Exact functors on perverse coherent sheaves

Clemens Koppensteiner

Abstract

Inspired by symplectic geometry and a microlocal characterizations of constructible perverse sheaves we consider an alternative definition of perverse coherent sheaves: we show that a coherent sheaf is perverse if and only if $R\Gamma_Z\mathcal{F}$ is concentrated in degree 0 for special subvarieties Z of X. These subvarieties Z are analogs of Lagrangians in the symplectic case.

1. Introduction

A general way to obtain insights about the heart of a t-structure is to study exact functors on the t-structure. For example, for the category of constructible perverse sheaves on a complex manifold, one can obtain a large amount of exact functors by taking vanishing cycles [KS94, Corollary 10.3.13].

Let $\mathcal{F}$ be a constructible (middle) perverse sheaf on an affine Kähler manifold X. Let $x \in X$ be point and $f\colon X \to \mathbb{C}$ a suitably chosen holomorphic Morse function with $f(x) = 0$ and single critical point x. Then the stalk $(\Phi_f\mathcal{F})_x$ is concentrated in cohomological degree 0. A more "geometric" formulation of this statement can be obtained in the following way. Let L be the stable manifold for the gradient of the Morse function $\Re f$. Write $i_x\colon \{x\} \hookrightarrow L$ and $i_L\colon L \hookrightarrow X$ for the inclusions. Then $i_x^* i_L^! \mathcal{F}$ is also concentrated in cohomological degree 0. Note that L is a Lagrangian with respect to the symplectic structure given by the Kähler form.

Now consider a symplectic variety X (in the sense of [Bea00]) with an action by a group G such that the G-orbits give a symplectic foliation of X. In this situation there is a middle perversity t-structure on the derived category $D_c^b(X)^G$ of coherent G-equivariant sheaves (see Section 2 for a review of the theory of perverse coherent sheaves). Let L be a Lagrangian on X, i.e. a smooth subvariety that intersects every symplectic leaf in a Lagrangian. Let $\mathcal{F} \in D_c^b(X)^G$ be a perverse coherent sheaf on X. Following the intuition obtained in the constructible case, is natural to ask whether the !-restriction $i_L^!\mathcal{F}$ of $\mathcal{F}$ to L is concentrated in degree 0.

For an arbitrary variety X with a G-action that has finitely many orbits, we define the notion of a *measuring subvariety* as an analog of a Lagrangian in the symplectic case (Definition 4). Our main theorem (Theorem 6) then states that a coherent sheaf $\mathcal{F} \in D_c^b(X)^G$ is perverse if and only if $i_Z^!\mathcal{F}$ is concentrated in cohomological degree 0 for all measuring subvarieties Z of X.

Example 1. Let N be the nilpotent cone in the complex Lie algebra $\mathfrak{sl}_n$ and let $G = \mathrm{SL}_n$ act on N adjointly. Then the dimensions of the G-orbits in N are known to be even dimensional. Thus there exists a middle perversity p with $p(O) = \frac{1}{2}\dim O$ for each G-orbit O. Let X be the flag variety for $\mathfrak{sl}_n$ and $\mu\colon T^*X \to N$ the Springer resolution. Choose a point $x \in X$. Then T_x^*X is a Lagrangian

---

2010 Mathematics Subject Classification 14F05 (primary), 18E30, 14B15 (secondary)
Keywords: perverse coherent sheaves, local cohomology

in T^*X and one can show explicitly that $\mu(T^*_xX)$ is a measuring subvariety of N. Thus a sheaf $\mathcal{F} \in D^b_c(N)^G$ is perverse if and only if $R\Gamma_{\mu(T^*_xX)}\mathcal{F}$ is concentrated in degree 0 for all $x \in X$.

Since the motivating observation about constructible perverse sheaves does not seem to be in the literature (though [MV07, Theorem 3.5] is is the same spirit), we give a direct proof of the statement in the appendix.

1.1 Setup and notation

Let X be a finite-dimensional Noetherian separated scheme over an algebraically closed field k. Let G be an algebraic group over k acting on X. Until Section 3 we include the possibility of G being trivial. We write X^{top} for the subset of the Zariski space of X consisting of generic points of G-invariant subschemes and equip X^{top} with the induced topology. To simplify notation, if $x \in X^{\mathrm{top}}$ is any point, we write $\overline{x}$ for the closure $\overline{\{x\}}$ and $\dim x = \dim \overline{x}$.

We write $D(X)$, $D_{qc}(X)$ and $D_c(X)$ for the derived category of $\mathcal{O}_X$-modules and its full subcategories consisting of complexes with quasi-coherent and coherent cohomology sheaves respectively. The corresponding categories of G-equivariant sheaves (i.e. the categories for the quotient stack $[X/G]$) are denoted $D(X)^G$, $D_{qc}(X)^G$ and $D_c(X)^G$. As usual, $D^b(X)$ (etc.) is the full subcategory of $D(X)$ consisting of complexes with cohomology in only finitely many degrees. All functors are derived, though we usually do not explicitly mention it in the notation.

