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Multi-period portfolio optimization for asset–liability management with bankrupt control ☆

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ABSTRACT

In this paper, we investigate a multi-period portfolio optimization problem for asset–liability management of an investor who intends to control the probability of bankruptcy before reaching the end of an investment horizon. We formulate the problem as a generalized mean–variance model that incorporates bankrupt control over intermediate periods. Based on the Lagrangian multiplier method, the embedding technique, the dynamic programming approach and the Lagrangian duality theory, we propose a method to solve the model. A numerical example is given to demonstrate our method and show the impact of bankrupt control and market parameters on the optimal portfolio strategy.

1. Introduction

It is necessary for investors to find optimal allocations of their wealth among various assets. The pioneering work was done by Markowitz [1], who developed the mean–variance analysis for single-period portfolio selection. After his work, Li and Ng [2] and Zhou and Li [3], for the first time, initiated an embedding technique to overcome the difficulty of nonseparability in the sense of dynamic programming in dynamic mean–variance models and derived analytical optimal solutions in a multi-period case and a continuous-time case respectively. Since then, dynamic mean–variance portfolio selection becomes a hot topic in academia. Zhou and Yin [4] used the mean–variance analysis to study continuous-time portfolio selection with regime switching. Bielecki et al. [5] studied continuous-time portfolio selection with bankrupt prohibition under the mean–variance framework. Zhu et al. [6] established a generalized mean–variance model for multi-period portfolio selection with bankrupt control over intermediate periods. Wei and Ye [7] investigated a mean–variance model for multi-period portfolio selection with bankrupt constraint in a Markov regime switching market.

For many financial institutions, such as pension funds or banks, both sides of the balance sheet have to be considered. So it is absolutely necessary for these institutions to take liability management into account. During the past decades, many authors have paid their attention to portfolio selection problems with liability. Leibowitz and Henriksson [8] demonstrated that traditional asset allocation tools can be adapted to a surplus framework. Sharpe and Tint [9] first investigated asset–liability (AL) management under the mean–variance criteria in a single-period setting. Keel and Müller [10] discussed the set of efficient portfolios and drew the conclusion that the liability leads to a parallel shift of the efficient set. Waring [11] developed a solution that duration-matches assets with the pension liability, and showed how to integrate it with surplus optimization, achieving better pension risk control in the process. Waring [12] developed an updated technology for calculating

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surplus efficient frontiers and surplus asset allocation that separately incorporates both systematic and unsystematic characteristics, and yields an economic view of the liability. Jones and Brown [13] presented an approach which incorporates mean–variance optimization problem within a multi-horizon AL risk management and allows an individual's portfolio to provide short-term cash flow. With respect to multi-period dynamic AL management problems, Leippold et al. [14] established a mean–variance model and derived its analytical optimal strategy and efficient frontier by making use of the embedding technique of Li and Ng [2]; Yi et al. [15] extended the work of Leippold et al. [14] to the case with uncertain investment horizon; Chen and Yang [16] considered an optimal portfolio selection problem under mean–variance criterion in a regime switching model and derived the analytical optimal strategy. As for continuous-time dynamic AL management problems, Chiu and Li [17] studied a mean–variance model by means of the stochastic linear-quadratic control technique; Chen et al. [18] and Xie [19] investigated a mean–variance model with Markov regime switching by adopting the technique of Zhou and Yin [4]; recently, Zeng and Li [20] considered continuous-time AL management problems under benchmark and mean–variance criteria in a jump diffusion market and derived analytical solutions of the two models.

To our knowledge, all the existing literature on dynamic mean–variance analysis focused either on pure portfolio selection without liability or on AL management without bankrupt control. However, it is essential for institutional investors to take account of liability and prevent themselves from bankruptcy. In this paper, we investigate a multi-period AL management problem with bankrupt control over intermediate periods. The problem is formulated as a generalized mean–variance model. A solution method for the model is proposed based on the Lagrangian multiplier method, the embedding technique, the dynamic programming approach and the Lagrangian duality theory.

The organization of this paper is as follows. In Section 2, we first establish a generalized mean–variance model for the multi-period AL management with bankrupt control over intermediate periods. Then we formulate the Lagrangian optimization problem for the model and construct an auxiliary optimization problem for the Lagrangian problem. In Section 3, we explicitly solve the auxiliary problem by using the dynamic programming approach. The relationship between the auxiliary problem and the Lagrangian problem is established in Section 4. Based on the results obtained in the previous two sections, we solve the Lagrangian problem in Section 5. Using the Lagrangian duality method, we find the optimal Lagrangian multipliers and then solve the generalized mean–variance model in Section 6. To demonstrate our method, a numerical example is given in Section 7. Section 8 concludes the paper. Comparing with the work of Zhu et al. [6], consideration of liability in our problem increases the dimension of the auxiliary problem and the difficulty of establishing the relationship between the Lagrangian problem and the auxiliary problem.

2. Problem formulation

We consider a market with two risky assets, one of which can be degenerated to the risk-less asset. It is assumed that an investor joins the market at time $t = 0$ with an initial wealth x_0 and an initial liability l_0. He intends to make a plan for allocating his wealth among the two assets within a time horizon of T periods. His portfolio can be rebalanced at the beginning of each subsequent period.

We denote the rates of return of the two assets and the liability in the tth period (from time $t-1$ to time t) as $r^0_{t-1},\ \tilde{r}_{t-1}$ and q_{t-1} respectively, and define the return vector $R_t = (r^0_t, \tilde{r}_t, q_t)'$. Here the superscript $'$ represents the transpose of a vector or matrix. We assume that the returns of the three financial instruments are statistically independent among different time periods and that the return vector R_t has known mean $\mathrm{E}[R_t] = (\mathrm{E}[r^0_t], \mathrm{E}[\tilde{r}_t], \mathrm{E}[q_t])'$ and covariance $\mathrm{Cov}(R_t),\ t = 0, 1, \ldots, T-1$. So

$$\mathrm{E}[R_t R_t'] = \mathrm{Cov}(R_t) + \mathrm{E}[R_t]\mathrm{E}[R_t'] \quad (t = 0, 1, \ldots, T-1).$$

Further we assume that all the matrices $\mathrm{E}[R_t R_t']$, $t = 0, 1, \ldots, T-1$, are positive definite.

Let u_t be the amount invested in the asset with return $\tilde{r}_t$ at time t. We call $u := \{u_0, \ldots, u_{T-1}\}$ a dynamic portfolio strategy. Let x_t be the aggregated value of assets at time t. Then the wealth dynamics can be given as

$$x_{t+1} = r^0_t(x_t - u_t) + \tilde{r}_t u_t = r^0_t x_t + r^1_t u_t \quad (t = 0, 1, \ldots, T-1), \tag{1}$$

where $r^1_t = \tilde{r}_t - r^0_t$. We assume that $\mathrm{E}[r^1_t] \neq 0$ for $t = 0, 1, \ldots, T-1$, which means that the two assets have different expected returns in each period.

As in Yi et al. [15], the liability in our paper is assumed to be exogenous, thus uncontrollable. In another word, the aggregated value of the liability is not affected by the portfolio of the investor. Denote the value of the liability at time t as l_t. The liability dynamics can be given as

$$l_{t+1} = q_t l_t \quad (t = 0, 1, \ldots, T-1). \tag{2}$$

It is noticeable that, the consideration of uncontrollable liability agrees with Sharpe and Tint [9], Leippold et al. [14], and Chiu and Li [17]. Particularly, Chiu and Li [17] stated that "the mean–variance AL problem reduces to a standard mean–variance portfolio selection problem if liabilities are controllable".

At last, we define the surplus s_t at time t as the difference between the assets value and the liability value at time t, i.e., $s_t = x_t - l_t$ $(t = 0, 1, \ldots, T)$.

The classical mean–variance model for the multi-period AL management problem (MV (ω_T)) can be formulated as

$$\begin{cases} \max\limits_{u} & \mathrm{E}[s_T] - \omega_T \mathrm{Var}(s_T) \\ \text{subject to} & (1) \text{ and } (2), \end{cases}$$

where $\omega_T > 0$ is a given parameter representing the degree of risk aversion of the investor. The problem $(\mathrm{MV}(\omega_T))$ is a special case of the one in Yi et al. [15] which considers an uncertain exit time.

To be more general, we consider a bankrupt control for the above dynamic AL management problem. A bankruptcy occurs when the total surplus falls below a predefined "disaster" level in any intermediate period. It is reasonable to require that the investor should not go bankrupt so as to continue his investment. To meet this requirement, we impose a control which, to some extent, prevents the surplus of each intermediate period from falling below the disaster level on the optimal problem $(\mathrm{MV}(\omega_T))$, and we refer to this control as bankrupt control.

Let b_t be the "disaster" level in period t for $t = 1, \ldots, T-1$. It is reasonable to assume $b_t < \mathrm{E}[s_t]$. The probability of the investor going bankrupt in period t is

$$\mathrm{P}(BR_t) = \mathrm{P}(s_t \leqslant b_t, s_j > b_j,\ j = 1, \ldots, t-1).$$

By the Tchebycheff inequality, $\mathrm{P}(BR_t) \leqslant \mathrm{P}(s_t \leqslant b_t) \leqslant \mathrm{Var}(s_t)/(\mathrm{E}[s_t] - b_t)^2$. Thus, we can control the risk of bankruptcy at period t by means of setting a small value $\alpha_t \in (0,1)$ to bound $\mathrm{Var}(s_t)/(\mathrm{E}[s_t] - b_t)^2$. Then, the multi-period AL management problem with risk control over bankruptcy can be represented as the following generalized mean–variance model (GMV (ω_T, α)):

$$\begin{cases} \max\limits_{u} & \mathrm{E}[s_T] - \omega_T \mathrm{Var}(s_T) \\ \text{subject to} & \mathrm{Var}(s_t) \leqslant \alpha_t (\mathrm{E}[s_t] - b_t)^2 \quad (t = 1, 2, \ldots, T-1),\ (1) \text{ and } (2). \end{cases}$$

Here, the parameters ω_T and $\alpha = (\alpha_1, \ldots, \alpha_{T-1})' \in R_+^{T-1}$ reflect the investor's attitude toward risk, and should be assigned by the investor before searching for an optimal dynamic portfolio strategy.

