

# Netspar DISCUSSION PAPERS

*Andrew Ang, Bingxu Chen and Suresh Sundaresan*  
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Downside Risk**

# Liability Driven Investment with Downside Risk\*

Andrew Ang

Bingxu Chen

Suresh Sundaresan

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## **Abstract**

We develop a liability driven investment framework that incorporates downside risk penalties for not meeting liabilities. The shortfall between the asset and liabilities can be valued as an option which swaps the value of the endogenously determined optimal portfolio for the value of the liabilities. The optimal portfolio selection exhibits endogenous risk aversion and as the funding ratio deviates from the fully funded case in both directions, effective risk aversion decreases. When funding is low, the manager “swings for the fences” to take on risk, betting on the chance that liabilities can be covered. Over-funded plans also can afford to take on more risk as liabilities are already well covered and so invest aggressively in risky securities.

# 1 Introduction

The fall in both interest rates and equity prices during the global financial crisis over 2007-2008 took a large toll on pension funds. Despite the rebound in asset values since 2009, funding ratios (asset values / projected benefit obligations) have not yet rebounded. In 2011, the funded status of the top 100 U.S. companies with the largest defined benefit pension assets was around 80% compared to above 100% in 2007 and was approximately 400 billion dollars lower than at year-end 2007.<sup>1</sup> The average funding ratios were close to 80% for these funds over 2009-2011. These downside risks are costly both for corporations, which need to pay higher insurance premiums, hold higher reserves, and transfer money to pension plans that could be used for other investments, and beneficiaries, who must bear higher default risk which is often highly correlated with their main source of labor income.<sup>2</sup>

We present a liability-driven investment (LDI) approach to take into account downside risk. The approach is different from the surplus management approach developed by Sharpe and Tint (1990), Ezra (1991), and Leibowitz and Kogelman and Bader (1992) as we include a penalty term associated with not meeting liabilities.<sup>3</sup> Downside risk has a large effect on pension investments: optimal asset allocation is affected both by a current shortfall, when the value of the pension assets falls below the liabilities today, but also by the risk of a potential shortfall in the future. We show that the shortfall penalty can be valued as an option to exchange the optimal portfolio for the random value of the liabilities. The optimal portfolio, however, must be solved *simultaneously* with the value of the option. When the liabilities exceed the assets, the option is in the money. A cost factor parameter which multiplies the value of the exchange option in the manager's utility function can be interpreted as a downside risk aversion parameter. As the cost factor decreases to zero, the standard mean-variance framework holds.

The downside risk we address here is the failure of meeting liability. This is different from just taking into account liabilities in standard Sharpe-Tint surplus management. There are significant penalties in failing to meet liabilities in the real world. The 2006 Pension Protection Act requires that plan funding should equal 100% of the plan's liabilities. Sponsors of severely

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<sup>1</sup> See Milliman 2012 Pension Funding Study.

<sup>2</sup> See, for example, Rauh (2006) for evidence that higher than expected contributions to pension plans reduces firm investment and Poterba (2003) on the excessive concentration of employer stock in pension plans.

<sup>3</sup> Some notable contributions in the area of optimal asset allocation of pension funds are Sundaresan and Zapatero (1997), Rudolf and Ziemba (2004), and van Binsbergen and Brandt (2009).

underfunded plans are required to fund their plans according to special rules that result in higher employer contributions to the plan. In addition, FAS 158, implemented in 2006, requires plan sponsors to “flow through” pension fund deficits into their financial statements. These have real impacts on earnings and stockholders’ equity. A case study is AT&T whose funding status changed from \$17 billion surplus in 2007 to a nearly \$4 billion dollar deficit in 2008. This played a role in the decline of AT&T’s equity from 2007 to 2008.

Taking into account downside risk leads to *endogenous* risk aversion. The funding ratio affects the likelihood of the assets being sufficient to cover the liabilities in the future, which affects the option value. There are pronounced non-linear effects of the funding ratio on risk taking. The fund manager’s risk aversion peaks when the plan is close to fully funded. As the funding ratio deviates from the fully funded position, risk aversion decreases. An under-funded plan investment manager displays much lower risk aversion than the manager of a fully funded plan leading to a “swing for the fences” effect. If the fund is poorly funded, then only by taking on risk can the manager hope to avoid the shortfall. Managers of over-funded plans also act in a less risk averse manner because they can afford to take on more risk as the probability of the option being exercise falls as the funding ratio increases.

Our framework of LDI with downside risk is related to a portfolio choice literature that specifies drawdown constraints, such as Grossman and Zhou (1993) and Chekhlov, Uryasev and Zabarankin (2005). Constraints that capture shortfall risk have also been employed in surplus optimization problems by Leibowitz and Henriksson (1989), Jaeger and Zimmermann (1996), Amenc et al. (2010), and Berkelaar and Kouwenberg (2010). These approaches do not directly take into account the downside risk in the utility function of the manager, but instead specify a constraint that the surplus or the portfolio needs to satisfy. This constraint is usually that the surplus or portfolio return must be above a certain threshold with some probability. As an extension of the Sharpe and Tint (1990) to a dynamic setting, Detemple and Rindisbacher (2008) allow for a fund sponsor to exhibit aversion over a shortfall when a plan terminates. Their shortfall has a utility cost, whereas ours has an actual real-world value through an option calculation. Our shortfall cost is determined simultaneously with the optimal portfolio.

## 2 Model

LDI models treat fund liabilities as a state variable and specify an objective function of assets relative to liabilities. The investor takes into account the correlation between the liabilities and assets in determining the optimal portfolio allocation. We start by reviewing the simple LDI model of Sharpe and Tint (1990). Then we present our model with downside risk and show how to value the shortfall risk as an option.

### 2.1 Sharpe and Tint (1990)

Sharpe and Tint (1990) define surplus,  $S_t$  to be  $S_t = A_t - L_t$ , where  $A_t$  represents the plan's market value of assets and  $L_t$  is the value of the liabilities. Normalizing by the assets at the beginning of the period, we can define the surplus return over assets,  $z$ , as

$$z = \frac{S_1}{A_0} = \frac{A_1 - L_1}{A_0} = \left(1 - \frac{L_0}{A_0}\right) + \left(r_A - \frac{L_0}{A_0}r_L\right), \quad (1)$$

where  $r_A = A_1/A_0 - 1$  is the return on assets,  $r_L = L_1/L_0 - 1$  is the return on the liabilities, and  $L_0/A_0$  is the inverse of the funding ratio.

The objective function is mean-variance over the surplus return:

$$\max_w E(z) - \frac{\lambda}{2} \text{var}(z), \quad (2)$$

where  $w$  is the portfolio of risky assets and  $\lambda$  is (standard) risk aversion in the mean-variance context. Sharpe and Tint show that this problem is equivalent to

$$\max_w E(r_A) - \frac{\lambda}{2} \text{var}(r_A) + \lambda \text{cov}(r_A, r_L), \quad (3)$$

which emphasizes that the correlation of the liabilities with the asset returns influences the optimal portfolio holdings.

If the assets are uncorrelated with the liabilities,  $\text{cov}(r_A, r_L) = 0$ , then the surplus problem in equation (3) is the standard mean-variance portfolio weight. Sharpe-Tint LDI does take into account downside covariance of assets and liabilities but it does this symmetrically with upside covariance through the  $\text{cov}(r_A, r_L)$  term. In our formulation, we will explicitly penalize *only*

shortfall loss.

## 2.2 Liability Driven Investment with Downside Risk

We now introduce a LDI framework to include a penalty if the manager fails to meet the liability of the fund.

