Luminy Geometric Langlands 2015

Toen 1

Plan: I. ∞-cats and derived stacks

II. de Rham theory and formal completion

III. "Applications" to Poisson structures and existence of def. quantization

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I.1. Language of ∞-categories

"Def": **An ∞-cat is like a** category, but Hom's are top. spaces (hom. types) rather than sets.

Concretely, an ∞-cat $\mathbb{T}$ consists of

- a set of objects
- $\forall x,y$ objects a space $T(x,y)$ (= "space of maps $x \to y$")
- $\forall x,y,z$ $T(x,y) \times T(y,z) \to T(x,z)$ a composition map

+ associativity + unit "condition/structure"

Depending on how this is defined this gives rise to different possible definitions of ∞-cat,

- Joyal's quasi-categories
- Simpson's Segal cats
- Grothendieck-Ara ∞-cats
- Rezk's complete Segal spaces

$\vdots$

Thm (Bergner): all these theories are equivalent.

Here we use the most naive one:

$\mathcal{S}$-categories: associativity and unit "on the nose".

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Top. categories

"

Cat enriched in spaces

Def: An ∞-cat T is a category enriched in spaces.

- If T is an ∞-category, it's homotopy category $[T]$ is:

object = objects of T

$[T](x,y) = \pi_0(T(x,y))$

$$[\ ] : \infty\text{-}Cat \rightleftarrows Cat$$

sets $\subseteq$ spaces

<u>Def</u>: $f: T \to T'$ a <u>strict ∞-functor</u> (enriched functor) is an <u>equivalence</u> if

*) $[T] \to [T']$ is an equivalence of categories

**) $\forall x, y$ $\quad T(x,y) \to T'(fx, fy)$ is a (weak) equivalence of spaces.

We are mainly interested in $Ho(\infty\text{-cat}) := (\text{equiv})^{-1}(\infty\text{-cat})$

An <u>important source of ∞-categories</u> is localization:

C cat, $W \subseteq Mor(C)$

<u>Fact</u>: $\exists$ an ∞-category $L_W C$ + a morphism $C \to L_W C$ in $Ho(\infty\text{-Cat})$ which is universal (in $Ho(\infty\text{-Cat})$) for $C \to T$ sending W to isos in $[T]$

$W \longmapsto Isom([T])$

$$\begin{tikzcd} C \arrow[r] \arrow[d] & T \\ L_W C \arrow[ur, "\exists"'] & \end{tikzcd}$$

This is called Dwyer-Kan **localization**

$\cdot$ $[L_W C] \simeq W^{-1} C \leftarrow$ usual Gabriel-Zisman localization

$\uparrow$

$L_W C$

Assume that $(\mathcal{C}, W)$ comes from a model category structure (+ $\mathcal{C}$ enriched over spaces)

Thm (Quillen, Dwyer-Kan): Under these condition exists a natural equivalence of ∞-cat

$$L_W \mathcal{C} = \underline{\mathcal{C}}^{\text{cf-fib}}$$

$\uparrow$ S-category of cofibrant-fibrant objects in $\mathcal{C}$

Examples:

*) ∞-cat of spaces: $\mathcal{C} = \mathrm{Top}$, $W = $ w. equiv

$$\mathbb{T}\mathrm{op} := L_W \mathcal{C} \simeq \text{S-cat of CW-complexes.}$$

**) A a ring (dg), $\mathcal{C} = C(A)$ complexes of A modules
$W = $ q.-iso

$$L(A) := L_W \mathcal{C} \simeq \text{S-cat of projective complexes}$$

$$L(A)(x,y) = DK(\underline{\mathrm{Hom}}^*(x', y))$$

$x' \to x$ a "proj. resolution"

$DK = $ Dold - Kan

$$\pi_i(L(A)(x,y)) = \mathrm{Ext}^{-i}_{D(A)}(x,y) \quad , \quad i \geq 0$$

$$[L(A)] = D(A)$$

∞-functors

<u>Fact</u>: $T, T' \in \mathrm{Ho}(\infty\text{-}\mathrm{Cat})$

$\exists$ ∞-cat $\mathrm{Fun}^\infty(T, T') \in \mathrm{Ho}(\infty\text{-}\mathrm{Cat})$

s.t. $[U, \mathrm{Fun}^\infty(T, T')] \simeq [U \times T, T']$

$\uparrow$ maps in $\mathrm{Ho}(\infty\text{-}\mathrm{Cat})$

$\mathrm{Ho}(\infty\text{-}\mathrm{Cat})$ is Cartesian closed.

