
        tend = len(angvel) - 1
        finenergy = 0.5 * np.sum(principal_intertia * angvel[tend] ** 2)
        accuracy = abs((finenergy - inenergy) / inenergy)
        angvels.append((accuracy, t, angvel, color))

print(f'Max Accuracy (worst) = {max([acc for acc, _, _, _ in angvels]): .4%}')

# Set of all spherical angles:
u = np.linspace(0, 2 * np.pi, 100)
v = np.linspace(0, np.pi, 100)

# Cartesian coordinates that correspond to the spherical angles:
# (this is the equation of an ellipsoid):
x = rx * np.outer(np.cos(u), np.sin(v))
y = ry * np.outer(np.sin(u), np.sin(v))
z = rz * np.outer(np.ones_like(u), np.cos(v))

# Plot:
fig = plt.figure(figsize=plt.figaspect(1))  # Square figure
ax = fig.add_subplot(111, projection='3d')
ax.set_xlabel('X')
ax.set_ylabel('Y')
ax.set_zlabel('Z')
ax.plot_surface(x, y, z, color='orange', alpha=0.4)
for _, _, angvel, color in angvels:
```

```
    ax.plot(angvel[:, 0], angvel[:, 1], angvel[:, 2], color=color) # Draw the␣
↪trajectories

# Adjustment of the axes, so that they all have the same span:
max_radius = max(rx, ry, rz)
for axis in 'xyz':
    getattr(ax, 'set_{}lim'.format(axis))((-max_radius, max_radius))

plt.show()
```

```
Max Accuracy (worst) =  0.0005%
```



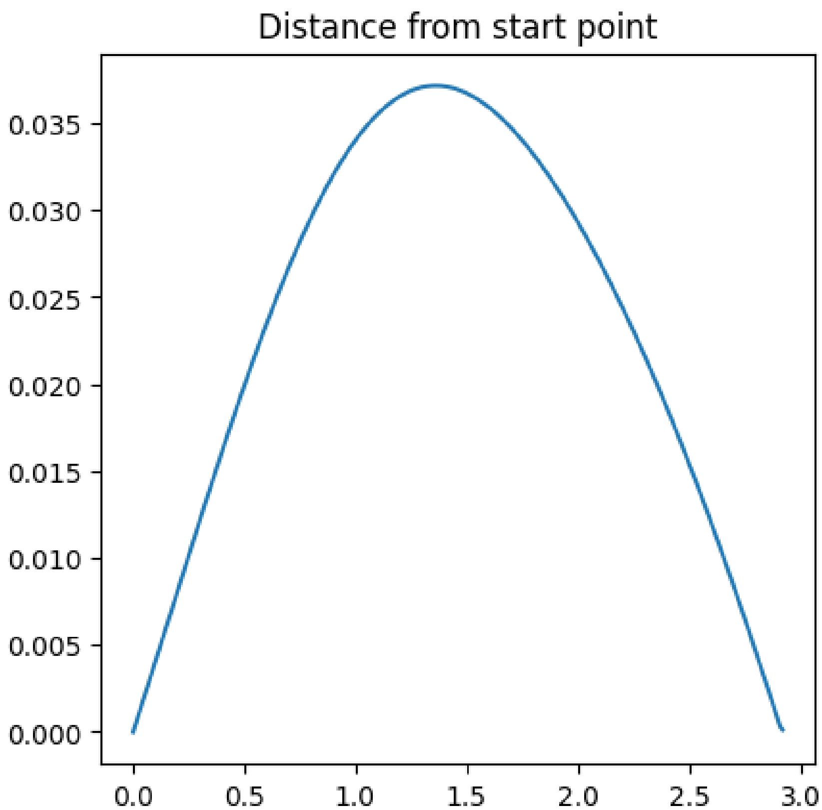
From the previous digram, it is clear that the most stable axis is the X axis and the less stable is the Y axis. Even really small rotation velocity around this axis will end up on high amplitude motion.

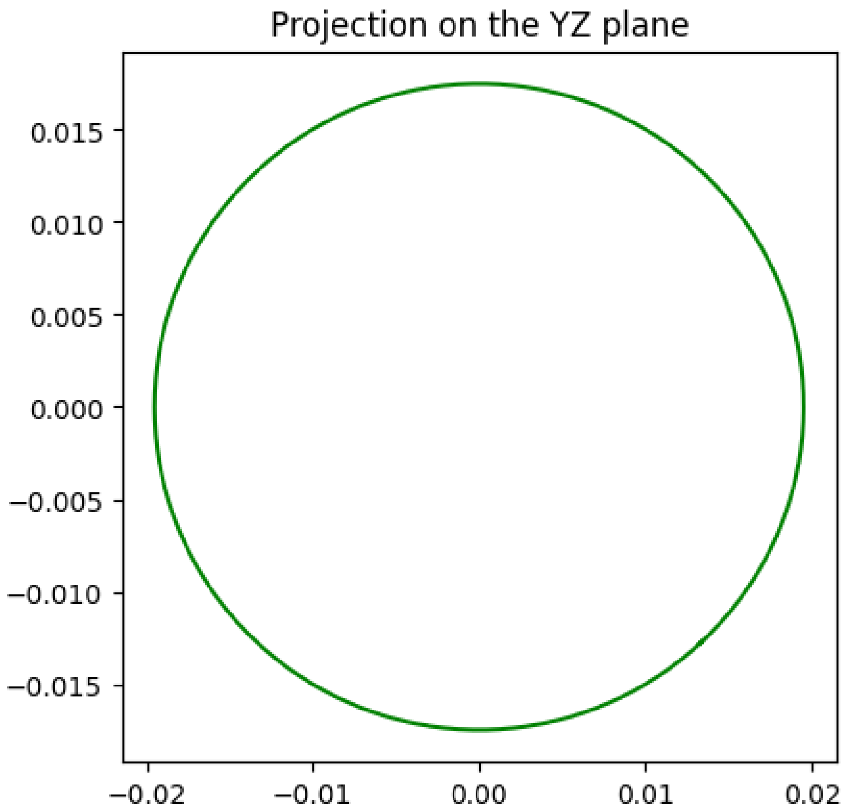
2.1 Period of motion

[9]:
```
from scipy.signal import argrelextrema
from numpy import less
_, t, av, _ = angvels[1]
distance_from_start = np.sqrt((av[:, 0]-av[0, 0])**2 + (av[:, 1]-av[0, 1])**2 +␣
↪(av[:, 2]-av[0, 2])**2)
idx = argrelextrema(distance_from_start, less)
```

```
first_loop = idx[0][0]
print(f"Period of motion: {t[first_loop]}") # Period of motion in seconds

plt.figure(figsize=plt.figaspect(1))  # Square figure
plt.title('Distance from start point')
plt.plot(t[:first_loop+1], distance_from_start[:first_loop+1])
plt.show()
plt.figure(figsize=plt.figaspect(1))  # Square figure
plt.title('Projection on the YZ plane')
plt.plot(av[:first_loop+1, 1], av[:first_loop+1, 2], color=color) # Draw the
 ↪trajectories
plt.show()
```