The main task of this paper is to solve problem $(\mathrm{GMV}(\omega_T, \alpha))$. We are going to do it by the Lagrangian dual approach which, assuming no duality gap, solves the primal problem indirectly by solving the dual problem. To solve the dual problem, we first need to solve the Lagrangian problem $(\mathrm{L}(\omega, \omega_T, \alpha))$, which is constructed by introducing nonnegative Lagrangian multipliers $\omega_1, \omega_2, \ldots, \omega_{T-1}$ associated with the bankrupt control of $(\mathrm{GMV}(\omega_T, \alpha))$, and attaching these constraints to the objective function, and formed as follows:

$$\begin{cases} \max\limits_{u} & \mathrm{E}[s_T] - \omega_T \mathrm{Var}(s_T) - \sum\limits_{t=1}^{T-1} \omega_t \left[\mathrm{Var}(s_t) - \alpha_t (\mathrm{E}[s_t] - b_t)^2\right] \\ \text{subject to} & (1) \text{ and } (2), \end{cases}$$

where $\omega = (\omega_1, \ldots, \omega_{T-1})'$.

The dynamic programming approach can not be directly applied to $(\mathrm{L}(\omega, \omega_T, \alpha))$ because the problem is not separable in the sense of dynamic programming. Motivated by Zhu et al. [6], we construct auxiliary problem $(\mathrm{A}(\lambda, \omega, \omega_T))$ as follows:

$$\begin{cases} \max\limits_{u} & \mathrm{E}\left[\sum\limits_{t=1}^{T} \left(\lambda_t s_t - \omega_t (s_t)^2\right)\right] \\ \text{subject to} & (1) \text{ and } (2), \end{cases}$$

where $\lambda = (\lambda_1, \ldots, \lambda_T)'$ is the vector of auxiliary parameters. If $\omega_t = 0$ for some $t \in \{1, 2, \ldots, T-1\}$, then the corresponding λ_t is taken as zero. This does not change our analysis. In the next two sections, we assume that all $\omega_1, \omega_2, \ldots, \omega_{T-1}$ are strictly positive.

Define the optimal solution sets of problems $(\mathrm{L}(\omega, \omega_T, \alpha))$ and $(\mathrm{A}(\lambda, \omega, \omega_T))$ as

$$\Phi_L(\omega, \omega_T, \alpha) = \{u(\omega) | u(\omega) \text{ is optimal to } (\mathrm{L}(\omega, \omega_T, \alpha))\},$$
$$\Phi_A(\lambda, \omega, \omega_T) = \{u(\lambda, \omega) | u(\lambda, \omega) \text{ is optimal to } (\mathrm{A}(\lambda, \omega, \omega_T))\}.$$

3. Solving the auxiliary problem

In this section we solve the auxiliary problem $(\mathrm{A}(\lambda, \omega, \omega_T))$.

For $t = 0, 1, \ldots, T-1$, we introduce the following notations:

$$z_t = \begin{pmatrix} x_t \\ l_t \end{pmatrix}, \quad e = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B_t = \begin{pmatrix} r_t^0 & 0 \\ 0 & q_t \end{pmatrix}, \quad A_t = \begin{pmatrix} r_t^1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $(\mathrm{A}(\lambda, \omega, \omega_T))$ can be transformed into the equivalent problem in matrix form:

$$\begin{cases} \max\limits_{u} & \mathrm{E}\left[\sum\limits_{t=1}^{T} (\lambda_t e' z_t - \omega_t z_t' e e' z_t)\right] \\ \text{subject to} & z_{t+1} = B_t z_t + A_t e_1 u_t \quad (t = 0, 1, \ldots, T-1). \end{cases}$$

The solution of $(\mathrm{A}(\lambda,\omega,\omega_T))$ can be derived analytically by the dynamic programming approach, which is summarized in the following theorem.

Theorem 3.1. *The optimal strategy of* $(\mathrm{A}(\lambda,\omega,\omega_T))$ *is given by*

$$u_t^*(\lambda,\omega)=\frac{1}{2}\frac{E[e_1'A_t'F_{t+1}]}{E[e_1'A_t'\overline{D}_{t+1}\overline{D}_{t+1}'A_te_1]}-\frac{E[e_1'A_t'\overline{D}_{t+1}\overline{D}_{t+1}'B_tz_t]}{E[e_1'A_t'\overline{D}_{t+1}\overline{D}_{t+1}'A_te_1]}, \tag{3}$$

for $t=0,1,\ldots,T-1$, *where*

$$\overline{D}_t=\left(\omega_T^{\frac{1}{2}}\overline{B}_t^{T-t-1}e,\quad \ldots,\quad \omega_s^{\frac{1}{2}}\overline{B}_t^{s-t-1}e,\quad \ldots,\quad \omega_{t+1}^{\frac{1}{2}}\overline{B}_t^{0}e,\quad \omega_t^{\frac{1}{2}}e\right), \tag{4}$$

$$F_t=\begin{pmatrix}\displaystyle\sum_{s=t+1}^{T}\bar{r}_t^{s-t-1}\lambda_s+\lambda_t\\ \displaystyle-\sum_{s=t+1}^{T}\bar{q}_t^{s-t-1}\lambda_s-\lambda_t\end{pmatrix},\quad \overline{B}_t^0=\begin{pmatrix}\bar{r}_t^0 & 0\\ 0 & \bar{q}_t^0\end{pmatrix},\quad \overline{B}_t^i=\begin{pmatrix}\bar{r}_t^i & 0\\ 0 & \bar{q}_t^i\end{pmatrix}, \tag{5}$$

$$\bar{r}_t^0=r_t^0-r_t^1\frac{\mathrm{E}\left[r_t^1r_t^0\right]}{\mathrm{E}\left[(r_t^1)^2\right]},\quad \bar{r}_t^i=r_t^0\bar{r}_{t+1}^{i-1}-r_t^1\bar{r}_{t+1}^{i-1}\frac{\mathrm{E}\left[r_t^1r_t^0\right]}{\mathrm{E}\left[(r_t^1)^2\right]}, \tag{6}$$

$$\bar{q}_t^0=q_t-r_t^1\frac{\mathrm{E}\left[r_t^1q_t\right]}{\mathrm{E}\left[(r_t^1)^2\right]}\frac{\mathrm{E}\left[\sum_{s=t+2}^{T}\bar{r}_{t+1}^{s-t-2}\bar{q}_{t+1}^{s-t-2}\omega_s+\omega_{t+1}\right]}{\mathrm{E}\left[\sum_{s=t+2}^{T}(\bar{r}_{t+1}^{s-t-2})^2\omega_s+\omega_{t+1}\right]}, \tag{7}$$

$$\bar{q}_t^i=q_t\bar{q}_{t+1}^{i-1}-r_t^1\bar{r}_{t+1}^{i-1}\frac{\mathrm{E}\left[r_t^1q_t\right]}{\mathrm{E}\left[(r_t^1)^2\right]}\frac{\mathrm{E}\left[\sum_{s=t+2}^{T}\bar{r}_{t+1}^{s-t-2}\bar{q}_{t+1}^{s-t-2}\omega_s+\omega_{t+1}\right]}{\mathrm{E}\left[\sum_{s=t+2}^{T}(\bar{r}_{t+1}^{s-t-2})^2\omega_s+\omega_{t+1}\right]} \tag{8}$$

for $i=1,\ldots,T-t-1$. *In order to simplify the notation we define* $\sum_{j=s}^{t}(\cdot)_j=0$ *and* $\prod_{j=s}^{t}(\cdot)_j=1$ *if* $s>t$.

Proof. See Appendix A. □

4. The relation between the Lagrangian and auxiliary problems

Motivated by the idea of Theorems 1 and 2 in Zhu et al. [6], this section establishes the relationship between the Lagrangian problem $(\mathrm{L}(\omega,\omega_T,\alpha))$ and the auxiliary problem $(\mathrm{A}(\lambda,\omega,\omega_T))$.