### Asset Returns

We assume that the portfolio managers can only allocate wealth between two assets, risky equities ( $E$ ) and risk-free bonds or cash ( $B$ ). We analyze the case of risky bonds in the appendix and also investigate asset allocation over risky equities and bonds in our empirical calibration. We denote the liabilities by  $L$ .

We assume a log normal process for equities and denote the risk-free rate as  $r_f$ :

$$\begin{aligned}\frac{dB}{B} &= r_f dt \\ \frac{dE}{E} &= \left(\mu - \frac{\sigma_E^2}{2}\right)dt + \sigma_E dW_t^E\end{aligned}\tag{4}$$

We also assume a log normal process for the liabilities:

$$\frac{dL}{L} = \left(\mu_L - \frac{\sigma_L^2}{2}\right)dt + \sigma_L dW_t^L.\tag{5}$$

The correlation between the diffusions of the risky asset and the liability is specified as  $\rho dt = [dW_t^L, dW_t^E]$ .

### Liability Shortfall

Following Sharpe and Tint (1990), we work in a one period setting and assume the asset weights are set at the beginning of the period, which we interpret as one year. The value of the assets at time 0 is denoted as  $A_0$ . The asset payoff at time 1 is a function of the equity weight,  $w$ :

$$A_1 = wA_0 \exp\left(\left(\mu - \frac{\sigma_E^2}{2}\right) + \sigma_E W_1^E\right) + (1 - w)A_0 \exp(r_f).\tag{6}$$

Note that  $w$  is chosen at time 0.

The value of the liabilities at time 1,  $L_1$ , is:

$$L_1 = L_0 \exp \left( \left( \mu_L - \frac{\sigma_L^2}{2} \right) + \sigma_L W_1^L \right). \quad (7)$$

The value of the shortfall is a put option on the terminal value of the assets at a strike price of  $L_1$ , which is unknown at time 0. The payoff of this option is

$$\max(L_1 - A_1, 0),$$

where  $A_1$  and  $L_1$  are given in equations (6) and (7), respectively. We denote the value of this option as  $P(w, L_0, A_0)$ . The notation emphasizes that the downside risk depends on the original funding level given by  $L_0$  and  $A_0$  and it also depends on the asset allocation policy,  $w$ , chosen by the fund manager.

### Downside Liability Risk

We specify the objective function of the fund as mean-variance over asset returns plus a downside risk penalty on the liability shortfall:<sup>4</sup>

$$\max_w E(r_A) - \frac{\lambda}{2} \text{var}(r_A) - \frac{c}{A_0} P(w, L_0, A_0), \quad (8)$$

where  $c$  is a penalty cost associated with the downside risk. The parameter  $c$  can be interpreted as a downside risk aversion parameter in the context of shortfall loss. We scale the funding cost by assets to keep everything on a per dollar return metric. As the shortfall risk increases on the downside, the investor's utility is decreased. It is important to note that the option price,  $P$ , is the value of the shortfall risk at time 0; it is the value the fund manager would pay today to insure against the shortfall risk tomorrow.

The standard Sharpe-Tint (1990) LDI framework recognizes the fact that the correlation of assets with liabilities plays a role in driving optimal asset allocation. Our objective function (8)

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<sup>4</sup> This defines the downside risk limit as a funding ratio of 100%. The 2006 Pension Protection Act (PPA) aims at a minimum funding ratio of 100% and in cases of underfunding requires shortfalls be amortized and increases in contributions over certain horizons. Other countries, however, have other minimum levels of funding such as the Netherlands where the minimum funding level is 105%. Plan sponsors may consider other terminal funding levels for defining the strike of the option. It is also possible to extend the methodology to introduce a second option with an additional penalty at another strike. For example, under the PPA a fund is deemed "at-risk" and subject to onerous restrictions and steep contribution increases if the funding ratio falls below 80%.



for LDI with downside risk replaces the Sharpe-Tint  $\lambda \text{cov}(r_A, r_L)$  term with a shortfall penalty term,  $cP/A_0$ . The value of the option is *endogenous* as the fund manager can reduce the value of this option by increasing the correlation of the optimal portfolio with the pension liabilities. Thus, the value of the insurance must be computed *simultaneously* with the optimal portfolio choice.

It is possible to extend equation (8) to include an additional term  $\lambda \text{cov}(r_A, r_L)$ . This would then nest the traditional Sharpe-Tint (1990) surplus optimization (see equation (3)). We do not take this route because by including the  $\text{cov}(r_A, r_L)$  term, we “double count” the effect of downside risk in both the covariance and the put option term. We purposely highlight the shortfall penalty in equation (8) to distinguish it from traditional Sharpe-Tint analysis in our calibrations below.

In their appendix, Amenc et al. (2010) in their appendix consider the related problem

$$\max_w E \left[ \frac{A_1}{L_1} \right] \quad \text{such that } A_t \geq kL_t \text{ for all } t,$$

which is similar to equation (8) in that the portfolio problem also involves an option. A major difference is that in our formulation the option valuation endogenously depends on the optimal portfolio strategy, and the optimal strategy simultaneously depends on the cost of the shortfall risk. In Amenc et al., the option value is stated *exogenously*.

## 2.3 Valuing the Shortfall Risk

The value of the shortfall risk is a put option. The shortfall process, however, does not follow a log-normal process and so we cannot use standard methods to value it. We can interpret the shortfall option as a spread option due to the stochastic evolution of both pension assets and pension liabilities. That is, we can write the payoff of the put option as

$$\max(L_1 - A_1, 0) = \max(S_{1,1} - S_{2,1} - K, 0),$$

which takes the form of a spread option on assets  $S_1$  and  $S_2$  with strike  $K$ . The assets are

$$\begin{aligned} S_{1,t} &= L_t \\ S_{2,t} &= wA_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t^E\right) \end{aligned}$$

with the strike  $K = (1 - w)A_0 \exp(r_f)$ . While the literature on spread options has not found a closed-form solution for valuation, there are very accurate approximations. The appendix shows how we can value the spread option following Alexander and Venkatramanan (2011) by representing the spread option as two compound exchange options.

The important economic concept is that the option value *endogenously* depends on the portfolio chosen by the pension plan. As the correlation between portfolio and the liabilities increases, the likelihood that there will be a shortfall decreases and the value of the option decreases. As the volatility increases, the option value increases. This property implies that when the penalty costs associated with downside risk increase, the optimal portfolio becomes more defensive. The correlation of the liabilities with the asset returns influences the portfolio weights, just as in a standard LDI problem, but the comovement of liabilities with assets also directly affects the risk of not meeting the liability schedules. As the penalty costs of downside risk increase ( $c$  increases), these effects dominate the standard Sharpe-Tint (1990) LDI effects.

## 2.4 Optimal Portfolios

Taking first order conditions of (8), we can solve for the optimal portfolio weight as

$$w^* = \frac{1}{\lambda} \sigma_E^{-2} \left[ (\mu - r_f) - \frac{c}{A_0} P_w \right], \quad (9)$$

where  $P_w = \partial P(w, L_0, A_0) / \partial w$ . Note that the option value depends explicitly on the portfolio weight chosen by the pension fund manager. Thus, equation (9) implicitly defines both the optimal portfolio weight and the cost of the shortfall insurance.

Clearly if there is no downside penalty ( $c = 0$ ), then the standard mean-variance efficient portfolio weight arises as a special case. If  $c \rightarrow \infty$ , the manager cares only about hedging the liability and does not care about mean-variance performance. As a result, we can define the

liability hedging portfolio as

$$w^{LH} = \arg \max_w -\frac{1}{A_0} P(w, L_0, A_0). \quad (10)$$

Although the shortfall process is not log normal, let us assume the time interval is small so that a log normal approximation holds. Then a Margrabe (1978) exchange option formula can be used to value the shortfall put option. This provides some intuition.