For a subset Y of a topological space X we write i_Y for the inclusion of Y into X. If $x \in X$ is a point, then we simply write i_x for $i_{\{x\}}$. Let Z be a closed subset of X. For an $\mathcal{O}_X$-module $\mathcal{F}$ let $\Gamma_Z\mathcal{F}$ be the subsheaf of $\mathcal{F}$ of sections with support in Z [Har66, Variation 3 in IV.1]. By abuse of notation, we simply write Γ_Z for the right-derived functor $R\Gamma_Z\colon D_{qc}(X) \to D_{qc}(X)$. Recall that Γ_Z only depends on the closed subset Z, and not on the structure of Z as a subscheme.

Let x be a (not necessarily closed) point of X and $\mathcal{F} \in D^b(X)$. Then $\mathbf{i}_x^*\mathcal{F} = \mathcal{F}_x \in D^b(\mathcal{O}_x\text{-}\mathbf{Mod})$ denotes the (derived) functor of talking stalks. We further set $\mathbf{i}_x^!\mathcal{F} = \mathbf{i}_x^*\Gamma_{\overline{x}}$, cf. [Har66, Variation 8 in IV.1].

We assume that X has a G-equivariant dualizing complex $\mathcal{R}$ (see [Bez00, Definition 1]) which we assume to be normalized, i.e. $\mathbf{i}_x^!\mathcal{R}$ is concentrated in degree $-\dim x$ for all $x \in X^{\mathrm{top}}$. For $\mathcal{F} \in D(X)$ (or $D(X)^G$) we write $\mathbb{D}\mathcal{F} = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{R})$ for its dual.

1.2 Acknowledgements

The motivating observation about constructible sheaves was suggested to me by my advisor, David Nadler. I would also like to thank Sam Gunningham for valuable comments and Geordie Williamson for answering a question on MathOverflow (see Lemma 11).

2. Perverse coherent sheaves

By *perversity* we mean a function $p\colon \{0, \dots, \dim X\} \to \mathbb{Z}$. For $x \in X^{\mathrm{top}}$ we abuse notation and set $p(x) = p(\dim x)$. Then $p\colon X^{\mathrm{top}} \to \mathbb{Z}$ is a perversity function in the sense of [Bez00]. Note that we insist that $p(x)$ only depends on the dimension of $\overline{x}$. A perversity is called *monotone* if it is decreasing and *comonotone* if the *dual perversity* $\overline{p}(n) :- -n - p(n)$ is decreasing. It is *strictly monotone* (resp. *strictly comonotone*) if for all $x, y \in X^{\mathrm{top}}$ with $\dim x < \dim y$ one has $p(x) > p(y)$ (resp. $\overline{p}(x) > \overline{p}(y)$). Note that a strictly monotone perversity is not necessarily strictly decreasing (e.g. if X only has even-dimensional G-orbits).

Recall that if p is a monotone and comonotone perversity then Bezrukavnikov (following

Deligne) defines a t-structure on $D^b_c(X)^G$ by taking the following full subcategories [Bez00; AB09]:

$$\begin{aligned} {}^pD^{\leqslant 0}(X) &= \left\{\mathcal{F} \in D^b_c(X)^G : \mathbf{i}_x^*\mathcal{F} \in D^{\leqslant p(x)}(\mathcal{O}_x\text{-}\mathbf{Mod}) \text{ for all } x \in X^{\mathrm{top}}\right\}, \\ {}^pD^{\geqslant 0}(X) &= \left\{\mathcal{F} \in D^b_c(X)^G : \mathbf{i}_x^!\mathcal{F} \in D^{\geqslant p(x)}(\mathcal{O}_x\text{-}\mathbf{Mod}) \text{ for all } x \in X^{\mathrm{top}}\right\}. \end{aligned}$$

The heart of this t-structure is called the category of perverse sheaves with respect to the perversity p.

In [Kas04], Kashiwara also gives a definition of a perverse t-structure on $D^b_c(X)$. While we work in Bezrukavnikov's setting (i.e. in the equivariant derived category on a potentially singular scheme), we need a description of the perverse t-structure that is closer to the one Kashiwara uses. This is accomplished in the following proposition.

Proposition 2. *Let $\mathcal{F} \in D^b_c(X)^G$ and let p be a monotone and comonotone perversity function.*

(i) *The following are equivalent:*
  (a) *$\mathcal{F} \in {}^pD^{\leqslant 0}(X)$, i.e. $\mathbf{i}_x^*\mathcal{F} \in D^{\leqslant p(x)}(\mathcal{O}_x\text{-}\mathbf{Mod})$ for all $x \in X^{\mathrm{top}}$;*
  (b) *$p(\dim \operatorname{supp} H^k(\mathcal{F})) \geqslant k$ for all k.*

(ii) *If p is strictly monotone, then the following are equivalent*
  (a) *$\mathcal{F} \in {}^pD^{\geqslant 0}(X)$, i.e. $\mathbf{i}_x^!\mathcal{F} \in D^{\geqslant p(x)}(\mathcal{O}_x\text{-}\mathbf{Mod})$ for all $x \in X^{\mathrm{top}}$;*
  (b) *$\Gamma_{\overline{x}}\mathcal{F} \in D^{\geqslant p(x)}(X)$ for all $x \in X^{\mathrm{top}}$;*
  (c) *$\Gamma_Y\mathcal{F} \in D^{\geqslant p(\dim Y)}(X)$ for all G-invariant closed subvarieties Y of X;*
  (d) *$\dim\left(\overline{x} \cap \operatorname{supp}\left(H^k(\mathbb{D}F)\right)\right) \leqslant -p(x) - k$ for all $x \in X^{\mathrm{top}}$ and all k.*

