

# EXACT FUNCTORS ON PERVERSE COHERENT SHEAVES

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Inspired by symplectic geometry and a microlocal characterizations of constructible perverse sheaves we consider an alternative definition of perverse coherent sheaves: we show that a coherent sheaf is perverse if and only if  $R\Gamma_Z \mathcal{F}$  is concentrated in degree 0 for special subvarieties  $Z$  of  $X$ . These subvarieties  $Z$  are analogs of Lagrangians in the symplectic case.

## 1. INTRODUCTION

A general way to obtain insights about the heart of a t-structure is to study exact functors on the t-structure. For example, for the category of constructible perverse sheaves on a complex manifold, one can obtain a large amount of exact functors by taking vanishing cycles [KS94, Corollary 10.3.13].

Let  $\mathcal{F}$  be a constructible (middle) perverse sheaf on an affine Kähler manifold  $X$ . Let  $x \in X$  be point and  $f : X \rightarrow \mathbb{C}$  a suitably chosen holomorphic Morse function with  $f(x) = 0$  and single critical point  $x$ . Then the stalk  $(\varphi_f \mathcal{F})_x$  is concentrated in cohomological degree 0. A more “geometric” formulation of this statement can be obtained in the following way. Let  $L$  be the stable manifold for the gradient of the Morse function  $\Re f$ . Write  $\iota_x : \{x\} \hookrightarrow L$  and  $\iota_L : L \hookrightarrow X$  for the inclusions. Then  $\iota_x^* \iota_L^! \mathcal{F}$  is also concentrated in cohomological degree 0. Note that  $L$  is a Lagrangian with respect to the symplectic structure given by the Kähler form.

Now consider a symplectic variety  $X$  (in the sense of [Bea00]) with an action by a group  $G$  such that the  $G$ -orbits give a symplectic foliation of  $X$ . In this situation there is a middle perversity t-structure on the derived category  $D_c^b(X_G)$  of coherent  $G$ -equivariant sheaves (see Section 2 for a review of the theory of perverse coherent sheaves). Let  $L$  be a Lagrangian on  $X$ , i.e. a smooth subvariety that intersects every symplectic leaf in a Lagrangian. Let  $\mathcal{F} \in D_c^b(X_G)$  be a perverse coherent sheaf on  $X$ . Following the intuition obtained in the constructible case, is natural to ask whether the  $!$ -restriction  $\iota_L^! \mathcal{F}$  of  $\mathcal{F}$  to  $L$  is concentrated in degree 0.

For an arbitrary variety  $X$  with a  $G$ -action that has finitely many orbits, we define the notion of a *measuring subvariety* as an analog of a Lagrangian in the symplectic case (Definition 4). Our main theorem (Theorem 6) then states that a coherent sheaf  $\mathcal{F} \in D_c^b(X_G)$  is perverse if and only if  $\iota_Z^! \mathcal{F}$  is concentrated in cohomological degree 0 for all measuring subvarieties  $Z$  of  $X$ .

*Example 1.* Let  $N$  be the nilpotent cone in the complex Lie algebra  $\mathfrak{sl}_n$  and let  $G = \mathrm{SL}_n$  act on  $N$  adjointly. Then the dimensions of the  $G$ -orbits in  $N$  are known to be even dimensional.

Thus there exists a middle perversity  $p$  with  $p(O) = \frac{1}{2} \dim O$  for each  $G$ -orbit  $O$ . Let  $X$  be the flag variety for  $\mathfrak{sl}_n$  and  $\mu: T^*X \rightarrow N$  the Springer resolution. Choose a point  $x \in X$ . Then  $T_x^*X$  is a Lagrangian in  $T^*X$  and one can show explicitly that  $\mu(T_x^*X)$  is a measuring subvariety of  $N$ . Thus a sheaf  $\mathcal{F} \in D_c^b(N^G)$  is perverse if and only if  $R\Gamma_{\mu(T_x^*X)} \mathcal{F}$  is concentrated in degree 0 for all  $x \in X$ .  $\triangleleft$

Since the motivating observation about constructible perverse sheaves does not seem to be in the literature (though [MV07, Theorem 3.5] is in the same spirit), we give a direct proof of the statement in the appendix.

### 1.1. SETUP AND NOTATION

Let  $X$  be a finite-dimensional Noetherian separated scheme over an algebraically closed field  $k$ . Let  $G$  be an algebraic group over  $k$  acting on  $X$ . For the moment we include the possibility of  $G$  being trivial (this will change in Section 3). We write  $X^{\text{top}}$  for the subset of the Zariski space of  $X$  consisting of generic points of  $G$ -invariant subschemes and equip  $X^{\text{top}}$  with the induced topology. To simplify notation, if  $x \in X^{\text{top}}$  is any point, we write  $\bar{x}$  for the closure  $\overline{\{x\}}$  and  $\dim x = \dim \bar{x}$ .

