Optimal Asset-Liability Management with Inflation Hedging and Regulatory Constraints

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Abstract

We examine how inflation risk affects the asset-liability investment allocation problem of a fund manager who is constrained by regulatory rules. The liability driven investment framework faces a time-varying investment opportunity set, characterized by a two-factor affine model of term structure, and is subject to annually-checked Value-at-Risk and maximum funding constraints required by regulatory authorities. We show that inflation-hedging is of particular importance for the long-term ALM investor and has an influential impact on asset allocation by interacting with liability-hedging and intertemporal hedge demands simultaneously. We also explain the practical fact that many managers tend to act myopically, instead of rebalancing strategically.

Keywords: Inflation risk, Liability risk, Hedging demand, Affine term structure, Risk premium

JEL: G11, G12, G23

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I. Introduction

Inflation hedging has become a concern of critical importance for fund managers. They consider inflation risk not only as a direct threat with respect to the protection of their purchasing power, but also, and perhaps more importantly, a threat to fund stability in terms of arising liabilities. For example, pension plans typically take the promised benefit payments as pension liabilities and then pay the liability stream by using accumulated investment gains and pension contributions. While these pension payments will be eroded by inflation over time. When pension payments are linked (conditional or full indexation) to consumer price or wage level indexes, an unexpected rise in inflation will result in an increase in pension liabilities, thereby leading to a deterioration of fund financial level. To this end, the hedging demand against inflation has received more attention from the fund managers when making optimal investment decisions.

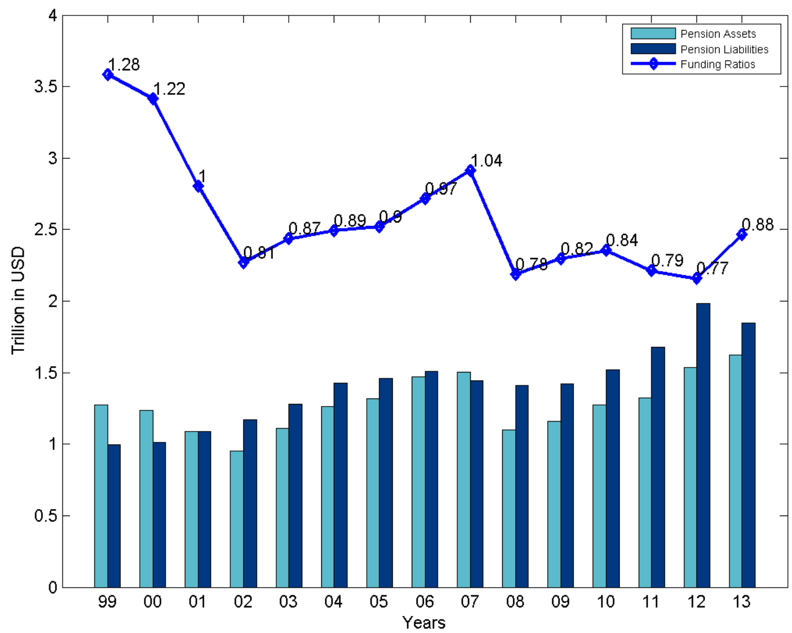
The focus on inflation hedging is consistent with the heightened focus on liability risk management that has emerged as a consequence of recent pension crisis. As Figure 1 illustrates, a growing number of pension plans in the US are unfortunately materially underfunded since 2002, reflecting the abnormal increase in liabilities. As a result of these trends, the funding ratio (i.e. asset to liability ratio) falls below 100%, the "funding floor", for more than a decade, with one exception in 2007 when the bullish financial market outperformed the liability stream. More recently, the Universities Superannuation Scheme (USS) in the UK reports that its liabilities exceeded assets by £2.9 billion in March 2011 and this deficit increased sharply to £7.9 billion in June 2013.[1] Generally, funding shortfalls are partly

[1] See employerspensionsforum.co.uk/en/pension-schemes/uss/briefing-on-the-uss--july-2014.

FIGURE 1

S&P500 Pension Fund Status

Figure 1 displays the S&P500 corporate pension status over the period 1999 to 2013. Source: S&P Dow Jones Indices LLC. Data and information obtained from individual company fiscal year-end reports.



attributable to the abnormal increase in liabilities resulting from the increasing longevity, inflation risk, and low interest rate since the financial crisis in 2008. We argue that the effect of inflation risk and asset-liability mismatching risk deserve more concerns.

In this paper, we take the institutional investor to be fund managers, who have a mandate to manage a portfolio for a mutual fund, a pension fund, a investment bank or insurance company. We focus on the widely-adopted Asset Liability Management (ALM) approach and attempt to manage assets and cash inflows to satisfy various obligations. ALM approach is not just about increasing investment profitability but focuses on offering solutions to

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mitigate or hedge the unexpected changes in risk factors affecting liability values, and most notably inflation risks. Our goal is to demonstrate how fund managers optimally choose their portfolio allocation to match liabilities in the future, and as oppose to the nominal economy, how different their investment strategies are in the real context with the aim of hedging against inflation risk, and importantly, how regulatory rules affect the manager's investment behaviour. To the best of our knowledge, we are the first to investigate the inflation effect on the optimal risk exposure in the presence of risky liabilities and regulatory rules.

To address these questions, we consider a dynamic asset allocation problem in the framework of the ALM. The investment opportunity set is characterized by a two-factor market model of affine term structure. We account for inflation risk by adopting an inflation-adjusted yield as the discount rate for calculating the present value of liabilities and correspondingly the *real funding ratio*. The real funding ratio is different from the nominal ratio, which uses a risk-free interest rate or a nominal yield, as it links the liabilities to the price level which ensures that the purchasing power of future liability payments rises with the price level. To demonstrate the impact of inflation risk, we examine portfolio weights, hedging demands and utility losses in the real and nominal economy respectively. Within each economy, we analyze two distinct investment strategies: a myopic strategy and a dynamic strategy with optimal rebalancing.

Meanwhile, with the aim of keeping a sustainable funding level, the manager has to follow two regulatory constraints on the funding ratio required by regulatory authorities. One constraint requires a number of subsequent and non-overlapping Value-at-Risk (VaR) checks on an annual basis. Another annually repeated constraint is the *maximum* funding constraint imposed by the tax authority, aiming at avoiding the situation where the investor deliberately overfunds to shelter more taxable corporate profits. The complicated problem of dynamic portfolio optimization in the incomplete market is solved by a state-of-the-art simulation-based method along with grid search technique (Brandt et al. 2005).

Using this framework, we first examine the impact of inflation on the ALM investment strategy. We show that inflation hedging is a dominant incentive for a dynamic fund manager with a long-term horizon. Intuitively, an investor who adopts a dynamic strategy tends to take full advantages of the mean-reverting feature in bond risk premium by investing more in bonds than the investor with a myopic policy. The extra bond holdings are intertemporal hedging demands, which indeed exist with a positive sign in our nominal economy. However, accounting for inflation into the framework alters this. We find that the dynamic investor reduces his holding for bonds and tilts his portfolio to equities instead, because stock returns are negatively correlated with inflation innovations and more importantly, offer higher expected returns. To this end, stock turns out to be a good inflation hedge in the ALM and the inflation-hedging effect is even larger when we exclude predictability in both bond and stock premia by setting them to be constants. Investors who ignore inflation risk would suffer significant welfare losses.

Moreover, we also compare portfolio allocations of an asset-liability problem to those of an asset-only problem. We find that the liability-driven risk induces a large allocation to bonds, especially for the myopic strategy in the ALM framework. In general, the fund manager spreads his wealth mainly between stocks and bonds in the presence of liabilities, whereas the asset-only investor has a well-diversified portfolio with a significant amount of bonds being replaced by cash. There is an interesting interplay between inflation and liability. On the one hand, a positive inflation-hedging demand induces a large component in stocks, which provides higher returns and maintain the financial position of funding ratio. This, in return, partly offsets investor' concern about liability risk; on the other hand, when the liability is completely excluded in the optimization problem, then inflation hedging demand for stocks becomes zero across different funding levels, suggesting that in the asset-only case inflation is not as important as in the ALM context.

In terms of regulatory effects, we find that both VaR and maximum funding ratio constraints reduce the optimal risk exposure. To be specific, when the fund is materially underfunded, it may be optimal to hold the entire portfolio in cash instead, due to the horizon disparity and the limit of additional cash inflows in our model. When the position is overfunded, the VaR constraints bind yet with a decreasing effect until the fund has a high surplus, by then the optimal strategy is less sensitive to the funding levels. With maximum funding ratio constraints, the ability of exploiting the time variation in the investment opportunities has been further limited, thus leading the dynamic strategy almost indifferent with the myopic strategy.