A crucial fact that we will implicitly use quite often in the following arguments is that the support of a coherent sheaf is always closed. In particular, this means that if x is a generic point and $\mathcal{F}$ a coherent sheaf, then $\mathbf{i}_x^*\mathcal{F} = 0$ if and only if $\mathcal{F}|_U = 0$ for some open set U intersecting $\overline{x}$.

Proof.

(i) First let $\mathcal{F} \in {}^pD^{\leqslant 0}(X)$ and assume for contradiction that there exists an integer k such that $p(\dim \operatorname{supp} H^k(\mathcal{F})) < k$. Let x be the generic point of an irreducible component of maximal dimension of $\operatorname{supp} H^k(\mathcal{F})$. Then $H^k(\mathbf{i}_x^*\mathcal{F}) \neq 0$. But on the other hand, $\mathbf{i}_x^*\mathcal{F} \in D^{\leqslant p(x)}(\mathcal{O}_x\text{-}\mathbf{Mod})$ and $p(x) = p(\dim \operatorname{supp} H^k(\mathcal{F})) < k$, yielding a contradiction.
Conversely assume that $p(\dim \operatorname{supp} H^k(\mathcal{F})) \geqslant k$ for all k and let $x \in X^{\mathrm{top}}$. If $H^k(\mathbf{i}_x^*\mathcal{F}) \neq 0$, then $\dim x \leqslant \dim \operatorname{supp} H^k(\mathcal{F})$. Thus monotonicity of the perversity implies that $\mathcal{F} \in {}^pD^{\leqslant 0}(X)$.

(ii) The implications from (iii) to (ii) and (ii) to (i) are trivial and the equivalence of (ii) and (iv) follows from Lemma 3 below. Thus we only need to show that (i) implies (iii). So assume that $\mathcal{F} \in {}^pD^{\geqslant 0}(X)$. We induct on the dimension of Y.
If $\dim Y = 0$, then $\Gamma(X, \Gamma_Y\mathcal{F}) = \bigoplus_{y \in Y^{\mathrm{top}}} \mathbf{i}_y^!\mathcal{F}$ and thus $\Gamma_Y\mathcal{F} \in D^{\geqslant p(0)}(X)$ by assumption.
Now let $\dim Y > 0$. We first assume that Y is irreducible with generic point $x \in X^{\mathrm{top}}$. Let k be the smallest integer such that $H^k(\Gamma_{\overline{x}}\mathcal{F}) \neq 0$ and assume that $k < p(x)$. We will show that this implies that $H^k(\Gamma_{\overline{x}}\mathcal{F}) = 0$, giving a contradiction.
We first show that $H^k(\Gamma_{\overline{x}}\mathcal{F})$ is coherent. Let $j\colon X \setminus \overline{x} \hookrightarrow X$ and consider the distinguished triangle

$$\Gamma_{\overline{x}}\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F} \xrightarrow{+1}.$$

Applying cohomology to it we get an exact sequence

$$H^{k-1}(j_*j^*\mathcal{F}) \to H^k(\Gamma_{\overline{x}}\mathcal{F}) \to H^k(\mathcal{F}).$$

By assumption, $k-1 \leqslant p(x)-2$, so that $H^{k-1}(j_*j^*\mathcal{F})$ is coherent by the Grothendieck Finiteness Theorem in the form of [Bez00, Corollary 3]. As $H^k(\mathcal{F})$ is coherent by definition, this implies that $H^k(\Gamma_{\overline{x}}\mathcal{F})$ also has to be coherent.
Set $Z = \operatorname{supp} H^k(\Gamma_{\overline{x}}\mathcal{F})$. Then, since $i_x^* H^k(\Gamma_{\overline{x}}\mathcal{F}) = H^k(\mathbf{i}_x^! \mathcal{F})$ vanishes, Z is a proper closed subset of $\overline{x}$. We consider the distinguished triangle

$$H^k(\Gamma_{\overline{x}}\mathcal{F})[-k] \to \Gamma_{\overline{x}}\mathcal{F} \to \tau_{>k}\Gamma_{\overline{x}}\mathcal{F} \xrightarrow{+1},$$

and apply Γ_Z to it:

$$\Gamma_Z H^k(\Gamma_{\overline{x}}\mathcal{F})[-k] = H^k(\Gamma_{\overline{x}}\mathcal{F})[-k] \to \Gamma_Z\mathcal{F} \to \Gamma_Z\tau_{>k}\Gamma_{\overline{x}}\mathcal{F} \xrightarrow{+1}.$$