We write  $D(X)$ ,  $D_{qc}(X)$  and  $D_c(X)$  for the derived category of  $\mathcal{O}_X$ -modules and its full subcategories consisting of complexes with quasi-coherent and coherent cohomology sheaves respectively. The corresponding categories of  $G$ -equivariant sheaves (i.e. the categories for the quotient stack  $[X/G]$ ) are denoted  $D(X_G)$ ,  $D_{qc}(X_G)$  and  $D_c(X_G)$ . As usual,  $D^b(X)$  (etc.) is the full subcategory of  $D(X)$  consisting of complexes with cohomology in only finitely many degrees. All functors are derived, though we usually do not explicitly mention it in the notation.

For a subset  $Y$  of a topological space  $X$  we write  $\iota_Y$  for the inclusion of  $Y$  into  $X$ . If  $x \in X$  is a point, then we simply write  $\iota_x$  for  $\iota_{\{x\}}$ . Let  $Z$  be a closed subset of  $X$ . For an  $\mathcal{O}_X$ -module  $\mathcal{F}$  let  $\Gamma_Z \mathcal{F}$  be the subsheaf of  $\mathcal{F}$  of sections with support in  $Z$  [Har66, Variation 3 in IV.1]. By abuse of notation, we simply write  $\Gamma_Z$  for the right-derived functor  $R\Gamma_Z: D_{qc}(X) \rightarrow D_{qc}(X)$ . Recall that  $\Gamma_Z$  only depends on the closed subset  $Z$ , and not on the structure of  $Z$  as a subscheme.

Let  $x$  be a (not necessarily closed) point of  $X$  and  $\mathcal{F} \in D^b(X)$ . Then  $i_x^* \mathcal{F} = \mathcal{F}_x \in D^b(\mathcal{O}_x\text{-Mod})$  denotes the (derived) functor of talking stalks. We further set  $i_x^! \mathcal{F} = i_x^* \Gamma_{\bar{x}} \mathcal{F}$ , cf. [Har66, Variation 8 in IV.1].

We assume that  $X$  has a  $G$ -equivariant dualizing complex  $\mathcal{R}$  (see [Bez00, Definition 1]) which we assume to be normalized, i.e.  $i_x^! \mathcal{R}$  is concentrated in degree  $-\dim x$  for all  $x \in X^{\text{top}}$ . For  $\mathcal{F} \in D(X)$  (or  $D(X_G)$ ) we write  $\mathbb{D}\mathcal{F} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{R})$  for its dual.

### 1.2. ACKNOWLEDGEMENTS

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## 2. PERVERSE COHERENT SHEAVES

A *perversity* is a function  $p : \{0, \dots, \dim X\} \rightarrow \mathbb{Z}$ . For  $x \in X^{\text{top}}$  we abuse notation and set  $p(x) = p(\dim x)$ . Then  $p : X^{\text{top}} \rightarrow \mathbb{Z}$  is a perversity function in the sense of [Bez00]. Note that we insist that  $p(x)$  only depends on the dimension of  $\bar{x}$ . A perversity is called *monotone* if it is decreasing and *comonotone* if the *dual perversity*  $\bar{p}(n) := -n - p(n)$  is decreasing. It is *strictly monotone* (resp. *strictly comonotone*) if for all  $x, y \in X^{\text{top}}$  with  $\dim x < \dim y$  one has  $p(x) > p(y)$  (resp.  $\bar{p}(x) > \bar{p}(y)$ ). Note that a strictly monotone perversity is not necessarily strictly decreasing (e.g. if  $X$  only has even-dimensional  $G$ -orbits).

Recall that if  $p$  is a monotone and comonotone perversity then Bezrukavnikov (following Deligne) defines a t-structure on  $D_c^b(X_G)$  by taking the following full subcategories (see [Bez00; AB09]):

$$\begin{aligned} {}^pD^{\leq 0}(X) &= \{\mathcal{F} \in D_c^b(X_G) : \iota_x^* \mathcal{F} \in D^{\leq p(x)}(\mathcal{O}_x\text{-}\mathbf{Mod}) \text{ for all } x \in X^{\text{top}}\}, \\ {}^pD^{\geq 0}(X) &= \{\mathcal{F} \in D_c^b(X_G) : \iota_x^! \mathcal{F} \in D^{\geq p(x)}(\mathcal{O}_x\text{-}\mathbf{Mod}) \text{ for all } x \in X^{\text{top}}\}. \end{aligned}$$

The heart of this t-structure is called the category of perverse sheaves with respect to the perversity  $p$ .

In [Kas04], Kashiwara also gives a definition of a perverse t-structure on  $D_c^b(X)$ . While we work in Bezrukavnikov's setting (i.e. in the equivariant derived category on a potentially singular scheme), we need a description of the perverse t-structure that is closer to the one Kashiwara uses. This is accomplished in the following proposition.

**Proposition 2.** *Let  $\mathcal{F} \in D_c^b(X_G)$  and let  $p$  be a monotone and comonotone perversity function.*

(a) *The following are equivalent:*

- (i)  $\mathcal{F} \in {}^pD^{\leq 0}(X)$ , i.e.  $\iota_x^* \mathcal{F} \in D^{\leq p(x)}(\mathcal{O}_x\text{-}\mathbf{Mod})$  for all  $x \in X^{\text{top}}$ ;
- (ii)  $p(\dim \text{supp } H^k(\mathcal{F})) \geq k$  for all  $k$ .