The contributions of this article are threefold from the perspective of the ALM model, implementation and insights into practice. To begin with, our paper contributes to the literature by investigating a theoretical ALM model in the constrained utility maximization problem, and modelling the real funding ratio in the context of affine term structure. Given that interest and inflation innovations are two main sources of risk affecting the financial market and the valuation of liabilities, we consider two common factors (state variables), i.e. the real interest rate and expected inflation. Their dynamics are characterized by a vector-valued mean-reverting process, which captures stylized predictability in the expected inflation rate, bond risk premium and interest rate.

To maintain the funding ratio in a healthy financial position, we consider a lower and an upper bound on funding ratio by repeatedly conducting VaR and maximum funding constraints on an annal basis. Empirical studies investigating the effect of regulatory requirements on optimal investment allocation are rather limited in both quantity and quality. These two constraints are separately taken into account by Martellini and Milhau (2012) and van Binsbergen and Brandt (2014), but none of them has explored the joint effects simultaneously.

More importantly, our results help to explain the stylized practice that many long-horizon managers do not strategically rebalance portfolio, but instead prefer to acting myopically. The reasons can be best summarized as follows: first, inflation hedging is of particular importance for an ALM investor who has risky liability streams over a long investment horizon, while it also adds values for the myopic strategy by reducing the probability of exploiting time-varying bond risk premium; second, both liability immunization and the time variation in bond market result in a large bond component in the myopic and dynamic portfolio in turn. In this sense, the difference between the dynamic and myopic investment strategy is relatively small; third, imposing regulatory constraints further restrict investors from acting dynamically. Base on these results, we conclude that it is plausible for the fund manager to take myopically optimal strategy, in which sense the investor can divide the long-term investment horizon into several relatively shorter terms, e.g. 1-3 years, and then optimizes the portfolio each time with a terminal wealth (or funding ratio) over a short horizon.

The rest of our paper is organized as follows. Section II motivates our investigation with a brief review of related literature. Section III describes the dynamic ALM framework we used for accounting for inflation risk and regulatory rules. Section IV analyzes the joint effects of inflation, regulatory constraints and time-varying bond premium on optimal portfolio choice and welfare benefits/losses. Section V concludes. Appendix A summarizes the numerical method and parameter values.

II. Related Literature

Our study is related to several strands of the literature. Foremost, our research is directly related to the ALM studies that link obligations with optimal asset-only investment decisions under regulatory constraints and time-varying expected returns. Our study of inflation-hedging investment also complements the ALM literature. In this section we summarize the related literature.

The first ALM application was presented by Merton (1992), who studies a university managing an endowment fund. Sundaresan and Zapatero (1997) examine the valuation of liabilities for a fund and derive the optimal trading strategy with constant parameters. Rudolf and Ziemba (2004) examine the optimal portfolio of managing the net surplus of assets and liabilities under time-varying investment opportunities with one state variable. Allowing a vector-valued diffusion process of state variables, Detemple and Rindisbacher (2008) study the ALM with intermediate dividends and a funding shortfall at the terminal date. Nevertheless, the above papers mainly focus on strategies without considering regulatory constraints. Conversely, we discuss the ALM with two regulatory rules under the two-factor affine model. Thus, we can provide new insights by revealing the joint impact of regulations, inflation and time-varying bond premium.

The VaR constraint that we consider has been widely imposed by regulators to control the risk of extreme losses and protect the interests of beneficiaries. The effects of VaR on asset allocation without liability are studied first by Basak and Shapiro (2001) who assume the regulatory horizon coincides with the investment horizon (i.e. static VaR). They find that a VaR-constrained investor is forced to choose a larger exposure to risky assets than the unconstrained one in some specific situations. Cuoco et al. (2008) address that such negative assessment on VaR arises from the static VaR assumption. When VaR is reevaluated at all time, the allocation to risky assets is always lower. Similar conclusions are obtained by Yiu (2004) who faces continuously-imposed VaR and uses dynamic programming technique. Using a pointwise optimization method under a complete market, Shi and Werker (2012) find that an annually repeated asset-only VaR reduces losses of long-term trading strategies, but it induces a large opportunity cost by limiting portfolio holdings of risky assets.

In addition to VaR, the maximum funding constraint is usually introduced by the tax authorities for preventing a deliberate or accidental build-up of excess funds to shelter more taxable corporate profits (see Pugh 2006). Surprisingly, little literature on ALM has paid attention to such constraint, except for Martellini and Milhau (2012). This constraint is required by regulations or customary practices for some funding schemes. For example, the pension assets should not exceed 105% of the accrued liabilities in the UK. The pension funding ratio is on average bounded by 130% in the Netherlands. Similar caps are imposed in Canada and Japan.[2]

Our paper is closely related to three studies in the recent literature - Hoevenaars et al.

[2]See also Pugh (2006) for a survey on regulation rules.

(2008), Martellini and Milhau (2012) and van Binsbergen and Brandt (2014). These papers focus on the funding ratio under time-varying investment opportunities. Hoevenaars et al. (2008) report the hedge qualities and welfare benefits of investing in a menu of seven alternative assets. They do not explicitly model inflation but infer the inflation hedging qualities of assets from the correlation between nominal asset returns and inflation. Martellini and Milhau (2012) derive optimal weights in a one-factor complete market setting with the constant bond risk premium and expected inflation rate. van Binsbergen and Brandt (2014) compare portfolio weights and certainty equivalents using different discount rates, while they do not consider inflation effects.

The differences between our paper and the above three are obvious. First, we develop our ALM model in a two-factor affine term structure framework rather than in the Gaussian vector autoregression (VAR) model adopted by Hoevenaars et al. (2008) and van Binsbergen and Brandt (2014). Although both affine and VAR models capture predictability in asset returns, the VAR model is relatively simple. It does not define the term structure well and ignores the extra information provided by the no-arbitrage restriction (see, e.g. Sangvinatsos and Wachter 2005; Piazzesi 2010). Indeed, affine models are advocated by, e.g., Cochrane and Piazzesi (2005) who point out that affine models exhibit a pricing kernel consistent with the bond price dynamics and it can predict prices for bonds with different maturities without breaking no-arbitrage arguments. It is also flexible to the time-series process for yields with different restrictions.

Second, the incomplete two-factor model characterizes market dynamics more realistically than the complete one-factor short rate model taken by Martellini and Milhau (2012). The former captures the time-variation in bond risk premium. It also takes the expected inflation as a second factor in addition to the real interest rate since both the factors affect the fundamentals in the ALM. It naturally models the dynamics of expected inflation rate as an affine function of the state variables, whereas the one-factor model and VAR approach fail in modeling the dynamics of expected inflation.

Third, we discount the liability by using the *real* bond yield, which complements the case discussed in van Binsbergen and Brandt (2014). Fourth, the above references do not examine changes of hedging demands against different funding levels. We fill the gap by comparing three types of hedging demands under various scenarios. Finally, we consider the maximum funding constraint along with VaR to ensure that the fund fulfills the requirements by tax authorities and fund regulators, though these constraints are separately taken into account by Martellini and Milhau (2012) and van Binsbergen and Brandt (2014) along with other constraints.

III. Analytical ALM Framework

A. Financial Market

We consider a fund manager with a T-year investment horizon and an asset menu including a stock index, a 10-year nominal bond and a nominal money market account. The financial market setting follows Koijen et al. (2010) who specify a two-factor term structure model based on a general latent three-factor model in Sangvinatsos and Wachter (2005). The two factors are the real interest rate and expected inflation as in Brennan and Xia (2002) and Campbell and Viceira (2001). The affine model characterizes the dynamics of time-varying bond risk premium and expected inflation.

The state variables are described by a vector-valued mean-reverting process around zeros, i.e.,

$$dX_t = -\kappa X_t dt + \sigma_X dz_t, \quad \kappa \in \mathbb{R}^{2\times 2}, \sigma_X \in \mathbb{R}^{2\times 4}, \tag{1}$$

where $X_t \equiv (X_{1t}, X_{2t})^\top$ denote the real interest rate and expected inflation respectively and $z \in \mathbb{R}^{4\times 1}$ is a vector of independent Brownian motions driving uncertainties in the financial market. Following the above literature, we set $\sigma_X = [I_{2\times 2} \quad 0_{2\times 2}]$, and normalize κ to be lower triangular.