The value of the exchange put option can be approximated by

$$P(w, L_0, A_0) = L_0 N(d_1(w)) - A_0 N(d_2(w)), \quad (11)$$

where  $N(\cdot)$  represents the normal cumulative density function, the parameters  $d_1$  and  $d_2$  are given by

$$\begin{aligned} d_1(w) &= \frac{\ln(L_0/A_0) + \Omega^2(w)/2}{\Omega(w)} \\ d_2(w) &= \frac{\ln(L_0/A_0) - \Omega^2(w)/2}{\Omega(w)}, \end{aligned}$$

where

$$\Omega(w) = \sqrt{w^2 \sigma_E^2 - 2w\rho\sigma_E\sigma_L + \sigma_L^2}$$

is the volatility of the portfolio relative to the liability.

Using the Margrabe (1978) approximation of the shortfall option in equation (11), we can write the optimal portfolio weight in equation (9) as a weighted average of the mean-variance efficient portfolio and the liability-hedging portfolio:

$$w^* = \frac{\lambda}{\lambda + \frac{cP_\Omega}{A_0\Omega}} w^{MV} + \frac{\frac{cP_\Omega}{A_0\Omega}}{\lambda + \frac{cP_\Omega}{A_0\Omega}} w^{LH}, \quad (12)$$

where

$$P_\Omega = A_0 n(d_2)$$

is the vega of the exchange option and  $n(\cdot)$  is the normal probability density function. Note that from the chain rule,  $P_w = P_\Omega \frac{\partial \Omega}{\partial w}$ .

Equation (12) illustrates the trade-off in the optimal portfolio weight between the mean-

variance performance seeking target and the liability hedge. When the downside risk penalty is very large ( $c$  is large), the liability hedging demand dominates in the optimal portfolio and the pension plan moves towards the liability hedging portfolio. When the shortfall penalty is close to zero, the optimal portfolio is just the mean-variance portfolio.

The weight placed on the liability-hedging portfolio is

$$\theta = \frac{cP_{\Omega}}{A_0\Omega}.$$

The funding ratio,  $A_0/L_0$ , enters the vega of the exchange option through  $d_2$  and therefore has a large influence on the liability hedging demand. It is easy to show that the weight on the liability-hedging portfolio increases with the downside risk penalty,  $c$ :

$$\frac{\partial \theta}{\partial c} = \frac{n(d_2)}{\Omega} > 0$$

. We can also show that

$$\frac{\partial \theta}{\partial (A_0/L_0)} = \frac{cn(d_2)}{A_0/L_0\Omega^4}$$

and setting  $\partial \theta / \partial (A_0/L_0) = 0$ , we obtain a maximum value when  $(A_0/L_0)^* = \exp(-\Omega^2/2)$ . On either side of the maximum,  $\theta$  declines with the funding ratio.

## 2.5 Endogenous Risk Aversion

In the mean-variance efficient portfolio, the risk aversion is  $\lambda$ . When the LDI objective function embodies downside risk ( $c > 0$ ), there is an effective overall increase in risk aversion, which is seen in the denominator in the term  $1/(\lambda + \frac{cP_{\Omega}}{A_0\Omega})$  in equation (12). With downside risk in equation (9), the risk aversion has increased by  $\theta = (cP_{\Omega})/(A_0\Omega)$ . Effective risk aversion increases as the downside penalty,  $c$ , increases and there is a *nonlinear* relation between the funding ratio,  $A_0/L_0$ , and risk aversion.

Suppose the funding ratio is very high so that the option value is close to zero. Mathematically this makes  $n(d_2) \approx 0$  in equation (12). The plan's assets are very far from liabilities, and so the pension fund manager effectively ignores the liabilities in setting the asset allocation policy – which is the mean-variance efficient portfolio,  $w^* = w^{MV}$ . Thus, for very high funding ratios, there will be little effect of downside risk even if the penalty  $c$  is large. On the other hand,

when the funding ratio is around one, and the option value is most sensitive to the volatility, the investor wishes to avoid the shortfall risk. This pushes the investor to place a larger weight on the liability-hedging portfolio when the funding ratio is close to unity.

Interestingly, when the funding ratio is far below one, the liability hedging portfolio also becomes less important. In these regions, the option value is not sensitive to the volatility because it is deep in the money. This causes the optimal portfolio weight to move towards the mean-variance efficient portfolio. Intuitively, as the funding ratio decreases, the pension fund manager has an incentive to “swing for the fences” to avoid the shortfall.

### 3 Empirical Application

Our calibrations are intended to convey intuition; they are deliberately simple. We start with a cash and equity case because all the intuition can be conveyed in this benchmark case. We then consider a two-asset case of equities and bonds without a risk-free asset to show that the intuition carries over to the setting where there are portfolio constraints. This is also a more realistic case. We leave to further research the more complex portfolios held in industry which include alternative asset classes.

We present results for a range of shortfall costs represented by the parameter  $c$ . Just as the Sharpe-Tint (1990) setting does not micro-found the risk aversion in the objective function (the parameter  $\lambda$ ), we do not discuss how the downside risk parameter,  $c$ , should be selected. How each fund selects its own set of parameters  $(c, \lambda)$  is beyond the scope of this article. The costs of downside risk, however, are real. Rauh (2006), for example, shows that higher than expected contributions to pension plans reduce firm investment in profitable business opportunities. These developments have dramatically increased the downside costs for sponsors of corporate pension plans.

#### 3.1 Parameters

We take the S&P 500 total return index to represent equity, and the Ibbotson U.S. long-term corporate bond total return index to represent the bond fund. We sample these at a monthly frequency from January 1952 to December 2011. Over this sample, the mean bond and equity

returns are 6.9% and 11.0%, respectively, with volatilities of 8.6% and 14.7%, respectively. We set the risk-free rate to be 4%. Following Leibowitz, Kogelman and Bader (1992) and Jaeger and Zimmermann (1996), we assume that the liability has the same expected return as the bond fund and set the volatility of the liability to be slightly higher than the volatility of the bond fund at 10%. We also follow Leibowitz, Kogelman and Bader and set the correlation of the liabilities with bonds and equities to be 0.98 and 0.35, respectively.<sup>5</sup> We report these summary statistics, which we use in our calibrations, in Table 1. In all our calibrations we assume a horizon of one year.

## 3.2 Cash and Equities

We first assume that the pension plan allocates between a risk-free asset and equities.

We set the risk aversion coefficient,  $\lambda$ , by taking the value of risk aversion such that the mean-variance efficient portfolio consists of a 60% equities/40% risk-free bond portfolio. This turns out to be  $\lambda = 5.88$ .

### 3.2.1 Comparison with Mean-Variance and Sharpe-Tint LDI

Table 2 reports the optimal portfolio weights held in equities in the mean-variance efficient portfolio, the Sharpe-Tint (1990) LDI portfolio, and our LDI portfolio that takes into account the downside shortfall risk. We assume a fully funded portfolio,  $A_0/L_0 = 1$ . By assumption of the value of  $\lambda$ , we start with a mean-variance efficient portfolio of 60% equities. The Sharpe-Tint portfolio places more weight on equities, at 84%, than the mean-variance portfolio because the liability is positively correlated with equities. Thus, equities serve to hedge the liabilities and the optimal Sharpe-Tint portfolio tilts towards the liability hedge portfolio. We report the effective risk aversion, which is the risk aversion required under mean-variance utility to produce the same portfolio as the optimal holding. An equity holding of 84% is produced by a risk aversion level of 4.21 in a mean-variance setting.