Since $\dim Z < \dim x$, we can use the induction hypothesis and monotonicity of p to deduce that $\Gamma_Z\mathcal{F}$ is in degrees at least $p(\dim Z) \geqslant p(x) > k$. Clearly $\Gamma_Z\tau_{>k}\Gamma_{\overline{x}}\mathcal{F}$ is also in degrees larger than k. Hence $H^k(\Gamma_{\overline{x}}\mathcal{F})$ has to vanish.
If Y is not irreducible, let Y_1 be an irreducible component of Y and Y_2 be the union of the other components. Then there is a Mayer-Vietoris distinguished triangle

$$\Gamma_{Y_1\cap Y_2}\mathcal{F} \to \Gamma_{Y_1}\mathcal{F} \oplus \Gamma_{Y_2}\mathcal{F} \to \Gamma_Y\mathcal{F} \xrightarrow{+1},$$

where $\Gamma_{Y_1\cap Y_2}\mathcal{F} \in D^{\geqslant p(\dim Y_1\cap Y_2)}(X) \subseteq D^{\geqslant p(\dim Y)+1}$ (by the induction hypothesis and strict monotonicity of p) and $\Gamma_{Y_1}\mathcal{F}$ and $\Gamma_{Y_2}\mathcal{F}$ are in $D^{\geqslant p(\dim Y)}(X)$ by induction on the number of components of Y. Thus $\Gamma_Y\mathcal{F} \in D^{\geqslant p(\dim Y)}$ as required. □

Lemma 3 [Kas04, Proposition 5.2]. *Let $\mathcal{F} \in D_c^b(X)$, Z a closed subset of X, and n an integer. Then $\Gamma_Z\mathcal{F} \in D_{qc}^{\geqslant n}(X)$ if and only if $\dim(Z \cap \operatorname{supp}(H^k(\mathbb{D}\mathcal{F}))) \leqslant -k-n$ for all k.*

This lemma extends [Kas04, Proposition 5.2] to singular varieties. The proof is same as for the smooth case, but we will include it here for completeness.

Proof. By [SGA2, Proposition VII.1.2], $\Gamma_Z\mathcal{F} \in D_{qc}^{\geqslant n}(X)$ if and only if

$$\underline{\operatorname{Hom}}(\mathcal{G},\mathcal{F}) \in D_c^{\geqslant n}(X) \tag{1}$$

for all $\mathcal{G} \in \mathbf{Coh}(X)$ with $\operatorname{supp}\mathcal{G} \subseteq Z$. Let $d(n) = -n$ be the dual standard perversity. Then by [Bez00, Lemma 5a], (1) holds if and only if $\mathbb{D}\underline{\operatorname{Hom}}(\mathcal{G},\mathcal{F}) \in {}^dD^{\leqslant -n}(X)$. By [Har66, Proposition V.2.6], $\mathbb{D}\underline{\operatorname{Hom}}(\mathcal{G},\mathcal{F}) = \mathcal{G} \otimes_{\mathcal{O}_X} \mathbb{D}\mathcal{F}$, so that by Proposition 2(a) we need to show that

$$\dim\operatorname{supp} H^k\left(\mathcal{G} \otimes_{\mathcal{O}_X} \mathbb{D}\mathcal{F}\right) \leqslant -k-n$$

for all k. By [Kas04, Lemma 5.3] this is equivalent to

$$\dim\left(Z \cap \operatorname{supp} H^k(\mathbb{D}\mathcal{F})\right) \leqslant -k-n$$

for all k, completing the proof. □

3. Measuring subvarieties

From now on we assume that the G-action has finitely many orbits.

Definition 4. Let p be a perversity. A *p-measuring subvariety* of X is an equidimensional subvariety Z of X such that the following conditions hold for each $x \in X^{\mathrm{top}}$ with $\overline{x} \cap Z \neq \emptyset$:

- $\dim(\overline{x} \cap Z) = p(x) + \dim x$;

– $\overline{x} \cap Z$ is the underlying variety of a regularly embedded subscheme in $\overline{x}$, i.e., up to radical $\overline{x} \cap Z$ it is locally defined in $\overline{x}$ by exactly $-p(x)$ functions.

We say that X has *enough p-measuring subvarieties* if for each $x \in X^{\mathrm{top}}$ there exists a p-measuring subvariety Z with $Z \cap \overline{x} \neq \emptyset$.

Remark 5. Let Z be a p-measuring subvariety. Then $\dim(\overline{x} \cap Z) = -\overline{p}(x)$. Thus comonotonicity of p ensures that if $\dim y \leqslant \dim x$ then $\dim(\overline{y} \cap Z) \leqslant \dim(\overline{x} \cap Z)$ for each p-measuring Z. Monotonicity of p then further says that $\dim(\overline{x} \cap Z) - \dim(\overline{y} \cap Z) \leqslant \dim x - \dim y$. We clearly have $0 \leqslant \dim(\overline{x} \cap Z) \leqslant \dim x$ and hence $-\dim x \leqslant p(x) \leqslant 0$. We will show in Theorem 9 that these condition are actually sufficient for the existence of enough p-measuring subvarieties, at least when X is affine.