The instantaneous nominal interest rate, R_t^0, is affine in the two state variables:

$$R_t^0 = \delta_{0R} + \delta_{1R}^\top X_t, \quad \delta_{0R} > 0, \quad \delta_{1R} \in \mathbb{R}^{2\times 1}. \tag{2}$$

The price of risk Λ_t is also linear in X_t:

$$\Lambda_t = \Lambda_0 + \Lambda_1 X_t, \quad \Lambda_0 \in \mathbb{R}^{4\times 1}, \quad \Lambda_1 \in \mathbb{R}^{4\times 2}. \tag{3}$$

Given a process for R_t^0 and Λ_t, the pricing kernel then follows

$$\frac{d\phi_t}{\phi_t} = -R_t^0 dt - \Lambda_t^\top dz_t. \tag{4}$$

Let $B(X_t, t, s)$ denote the nominal price of a bond maturing at $s > t$. It is assumed to be an exponential function of time t and of state variables X_t (e.g. Dai and Singleton 2000; Sangvinatsos and Wachter 2005), i.e.

$$B(X_t, t, s) = \exp\{A_2(\tau) X_t + A_1(\tau)\}, \quad A_2(\tau) \in \mathbb{R}^{1\times 2}, \quad \tau = s - t, \tag{5}$$

where $A_1(\tau)$ and $A_2(\tau)$ solve a system of ordinary differential equations given by:

(6)
$$\begin{aligned} A_1'(\tau) &= -A_2(\tau)\sigma_X\Lambda_0 + \frac{1}{2}A_2(\tau)\sigma_X\sigma_X^\top A_2^\top(\tau) - \delta_{0R}, \\ A_2'(\tau) &= -A_2(\tau)[\kappa + \sigma_X\Lambda_1] - \delta_{1R}^\top. \end{aligned}$$

The resulting bond yield is $y_{\tau,t} = -\frac{1}{\tau}[A_2(\tau)X_t + A_1(\tau)]$. By applying Itô's lemma, the instantaneous bond price is given by

(7)
$$\frac{dB_t}{B_t} = [R_t^0 + \sigma_B^\top\Lambda_t]dt + \sigma_B^\top dz_t, \quad \sigma_B \in \mathbb{R}^{4\times 1},$$

where $\sigma_B^\top = A_2(\tau)\sigma_X$. Equation (7) shows that the bond risk premium varies with the state vector X_t. The correlation between the bond risk premium and the state variables depends on the maturity of the bond via the function $A_2(\tau)$.

There exists a stock index with the nominal price following the dynamics

(8)
$$\frac{dS_t}{S_t} = (R_t^0 + \eta_S)dt + \sigma_S^\top dz_t, \quad \sigma_S \in \mathbb{R}^{4\times 1},$$

where the equity risk premium η_S is a constant. The row vector $\sigma_S^\top$ is assumed to be linearly independent of the rows of σ_X, so that the stock is not spanned by the bond.

The nominal gross return on the single-period cash account, R_t^f, is given by

(9)
$$R_t^f = \exp(R_{t-1}^0).$$

We have described the nominal investment opportunity set so far. To address inflation risk for the fund manager who pays attention to the real funding ratio, it is necessary to define a stochastic price level Π_t,

(10)
$$\frac{d\Pi_t}{\Pi_t} = \pi_t dt + \sigma_\Pi^\top dz_t, \quad \sigma_\Pi \in \mathbb{R}^{4\times 1}.$$

The instantaneous expected inflation π_t is affine in the state variables

(11)
$$\pi_t = \delta_{0\pi} + \delta_{1\pi}^\top X_t, \quad \delta_{0\pi} > 0, \quad \delta_{1\pi} \in \mathbb{R}^{2\times 1}.$$

B. Asset-Liability Optimization Problem

Let R_t denote the nominal gross asset returns, the dynamics of financial wealth W_t follows

$$W_{t+1} = W_t[\theta_t^\top (R_{t+1} - R_{t+1}^f) + R_{t+1}^f], \tag{12}$$

in which θ_t is the fraction of wealth invested in the risky assets at time t, and the reminder $1 - \theta_t^\top \imath$ is allocated to the cash account ($\imath$ is a 2×1 vector of ones).

A fund typically has long-term liability streams in the future which are often modeled as coupon payments of long-term bonds. Assuming the duration of the liability is T years, one can obtain the present value of liabilities by using the government bond yield as the discount rate (see van Binsbergen and Brandt (2014)), i.e.,

$$L_t^N = \exp(-T\, y_{\tau,t}). \tag{13}$$

Generally, a fund manager focuses on the relative value of assets with respect to liabilities. Therefore, we approach the ALM problem from the perspective of a funding ratio. Following van Binsbergen and Brandt (2014), the funding ratio in nominal terms is defined as

$$F_t^N = \frac{W_t}{L_t^N} = \frac{W_{t-1}[\theta_{t-1}^\top (R_t - R_t^f) + R_t^f]}{L_t^N} = F_{t-1}^N \frac{[\theta_{t-1}^\top (R_t - R_t^f) + R_t^f]}{L_t^N / L_{t-1}^N}. \tag{14}$$

Note that, unlike van Binsbergen and Brandt (2014) who do not consider inflation risk, we also calculate bond yield $y_{\tau,t}^{real}$ in real terms by subtracting the inflation rate from the bond yield $y_{\tau,t}$ in nominal terms, i.e.

$$y_{\tau,t}^{real} = y_{\tau,t} - \log(\frac{\Pi_t}{\Pi_{t-1}}). \tag{15}$$

Then, we discount the present value of liabilities using $y_{\tau,t}^{real}$:

$$L_t^{real} = \exp(-T\, y_{\tau,t}^{real}). \tag{16}$$

By doing this we link the liabilities to the price level which ensures that the purchasing power of future liability payments rises with the price level.

The financial wealth and asset returns in real terms are denoted by small letters, i.e.,

$$w_t = \frac{W_t}{\Pi_t}, r_t = \frac{R_t \Pi_{t-1}}{\Pi_t}, r_t^f = \frac{R_t^f \Pi_{t-1}}{\Pi_t}. \tag{17}$$

Then, the financial wealth w_t in real terms follows:

$$w_t = w_{t-1}[\theta_{t-1}^\top (r_t - r_t^f) + r_t^f]. \tag{18}$$

As a contribution to the literature, we incorporate inflation risk into the *real* funding ratio defined as

$$F_t = \frac{w_t}{L_t^{real}} = \frac{w_{t-1}[\theta_{t-1}^\top (r_t - r_t^f) + r_t^f]}{L_t^{real}} = F_{t-1} \frac{[\theta_{t-1}^\top (r_t - r_t^f) + r_t^f]}{L_t^{real}/L_{t-1}^{real}}. \tag{19}$$

Finally, we consider a fund manager who has a power utility function defining over the funding ratio F_t (resp. F_t^N) in real (resp. nominal) terms with constant relative risk-aversion (CRRA). The manager then solves

$$J(F, X, t, T) = \max_{\theta_t \in \mathcal{K}_t} \mathbb{E} \left(\frac{F_T^{1-\gamma}}{1-\gamma} \right), \tag{20}$$

where $\mathcal{K}_t$ is the constraint set described below.

C. Investment Constraints

First, we impose the standard borrowing and short-selling constraints, i.e., $\theta_t \geq 0$ and $1 - \imath^\top \theta_t \geq 0$. Second, the fund manager is required to conduct VaR checks repeatedly for each regulatory horizon, i.e. one year, which is shorter than the 10-year investment horizon. The manager has to ensure that the probability of a shortfall in funding is not more than the probability δ given by the regulators. We refer to shortfall as the situation where the asset value of a fund falls below its liability value. Formally, the manager has the VaR constraint[3]

$$\text{VaR}: \quad \mathbb{P}_t(F_{t+1} < 1) \leq \delta, \quad \delta \in [0, 1], \tag{21}$$

where each sub-period equals to the regulatory horizon.

In addition, in the financial market that we consider, it is difficult to fulfill the VaR constraint at $t+1$ when the funding ratio F_t is not greater than 1 at the beginning of the period $[t, t+1]$. This situation has to be compromised so that the investor is allowed to continue his trading strategy. To this end, van Binsbergen and Brandt (2014) suggest that the VaR constraint can be adapted as the probability of a decrease in the funding ratio is not greater than δ:

$$\text{VaR}': \quad \mathbb{P}_t(F_{t+1} < F_t | F_t \leq 1) \leq \delta, \quad \delta \in [0, 1]. \tag{22}$$

In other words, the VaR constraint is evaluated as if $F_t = 1$ in this case. Note that (21) and (22) are equivalent when $F_t = 1$.