⚠ $\mathrm{Fun}^\infty(T, T')$ is <u>not</u> the naive ∞-cat of <u>strict</u> ∞-functors $T \to T'$

Concretely can be understood using the "strictification" theorem

<u>Thm</u> (Simpson): If $T' = L_W C$ for C a model cat then $\mathrm{Fun}^\infty(T, T') \simeq L_W(C^T)$, where C^T is model category of diagrams of shape T in C

The ∞-cat of ∞-categories is

$$\infty\text{-}\underline{\mathrm{cat}} := L_{equiv}(\infty\text{-}\mathrm{cat})$$

- It is Cartesian closed as an ∞-category
- The theory can be pursued
  - notion of limits & colimits
  - notion of adjoint ∞-functors

- ∞-Cat has limits & colimits
- $\exists$ notion of (∞,∞)-categories
- ...

2) Derived stacks

k is a base commutative ring of char 0 (not necessary)

We are going to define an ∞-category dSt_k of derived stacks over k

$\uparrow$ ∞-cat version of the tops of (affine) k-schemes + etale top

$cdga_k$ = commutative dg algebras$/k$ non-positively graded
∞-cat by localization along quasi-isos.

(Note: $cdga_k$ has a model structure)

Thm $\Rightarrow$ $cdga_k \underset{\infty\text{-cat}}{\simeq} \begin{cases} \text{obj: } \underline{\text{cofibrant}} \text{ cdga}_k \\ \underline{\text{Hom}}(A,B) = \text{Hom}(A, B\otimes\Omega_{\Delta}) \end{cases}$

"cellular"

<u>Def</u>: The ∞-category $dAff_k$ of derived k-schemes is $(cdga_k)^{op}$

<u>Notation</u>: $A\in cdga_k$, $\operatorname{Spec} A$ denotes the corresponding object in $dAff_k$.

$dAff_k$ comes equipped with an étale topology

$\Big\updownarrow$ def

topology on $[dAff_k]$

$A \longrightarrow B$ in $cdga_k$ is étale if

$$\begin{cases} \pi_0(A) \longrightarrow \pi_0(B) \text{ is étale} \\ \pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \xrightarrow{\sim} \pi_*(B) \end{cases}$$

A family of maps $\{A \to A_i\}_{i \in I}$ is an étale covering if each map is étale and $\{\pi_0(A) \to \pi_0(A_i)\}_{i \in I}$ is a covering.

This defines a Grothendieck top. on $dAff_k$.

An ∞-prestack is an ∞-functor $F : dAff_k^{op} \longrightarrow Top$:

$$d\widehat{Aff}_k := Fun^\infty(dAff_k^{op}, Top)$$
$$\parallel$$
$$Fun^\infty(cdga_k, Top)$$

<u>Def</u>: The <u>∞-cat of derived stacks</u> is the full sub-∞-cat of $d\widehat{Aff}_k$ consisting of $F : dAff_k^{op} \to Top$ satisfying étale descent

<u>descent</u>: many possible meaning

<u>two standard:</u>

- Čech descent: $X \to U$ étale covering in $dAff$

  $X_\bullet$: Nerve of $X \to U$

  then: $F(U) \xrightarrow{\sim} \lim_{n \in \Delta} F(X_n)$ in the ∞-cat $\mathbb{T}op$