```
Period of motion: 2.9202920292029204
```





[10]:
```
# Farther from the center point
_, t, av, _ = angvels[5]
distance_from_start = np.sqrt((av[:, 0]-av[0, 0])**2 + (av[:, 1]-av[0, 1])**2 +
 ↪(av[:, 2]-av[0, 2])**2)
idx = argrelextrema(distance_from_start, less)
first_loop = idx[0][0]
print(f"Period of motion: {t[first_loop]}") # Period of motion in seconds
```

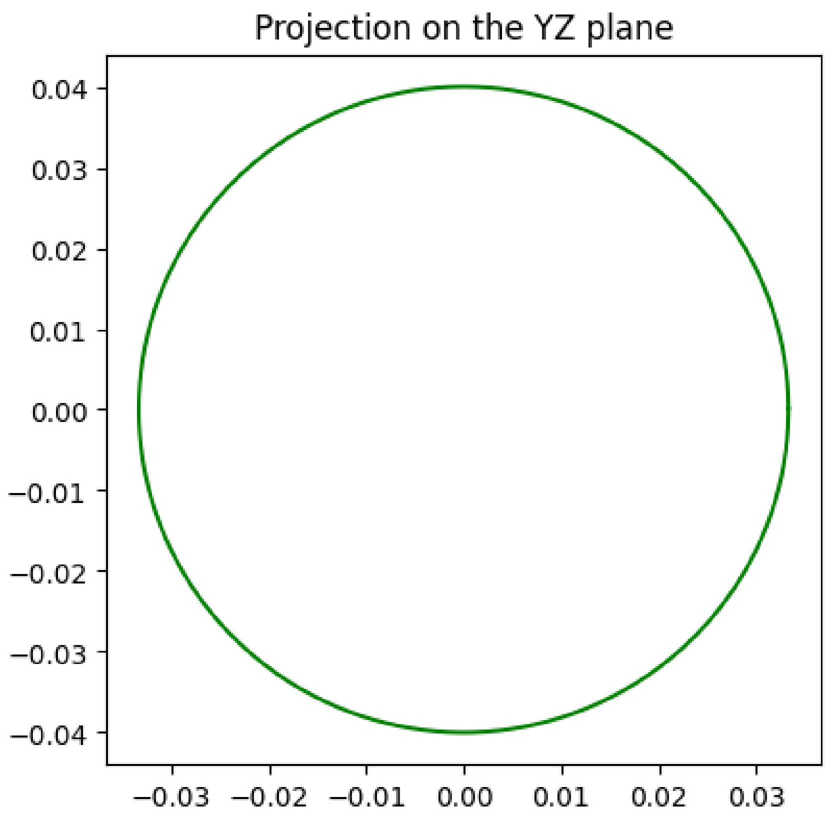
```
Period of motion: 3.0303030303030303
```

Here is the period of motion when relatively close to the center point of motion along the X axis. As we repeat this calculation farther from the axis, the period increase. This can be repeat for the other stable axis Z.

[11]:
```
_, t, av, _ = angvels[number_of_trajectories*2+1]
distance_from_start = np.sqrt((av[:, 0]-av[0, 0])**2 + (av[:, 1]-av[0, 1])**2 +
 ↪(av[:, 2]-av[0, 2])**2)
idx = argrelextrema(distance_from_start, less)
first_loop = idx[0][0]
print(f"Period of motion: {t[first_loop]}") # Period of motion in seconds
```

```
plt.figure(figsize=plt.figaspect(1))  # Square figure
plt.title('Projection on the YZ plane')
plt.plot(av[:first_loop+1, 0], av[:first_loop+1, 1], color=color) # Draw the␣
 ↪trajectories
plt.show()
```

```
Period of motion: 8.22082208220822
```



The period of motion for the Y axis is much greater than the X axis. This is due to the shape being more flat on that axis meaning that the distance is greater to travel for the oscillator.

2.2 Euler's equations period

Assuming the following Euler's equation:

$$\frac{d\Omega_x}{dt} + \frac{I_z - I_y}{I_x}\Omega_y\Omega_z = 0$$
$$\frac{d\Omega_y}{dt} + \frac{I_x - I_z}{I_y}\Omega_x\Omega_z = 0$$
$$\frac{d\Omega_z}{dt} + \frac{I_y - I_x}{I_z}\Omega_y\Omega_x = 0$$

It is possible to derive the period of motion from these.

2.2.1 Axis X

For a motion closer as possible from this axis. It can be assumed that $\frac{\Omega_x}{dt} = 0$. From that we differentiate the equations for Y and Z axis.

$$\begin{cases} \frac{d^2\Omega_y}{dt^2} + \frac{I_x - I_z}{I_y}\Omega_x\frac{d\Omega_z}{dt} = 0 \\ \frac{d^2\Omega_z}{dt^2} + \frac{I_y - I_x}{I_z}\Omega_x\frac{d\Omega_y}{dt} = 0 \end{cases}$$

$$\begin{cases} \frac{d^2\Omega_y}{dt^2} - \frac{I_x - I_z}{I_y}\Omega_x^2\frac{I_y - I_x}{I_z}\Omega_y = 0 \\ \frac{d^2\Omega_z}{dt^2} - \frac{I_y - I_x}{I_z}\Omega_x^2\frac{I_x - I_z}{I_y}\Omega_z = 0 \end{cases}$$

Now, this is clear that Ω_y and Ω_z follow an harmonic oscillator equation of type $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$. Then, the period can be define as $T = \frac{2\pi}{\sqrt{\frac{k}{m}}}$.

In this case, $\frac{k}{m} = -\frac{(I_y - I_x)(I_x - I_z)}{I_yI_z}\Omega_x^2$. Hence,

$$T = \frac{2\pi}{\sqrt{-\frac{(I_y - I_x)(I_x - I_z)}{I_yI_z}\Omega_x^2}}$$

[12]:
```
x0 = rx
km = -x0**2/(w[1]*w[2])*(w[0]-w[2])*(w[1]-w[0])
period = 2*np.pi/np.sqrt(km)
print(period) # Period of motion in seconds
```

```
2.916776136151734
```

As expected the period is really close to the period previously calculate with the simulated motion.

2.2.2 Axis Z

Now, we are doing the same calculation with the Z axis. In this case, it can be assumed that $\frac{\Omega_z}{dt} = 0$. From that we differentiate the equations for X and Y axis.

$$\begin{cases} \frac{d^2\Omega_x}{dt^2} + \frac{I_z - I_y}{I_x}\Omega_z\frac{d\Omega_y}{dt} = 0 \\ \frac{d^2\Omega_y}{dt^2} + \frac{I_x - I_z}{I_y}\Omega_z\frac{d\Omega_x}{dt} = 0 \end{cases}$$

$$\begin{cases} \frac{d^2\Omega_x}{dt^2} - \frac{I_z - I_y}{I_x}\Omega_z^2\frac{I_x - I_z}{I_y}\Omega_x = 0 \\ \frac{d^2\Omega_y}{dt^2} + \frac{I_x - I_z}{I_y}\Omega_z^2\frac{I_z - I_y}{I_x}\Omega_y = 0 \end{cases}$$

Now, this is clear that Ω_y and Ω_z follow an harmonic oscillator equation of type $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$. Then, the period can be define as $T = \frac{2\pi}{\sqrt{\frac{k}{m}}}$.

In this case, $\frac{k}{m} = -\frac{(I_x - I_z)(I_z - I_y)}{I_y I_x}\Omega_z^2$. Hence,

$$T = \frac{2\pi}{\sqrt{-\frac{(I_x - I_z)(I_z - I_y)}{I_y I_x}\Omega_z^2}}$$

[13]:
```
z0 = rz
km = -z0**2/(w[1]*w[0])*(w[0]-w[2])*(w[2]-w[1])
period = 2*np.pi/np.sqrt(km)
print(period) # Period of motion in seconds
```

```
8.22058023422136
```

Finally, for the Z axis the results are also similare to what we should expect.

3 Ex 13: Kuiper belt object interception

We are focusing on intercepting $1994GV_9$. We are assuming that the asteroid is evolving on the same plane as the earth (the eccliptic) and orbiting the sun at 43.6AU in a circular motion.

[14]:
```
import matplotlib.pyplot as plt

from space_base import GravBody, Probe
import numpy as np

# Define constants
G = 6.67e-11  # Gravitational constant
g0 = 9.80665
sun = GravBody(name="Sun", mass=1_988_500e24, radius=695_700e3)  # Sun as an
 ↪object with mass and radius

# Define conversion function
def UA_to_meters(UA):
    return UA * 1.496e11
def meters_to_UA(meters):
    return meters / 1.496e11
```

3.1 Long duration burn simulation

To be more realistic with our ion rocket we are not going to simulate a impulsive burn but a continuous burn. For that, the differential funcion use to simulate the probe needs some changes.

Z and vz will be use to store the mass of the probe as space_base do not support 4D inputs.

[15]:
```
mass_lost_rate = 10e-6 # kg/s
dry_mass = 240 # dry mass of the probe
def probeqns_rocket(_, posvelmass):
    Isp = 3400 # in seconds
```

