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Derived Schemes (Representability Theorems)

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Artin's Theorem

R an excellent Noetherian ring (e.g. $\mathbb{Z}$)

$$\mathrm{Alg}_{R/-} \xrightarrow{F} (\mathrm{Grpd})$$

F is representable by an Artin stack locally of fin. presentation $/R$ if:

(a) $F(\operatorname{colim} A_i) = \operatorname{colim}_i F(A_i)$ for filtered diagrams $i \mapsto A_i$

(b) F is a sheaf for the étale topology

(i.e. If $\{U_i \to X\}$ an étale cover of affines, then

$$F(X) \xrightarrow{\simeq} \operatorname{holim} \left(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_X U_j) \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{i,j,k} F(U_i \underset{X}{\times} U_j \underset{X}{\times} U_k) \right)$$

(c) $F(B) \xrightarrow{\simeq} \operatorname{holim}_n F(B/m^n)$ for complete Noeth. local R-alg. (B, m)

(Grothendieck Formal GAGA)

(d) F admits a deformation/obstruction theory satisfying Schlessinger's conditions

(e) $\underset{\text{diagonal functor}}{\Delta}: F \to F \times F$ is representable by algebraic spaces

Examples ① $F(A) = \left\{ \begin{matrix} \text{Smooth proper curves}/A \\ \text{of genus } g \end{matrix} \right\} = M_g(A)$

② Fix $X/_{\mathrm{Spec}(R)}$ which is proper and flat

$$F(A) = \left\{ \begin{matrix} \text{vector bundles} \\ \text{on } X \times_R A \end{matrix} \right\} = \mathrm{Vect}(X/_S)(A)$$

Lurie's Representability - Theorem

(See Lurie's thesis / Pridham's
and DAG XIV / Papers)

Theorem R is a derived G-ring ($\pi_0(R)$ Noeth. ....)
$F: sCR_{R/-} \longrightarrow$ Spaces (i.e. Kan Complexes)

F is representable by a derived DM stack almost of fin. presentation $/R$, if:

(a) F commutes with filtered colimits on $sCR_{R/-, \leq K}$ (for all $K \in \mathbb{Z}_{\geq 0}$) $\{0 \leq \deg \leq K\}$

(b) F is an étale sheaf

(c) $F(B) = \operatorname{holim}_n F(B/m^n)$ for discrete complete Noeth. local R-algebras

(d_1) F has a connective cotangent complex

(d_2) F is infitesimally cohesive

(e) F is nilcomplete.

$\boxed{\text{Set up}}$ $F \in \mathrm{Fun}(sCR_{R/-}, \text{Spaces})$

(a) <u>Commuting with filtered colimits</u>

If $I \xrightarrow{A} sCR_{R/-, \leq K}$ filtered, then

$$F(\operatorname{colim} A(i)) \xleftarrow{\sim} \operatorname{Colim}_i F(A(i))$$

Example (1) If $B \in sCR_{R/-}$, $F = \underline{\mathrm{Hom}}_{sCR}(B, -)$ satisfies (a) if B is almost fin. presented $/R$

(2) $F(A) = \mathrm{Mod}_A^{f.p.} \subset \mathrm{Mod}_A$

(b) being an étale sheaf

F is an étale sheaf if:

(i) F commutes with finite products

(ii) If $A \longrightarrow B$ is an étale cover in $sCR_{R/-}$, then

$$F(A) \xrightarrow{\sim} \operatorname{holim}_n F(B^{\otimes(n+1)})$$

<u>Example</u> $F = \underline{\mathrm{Hom}}_{sCR_{R/-}}(C, -)$ is an étale sheaf

<u>Key Point</u> $A \simeq \operatorname{holim}_n (B^{\otimes(n+1)})$

<u>Cotangent Complex on a Functor</u>

Given $sCR \xrightarrow{F}$ Spaces

a) $\mathrm{QC}_F^{\geq 0} = \operatorname{holim}_{\substack{A \in sCR \\ \eta \in F(A)}} \mathrm{Mod}_A \rightsquigarrow$ For each (A, η) we get

quasi-coherent sheaf

$M(\eta) \in \mathrm{Mod}_A$

For $A \xrightarrow{\varphi} B$,

$M(\eta) \otimes_A B \xrightarrow{\sim} M(\varphi_* \eta)$

<u>Example</u> $F = \underline{\mathrm{Hom}}(C, -) \rightsquigarrow \mathrm{QC}_F^{\geq 0} = \mathrm{Mod}_A$

b) F has connective cotangent complex

$$L_F \in \mathrm{QC}_F^{\geq 0}$$

if $\exists$ equiv $\mathrm{Hom}_{\mathrm{Mod}_A}(L_F(\eta), N) =$ fiber of $F(A \oplus N) \to F(A)$ over η

(where $A \in sCR$, $\eta \in F(A)$)

(functorially in A, η, N)

Infinitesimally Cohesive

- glue across closed subschemes

Nil complete

$$F(B) \xrightarrow{\sim} \operatorname{holim}_n F(\tau_{\leq n} B)$$

$\boxed{\overline{M}_g}$ Definition A map $X \longrightarrow \operatorname{Spec} A$ is a stable curve of genus $g \geq 2$ if

a) f is flat, qcqs, almost fin presented

b) $X \times_A \pi_0 A \longrightarrow \operatorname{Spec}(\pi_0 A)$ is a stable curve (in usual sense)

Theorem $\overline{M}_g : sCR \longrightarrow Spaces$

$$A \longmapsto \left\{ \begin{array}{c} \text{all stable curves } X \to \operatorname{Spec} A \\ \text{of genus } g \geq 2 \end{array} \right\}$$

$\overline{M}_g$ is representable by $\overline{M}_g^{cl}$

Proposition If $X \xrightarrow{f} \operatorname{Spec}(A)$ is a derived scheme and $M \in \operatorname{Mod}_A$, then

$$\operatorname{Hom}_{QC_X}(L_{X/A}, f^* M[1]) = \left\{ \begin{array}{c} \text{deformations of } X \\ \text{over } \operatorname{Spec}(A \oplus M) \end{array} \right\}$$

Show $\overline{M}_g$ has connective cotangent complex

$$\overline{M}_g(A \oplus M) \longrightarrow \overline{M}_g(A)$$

$$\psi$$

$$\operatorname{Hom}_X(L_{X/A}, f^* M[1]) \longmapsto \eta : (X \xrightarrow{f} \operatorname{Spec}(A))$$

we want $\|$

$$\operatorname{Hom}_A(\eta^* L_{\overline{M}_g}, M)$$

Check $\eta^*(L_{\bar{M}_g/\mathbb{Z}}) = (Rf_* L_{X/A}^{\vee})^{\vee}[-1]$

$$\bar{M}_g(A) \ni \eta : X \longrightarrow \operatorname{Spec}(A)$$