Theorem 4.1. *Problems* $(\mathrm{L}(\omega,\omega_T,\alpha))$ *and* $(\mathrm{A}(\lambda,\omega,\omega_T))$ *have the following relations:*

(i) *If* $u^*(\omega)\in\Phi_L(\omega,\omega_T,\alpha)$, *then* $u^*(\omega)\in\Phi_A(\lambda^*,\omega,\omega_T)$, *where* $\lambda^*=(\lambda_1^*,\ldots,\lambda_T^*)$ *satisfies*

$$\lambda_t^*=-2\omega_t\alpha_tb_t+2\omega_t(1+\alpha_t)\mathrm{E}[s_t]|_{u^*(\omega)}\quad (t=1,2,\ldots,T-1),$$
$$\lambda_T^*=1+2\omega_T\mathrm{E}[s_T]|_{u^*(\omega)}.$$

(ii) *Suppose* $u^*(\lambda^*,\omega)\in\Phi_A(\lambda^*,\omega,\omega_T)$. *A necessary condition for* $u^*(\lambda^*,\omega)\in\Phi_L(\omega,\omega_T,\alpha)$ *is*

$$\lambda_t^*=-2\omega_t\alpha_tb_t+2\omega_t(1+\alpha_t)\mathrm{E}[s_t]|_{u^*(\lambda^*,\omega)}\quad (t=1,2,\ldots,T-1), \tag{9}$$
$$\lambda_T^*=1+2\omega_T\mathrm{E}[s_T]|_{u^*(\lambda^*,\omega)}. \tag{10}$$

Proof. See Appendix B. □

Remark 4.1. If we do not consider the bankrupt control, the Lagrange multiplier vector $\omega=(\omega_1,\ldots,\omega_{T-1})$ is equal to zero. By substituting it into (9) and (10), the appropriate λ which enables the solution of $(\mathrm{A}(\lambda,\omega,\omega_T))$ to be the optimal solution of $(\mathrm{L}(\omega,\omega_T,\alpha))$ can be obtained. As a result, the optimal strategy of the AL management without bankrupt control is derived by substituting the ω and λ into the Eq. (3). In this case, the problems $(\mathrm{MV}(\omega_T))$, $(\mathrm{GMV}(\omega_T))$ and $(\mathrm{L}(\omega_T))$ are the same and are the one studied in Yi et al. [15]. Hence, our results include the one in Yi et al. [15] as a special case.

The expressions of $\mathrm{E}[s_t^*]$ and $\mathrm{E}[(s_t^*)^2]$, under the optimal strategy of problem $(\mathrm{A}(\lambda,\omega,\omega_T))$, are given by Eq. (B.6) and Appendix C. Noted that the expressions of $\mathrm{E}[s_t^*]$ and $\mathrm{E}[(s_t^*)^2]$, which all depend on the parameters λ and ω, enable us to express the optimal objective value of $(\mathrm{L}(\omega,\omega_T,\alpha))$ when appropriate λ is given, and hence to express the optimal objective value and the efficient frontier of $(\mathrm{GMV}(\omega_T,\alpha))$ when appropriate λ and ω are given.

5. Solving the Lagrangian problem

In this section we solve $(\mathrm{L}(\omega,\omega_T,\alpha))$ after having established the relation between the Lagrangian problem $(\mathrm{L}(\omega,\omega_T,\alpha))$ and the auxiliary problem $(\mathrm{A}(\lambda,\omega,\omega_T))$ and solved the latter.

In view of Theorem 4.1, $\Phi_L(\omega,\omega_T,\alpha)\subseteq\cup_\lambda\Phi_A(\lambda,\omega,\omega_T)$. While Theorem 3.1 gives the optimal solution of problem $(\mathrm{A}(\lambda,\omega,\omega_T))$ for any λ, we need only to find the appropriate λ that enables the optimal solution of $(\mathrm{A}(\lambda,\omega,\omega_T))$ to be the solution of problem $(\mathrm{L}(\omega,\omega_T,\alpha))$.

Define the matrix $\Psi=\mathrm{diag}(2\omega_1(1+\alpha_1),\ldots,2\omega_{T-1}(1+\alpha_{T-1}),2\omega_T)$. From Eqs. (9), (10), (B.3), (B.7), and noted that Ψ is invertible, we have

$$(\Lambda-\Psi^{-1})\begin{pmatrix}\lambda_1^*\\ \vdots\\ \lambda_{T-1}^*\\ \lambda_T^*\end{pmatrix}=-\begin{pmatrix}\mathrm{E}[e'\overline{B}_0^0 z_0]\\ \vdots\\ \mathrm{E}[e'\overline{B}_0^{T-1}z_0]\end{pmatrix}-\Psi^{-1}\begin{pmatrix}-2\omega_1\alpha_1 b_1\\ \vdots\\ -2\omega_{T-1}\alpha_{T-1}b_{T-1}\\ 1\end{pmatrix}. \tag{11}$$

Therefore, the λ^* we need can be obtained by the following ways:

(1) If $\Lambda-\Psi^{-1}$ is nonsingular, the λ^* can be found uniquely by solving the system of linear equations (11).
(2) If $\Lambda-\Psi^{-1}$ is singular, the λ^* can be found by solving the problem

$$\begin{aligned}&\max_{\lambda} && \left\{\mathrm{E}[s_T]-\omega_T\mathrm{Var}(s_T)-\sum_{t=1}^{T-1}\omega_t\left[\mathrm{Var}(s_t)-\alpha_t(\mathrm{E}[s_t]-b_t)^2\right]\right\}\Big|_{u^*}\\ &\text{subject to} && \lambda \text{ satisfies } (11),\end{aligned}$$

where $u^*=u^*(\lambda,\omega)\in\Phi_A(\lambda,\omega,\omega_T)$, and the objective function is the value of the objective function of $(\mathrm{L}(\omega,\omega_T,\alpha))$ under strategy $u^*(\lambda,\omega)$. In particular, if $\mathrm{rank}\left(\Lambda-\Psi^{-1}\right)=T-1$, we can calculate the optimal λ^* by a line search method. If $\mathrm{rank}\left(\Lambda-\Psi^{-1}\right)<T-1$, then the Lagrangian multiplier method can be used to get the optimal λ^*.

After the λ^* is obtained, substituting it back to (3) yields the optimal strategy for problem $(\mathrm{L}(\omega,\omega_T,\alpha))$.

6. Solving the generalized mean–variance model

Finally, in this section we solve the generalized mean–variance model (GMV (ω_T,α)).

Because $(\mathrm{L}(\omega,\omega_T,\alpha))$ is the Lagrangian problem of $(\mathrm{GMV}(\omega_T,\alpha))$ and we have solved the problem $(\mathrm{L}(\omega,\omega_T,\alpha))$, the remaining work is to search for an appropriate Lagrangian multiplier vector such that the optimal solution of the primal problem $(\mathrm{GMV}(\omega_T,\alpha))$ can be given by the optimal solution of the Lagrangian problem $(\mathrm{L}(\omega,\omega_T,\alpha))$. Similar to Zhu et al. [6], we use the primal–dual method to solve the problem.

Consider the following Lagrangian dual problem LD (ω_T,α):

$$\min_{\omega\geqslant 0}\quad H(\omega)=\max_u \mathrm{E}[s_T]-\omega_T\mathrm{Var}(s_T)-\sum_{t=1}^{T-1}\omega_t\left[\mathrm{Var}(s_t)-\alpha_t(\mathrm{E}[s_t]-b_t)^2\right],$$

where the dual function $H(\omega)$ is the maximum value of $(\mathrm{L}(\omega,\omega_T,\alpha))$ for given ω. It can be verified that $H(\omega)$ is a convex function. By the primal–dual method, we indirectly solve the primal problem (GMV (ω_T,α)) by solving the dual problem (LD (ω_T,α)).

Given $\omega=\bar{\omega}$, suppose that $\bar{u}(\bar{\omega})$ is the optimal strategy to $(\mathrm{L}(\bar{\omega},\omega_T,\alpha))$. Denote $g(\bar{\omega};\bar{u})=(g_1(\bar{\omega};\bar{u}),\ldots,g_{T-1}(\bar{\omega};\bar{u}))'$, where, for $t=1,\ldots,T-1$,

$$g_t(\bar{\omega};\bar{u})=\begin{cases}\left[\mathrm{Var}(s_t)-\alpha_t(\mathrm{E}[s_t]-b_t)^2\right]|_{\bar{u}(\bar{\omega})}, & \bar{\omega}_t>0,\\ \max\left\{0,\left[\mathrm{Var}(s_t)-\alpha_t(\mathrm{E}[s_t]-b_t)^2\right]|_{\bar{u}(\bar{\omega})}\right\}, & \bar{\omega}_t=0.\end{cases}$$

According to Theorem 6.4.1 of Bazaraa and Shetty [21], if $g(\bar{\omega};\bar{\pi})\neq 0$, then $g(\bar{\omega};\bar{\pi})$ is a feasible descent direction of $H(\omega)$ at $\bar{\omega}$; if $g(\bar{\omega};\bar{\pi})=0$, then $\bar{\omega}$ is an optimal solution to (LD (ω_T,α)). In the latter case, by means of the Lagrangian dual theory (see Theorem 6.5.1 of Bazaraa and Shetty [21]), $\bar{\pi}(\bar{\omega})$ is the optimal portfolio strategy of $(\mathrm{GMV}(\omega_T,\alpha))$.

Based on the above discussion, a primal–dual iterative algorithm is proposed as follows.

Algorithm 6.1. Step 0. Choose an initial point $\omega^0\geqslant 0$ and a very small positive number ε. Let $k=0$;

Step 1. Solve subproblem $(\mathrm{L}(\omega^k,\omega_T,\alpha))$ via $(\mathrm{A}(\lambda,\omega,\omega_T))$. Denote its optimal strategy as $u^k(\omega^k)$. If $\left|g_t(\omega^k;u^k)\right|\leqslant\varepsilon$ for all $t=1,\ldots,T-1$, then stop the algorithm and export ω^k and $u^k(\omega^k)$ as the optimal solutions of the Lagrangian dual problem (LD (ω_T,α)) and Lagrangian problem $(\mathrm{L}(\omega^k,\omega_T,\alpha))$ respectively. Else, go to Step 2;

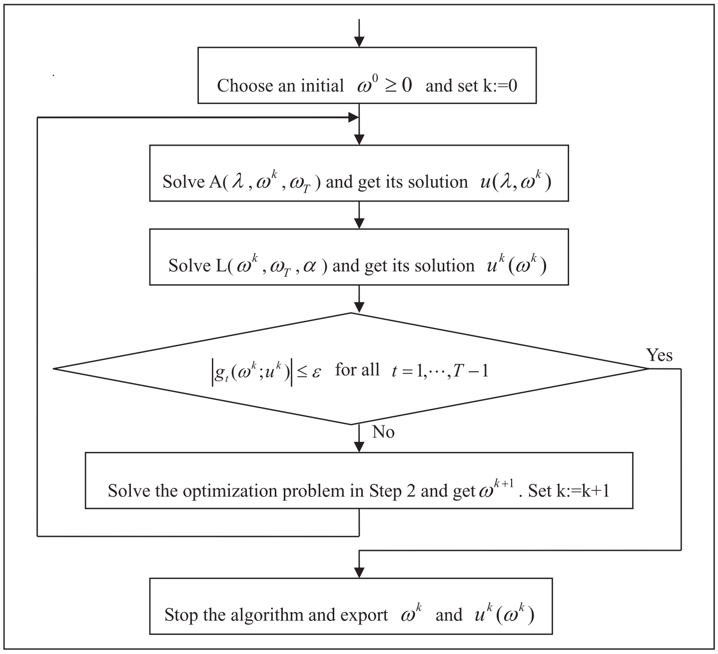

Fig. 1. Flow chart of Algorithm 1.

