

Exact functors on perverse coherent sheaves

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ABSTRACT

Inspired by symplectic geometry and a microlocal characterizations of constructible perverse sheaves we consider an alternative definition of perverse coherent sheaves: we show that a coherent sheaf is perverse if and only if $R\Gamma_Z \mathcal{F}$ is concentrated in degree 0 for special subvarieties Z of X . These subvarieties Z are analogs of Lagrangians in the symplectic case.

1. Introduction

A general way to obtain insights about the heart of a t-structure is to study exact functors on the t-structure. For example, for the category of constructible perverse sheaves on a complex manifold, one can obtain a large amount of exact functors by taking vanishing cycles [KS94, Corollary 10.3.13].

Let \mathcal{F} be a constructible (middle) perverse sheaf on an affine Kähler manifold X . Let $x \in X$ be point and $f: X \rightarrow \mathbb{C}$ a suitably chosen holomorphic Morse function with $f(x) = 0$ and single critical point x . Then the stalk $(\Phi_f \mathcal{F})_x$ is concentrated in cohomological degree 0. A more “geometric” formulation of this statement can be obtained in the following way. Let L be the stable manifold for the gradient of the Morse function $\Re f$. Write $i_x: \{x\} \hookrightarrow L$ and $i_L: L \hookrightarrow X$ for the inclusions. Then $i_x^* i_L^! \mathcal{F}$ is also concentrated in cohomological degree 0. Note that L is a Lagrangian with respect to the symplectic structure given by the Kähler form.

Now consider a symplectic variety X (in the sense of [Bea00]) with an action by a group G such that the G -orbits give a symplectic foliation of X . In this situation there is a middle perversity t-structure on the derived category $D_c^b(X)^G$ of coherent G -equivariant sheaves (see Section 2 for a review of the theory of perverse coherent sheaves). Let L be a Lagrangian on X , i.e. a smooth subvariety that intersects every symplectic leaf in a Lagrangian. Let $\mathcal{F} \in D_c^b(X)^G$ be a perverse coherent sheaf on X . Following the intuition obtained in the constructible case, is natural to ask whether the $!$ -restriction $i_L^! \mathcal{F}$ of \mathcal{F} to L is concentrated in degree 0.

For an arbitrary variety X with a G -action that has finitely many orbits, we define the notion of a *measuring subvariety* as an analog of a Lagrangian in the symplectic case (Definition 4). Our main theorem (Theorem 6) then states that a coherent sheaf $\mathcal{F} \in D_c^b(X)^G$ is perverse if and only if $i_Z^! \mathcal{F}$ is concentrated in cohomological degree 0 for all measuring subvarieties Z of X .

Example 1. Let N be the nilpotent cone in the complex Lie algebra \mathfrak{sl}_n and let $G = \mathrm{SL}_n$ act on N adjointly. Then the dimensions of the G -orbits in N are known to be even dimensional. Thus there exists a middle perversity p with $p(O) = \frac{1}{2} \dim O$ for each G -orbit O . Let X be the flag variety for \mathfrak{sl}_n and $\mu: T^*X \rightarrow N$ the Springer resolution. Choose a point $x \in X$. Then $T_x^* X$ is a Lagrangian

in T^*X and one can show explicitly that $\mu(T_x^*X)$ is a measuring subvariety of N . Thus a sheaf $\mathcal{F} \in D_c^b(N)^G$ is perverse if and only if $R\Gamma_{\mu(T_x^*X)}\mathcal{F}$ is concentrated in degree 0 for all $x \in X$.

Since the motivating observation about constructible perverse sheaves does not seem to be in the literature (though [MV07, Theorem 3.5] is the same spirit), we give a direct proof of the statement in the appendix.

1.1 Setup and notation

Let X be a finite-dimensional Noetherian separated scheme over an algebraically closed field k . Let G be an algebraic group over k acting on X . Until Section 3 we include the possibility of G being trivial. We write X^{top} for the subset of the Zariski space of X consisting of generic points of G -invariant subschemes and equip X^{top} with the induced topology. To simplify notation, if $x \in X^{\text{top}}$ is any point, we write \bar{x} for the closure $\{x\}$ and $\dim x = \dim \bar{x}$.

We write $D(X)$, $D_{qc}(X)$ and $D_c(X)$ for the derived category of \mathcal{O}_X -modules and its full subcategories consisting of complexes with quasi-coherent and coherent cohomology sheaves respectively. The corresponding categories of G -equivariant sheaves (i.e. the categories for the quotient stack $[X/G]$) are denoted $D(X)^G$, $D_{qc}(X)^G$ and $D_c(X)^G$. As usual, $D^b(X)$ (etc.) is the full subcategory of $D(X)$ consisting of complexes with cohomology in only finitely many degrees. All functors are derived, though we usually do not explicitly mention it in the notation.