In the last column of Table 2, we report the optimal portfolio of the LDI with downside risk. We assume that  $c = 1$ . The LDI with downside risk portfolio undoes the higher equity position

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<sup>5</sup> Leibowitz, Kogelman and Bader (1992) assume the correlation between the pension liability and bonds to be 1.00, but we set it at 0.98 as liabilities of pension funds are generally not tradeable and cannot be hedged perfectly. They also incorporate longevity and other risks as well as credit spread risk.

in the traditional Sharpe-Tint position. Taking into account the liability shortfall pushes the manager toward a lower holding in equities. The LDI position holds an equity proportion of 48%, which is lower than the mean-variance efficient portfolio. This is exactly the opposite of the Sharpe-Tint advice! The LDI with the downside penalty recognizes that although equities are positively correlated with the liabilities, there can be instances of substantial underperformance when investing in equities. This is costly, and reflected in the value of the put option. Thus, the downside risk averse manager cuts back on equities.

In Figure 1, we show further the effect of the downside penalty,  $c$ , on the optimal portfolio weight in equities. The portfolio corresponding to  $c = 0$  on the  $y$ -axis is the mean-variance efficient weight of 60% in Table 2. As  $c$  increases, the downside LDI strategy allocates less weight on equities. The dash dotted line draws the Sharpe-Tint LDI optimal weight on equity of 84%. As  $c$  increases, the optimal weight asymptotes to the liability-hedging portfolio, which holds 24% in equity (see equation (10)).

### 3.2.2 Funding Ratios and Endogenous Risk Aversion

In Figure 2, we compare the traditional Sharpe-Tint LDI with our LDI with downside risk. The top panel plots the optimal equity weight as a function of the initial funding ratio,  $A_0/L_0$ . We show four cases: the horizontal dotted line represents the case of no downside penalty, which is the regular mean-variance portfolio (60%); the dashed-dotted line shows the Sharpe-Tint LDI portfolio; the solid line plots the downside risk LDI with  $c = 1$ , and the dashed line plots the downside risk LDI with  $c = 2$ . The Sharpe-Tint portfolio holds more equities as the funding ratio decreases because the funding ratio decreases, the hedging demand increases. As equities are correlated with the liabilities, the fund holds more equities to hedge the liabilities as the funding drops. But this effect is a *linear* effect due to the liabilities entering surplus one-for-one. The downside risk induced by the put option ( $c > 0$ ) is highly *nonlinear* and totally different from the Sharpe-Tint advice.

When downside risk is taken into account, the weight on equities reaches a minimum close to the fully funded case. At this point, the downside risk-averse manager places much less weight on the equities. In the case of  $c = 1$ , the equity weight reaches its minimum at 45% when the funding ratio is 1.03. The intuition is that at full funding, the fund value could easily

just cover, or just be below, the liabilities at the end of the period. The manager dislikes this sensitivity and hedges by moving the portfolio towards lower holdings on equity.

As the initial funding ratio increases, it is less likely there will be a liability shortfall and the option value falls. For highly over-funded plans, the value of the option is negligible and the asset allocation problem is equivalent to mean-variance optimization.

When the plan is very underfunded, the shortfall option is deep in the money. As a result, the objective function starts to put less weight on the shortfall risk because there is less ability of the manager to alter the portfolio choice to meet the liabilities. In extreme cases of chronic underfunding, the liabilities cannot be met in most states of the world and then the liabilities become irrelevant to the portfolio choice problem. In the limit as the funding ratio goes to zero, the fund just moves towards the mean-variance efficient portfolio to maximize performance.

Thus, there is an overall U-shaped equity weight as a function of the funding ratio, with a minimum weight on equities at a funding ratio close to one.

The bottom panel of Figure 2 plots the effective risk aversion, which is the risk aversion required in a standard mean-variance optimization over asset returns to yield the same portfolio weight in equities. By construction, as the equity weight decreases, effective risk aversion increases. The pension fund manager is most risk averse at the fully funded case where the option value reaches a maximum. For  $c = 1$ , the maximum of effective risk aversion is achieved when the funding ratio is 1.03. The corresponding effective risk aversion is 7.83. The manager is highly sensitive to the shortfall risk at this point and tilts the optimal portfolio resolutely towards holding more risk-free assets to minimize the cost of the shortfall.

In summary, both highly under-funded and over-funded plans are less risk averse than fully funded plans under LDI with downside risk and hold more equities. In particular, the manager “swings for the fences” as funding ratios decrease. There is some empirical evidence for this behavior, as Addoum, van Binsbergen and Brandt (2010) and Pennachi and Rastad (2011) find. Addoum, van Binsbergen and Brandt (2010) show that pension plans approaching a funding ratio of 80%, which subject plan sponsors to severe mandatory additional contributions, increase the risk of their portfolios. They also find similar, but weaker, results at a threshold of 100% where there are milder forms of required contributions. Pennachi and Rastad (2011) find that public pensions funds choose riskier portfolios following periods of relatively poor investment



performance after their funding ratios have declined.

### 3.2.3 Funding Ratios and Option Values

Figure 3 characterizes the shortfall option value as a function of the funding ratio. We take  $c = 1$  for both plots. The top panel plots the option value. Not surprisingly, the option value decreases as the funding ratio increases as the higher the funding level, the lower the probability of a shortfall at the end of the period. In the bottom panel, we graph the shortfall option's sensitivity to the optimal weight on the equity,  $P_w$ , as a function of the funding ratio. The sensitivity is concave in  $A_0/L_0$ , and reaches a maximum when  $A_0/L_0 = 1.04$ . The option's sensitivity is highest when the pension plan is close to fully funded because the probability of the shortfall risk is highest at this point.

## 3.3 Equities and Bonds

Our second example is allocation over equities and bonds without a risk-free asset. In this case, we set the risk aversion coefficient,  $\lambda$ , by choosing its value so that it corresponds to a 60% equity/40% risky bond mean-variance efficient portfolio. This value is  $\lambda = 4.37$ .

Table 3 reports the optimal portfolios for the equities and bond allocation case. All three optimal portfolios are computed under the fully funded condition,  $A_0/L_0 = 1$ . The first column lists the mean-variance efficient portfolio of 60% in equities, and its effective risk aversion of  $\lambda = 4.37$  by construction. The Sharpe-Tint (1990) LDI portfolio holds 45% equities, which corresponds to an effective mean-variance efficient risk aversion of 6.73. There is a decrease in the equity weight here compared to an increase in equities when only equities and cash were held (see Table 2) because now both equities and bonds are correlated with the liability, and the bond is a much better liability-hedging instrument than equities (bonds have a correlation of 0.98 with the liabilities).

The last column in Table 3 reports the optimal portfolio under LDI with downside risk, which has a weight of just 18% in equities. This does not have a corresponding effective mean-variance risk aversion coefficient. The mean-variance efficient portfolios are bounded below by 23%, which corresponds to a risk aversion of infinity. Taking into account downside risk tilts the optimal portfolio markedly towards bonds, rather than equities.

We plot the optimal equity portfolio weight in the top panel of Figure 4 as a function of the downside risk parameter,  $c$ . The Sharpe-Tint portfolio corresponds to the 45% horizontal dashed-dotted line. Like the equities-cash case in Figure 1, the optimal downside LDI portfolio weight decreases with  $c$  and asymptotes to the downside risk liability hedging portfolio of 4% equity/96% bond, as the weight on the shortfall risk increases. As the Sharpe-Tint portfolio holds less equity than the mean-variance efficient portfolio, when  $c$  is small ( $c \leq 0.25$ ), the downside risk LDI put more weight on wealth on equities than the Sharpe-Tint portfolio. When meeting the liabilities is not so important ( $c$  is small), the downside risk LDI objective function seeks mean-variance performance. Even for modest  $c$ , there are marked reductions in the equity holdings.