Step 2. Solve

$$\begin{aligned} \min_{\delta} \quad & H(\omega^k + \delta g(\omega^k; u^k)) \\ \text{subject to} \quad & \omega^k + \delta g(\omega^k; u^k) \geqslant 0, \\ & \delta \geqslant 0. \end{aligned}$$

Let δ^k be the optimal solution, and let $\omega^{k+1} = \omega^k + \delta^k g(\omega^k; u^k)$. Set $k := k + 1$, and go back to Step 1.

The flow chart of the algorithm is depicted in Fig. 1.

Remark 6.1. Based on the discussion of Section 6.4 in Bazaraa and Shetty [21], if a finite optimal solution δ^k in Step 2 of Algorithm 6.1 does not exist, either the optimal objective value is unbounded from below, or else it is bounded but can not be achieved at any particular δ. In the first case, we stop with the conclusion that the dual problem is unbounded and the primal is infeasible. In the latter case, δ^k could be taken as a sufficiently large number. We apply the Penalty Function Method to solve the optimization problem in Step 2. The Penalty Function Method can be regarded as a special case of Tikhonov Regularization. In fact, by choosing penalty parameter large enough, the solution of penalty problem can be made arbitrarily close to the solution of the original problem, and the selected penalty parameter is a special Tikhonov factor. For more details of the method, we refer the reader to Bazaraa and Shetty [21].

In summary, we obtain the relationship among problems (L(ω, ω_T, α)), (GMV(ω_T, α)) and (LD(ω_T, α)) as follows.

Theorem 6.1. *Suppose that $\bar{\omega}$ is the optimal solution of (LD(ω_T, α)), and suppose that $\bar{u}(\bar{\omega})$ is the optimal solution of (L($\bar{\omega}, \omega_T, \alpha$)). Then $\bar{u}(\bar{\omega})$ is an optimal solution of (GMV(ω_T, α)).*

7. A numerical example

In this section, we provide a numerical example to analyze how the bankrupt control impacts on the optimal investment strategy and how the market parameters impact on the efficient frontier.

In order to investigate the impact of bankrupt control which is introduced in our model, we consider the example in Yi et al. [15] and compare the results of our model and their model. The basic parameter settings are the same as the ones in Yi et al. [15]. Concretely, the investor has 2 units of assets and 1 unit of liability and hence 1 unit surplus at the beginning of time horizon, that is, $x_0 = 2,\ l_0 = 1, s_0 = 1$. He is trying to find an optimal wealth allocation among two assets A and B under the influence of an exogenous and uncontrollable liability C. The length of time horizon is $T = 4$. The expected returns for assets A and B and liability C are $E[r_t^A] = 1.159,\ E[r_t^B] = 1.243$ and $E[q_t^C] = 1.224$, for $t = 0, 1, 2, 3$, while the covariance of $R_t = (r_t^A, r_t^B, q_t^C)'$ is given as

$$\mathrm{Cov}(R_t) = \begin{pmatrix} 0.0148 & 0.0185 & 0.0146 \\ 0.0185 & 0.0855 & 0.0105 \\ 0.0146 & 0.0105 & 0.0288 \end{pmatrix} \quad (t = 0, 1, 2, 3).$$

Taking asset A as the benchmark, we have

$$\mathrm{E}[r_t^0] = \mathrm{E}[r_t^A] = 1.159, \mathrm{E}[r_t^1] = \mathrm{E}[r_t^B - r_t^A] = 0.084, \quad \mathrm{E}[q_t] = \mathrm{E}[q_t^C] = 1.224.$$

Unlike the one in Yi et al. [15], the investor in our example intends not only to maximize $\mathrm{E}[s_T] - \omega_T \mathrm{Var}(s_T)$, but also to prevent surplus from falling below a predefined disaster level in any intermediate period. Let $\omega_4 = 0.2$, $b_t = 0$ for $t = 1, 2, 3$, $\alpha = (\alpha_1, \alpha_2, \alpha_3) = (0.12, 0.15, 0.15)$. Using the method described in this paper, (GMV(ω_T, α)), i.e. the multi-period AL management problem with bankrupt control, is solved.

(1) Impact of bankrupt control on the optimal strategy

In order to examine the impact of the bankrupt control on the optimal investment strategy, we compare the mean–variance efficient frontiers in $\mathrm{Var}(s_T)$-$\mathrm{E}[s_T]$ plane, generated by solving (MV (ω_T)) and (GMV (ω_T, α)) with $\alpha = (0.12, 0.15, 0.15)$ respectively.

In Fig. 2, the subgraph 2–1 and 2–2 demonstrate respectively the efficient frontier of problem (MV(ω_T)), denoted by EF-MV, and the efficient frontier of problem (GMV(ω_T, α)), denoted by EF-GMV. Because of the bankrupt risk control, we can not obtain the analytical solution of problem (GMV(ω_T, α)). The efficient frontiers has been computed for ω_T going from 0.1 to 5 with a step size of 0.05. The series of means and variances of terminal surplus are obtained corresponding to the different values of ω_T. In the subgraph 2–3, it is observed that the efficient frontier of problem (MV(ω_T)) is higher than that of problem (GMV(ω_T, α)). So, the investor gets a smaller expected terminal surplus from problem (GMV(ω_T, α)) with consideration of bankrupt control than the one from problem (MV(ω_T)).

In order to investigate the effect of bankrupt control, a Monte Carlo simulation of $SN (= 10000)$ samples was performed. We consider the above example with $\omega_T = 0.2$ and $\alpha = (0.10, 0.10, 0.10)$. The corresponding results are shown in Table 1, where $\mathrm{E}^M[s_t]$ and $\mathrm{E}^G[s_t]$ denote the theoretical expected value of s_t, and the subscripts M and G represent the model MV and GMV. $\mathrm{Var}^M(s_t)$ and $\mathrm{Var}^G(s_t)$ denote the theoretical variance of s_t, BN_t^M and BN_t^G are numbers of going bankrupt for the investor at period t in the simulation, and $\mathrm{P}^M(BF_t)$ as well as $\mathrm{P}^G(BF_t)$ are the estimations of bankrupt frequency, which are defined by BN_t^M/SN and BN_t^G/SN respectively. It can be observed from Table 1 that the bankrupt frequency $\mathrm{P}^G(BF_t)$ corre-

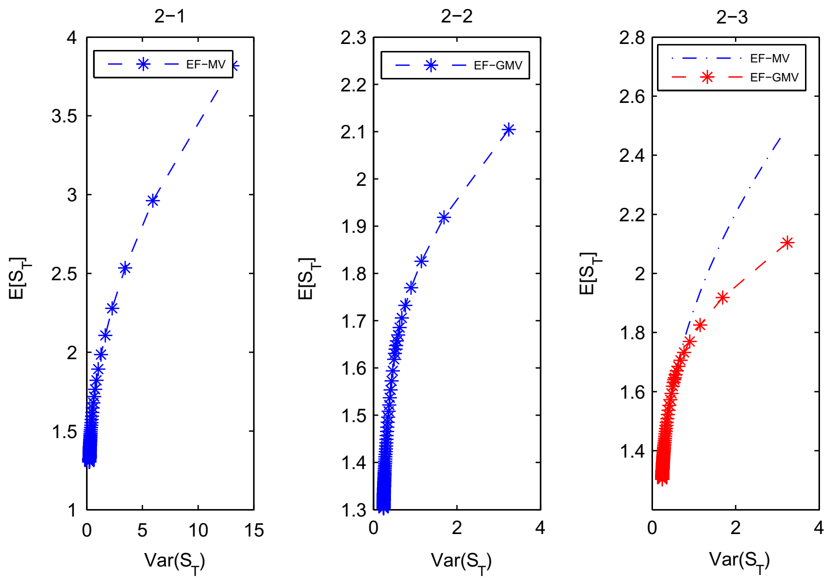

Fig. 2. Efficient frontiers of the example with MV and GMV models.

Table 1
Bankrupt frequency in the example with MV and GMV models.

Period t	$\mathrm{E}^M[s_t]$	$\mathrm{Var}^M(s_t)$	BN_t^M	$\mathrm{P}^M(BF_t)$	$\mathrm{E}^G[s_t]$	$\mathrm{Var}^G(s_t)$	BN_t^G	$\mathrm{P}^G(BF_t)$
1	1.3253	0.5475	376	0.0376	1.1032	0.0330	0	0
2	1.6977	1.4907	1716	0.1716	1.2086	0.0847	32	0.0032
3	2.1220	2.2816	2482	0.2482	1.3133	0.1751	530	0.0530
4	2.6032	3.6057	2884	0.2884	1.6677	0.9923	1637	0.1637

Table 2
The mean and variance of surplus of each period in GMV model with different parameters ω_4 and α.