For a subset Y of a topological space X we write i_Y for the inclusion of Y into X . If $x \in X$ is a point, then we simply write i_x for $i_{\{x\}}$. Let Z be a closed subset of X . For an \mathcal{O}_X -module \mathcal{F} let $\Gamma_Z\mathcal{F}$ be the subsheaf of \mathcal{F} of sections with support in Z [Har66, Variation 3 in IV.1]. By abuse of notation, we simply write Γ_Z for the right-derived functor $R\Gamma_Z: D_{qc}(X) \rightarrow D_{qc}(X)$. Recall that Γ_Z only depends on the closed subset Z , and not on the structure of Z as a subscheme.

Let x be a (not necessarily closed) point of X and $\mathcal{F} \in D^b(X)$. Then $\mathbf{i}_x^*\mathcal{F} = \mathcal{F}_x \in D^b(\mathcal{O}_x\text{-Mod})$ denotes the (derived) functor of talking stalks. We further set $\mathbf{i}_x^!\mathcal{F} = \mathbf{i}_x^*\Gamma_{\bar{x}}\mathcal{F}$, cf. [Har66, Variation 8 in IV.1].

We assume that X has a G -equivariant dualizing complex \mathcal{R} (see [Bez00, Definition 1]) which we assume to be normalized, i.e. $\mathbf{i}_x^!\mathcal{R}$ is concentrated in degree $-\dim x$ for all $x \in X^{\text{top}}$. For $\mathcal{F} \in D(X)$ (or $D(X)^G$) we write $\mathbb{D}\mathcal{F} = \underline{\text{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{R})$ for its dual.

1.2 Acknowledgements

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2. Perverse coherent sheaves

By *perversity* we mean a function $p: \{0, \dots, \dim X\} \rightarrow \mathbb{Z}$. For $x \in X^{\text{top}}$ we abuse notation and set $p(x) = p(\dim x)$. Then $p: X^{\text{top}} \rightarrow \mathbb{Z}$ is a perversity function in the sense of [Bez00]. Note that we insist that $p(x)$ only depends on the dimension of \bar{x} . A perversity is called *monotone* if it is decreasing and *comonotone* if the *dual perversity* $\bar{p}(n) := -n - p(n)$ is decreasing. It is *strictly monotone* (resp. *strictly comonotone*) if for all $x, y \in X^{\text{top}}$ with $\dim x < \dim y$ one has $p(x) > p(y)$ (resp. $\bar{p}(x) > \bar{p}(y)$). Note that a strictly monotone perversity is not necessarily strictly decreasing (e.g. if X only has even-dimensional G -orbits).

Recall that if p is a monotone and comonotone perversity then Bezrukavnikov (following

Deligne) defines a t-structure on $D_c^b(X)^G$ by taking the following full subcategories [Bez00; AB09]:

$$\begin{aligned} {}^pD^{\leq 0}(X) &= \{\mathcal{F} \in D_c^b(X)^G : \mathbf{i}_x^* \mathcal{F} \in D^{\leq p(x)}(\mathcal{O}_x\text{-}\mathbf{Mod}) \text{ for all } x \in X^{\text{top}}\}, \\ {}^pD^{\geq 0}(X) &= \{\mathcal{F} \in D_c^b(X)^G : \mathbf{i}_x^! \mathcal{F} \in D^{\geq p(x)}(\mathcal{O}_x\text{-}\mathbf{Mod}) \text{ for all } x \in X^{\text{top}}\}. \end{aligned}$$

The heart of this t-structure is called the category of perverse sheaves with respect to the perversity p .

In [Kas04], Kashiwara also gives a definition of a perverse t-structure on $D_c^b(X)$. While we work in Bezrukavnikov's setting (i.e. in the equivariant derived category on a potentially singular scheme), we need a description of the perverse t-structure that is closer to the one Kashiwara uses. This is accomplished in the following proposition.