THEOREM 6. *Let p be a strictly monotone and (not necessarily strictly) comonotone perversity and assume that X has enough p-measuring subvarieties. Then,*

(i) ${}^pD^{\leqslant 0}(X) = \{\mathcal{F} \in D^b_c(X)^G : \Gamma_Z\mathcal{F} \in D^{\leqslant 0}(X)$ *for all p-measuring subvarieties Z*$\}$;
(ii) ${}^pD^{\geqslant 0}(X) = \{\mathcal{F} \in D^b_c(X)^G : \Gamma_Z\mathcal{F} \in D^{\geqslant 0}(X)$ *for all p-measuring subvarieties Z*$\}$.

Therefore the sheaf $\mathcal{F} \in D^b_c(X)^G$ is perverse with respect to p if and only if $\Gamma_Z\mathcal{F}$ is cohomologically concentrated in degree 0 for each p-measuring subvariety Z.

The following lemma encapsulates the central argument of the proof of the first part of the theorem.

LEMMA 7. *Let $\mathcal{F} \in \mathbf{Coh}(X)^G$ be a G-equivariant coherent sheaf on X, let p be a monotone perversity and let n be an integer. Assume that X has enough p-measuring subvarieties. Then the following are equivalent:*

(i) *$p(\dim \operatorname{supp} \mathcal{F}) \geqslant n$;*
(ii) *$H^i(\Gamma_Z\mathcal{F}) = 0$ for all $i \geqslant -n+1$ and all measuring subvarieties Z.*

Proof. Since $\operatorname{supp}\mathcal{F}$ is always a union of the closure of orbits, we can restrict to the support and assume that $\operatorname{supp}\mathcal{F} = X$.

First assume that $p(\dim X) = p(\dim \operatorname{supp}\mathcal{F}) \geqslant n$. By the definition of a p-measuring subvariety, this means that, up to radical, Z can be locally defined by at most $-n$ equations. Thus $H^i(\Gamma_Z\mathcal{F}) = 0$ for $i > -n$ [BS98, Theorem 3.3.1].

Now assume conversely that $H^i(\Gamma_Z\mathcal{F}) = 0$ for all $i \geqslant -n+1$ and all measuring subvarieties Z. We have to show that $p(\dim X) \geqslant n$. Set $d = \dim X$. Choose any p-measuring subvariety Z. Then $\operatorname{codim}_X Z = -p(d)$. We will show that $H^{-p(d)}(\Gamma_Z\mathcal{F}) \neq 0$ and hence $p(d) \geqslant n$ by assumption. Take some affine open subset U of X such that $U \cap Z$ is non-empty and irreducible in U. It suffices to show that the cohomology is non-zero in U. Thus we can assume without loss of generality that X is affine, say $X = \operatorname{Spec} A$, and Z is irreducible. Write $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ of A. By flat base change [BS98, Theorem 4.3.2],

$$\Gamma(X, H^{-p(d)}(\Gamma_Z\mathcal{F}))_{\mathfrak{p}} = \Big(H^{-p(d)}_{\mathfrak{p}}(\Gamma(X,\mathcal{F}))\Big)_{\mathfrak{p}} = H^{-p(d)}_{\mathfrak{p}_{\mathfrak{p}}}(\Gamma(X,\mathcal{F})_{\mathfrak{p}})$$

Since $\dim \operatorname{supp}\mathcal{F} = \dim X = d$, the dimension of the $A_{\mathfrak{p}}$-module $\Gamma(X,\mathcal{F})_{\mathfrak{p}}$ is $-p(d)$. Thus by the Grothendieck non-vanishing theorem [BS98, Theorem 6.1.4] $H^{-p(d)}_{\mathfrak{p}_{\mathfrak{p}}}(\Gamma(X,\mathcal{F})_{\mathfrak{p}}) \neq 0$ and hence $\Gamma(X, H^{-p(d)}(\Gamma_Z\mathcal{F})) \neq 0$ as required. $\square$

Proof of Theorem 6.

(i) We use the description of ${}^pD^{\leqslant 0}(X)$ given by Proposition 2, i.e.

$$ {}^pD^{\leqslant 0}(X) = \left\{ \mathcal{F} \in D^b_c(X)^G : p\left(\dim\left(\operatorname{supp} H^n(\mathcal{F})\right)\right) \geqslant n \text{ for all } n \right\}. $$

We induct on the largest k such that $H^k(\mathcal{F}) \neq 0$ to show that $\mathcal{F} \in {}^pD^{\leqslant 0}(X)$ if and only if $\Gamma_Z \mathcal{F} \in D^{\leqslant 0}(X)$ for all p-measuring subvarieties Z.
The equivalence is trivial for $k \ll 0$. For the induction step note that there is a distinguished triangle

$$ \tau_{<k}\mathcal{F} \to \mathcal{F} \to H^k(\mathcal{F})[-k] \xrightarrow{+1}. $$

Applying the functor Γ_Z and taking cohomology we obtain an exact sequence

$$ \cdots \to H^1(\Gamma_Z(\tau_{<k}\mathcal{F})) \to H^1(\Gamma_Z\mathcal{F}) \to H^{k+1}(\Gamma_Z(H^k(\mathcal{F}))) \to \\ H^2(\Gamma_Z(\tau_{<k}\mathcal{F})) \to H^2(\Gamma_Z\mathcal{F}) \to H^{k+2}(\Gamma_Z(H^k(\mathcal{F}))) \to \cdots. $$

By induction, $H^j(\Gamma_Z(\tau_{<k}\mathcal{F}))$ vanishes for $j \geqslant 1$ so that $H^j(\Gamma_Z\mathcal{F}) \cong H^{k+j}(\Gamma_Z(H^k(\mathcal{F})))$ for $j \geqslant 1$. Thus the statement follows from Lemma 7.