Parameters	$E[s_1]$	$Var(s_1)$	$E[s_2]$	$Var(s_2)$	$E[s_3]$	$Var(s_3)$	$E[s_4]$	$Var(s_4)$
$\omega_4 = 1$, $\alpha = (0.10, 0.10, 0.10)$	1.1015	0.0322	1.2047	0.0826	1.3064	0.1716	1.4253	0.2982
$\omega_4 = 1$, $\alpha = (0.12, 0.15, 0.15)$	1.1213	0.0438	1.2482	0.1109	1.3784	0.2257	1.5084	0.3640
$\omega_4 = 2$, $\alpha = (0.12, 0.15, 0.15)$	1.0975	0.0307	1.1957	0.0789	1.2913	0.1645	1.3800	0.2678

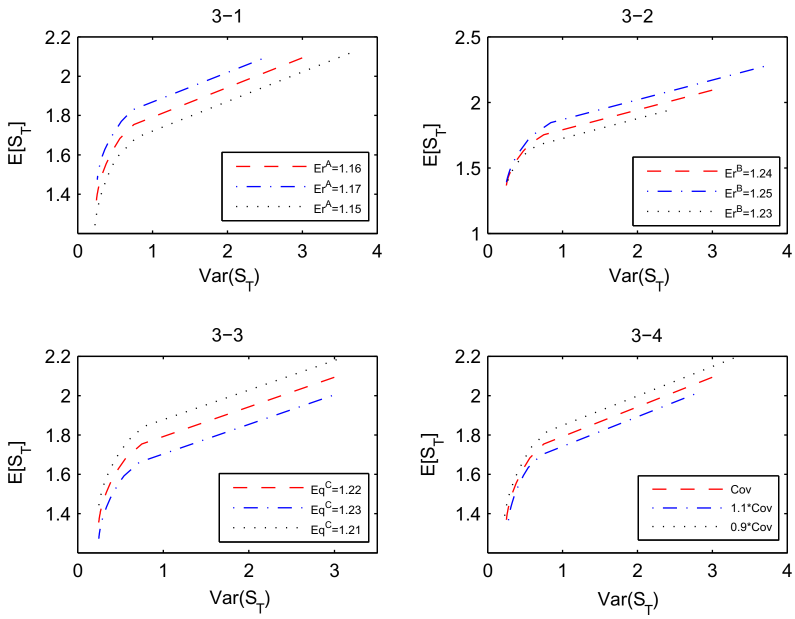

Fig. 3. The impact of market parameters on the efficient frontier of GMV model.

sponding to the model GMV decreases significantly when compared to $P^M(BF_t)$ of model MV. Note that the bankrupt frequency in the terminal period of model GMV is 0.1637, which is much higher than the one in previous period. The reason is that the bankrupt control is imposed only on the intermediate periods, rather than the terminal period.

Table 2 shows how different parameters ω_T and α affect the optimal strategy of problem (GMV (ω_T, α)). In the above example, we take $\omega_4 = 1$ and $\alpha = (0.12, 0.15.0.15)$ as benchmark parameters. From Table 2, we can see that, when ω_4 increases or α decreases, both the expectation and the variance of surplus of each intermediate period as well as terminal period decrease. So increasing ω_T or reducing α results in a comparatively conservative optimal strategy.

(2) Impact of market parameters on the efficient frontier

The efficient frontier in each subgraph of Fig. 3 is computed by the same method as the one for Fig. 2. Fig. 3 shows that when the market parameters are perturbed, Algorithm 6.1 still works well. Furthermore, the subgraph 3–1 and 3–2 show that the efficient frontier of problem (GMV (ω_T, α)) moves up as the expected rate of return of asset A or asset B increases; the subgraph 3–3 and 3–4 show that the efficient frontier of problem (GMV (ω_T, α)) moves down as the expected rate of return of liability C or the variances and covariances of financial instruments increases. Thus, the higher the expected rates of return of assets, the bigger the expected terminal surplus with the same variance. With regards to the expected rate of return of liability and the variances and covariances of financial instruments, the conclusion is just the opposite.

8. Conclusion

This paper investigates a generalized mean–variance model for multi-period AL management with bankrupt control over intermediate periods. Although the model is much more complicated than the case without bankrupt control, we develop a method to solve it. The outline of the solution method is that: (i) constructing the Lagrangian problem and the Lagrangian dual problem of the primal model, and the auxiliary problem of the Lagrangian problem; (ii) for a given Lagrangian multiplier ω, explicitly solving the auxiliary problem for any auxiliary parameter λ by using the dynamic programming approach; (iii) getting the appropriate auxiliary parameter λ by using the method in Section 5 and hence finding the optimal solution u of the Lagrangian problem; (iv) judging whether the (ω, u) solves the Lagrangian dual problem and hence deciding to go on or to stop. When the algorithm stops, u is the optimal solution of the primal generalized mean–variance model. To illustrate our method and results, we give a numerical example. The numerical analysis shows that the bankrupt control has a significant impact on the optimal portfolio strategy.

In our model, we use the Tchebycheff inequality to substitute for the original bankrupt control. The consequence is that the corresponding strategy is too conservative for the investor. It is worth to consider other methods instead of Tchebycheff inequality to handle the bankrupt control. Also it would be interesting to extend the discrete-time model of the AL management with bankrupt control to the continuous-time model.

Appendix A. The proof of Theorem 3.1

We prove this theorem by the dynamic programming approach. The value functions $f_t(z_t) = \max_{u_t,\ldots,u_{T-1}} \mathrm{E}\left[\sum_{s=t}^{T}(\lambda_s e'z_s - \omega_s z_s' ee' z_s)\right]$, for $t = 0, 1, \ldots, T$. Then we have the Bellman equations

$$f_t(z_t) = \max_{u_t} \mathrm{E}\left[\lambda_t e'z_t - \omega_t z_t' ee' z_t + f_{t+1}(z_{t+1})\right] \quad (t = 0, 1, \ldots, T-1)$$

with the terminal condition $f_T(z_T) = \lambda_T e'z_T - \omega_T z_T' ee' z_T$.

We begin from stage $T-1$. When $t = T-1$, one has

$$\begin{aligned} f_{T-1}(z_{T-1}) = \max_{u_{T-1}} \mathrm{E}\big[&\lambda_{T-1} e'z_{T-1} - \omega_{T-1} z_{T-1}' ee' z_{T-1} + \lambda_T e'(B_{T-1}z_{T-1} + A_{T-1}e_1 u_{T-1}) \\ &-\omega_T (B_{T-1}z_{T-1} + A_{T-1}e_1 u_{T-1})' ee' (B_{T-1}z_{T-1} + A_{T-1}e_1 u_{T-1})\big]. \end{aligned}$$

Its first-order condition gives its optimal solution

$$u_{T-1}^* = \frac{\lambda_T}{2\omega_T} \frac{\mathrm{E}[e_1' A_{T-1}' e]}{\mathrm{E}[e_1' A_{T-1}' ee' A_{T-1} e_1]} - \frac{\mathrm{E}[e_1' A_{T-1}' ee' B_{T-1} z_{T-1}]}{\mathrm{E}[e_1' A_{T-1}' ee' A_{T-1} e_1]}.$$

Thus $s_T = e'z_T = e'\overline{B}_{T-1}^0 z_{T-1} + \frac{\lambda_T}{2\omega_T} r_{T-1}^1 Y_{T-1}$, where

$$Y_{T-1} = \frac{\mathrm{E}[e_1' A_{T-1}' e]}{\mathrm{E}[e_1' A_{T-1}' ee' A_{T-1} e_1]} = \frac{\mathrm{E}[r_{T-1}^1]}{\mathrm{E}[(r_{T-1}^1)^2]}, \quad \overline{B}_{T-1}^0 = \begin{pmatrix} \bar{r}_{T-1}^0 & 0 \\ 0 & \bar{q}_{T-1}^0 \end{pmatrix}, \quad \bar{r}_{T-1}^0 = r_{T-1}^0 - r_{T-1}^1 \frac{\mathrm{E}[r_{T-1}^1 r_{T-1}^0]}{\mathrm{E}[(r_{T-1}^1)^2]},$$

$$\bar{q}_{T-1}^0 = q_{T-1} - r_{T-1}^1 \frac{\mathrm{E}[r_{T-1}^1 q_{T-1}]}{\mathrm{E}[(r_{T-1}^1)^2]}.$$

Furthermore, noting that $r_{T-1}^1 = e_1' A_{T-1} e$, we have

$$\mathrm{E}\left[(e'\overline{B}_{T-1}^0 z_{T-1})(r_{T-1}^1 Y_{T-1})\right] = 0, \quad \mathrm{E}[(r_{T-1}^1 Y_{T-1})^2] = \mathrm{E}[r_{T-1}^1 Y_{T-1}].$$