In the bottom panel of Figure 4, we investigate the relation between the optimal portfolio weight and the initial funding ratio. The Sharpe-Tint LDI portfolio in the equity-bond case is upward sloping, and lies below the mean-variance portfolio. This is different from the equity-cash case because of the high correlation of the liability with bonds. The downside risk LDI portfolio weight is highly nonlinear and the manager holds the largest amount of equities at very low and high funding ratios.

The manager is most risk averse around the fully funded case and holds the minimum amount of equities at this level. The minimum equity holdings are 18% for  $c = 1$  and 11% for  $c = 2$  and are both reached at  $A_0/L_0 = 1.0$ . As the funding ratio decreases, the manager “swings for the fences” and holds more equity because the manager’s best option is to hold the mean-variance portfolio. When funding ratio is large, the manager can afford to take on risk because the option value is small. Only when the funding ratio is around one does the downside risk LDI optimally recommend a position heavily tilted towards risky bonds to hedge the liability.

## 4 Conclusion

We extend the liability driven investment (LDI) framework to incorporate downside risk. We include a penalty term for the liability shortfall. This can reflect penalties on a plan sponsor, which could be imposed by a regulatory agency, or represent the opportunity cost of capital of a firm required to be diverted to the pension plan if the assets are not sufficient to meet

the liabilities. It can also reflect the additional, asymmetric, risk borne by plan participants in shortfall situations.

We show the shortfall between assets and liabilities can be valued as an option. The option pays the difference between liability and asset value at maturity, if the liabilities are greater than the assets, and zero otherwise. The value of the option is determined *simultaneously* with the optimal portfolio, since an optimally chosen portfolio affects the probability of a liability shortfall in the future. The exposure to shortfall risk is controlled by a downside risk parameter. Optimal portfolio allocation with downside risk is very different from traditional Sharpe-Tint (1990) surplus optimization, which produces portfolio weights that are monotonic in funding ratios.

Under LDI with downside risk, the optimal portfolio exhibits *endogenous* risk aversion. Risk aversion peaks when the plan is approximately fully funded. At this point the optimal portfolio holds the *lowest* proportion in equities. The manager wishes to minimize the sensitivity of the portfolio to a shortfall event and the optimal portfolio is heavily tilted to the liability hedging portfolio, which has a low proportion of equities. As the funding ratio moves away from one in both directions, endogenous risk aversion decreases and the manager takes on more risk. Under-funded plans “swing for the fences” on the chance that the portfolio return may be sufficiently high to avoid a shortfall. When the plan is drastically under-funded, the shortfall option is way in the money and the manager has little ability to avoid the shortfall. In this case, the manager’s best option is to hold the traditional mean-variance portfolio. For over-funded plans, the probability of a shortfall event is small and this allows the pension plan seek mean-variance performance and take on more risk.

We illustrate the LDI with downside risk framework with allocations only over equities and cash, and an equities and bond case. More practical application would require extension to many more asset classes. The same asymmetries for shortfall risk arise in many other asset management contexts like central bank reserves, sovereign wealth funds, and stabilization funds. These funds also bear downside risk. We also deliberately restrict our analysis to a simple two-date setting in order to compare the implications of our downside risk analysis with two well-known benchmarks in the pension fund industry that do not take into account downside risk: the mean-variance efficient portfolio and the Sharpe-Tint (1990) portfolio. Much of the

economic intuition that we develop in this simplified static setting will carry over to an intertemporal setting, albeit with considerably more notational complexity. Developing our ideas into an intertemporal setting like Rudolf and Ziemba (2004) is a fruitful direction for future research.

# A Appendix

In this appendix, we value the shortfall option. Our approach follows the analytical approximation in Alexander and Venkatramanan (2011). The time to maturity of the option is assumed to be one year.

## A.1 Spread Option Interpretation

In the first place, we study the asset allocation between risky equity and risk-free cash, as a benchmark case. The case of allocation between equities and risky bonds will be addressed in the next section. We denote the market value of liability and the asset portfolio by  $L_t$  and  $A_t$  respectively, and the payoff of the option by  $\max\{L_1 - A_1, 0\}$ .

The market value of the liability at the end of the year is

$$L_1 = L_0 \exp\left(\left(\mu_L - \frac{\sigma_L^2}{2}\right) + \sigma_L W_1^L\right)$$

We assume that the weight on equity and cash are chosen at the beginning of the period and not rebalanced during the year. The market value of the portfolio managed by the fund is

$$A_1 = wA_0 \exp\left(\left(\mu - \frac{\sigma_E^2}{2}\right) + \sigma_E W_1^E\right) + (1 - w)A_0 \exp(r_f)$$

Note that  $A_1$  does not satisfy the assumption of a log-normal diffusion. Thus, the exchange option pricing formulas of Fisher (1978) and Margrabe (1978) do not apply for valuing  $P(w, A_0, L_0) = \mathbf{E}^{\mathbb{Q}}[\max(L_1 - A_1, 0)]$ , where  $\mathbb{Q}$  is the risk-neutral measure. The exchange options are good approximations when option maturities are very short, as Alexander and Venkatramanan (2011) comment.

Let us define

$$\begin{aligned} S_{1,t} &= L_t \\ S_{2,t} &= wA_0 \exp\left(\left(\mu - \frac{\sigma_E^2}{2}\right)t + \sigma W_t^E\right) \\ K &= (1 - w)A_0 \exp(r_f). \end{aligned} \tag{A.1}$$

As both  $S_1$  and  $S_2$  are log-normally distributed, we can transform the problem into pricing a spread option with the underlying assets being  $S_1$  and  $S_2$ , and the strike spread being  $K$ :

$$P(w, A_0, L_0) = \mathbf{E}^{\mathbb{Q}}[\max(L_1 - A_1, 0)] = \mathbf{E}^{\mathbb{Q}}[\max(S_{1,1} - S_{2,1} - K, 0)]. \tag{A.2}$$

We employ the analytical approximation of spread options as compound exchange options following Alexander and Venkatramanan (2011). The compound exchange option representation appears to provide the most precise estimate of the value of spread options.

## A.2 Valuation of the Shortfall Option

Define  $m$  to be a real number such that  $m \geq 1$ . Let us define the regions:

$$\begin{aligned}\mathbf{L} &= \{S_{1,1} - S_{2,1} - K \geq 0\} \\ \mathbf{A} &= \{S_{1,1} - mK \geq 0\} \\ \mathbf{B} &= \{S_{2,1} - (m-1)K \geq 0\}.\end{aligned}\tag{A.3}$$

Then the spread option's payoff of strike  $K$  at year end can be written as:

$$\begin{aligned}\mathbf{1}_{\mathbf{L}}[S_{1,1} - S_{2,1} - K] &= \mathbf{1}_{\mathbf{L}}(\mathbf{1}_{\mathbf{A}}[S_{1,1} - mK] - \mathbf{1}_{\mathbf{B}}[S_{2,1} - (m-1)K]) \\ &\quad + (1 - \mathbf{1}_{\mathbf{A}})[S_{1,1} - mK] - (1 - \mathbf{1}_{\mathbf{B}})[S_{2,1} - (m-1)K].\end{aligned}\tag{A.4}$$

With some algebraic manipulation, Alexander and Venaramanan (2011) show that

$$P(w, L_0, A_0) = e^{-r_f} (\mathbf{E}^{\mathbb{Q}}\{[(U_{1,1} - U_{2,1})^+]\} + \mathbf{E}^{\mathbb{Q}}\{[(V_{2,1} - V_{1,1})^+]\})\tag{A.5}$$

where  $U_{1,1}, V_{1,1}$  are payoffs to European call and put options on  $S_1$  with strike  $mK$ , respectively. Likewise,  $U_{2,1}, V_{2,1}$  are European call and put options on  $S_2$  with strike  $(m-1)K$ , respectively. The spread option is thus equivalent to compound exchange options on two calls and two puts. The parameter  $m$  is chosen such that the single-asset call options are deep-in-the-money. For our calibrations we choose  $m = 5$  which satisfies the approximation conditions in Alexander and Venaramanan (2011).