```
    if posvelmass[2] <= dry_mass:
        posvelmass[5] = 0.0
    else:
        posvelmass[5] = -mass_lost_rate

    r = np.sqrt(posvelmass[0] ** 2 + posvelmass[1] ** 2)
    f = -G * sun.mass / r ** 3
    gravity_force = f * posvelmass[0:2]
    axy = gravity_force - (g0*Isp*np.abs(posvelmass[5])/
 ↪posvelmass[2])*(-posvelmass[3:5]/np.linalg.norm(posvelmass[3:5]))

    return posvelmass[3], posvelmass[4], posvelmass[5], axy[0], axy[1], 0.0
```

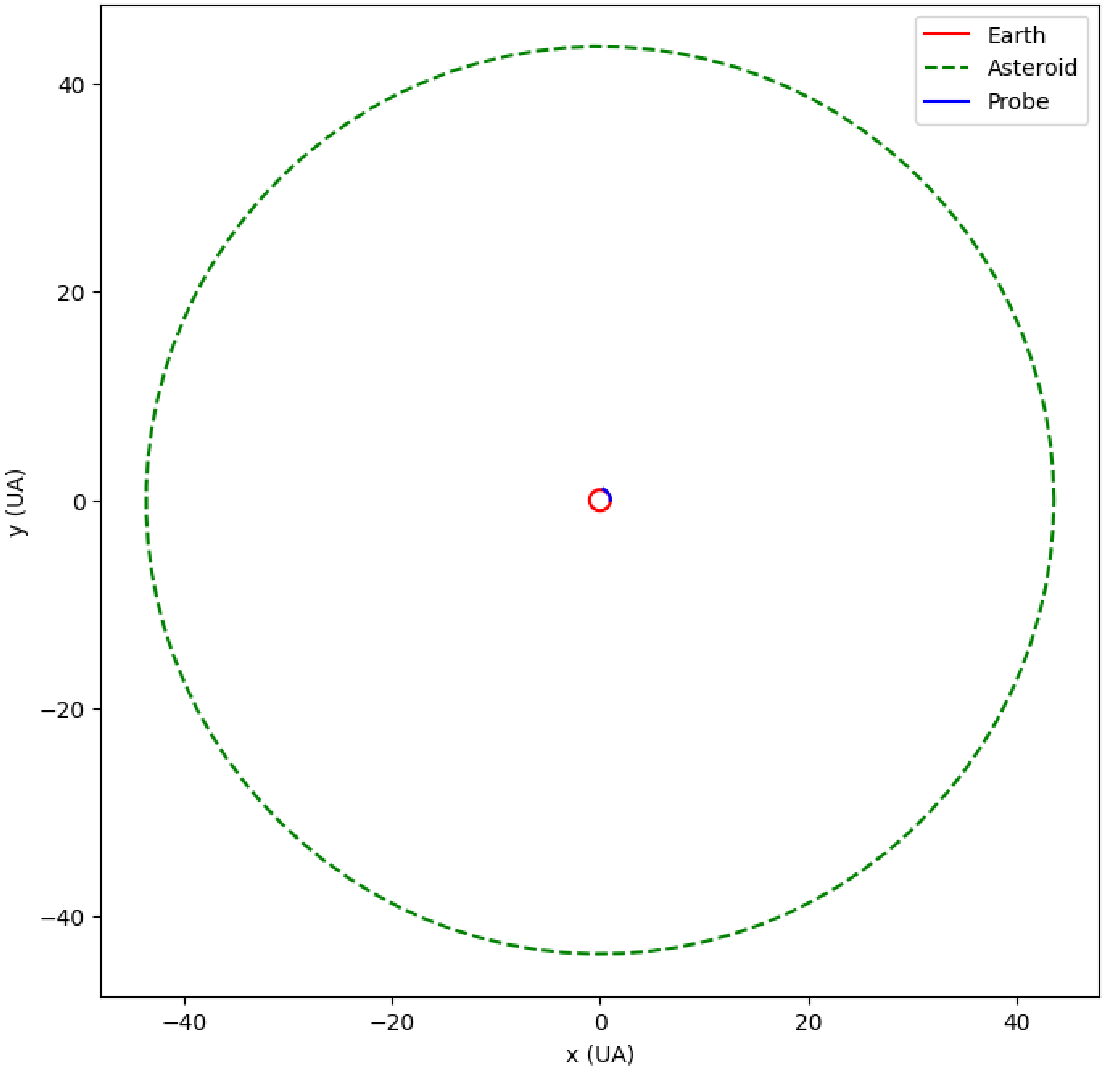
We then initialize our probe at Earth's orbit around sun as the L4 point is on this orbit.

[16]:
```
v0 = np.sqrt(G * sun.mass / UA_to_meters(1))  # initial speed
xymass0 = [UA_to_meters(1), 0, 300]  # start position
vxy0 = [0, v0]  # start vertical speed
tf = (300-dry_mass)/mass_lost_rate  # Max burn time

probe = Probe(probeqns_rocket, tf, tf/3600, x0=xymass0[0], vx0=vxy0[0],
            y0=xymass0[1], vy0=vxy0[1], z0=xymass0[2], vz0=mass_lost_rate) #␣
 ↪probe as an object
t, posvel = probe.odesolve() # solve the differential equations
```

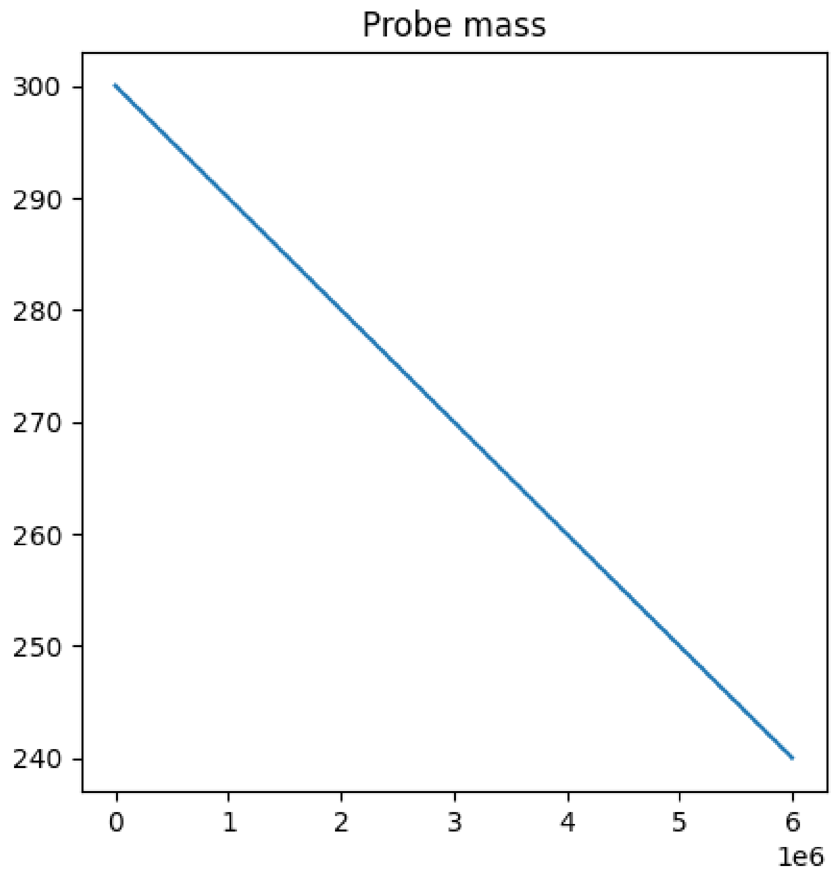
At the end of the burn this is the probe trajectory. However it is difficult to conclude on the final orbit with only the burn data.

[17]:
```
plt.figure(figsize=(8, 8))  # create figure, figsize can be changed as preferred

