VI. (Alex Perry) Cotangent Complex (2)

Plan ① Review
② Connectivity Results
③ Deformation theory

fix $f : A \to B$

① Quillen adjoint pair

$$F : SCR_{A//B} \longrightarrow Mod_B$$
$$X \longmapsto \Omega_{X/A} \otimes_X B$$

where $X \in SCR_{A//B}$ means X is equipped with $A \to X \to B$

$$G : Mod_B \longrightarrow SCR_{A//B}$$
$$M \longmapsto B \oplus M$$

$\Rightarrow$ Total derived functors exist and are adjoint.

in this notation, $L_{B/A} := LF(B)$

$$Hom_{h(Mod_B)}(L_{B/A}, M) \underset{(*)}{=} Hom_{h(SCR_{A//B})}(B, \overset{B\oplus M}{\overset{\shortparallel}{RG(M)}})$$

② Theorem. $f : A \to B \in SCR$. $cofib(f)$ is n-connected $\Rightarrow$ $\varepsilon : cofib(f) \otimes_A B \longrightarrow L_{B/A}$ is $(2n+1)$-connected

where $cofib(f)$ is the pushout of

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ 0 & \end{tikzcd}$$

$fib(f)$ is the pullback of

$$\begin{tikzcd} & X \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$

when we say a map is m-connected, we mean that its fiber is.

We will not prove this, but we say what ε is.

$B \longrightarrow L_{B/A}$ comes from $(*)$ applied to $M = L_{B/A}$

then by universal property, get $cofib(f) \otimes_A B \to L_{B/A}$.

<u>Corollary</u>: $f : A \to B$

$cofib(f)$ is n-conn. $\Rightarrow$ $L_{B/A}$ n-conned

$\Leftarrow$ true if $\pi_0(A) \xrightarrow{\cong} \pi_0(B)$

<u>Pf</u> "$\Rightarrow$" $fib(\varepsilon) \to cofib(f) \underset{A}{\otimes} B \to L_{B/A}$

"$\Leftarrow$" pick smallest k s.t. $\pi_k(cofib(f)) \neq 0$.

$$\pi_k(cofib(f)) \xrightarrow{\cong} \pi_k(cofib(f) \underset{A}{\otimes} B) \xrightarrow{\cong} \pi_k(L_{B/A})$$

follows from the theorem.

<u>Corollary</u>: $f : A \to B$ is an equivalence iff

(1) $\pi_0(A) \xrightarrow{\cong} \pi_0(B)$

(2) $L_{B/A} \simeq 0$

<u>Corollary</u> $A \to \pi_0(A)$ is an equivalence iff $L_{\pi_0(A)/A} \simeq 0$

so "the cotangent complex controls the homotopy information of a derived rin

③ A, B, M

$$Ext^1(L_{B/A}, M) = Ext^0(L_{B/A}, M[1]) = Hom(B, B \oplus M[1])$$

"$Spec(B \oplus M[1]) \to Spec(B)$"

schemes $\begin{matrix} X \\ \downarrow \\ Y \end{matrix} \rightsquigarrow L_{X/Y}$

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d, "f"'] \arrow[dr, phantom, "?"] & X' \arrow[d, dashed] \\ Y \arrow[r, hook, "i", "sq\text{-}0"'] \arrow[dr] & Y' \arrow[d] \\ & S \end{tikzcd}$$

flat f

<u>thm</u> 1) $\exists\ ob(f) \in Ext^2(L_{X/Y}, f^*M)$

$ob = 0 \Leftrightarrow$ def exists.

2) if $ob = 0$, then $\{defs\}/_{\simeq} = Ext^1(L_{X/Y},$

<u>Remarks</u> 1) $Ext^1(L_{Y/S}, M)$ classifies sq. zero extensions of Y by M over S

2) $\underset{\underset{\displaystyle Ext^2(L_{X/Y}, f^*M)}{\cap}}{ob}$ = composition of $f^*[j]$ w/ $\underset{\underset{\displaystyle Ext^1(f^*L_{Y/S}, f^*M)}{\cap}}{}$ $KS(X \to Y \to S) \in Ext^1(L_{X/Y}, f^*L_{Y/S})$

after changing Ext^1's to E
introducing a degree shift.