Therefore, substituting u_{T-1}^* back to $f_{T-1}(z_{T-1})$, we have

$$f_{T-1}(z_{T-1}) = -z_{T-1}' \mathrm{E}[D_{T-1}] z_{T-1} + \mathrm{E}[F_{T-1}'] z_{T-1} + C_{T-1},$$

where

$$\begin{aligned} &\widetilde{F}_{T-1} = \overline{B}_{T-1}^{0\prime} e \lambda_T, \quad F_{T-1} = e\lambda_{T-1} + \widetilde{F}_{T-1} = \begin{pmatrix} \lambda_{T-1} + \lambda_T \bar{r}_{T-1}^0 \\ -\lambda_{T-1} - \lambda_T \bar{q}_{T-1}^0 \end{pmatrix}, \\ &D_{T-1} = \overline{D}_{T-1} \overline{D}_{T-1}' = \omega_{T-1} ee' + \omega_T \overline{B}_{T-1}^{0\,\prime} ee' \overline{B}_{T-1}^0, \\ &\overline{D}_{T-1}' = \begin{pmatrix} \bar{r}_{T-1}^0 \omega_T^{\frac{1}{2}} & -\bar{q}_{T-1}^0 \omega_T^{\frac{1}{2}} \\ \omega_{T-1}^{\frac{1}{2}} & -\omega_{T-1}^{\frac{1}{2}} \end{pmatrix}, \quad C_{T-1} = \frac{\lambda_T^2}{4\omega_T} \mathrm{E}[r_{T-1}^1 Y_{T-1}]. \end{aligned}$$

Suppose that the optimal strategy and the value function at time $t < T-1$ are given by (3) and

$$f_t(z_t) = -z_t' \mathrm{E}[\overline{D}_t \overline{D}_t'] z_t + \mathrm{E}[F_t'] z_t + \sum_{i=t}^{T-1} C_i,$$

where $\overline{D}_k,\ F_k,\ \overline{B}_k^0,\ \overline{B}_k^i,\ \bar{r}_k^0,\ \bar{r}_k^i,\ \bar{q}_k^0,\ \bar{q}_k^i$ are given by Eqs. (4)–(8), for $k = t, t+1, \ldots, T-1$ and $i = 1, \ldots, T-k-1$. In addition,

$$\begin{aligned} &\widetilde{D}_k = \left(\omega_T^{\frac{1}{2}} \overline{B}_k^{T-k-1} e, \quad \ldots, \quad \omega_s^{\frac{1}{2}} \overline{B}_k^{s-k-1} e, \quad \ldots, \quad \omega_{k+1}^{\frac{1}{2}} \overline{B}_k^0 e\right), \\ &\widetilde{F}_k' = \sum_{s=k+1}^{T} e' \overline{B}_k^{s-k-1} \lambda_s, \quad C_k = \frac{1}{4} \mathrm{E}[R_k^2 Y_k], \\ &Y_k = \frac{\mathrm{E}[r_k^1]}{\mathrm{E}[(r_k^1)^2]} \frac{\mathrm{E}\left[\sum_{s=k+2}^{T} \bar{r}_{k+1}^{s-k-2} \lambda_s + \lambda_{k+1}\right]}{\mathrm{E}\left[\sum_{s=k+2}^{T} (\bar{r}_{k+1}^{s-k-2})^2 \omega_s + \omega_{k+1}\right]}, \quad R_k^2 = r_k^1 \left(\sum_{s=k+2}^{T} \bar{r}_{k+1}^{s-k-2} \lambda_s + \lambda_{k+1}\right), \\ &R_k^{1\prime} = \left(r_k^1 \bar{r}_{k+1}^{T-k-2} \omega_T^{\frac{1}{2}}, \quad \ldots, \quad r_k^1 \bar{r}_{k+1}^{s-k-2} \omega_s^{\frac{1}{2}}, \quad \ldots, \quad r_k^1 \bar{r}_{k+1}^0 \omega_{k+2}^{\frac{1}{2}}, \quad r_k^1 \omega_{k+1}^{\frac{1}{2}}\right) \end{aligned}$$

for $k = t, t+1, \ldots, T-1$ and $i = 1, \ldots, T-k-1$. Then, the optimization problem at time $t-1$ for a given z_{t-1} is

$$f_{t-1}(z_{t-1}) = \max_{u_{t-1}} \mathrm{E}\Bigg[\lambda_{t-1}e'z_{t-1} - \omega_{t-1}z'_{t-1}ee'z_{t-1} + F'_t(B_{t-1}z_{t-1} + A_{t-1}e_1u_{t-1}) + \sum_{i=t}^{T-1} C_i - (B_{t-1}z_{t-1} + A_{t-1}e_1u_{t-1})'\overline{D}_t\overline{D}'_t(B_{t-1}z_{t-1} + A_{t-1}e_1u_{t-1})\Bigg].$$

The first-order condition of the optimization problem gives its optimal solution

$$u^*_{t-1} = \frac{1}{2}\frac{\mathrm{E}[e'_1A'_{t-1}F_t]}{\mathrm{E}[e'_1A'_{t-1}\overline{D}_t\overline{D}'_tA_{t-1}e_1]} - \frac{\mathrm{E}[e'_1A'_{t-1}\overline{D}_t\overline{D}'_tB_{t-1}z_{t-1}]}{\mathrm{E}[e'_1A'_{t-1}\overline{D}_t\overline{D}'_tA_{t-1}e_1]}.$$

Hence,

$$s_t = e'\overline{B}^0_{t-1}z_{t-1} + \frac{1}{2}r^1_{t-1}Y_{t-1}, \quad \overline{D}'_tz_t = \widetilde{D}'_{t-1}z_{t-1} + \frac{1}{2}R^1_{t-1}Y_{t-1},$$
$$F'_tz_t = \widetilde{F}'_{t-1}z_{t-1} + \frac{1}{2}R^2_{t-1}Y_{t-1},$$

where

$$\widetilde{D}_{t-1} = \left(\omega_T^{\frac{1}{2}}\overline{B}^{T-t}_{t-1}e, \quad \ldots, \quad \omega_s^{\frac{1}{2}}\overline{B}^{s-t}_{t-1}e, \quad \ldots, \quad \omega_t^{\frac{1}{2}}\overline{B}^0_{t-1}e\right), \quad \widetilde{F}'_{t-1} = \sum_{s=t}^{T} e'\overline{B}^{s-t}_{t-1}\lambda_s,$$

$$R^1_{t-1}{}' = \left(r^1_{t-1}\bar{r}^{T-t-1}_t\omega_T^{\frac{1}{2}}, \quad \ldots, \quad r^1_{t-1}\bar{r}^{s-t-1}_t\omega_s^{\frac{1}{2}}, \quad \ldots, \quad r^1_{t-1}\bar{r}^0_t\omega_{t+1}^{\frac{1}{2}}, \quad r^1_{t-1}\omega_t^{\frac{1}{2}}\right),$$

$$R^2_{t-1} = r^1_{t-1}\left(\sum_{s=t+1}^{T}\bar{r}^{s-t-1}_t\lambda_s + \lambda_t\right), \quad Y_{t-1} = \frac{\mathrm{E}[r^1_{t-1}]}{\mathrm{E}[(r^1_{t-1})^2]}\frac{\mathrm{E}\left[\sum_{s=t+1}^{T}\bar{r}^{s-t-1}_t\lambda_s + \lambda_t\right]}{\mathrm{E}\left[\sum_{s=t+1}^{T}(\bar{r}^{s-t-1}_t)^2\omega_s + \omega_t\right]}$$

with $\overline{B}^0_{t-1}, \ \overline{B}^i_{t-1}, \ \bar{r}^0_{t-1}, \bar{r}^i_{t-1}, \ \bar{q}^0_{t-1}, \ \bar{q}^i_{t-1}$ are given by Eqs. (5)–(8), for $i = 1, 2, \ldots, T-t$. Furthermore,

$$\mathrm{E}\left[(R^1_{t-1}Y_{t-1})'(\widetilde{D}'_{t-1}z_{t-1})\right] = 0, \quad \mathrm{E}\left[(R^1_{t-1}Y_{t-1})'(R^1_{t-1}Y_{t-1})\right] = \mathrm{E}[R^2_{t-1}Y_{t-1}].$$

Substituting above equations into $f_{t-1}(z_{t-1})$, we have

$$f_{t-1}(z_{t-1}) = -z'_{t-1}\mathrm{E}[D_{t-1}]z_{t-1} + \mathrm{E}[F'_{t-1}]z_{t-1} + \sum_{i=t-1}^{T-1} C_i,$$

where

$$D_{t-1} = \overline{D}_{t-1}\overline{D}'_{t-1} = \omega_{t-1}ee' + \widetilde{D}_{t-1}\widetilde{D}'_{t-1}, \quad \overline{D}'_{t-1} = \left(\tilde{D}_{t-1}, \omega_{t-1}^{\frac{1}{2}}e\right),$$

$$F_{t-1} = \lambda_{t-1}e + \widetilde{F}_{t-1} = \begin{pmatrix} \sum_{s=t}^{T}\bar{r}^{s-t}_{t-1}\lambda_s + \lambda_{t-1} \\ -\sum_{s=t}^{T}\bar{q}^{s-t}_{t-1}\lambda_s - \lambda_{t-1} \end{pmatrix}, \quad C_{t-1} = \frac{1}{4}\mathrm{E}[R^2_{t-1}Y_{t-1}].$$

By induction, the theorem is proved.

Appendix B. The proof of Theorem 4.1

(i) Denote $\Gamma(u) = \left(\mathrm{E}[s_1^2], \ldots, \mathrm{E}[s_T^2], \mathrm{E}[s_1], \ldots, \mathrm{E}[s_T]\right)'|_u$, where, for $t = 1, \ldots, T, \mathrm{E}[s_t^2]$ and $\mathrm{E}[s_t]$ are computed under the strategy u. Denote $U(\Gamma(u))$ as the objective function of $(\mathrm{L}(\omega, \omega_T, \alpha))$, then

$$U(\Gamma(u)) = \left[\mathrm{E}[s_T] - \sum_{t=1}^{T-1} 2\omega_t\alpha_tb_t\mathrm{E}[s_t] + \omega_T(\mathrm{E}[s_T])^2 + \sum_{t=1}^{T-1}\omega_t\alpha_tb_t^2 + \sum_{t=1}^{T-1}\omega_t(1+\alpha_t)(\mathrm{E}[s_t])^2 - \omega_T\mathrm{E}[s_T^2] - \sum_{t=1}^{T-1}\omega_t\mathrm{E}[s_t^2]\right]\Bigg|_u. \tag{B.1}$$

It is obvious that U is a linear function of $\mathrm{E}[s_1^2], \ldots, \mathrm{E}[s_T^2]$, and a quadratic function of $\mathrm{E}[s_1], \ldots, \mathrm{E}[s_T]$ with positive coefficients of the quadratic terms.