# Plotting Earth's orbit
uang = np.linspace(0, 2 * np.pi, 100)
x = np.cos(uang)
y = np.sin(uang)
plt.plot(x, y, color='red', label='Earth')
# Plotting Asteroid's orbit
uang = np.linspace(0, 2 * np.pi, 100)
x = 43.6 * np.cos(uang)
y = 43.6 * np.sin(uang)
plt.plot(x, y, color='green', linestyle="--", label='Asteroid')

plt.plot(meters_to_UA(posvel[:, 0]), meters_to_UA(posvel[:, 1]), color='blue',␣
 ↪label="Probe") # plot the probe's orbit

plt.xlabel('x (UA)')
plt.ylabel('y (UA)')
plt.axis('equal')
plt.legend()
```

```
plt.show() # make plot appear
```



[18]:
```
plt.figure(figsize=(5, 5))  # create figure, figsize can be changed as preferred
plt.title("Probe mass")
plt.plot(t, posvel[:, 2])
plt.show()
```



3.2 Burn aftermath

To calculate the orbit characteristic after the burn we can use the energy formula to have the semi-major axis:

$$-\frac{GM_{sun}}{2a} = \frac{1}{2}V^2 - \frac{GM_{sun}}{r}$$
$$a = \frac{GM_{sun}}{V^2 - 2\frac{GM_{sun}}{r}}$$

[19]:
```
last_v = np.linalg.norm(posvel[-1, 3:5])
last_r = np.linalg.norm(posvel[-1, 0:2])
a = np.abs(G * sun.mass / (last_v ** 2 - 2*G*sun.mass/last_r))
meters_to_UA(a) # semi-major axis of the probe's orbit in UA
```

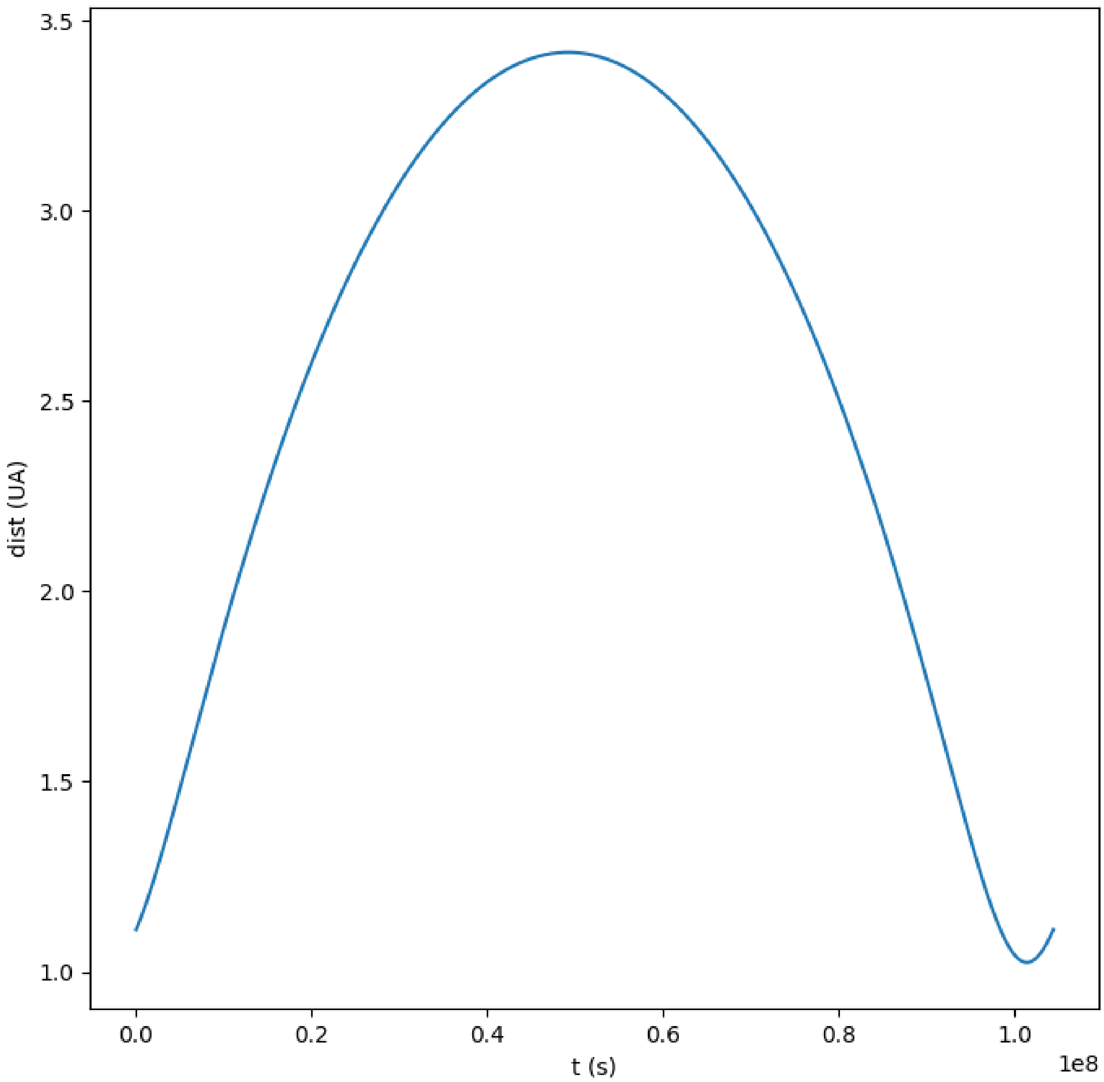
[19]: 2.2199916423314416

We can also simulate the probe with the initial condition being the output of the burn simulation to understand the trajectory that the probe would have.

[20]:
```python
def probeqns(_, posvel):
    r = np.sqrt(posvel[0] ** 2 + posvel[1] ** 2)
    f = -G * sun.mass / r ** 3
    gravity_force = f * posvel[0:2]
    axy = gravity_force

    return posvel[2], posvel[3], axy[0], axy[1]