(b) *If  $p$  is strictly monotone, then the following are equivalent*

- (i)  $\mathcal{F} \in {}^pD^{\geq 0}(X)$ , i.e.  $\iota_x^! \mathcal{F} \in D^{\geq p(x)}(\mathcal{O}_x\text{-}\mathbf{Mod})$  for all  $x \in X^{\text{top}}$ ;
- (ii)  $\Gamma_{\bar{x}} \mathcal{F} \in D^{\geq p(x)}(X)$  for all  $x \in X^{\text{top}}$ ;
- (iii)  $\Gamma_Y \mathcal{F} \in D^{\geq p(\dim Y)}(X)$  for all  $G$ -invariant closed subvarieties  $Y$  of  $X$ ;
- (iv)  $\dim(\bar{x} \cap \text{supp}(H^k(\mathbb{D}F))) \leq -p(x) - k$  for all  $x \in X^{\text{top}}$  and all  $k$ .

A crucial fact that we will implicitly use quite often in the following arguments is that the support of a coherent sheaf is always closed. In particular, this means that if  $x$  is a generic point and  $\mathcal{F}$  a coherent sheaf, then  $\iota_x^* \mathcal{F} = 0$  if and only if  $\mathcal{F}|_U = 0$  for some open set  $U$  intersecting  $\bar{x}$ .

*Proof.*

- (a) First let  $\mathcal{F} \in {}^pD^{\leq 0}(X)$  and assume for contradiction that there exists an integer  $k$  such that  $p(\dim \operatorname{supp} H^k(\mathcal{F})) < k$ . Let  $x$  be the generic point of an irreducible component of maximal dimension of  $\operatorname{supp} H^k(\mathcal{F})$ . Then  $H^k(\iota_x^* \mathcal{F}) \neq 0$ . But on the other hand,  $\iota_x^* \mathcal{F} \in D^{\leq p(x)}(\mathcal{O}_x\text{-Mod})$  and  $p(x) = p(\dim \operatorname{supp} H^k(\mathcal{F})) < k$ , yielding a contradiction.

Conversely assume that  $p(\dim \operatorname{supp} H^k(\mathcal{F})) \geq k$  for all  $k$  and let  $x \in X^{\text{top}}$ . If  $H^k(\iota_x^* \mathcal{F}) \neq 0$ , then  $\dim x \leq \dim \operatorname{supp} H^k(\mathcal{F})$ . Thus monotonicity of the perversity implies that  $\mathcal{F} \in {}^pD^{\leq 0}(X)$ .

- (b) The implications from (iii) to (ii) and (ii) to (i) are trivial and the equivalence of (ii) and (iv) follows from Lemma 3 below. Thus we only need to show that (i) implies (iii). So assume that  $\mathcal{F} \in {}^pD^{\geq 0}(X)$ . We induct on the dimension of  $Y$ .

If  $\dim Y = 0$ , then  $\Gamma_Y^! \mathcal{F} = \Gamma(X, \Gamma_Y F)$  and thus  $\Gamma_Y \mathcal{F} \in D^{\geq p(0)}(X)$  by assumption.

Now let  $\dim Y > 0$ . We first assume that  $Y$  is irreducible, i.e.  $Y = \bar{x}$  for some  $x \in X^{\text{top}}$ . Let  $k$  be the smallest integer such that  $H^k(\Gamma_{\bar{x}} \mathcal{F}) \neq 0$  and assume that  $k < p(x)$ . We will show that this implies that  $H^k(\Gamma_{\bar{x}} \mathcal{F}) = 0$ , giving a contradiction.

We first show that  $H^k(\Gamma_{\bar{x}} \mathcal{F})$  is coherent. Let  $j : X \setminus \bar{x} \hookrightarrow X$  and consider the distinguished triangle

$$\Gamma_{\bar{x}} \mathcal{F} \rightarrow \mathcal{F} \rightarrow j_* j^* \mathcal{F} \xrightarrow{+1}.$$

Applying cohomology to it we get an exact sequence

$$H^{k-1}(j_* j^* \mathcal{F}) \rightarrow H^k(\Gamma_{\bar{x}} \mathcal{F}) \rightarrow H^k(\mathcal{F}).$$

By assumption,  $k - 1 \leq p(x) - 2$ , so that  $H^{k-1}(j_* j^* \mathcal{F})$  is coherent by the Grothendieck finiteness theorem in the form of [Bez00, Corollary 3]. As  $H^k(\mathcal{F})$  is coherent by definition, this implies that  $H^k(\Gamma_{\bar{x}} \mathcal{F})$  also has to be coherent.