PROPOSITION 2. *Let $\mathcal{F} \in D_c^b(X)^G$ and let p be a monotone and comonotone perversity function.*

- (i) *The following are equivalent:*
 - (a) $\mathcal{F} \in {}^pD^{\leq 0}(X)$, i.e. $\mathbf{i}_x^* \mathcal{F} \in D^{\leq p(x)}(\mathcal{O}_x\text{-}\mathbf{Mod})$ for all $x \in X^{\text{top}}$;
 - (b) $p(\dim \text{supp } H^k(\mathcal{F})) \geq k$ for all k .
- (ii) *If p is strictly monotone, then the following are equivalent*
 - (a) $\mathcal{F} \in {}^pD^{\geq 0}(X)$, i.e. $\mathbf{i}_x^! \mathcal{F} \in D^{\geq p(x)}(\mathcal{O}_x\text{-}\mathbf{Mod})$ for all $x \in X^{\text{top}}$;
 - (b) $\Gamma_{\bar{x}} \mathcal{F} \in D^{\geq p(x)}(X)$ for all $x \in X^{\text{top}}$;
 - (c) $\Gamma_Y \mathcal{F} \in D^{\geq p(\dim Y)}(X)$ for all G -invariant closed subvarieties Y of X ;
 - (d) $\dim(\bar{x} \cap \text{supp}(H^k(\mathbb{D}\mathcal{F}))) \leq -p(x) - k$ for all $x \in X^{\text{top}}$ and all k .

A crucial fact that we will implicitly use quite often in the following arguments is that the support of a coherent sheaf is always closed. In particular, this means that if x is a generic point and \mathcal{F} a coherent sheaf, then $\mathbf{i}_x^* \mathcal{F} = 0$ if and only if $\mathcal{F}|_U = 0$ for some open set U intersecting \bar{x} .

Proof.

- (i) First let $\mathcal{F} \in {}^pD^{\leq 0}(X)$ and assume for contradiction that there exists an integer k such that $p(\dim \text{supp } H^k(\mathcal{F})) < k$. Let x be the generic point of an irreducible component of maximal dimension of $\text{supp } H^k(\mathcal{F})$. Then $H^k(\mathbf{i}_x^* \mathcal{F}) \neq 0$. But on the other hand, $\mathbf{i}_x^* \mathcal{F} \in D^{\leq p(x)}(\mathcal{O}_x\text{-}\mathbf{Mod})$ and $p(x) = p(\dim \text{supp } H^k(\mathcal{F})) < k$, yielding a contradiction. Conversely assume that $p(\dim \text{supp } H^k(\mathcal{F})) \geq k$ for all k and let $x \in X^{\text{top}}$. If $H^k(\mathbf{i}_x^* \mathcal{F}) \neq 0$, then $\dim x \leq \dim \text{supp } H^k(\mathcal{F})$. Thus monotonicity of the perversity implies that $\mathcal{F} \in {}^pD^{\leq 0}(X)$.
- (ii) The implications from (iii) to (ii) and (ii) to (i) are trivial and the equivalence of (ii) and (iv) follows from Lemma 3 below. Thus we only need to show that (i) implies (iii). So assume that $\mathcal{F} \in {}^pD^{\geq 0}(X)$. We induct on the dimension of Y . If $\dim Y = 0$, then $\Gamma(X, \Gamma_Y \mathcal{F}) = \bigoplus_{y \in Y^{\text{top}}} \mathbf{i}_y^! \mathcal{F}$ and thus $\Gamma_Y \mathcal{F} \in D^{\geq p(0)}(X)$ by assumption. Now let $\dim Y > 0$. We first assume that Y is irreducible with generic point $x \in X^{\text{top}}$. Let k be the smallest integer such that $H^k(\Gamma_{\bar{x}} \mathcal{F}) \neq 0$ and assume that $k < p(x)$. We will show that this implies that $H^k(\Gamma_{\bar{x}} \mathcal{F}) = 0$, giving a contradiction. We first show that $H^k(\Gamma_{\bar{x}} \mathcal{F})$ is coherent. Let $j: X \setminus \bar{x} \hookrightarrow X$ and consider the distinguished triangle

$$\Gamma_{\bar{x}} \mathcal{F} \rightarrow \mathcal{F} \rightarrow j_* j^* \mathcal{F} \xrightarrow{+1}.$$

Applying cohomology to it we get an exact sequence

$$H^{k-1}(j_* j^* \mathcal{F}) \rightarrow H^k(\Gamma_{\bar{x}} \mathcal{F}) \rightarrow H^k(\mathcal{F}).$$

By assumption, $k - 1 \leq p(x) - 2$, so that $H^{k-1}(j_* j^* \mathcal{F})$ is coherent by the Grothendieck Finiteness Theorem in the form of [Bez00, Corollary 3]. As $H^k(\mathcal{F})$ is coherent by definition, this implies that $H^k(\Gamma_{\bar{x}} \mathcal{F})$ also has to be coherent.

