1 What is this?

Post by Terence Tao from Google Buzz about foundations of mathematics. Original link is down and it's available only via Wayback Machine. I exported post to pdf, because, unfortunately, archived version does not display symbols. I don't have time to polish it, at least it's readable. Have a nice read!

2 Post

In the foundations of mathematics, the standard construction of the classical number systems (the natural numbers N, the integers Z, the rationals Q, the reals R, and the complex numbers C) starting from the natural numbers N is conceptually simple (Z is the additive completion of N, Q is the multiplicative completion of Z, R is the metric completion of Q, and C is the algebraic completion of R), but the actual technical details of the construction are lengthy and somewhat inelegant. Here is a typical instance of this construction (as given for instance in my own real analysis textbook):
* Z is constructed as the space of formal differences a-b of natural numbers a, b, quotiented by additive equivalence (thus a-b   c-d iff a+d=b+c), with the arithmetic operations extended in a manner consistent with the laws of algebra.
* Q is constructed as the space of formal quotients a/b of an integer a and a non-zero integer b, quotiented by multiplicative equivalence (thus a/b   c/d iff ad=bc), with the arithmetic operations extended in a manner consistent with the laws of algebra.
* R is constructed as the space of formal limits $\lim_{n\to\infty} a_n$ of Cauchy sequences a_n of rationals, quotiented by Cauchy equivalence (thus $\lim_{n\to\infty} a_n \sim \lim_{n\to\infty} b_n$ iff $a_n - b_n$ converges to zero as n goes to infinity), with the arithmetic operations extended by continuity.
* C is constructed as the space of formal sums a+bi of two reals a,b, with the arithmetic operations extended in a manner consistent with the laws of algebra and the identity $i^2 = -1$.
(Note that one can also perform these completions in a different order, leading to other important number systems such as the positive rationals Q^+, the positive reals R^+, the Gaussian integers Z[i], the algebraic numbers $\bar{\mathbf{Q}}$, or the algebraic integers O.)
There is just one slight problem with all this: technically, with these constructions, it is not quite true that the natural numbers are a subset of the integers, the integers are a subset of the rationals, the rationals are a subset of the reals, or the reals are a subset of the complex numbers! For instance, with the above definitions, an integer is an equivalence class of formal differences a-b of natural numbers. A natural number such as 3 is not then an integer. Instead, there is a canonical embedding of the natural numbers into the integers, which for instance identifies 3 with the equivalence class
3-0, 4-1, 5-2, ...

Similarly for the other number systems. So, rather than having a sequence of inclusions
$\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R} \subset \mathbf{C}$,
what we have here is a sequence of canonical embeddings
$\mathbf{N} \hookrightarrow \mathbf{Z} \hookrightarrow \mathbf{Q} \hookrightarrow \mathbf{R} \hookrightarrow \mathbf{C}$.
In practice, of course, this is not a problem, because we simply identify a natural number with its integer counterpart, and similarly for the rest of the chain of embeddings. At an ontological level, this may seem a bit messy - the number 3, for instance is now simultaneously a natural number, an equivalence class of formal differences of natural numbers, and equivalence class of formal quotients of equivalence classes of formal differences of natural numbers, and so forth; but the beauty of the axiomatic approach to mathematics is that it is almost completely irrelevant exactly how one chooses to model a mathematical object such as 3, so long as all the relevant axioms concerning one's objects are verified, and so one can ignore such questions as what a number actually is once the foundations of one's mathematics have been completed.
[Alternatively, one can carefully keep all the number systems disjoint by using distinct notation for each; for instance, one could distinguish between the natural number 3, the integer +3, the rational 3/1, the real number 3.0, and the complex number 3.0 + i 0.0. This type of distinction is useful in some situations, for instance when writing mathematical computer code, but in most cases it is more convenient to collapse all these distinctions and perform the identifications mentioned above.]
Another way of thinking about this is to define a (classical) number to be an element not of any one of the above number systems per se, but rather of the direct limit
$\lim_{\rightarrow}(\mathbf{N} \hookrightarrow \mathbf{Z} \hookrightarrow \mathbf{Q} \hookrightarrow \mathbf{R} \hookrightarrow \mathbf{C})$
of the canonical embeddings. Recall that the direct limit
$\lim_{\rightarrow}(\ldots \rightarrow A_{n-1} \rightarrow A_n \rightarrow A_{n+1} \rightarrow \ldots)$
of a sequence of sets (or objects in set-like categories, e.g. groups, vector spaces, etc.) chained together by maps (or morphisms, for more general categories) $f_n : A_n \rightarrow A_{n+1}$ is the space of sequences $(a_{n_0}, a_{n_0+1}, \ldots)$ of elements of some terminal segment $A_{n_0} \rightarrow A_{n_0+1} \rightarrow \ldots$ of the sequence of sets, such that the sequence of elements is compatible with the maps (i.e. $a_{n+1} = f_n(a_n)$ for all $n \geq n_0$), and then quotiented by tail equivalence: two sequences $(a_{n_0}, a_{n_0+1}, \ldots)$ and $(a_{1_0}, b_{n_1+1}, \ldots)$ are equivalent iff they eventually agree (i.e. $a_n = b_n$ for all sufficiently large n).
Direct limits also have an elegant category-theoretic definition; the direct limit A of the above sequence can be defined (up to isomorphism) as a universal object for the commutative diagram
$\ldots \rightarrow A_{n-1} \rightarrow A_n \rightarrow A_{n+1} \rightarrow \ldots \rightarrow A$,
which means that every other competitor B to the direct limit (i.e. any commutative diagram of the form
$\ldots \rightarrow A_{n-1} \rightarrow A_n \rightarrow A_{n+1} \rightarrow \ldots \rightarrow B$
) factors uniquely through A.
There is also an important dual notion of a direct limit, namely the inverse limit

$\lim_{\leftarrow}(\ldots \to A_{n-1} \to A_n \to A_{n+1} \to \ldots)$
of a sequence, which is defined similarly to the direct limits but using initial segments of the sequence rather than terminal segments. Whereas direct limits seek to build a canonical space in which all the elements of the sequence embed, inverse limits seek to build a canonical space for which all the elements of the sequence are projections. A classic example of an inverse limit is the p-adic number system $\mathbf{Z}_p$, which is the inverse limit of the cyclic groups $\mathbf{Z}/p^n\mathbf{Z}$. Another example is the real number system $\mathbf{R}$, which can be viewed as the inverse limit of the finite-precision number systems $10^{-n}\mathbf{Z}$ (using arithmetic operations with rounding, and using rounding to map each finite precision number system to the next coarsest system).
In the foundations of mathematics, the standard construction of the classical number systems (the natural numbers N, the integers Z, the rationals Q, the reals R, and the complex numbers C) starting from the natural numbers N is conceptually simple (Z is the additive completion of N, Q is the multiplicative completion of Z, R is the metric completion of Q, and C is the algebraic completion of R), but the actual technical details of the construction are lengthy and somewhat inelegant. Here is a typical instance of this construction (as given for instance in my own real analysis textbook):
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Direct limits and inverse limits can be generalised even further; in category theory, one can often take limits and colimits of more general diagrams than sequences. This gives a rich source of constructions of abstract spaces (e.g. direct sums or direct products) that are convenient places to do mathematics in, as they can connect to many otherwise distinct classes of mathematical structures simultaneously. For instance, the adele ring, which is the direct product of the reals and the p-adics, is a useful universal number system in algebraic number theory, which among other things can be used to greatly clarify the nature of the functional equation of the Riemann zeta function.