Third, some fund managers may consciously or accidentally over fund too much by following their favorable investment experiences, e.g. building overfunded funds to buffer themselves against future shortfalls meanwhile taking advantage of more tax shields. To avoid such situations, the tax authorities limit the overfunded level. We assume that the maximum funding constraint is imposed on an annual basis, which is consistent with the regulatory horizon used in the VaR constraint. Specifically, for a given value of F_t, the maximum

[3] We describe all regulatory constraints in real terms only, the expressions in nominal terms are similar.

funding level at the next period $t+1$ should not be greater than a constant k, namely,

$$\text{maximum funding constraint}: \quad F_{t+1,t} \le k, \tag{23}$$

implying that the maximum funding constraint is dynamically respected at all periods. This is similar to the case of Martellini and Milhau (2012) when their particular conditions are satisfied.

The set $\mathcal{K}_t$ below summarizes the constraints on the portfolio weights and funding ratio.

$$\mathcal{K}_t = \{(\theta, F) : \ \theta_t \ge 0, 1 - \imath^\top \theta_t \ge 0, \text{VaR or VaR}', \text{ and } F_{t+1,t} \le k\}. \tag{24}$$

IV. Strategic Portfolio Choice

This section presents the main insights into ALM by evaluating the joint effects of inflation, time-varying bond premium, liability, and regulatory constraints on the fund manager's portfolio policies and welfare losses. In our analysis, we consider a fund manager who has a $T = 10$ years investment horizon with a risk aversion $\gamma = 5$, and who can take either myopic strategy or dynamic strategy. To be specific, for myopically optimal strategies, the investor solves 10 sequential one-year optimizations and does not use any of the new information, whereas the dynamic investor strategically optimizes the T-year expected utility function by using the new information at each rebalancing date.

Moreover, the dynamic asset allocation strategy is driven by three types of hedging demands. First, the portfolio difference between the dynamic and myopic strategies defines the *intertemporal hedging demand*, which hedges against the risks of time-variation in the investment opportunity set. For example, the dynamic strategy increases bond holdings to take advantage of mean-reverting predictability in the time-varying bond risk premium.

Second, the portfolio difference between the real and nominal economies indicates the *inflation-hedging demand*, which characterizes the effects of inflation risk. This hedging demand is shown by comparing the manager's ALM problems of maximizing the expected utility of terminal funding ratio in real and nominal terms respectively. A fundamental finding is that the stock can provide a good inflation hedge.

Third, the difference between the strategies of ALM and of asset-only management implies the *liability-hedging demand*, which arises from assets/liabilities mismatch. Another fundamental finding is that the liability risk overwhelmingly affects the myopic strategy by resulting in a large allocation to bonds. In a nutshell, the three hedging demands are not parallel but interacting with each other. We will take a closer look at their joint effects on portfolio choice in the following sections.

For the financial market, we adopt the values of parameters estimated by Koijen et al. (2010) as reported by our Table 2 in Appendix A. We briefly summarize the stylized facts of the investment market as follows. First, stock risk premium (5.38%) is higher than the unconditional/long-run mean of the 10-year bond risk premium (2.06%), and the volatilities of stock and bond are 15.16% and 11.7% respectively.[4] Correspondingly, the Sharpe ratio of stock (0.35) is almost twice as high as the unconditional mean of bond Sharpe ratio (0.176), which indicates that generally the stock provides higher excess returns for the same risk.

[4]The term structure factors are set to their unconditional/long-run expectation ($X_t = 0_{2\times 1}$) when we calculate the unconditional mean of the bond risk premium. The volatility of the bond is given by $\sqrt{\sigma_B^\top \sigma_B}$.

Second, both stock and bond returns are negatively correlated with inflation innovations. Third, the 10-year bond returns are negatively correlated with its risk premium, which is critical because a long position in the bond can hedge against future changes in investment environment.

A. Asset Allocation without Regulatory Constraints

In this subsection, we temporarily set aside VaR and maximum funding constraints. The *unconstrained* ALM problem has the merits of both simplicity,[5] for it excludes regulatory constraints, and of highlighting the role of various hedging demands. The homothetic CRRA utility removes the wealth (or funding ratio) effect, we therefore normalize the initial funding ratio to $F_0 = 1$ and Table 1 reports the optimal portfolio weights.

Panel A of Table 1 shows that both myopic and dynamic investors with higher risk aversions ($\gamma = 5, 8$) make a substantial allocation to 10-year bonds, reflecting a strong liability-driven hedging demand. This is consistent with findings in van Binsbergen and Brandt (2014). Moreover, relatively to the myopic strategy, the dynamic strategy also includes the intertemporal hedging demand, which allows the dynamic investor to strategically exploit the time-varying investment opportunities. We find that the dynamic investor allocates more fund in bonds, since the mean-reverting nature in bond risk premium lowers the variance of bond returns in the long run. To illustrate, when $\gamma = 8$, there is an intertemporal hedging

[5] We impose the borrowing and short-selling constraints throughout the paper. We refer to the situation here as the *unconstrained* case for ease of exposition. This simplicity should not bring confusion since we do not focus on the effect of borrowing and short-selling constraints.

TABLE 1

Optimal Portfolio Weights for the Unconstrained ALM

Panel A (resp. Panel B) gives the portfolio weights and standard deviations (between brackets) at time 0 for a 10-year nominal (resp. real) ALM under the risk aversion $\gamma = 2, 5, 8$ respectively, ignoring the VaR and maximum funding constraints.

Panel A: Nominal ALM

	Myopic			Dynamic		
γ	Stocks	Cash	Bonds	Stocks	Cash	Bonds
2	0.69 (0.039)	0.00 (0.000)	0.31 (0.034)	0.70 (0.046)	0.00 (0.000)	0.30 (0.044)
5	0.28 (0.016)	0.03 (0.037)	0.69 (0.021)	0.27 (0.034)	0.02 (0.025)	0.71 (0.009)
8	0.17 (0.007)	0.02 (0.049)	0.81 (0.041)	0.13 (0.019)	0.00 (0.000)	0.87 (0.013)

Panel B: Real ALM

	Myopic			Dynamic		
γ	Stocks	Cash	Bonds	Stocks	Cash	Bonds
2	0.70 (0.013)	0.00 (0.000)	0.30 (0.013)	0.73 (0.020)	0.00 (0.000)	0.27 (0.020)
5	0.29 (0.006)	0.03 (0.003)	0.68 (0.006)	0.33 (0.002)	0.04 (0.001)	0.63 (0.003)
8	0.18 (0.004)	0.03 (0.003)	0.79 (0.004)	0.20 (0.003)	0.03 (0.002)	0.77 (0.003)

demand of 6%, calculated by taking the difference of the dynamic and myopic allocation in bonds, implying that in the nominal economy long-term bond offers a positive hedging demand.

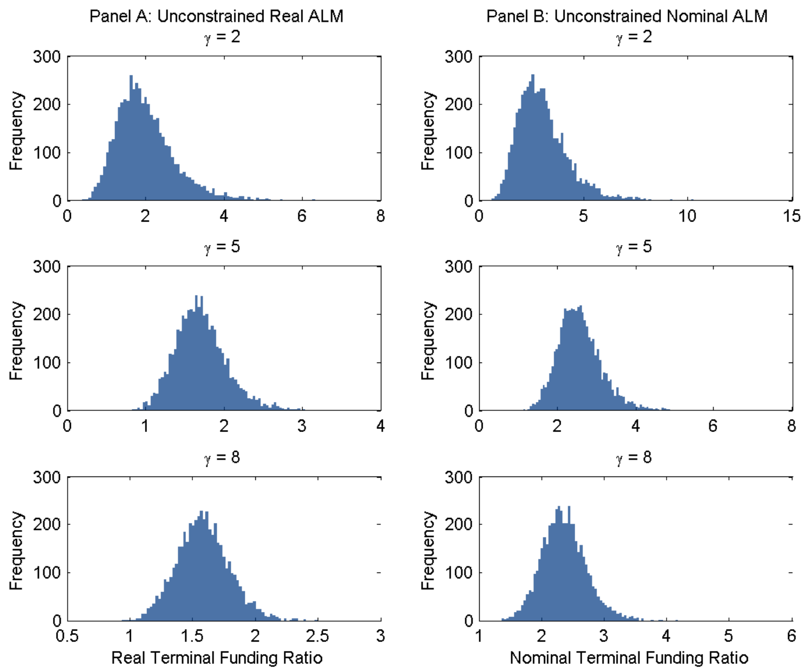
The picture is remarkably different when it is in the *real* economy as shown in Panel B. When taking inflation risk into account, the hedging demand for bonds becomes negative, suggesting that inflation hedging dominates the effect induced by the time variation in investment opportunities. We show that the investor in this case tilts his portfolio towards the equity and the dynamic strategy always holds more stocks than the myopic. As explained in Campbell and Shiller (1988), inflation increases future dividends and thus leads to a boost in stock prices. In our model, both stock and bond returns have negative correlations with inflation innovations, but the stock's Sharpe ratio of 0.35 is higher than the unconditional mean of the bond's Sharpe ratio (0.176). A negative shock in inflation innovation brings a higher positive shock in stock return. In other words, the stock investment can hedge inflation risk more effectively than the bond by offering larger expected returns. Thus, in comparison with the nominal economy, inflation risk leads to an increase (decrease) in the stock (bond) portfolio weight in the dynamic strategy. We also note that the difference of investing myopically between nominal and real economy are rather small. It is because inflation effect in one-step-ahead efficient static portfolio is relatively less significant, while it accumulates when investing dynamically at a longer horizon.