Set $Z = \text{supp } H^k(\Gamma_{\bar{x}} \mathcal{F})$. Then, since $i_x^* H^k(\Gamma_{\bar{x}} \mathcal{F}) = H^k(i_x^! \mathcal{F})$ vanishes, Z is a proper closed subset of \bar{x} . We consider the distinguished triangle

$$H^k(\Gamma_{\bar{x}} \mathcal{F})[-k] \rightarrow \Gamma_{\bar{x}} \mathcal{F} \rightarrow \tau_{>k} \Gamma_{\bar{x}} \mathcal{F} \xrightarrow{+1},$$

and apply Γ_Z to it:

$$\Gamma_Z H^k(\Gamma_{\bar{x}} \mathcal{F})[-k] = H^k(\Gamma_{\bar{x}} \mathcal{F})[-k] \rightarrow \Gamma_Z \mathcal{F} \rightarrow \Gamma_Z \tau_{>k} \Gamma_{\bar{x}} \mathcal{F} \xrightarrow{+1}.$$

Since $\dim Z < \dim x$, we can use the induction hypothesis and monotonicity of p to deduce that $\Gamma_Z \mathcal{F}$ is in degrees at least $p(\dim Z) \geq p(x) > k$. Clearly $\Gamma_Z \tau_{>k} \Gamma_{\bar{x}} \mathcal{F}$ is also in degrees larger than k . Hence $H^k(\Gamma_{\bar{x}} \mathcal{F})$ has to vanish.

If Y is not irreducible, let Y_1 be an irreducible component of Y and Y_2 be the union of the other components. Then there is a Mayer-Vietoris distinguished triangle

$$\Gamma_{Y_1 \cap Y_2} \mathcal{F} \rightarrow \Gamma_{Y_1} \mathcal{F} \oplus \Gamma_{Y_2} \mathcal{F} \rightarrow \Gamma_Y \mathcal{F} \xrightarrow{+1},$$

where $\Gamma_{Y_1 \cap Y_2} \mathcal{F} \in D^{\geq p(\dim Y_1 \cap Y_2)}(X) \subseteq D^{\geq p(\dim Y)+1}(X)$ (by the induction hypothesis and strict monotonicity of p) and $\Gamma_{Y_1} \mathcal{F}$ and $\Gamma_{Y_2} \mathcal{F}$ are in $D^{\geq p(\dim Y)}(X)$ by induction on the number of components of Y . Thus $\Gamma_Y \mathcal{F} \in D^{\geq p(\dim Y)}$ as required. \square

LEMMA 3 [Kas04, Proposition 5.2]. *Let $\mathcal{F} \in D_c^b(X)$, Z a closed subset of X , and n an integer. Then $\Gamma_Z \mathcal{F} \in D_{qc}^{\geq n}(X)$ if and only if $\dim(Z \cap \text{supp}(H^k(\mathbb{D}\mathcal{F}))) \leq -k - n$ for all k .*

This lemma extends [Kas04, Proposition 5.2] to singular varieties. The proof is same as for the smooth case, but we will include it here for completeness.

Proof. By [SGA2, Proposition VII.1.2], $\Gamma_Z \mathcal{F} \in D_{qc}^{\geq n}(X)$ if and only if

$$\underline{\text{Hom}}(\mathcal{G}, \mathcal{F}) \in D_c^{\geq n}(X) \tag{1}$$

for all $\mathcal{G} \in \mathbf{Coh}(X)$ with $\text{supp } \mathcal{G} \subseteq Z$. Let $d(n) = -n$ be the dual standard perversity. Then by [Bez00, Lemma 5a], (1) holds if and only if $\mathbb{D}\underline{\text{Hom}}(\mathcal{G}, \mathcal{F}) \in {}^d D^{\leq -n}(X)$. By [Har66, Proposition V.2.6], $\mathbb{D}\underline{\text{Hom}}(\mathcal{G}, \mathcal{F}) = \mathcal{G} \otimes_{\mathcal{O}_X} \mathbb{D}\mathcal{F}$, so that by Proposition 2(a) we need to show that

$$\dim \text{supp } H^k(\mathcal{G} \otimes_{\mathcal{O}_X} \mathbb{D}\mathcal{F}) \leq -k - n$$

for all k . By [Kas04, Lemma 5.3] this is equivalent to

$$\dim(Z \cap \text{supp } H^k(\mathbb{D}\mathcal{F})) \leq -k - n$$

for all k , completing the proof. \square

3. Measuring subvarieties

From now on we assume that the G -action has finitely many orbits.

DEFINITION 4. Let p be a perversity. A p -measuring subvariety of X is an equidimensional subvariety Z of X such that the following conditions hold for each $x \in X^{\text{top}}$ with $\bar{x} \cap Z \neq \emptyset$:

- $\dim(\bar{x} \cap Z) = p(x) + \dim x$;

- $\bar{x} \cap Z$ is the underlying variety of a regularly embedded subscheme in \bar{x} , i.e., up to radical $\bar{x} \cap Z$ it is locally defined in \bar{x} by exactly $-p(x)$ functions.