(ii) By Proposition 2(b), $\mathcal{F} \in {}^pD^{\geqslant 0}$ if and only if

$$ \dim\left(\overline{x} \cap \operatorname{supp}\left(H^k(\mathbb{D}F)\right)\right) \leqslant -p(x) - k \qquad \text{for all } x \in X^{\mathrm{top}} \text{ and all } k. \tag{2} $$

Using Lemma 3 for $\Gamma_Z\mathcal{F} \in D^{\geqslant 0}(X)$, we see that we have to show the equivalence of (2) with

$$ \dim\left(Z \cap \operatorname{supp}\left(H^k(\mathbb{D}F)\right)\right) \leqslant -k \qquad \text{for all } k \text{ and } p\text{-measuring } Z. $$

Since there are only finitely many orbits, this is in turn equivalent to

$$ \dim\left(Z \cap \overline{x} \cap \operatorname{supp}\left(H^k(\mathbb{D}F)\right)\right) \leqslant -k \qquad \forall x \in X^{\mathrm{top}}, k \text{ and } p\text{-measuring } Z. \tag{3} $$

We will show the equivalence for each fixed k separately. Let us first show the implication from (2) to (3). Since $H^k(\mathbb{D}\mathcal{F})$ is G-equivariant and there are only finitely many G-orbits, it suffices to show (3) assuming that $\dim x \leqslant \dim \operatorname{supp} H^k(\mathbb{D}F)$ and $\overline{x} \cap \operatorname{supp} H^k(\mathbb{D}F) \neq \emptyset$. Then $\dim\left(\overline{x} \cap \operatorname{supp}\left(H^k(\mathbb{D}F)\right)\right) = \dim \overline{x}$. Thus,

$$ \dim\left(Z \cap \overline{x} \cap \operatorname{supp}\left(H^k(\mathbb{D}F)\right)\right) \leqslant \dim(Z \cap \overline{x}) = p(x) + \dim x = \\ p(x) + \dim\left(\overline{x} \cap \operatorname{supp}\left(H^k(\mathbb{D}F)\right)\right) \leqslant p(x) - p(x) - k = -k. $$

Conversely, assume that (3) holds for k. If $\overline{x} \cap \operatorname{supp} H^k(\mathbb{D}F) = \emptyset$, then (2) is trivially true. Otherwise choose a p-measuring Z that intersects $\operatorname{supp} H^k(\mathbb{D}F)$. First assume that $\overline{x}$ is contained in $\operatorname{supp} H^k(\mathbb{D}F)$. Then

$$ \dim\left(\overline{x} \cap \operatorname{supp}\left(H^k(\mathbb{D}F)\right)\right) = \dim x = -p(x) + \dim(Z \cap \overline{x}) = \\ -p(x) + \dim\left(Z \cap \overline{x} \cap \operatorname{supp}\left(H^k(\mathbb{D}F)\right)\right) \leqslant -p(x) - k. $$

Otherwise $\overline{x} \cap \operatorname{supp}\left(H^k(\mathbb{D}F)\right) = \overline{y}$ for some $y \in X^{\mathrm{top}}$ with $\dim y < \dim x$. Then (2) holds for y in place of x and hence

$$ \dim\left(\overline{x} \cap \operatorname{supp}\left(H^k(\mathbb{D}F)\right)\right) = \dim\left(\overline{y} \cap \operatorname{supp}\left(H^k(\mathbb{D}F)\right)\right) \leqslant -p(y) - k \leqslant -p(x) - k $$

by monotonicity of p. $\square$

Example 8. For the dual standard perversity $p(n) = -n$ (i.e. $p(x) = -\dim x$), we recover the definition of Cohen-Macaulay sheaves [Har66, Section IV.3].

Of course, for the theorem to have any content, one needs to show that X has enough p-measuring subvarieties. The next theorem shows that at least for affine varieties there are always enough measuring subvarieties whenever p satisfies the obvious conditions (see Remark 5).

Theorem 9. *Assume that X is affine and the perversity p satisfies $-n \leqslant p(n) \leqslant 0$ and is monotone and comonotone. Then X has enough p-measuring subvarieties.*

Proof. Let $X = \operatorname{Spec} A$. We induct on the dimension d. More precisely, we induct on the following statement:

> There exists a closed equidimensional subvariety Z_d of X such that for all $x \in X^{\text{top}}$ the following holds:
> - $Z_d \cap \overline{x} \neq \emptyset$ and $Z_d \cap \overline{x}$ is regularly embedded in $\overline{x}$;
> - if $\dim x \leqslant d$, then $\dim(\overline{x} \cap Z_d) = p(x) + \dim x$;
> - if $\dim x > d$, then $\dim(\overline{x} \cap Z_d) = p(d) + \dim x$.