The calls and puts can be described as:

$$\begin{aligned}dU_{i,t} &= r_f U_{i,t} dt + \xi_i U_{i,t} dW_{i,t}^{\mathbb{Q}} \\ dV_{i,t} &= r_f V_{i,t} dt + \eta_i V_{i,t} dW_{i,t}^{\mathbb{Q}}.\end{aligned}\tag{A.6}$$

Note that  $U_{1,0}$  and  $V_{1,0}$  are the Black-Scholes prices at  $t = 0$ . By applying Ito's theorem on the calls and the puts, the parameters  $\xi$  and  $\eta$  are:

$$\begin{aligned}\xi_i &= \sigma_i \frac{S_{i,t}}{U_{i,t}} \frac{\partial U_{i,t}}{\partial S_{i,t}} \\ \eta_i &= \sigma_i \frac{S_{i,t}}{V_{i,t}} \left| \frac{\partial V_{i,t}}{\partial S_{i,t}} \right|\end{aligned}\tag{A.7}$$

Under our assumption that  $S_1$  and  $S_2$  are driven by geometrical Brownian motion processes, the calls and puts in equation (A.5) can be approximated as log normal even though the spread option is not log normal by suitable choice of  $m$ .

We can now apply the Margrabe's (1978) formula for exchange options on equation (A.5):

$$P(w, L_0, A_0) = e^{-r_f} [U_{1,0}N(d_{1U}) - U_{2,0}N(d_{2U}) - (V_{1,0}N(-d_{1V}) - V_{2,0}N(-d_{2V}))]\tag{A.8}$$

where  $N(\cdot)$  represents the normal cumulative density function, and the parameters  $d_{1X}$  and  $d_{2X}$  for  $X \in \{U, V\}$

are given by

$$\begin{aligned} d_{1X} &= \frac{\ln\left(\frac{X_{10}}{X_{20}}\right) + \frac{1}{2}\sigma_X^2}{\sigma_X} \\ d_{2X} &= d_{1X} - \sigma_X \end{aligned}$$

The volatilities of the call and put are given by

$$\begin{aligned} \sigma_U &= \sqrt{\xi_1^2 + \xi_2^2 - 2\rho\xi_1\xi_2} \\ \sigma_V &= \sqrt{\eta_1^2 + \eta_2^2 - 2\rho\eta_1\eta_2} \end{aligned}$$

The correlation used to compute the exchange option volatility is the implied correlation between two vanilla calls or puts. They are the same as the correlation between the underlying prices of the two assets, as the options and the underlying prices are driven by the same Brownian motions. Note also that as the underlying asset 1 is the liability,  $S_{1,0} = L_0$  and  $\sigma_1 = \sigma_L$ . Similarly as the underlying asset 2 is the equity portion of the portfolio,  $S_{2,0} = wA_0$  and  $\sigma_2 = \sigma_E$ .

### A.3 Equity and Risky Bond Case

In this section, we value the shortfall option when the bond is risky. The risky bond has log normal price process:

$$\frac{dB}{B} = (\mu_B - \frac{\sigma_B^2}{2})dt + \sigma_B dW_{B,t} \quad (\text{A.9})$$

We choose the risky bond as the numeraire, and the equivalent martingale measure associated with this numeraire is denoted as  $\mathbb{R}$ . The liability and equities have the following normalized price processes:

$$\begin{aligned} \frac{d(L/B)}{L/B} &= \sigma_L dW_{L,t}^{\mathbb{R}} - \sigma_B dW_{B,t}^{\mathbb{R}} \\ \frac{d(E/B)}{E/B} &= \sigma_E dW_{E,t}^{\mathbb{R}} - \sigma_B dW_{B,t}^{\mathbb{R}} \end{aligned}$$

The contingent claim  $[L_1 - A_1]^+$  can be priced as

$$P(w, A_0, L_0, B_0) = B_0 \mathbf{E}^{\mathbb{R}}\{[L_1 - A_1]^+ / B_1\}$$

We can then write  $[L_1 - A_1]^+ / B_1$  in the form of  $(S_{1,1} - S_{2,1} - K)^+$ , where

$$\begin{aligned} S_{1,t} &= L_t / B_t \\ S_{2,t} &= w \frac{A_0}{B_0} \exp\left(-\frac{\sigma_E^2 - 2\rho_{EB}\sigma_E\sigma_B - \sigma_B^2}{2}t + \sigma_E W_{E,t}^{\mathbb{R}} - \sigma_B W_{B,t}^{\mathbb{R}}\right) \\ K &= (1-w)A_0/B_0 \end{aligned}$$

The rest of the pricing approach is identical to what we outlined in the previous section.

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Table 1: Data Summary Statistics

	Mean	Volatility	Correlations		
			Bond	Equity	Liability
Bond	6.92%	8.60%	1.00		
Equity	11.04%	14.69%	0.25	1.00	
Liability	6.92%	10.00%	0.98	0.35	1.00
Risk-Free	4.00%				

The table reports annualized expected returns, volatilities, and correlations of bonds and equities, which are total returns of the Ibbotson U.S. Long-Term Corporate Debt Index and the S&P 500 Index between January 1952 and December 2011. These are monthly frequency series and we annualize the means and volatilities by multiplying the monthly frequency mean and volatility by 12 and  $\sqrt{12}$ , respectively. Parameters for the liability and the risk-free rate are set by assumption and follow closely those set by Leibowitz, Kogelman and Bader (1992) and Jaeger and Zimmermann (1996).

Table 2: Optimal Portfolio Choice Over Equities and Risk-Free Cash

	MV Efficient	Sharpe-Tint LDI	LDI with Downside Risk
Equity Portfolio Weight	0.60	0.84	0.48
Effective Risk Aversion	5.88	4.21	7.30