- hyperdescent: $\forall\, X_\bullet \to U$ ét. hypercovering in $dAff$,

  $F(U) \xrightarrow{\sim} \lim_{n \in \Delta} F(X_n)$

Notation: $\widehat{dAff}_k \supseteq dAff^\sim = \infty$-cat of derived stacks

$$\begin{tikzcd} \widehat{dAff}_k & dAff^\sim \\ dAff \arrow[u, "\text{Yoneda}"] \arrow[ur, "\text{full embedding}"'] & \end{tikzcd}$$

<u>Example:</u> G - reductive$/k$

X smooth proper scheme$/k$

The derived stack of G-bundles on X is

$$\mathbb{R}Bun_G : dAff_k^{op} \longrightarrow \mathbb{T}op$$
$$S \longmapsto Hom_{dSt_k}(X \times S, BG)$$

Some comments: $X \in$ Schemes $\subset Aff^\sim \hookrightarrow dAff^\sim$

full embedding of ∞-cat "left Kan ext"

$G \in$ Groupschemes $\subseteq Gp$ inside $dAff^\sim$

$BG = \operatorname{colim}_{n \in \Delta^{op}} G^n \in dAff^\sim_k$

Algebraic derived stack:

smooth maps: $A \to B$ in $cdga_k$ is <u>smooth</u> if

$$\begin{cases} \pi_0(A) \to \pi_0(B) \text{ smooth} \\ \pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \xrightarrow{\sim} \pi_*(B) \end{cases}$$

<u>Def</u>: *) $F \in dAff^{\sim}$ is <u>0-geometric</u> if it is affine (in the image of Yoneda)

*) F is <u>n-geometric</u> if $\exists$ affine scheme X_i and $\coprod X_i \to X$ $(n-1)$-smooth & surjective

**) $f: F \to G$ is n-smooth if $\forall$ affine X, $\forall X \to G$

1) $F \times_G X$ is n-geometric

2) $F \times_G X \to X$ is smooth

i.e. $\exists U_i$ affine & $\coprod U_i \to F \times_G X$ is surj.

$U_i \to X$ is smooth map of affines

<u>Thm</u>: X sm. proper, G reductive

Then $\mathbb{R}Bun_G(X)$ is n-geometric for some n.

Gaitsgory I

X – curve (smooth & proper)

G – reductive groupe

$$\text{D-mod}(\mathrm{Bun}_G) \overset{\mathbb{L}_G}{\simeq} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{G^\vee})$$

$$\begin{tikzcd}
\boxed{\begin{matrix} \mathrm{Whit}^{\mathrm{ext}} \\ \uparrow \\ \text{D-mod}(\mathrm{Bun}_G) \end{matrix}} & ? \arrow[l] \\
& \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{G^\vee}) \arrow[u]
\end{tikzcd}$$

today

$$\mathrm{Bun}_G(k) = G_k \backslash G_{\mathbb{A}} / G_{\mathcal{O}}$$

For finite fields:

$$\begin{tikzcd}
\mathrm{Funct}(G_k \backslash G_{\mathbb{A}} / G_{\mathcal{O}}) \arrow[d] & \\
\mathrm{Funct}(B_k \backslash G_{\mathbb{A}} / G_{\mathcal{O}}) \arrow[r] & \mathrm{Funct}\Big(T_k \backslash \{\text{Char. } N_{\mathbb{A}} \to \mathbb{C}^* \text{ trivial on } N_k\} \underset{N_k}{\times} G_{\mathbb{A}} / G_{\mathcal{O}}\Big)
\end{tikzcd}$$

$$\hat{f}(ng) = \chi(n)\,\hat{f}(g)$$

$N_{\mathbb{A}} \curvearrowright G_{\mathbb{A}} / G_{\mathcal{O}} \qquad f \longmapsto \hat{f}$

$$\hat{f}(\chi, g) = \int_{N_{\mathbb{A}}/N_k} f(ng)\,\chi^{-1}(n)\,dn$$

Do an analogue of this:

$$\begin{tikzcd} \mathrm{Bun}_G \\ \{G \text{ bundle on } X + \text{generic reduction to } B\} = \mathrm{Bun}_G^{B\text{-gen}} \arrow[u, "p"] \\ Q^{ext} \arrow[u, bend left, "r"] \end{tikzcd}$$