We set $p(-1) = 0$. The statement is trivially true for $d = -1$, e.g. take $Z = X$. Assume that the statement is true for some $d \geqslant -1$. We want to show it for $d + 1 \leqslant \dim X$.

If $p(d) = p(d+1)$, then $Z_{d+1} = Z_d$ works. Otherwise, by (co)monotonicity, $p(d+1) = p(d) - 1$. Set $S = \bigcup\{\overline{x} \in X^{\text{top}} : \dim x \leqslant d\}$. Since there are only finitely many orbits, we can choose a function f such that f vanishes identically on S, $V(f)$ does not share a component with Z_d and $V(f)$ intersects every $\overline{x}$ with $\dim x > d$. Then $Z_{d+1} = Z_d \cap V(f)$ satisfies the conditions. □

Appendix. Constructible sheaves

We return now to the claim about exact functors on the t-structure of constructible perverse sheaves made in the introduction. Let X be a complex manifold and $\mathfrak{S}$ a finite stratification of X by complex submanifolds. We write $D^b_{\mathfrak{S}}(X)$ for the bounded derived category of $\mathfrak{S}$-constructible sheaves on X. We call a sheaf $\mathcal{F} \in D^b_{\mathfrak{S}}(X)$ perverse if it is perverse with respect to the middle perversity function on $\mathfrak{S}$. We are going to formulate and prove an analog of Theorem 6 in this situation.

A closed real submanifold Z of X is called a *measuring submanifold* if for each stratum S of X either $Z \cap \overline{S} = \emptyset$ or $\dim_{\mathbb{R}} Z \cap S = \dim_{\mathbb{C}} S$.

Theorem 10. *A sheaf $\mathcal{F} \in D^b_{\mathfrak{S}}(X)$ is perverse if and only if $i^!_Z\mathcal{F}$ is concentrated in cohomological degree 0 for each measuring submanifold Z of X.*

The proof of the following lemma is based on a MathOverflow post by Geordie Williamson [Wil13]. The author takes responsibility for possible mistakes.

Lemma 11. *Let X be a real manifold, $\mathcal{F}$ be a constructible sheaf (concentrated in degree 0) on X and let $i\colon Z \hookrightarrow X$ be the inclusion of a closed submanifold. Then $H^j(i^!\mathcal{F}) = 0$ for $j > \operatorname{codim}_X Z$.*

Proof. By taking normal slices we can reduce to the case that $Z = \{z\}$ is a point. Let j be the inclusion of $X \setminus \{z\}$ into X and consider the distinguished triangle

$$i_! i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F} \xrightarrow{+1} .$$

By [KS94, Lemma 8.4.7] we have

$$H^j(j_*j^*\mathcal{F})_z = H^j(S_\epsilon^{\dim X-1}, \mathcal{F})$$

for a sphere $S_\epsilon^{\dim X-1}$ around x of sufficiently small radius. The latter cohomology vanishes for $j \geqslant \dim X$ and hence $H^j(i^!\mathcal{F}) = 0$ for $j > \dim X$ as required. □

Proof of Theorem. Clearly it is enough to check the condition on a collection of measuring submanifolds $\{Y_i\}$ such that each connected component of each stratum has non-empty intersection with at least one Y_i. Similarly to Theorem 9, one shows inductively that such a collection of submanifolds exists.

Define two full subcategories ${}^LD^{\leqslant 0}(X)$ and ${}^LD^{\geqslant 0}(X)$ of $D^b_{\mathfrak{S}}(X)$ by

$$\begin{aligned} {}^LD^{\leqslant 0}(X) &= \left\{\mathcal{F} \in D^b_{\mathfrak{S}}(X) : i^!_Z\mathcal{F} \in D^{\leqslant 0}(Z) \text{ for all measuring submanifolds } Z\right\}, \\ {}^LD^{\geqslant 0}(X) &= \left\{\mathcal{F} \in D^b_{\mathfrak{S}}(X) : i^!_Z\mathcal{F} \in D^{\geqslant 0}(Z) \text{ for all measuring submanifolds } Z\right\}. \end{aligned}$$

We will show that these categories are the same as the categories ${}^pD^{\leqslant 0}(X)$ and ${}^pD^{\geqslant 0}(X)$ defining the perverse t-structure on $D^b_{\mathfrak{S}}(X)$.

We induct on the number of strata. If X consists of only one stratum and Z is a measuring submanifold, then $i^!_Z\mathcal{F} \cong \omega_{Z/X} \otimes i^*_Z\mathcal{F}$ and hence $i^!_Z\mathcal{F}$ is in degree 0 if and only if $\mathcal{F}$ is in degree $-\frac{1}{2}\dim_{\mathbb{R}} X$. So assume that X has more then one stratum. Without loss of generality we can assume that X is connected. Let U be the union of all open strata and F its complement. Both U and F are unions of strata of X. Let j be the inclusion of U and i the inclusion of X.