In addition, we examine the distribution of the optimal terminal funding ratio F_T in real and nominal terms respectively. Figure 2 displays that the dispersion of the terminal funding ratio decreases with γ. Indeed, a lower risk aversion implies larger allocation to risky assets for high expected returns, and thus resulting in a larger probability of generating a higher terminal funding ratio. For example, when $\gamma = 2$, the 97.5% quantile of the real (nominal) terminal funding ratio given by the dynamic strategy can be exceedingly high, reaching 3.98 (6.29). This feature provides empirical evidence for imposing a right truncation for the terminal funding ratio distribution, in order to avoid the situation where the investor deliberately overfunds to shelter more taxable corporate profits. The maximum funding

FIGURE 2

The Distributions of the Terminal Funding Ratio

Figure 2 displays the histograms of the terminal funding ratio F_T using the dynamic strategy in real and nominal terms respectively, without the VaR and maximum funding constraints.



ratio constraints required by tax authorities are therefore being considered as well in our optimization problem. The new insight into ALM implied by this feature is that less risk-averse managers are affected by the maximum funding constraint more considerably since the constraint forces them to discard more risky assets.

Overall, these findings provide a first glimpse at the joint impact on the optimal ALM decision driven by inflation, liability and time variation, which we analyze in greater detail in the next subsection. We demonstrate that long-term bonds can best "immunizes" liability risk in both nominal and real economy. More importantly, without considering inflation risk into investor's decision-making problem may result in an overallocation to bonds, which provide a poor hedge against inflation. The results suggest that inflation effect dominates the effect of time variation in the investment opportunities. Conversely, inflation risk affects the unconstrained myopic investor slightly, while later we show that the effect of inflation could turn to be overwhelming if the regulatory VaR constraint is imposed.

B. Asset Allocation with VaR

B.1. Sensitivity to Funding Levels

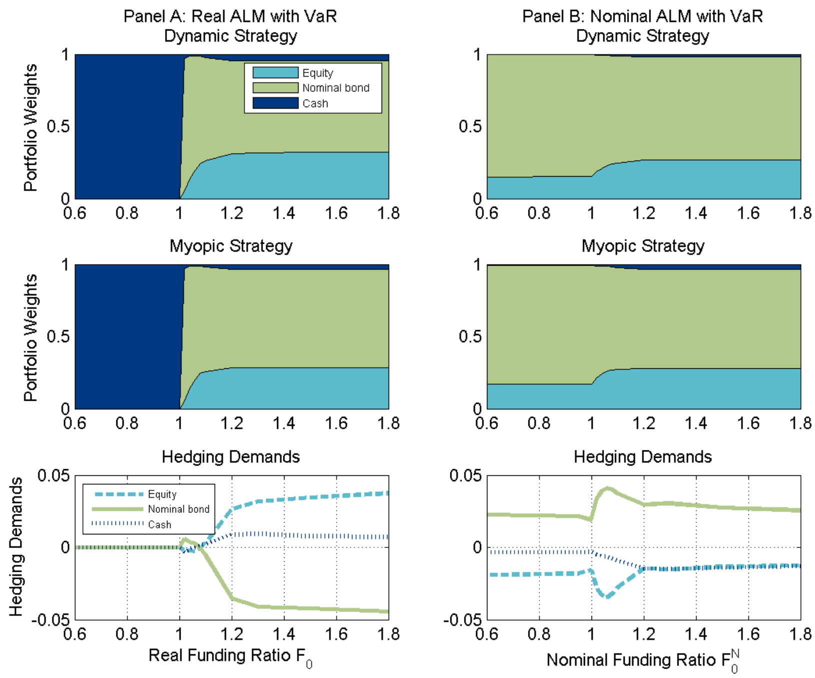
To this point, we have illustrated the optimal investment allocation in the absence of regulatory constraints. We are now ready to reveal the impact of annually-repeated VaR constraints under various scenarios. Figure 3 plots the optimal portfolio weights against different initial funding levels.

Panel A shows the dynamic and myopic investment strategies in the real economy. To begin with, we focus on the under- or fully-funded case ($F_0 \leq 1$). A striking feature is that cash takes the whole portfolio proportion under both investment strategies. Note that, when the initial funding ratio is not greater than one, the VaR constraint requires the probability of $F_1 < F_0$ being no larger than 0.025. In this case, risky assets are not likely to fulfill the VaR requirement by increasing the funding ratio within one period. Given that we do not consider any net additional cash inflows throughout the paper and there is no funding surplus, the manager has no other choice but has to allocate 100% of his fund into riskless cash. This is the adverse result in real economy caused by the horizon disparity between the regulatory horizon (one year) and investment horizon (ten years). The intuition behind this finding is that a shortfall in the initial funding ratio, especially a severe shortfall, could be an indicator of an economic downturn in a gloomy market, such as low interest rate and high

FIGURE 3

ALM Portfolio Choice with VaR

Figure 3 depicts the optimal portfolio weights and hedging demands versus the funding ratio at time 0 for a 10-year ALM with the VaR ($\delta = 0.025$) constraint. Panel A (resp. Panel B) summarizes the dynamic and myopic strategies in the real (resp. nominal) economy under $\gamma = 5$.



inflation rate. If there is no net cash inflow to offset the shortfall, the manager probably has to hold portfolio weights mainly in cash until the financial market recovers. In this sense, constrained by the borrowing and short-selling limits, the fund manager with lower level of real funding ratio has to give up potential high returns from risky assets and choose less risk exposure by holding a large amount of cash for the short-term VaR constraint.

For an over-funded case ($F_0 > 1$), there is a trend of increasing equity allocation in both the dynamic and myopic strategies, although this pattern has been dampened by VaR constraints. For instance, the dynamic investor's stock weight is only 14% (when $F_0 = 1.04$) in Panel A, whereas the weight reaches up to 33% in the unconstrained ALM in Table 1.

Nevertheless, when the fund level increases, the effect of VaR gradually weakens, which reduces the possibility of being constrained by VaR in the future. This situation is referred to as the "less binding"VaR constraints by van Binsbergen and Brandt (2014). We find that the dynamic optimal stock weight is the same as that in the unconstrained setting when the fund has a large surplus, i.e. F_0 is beyond 1.3. To this end, the manager is not restricted by the regulatory rule and chooses his favourable trading policy as if there was no VaR constraint.

We complement the literature by investigating hedging demands in real terms across different initial funding levels. Comparing the dynamic and myopic strategies in Panel A shows that there is an up to 4% intertemporal hedge demand for stocks. In fact, this positive intertemporal hedging demand is largely induced by the positive inflation-hedging demand for the stock as discussed in Section A. By contrast, the dynamic strategy has negative intertemporal hedging demands for the bond in real terms. It is opposite to the finding in nominal terms, as discussed in van Binsbergen and Brandt (2014). The negative hedging demands can be is explained by the interplay driven by three types of hedging demands. First, the inflation-hedging demand raises stock holdings while at the same time reduces the bond allocation in the dynamic dynamic; second, the liability-hedging demand makes the myopic investor to invest largely in the bond, aiming at hedging against liability risk; third, the effect of liability risk on the dynamic manager is partly offset by the positive inflation-hedging and intertemporal hedging demands for the stock. The higher expected returns of stock in the time-varying investment environment will provide a reliable funding status in the long run and further reduces the dynamic investor's concern about liability risk, which then diereses the bond allocation.