period = np.sqrt(4 * np.pi**2 * np.abs(a)**3 / (G * sun.mass))  # Orbital period
probe = Probe(probeqns, period, period/3600, x0=posvel[-1, 0], vx0=posvel[-1,␣
↪3],
            y0=posvel[-1, 1], vy0=posvel[-1, 4]) # probe as an object
t_after, posvel_after = probe.odesolve() # solve the differential equations
final_v = np.linalg.norm(posvel_after[-1, 2:4])
final_v / 1e3 # final speed of the probe in km/s
```

[20]: 34.58945341033933

[21]:
```python
dist_to_sun = np.linalg.norm(posvel_after[:, 0:2], axis=1)
plt.figure(figsize=(8, 8))  # create figure, figsize can be changed as preferred
plt.plot(t_after, meters_to_UA(dist_to_sun))
plt.xlabel('t (s)')
plt.ylabel('dist (UA)')
plt.show() # make plot appear
```



[22]:
```
r_perihelion = np.min(dist_to_sun)
meters_to_UA(r_perihelion) # Perihelion distance in UA
```

[22]: 1.024996824575165

[23]:
```
r_aphelion = np.max(dist_to_sun)
meters_to_UA(r_aphelion) # Aphelion distance in UA
```

[23]: 3.414986453107802

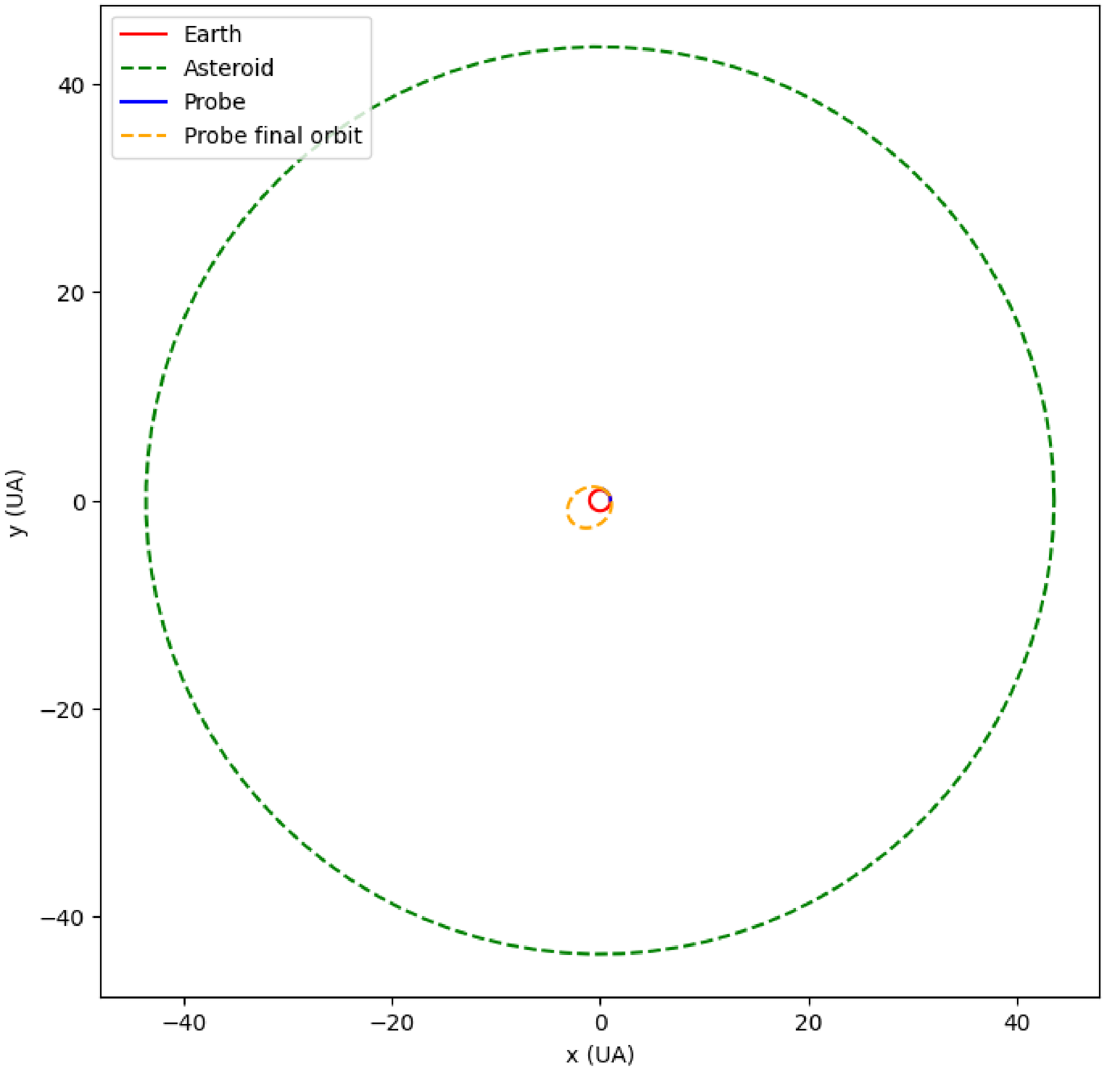
[24]:
```
a = (r_perihelion + r_aphelion) / 2
meters_to_UA(a) # semi-major axis of the probe's orbit in UA
```

[24]: 2.2199916388414835

[25]:
```python
e = (r_aphelion - r_perihelion) / (r_aphelion + r_perihelion)
e # eccentricity of the probe's orbit
```

[25]: 0.5382879797195695

[26]:
```python
plt.figure(figsize=(8, 8))  # create figure, figsize can be changed as preferred

# Plotting Earth's orbit
uang = np.linspace(0, 2 * np.pi, 100)
x = np.cos(uang)
y = np.sin(uang)
plt.plot(x, y, color='red', label='Earth')
# Plotting Asteroid's orbit
uang = np.linspace(0, 2 * np.pi, 100)
x = 43.6 * np.cos(uang)
y = 43.6 * np.sin(uang)
plt.plot(x, y, color='green', linestyle="--", label='Asteroid')

plt.plot(meters_to_UA(posvel[:, 0]), meters_to_UA(posvel[:, 1]), color='blue',␣
 ↪label="Probe") # plot the probe's orbit

# Plot probe's final orbit
plt.plot(meters_to_UA(posvel_after[:, 0]), meters_to_UA(posvel_after[:, 1]),␣
 ↪color='orange', label='Probe final orbit', linestyle="--")

plt.xlabel('x (UA)')
plt.ylabel('y (UA)')
plt.axis('equal')
plt.legend()
plt.show() # make plot appear
```



3.3 Fuel calculation

As the starting amount of fuel will determine the final orbit the probe will reach, it is important to tune this parameter so that our probe reach the desired orbit. For that we could use a loop that will find the right starting fuel mass to reach the desire aphelion.

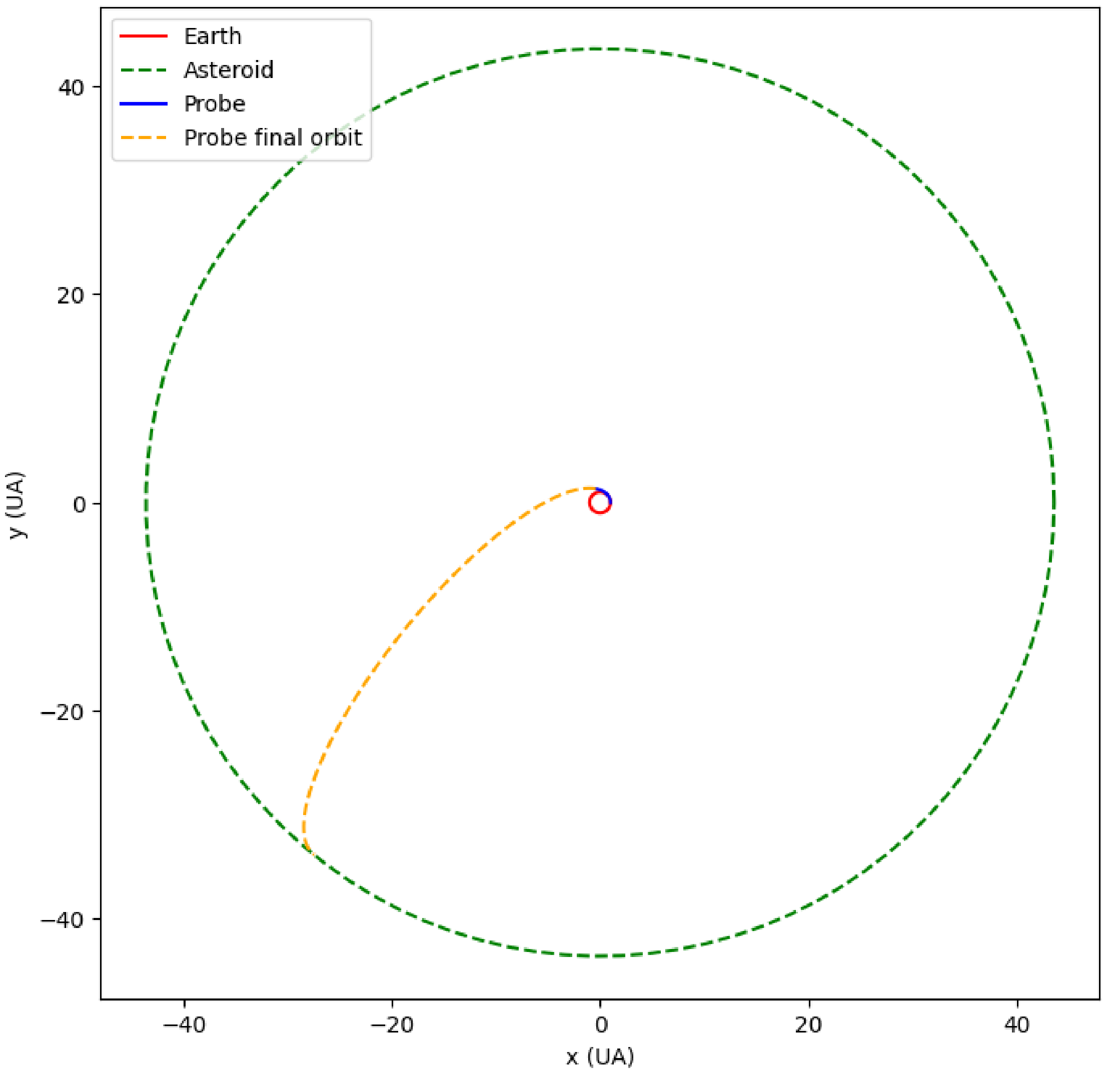
[27]:
```
import pandas as pd

dist_to_sun = 1
dry_mass_dist_cache = pd.DataFrame({'d': [1.0, 500.0], 'dry_mass': [300, 0]})
dry_mass_dist_cache.set_index('d', inplace=True)
while np.abs(dist_to_sun - 43.6) >= 0.01:
    v0 = np.sqrt(G * sun.mass / UA_to_meters(1))  # initial speed
```