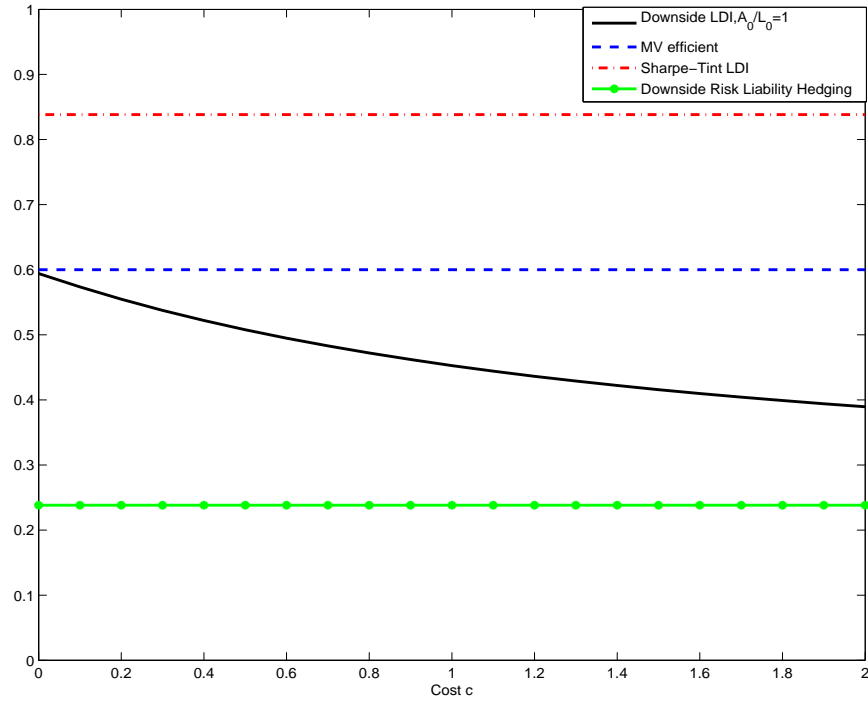
The table reports the portfolio weights for the mean-variance (MV) efficient, Sharpe-Tint (1990) LDI and the LDI with downside risk optimizations for a risk-free asset (cash) and equities. The “portfolio weight” row lists the proportion of the portfolio held in equities. We compute these using the expected returns, volatilities, and correlations given in Table 1. We use the parameters  $\lambda = 5.88$ ,  $c = 1$ , and  $A_0/L_0 = 1$  with a one-year horizon. The “effective risk aversion” is the risk aversion required in the mean-variance efficient portfolio weight to give the same weight in equities as the optimal portfolio weight.

Table 3: Optimal Portfolio Choice Over Stocks and Bonds

	MV Efficient	Sharpe-Tint LDI	LDI with Downside Risk
Equity Portfolio Weight	0.60	0.45	0.18
Effective Risk Aversion	4.37	6.73	–

The table reports the portfolio weights for mean-variance (MV) efficient, Sharpe-Tint (1990) LDI and the LDI with downside risk optimizations, respectively, for bonds and equities. The “portfolio weight” row lists the proportion of the portfolio held in equities. We compute these using the expected returns, volatilities, and correlations given in Table 1. We use the parameters  $\lambda = 4.37$ ,  $c = 1$ , and  $A_0/L_0 = 1$  with a one-year horizon. The “Effective risk aversion” is the risk aversion required in the mean-variance efficient portfolio weight to give the same weight in equities as the optimal portfolio weight. The LDI with downside risk portfolio does not have a corresponding effective mean-variance risk aversion coefficient because the mean-variance efficient portfolios are bounded below by 23%, which corresponds to a risk aversion of infinity.

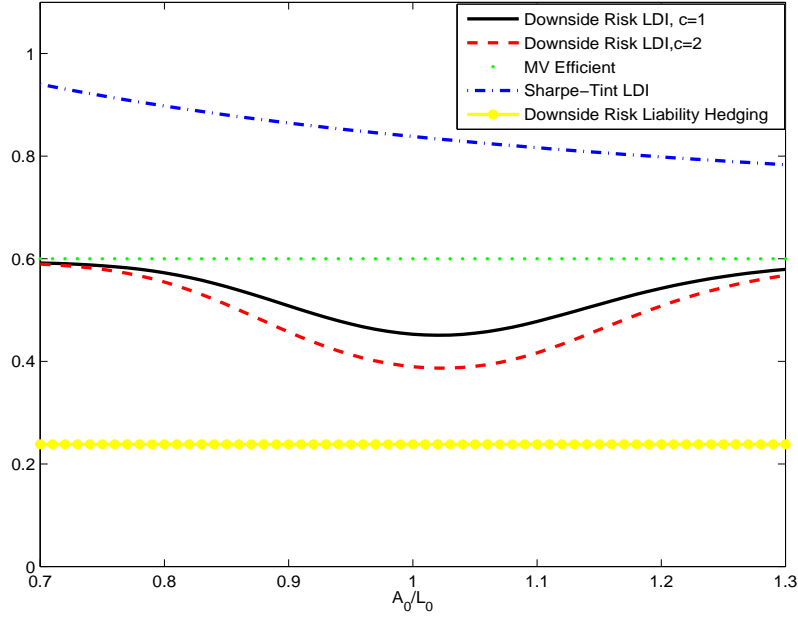
Figure 1: Downside Risk Penalty: Stocks and Cash



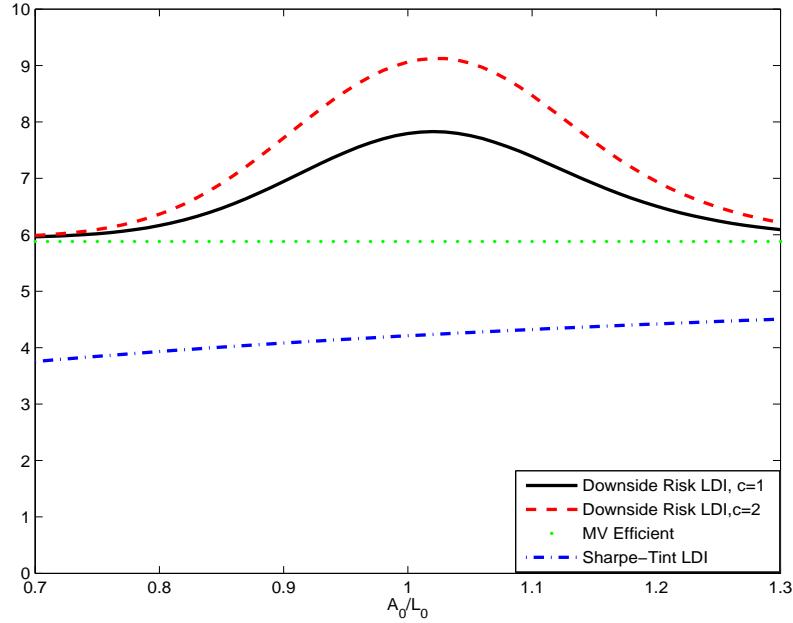
The figure plots the optimal weight in equities as a function of the penalty cost,  $c$ , on downside shortfall risk for the LDI problem with downside risk with only a risk-free asset and equities. We use the expected returns, volatilities, and correlations given in Table 1 and the parameters  $\lambda = 5.88$  and  $A_0/L_0 = 1$  with a one-year horizon.