- If $\mathcal{F} \in {}^pD^{\leqslant 0}(X)$, then $\mathcal{F} \in {}^LD^{\leqslant 0}$ follows in exactly the same way as in the coherent case, using Lemma 11.
- Let $\mathcal{F} \in {}^pD^{\geqslant 0}(X)$. Then $i^!\mathcal{F} \in {}^pD^{\geqslant 0}(F)$ and $j^*\mathcal{F} \in {}^pD^{\geqslant 0}(U)$. Let Z be a measuring subvariety. Consider the distinguished triangle
$$i_*i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F}.$$
Using base change, induction and the (left)-exactness of the push-forward functors one sees that $i^!_Z$ of the outer sheaves in the triangle are concentrated in non-negative degrees. Thus so is $i^!_Z\mathcal{F}$.
- Let $\mathcal{F} \in {}^LD^{\geqslant 0}(X)$. Since all measurements are local this implies that $j^*\mathcal{F} \in {}^LD^{\geqslant 0}(U) = {}^pD^{\geqslant 0}(U)$. Using the same triangle and argument as in the last point, this implies that also $i^!\mathcal{F} \in {}^LD^{\geqslant 0}(F) = {}^pD^{\geqslant 0}(F)$. Hence, by recollement, $\mathcal{F} \in {}^pD^{\geqslant 0}(X)$.
- Finally, let $\mathcal{F} \in {}^LD^{\leqslant 0}(X)$. Again this immediately implies that $j^*\mathcal{F} \in {}^LD^{\leqslant 0}(U) = {}^pD^{\leqslant 0}(U)$. Thus $j_!j^*\mathcal{F} \in {}^pD^{\leqslant 0}(X)$. Let Z be a measuring submanifold and consider the distinguished triangle
$$i^!_Zj_!j^*\mathcal{F} \to i^!_Z\mathcal{F} \to i^!_Zi_*i^*\mathcal{F}.$$
By what we already know, the first sheaf is concentrated in non-positive degrees and hence so is $i^!_Zi_*i^*\mathcal{F}$. By base change and the exactness of i_* this implies that $i^*\mathcal{F} \in {}^LD^{\leqslant 0}(F) = {}^pD^{\leqslant 0}(F)$. Hence, by recollement, $\mathcal{F} \in {}^pD^{\leqslant 0}(X)$. □

Remark 12. The equality ${}^pD^{\geqslant 0}(X) = {}^LD^{\geqslant 0}(X)$ could also be proved in exactly the same way as in the coherent case, using [KS94, Exercise X.10].

References

[AB09] Dmitry Arinkin and Roman Bezrukavnikov. “Perverse coherent sheaves”. In: *ArXiv e-prints* (Feb. 2009). arXiv: 0902.0349 [math.AG].

[Bea00] Arnaud Beauville. “Symplectic singularities”. In: *Inventiones mathematicae* 139.3 (2000), pp. 541–549. doi: 10.1007/s002229900043.

[Bez00] Roman Bezrukavnikov. “Perverse coherent sheaves (after Deligne)”. In: *ArXiv e-prints* (May 2000). arXiv: math/0005152.

[BS98] M. P. Brodmann and R. Y. Sharp. *Local cohomology: an algebraic introduction with geometric applications*. Cambridge Studies in Advanced Mathematics 60. Cambridge: Cambridge University Press, 1998. isbn: 0-521-37286-0. doi: 10.1017/CBO9780511629204.

[Har66] Robin Hartshorne. *Residues and Duality*. Lecture Notes in Mathematics 20. Berlin: Springer-Verlag, 1966. vii+423.

[Kas04] Masaki Kashiwara. “t-structures on the derived categories of holonomic $\mathcal{D}$-modules and coherent $\mathcal{O}$-modules”. In: *Moscow Mathematical Journal* 4.4 (2004), pp. 847–868. url: http://mi.mathnet.ru/eng/mmj173.

[KS94] Masaki Kashiwara and Pierre Schapira. *Sheaves on manifolds*. Grundlehren der Mathematischen Wissenschaften 292. With a chapter in French by Christian Houzel, Corrected reprint of the 1990 original. Berlin: Springer-Verlag, 1994. x+512. isbn: 3-540-51861-4.

[MV07] I. Mirković and K. Vilonen. “Geometric Langlands duality and representations of algebraic groups over commutative rings”. In: *Annals of Mathematics. Second Series* 166.1 (2007), pp. 95–143. issn: 0003-486X. doi: 10.4007/annals.2007.166.95.

[SGA2] Alexander Grothendieck and Michele Raynaud. *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)*. Ed. by Yves Laszlo. Documents Mathématiques (Paris) 4. Séminaire de Géométrie Algébrique du Bois Marie, 1962. Augmenté d’un exposé de Michèle Raynaud. Revised reprint of the 1968 French original. Paris: Société Mathématique de France, 2005. x+208. isbn: 2-85629-169-4. arXiv: math/0511279.

[Wil13] Geordie Williamson. *Vanishing of !-restriction of constructible sheaves*. MathOverflow. Apr. 30, 2013. url: http://mathoverflow.net/a/129244 (visited on 11/20/2013).

Clemens Koppensteiner clemens@math.northwestern.edu
Department of Mathematics, Northwestern University, 2033 Sheridan Rd, Evanston, IL 60208, USA