We say that X has *enough p -measuring subvarieties* if for each $x \in X^{\text{top}}$ there exists a p -measuring subvariety Z with $Z \cap \bar{x} \neq \emptyset$.

Remark 5. Let Z be a p -measuring subvariety. Then $\dim(\bar{x} \cap Z) = -\bar{p}(x)$. Thus comonotonicity of p ensures that if $\dim y \leq \dim x$ then $\dim(\bar{y} \cap Z) \leq \dim(\bar{x} \cap Z)$ for each p -measuring Z . Monotonicity of p then further says that $\dim(\bar{x} \cap Z) - \dim(\bar{y} \cap Z) \leq \dim x - \dim y$. We clearly have $0 \leq \dim(\bar{x} \cap Z) \leq \dim x$ and hence $-\dim x \leq p(x) \leq 0$. We will show in Theorem 9 that these condition are actually sufficient for the existence of enough p -measuring subvarieties, at least when X is affine.

THEOREM 6. *Let p be a strictly monotone and (not necessarily strictly) comonotone perversity and assume that X has enough p -measuring subvarieties. Then,*

- (i) ${}^p D^{\leq 0}(X) = \{\mathcal{F} \in D_c^b(X)^G : \Gamma_Z \mathcal{F} \in D^{\leq 0}(X) \text{ for all } p\text{-measuring subvarieties } Z\};$
- (ii) ${}^p D^{\geq 0}(X) = \{\mathcal{F} \in D_c^b(X)^G : \Gamma_Z \mathcal{F} \in D^{\geq 0}(X) \text{ for all } p\text{-measuring subvarieties } Z\}.$

Therefore the sheaf $\mathcal{F} \in D_c^b(X)^G$ is perverse with respect to p if and only if $\Gamma_Z \mathcal{F}$ is cohomologically concentrated in degree 0 for each p -measuring subvariety Z .

The following lemma encapsulates the central argument of the proof of the first part of the theorem.

LEMMA 7. *Let $\mathcal{F} \in \mathbf{Coh}(X)^G$ be a G -equivariant coherent sheaf on X , let p be a monotone perversity and let n be an integer. Assume that X has enough p -measuring subvarieties. Then the following are equivalent:*

- (i) $p(\dim \text{supp } \mathcal{F}) \geq n$;
- (ii) $H^i(\Gamma_Z \mathcal{F}) = 0$ for all $i \geq -n + 1$ and all measuring subvarieties Z .

Proof. Since $\text{supp } \mathcal{F}$ is always a union of the closure of orbits, we can restrict to the support and assume that $\text{supp } \mathcal{F} = X$.

First assume that $p(\dim X) = p(\dim \text{supp } \mathcal{F}) \geq n$. By the definition of a p -measuring subvariety, this means that, up to radical, Z can be locally defined by at most $-n$ equations. Thus $H^i(\Gamma_Z \mathcal{F}) = 0$ for $i > -n$ [BS98, Theorem 3.3.1].

Now assume conversely that $H^i(\Gamma_Z \mathcal{F}) = 0$ for all $i \geq -n + 1$ and all measuring subvarieties Z . We have to show that $p(\dim X) \geq n$. Set $d = \dim X$. Choose any p -measuring subvariety Z . Then $\text{codim}_X Z = -p(d)$. We will show that $H^{-p(d)}(\Gamma_Z \mathcal{F}) \neq 0$ and hence $p(d) \geq n$ by assumption. Take some affine open subset U of X such that $U \cap Z$ is non-empty and irreducible in U . It suffices to show that the cohomology is non-zero in U . Thus we can assume without loss of generality that X is affine, say $X = \text{Spec } A$, and Z is irreducible. Write $Z = V(\mathfrak{p})$ for some prime ideal \mathfrak{p} of A . By flat base change [BS98, Theorem 4.3.2],

$$\Gamma(X, H^{-p(d)}(\Gamma_Z \mathcal{F}))_{\mathfrak{p}} = \left(H_{\mathfrak{p}}^{-p(d)}(\Gamma(X, \mathcal{F})) \right)_{\mathfrak{p}} = H_{\mathfrak{p}_{\mathfrak{p}}}^{-p(d)}(\Gamma(X, \mathcal{F})_{\mathfrak{p}})$$

Since $\dim \text{supp } \mathcal{F} = \dim X = d$, the dimension of the $A_{\mathfrak{p}}$ -module $\Gamma(X, \mathcal{F})_{\mathfrak{p}}$ is $-p(d)$. Thus by the Grothendieck non-vanishing theorem [BS98, Theorem 6.1.4] $H_{\mathfrak{p}_{\mathfrak{p}}}^{-p(d)}(\Gamma(X, \mathcal{F})_{\mathfrak{p}}) \neq 0$ and hence $\Gamma(X, H^{-p(d)}(\Gamma_Z \mathcal{F})) \neq 0$ as required. \square

Proof of Theorem 6.