By contradiction, assume that $u^*(\omega)$ is not in $\Phi_A(\lambda^*, \omega, \omega_T)$. Then there exists a strategy $u(\omega)$ such that

$$[-\omega', -\omega_T, \lambda^*]\Gamma(u(\omega)) > [-\omega', -\omega_T, \lambda^*]\Gamma(u^*(\omega)).$$

By the definition of convex functions, we have

$$U(\Gamma(u(\omega))) \geqslant U(\Gamma(u^*(\omega))) + [-\omega', -\omega_T, \lambda^*](\Gamma(u(\omega)) - \Gamma(u^*(\omega)).$$

Combining the above two inequalities yields $U(\Gamma(u(\omega))) > U(\Gamma(u^*(\omega)))$. This contradicts the assumption $u^*(\omega) \in \Phi_L(\omega, \omega_T, \alpha)$.

(ii) By (i), $\Phi_L(\omega, \omega_T, \alpha) \subseteq \bigcup_\lambda \Phi_A(\lambda, \omega, \omega_T)$. That is, the optimal solution of (L(ω, ω_T, α)) must be the optimal solution of (A($\lambda, \omega, \omega_T$)) for some λ. Hence, the λ solves the following optimization problem:

$$\max_{\lambda} \quad U(\Gamma(u(\lambda, \omega))), \tag{B.2}$$

where $u(\lambda, \omega)$ is the optimal strategy of (A($\lambda, \omega, \omega_T$)). On the one hand, from (B.1), the first-order necessary condition for the optimal solution λ^* to (B.2) is

$$(1 + 2\omega_T \mathrm{E}[s_T(\lambda^*, \omega)]) \frac{\partial \mathrm{E}[s_T(\lambda^*, \omega)]}{\partial \lambda_k} - \sum_{t=1}^{T} \omega_t \frac{\partial \mathrm{E}\left[s_t^2(\lambda^*, \omega)\right]}{\partial \lambda_k} + \sum_{t=1}^{T-1} (-2\omega_t \alpha_t b_t + 2\omega_t(1 + \alpha_t)\mathrm{E}[s_t(\lambda^*, \omega)]) \frac{\partial \mathrm{E}[s_t(\lambda^*, \omega)]}{\partial \lambda_k} = 0$$
$$(k = 1, \ldots, T).$$

On the other hand, since $u^*(\lambda^*, \omega) \in \Phi_A(\lambda^*, \omega, \omega_T)$, we have the first-order condition

$$\sum_{t=1}^{T} \lambda_t^* \frac{\partial \mathrm{E}[s_t(\lambda^*, \omega)]}{\partial \lambda_k} - \sum_{t=1}^{T} \omega_t \frac{\partial \mathrm{E}\left[s_t^2(\lambda^*, \omega)\right]}{\partial \lambda_k} = 0 \quad (k = 1, \ldots, T).$$

Combining the pervious two equations yields

$$\left(\lambda_T^* - 1 - 2\omega_T \mathrm{E}[s_T(\lambda^*, \omega)]\right) \frac{\partial \mathrm{E}[s_T(\lambda^*, \omega)]}{\partial \lambda_k} + \sum_{t=1}^{T-1} \left(\lambda_t^* + 2\omega_t \alpha_t b_t - 2\omega_t(1 + \alpha_t)\mathrm{E}[s_t(\lambda^*, \omega)]\right) \frac{\partial \mathrm{E}[s_t(\lambda^*, \omega)]}{\partial \lambda_k} = 0$$

for $k = 1, \ldots, T$. For simplicity, we write $s_t(\lambda^*, \omega)$ as $s_t|_{u^*(\lambda^*, \omega)}$ or as s_t if not causing confusion. To complete the proof, we need only to prove that the following T vectors

$$\left.\left(\frac{\partial \mathrm{E}[s_1]}{\partial \lambda_k}, \quad \frac{\partial \mathrm{E}[s_2]}{\partial \lambda_k}, \quad \ldots, \quad \frac{\partial \mathrm{E}[s_T]}{\partial \lambda_k}\right)\right|_{u^*(\lambda^*, \omega)} \quad (k = 1, \ldots, T)$$

are linearly independent, or equivalently to show that the matrix

$$\Lambda = \left.\begin{pmatrix} \frac{\partial}{\partial \lambda_1}\mathrm{E}[s_1] & \ldots & \frac{\partial}{\partial \lambda_T}\mathrm{E}[s_1] \\ \cdots & \cdots & \cdots \\ \frac{\partial}{\partial \lambda_1}\mathrm{E}[s_T] & \ldots & \frac{\partial}{\partial \lambda_T}\mathrm{E}[s_T] \end{pmatrix}\right|_{u^*(\lambda^*, \omega)} \tag{B.3}$$

is nonsingular. To this end, we first derive the expression of $\mathrm{E}[s_t]$ for $t = 1, 2, \ldots, T$. Similar to the proof of Theorem 3.1, let

$$Y_t = \frac{\mathrm{E}[r_t^1]}{\mathrm{E}[(r_t^1)^2]} \frac{\mathrm{E}\left[\sum_{s=t+2}^{T} \bar{r}_{t+1}^{s-t-2} \lambda_s + \lambda_{t+1}\right]}{\mathrm{E}\left[\sum_{s=t+2}^{T} (\bar{r}_{t+1}^{s-t-2})^2 \omega_s + \omega_{t+1}\right]} \tag{B.4}$$

for $t = 0, 1, \ldots, T-1$. Evidently, Y_t is a linear function of $\lambda_{t+1}, \ldots, \lambda_T$ and has nothing to do with $\lambda_1, \ldots, \lambda_t$. Under the optimal strategy given in Theorem 3.1, $e'z_t, e'\overline{B}_{t-1}^0 z_{t-1}, \ldots, e'\overline{B}_0^{t-1} z_0$ can be expressed as

$$\begin{aligned} e'z_t &= e'\overline{B}_{t-1}^0 z_{t-1} + \frac{1}{2} Y_{t-1} r_{t-1}^1, \quad e'\overline{B}_{t-1}^0 z_{t-1} = e'\overline{B}_{t-2}^1 z_{t-2} + \frac{1}{2} Y_{t-2} r_{t-2}^1 \bar{r}_{t-1}^0, \ldots, e'\overline{B}_j^{t-1-j} z_j \\ &= e'\overline{B}_{j-1}^{t-j} z_{j-1} + \frac{1}{2} Y_{j-1} r_{j-1}^1 \bar{r}_j^{t-1-j}, \ldots, e'\overline{B}_1^{t-2} z_1 = e'\overline{B}_0^{t-1} z_0 + \frac{1}{2} Y_0 r_0^1 \bar{r}_1^{t-2}, \quad e'\overline{B}_0^{t-1} z_0 = \bar{r}_0^{t-1} x_0 - \bar{q}_0^{t-1} l_0, \end{aligned}$$

for $t = 1, 2, \ldots, T$. Recall that $s_t = e'z_t$. After some operation, s_t and $e'\overline{B}_j^{t-j-1} z_j$ can be expressed in terms of Y_i, r_i^1 and $\bar{r}_i^j$:

$$s_t = e'\overline{B}_0^{t-1} z_0 + \frac{1}{2}\left(Y_{t-1} r_{t-1}^1 + \sum_{i=0}^{t-2} Y_i r_i^1 \bar{r}_{i+1}^{t-2-i}\right), e'\overline{B}_j^{t-j-1} z_j = e'\overline{B}_0^{t-1} z_0 + \frac{1}{2} \sum_{i=0}^{j-1} Y_i r_i^1 \bar{r}_{i+1}^{t-2-i}. \tag{B.5}$$

Furthermore, the expectation of s_t is

$$\mathrm{E}[s_t] = \mathrm{E}[e'\overline{B}_0^{t-1} z_0] + \frac{1}{2} \mathrm{E}\left[Y_{t-1} r_{t-1}^1 + \sum_{i=0}^{t-2} Y_i r_i^1 \bar{r}_{i+1}^{t-2-i}\right] \tag{B.6}$$

for all $t = 1, \ldots, T$. From its definition, each $\overline{B}_0^t$ $(t = 0, 1, \ldots, T-1)$ is independent of λ_k for $k = 1, \ldots, T$. According to (B.6), we can see that $\mathrm{E}[s_t]$ is a linear function of $Y_0, \ldots, Y_{t-1}$, and Y_i is linear with respect to $\lambda_{i+1}, \ldots, \lambda_T$, for all $i = 0, 1, \ldots, t-1,\ t = 1, \ldots, T$. So $\mathrm{E}[s_t]$ is a linear function of $\lambda = (\lambda_1, \ldots, \lambda_T)'$. Hence we can express $\mathrm{E}[s_t]$ in the following way:

$$\mathrm{E}[s_t] = \mathrm{E}[e'\overline{B}_0^{t-1}z_0] + \lambda'\left(\frac{\partial \mathrm{E}[s_t]}{\partial\lambda_1},\ldots,\frac{\partial \mathrm{E}[s_t]}{\partial\lambda_T}\right). \tag{B.7}$$