Set  $Z = \operatorname{supp} H^k(\Gamma_{\bar{x}} \mathcal{F})$ . Then, since  $\iota_x^* H^k(\Gamma_{\bar{x}} \mathcal{F})$  vanishes,  $Z$  is a proper closed subset of  $\bar{x}$ . Consider the distinguished triangle

$$H^k(\Gamma_{\bar{x}} \mathcal{F})[-k] \rightarrow \Gamma_{\bar{x}} \mathcal{F} \rightarrow \tau_{>k} \Gamma_{\bar{x}} \mathcal{F} \xrightarrow{+1},$$

and apply  $\Gamma_Z$  to it:

$$\Gamma_Z H^k(\Gamma_{\bar{x}} \mathcal{F})[-k] = H^k(\Gamma_{\bar{x}} \mathcal{F})[-k] \rightarrow \Gamma_Z \mathcal{F} \rightarrow \Gamma_Z \tau_{>k} \Gamma_{\bar{x}} \mathcal{F} \xrightarrow{+1}.$$

Since  $\dim Z < \dim x$ , we can use the induction hypothesis and monotonicity of  $p$  to deduce that  $\Gamma_Z \mathcal{F}$  is in degrees at least  $p(\dim Z) \geq p(x) > k$ . Clearly  $\Gamma_Z \tau_{>k} \Gamma_{\bar{x}} \mathcal{F}$  is also in degrees larger than  $k$ . Hence  $H^k(\Gamma_{\bar{x}} \mathcal{F})$  has to vanish.

If  $Y$  is not irreducible, let  $Y_1$  be an irreducible component of  $Y$  and  $Y_2$  be the union of the other components. Then there is a Mayer-Vietoris distinguished triangle

$$\Gamma_{Y_1 \cap Y_2} \mathcal{F} \rightarrow \Gamma_{Y_1} \mathcal{F} \oplus \Gamma_{Y_2} \mathcal{F} \rightarrow \Gamma_Y \mathcal{F} \xrightarrow{+1},$$

where  $\Gamma_{Y_1 \cap Y_2} \mathcal{F} \in D^{\geq p(\dim Y_1 \cap Y_2)}(X) \subseteq D^{\geq p(\dim Y)+1}(X)$  (by the induction hypothesis and strict monotonicity of  $p$ ) and  $\Gamma_{Y_1} \mathcal{F}$  and  $\Gamma_{Y_2} \mathcal{F}$  are in  $D^{\geq p(\dim Y)}(X)$  by induction on the number of components of  $Y$ . Thus  $\Gamma_Y \mathcal{F} \in D^{\geq p(\dim Y)}(X)$  as required.  $\square$

**Lemma 3** ([Kas04, Proposition 5.2]). *Let  $\mathcal{F} \in D_c^b(X)$ ,  $Z$  a closed subset of  $X$ , and  $n$  an integer. Then  $\Gamma_Z \mathcal{F} \in D_{qc}^{\geq n}(X)$  if and only if  $\dim(Z \cap \text{supp}(H^k(\mathbb{D}\mathcal{F}))) \leq -k - n$  for all  $k$ .*

This lemma extends [Kas04, Proposition 5.2] to singular varieties. The proof is same as for the smooth case, but we will include it here for completeness.

*Proof.* By [SGA2, Proposition VII.1.2],  $\Gamma_Z \mathcal{F} \in D_{qc}^{\geq n}(X)$  if and only if

$$\mathcal{H}om(\mathcal{G}, \mathcal{F}) \in D_c^{\geq n}(X) \quad (1)$$

for all  $\mathcal{G} \in \mathbf{Coh}(X)$  with  $\text{supp } \mathcal{G} \subseteq Z$ . Let  $d(n) = -n$  be the dual standard perversity. Then by [Bez00, Lemma 5a], (1) holds if and only if  $\mathbb{D}\mathcal{H}om(\mathcal{G}, \mathcal{F}) \in {}^d D^{\leq -n}(X)$ . By [Har66, Proposition V.2.6],  $\mathbb{D}\mathcal{H}om(\mathcal{G}, \mathcal{F}) = \mathcal{G} \otimes_{\mathcal{O}_X} \mathbb{D}\mathcal{F}$ , so that by Proposition 2(a) we need to show that

$$\dim \text{supp } H^k(\mathcal{G} \otimes_{\mathcal{O}_X} \mathbb{D}\mathcal{F}) \leq -k - n$$

for all  $k$ . By [Kas04, Lemma 5.3] this is equivalent to

$$\dim(Z \cap \text{supp } H^k(\mathbb{D}\mathcal{F})) \leq -k - n$$

for all  $k$ , completing the proof.  $\square$

### 3. MEASURING SUBVARIETIES

From now on we assume that the  $G$ -action has finitely many orbits.

**Definition 4.** Let  $p$  be a perversity. A  $p$ -measuring subvariety of  $X$  is an equidimensional subvariety  $Z$  of  $X$  such that the following conditions hold for each  $x \in X^{\text{top}}$  with  $\bar{x} \cap Z \neq \emptyset$ :

- $\dim(\bar{x} \cap Z) = p(x) + \dim x$ ;
- $\bar{x} \cap Z$  is the underlying variety of a regularly embedded subscheme in  $\bar{x}$ , i.e., up to radical  $\bar{x} \cap Z$  it is locally defined in  $\bar{x}$  by exactly  $-p(x)$  functions.