```
    dry_mass = np.interp(43.6, dry_mass_dist_cache.index,␣
 ↪dry_mass_dist_cache["dry_mass"])
    xymass0 = [UA_to_meters(1), 0, 300]  # start position
    vxy0 = [0, v0]  # start vertical speed
    tf = (300-dry_mass)/mass_lost_rate  # Max burn time

    probe = Probe(probeqns_rocket, tf, tf/3600, x0=xymass0[0], vx0=vxy0[0],
                y0=xymass0[1], vy0=vxy0[1], z0=xymass0[2], vz0=mass_lost_rate)␣
 ↪# probe as an object
    t, posvel = probe.odesolve() # solve the differential equations

    last_v = np.linalg.norm(posvel[-1, 3:5])
    last_r = np.linalg.norm(posvel[-1, 0:2])
    a = np.abs(G * sun.mass / (last_v ** 2 - 2*G*sun.mass/last_r))

    period = np.sqrt(4 * np.pi**2 * np.abs(a)**3 / (G * sun.mass))/2  # Orbital␣
 ↪period
    probe = Probe(probeqns, period, period/3600, x0=posvel[-1, 0],␣
 ↪vx0=posvel[-1, 3],
                y0=posvel[-1, 1], vy0=posvel[-1, 4]) # probe as an object
    t_after, posvel_after = probe.odesolve() # solve the differential equations
    dist_to_sun = meters_to_UA(max(np.linalg.norm(posvel_after[:, 0:2],␣
 ↪axis=1)))
    new_data = {'d': dist_to_sun, 'dry_mass': dry_mass}
    dry_mass_dist_cache.loc[dist_to_sun] = dry_mass
    dry_mass_dist_cache = dry_mass_dist_cache.sort_index()
    # print(f"d:{dist_to_sun}, m={dry_mass}")
300 - dry_mass # Fuel mass of the probe to reach the asteroid in kg
```

[27]: 94.36523841632365

[28]:
```
plt.figure(figsize=(8, 8))  # create figure, figsize can be changed as preferred

# Plotting Earth's orbit
uang = np.linspace(0, 2 * np.pi, 100)
x = np.cos(uang)
y = np.sin(uang)
plt.plot(x, y, color='red', label='Earth')
# Plotting Asteroid's orbit
uang = np.linspace(0, 2 * np.pi, 100)
x = 43.6 * np.cos(uang)
y = 43.6 * np.sin(uang)
plt.plot(x, y, color='green', linestyle="--", label='Asteroid')

plt.plot(meters_to_UA(posvel[:, 0]), meters_to_UA(posvel[:, 1]), color='blue',␣
 ↪label="Probe") # plot the probe's orbit
```

```
# Plot probe's final orbit
plt.plot(meters_to_UA(posvel_after[:, 0]), meters_to_UA(posvel_after[:, 1]),␣
↪color='orange', label='Probe final orbit', linestyle="--")

plt.xlabel('x (UA)')
plt.ylabel('y (UA)')
plt.axis('equal')
plt.legend()
plt.show() # make plot appear
```



[29]:
```
dist_to_sun = np.linalg.norm(posvel_after[:, 0:2], axis=1)
r_perihelion = np.min(dist_to_sun)
meters_to_UA(r_perihelion) # Perihelion distance in UA
```

[29]: 1.389733146740658

[30]:
```
r_aphelion = np.max(dist_to_sun)
meters_to_UA(r_aphelion) # Aphelion distance in UA
```

[30]: 43.592217259928866

[31]:
```
a = (r_perihelion + r_aphelion) / 2
meters_to_UA(a) # semi-major axis of the probe's orbit in UA
```

[31]: 22.490975203334763

[32]:
```
e = (r_aphelion - r_perihelion) / (r_aphelion + r_perihelion)
e # eccentricity of the probe's orbit
```

[32]: 0.9382092979883504

4 Ex 14: Fast track to the Moon

Our goal is to transfer from a parking orbit to the moon. It is possible to use a Hohmann transfer orbit, however, even it is often use because it requires the least amount of impulse, it is also the slowest. To improve the transfer time, an other arbitary point (not the periapsis of the future orbit) could be use as impulse point. In this case, the orbit after the impulse is characterise by the polar coordinates of the burn and its angle.

[33]:
```
from space_base import GravBody, Probe
import matplotlib.pyplot as plt
import numpy as np

# Constants
G = 6.67e-11  # Gravitational constant
earth = GravBody.earth()  # Earth as an object with mass and radius
```

The initial given condition are:

- Speed after burn $V_0 = 10.85 kms^{-1}$
- Altitude $z_0 = 300 km$
- Angle $\psi_0 = 6°$ (angle between $\vec{V}$ and $\hat{\theta}$ ($\hat{\theta} \perp \vec{r}$))

[34]:
```
z0 = 300e3  # Initial altitude
r0 = earth.radius + z0  # Initial distance from center of Earth
v0 = 10.85e3  # Initial velocity
psi0 = np.deg2rad(6) # Initial angle
r_moon = 384_400e3