$\mathrm{Hom}(S, \mathrm{Bun}_G^{B\text{-gen}}) = \{ P_G = G\text{-bundle on } S \times X +$ reduction β to B defined on $U \subseteq S \times X$ s.t. $U \cap \{s \times X\}$ is dense/non-empty $\forall s \in S \}/\sim$

$Q^{ext} = \{ P_G, \beta, \ \forall$ simple root α_i: $\alpha(P_{T_i}) \longrightarrow \omega_X \}$

$$\begin{tikzcd} \text{D-mod}(\mathrm{Bun}_G) \arrow[d, "p^!"] \arrow[ddr, "\mathrm{coeff}^{ext}"] & & \\ \text{D-mod}(\mathrm{Bun}_G^{B\text{-gen}}) \arrow[d, "r^!"] & \mathrm{Whit}^{ext} \arrow[r, shift left, hook] & \text{D-mod}(Q^{ext}) \arrow[l, shift left, "\mathrm{Av}"] \\ \text{D-mod}(Q^{ext}) \arrow[r, "\mathrm{Av}"] & \mathrm{Whit}^{ext} & \end{tikzcd}$$

equivariance condition

Thm (Barlev): $p^!$ is fully faithful

Conj: coeff^{ext} is fully faithful

Thm (Beraldo): Conjecture holds for $G = GL_n$.

$Q^{ext} = \coprod Q^P$

$Q^P \subseteq Q^{ext}$ corresponds to $(P_G, \beta, \alpha_i(P_T) \xrightarrow{\gamma_i} \omega)$ s.t.

$$\gamma_i = \begin{cases} 0, & \alpha_i \text{ not a root of the Levi of } P \\ \text{non-zero}, & \alpha_i \text{ a — " —} \end{cases}$$

$Whit^P$ = portion of $Whit^{ext}$ that lives over Q^P,

$Whit^P \subseteq D\text{-mod}(Q^P)$

Case 1 $P = G$

$Whit^G = Whit$ ($\alpha_i(P_T) \to \omega_x$ is non-zero for all simple roots)

Thm (Beraldo): $Whit (= Whit^G)$

$\wr\wr$ (G adjoint for simplicity)

$Rep(G^\vee)_{Ran(X)}$

Q^G - stratify it more

(P_G, β - generic reduction to B, $\alpha_T(P_T) \to \omega_x$)

on a locally closed sub-prestack

(P_G, β - actual reduction to B, $P_T = \rho(\omega_x)(\sum \nu_i x_i)$)

$x_i \in X$, ν_i - coweights

Lem: The portion of Whit on each such stratum is 0, unless $\forall i\ \nu_i$ is dominant and is equivalent to Vect if $\forall i\ \nu_i$ is dominant

$$\rightsquigarrow \qquad \bigotimes V^{\nu_i}$$

$\uparrow$

irred rep of $\check{G}$ w/ highest weight ν_i.

$Q^{B} = (\mathbb{P}^1, \beta)$

$\|$

$\mathrm{Bun}_G^{B\text{-gen}}$

$$\text{D-mod}(\mathrm{Bun}_G^{B\text{-gen}}) \simeq \mathrm{Whit}^{B} = I(G, B)$$

$\uparrow$

the principal series category

$$\mathrm{Bun}_B \xrightarrow{i} \mathrm{Bun}_G^{B\text{-gen}}$$

$$\begin{array}{ccc} \text{D-mod}(\mathrm{Bun}_G^{B\text{-gen}}) & \xrightarrow{i^!} & \text{D-mod}(\mathrm{Bun}_B) \\ \uparrow & \square & \uparrow \\ I(G, B) & \underset{i_!}{\overset{i^!}{\rightleftarrows}} & \text{D-mod}(\mathrm{Bun}_T) \end{array}$$

$$I(G,B) \underset{i_!}{\overset{i^!}{\rightleftarrows}} \text{D-mod}(\text{Bun}_T)$$

$$\text{D-mod}(\text{Bun}_T) \underset{\text{F-M}}{\simeq} \text{QCoh}(\text{LocSys}_T)$$

$I(G,B)$ = Modules for the monad $i^! i_!$ action on D-mod(Bun_T)

Thm-in-progress (Raskin)

describes $i^! i_!$ in terms of QCoh(LocSys_T)

$\rightsquigarrow$ Whit^P is "simpler"

What about Whit^{ext}?