We say that  $X$  has *enough  $p$ -measuring subvarieties* if for each  $x \in X^{\text{top}}$  there exists a  $p$ -measuring subvariety  $Z$  with  $Z \cap \bar{x} \neq \emptyset$ .

*Remark 5.* Let  $Z$  be a  $p$ -measuring subvariety. Then  $\dim(\bar{x} \cap Z) = -\bar{p}(x)$ . Thus comonotonicity of  $p$  ensures that if  $\dim y \leq \dim x$  then  $\dim(\bar{y} \cap Z) \leq \dim(\bar{x} \cap Z)$  for each  $p$ -measuring  $Z$ . Monotonicity of  $p$  then further says that  $\dim(\bar{x} \cap Z) - \dim(\bar{y} \cap Z) \leq \dim x - \dim y$ . We clearly have  $0 \leq \dim(\bar{x} \cap Z) \leq \dim x$  and hence  $-\dim x \leq p(x) \leq 0$ . We will show in Theorem 9 that these condition are actually sufficient for the existence of enough  $p$ -measuring subvarieties, at least when  $X$  is affine.

**Theorem 6.** *Let  $p$  be a strictly monotone and (not necessarily strictly) comonotone perversity and assume that  $X$  has enough  $p$ -measuring subvarieties. Then,*

- (i)  ${}^pD^{\leq 0}(X) = \{ \mathcal{F} \in D_c^b(X_G) : \Gamma_Z \mathcal{F} \in D^{\leq 0}(X) \text{ for all } p\text{-measuring subvarieties } Z \};$
- (ii)  ${}^pD^{\geq 0}(X) = \{ \mathcal{F} \in D_c^b(X_G) : \Gamma_Z \mathcal{F} \in D^{\geq 0}(X) \text{ for all } p\text{-measuring subvarieties } Z \}.$

*Therefore the sheaf  $\mathcal{F} \in D_c^b(X_G)$  is perverse with respect to  $p$  if and only if  $\Gamma_Z \mathcal{F}$  is cohomologically concentrated in degree 0 for each  $p$ -measuring subvariety  $Z$ .*

The following lemma encapsulates the central argument of the proof of the first part of the theorem.

**Lemma 7.** *Let  $\mathcal{F} \in \mathbf{Coh}(X_G)$  be a  $G$ -equivariant coherent sheaf on  $X$ , let  $p$  be a monotone perversity and let  $n$  be an integer. Assume that  $X$  has enough  $p$ -measuring subvarieties. Then the following are equivalent:*

- (i)  $p(\dim \operatorname{supp} \mathcal{F}) \geq n;$
- (ii)  $H^\ell(\Gamma_Z \mathcal{F}) = 0$  for all  $\ell \geq -n + 1$  and all measuring subvarieties  $Z$ .

*Proof.* Since  $\operatorname{supp} \mathcal{F}$  is always a union of the closure of orbits, we can restrict to the support and assume that  $\operatorname{supp} \mathcal{F} = X$ .

First assume that  $p(\dim X) = p(\dim \operatorname{supp} \mathcal{F}) \geq n$ . By the definition of a  $p$ -measuring subvariety, this means that, up to radical,  $Z$  can be locally defined by at most  $-n$  equations. Thus  $H^\ell(\Gamma_Z \mathcal{F}) = 0$  for  $\ell > -n$  [BS98, Theorem 3.3.1].

Now assume conversely that  $H^\ell(\Gamma_Z \mathcal{F}) = 0$  for all  $\ell \geq -n + 1$  and all measuring subvarieties  $Z$ . We have to show that  $p(\dim \operatorname{supp} \mathcal{F}) \geq n$ . Set  $d = \dim \operatorname{supp} \mathcal{F}$ . Choose any  $p$ -measuring subvariety  $Z$ . Then  $\operatorname{codim}_Z X = -p(d)$ . We will show that  $H^{-p(d)}(\Gamma_Z \mathcal{F}) \neq 0$  and hence  $p(d) \geq n$  by assumption. Take some affine open subset  $U$  of  $X$  such that  $U \cap Z$  is non-empty and irreducible in  $U$ . It suffices to show that the cohomology is non-zero in  $U$ . Thus we can assume without loss of generality that  $X$  is affine, say  $X = \operatorname{Spec} A$ , and  $Z$  is irreducible. Write  $Z = V(\mathfrak{p})$  for some prime ideal  $\mathfrak{p}$  of  $A$ . By flat base change [BS98, Theorem 4.3.2],

$$\Gamma(X, H^{-p(d)}(\Gamma_Z \mathcal{F}))_{\mathfrak{p}} = \left( H_{\mathfrak{p}}^{-p(d)}(\Gamma(X, \mathcal{F})) \right)_{\mathfrak{p}} = H_{\mathfrak{p}}^{-p(d)}(\Gamma(X, \mathcal{F})_{\mathfrak{p}})$$