Collectively, the negative hedging demand for the bond when $F_0 > 1$ under the real economy reveals the following new insight into the ALM. Hedging liability risk is crucial for the myopic investor, while the dynamic investor can partly offset the liability effect by using positive hedging demands for the stock to dynamically hedge inflation risk, and meanwhile to exploit the time-varying investment opportunity for higher gains.

Presenting the graphs side by side, our strategy to reveal the inflation effects under VaR-constrained portfolio choice is to compare the results under the real and nominal economy presented in Panel A and Panel B of Figure 3. We find that, without introducing inflation, the stock demand by dynamic investor in Panel B is less than that in the real economy in Panel A, since there is no incentive to hedge inflation in the nominal setting. Consequently, liability effect dominates again and the dynamic investor's intertemporal hedging demands turn out to be long in the bond and short in the stock. In addition, both the dynamic and myopic managers allocate a large amount of their wealth between the long-term bond and stock for relatively high nominal returns. In this way, they maintain a large probability of being over funded or having an increase in funding to satisfy the VaR requirement. In terms of the regulatory effect, our investment strategy in the nominal economy is similar to the finding of van Binsbergen and Brandt (2014) who argue that the dynamic investor has to be away from his preferred weight in the stock because of VaR constraints.

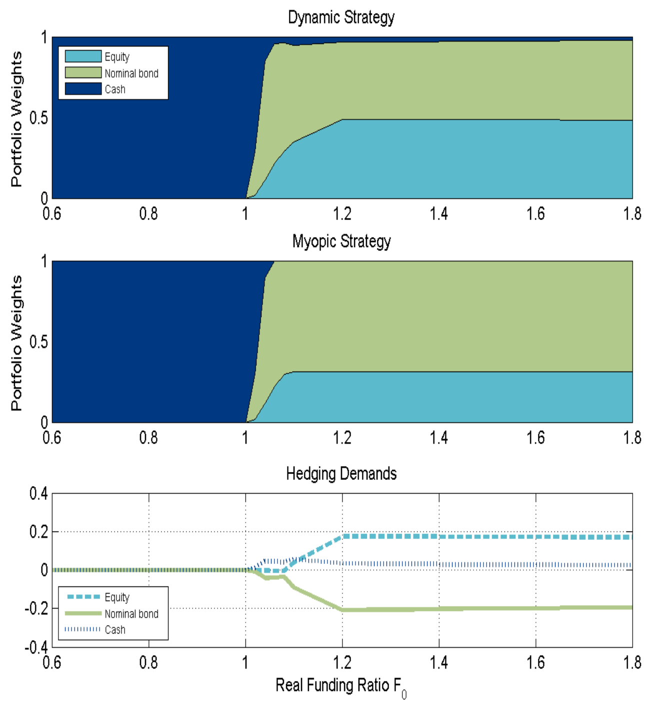
B.2. Sensitivity to Time Variation

Admittedly, the magnitude of hedging demands in previous case is rather small, generally less than 5%. We argue that it is due to the time variation in the dynamic strategy and the liability hedging in the myopic strategy. We are now set the bond risk premium to be its

FIGURE 4

ALM Portfolio Choice with VaR and Constant Bond Risk Premium

Figure 4 depicts the optimal portfolio weights and hedging demands versus the funding ratio at time 0 for a 10-year ALM with the VaR and the constant bond risk premium. The results are derived in the real economy under $\gamma = 5$. The main text provides further details.



unconditional mean 2.06% ($X_t = 0_{2\times1}$). To this end, the risk premia in stock and bond are constants in this case. Figure 4 shows the adjustment to the optimal strategies for the ALM problem.

In comparisons of the time-varying bond risk premium, the bond with the constant risk premium in this case becomes less appealing to the dynamic investor, resulting in a drop of 15% from 65% in Panel A of in Figure 3. Intuitively, the negative correlation (-0.275) between the bond return and bond risk premium implies that if there is a drop in bond risk premium in the current period, then the return on bond is expected to increase. Given this stylized fact, the dynamic investor substantially raises bond holdings to take advantage of the time-varying bond risk premium as shown in Figure 3 and in the asset-only setting by Sangvinatsos and Wachter (2005). Nevertheless, the constant bond premium provides the dynamic investor very limited opportunity to strategically exploit the time variation, and thereby tilting the portfolio weight to the stock for a higher risk premium (5.38%). As a consequence, this change produces relatively larger intertemporal hedging demands for bonds and stocks. We therefore conclude that in the framework of the ALM, time-varying opportunities in long-term bond market would lead the asset allocation nearly indifference between the dynamic and myopic trading strategies, indicating that in the presence of liability risk, myopically optimal strategy might be preferred by the investor.

B.3. Sensitivity to Liability Risk

To see more exactly how liability alters investor's allocation, our strategy is to compare the portfolio choice of an ALM investor to that of a traditional asset-only investor. In the latter case the investor optimizes his utility over the terminal wealth instead of the funding ratio, and at the same time, his investment behavior is limited by annual VaR constraints with the "floor" $\underline{w} = 1$ and probability $\delta = 0.025$. More specifically,

$$\begin{aligned} &\mathrm{VaR}_w: \quad \mathbb{P}_t(w_{t+1} < \underline{w}) \leq \delta, \\ &\mathrm{VaR}'_w: \quad \mathbb{P}_t(w_{t+1} < w_t | w_t \leq \underline{w}) \leq \delta. \end{aligned} \tag{25}$$

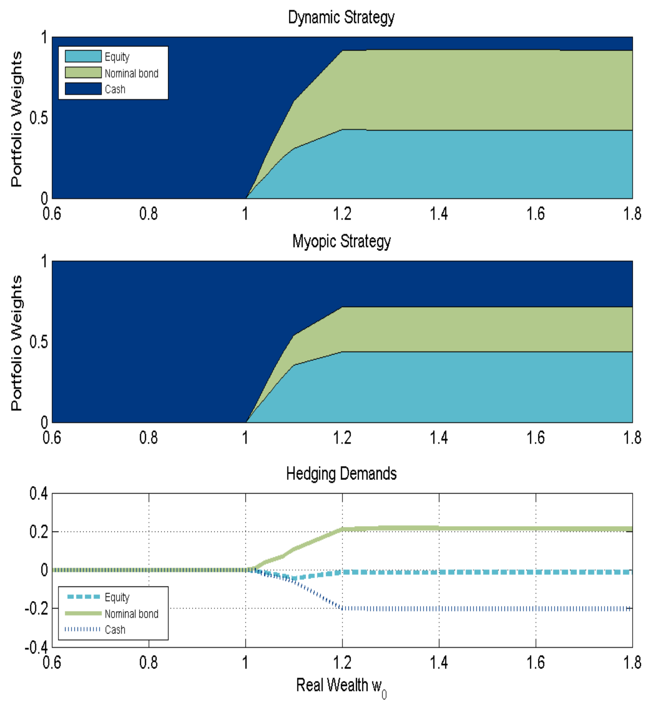
Then the asset-only investor's objective is to maximize the expected utility of terminal wealth

$$J(w, X, t, T) = \max_{\theta_t \in \mathcal{K}_t^w} \mathbb{E}\left(\frac{w_T^{1-\gamma}}{1-\gamma}\right), \tag{26}$$

FIGURE 5

Asset-Only Portfolio Choice with VaR

Figure 5 depicts the optimal portfolio weights and hedging demands versus different initial wealth levels for a 10-year asset-only problem with VaR constraints. The results are derived in the real economy under $\gamma = 5$. The main text provides further details.



subject to the constraint set $\mathcal{K}_t^w$

$$\mathcal{K}_t^w = \{(\theta, w) : \theta_t \geq 0, 1 - \imath^\top \theta_t \geq 0, \mathrm{VaR}_w \text{ or } \mathrm{VaR}'_w\}. \tag{27}$$

Figure 5 plots the optimal portfolio weights in real terms and hedging demands at time 0 when $\gamma = 5$.

Recall that in the presence of liability the fund has a large bond component as shown in Figure 3. However, the asset-only investor in Figure 5 allocates much lower in long-term bonds, particularly in myopic strategy. As a result, the hedge demand for long-term bonds increases significantly from -4% in Figure 3 to 20% here. Similar results can be also found in

Sangvinatsos and Wachter (2005) who study predictability in bond returns in the asset-only framework without constraints. They claim that time-varying risk premia generate large hedging demands for long-term bonds.

Another noticeable feature in Figure 5 is that when the bond allocation decreases, cash serves as an effective takeover and makes the portfolio well-diversified. Note that cash in the financial market that we consider provides an average 2% real return, which is roughly equal to the real bond yield, with much higher liquidity and lower risk. Hence, asset-only managers, especially those who behave myopically and/or those with shortfalls in wealth, tilt the optimal portfolio weights to riskless asset, which leads to considerable negative intertemporal hedging demands for cash in the dynamic strategy.