Gluing

$Y = \bigsqcup Y_\alpha$

s.t. $\overline{Y_{\alpha_1}} \cap Y_{\alpha_2} \neq \phi \Rightarrow \alpha_1 \geq \alpha_2$

$Y_a \xrightarrow{i_a} Y$

want to understand $\text{Sh}(Y)$ in terms of $\text{Sh}(Y_\alpha)$

$$i_{\alpha_1}^! i_{\alpha_2,!} : \text{Sh}(Y_{\alpha_2}) \to \text{Sh}(Y_{\alpha_1})$$

$$= F_{\alpha_2 \to \alpha_1}$$

$\alpha_1 \geq \alpha_2 \geq \alpha_3$

$$F_{\alpha_2 \to \alpha_1} \circ F_{\alpha_3 \to \alpha_2} \longrightarrow F_{\alpha_3 \to \alpha_1}$$

Lem: $Sh(Y) = \{ M_\alpha \in Sh(Y_\alpha)$ + maps $F_{\alpha_2 \to \alpha_1}(M_{\alpha_2}) \to M_{\alpha_1}$ that are compatible $\}$

$\forall \alpha \in A \rightsquigarrow C_\alpha$

$\alpha_2 \to \alpha_1$: $F_{\alpha_2 \to \alpha_1} : C_{\alpha_2} \to C_{\alpha_1}$

$F_{\alpha_2 \to \alpha_1} \circ F_{\alpha_3 \to \alpha_2} \longrightarrow F_{\alpha_3 \to \alpha_1}$ + assoc condition(s)

$Glue(C_\alpha, F_{\alpha_2 \to \alpha_1}) = \{ \forall \alpha \ c_\alpha \in C_\alpha, F_{\alpha_2 \to \alpha_1}(c_{\alpha_2}) \to c_{\alpha_1} \}$ + compatibility

(lax limit)

$\to$ $Whit^{ext} = Glue(Whit^P, F_{P_2 \to P_1})$

Conclusion: If you understand $Whit^P$ in spectral terms (i.e. in terms of $G^\vee$) and also the functors $F_{P_2 \to P_1}$ $\Longrightarrow$ you can go from the spectral side of Langlands to $Whit^{ext}$

$\cup\!\!\!\vert$

$Dmod(Bun_G)$

For each parabolic P we will introduce

- $I(G^\vee, P^\vee)_{spec} \xhookrightarrow{\text{Rashid}} \mathrm{Whit}^P$
- $\mathrm{Glue}\left(I(G^\vee, P^\vee)_{spec} \right)$
- $\mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}_{G^\vee}) \hookrightarrow \mathrm{Glue}(I(G^\vee, P^\vee)_{spec})$

Raskin I

First idea of Serre:

Serre duality thm

X/k sm projective variety of dim. n

$\mathcal{E}$ v.b. on $X \Rightarrow H^i(X, \mathcal{E})^* = H^{n-i}(X, \mathcal{E}^\vee \otimes \Omega^n_X)$

History (?)

Serre first proved this for $k = \mathbb{C}$

pf was essentially local in nature

Grothendieck suggested a formal to give a local (sheaf-theoretic?) proof.