Since  $\dim \operatorname{supp} \mathcal{F} = \dim X = d$ , the dimension of the  $A_{\mathfrak{p}}$ -module  $\Gamma(X, \mathcal{F})_{\mathfrak{p}}$  is  $-p(d)$ . Thus by the Grothendieck non-vanishing theorem [BS98, Theorem 6.1.4]  $H_{\mathfrak{p}}^{-p(d)}(\Gamma(X, \mathcal{F})_{\mathfrak{p}}) \neq 0$  and hence  $\Gamma(X, H^{-p(d)}(\Gamma_Z \mathcal{F})) \neq 0$  as required.  $\square$

*Proof of Theorem 6.*

- (i) We use the description of  ${}^pD^{\leq 0}(X)$  given by Proposition 2, i.e.

$${}^pD^{\leq 0}(X) = \{ \mathcal{F} \in D_c^b(X_G) : p(\dim(\operatorname{supp} H^n(\mathcal{F}))) \geq n \text{ for all } n \}.$$

We induct on the largest  $k$  such that  $H^k(\mathcal{F}) \neq 0$  to show that  $\mathcal{F} \in {}^pD^{\leq 0}$  if and only if  $\Gamma_Z \mathcal{F} \in D^{\leq 0}(X)$  for all  $p$ -measuring subvarieties  $Z$ .

The equivalence is trivial for  $k \ll 0$ . For the induction step note that there is a distinguished triangle

$$\tau_{<k} \mathcal{F} \rightarrow \mathcal{F} \rightarrow H^k(\mathcal{F})[-k] \xrightarrow{+1}.$$

Applying the functor  $\Gamma_Z$  and taking cohomology we obtain an exact sequence

$$\begin{aligned} \cdots \rightarrow H^1(\Gamma_Z(\tau_{<k} \mathcal{F})) \rightarrow H^1(\Gamma_Z \mathcal{F}) \rightarrow H^{k+1}(\Gamma_Z(H^k(\mathcal{F}))) \rightarrow \\ H^2(\Gamma_Z(\tau_{<k} \mathcal{F})) \rightarrow H^2(\Gamma_Z \mathcal{F}) \rightarrow H^{k+2}(\Gamma_Z(H^k(\mathcal{F}))) \rightarrow \cdots \end{aligned}$$

By induction,  $H^\ell(\Gamma_Z(\tau_{<k} \mathcal{F}))$  vanishes for  $\ell \geq 1$  so that  $H^\ell(\Gamma_Z \mathcal{F}) \cong H^{k+\ell}(\Gamma_Z(H^k(\mathcal{F})))$  for  $\ell \geq 1$ . Thus the statement follows from Lemma 7.

(ii) By Proposition 2(b),  $\mathcal{F} \in {}^p D^{\geq 0}$  if and only if

$$\dim(\bar{x} \cap \text{supp}(H^k(\mathbb{D}F))) \leq -p(x) - k \quad \text{for all } x \in X^{\text{top}} \text{ and all } k. \quad (2)$$

Using Lemma 3 for  $\Gamma_Z \mathcal{F} \in D^{\geq 0}(X)$ , we see that we have to show the equivalence of (2) with

$$\dim(Z \cap \text{supp}(H^k(\mathbb{D}F))) \leq -k \quad \text{for all } k \text{ and } p\text{-measuring } Z.$$

Since there are only finitely many orbits, this is in turn equivalent to

$$\dim(Z \cap \bar{x} \cap \text{supp}(H^k(\mathbb{D}F))) \leq -k \quad \forall x \in X^{\text{top}}, k \text{ and } p\text{-measuring } Z. \quad (3)$$

We will show the equivalence for each fixed  $k$  separately. Let us first show the implication from (2) to (3). Since  $H^k(\mathbb{D}F)$  is  $G$ -equivariant and there are only finitely many  $G$ -orbits, it suffices to show (3) assuming that  $\dim x \leq \dim \text{supp } H^k(\mathbb{D}F)$  and  $\bar{x} \cap \text{supp } H^k(\mathbb{D}F) \neq \emptyset$ . Then  $\dim(\bar{x} \cap \text{supp}(H^k(\mathbb{D}F))) = \dim \bar{x}$ . Thus,

$$\begin{aligned} \dim(Z \cap \bar{x} \cap \text{supp}(H^k(\mathbb{D}F))) &\leq \dim(Z \cap \bar{x}) = p(x) + \dim x = \\ &p(x) + \dim(\bar{x} \cap \text{supp}(H^k(\mathbb{D}F))) \leq p(x) - p(x) - k = -k. \end{aligned}$$

Conversely, assume that (3) holds for  $k$ . If  $\bar{x} \cap \text{supp } H^k(\mathbb{D}F) = \emptyset$ , then (2) is trivially true. Otherwise choose a  $p$ -measuring  $Z$  that intersects  $\text{supp } H^k(\mathbb{D}F)$ . First assume that  $\bar{x}$  is contained in  $\text{supp } H^k(\mathbb{D}F)$ . Then

$$\begin{aligned} \dim(\bar{x} \cap \text{supp}(H^k(\mathbb{D}F))) &= \dim x = -p(x) + \dim(Z \cap \bar{x}) = \\ &-p(x) + \dim(Z \cap \bar{x} \cap \text{supp}(H^k(\mathbb{D}F))) \leq -p(x) - k. \end{aligned}$$