(i) We use the description of ${}^pD^{\leq 0}(X)$ given by Proposition 2, i.e.

$${}^pD^{\leq 0}(X) = \left\{ \mathcal{F} \in D_c^b(X)^G : p(\dim(\operatorname{supp} H^n(\mathcal{F}))) \geq n \text{ for all } n \right\}.$$

We induct on the largest k such that $H^k(\mathcal{F}) \neq 0$ to show that $\mathcal{F} \in {}^pD^{\leq 0}(X)$ if and only if $\Gamma_Z \mathcal{F} \in D^{\leq 0}(X)$ for all p -measuring subvarieties Z .

The equivalence is trivial for $k \ll 0$. For the induction step note that there is a distinguished triangle

$$\tau_{<k} \mathcal{F} \rightarrow \mathcal{F} \rightarrow H^k(\mathcal{F})[-k] \xrightarrow{+1}.$$

Applying the functor Γ_Z and taking cohomology we obtain an exact sequence

$$\begin{aligned} \cdots \rightarrow H^1(\Gamma_Z(\tau_{<k} \mathcal{F})) \rightarrow H^1(\Gamma_Z \mathcal{F}) \rightarrow H^{k+1}(\Gamma_Z(H^k(\mathcal{F}))) \rightarrow \\ H^2(\Gamma_Z(\tau_{<k} \mathcal{F})) \rightarrow H^2(\Gamma_Z \mathcal{F}) \rightarrow H^{k+2}(\Gamma_Z(H^k(\mathcal{F}))) \rightarrow \cdots \end{aligned}$$

By induction, $H^j(\Gamma_Z(\tau_{<k} \mathcal{F}))$ vanishes for $j \geq 1$ so that $H^j(\Gamma_Z \mathcal{F}) \cong H^{k+j}(\Gamma_Z(H^k(\mathcal{F})))$ for $j \geq 1$. Thus the statement follows from Lemma 7.

(ii) By Proposition 2(b), $\mathcal{F} \in {}^pD^{\geq 0}$ if and only if

$$\dim(\bar{x} \cap \operatorname{supp}(H^k(\mathbb{D}F))) \leq -p(x) - k \quad \text{for all } x \in X^{\text{top}} \text{ and all } k. \quad (2)$$

Using Lemma 3 for $\Gamma_Z \mathcal{F} \in D^{\geq 0}(X)$, we see that we have to show the equivalence of (2) with

$$\dim(Z \cap \operatorname{supp}(H^k(\mathbb{D}F))) \leq -k \quad \text{for all } k \text{ and } p\text{-measuring } Z.$$

Since there are only finitely many orbits, this is in turn equivalent to

$$\dim(Z \cap \bar{x} \cap \operatorname{supp}(H^k(\mathbb{D}F))) \leq -k \quad \forall x \in X^{\text{top}}, k \text{ and } p\text{-measuring } Z. \quad (3)$$

We will show the equivalence for each fixed k separately. Let us first show the implication from (2) to (3). Since $H^k(\mathbb{D}F)$ is G -equivariant and there are only finitely many G -orbits, it suffices to show (3) assuming that $\dim x \leq \dim \operatorname{supp} H^k(\mathbb{D}F)$ and $\bar{x} \cap \operatorname{supp} H^k(\mathbb{D}F) \neq \emptyset$. Then $\dim(\bar{x} \cap \operatorname{supp}(H^k(\mathbb{D}F))) = \dim \bar{x}$. Thus,

$$\begin{aligned} \dim(Z \cap \bar{x} \cap \operatorname{supp}(H^k(\mathbb{D}F))) &\leq \dim(Z \cap \bar{x}) = p(x) + \dim x = \\ &p(x) + \dim(\bar{x} \cap \operatorname{supp}(H^k(\mathbb{D}F))) \leq p(x) - p(x) - k = -k. \end{aligned}$$

Conversely, assume that (3) holds for k . If $\bar{x} \cap \operatorname{supp} H^k(\mathbb{D}F) = \emptyset$, then (2) is trivially true. Otherwise choose a p -measuring Z that intersects $\operatorname{supp} H^k(\mathbb{D}F)$. First assume that \bar{x} is contained in $\operatorname{supp} H^k(\mathbb{D}F)$. Then

$$\begin{aligned} \dim(\bar{x} \cap \operatorname{supp}(H^k(\mathbb{D}F))) &= \dim x = -p(x) + \dim(Z \cap \bar{x}) = \\ &-p(x) + \dim(Z \cap \bar{x} \cap \operatorname{supp}(H^k(\mathbb{D}F))) \leq -p(x) - k. \end{aligned}$$