Interestingly, it is noted that ignoring liability leads the asset-only investor to be less concerned about inflation risk. Given the fact that stocks serve as a good inflation hedge with a positive intertemporal demand in Panel A of Figure 3, we find the hedging demand for stock is nearly zero across different initial wealth, implying that liability might be one of the main driving force for inflation hedging.

C. Asset Allocation with VaR and Maximum Funding Limits

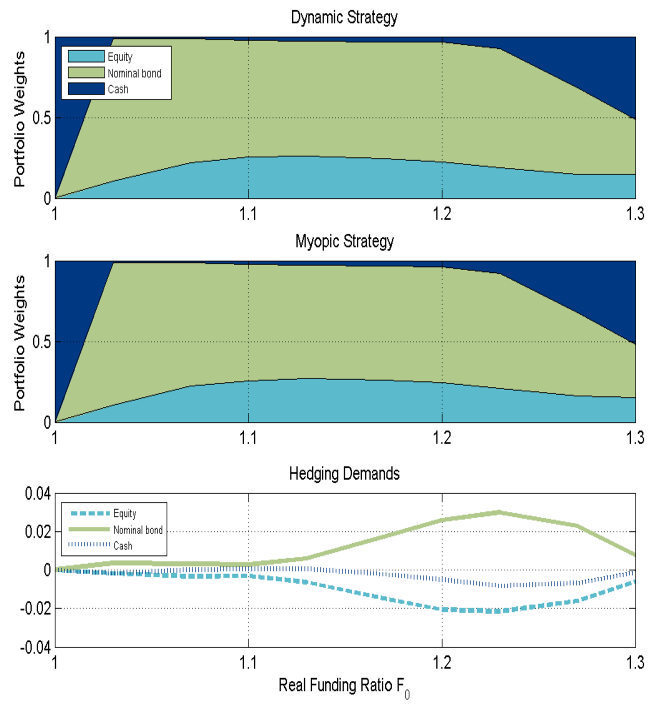
Up to this point, we have illustrated the impact of VaR on portfolio allocation under various settings. Next we examine the joint effect of regulatory requirements by introducing the maximum funding constraint, where the upper-bound level of funding ratio is $k = 130\%$.[6]

[6] We consider the upper-bound $k = 130\%$ since, as indicated by our results, the VaR constraint will not bind beyond this level. This bound is also in line with the practice in, e.g., the Netherlands, see Pugh (2006).

FIGURE 6

ALM Portfolio Choice with VaR and Maximum Funding Constraints

Figure 6 depicts the optimal portfolio weights versus the funding ratio at time 0 for a 10-year ALM with the VaR ($\delta = 0.025$) and maximum funding ratio constraints. The results are derived in the real economy under $\gamma = 5$. The main text provides further details.



To emphasize the effect of this constraint, we focus on the case with the initial funding ratio F_0 within the domain [1, 1.3], i.e. the fully-/over-funded case. Intuitively, the investor with a moderate risk aversion ($\gamma = 5$) would not build up an excess fund higher than the upper bound at the initial stage, otherwise he has to carry out some corrective actions required by the tax authorities.

Figure 6 demonstrates that the allocation to the bond and stock are hump-shaped when the maximum funding ratio constraint is introduced along with VaR. For instance, the optimal allocation to the stock reaches its peak at 27% when $F_0 = 1.13$ and then the weight is lowered by the manager to about 14% when $F_0 = 1.3$. At the same time, cash holdings go up dramatically, especially in the nominal economy. The hump-shaped allocations are attributed to the weakening effect of VaR and the strengthening effect of maximum funding constraint when F_0 rises. Namely, the maximum funding constraint is binding gradually to prevent the manager from building up an exceedingly large surplus. The new insight into the ALM with such constraints is that the manager should reduce the allocation to risky assets with higher expected returns and meanwhile increase cash holdings in order to avoid the penalty from tax laws.

The hump-shape (U-shape) hedging demand for the bond (stock) reveals our new insight into the ALM that the myopic investor reduces the portfolio weight of bond more than the dynamic investor the under maximum funding constraint. Interestingly, these hedging demands indicate that under such a constraint, the dynamic and myopic strategy tend to be coincide to each other when the funding level is high. Hence, we conjecture that such regulatory rules could even entirely remove the gains of dynamic strategy by limiting the dynamic investor's ability to exploit time-variation in the investment opportunity set. The following analysis on the welfare loss of acting myopically confirms our conjecture.

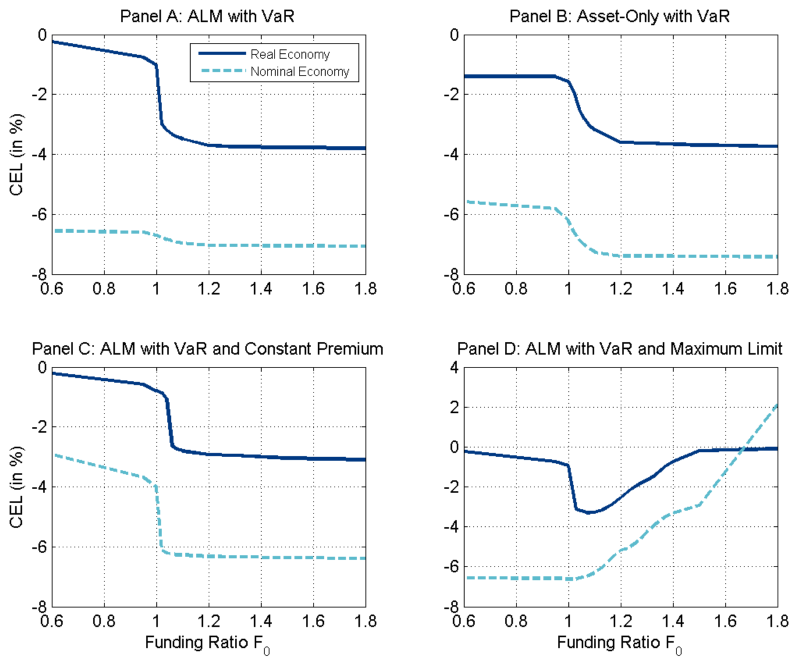
D. Welfare Losses under Real and Nominal Economies

We examine the welfare loss of behaving myopically in the presence of VaR and maximum funding constraints, and address the effect of inflation on the economic value. We evaluate such welfare loss of deviating from the dynamic trading strategy by calculating the annualized certainty equivalent loss (CEL) of the myopic strategy. We denote that, at time 0, the

FIGURE 7

Certainty Equivalent Loss of Behaving Myopically

Figure 7 depicts the certainty equivalent loss of behaving myopically under four scenarios when $\gamma = 5$. The vertical axes depicts the CEL in percentage per year. The main text provides further details.



value function resulting from the optimal policy by J_1 and that from the suboptimal by J_2. Following Koijen et al. (2009), the annualized CEL is given by

$$CEL = \left(\frac{J_2}{J_1}\right)^{1/(T(1-\gamma))} - 1, \tag{28}$$

where T is the investment horizon in years. Under different scenarios we plot the CEL of behaving myopically in Figure 7 when $\gamma = 5$. The results for other degrees of risk aversion are similar.

When we look at the results for the VaR-only constraint in Panel A, B, and C, the welfare gain to the dynamic strategy rises, i.e. the myopic investor's absolute value $|CEL|$ of welfare loss enlarges, when the funding ratio increases in both the real and nominal economies. The reason is that the restriction of VaR on the dynamic investor's access to the time-varying investment opportunities is becoming weak when the fund ratio goes up. This finding is in line with van Binsbergen and Brandt (2014) in the nominal economy.

Furthermore, we find the losses of behaving myopically in the real economy are smaller than those in the nominal economy. For example, the manager in Panel A with the initial funding ratio $F_0 = 1$ would incur a utility loss of 1% (6.3%) per annum in the real (nominal) economy, if he gave up exploiting time-variation in the investment opportunity set. The less welfare loss in the real economy implies our new insight that taking account of inflation risk into the ALM adds values to the myopic investor.

Put differently, inflation-hedging under the real economy restricts more benefits in the dynamic strategy. Intuitively, the accumulated effects of inflation risk in the long run could damage payoffs to the dynamic investor. This result is attributed to our aforementioned fundamental finding that inflation risk leads to an increase (decrease) in the dynamic investor's intertemporal hedging demand for the stock (bond). Therefore, the dynamic investor cannot fully exploit time-variation in the bond risk premium because of the raising stock holdings. In short, the new insight is that the manager loses more benefits of dynamic strategy when he encounters inflation risk.