Otherwise  $\bar{x} \cap \text{supp}(H^k(\mathbb{D}F)) = \bar{y}$  for some  $y \in X^{\text{top}}$  with  $\dim y < \dim x$ . Then (2) holds for  $y$  in place of  $x$  and hence

$$\dim(\bar{x} \cap \text{supp}(H^k(\mathbb{D}F))) = \dim(\bar{y} \cap \text{supp}(H^k(\mathbb{D}F))) \leq -p(y) - k \leq -p(x) - k$$

by monotonicity of  $p$ .  $\square$

*Example 8.* For the dual standard perversity  $p(n) = -n$  (i.e.  $p(x) = -\dim x$ ), we recover the definition of Cohen-Macaulay sheaves [Har66, Section IV.3].  $\triangleleft$

Of course, for the theorem to have any content, one needs to show that  $X$  has enough  $p$ -measuring subvarieties. The next theorem shows that at least for affine varieties there are always enough measuring subvarieties whenever  $p$  satisfies the obvious conditions (see Remark 5).

**Theorem 9.** *Assume that  $X$  is affine and the perversity  $p$  satisfies  $-n \leq p(n) \leq 0$  and is monotone and comonotone. Then  $X$  has enough  $p$ -measuring subvarieties.*

*Proof.* Let  $X = \operatorname{Spec} A$ . We induct on the dimension  $d$ . More precisely, we induct on the following statement:

There exists a closed equidimensional subvariety  $Z_d$  of  $X$  such that for all  $x \in X^{\text{top}}$  the following holds:

- $Z_d \cap \bar{x} \neq \emptyset$  and  $Z_d \cap \bar{x}$  is regularly embedded in  $\bar{x}$ ;
- if  $\dim x \leq d$ , then  $\dim(\bar{x} \cap Z_d) = p(x) + \dim x$ ;
- if  $\dim x > d$ , then  $\dim(\bar{x} \cap Z_d) = p(d) + \dim x$ .

We set  $p(-1) = 0$ . The statement is trivially true for  $d = -1$ , e.g. take  $Z = X$ . Assume that the statement is true for some  $d \geq -1$ . We want to show it for  $d + 1 \leq \dim X$ .

If  $p(d) = p(d + 1)$ , then  $Z_{d+1} = Z_d$  works. Otherwise, by (co)monotonicity,  $p(d + 1) = p(d) - 1$ . Set  $S = \bigcup \{\bar{x} \in X^{\text{top}} : \dim x \leq d\}$ . Since there are only finitely many orbits, we can choose a function  $f$  such that  $f$  vanishes identically on  $S$ ,  $V(f)$  does not share a component with  $Z_d$  and  $V(f)$  intersects every  $\bar{x}$  with  $\dim x > d$ . Then  $Z_{d+1} = Z_d \cap V(f)$  satisfies the conditions.  $\square$

## APPENDIX. CONSTRUCTIBLE SHEAVES

We return now to the claim about exact functors on the t-structure of constructible perverse sheaves made in the introduction. Let  $X$  be a complex manifold and  $\mathfrak{S}$  a finite stratification of  $X$  by complex submanifolds. We write  $D_{\mathfrak{S}}^b(X)$  for the bounded derived category of  $\mathfrak{S}$ -constructible sheaves on  $X$ . We call a sheaf  $\mathcal{F} \in D_{\mathfrak{S}}^b(X)$  perverse if it is perverse with respect to the middle perversity function on  $\mathfrak{S}$ . We are going to formulate and prove an analog of Theorem 6 in this situation.

A closed real submanifold  $Z$  of  $X$  is called a *measuring submanifold* if for each stratum  $S$  of  $X$  either  $Z \cap \bar{S} = \emptyset$  or  $\dim_{\mathbb{R}} Z \cap S = \dim_{\mathbb{C}} S$ .

**Theorem 10.** *A sheaf  $\mathcal{F} \in D_{\mathfrak{S}}^b(X)$  is perverse if and only if  $i_Z^! \mathcal{F}$  is concentrated in cohomological degree 0 for each measuring submanifold  $Z$  of  $X$ .*

**Lemma 11.** *Let  $X$  be a real manifold,  $\mathcal{F}$  be a constructible sheaf (concentrated in degree 0) on  $X$  and let  $i : Z \hookrightarrow X$  be the inclusion of a closed submanifold. Then  $H^j(i^! \mathcal{F}) = 0$  for  $j > \operatorname{codim}_X Z$ .*



*Proof.*<sup>1</sup> By taking normal slices we can reduce to the case that  $Z = \{z\}$  is a point. Let  $j$  be the inclusion of  $X \setminus \{z\}$  into  $X$  and consider the distinguished triangle

$$i_! i^! \mathcal{F} \rightarrow \mathcal{F} \rightarrow j_* j^* \mathcal{F} \xrightarrow{+1}.$$

By [KS94, Lemma 8.4.7] we have

$$H^j(j_* j^* \mathcal{F})_z = H^j(S_\epsilon^{\dim X - 1}, \mathcal{F})$$

for a sphere  $S_\epsilon^{\dim X - 1}$  around  $x$  of sufficiently small radius. The latter cohomology vanishes for  $j \geq \dim X$  and hence  $H^j(i^! \mathcal{F}) = 0$  for  $j > \dim X$  as required.  $\square$

*Proof of Theorem.* Clearly it is enough to check the condition on a collection of measuring submanifolds  $\{Y_i\}$  such that each connected component of each stratum has non-empty intersection with at least one  $Y_i$ . Similarly to Theorem 9, one easily shows inductively that such a collection of submanifolds exists.