Now, upgrade everything to simplicial comm. rings.

what is a sq. zero extension in DAG? There are no ideals.

<u>Construction</u>. (in the absolute setting now) $A \in sCR$, $M \in Mod_A$

$$\begin{tikzcd} A \oplus M[1] \arrow[d] \\ A \arrow[u, bend left, "s"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} A' \arrow[r, "\varphi_s"] \arrow[d] & A \arrow[d, "s"] \\ A \arrow[r, "0"'] & A \oplus M[1] \end{tikzcd}$$

pullback diagram

<u>Def</u> φ_s is called a sq. zero extension of A by M.

e.g. $A \oplus M$ is indeed a sq. zero extension
$k \oplus k[1]$

<u>Prop</u> $A \in sCR$, then every map in

$$\cdots \to \tau_{\le 2} A \to \tau_{\le 1} A \to \tau_{\le 0} A \to \pi_0(A)$$

(Postnikov Tower)

is a sq. zero extension.

<u>Consequence</u>

$$\begin{tikzcd} \tau_{\le n} A \arrow[r] \arrow[d] & \tau_{\le n-1} A \arrow[d, "k_n"] \\ \tau_{\le n-1} A \arrow[r] & \tau_{\le n-1} A \oplus \pi_n(A)[n+1] \end{tikzcd}$$

$k_n \in \mathrm{Hom}(L_{\tau_{\le n-1} A}, \pi_n(A)[n+1])$

$$\begin{tikzcd} B & B' \arrow[l, dashed] \\ A \arrow[u, "f"] & A' \arrow[l, "s.z.e."] \arrow[u, dotted] \end{tikzcd}$$

<u>thm</u> (1) $\exists\, ob(f) \in Ext^2(L_{B/A}, B \otimes_A M)$
$ob = 0 \Rightarrow$ def exists
(2) $ob = 0 \Rightarrow \{\text{defs}\}/\simeq \; = Ext^1(L_{B/A}, B \otimes_A M)$

This is the DAG version (everything is simplicial)

Explicitly write down what ob is.

$$L_{B/A}[-1] \to B \otimes L_A \to B \otimes M[1]$$

$$
\begin{tikzcd}
X = X_0 \arrow[r] \arrow[d, "f", "\text{sm. proj.}"'] & X_1 \arrow[r, dashed] \arrow[d, "f_1"'] & X_2 \arrow[d, dashed] \\
\operatorname{Spec} k \arrow[r] & \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \arrow[r] & \operatorname{Spec} k[\varepsilon]/\varepsilon^3
\end{tikzcd}
$$

$$ob(f_1) \in H^2(T_X) = Ext^2(L_X, k) = Ext^1(L_X, k[1])$$

classifies
$$
\begin{tikzcd}
X \arrow[r] \arrow[d] & X' \arrow[d] \\
\operatorname{Spec} k \arrow[r] & \operatorname{Spec}(k \oplus k[1])
\end{tikzcd}
$$
as long as the ob vanishes, which it does.

$$
\begin{tikzcd}
k[\varepsilon]/\varepsilon^3 \arrow[r] \arrow[d] & k[\varepsilon]/\varepsilon^2 \arrow[d] \\
k \arrow[r] & k \oplus k[1]
\end{tikzcd}
$$
pullback diagram

$\Rightarrow$
$$
\begin{tikzcd}
 & X_1^{ob} \arrow[rr] \arrow[d, "ob(f_1)"] & & X_1 \arrow[d, "f_1"] \\
X^{triv} \arrow[r] \arrow[d] & \operatorname{Spec}(k \oplus k[1]) \arrow[rr, "\text{pushout}"'] \arrow[d] & & \operatorname{Spec}(k[\varepsilon]/\varepsilon^2) \arrow[d] \\
X \arrow[r] & \operatorname{Spec}(k) \arrow[rr] & & \operatorname{Spec}(k[\varepsilon]/\varepsilon^3)
\end{tikzcd}
$$

This can be formalized by a deformation functor.

$$Ext^0(L_X, k[1])$$