Figure 2: Funding Ratios

Optimal Weight on Equities

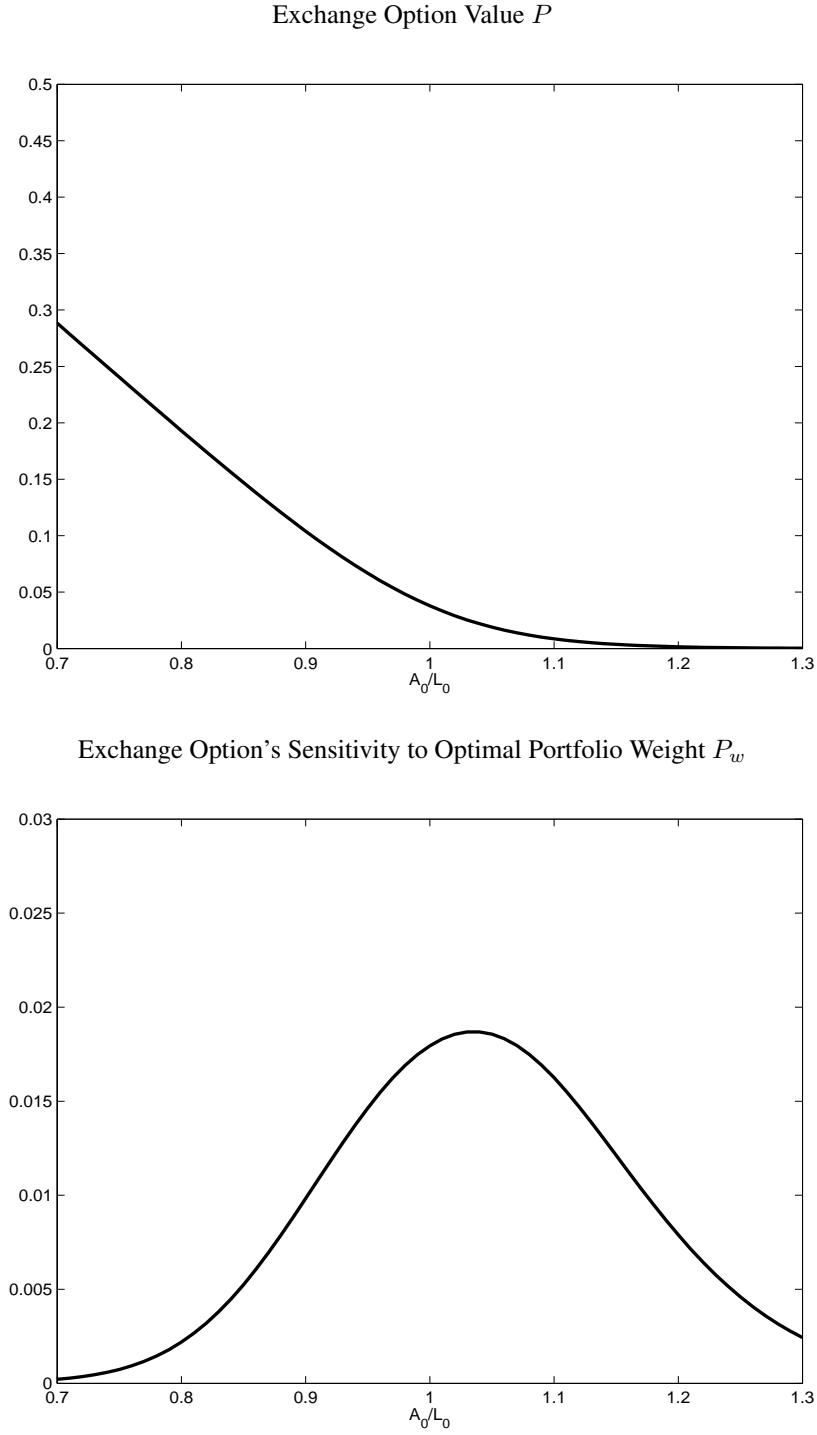


Effective Risk Aversion



We consider a LDI problem with downside risk with only a risk-free asset and equities. Both plots are functions of the initial funding ratio,  $A_0/L_0$ . In the top panel, we plot the optimal weight in equities. The effective risk aversion in the bottom panel is the risk aversion required in the mean-variance efficient portfolio weight to give the same weight in equities as the optimal portfolio weight. We plot the portfolio in the bottom panel. We use the expected returns, volatilities, and correlations listed in Table 1 and the parameters  $\lambda = 5.88$ ,  $c = 1$ , and  $c = 2$  with a one-year horizon.

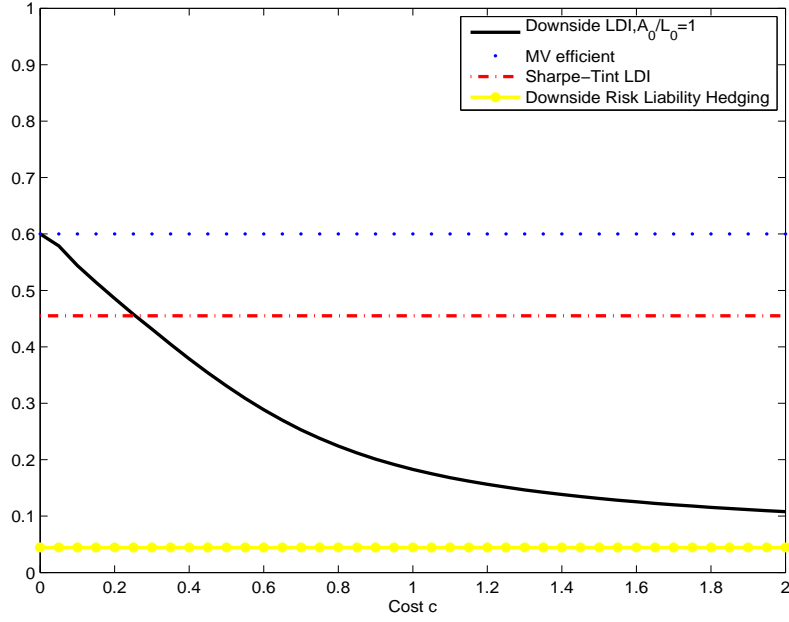
Figure 3: Shortfall Penalty Option Value and Option's Sensitivity to Optimal Portfolio Weight



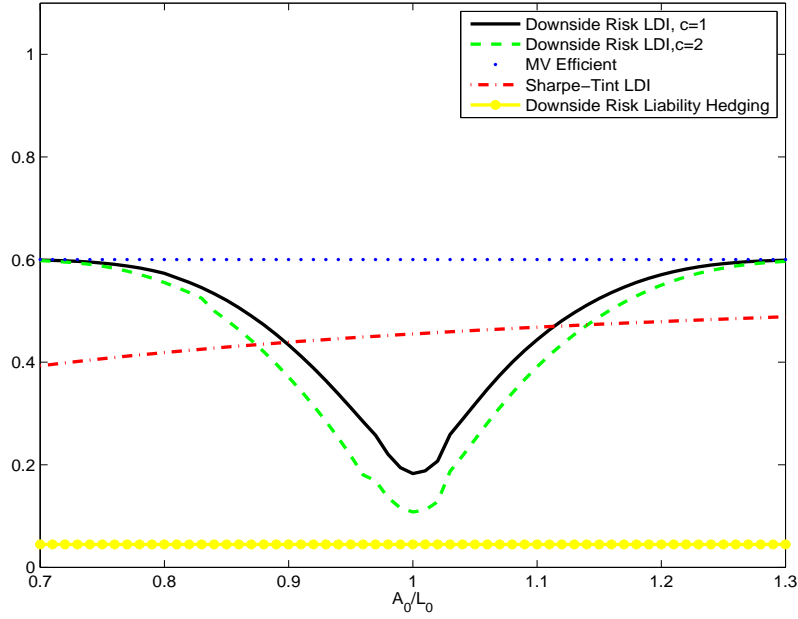
We consider an LDI problem with downside risk with only a risk-free asset and equities. Both plots are functions of the initial funding ratio,  $A_0/L_0$ . In the top panel, we plot the option value,  $P(w, A_0, L_0)$ . We plot  $\partial P(w, A_0, L_0)/\partial w$  in the bottom panel. We use the expected returns, volatilities, and correlations listed in Table 1 and the parameters  $\lambda = 5.88$  and  $c = 1$  with a one-year horizon.

Figure 4: Stocks and Bonds

Equity Weight as a Function of  $c$



Equity Weight as a Function of  $A_0/L_0$



We consider an LDI problem with downside risk, investing in equities and risky bonds. The top panel plots the optimal weight in equities as a function of the penalty cost,  $c$ , while fixing the parameter of funding ratio  $A_0/L_0 = 1$ . The bottom panel plots the optimal equity weight as a function of funding ratio  $A_0/L_0$ , the mean-variance efficient portfolio, the Sharpe-Tint (1990) LDI, and the LDI with downside risk. . We use the expected returns, volatilities, and correlations listed in Table 2 and the parameters  $c = 1$ ,  $c = 2$ , and  $\lambda = 4.37$  with a one-year horizon.