Define two full subcategories  ${}^L D^{\leq 0}(X)$  and  ${}^L D^{\geq 0}(X)$  of  $D_{\mathcal{C}}^b(X)$  by

$$\begin{aligned} {}^L D^{\leq 0}(X) &= \{ \mathcal{F} \in D_{\mathcal{C}}^b(X) : i_Z^! \mathcal{F} \in D^{\leq 0}(Z) \text{ for all measuring submanifolds } Z \}, \\ {}^L D^{\geq 0}(X) &= \{ \mathcal{F} \in D_{\mathcal{C}}^b(X) : i_Z^! \mathcal{F} \in D^{\geq 0}(Z) \text{ for all measuring submanifolds } Z \}. \end{aligned}$$

We will show that these categories are the same as the categories  ${}^p D^{\leq 0}(X)$  and  ${}^p D^{\geq 0}(X)$  defining the perverse t-structure on  $D_{\mathcal{C}}^b(X)$ .

We induct on the number of strata. If  $X$  consists of only one stratum and  $Z$  is a measuring submanifold, then  $i_Z^! \mathcal{F} \cong \omega_{Z/X} \otimes i_Z^* \mathcal{F}$  and hence  $i_Z^! \mathcal{F}$  is in degree 0 if and only if  $\mathcal{F}$  is in degree  $-\frac{1}{2} \dim_{\mathbb{R}} X$ . So assume that  $X$  has more than one stratum. Without loss of generality we can assume that  $X$  is connected. Let  $U$  be the union of all open strata and  $F$  its complement. Both  $U$  and  $F$  are unions of strata of  $X$ . Let  $j$  be the inclusion of  $U$  and  $i$  the inclusion of  $X$ .

- If  $\mathcal{F} \in {}^p D^{\leq 0}(X)$ , then  $\mathcal{F} \in {}^L D^{\leq 0}$  follows in exactly the same way as in the coherent case, using Lemma 11.
- Let  $\mathcal{F} \in {}^p D^{\geq 0}(X)$ . Then  $i^! \mathcal{F} \in {}^p D^{\geq 0}(F)$  and  $j^* \mathcal{F} \in {}^p D^{\geq 0}(U)$ . Let  $Z$  be a measuring subvariety. Consider the distinguished triangle

$$i_* i^! \mathcal{F} \rightarrow \mathcal{F} \rightarrow j_* j^* \mathcal{F}.$$

Using base change, induction and the (left)-exactness of the push-forward functors one sees that  $i_Z^!$  of the outer sheaves in the triangle are concentrated in non-negative degrees. Thus so is  $i_Z^! \mathcal{F}$ .

- Let  $\mathcal{F} \in {}^L D^{\geq 0}(X)$ . Since all measurements are local this implies that  $j^* \mathcal{F} \in {}^L D^{\geq 0}(U) = {}^p D^{\geq 0}(U)$ . Using the same triangle and argument as in the last point, this implies that also  $i^! \mathcal{F} \in {}^L D^{\geq 0}(F) = {}^p D^{\geq 0}(F)$ . Hence, by recollement,  $\mathcal{F} \in {}^p D^{\geq 0}(X)$ .

<sup>1</sup>Following <http://mathoverflow.net/questions/129244> by Geordie Williamson.

- Finally, let  $\mathcal{F} \in {}^L D^{\leq 0}(X)$ . Again this immediately implies that  $j^* \mathcal{F} \in {}^L D^{\leq 0}(U) = {}^p D^{\leq 0}(U)$ . Thus  $j_! j^* \mathcal{F} \in {}^p D^{\leq 0}(X)$ . Let  $Z$  be a measuring submanifold and consider the distinguished triangle

$$i_Z^! j_! j^* \mathcal{F} \rightarrow i_Z^! \mathcal{F} \rightarrow i_Z^! i_* i^* \mathcal{F}.$$

By what we already know, the first sheaf is concentrated in non-positive degrees and hence so is  $i_Z^! i_* i^* \mathcal{F}$ . By base change and the exactness of  $i_*$  this implies that  $i^* \mathcal{F} \in {}^L D^{\leq 0}(F) = {}^p D^{\leq 0}(F)$ . Hence, by recollement,  $\mathcal{F} \in {}^p D^{\leq 0}(X)$ .  $\square$

*Remark 12.* The equality  ${}^p D^{\geq 0}(X) = {}^L D^{\geq 0}(X)$  could also be proved in exactly the same way as in the coherent case, using [KS94, Exercise X.10].

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