More importantly, when the maximum funding constraint is imposed along with the VaR constraint, the myopic investor experiences U-shaped welfare losses in both the real and nominal economies, as shown in Panel D. Surprisingly, the myopic strategy could even provide more benefits than the dynamic strategy, i.e. the welfare loss turns to a positive profit $CEL > 0$, when the initial funding ratio is highly over funded more than 1.7%. It indicates that a relatively tight upper-bound limit would lead the manager has to behave myopically and be less efficient with respect to changes in the opportunities set. Our insight provides an argument for justifying some current practices of using a myopic allocation policy and ignoring time-varying opportunities in the presence of strict regulatory rules.

V. Conclusion

Inflation risk and regulatory rules feature dominantly in an asset-liability management problem, but they have been largely ignored in the financial literature. In this paper, we analyze the optimal risk exposure for an ALM optimization problem in the presence of interactive hedging demands driven by inflation, liability and the time variation in the investment opportunity set. The fund manager chooses the optimal portfolio allocation with respect to the annually repeated VaR and maximum funding constraints in a two-factor affine model of term structure.

Our study provides the fund manager with new insights into the ALM problem. First, long-term bonds can best immunizes liability risk in both nominal and real economy. More importantly, without considering inflation risk into investor's decision-making problem may result in an overallocation to bonds, which provide a poor hedge against inflation. The results suggest that inflation effect dominates the effect of time variation in the investment opportunities. Second, the regulatory constraints remarkably reduce the welfare of dynamic strategy, especially under inflation risk. Furthermore, under the strict maximum funding constraint, the profit of dynamic strategy even completely vanishes and the myopic strategy of asset allocation is favored.

Given the novelty of our analysis, we believe there are several promising directions for future research. While we assume that the lower and upper thresholds of constraints are given exogenously by the regulators, it would be valuable to investigate an equilibrium that allows the risk-taking manager to exploit time variations in the market as much as possible, and meanwhile, maintains the funding ratio in a desirable range. It would also be of interest to consider a setting where managers are specialists, each of whom manages one asset class, instead of well-balanced managers who manage all assets. Moreover, these specialists compete each other. In practice, managers tend to allocate assets based on the performance of peer colleagues, because of career concerns, reputations or their annual bonuses. Analyzing how different such managerial agency behavior is would be worthwhile.

Appendix A Parameter Values & Numerical Method

TABLE 2

Parameter Values and Asset Return Correlations

The parameter values are taken from Koijen et al. (2010) in annual terms. $\Lambda_{0(3)}$ and $\Lambda_{1(3,:)}$ are set to zeros for technical reasons. $\Lambda_{0(4)}$ and $\Lambda_{1(4,:)}$ are calculated from $\sigma_S'\Lambda_0 = \eta_S$ and $\sigma_S'\Lambda_1 = 0_{1\times 2}$.

Expected inflation	$\delta_{0\pi}$	4.20%	Nominal interest rate	δ_{0R}	5.89%
	$\delta_{1\pi(1)}$	1.69%		$\delta_{1R(1)}$	1.92%
	$\delta_{1\pi(2)}$	0.50%		$\delta_{1R(2)}$	1.03%
Term structure factor	$\kappa_{(1,1)}$	0.687	Price level	$\sigma_{\Pi(1)}$	0.02%
	$\kappa_{(2,1)}$	-0.350		$\sigma_{\Pi(2)}$	0.11%
	$\kappa_{(2,2)}$	0.172		$\sigma_{\Pi(3)}$	0.98%
Equity	η_S	5.38%	Price of risk	$\Lambda_{0(1)}$	-0.293
	$\sigma_{S(1)}$	-1.98%		$\Lambda_{0(2)}$	-0.158
	$\sigma_{S(2)}$	-1.79%		$\Lambda_{1(1,1)}$	-0.103
	$\sigma_{S(3)}$	-1.74%		$\Lambda_{1(1,2)}$	-0.101
	$\sigma_{S(4)}$	14.82%		$\Lambda_{1(2,1)}$	0.504
				$\Lambda_{1(2,2)}$	-0.168

Correlations	Stock return	10-year bond return
10-year bond return	0.125	1
Risk premium on 10-year nominal bond	0.086	-0.275

Numerical Method To solve the complicated and path-dependent dynamic portfolio optimization problem with constraints, we use the simulation-based method proposed by Brandt et al. (2005). The main idea of their method is to approximate the conditional expectations by regressing the variables of interest on a polynomial basis of state variables. The regression coefficients are estimated by cross-sectional regressions across simulated paths of asset returns and state variables.[7]

To begin with, we construct a grid of the funding ratio indicated by $F_t(j), j = 1, \dots, M$, since the optimal portfolio weights depend on the given values of funding ratio. We also discretize the constrained space $[0, 1] \times [0, 1]$ for portfolio weights $\theta_{t,s}(i)$-grid, where $i = 1, \dots, L$, and risky assets $s = 1, 2$. We then choose the optimal portfolio weights in each period and along each path by searching over this $\theta_{t,s}(i)$-grid. The grid-search is robust and avoids several convergence issues that could occur when iterating to a solution based on the first-order conditions (van Binsbergen and Brandt 2007). We simulate $N = 5,000$ paths of asset returns and state variables with T years, and repeat this optimization process 10 times. Raising the number of paths and simulations does not improve the accuracy of our results at a reported level. We solve the dynamic optimization problem backward recursively and rebalance the portfolio at an annual basis. The following description is similar to van Binsbergen and Brandt (2014).

•**Time T-1:** The investment problem at time $T - 1$ can be summarized as

$$\text{(A.1)} \qquad \max_{\theta_{T-1} \in \mathcal{K}_{T-1}} \mathbb{E}_{T-1} \left(\frac{F_T^{1-\gamma}}{1-\gamma} | F_{T-1} = F_{T-1}(j) \right).$$

[7] This approach is inspired by the simulation method for pricing American options in Longstaff and Schwartz (2001).

For each given value of $F_{T-1}(j)$ and each point $\theta_{t,s}(i)$, we take simple OLS regression of N simulated $F_T^{1-\gamma}/(1-\gamma)$ on a basis function $f(X_{T-1})$ defined as

$$f(X_{T-1}) = \begin{bmatrix} 1 & X_{1,1} & X_{2,1} & (X_{1,1})^2 & (X_{2,1})^2 & (X_{1,1})(X_{2,1}) \\ 1 & X_{1,2} & X_{2,2} & (X_{1,2})^2 & (X_{2,2})^2 & (X_{1,2})(X_{2,2}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & X_{1,N} & X_{2,N} & (X_{1,N})^2 & (X_{2,N})^2 & (X_{1,N})(X_{2,N}) \end{bmatrix}_{T-1}.$$

We then have

$$\mathbb{E}_{T-1}\left(\frac{F_T^{1-\gamma}}{1-\gamma}|F_{T-1}\right) \simeq \alpha(\theta_{T-1,s}(i), F_{T-1}(j))^\top f(X_{T-1}), \tag{A.2}$$

where $\alpha(\theta_{T-1,s}(i), F_{T-1}(j))$ is the regression coefficient. The fitted value of this regression provides the approximated conditional expectations. We then repeat this procedure for a grid search of portfolio weights $\theta_{T-1,s}(i)$, and collect the optimal weights that maximize the expected utility in each path for each $j \in M$.

•**Time t = T-2, . . . , 0:** Suppose we have stored the mapping from F_{t+1} to the optimal weight $\theta^*_{t+1,s}$. Then we iterate backwards through time and follow the similar procedure as $T-1$ with some additional work. The addition work at $T-2$ is described as an example and it applies to $T-3, T-4, \ldots, 0$. For a portfolio return at $T-1$, one can compute the portfolio return from $T-1$ to T by interpolating over the mapping from F_{T-1} to $\theta^*_{T-1,s}$ stored at the previous step. Finally, the backward recursion results in the optimal portfolio weights at all $N \times M$ grid points at each time point.

For the regulatory constraints, we estimate the conditional mean and variance of funding ratio at $t+1$ for each given value of $F_t(j)$ and each $\theta_{t,s}(i)$, by assuming log-normally distribution. We evaluate whether the probability of F_{t+1} being underfunded is less than δ.

Those portfolio weights that do not satisfy this VaR requirement (or the maximum funding constraint) will be excluded from the investor's choice set.

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