By induction, it is not hard to verify that

$$\bar{r}_t^i = \bar{r}_t^0\bar{r}_{t+1}^0\ldots\bar{r}_{t+i}^0 \quad (i = 1,\ldots,T-1-t,\ t = 0,1,\ldots,T-2).$$

Notice that each $\bar{r}_t^0$ depends only on the returns of the two assets in period $t+1$ and that the returns are assumed to be statistically independent among different time periods. Hence we have

$$\mathrm{E}[\bar{r}_t^i] = \mathrm{E}[\bar{r}_t^0]\mathrm{E}[\bar{r}_{t+1}^0]\ldots\mathrm{E}[\bar{r}_{t+i}^0] \quad (i = 1,\ldots,T-t-1,\ t = 0,1,\ldots,T-2).$$

Thus, according to Eq. (B.6), it follows that

$$\frac{\partial \mathrm{E}[s_1]}{\partial\lambda_k} = \frac{1}{2}\mathrm{E}[r_0^1]\frac{\partial Y_0}{\partial\lambda_k},\ \frac{\partial \mathrm{E}[s_2]}{\partial\lambda_k} = \frac{1}{2}\mathrm{E}[r_0^1]\mathrm{E}[\bar{r}_1^0]\frac{\partial Y_0}{\partial\lambda_k} + \frac{1}{2}\mathrm{E}[r_1^1]\frac{\partial Y_1}{\partial\lambda_k},\ldots,\frac{\partial \mathrm{E}[s_T]}{\partial\lambda_k} = \frac{1}{2}\sum_{i=0}^{T-2}\mathrm{E}[r_i^1]\left(\prod_{j=i+1}^{T-1}\mathrm{E}[\bar{r}_j^0]\right)\frac{\partial Y_i}{\partial\lambda_k} + \frac{1}{2}\mathrm{E}[r_{T-1}^1]\frac{\partial Y_{T-1}}{\partial\lambda_k}.$$

Denote row_i as the ith row of matrix Λ. We do the row operation $\mathrm{row}_2 - \mathrm{row}_1\mathrm{E}[\bar{r}_1^0]$ and obtain a new row row_2'. And then, we do the row operation $\mathrm{row}_3 - \mathrm{row}_2'\mathrm{E}[\bar{r}_2^0] - \mathrm{row}_1\mathrm{E}[\bar{r}_1^0]\mathrm{E}[\bar{r}_2^0]$ and obtain a new row row_3'. Continue this procedure up to the last row. We get

$$|\Lambda| = \begin{vmatrix} \frac{1}{2}\mathrm{E}[r_0^1]\frac{\partial Y_0}{\partial\lambda_1} & \frac{1}{2}\mathrm{E}[r_0^1]\frac{\partial Y_0}{\partial\lambda_2} & \ldots & \frac{1}{2}\mathrm{E}[r_0^1]\frac{\partial Y_0}{\partial\lambda_T} \\ \frac{1}{2}\mathrm{E}[r_1^1]\frac{\partial Y_1}{\partial\lambda_1} & \frac{1}{2}\mathrm{E}[r_1^1]\frac{\partial Y_1}{\partial\lambda_2} & \ldots & \frac{1}{2}\mathrm{E}[r_1^1]\frac{\partial Y_1}{\partial\lambda_T} \\ \vdots & \vdots & \ldots & \vdots \\ \frac{1}{2}\mathrm{E}[r_{T-1}^1]\frac{\partial Y_{T-1}}{\partial\lambda_1} & \frac{1}{2}\mathrm{E}[r_{T-1}^1]\frac{\partial Y_{T-1}}{\partial\lambda_2} & \ldots & \frac{1}{2}\mathrm{E}[r_{T-1}^1]\frac{\partial Y_{T-1}}{\partial\lambda_T} \end{vmatrix}.$$

Denote

$$Z_t = \mathrm{E}[r_t^1]\bigg/\left(\mathrm{E}[(r_t^1)^2]\mathrm{E}\left[\sum_{s=t+2}^{T}(\bar{r}_{t+1}^{s-t-2})^2\omega_s + \omega_{t+1}\right]\right) \quad (t = 0,1,\ldots,T-1).$$

According to (B.4), for $t = 0,1,\ldots,T-1$,

$$\frac{\partial Y_t}{\partial\lambda_k} = 0\ (k = 1,\ldots,t), \quad \frac{\partial Y_t}{\partial\lambda_{t+1}} = Z_t,$$
$$\frac{\partial Y_t}{\partial\lambda_k} = Z_t\mathrm{E}[\bar{r}_{t+1}^{k-t-2}] = Z_t\prod_{j=t+1}^{k-1}\mathrm{E}[\bar{r}_j^0] \quad (k = t+2,\ldots,T).$$

Consequently,

$$|\Lambda| = \begin{vmatrix} \frac{1}{2}\mathrm{E}[r_0^1]Z_0 & \frac{1}{2}\mathrm{E}[r_0^1]\mathrm{E}[\bar{r}_1^0]Z_0 & \ldots & \frac{1}{2}\mathrm{E}[r_0^1]\prod_{j=1}^{T-2}\mathrm{E}[\bar{r}_j^0]Z_0 & \frac{1}{2}\mathrm{E}[r_0^1]\prod_{j=1}^{T-1}\mathrm{E}[\bar{r}_j^0]Z_0 \\ 0 & \frac{1}{2}\mathrm{E}[r_1^1]Z_1 & \ldots & \frac{1}{2}\mathrm{E}[r_1^1]\prod_{j=2}^{T-2}\mathrm{E}[\bar{r}_j^0]Z_1 & \frac{1}{2}\mathrm{E}[r_1^1]\prod_{j=2}^{T-1}\mathrm{E}[\bar{r}_j^0]Z_1 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & \frac{1}{2}\mathrm{E}[r_{T-2}^1]Z_{T-2} & \frac{1}{2}\mathrm{E}[r_{T-2}^1]\mathrm{E}[\bar{r}_{T-1}^0]Z_{T-2} \\ 0 & 0 & \ldots & 0 & \frac{1}{2}\mathrm{E}[r_{T-1}^1]Z_{T-1} \end{vmatrix} = \frac{1}{2^T}\prod_{i=0}^{T-1}\mathrm{E}[r_i^1]Z_i \neq 0.$$

Therefore, the matrix Λ is nonsingular.

Appendix C. The computational process of $\mathbf{E}[(s_t^*)^2]$

For each $t = 1,\ldots,T$,

$$(s_t^*)^2 = z_{t-1}'\overline{B}_{t-1}^{0\prime}ee'\overline{B}_{t-1}^0 z_{t-1} + \frac{1}{4}(Y_{t-1}r_{t-1}^1)^2 + Y_{t-1}r_{t-1}^1 e'\overline{B}_{t-1}^0 z_{t-1},\ldots,z_j'\overline{B}_j^{t-1-j\prime}ee'\overline{B}_j^{t-1-j}z_j$$
$$= z_{j-1}'\overline{B}_{j-1}^{t-j\prime}ee'\overline{B}_{j-1}^{t-j}z_{j-1} + \frac{1}{4}(Y_{j-1}r_{j-1}^1\bar{r}_j^{t-1-j})^2 + Y_{j-1}r_{j-1}^1\bar{r}_j^{t-1-j}e'\overline{B}_{j-1}^{t-j}z_{j-1},\ldots,z_0'\overline{B}_0^{t-1\prime}ee'\overline{B}_0^{t-1}z_0 = (\bar{r}_0^{t-1}x_0 - \bar{q}_0^{t-1}l_0)^2,$$

by using (B.5) we have

$$
\begin{aligned}
\mathrm{E}[(s_t^*)^2] &= \mathrm{E}\left[z_0'\overline{B}_0^{t-1\prime}ee'\overline{B}_0^{t-1}z_0 + \frac{1}{4}\left[(Y_{t-1}r_{t-1}^1)^2 + \sum_{i=0}^{t-2}(Y_i r_i^1 \bar{r}_{i+1}^{t-2-i})^2\right] + Y_{t-1}r_{t-1}^1 e'\overline{B}_{t-1}^0 z_{t-1} + \sum_{i=0}^{t-2} Y_i r_i^1 \bar{r}_{i+1}^{t-2-i} e'\overline{B}_i^{t-1-i} z_i\right] \\
&= \frac{1}{4}M_t + N_t + Q_t,
\end{aligned}
$$

where

$$M_t = 2\mathrm{E}\left[\sum_{i=0}^{t-3} Y_i r_i^1 \bar{r}_{i+1}^{t-2-i} \sum_{j=i+1}^{t-2} Y_j r_j^1 \bar{r}_{j+1}^{t-2-j} + Y_{t-1}r_{t-1}^1 \sum_{i=0}^{t-2} Y_i r_i^1 \bar{r}_{i+1}^{t-2-i}\right] + \mathrm{E}\left[(Y_{t-1}r_{t-1}^1)^2 + \sum_{i=0}^{t-2}(Y_i r_i^1 \bar{r}_{i+1}^{t-2-i})^2\right], \tag{C.1}$$

$$N_t = \mathrm{E}\left[\left(Y_{t-1}r_{t-1}^1 + \sum_{i=0}^{t-2} Y_i r_i^1 \bar{r}_{i+1}^{t-2-i}\right)e'\overline{B}_0^{t-1}z_0\right], \tag{C.2}$$

$$Q_t = \mathrm{E}[z_0'\overline{B}_0^{t-1\prime}ee'\overline{B}_0^{t-1}z_0]. \tag{C.3}$$

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