Otherwise $\bar{x} \cap \operatorname{supp}(H^k(\mathbb{D}F)) = \bar{y}$ for some $y \in X^{\text{top}}$ with $\dim y < \dim x$. Then (2) holds for y in place of x and hence

$$\dim(\bar{x} \cap \operatorname{supp}(H^k(\mathbb{D}F))) = \dim(\bar{y} \cap \operatorname{supp}(H^k(\mathbb{D}F))) \leq -p(y) - k \leq -p(x) - k$$

by monotonicity of p . \square

Example 8. For the dual standard perversity $p(n) = -n$ (i.e. $p(x) = -\dim x$), we recover the definition of Cohen-Macaulay sheaves [Har66, Section IV.3].

Of course, for the theorem to have any content, one needs to show that X has enough p -measuring subvarieties. The next theorem shows that at least for affine varieties there are always enough measuring subvarieties whenever p satisfies the obvious conditions (see Remark 5).

THEOREM 9. *Assume that X is affine and the perversity p satisfies $-n \leq p(n) \leq 0$ and is monotone and comonotone. Then X has enough p -measuring subvarieties.*

Proof. Let $X = \operatorname{Spec} A$. We induct on the dimension d . More precisely, we induct on the following statement:

There exists a closed equidimensional subvariety Z_d of X such that for all $x \in X^{\operatorname{top}}$ the following holds:

- $Z_d \cap \bar{x} \neq \emptyset$ and $Z_d \cap \bar{x}$ is regularly embedded in \bar{x} ;
- if $\dim x \leq d$, then $\dim(\bar{x} \cap Z_d) = p(x) + \dim x$;
- if $\dim x > d$, then $\dim(\bar{x} \cap Z_d) = p(d) + \dim x$.

We set $p(-1) = 0$. The statement is trivially true for $d = -1$, e.g. take $Z = X$. Assume that the statement is true for some $d \geq -1$. We want to show it for $d+1 \leq \dim X$.

If $p(d) = p(d+1)$, then $Z_{d+1} = Z_d$ works. Otherwise, by (co)monotonicity, $p(d+1) = p(d) - 1$. Set $S = \bigcup \{\bar{x} \in X^{\operatorname{top}} : \dim x \leq d\}$. Since there are only finitely many orbits, we can choose a function f such that f vanishes identically on S , $V(f)$ does not share a component with Z_d and $V(f)$ intersects every \bar{x} with $\dim x > d$. Then $Z_{d+1} = Z_d \cap V(f)$ satisfies the conditions. \square

Appendix. Constructible sheaves

We return now to the claim about exact functors on the t-structure of constructible perverse sheaves made in the introduction. Let X be a complex manifold and \mathfrak{S} a finite stratification of X by complex submanifolds. We write $D_{\mathfrak{S}}^b(X)$ for the bounded derived category of \mathfrak{S} -constructible sheaves on X . We call a sheaf $\mathcal{F} \in D_{\mathfrak{S}}^b(X)$ perverse if it is perverse with respect to the middle perversity function on \mathfrak{S} . We are going to formulate and prove an analog of Theorem 6 in this situation.

A closed real submanifold Z of X is called a *measuring submanifold* if for each stratum S of X either $Z \cap \bar{S} = \emptyset$ or $\dim_{\mathbb{R}} Z \cap S = \dim_{\mathbb{C}} S$.

THEOREM 10. *A sheaf $\mathcal{F} \in D_{\mathfrak{S}}^b(X)$ is perverse if and only if $i_Z^! \mathcal{F}$ is concentrated in cohomological degree 0 for each measuring submanifold Z of X .*

The proof of the following lemma is based on a MathOverflow post by Geordie Williamson [Wil13]. The author takes responsibility for possible mistakes.

LEMMA 11. *Let X be a real manifold, \mathcal{F} be a constructible sheaf (concentrated in degree 0) on X and let $i : Z \hookrightarrow X$ be the inclusion of a closed submanifold. Then $H^j(i^! \mathcal{F}) = 0$ for $j > \operatorname{codim}_X Z$.*

Proof. By taking normal slices we can reduce to the case that $Z = \{z\}$ is a point. Let j be the inclusion of $X \setminus \{z\}$ into X and consider the distinguished triangle

$$i_! i^! \mathcal{F} \rightarrow \mathcal{F} \rightarrow j_* j^* \mathcal{F} \xrightarrow{+1}.$$

By [KS94, Lemma 8.4.7] we have

$$H^j(j_*j^*\mathcal{F})_z = H^j(S_\epsilon^{\dim X-1}, \mathcal{F})$$

for a sphere $S_\epsilon^{\dim X-1}$ around x of sufficiently small radius. The latter cohomology vanishes for $j \geq \dim X$ and hence $H^j(i^!\mathcal{F}) = 0$ for $j > \dim X$ as required. \square

Proof of Theorem. Clearly it is enough to check the condition on a collection of measuring submanifolds $\{Y_i\}$ such that each connected component of each stratum has non-empty intersection with at least one Y_i . Similarly to Theorem 9, one shows inductively that such a collection of submanifolds exists.