# Initial position and velocity vectors
xy0 = [-r0, 0] # Start at left of the graph
vxy0 = [-v0*np.sin(psi0), -v0*np.cos(psi0)]
```

Knowing this, it is possible to calculate the specific energy ϵ using:

$$\epsilon = -\frac{GM_{earth}}{2a} = \frac{1}{2}V^2 - \frac{GM_{earth}}{r}$$

And the specific angular momentum h,

$$h = r^2\dot{\theta} = r_0 v_0 \cos(\psi_0)$$

[35]:
```
energy0 = 0.5*v0**2-(G*earth.mass)/r0
energy0 # Initial energy (should be constant throughout the simulation)
```

[35]: `-851797.5191125795`

[36]:
```
h0 = r0*v0*np.cos(psi0)
h0 / 1e6 # Initial angular momentum (km^2/s)
```

[36]: `71983.84286941901`

Then, the semi-major axis can be calculate from the previous energy equation.

$$a = -\frac{GM_{earth}}{2\epsilon}$$

[37]:
```
a = -G*earth.mass/(2*energy0)
a / 1e3 # Semi-major axis (km)
```

[37]: `233826.54390388748`

Finally, it is possible to use the polar equation of the orbit to find the eccentricity:

$$r = \frac{a(1-e^2)}{1+e\cos(\theta)}$$
$$ae^2 + er\cos(\theta) + r - a = 0$$

Hence,

$$\Delta = (r\cos(\theta))^2 - 4a(r-a)$$
$$e = \frac{-r\cos(\theta) \pm \sqrt{\Delta}}{2a}$$

[38]:
```
discriminant = r0**2 - 4*a*(r0-a)
em = (-r0+np.sqrt(discriminant))/(2*a)
ep = (-r0-np.sqrt(discriminant))/(2*a)
e = max(em, ep) # Eccentricity
e
```

[38]: 0.971470304916528

Finally, the time of traveling from a point (r_0, θ_0) to an other point (r_0, θ_1) is given using the flight time formula:

$$t(r) = \frac{GM_{earth}}{(-2\epsilon)^{3/2}} \left[\sin^{-} 1(\left\{ \frac{GM_{earth} + 2\epsilon r}{\sqrt{(GM_{earth})^2 + 2\epsilon h^2}} \right\}) - \left\{ \frac{\sqrt{2\epsilon (h^2 - 2GM_{earth} r - 2\epsilon r^2)}}{GM_{earth}} \right\} \right]$$

Hence, the transfer time should be $t(r_1) - t(r_0)$

Additionally, because we want that the angle increase as the distance r increase we need to add some modification on the $sin^{-}1$ block. First, result of the funcion is expected to be on the range $[-\frac{\pi}{2}; \frac{\pi}{2}]$, it is needed to switch it to $[0; \pi]$. Secondly, $\frac{GM_{earth}+2\epsilon r}{\sqrt{(GM_{earth})^2+2\epsilon h^2}}$ is decreasing as r increase. Therefore, it is needed to invert it so that the inverted sin function will increase as r increase.

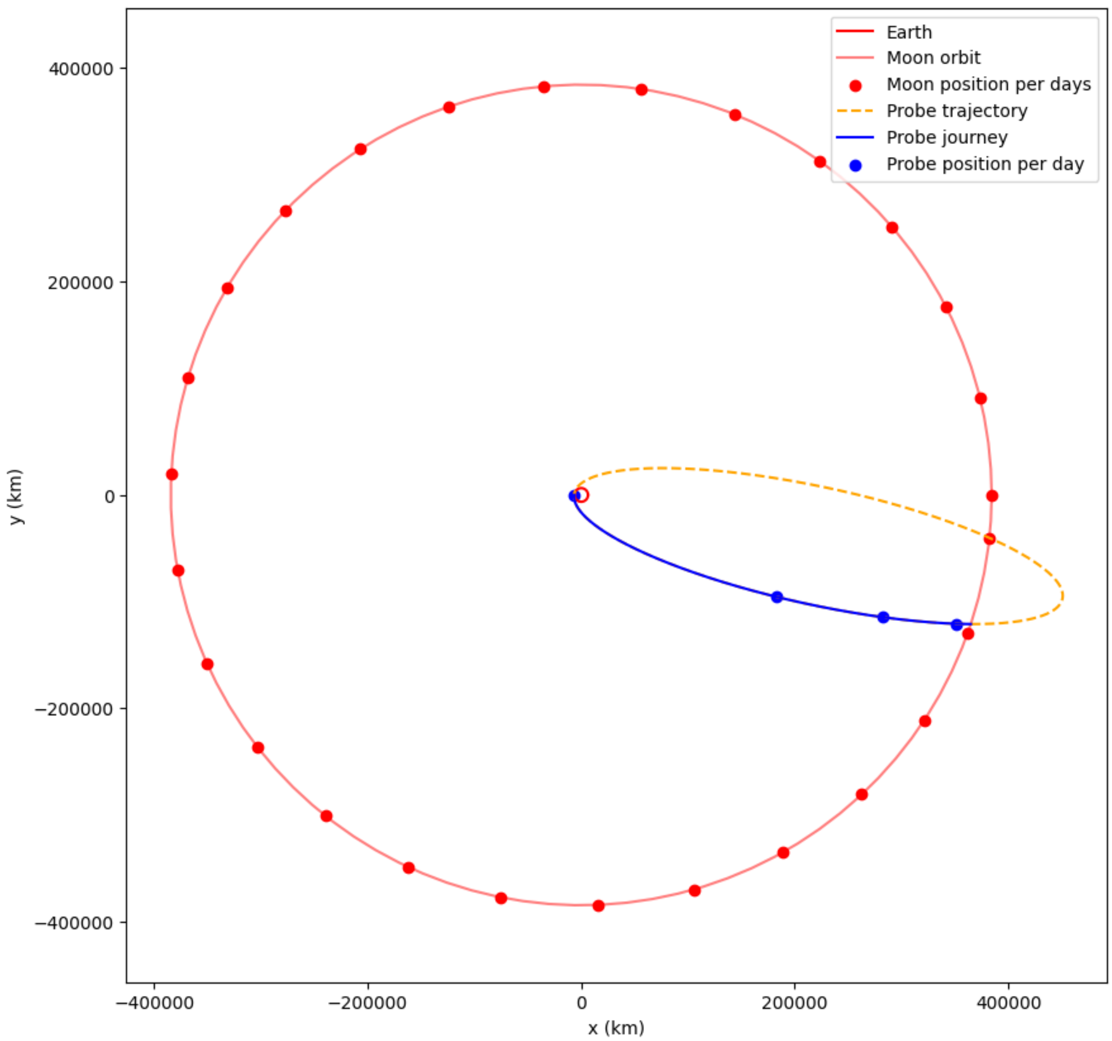
[39]:
```
def flight_time(r):
    A = np.arcsin(-(G*earth.mass+2*energy0*r)/np.sqrt((G*earth.
 ↪mass)**2+2*energy0*h0**2)) + np.pi/2
    B = np.sqrt(2*energy0*(h0**2-2*G*earth.mass*r - 2*energy0*r**2))/(G*earth.
 ↪mass)
    return (G*earth.mass/(-2*energy0)**1.5) * (A - B)

# Time of flight
travel_time = abs(flight_time(r_moon) - flight_time(r0))
travel_time / (24*3600) # Time of flight in days
```

[39]: 3.2483424316374734

[40]:
```
def probeqns(_, posvel):
    r = np.sqrt(posvel[0] ** 2 + posvel[1] ** 2)
    f = -G * earth.mass / r ** 3
    gravity_force = f * posvel[0:2]
    axy = gravity_force

    return posvel[2], posvel[3], axy[0], axy[1]

probe = Probe(probeqns, travel_time, travel_time / 60, x0=xy0[0], vx0=vxy0[0],
            y0=xy0[1], vy0=vxy0[1]) # probe as an object
t, posvel = probe.odesolve() # solve the differential equations
```

[41]:
```
# Plot the trajectory
plt.figure(figsize=(10,10))

# Plotting Earth
uang = np.linspace(0, 2 * np.pi, 100)
x = (earth.radius / 1e3) * np.cos(uang)
y = (earth.radius / 1e3) * np.sin(uang)
plt.plot(x, y, color='red', label='Earth')
```

```python
# Moon orbit
x = (r_moon / 1e3) * np.cos(uang)
y = (r_moon / 1e3) * np.sin(uang)
plt.plot(x, y, color="red", label='Moon orbit', alpha=0.5)
moon_day_period = np.sqrt(4*np.pi**2*r_moon**3/(G*earth.mass)) / (24*3600)
day_angle = 2*np.pi / moon_day_period
moon_days_round = np.floor(moon_day_period)
uang = np.linspace(0, moon_days_round * day_angle, int(moon_days_round))
x = (r_moon / 1e3) * np.cos(uang)
y = (r_moon / 1e3) * np.sin(uang)
plt.scatter(x, y, color="red", label='Moon position per days')

# Plotting entire orbit
probe = Probe(probeqns, travel_time*4, travel_time*4 / 60, x0=posvel[-1, 0],␣
 ↪vx0=posvel[-1, 2],
            y0=posvel[-1, 1], vy0=posvel[-1, 3]) # probe as an object
t_after, posvel_after = probe.odesolve() # solve the differential equations
plt.plot(posvel_after[:, 0] / 1e3, posvel_after[:, 1] / 1e3, color='orange',␣
 ↪linestyle="--", label="Probe trajectory") # plot the probe's orbit

plt.plot(posvel[0:, 0] / 1e3, posvel[0:, 1] / 1e3, color='blue', label="Probe␣
 ↪journey") # plot the probe's orbit
plt.scatter(posvel[0::60*24, 0] / 1e3, posvel[0::60*24, 1] / 1e3, color='blue',␣
 ↪label="Probe position per day") # plot the probe's orbit

plt.xlabel('x (km)')
plt.ylabel('y (km)')
plt.axis('equal')
plt.legend()
plt.show() # make plot appear
```



[42]:
```python
r = np.sqrt(posvel_after[:, 0] ** 2 + posvel_after[:, 1] ** 2) # distance from
  ↪the center of the Earth
r_per = np.min(r) # perigee
r_ap = np.max(r) # apogee
r_per / 1e3, r_ap / 1e3 # in km
```

[42]: (6599.827350109479, 461056.0972096128)

After the simulation we can check that real values match previous calculation.

[43]:
```python
a_real = (r_per + r_ap) / 2 # real semi-major axis
a_real / 1e3, (a - a_real) / 1e3 # in km
```

[43]: (233827.96227986112, -1.4183759736418724)

[44]:
```
e_real = (r_ap - r_per) / (r_ap + r_per) # real eccentricity
e_real, e - e_real
```

[44]: (0.9717748583798103, -0.0003045534632822866)

4.1 Comparing to Hohmann transfer

To compare our previous result we will assume a Hohmann transfer starting at an altitude 300km and with an apoapsis matching moon orbit altitude.

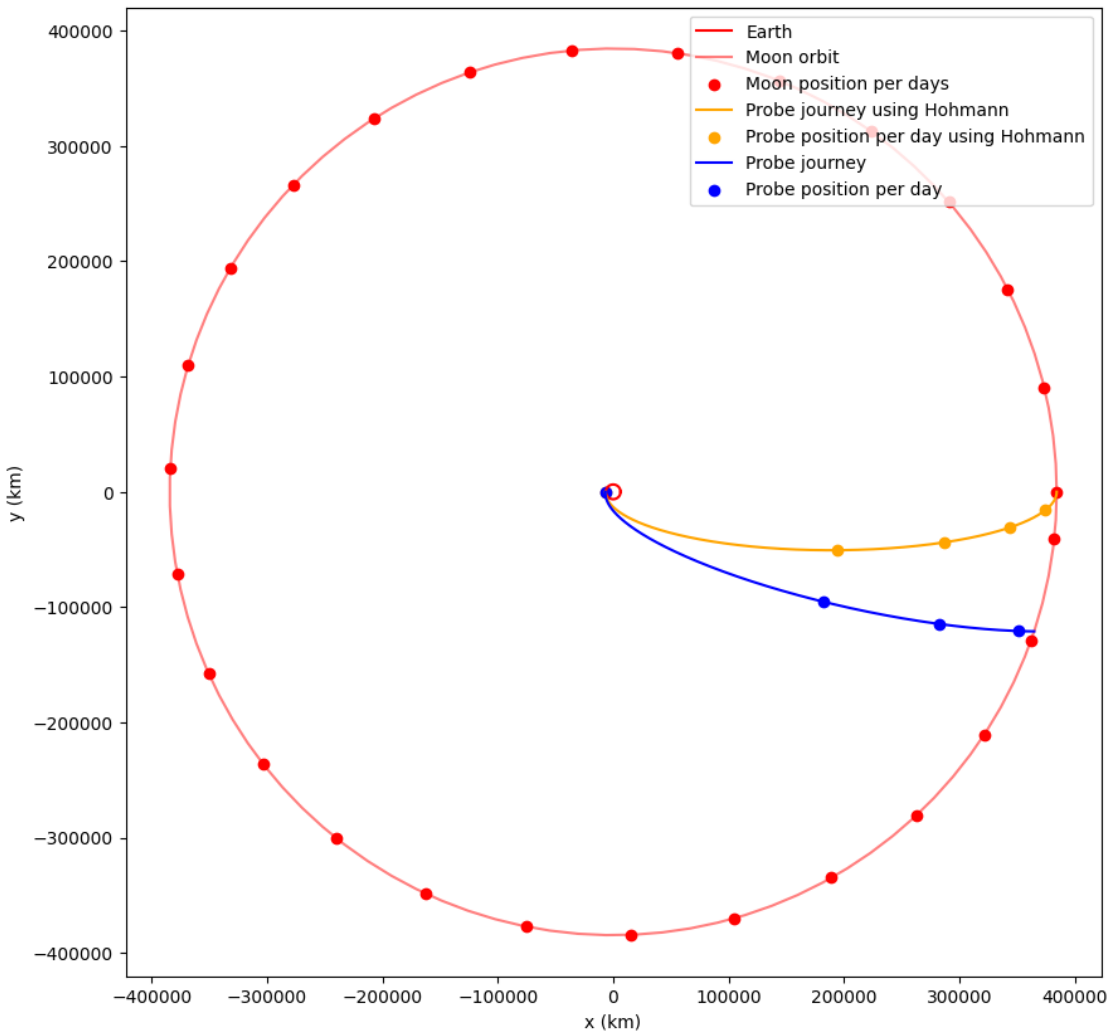
[45]:
```
r_per = r0
r_ap = r_moon
a = (r_per + r_ap) / 2
e = (r_ap - r_per) / (r_ap + r_per)
journey_time = np.sqrt(4 * np.pi**2 * a**3 / (G * earth.mass)) / 2
journey_time / (24*3600) # Time of flight in days
```

[45]: 4.981320278873557

The transfer time using an Hohmann transfer is greater than the previous technique. To better understand the difference it can be interesting to plot the two trajectories.

[46]:
```
xy0 = [-r0, 0] # Start at left of the graph
v0 = np.sqrt(G * earth.mass * (2 / r_per - 1 / a))
vxy0 = [0, -v0]

probe = Probe(probeqns, journey_time, journey_time / 60, x0=xy0[0], vx0=vxy0[0],
            y0=xy0[1], vy0=vxy0[1]) # probe as an object
t_hohmann, posvel_hohmann = probe.odesolve() # solve the differential equations

# Plot the trajectory
plt.figure(figsize=(10,10))

# Plotting Earth
uang = np.linspace(0, 2 * np.pi, 100)
x = (earth.radius / 1e3) * np.cos(uang)
y = (earth.radius / 1e3) * np.sin(uang)
plt.plot(x, y, color='red', label='Earth')
# Moon orbit
x = (r_moon / 1e3) * np.cos(uang)
y = (r_moon / 1e3) * np.sin(uang)
plt.plot(x, y, color="red", label='Moon orbit', alpha=0.5)
moon_day_period = np.sqrt(4*np.pi**2*r_moon**3/(G*earth.mass)) / (24*3600)
day_angle = 2*np.pi / moon_day_period
moon_days_round = np.floor(moon_day_period)
uang = np.linspace(0, moon_days_round * day_angle, int(moon_days_round))
x = (r_moon / 1e3) * np.cos(uang)
y = (r_moon / 1e3) * np.sin(uang)
```

```python
plt.scatter(x, y, color="red", label='Moon position per days')

plt.plot(posvel_hohmann[0:, 0] / 1e3, posvel_hohmann[0:, 1] / 1e3, color='orange', label="Probe journey using Hohmann") # plot the probe's orbit
plt.scatter(posvel_hohmann[0::60*24, 0] / 1e3, posvel_hohmann[0::60*24, 1] / 1e3, color='orange', label="Probe position per day using Hohmann") # plot the probe's orbit

plt.plot(posvel[0:, 0] / 1e3, posvel[0:, 1] / 1e3, color='blue', label="Probe journey") # plot the probe's orbit
plt.scatter(posvel[0::60*24, 0] / 1e3, posvel[0::60*24, 1] / 1e3, color='blue', label="Probe position per day") # plot the probe's orbit

plt.xlabel('x (km)')
plt.ylabel('y (km)')
plt.axis('equal')
plt.legend(loc='upper right')
plt.show() # make plot appear
```


Earth
Moon orbit
Moon position per days
Probe journey using Hohmann
Probe position per day using Hohmann
Probe journey
Probe position per day
400000
300000
200000
100000
0
−100000
−200000
−300000
−400000
y (km)
−400000
−300000
−200000
−100000
0
100000
200000
300000
400000
x (km)