Define two full subcategories ${}^L D^{\leq 0}(X)$ and ${}^L D^{\geq 0}(X)$ of $D_\mathbb{G}^b(X)$ by

$$\begin{aligned} {}^L D^{\leq 0}(X) &= \left\{ \mathcal{F} \in D_\mathbb{G}^b(X) : i_Z^!\mathcal{F} \in D^{\leq 0}(Z) \text{ for all measuring submanifolds } Z \right\}, \\ {}^L D^{\geq 0}(X) &= \left\{ \mathcal{F} \in D_\mathbb{G}^b(X) : i_Z^!\mathcal{F} \in D^{\geq 0}(Z) \text{ for all measuring submanifolds } Z \right\}. \end{aligned}$$

We will show that these categories are the same as the categories ${}^p D^{\leq 0}(X)$ and ${}^p D^{\geq 0}(X)$ defining the perverse t-structure on $D_\mathbb{G}^b(X)$.

We induct on the number of strata. If X consists of only one stratum and Z is a measuring submanifold, then $i_Z^!\mathcal{F} \cong \omega_{Z/X} \otimes i_Z^*\mathcal{F}$ and hence $i_Z^!\mathcal{F}$ is in degree 0 if and only if \mathcal{F} is in degree $-\frac{1}{2}\dim_\mathbb{R} X$. So assume that X has more than one stratum. Without loss of generality we can assume that X is connected. Let U be the union of all open strata and F its complement. Both U and F are unions of strata of X . Let j be the inclusion of U and i the inclusion of X .

- If $\mathcal{F} \in {}^p D^{\leq 0}(X)$, then $\mathcal{F} \in {}^L D^{\leq 0}$ follows in exactly the same way as in the coherent case, using Lemma 11.
- Let $\mathcal{F} \in {}^p D^{\geq 0}(X)$. Then $i^!\mathcal{F} \in {}^p D^{\geq 0}(F)$ and $j^*\mathcal{F} \in {}^p D^{\geq 0}(U)$. Let Z be a measuring subvariety. Consider the distinguished triangle

$$i_*i^!\mathcal{F} \rightarrow \mathcal{F} \rightarrow j_*j^*\mathcal{F}.$$

Using base change, induction and the (left)-exactness of the push-forward functors one sees that $i_Z^!$ of the outer sheaves in the triangle are concentrated in non-negative degrees. Thus so is $i_Z^!\mathcal{F}$.

- Let $\mathcal{F} \in {}^L D^{\geq 0}(X)$. Since all measurements are local this implies that $j^*\mathcal{F} \in {}^L D^{\geq 0}(U) = {}^p D^{\geq 0}(U)$. Using the same triangle and argument as in the last point, this implies that also $i^!\mathcal{F} \in {}^L D^{\geq 0}(F) = {}^p D^{\geq 0}(F)$. Hence, by recollement, $\mathcal{F} \in {}^p D^{\geq 0}(X)$.
- Finally, let $\mathcal{F} \in {}^L D^{\leq 0}(X)$. Again this immediately implies that $j^*\mathcal{F} \in {}^L D^{\leq 0}(U) = {}^p D^{\leq 0}(U)$. Thus $j_!j^*\mathcal{F} \in {}^p D^{\leq 0}(X)$. Let Z be a measuring submanifold and consider the distinguished triangle

$$i_Z^!j_!j^*\mathcal{F} \rightarrow i_Z^!\mathcal{F} \rightarrow i_Z^!i_*i^*\mathcal{F}.$$

By what we already know, the first sheaf is concentrated in non-positive degrees and hence so is $i_Z^!i_*i^*\mathcal{F}$. By base change and the exactness of i_* this implies that $i^*\mathcal{F} \in {}^L D^{\leq 0}(F) = {}^p D^{\leq 0}(F)$. Hence, by recollement, $\mathcal{F} \in {}^p D^{\leq 0}(X)$. \square

Remark 12. The equality ${}^p D^{\geq 0}(X) = {}^L D^{\geq 0}(X)$ could also be proved in exactly the same way as in the coherent case, using [KS94, Exercise X.10].

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