$f : X \to Y$

$p : X \to \operatorname{Spec}(k)$

f_* for coherent sheaves

$f^!$ $\qquad \omega_X := p^!(k) \in \operatorname{Coh}(X)$

X smooth $\Rightarrow \omega_X = \Omega^n_X[\dim X]$

$(f_*, f^!)$ adjunction for f proper

First problem [f_* does not preserve coherence]

Want to work cocomplete dg-categories

Suggested solution $QCoh(X)$

$X = \operatorname{Spec} A \rightsquigarrow QCoh(X) = A\text{-mod}$

& in general: homotopy descent

$X = U \cup V$: $QCoh(X) = QCoh(U) \underset{QCoh(U \cap V)}{\times} QCoh(V)$

<u>Def</u>: $\mathcal{C}$ cocomplete dg-compact

$\mathcal{F} \in \mathcal{C}$ <u>compact</u> if

$Hom(\mathcal{F}, -) : \mathcal{C} \longrightarrow Vect$

commutes with colimits

$\mathcal{C}$ is <u>compactly generated</u> if $\forall \mathcal{G} \neq 0$ exists $\mathcal{F} \in \mathcal{C}$

compact with $Hom(\mathcal{F}, \mathcal{G}) \neq 0$

dg subcat of cpt objects

$\mathcal{C}$ compactly generated $\Leftrightarrow$ $Ind(\mathcal{C}^c) \xrightarrow{\sim} \mathcal{C}$

free cocomplete dg category generated by $\mathcal{C}^c$

$\mathcal{C}$ compactly generated, $F : \mathcal{C} \to \mathcal{D}$ is continuous.
Then F admits a continuous right adjoint G
iff F preserves compactness

($\mathcal{C}, \mathcal{D}$ cocomplete)

X (almost) finite type dg scheme

locally $X = \operatorname{Spec} A$, $H^0(A)$ f.t. $/k$
$H^i(A)$ f.g. $/ H^0(A)$

finite cover by Zariski open affines

Thm (Thomason - Trobaugh)

1) $QCoh(X)$ is compactly generated

2) compact objects are exactly perfect complexes

i.e. $\mathcal{F}$ compact $\iff$ $\forall \underbrace{\operatorname{Spec} A}_{=U} \subseteq X$, $\mathcal{F}|_U$ lies in the subcategory of A-mod generated by A under cones and direct summands

First question:

$f : X \to Y$ proper

Does $f_* : QCoh(X) \to QCoh(Y)$ admit a cts right adjoint $f^!$?

A: No

Def: $Coh(X) \subseteq QCoh(X)$ bounded complexes with locally finitely generated cohomologies

Thm (Serre): $Coh(X) = Perf(X)$ iff X is smooth

$\rightsquigarrow$ Problem: $x \xhookrightarrow{i} X$ singular pt

$i_*(k)$ is not perfect

Thm (Grauert): f proper: $f_* : Coh(X) \to Coh(Y)$

$\rightsquigarrow$ Def: $IndCoh(X) := Ind(Coh(X))$

Upshot: f proper

$\to f_* : Coh(X) \to Coh(Y)$ extends to a continuous functor

$f_* : IndCoh(X) \to IndCoh(Y)$ admits a continuous right adjoint $f^!$

$$\mathrm{Coh}(X) \hookrightarrow \mathrm{QCoh}(X)$$

$$\Psi = \Psi_X : \mathrm{IndCoh}(X) \longrightarrow \mathrm{QCoh}(X)$$

<u>Fact</u>: $\exists!$ t-structure on $\mathrm{IndCoh}(X)$ compatible with filtered colimits such that Ψ is t-exact

Moreover $\Psi : \mathrm{IndCoh}(X)^+ \xrightarrow{\sim} \mathrm{QCoh}(X)^+$

Let's us construct

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X) \longrightarrow \mathrm{IndCoh}(Y)$$

$$\begin{array}{ccc} \mathrm{Coh}(X) & \longrightarrow & \mathrm{IndCoh}(Y) \\ & & \cup \\ & & \mathrm{IndCoh}(Y)^+ = \mathrm{QCoh}(Y)^+ \end{array}$$

<u>Claim</u>: The functor $f^!$ exists in general.

<u>Principle 1</u>: We know $f^!$ for proper maps

<u>Principle 2</u>: $f = j$ is an open embedding $f^! = f^{*,\mathrm{IndCoh}}$

Use Nagata's Thm to

$$\begin{tikzcd} & \bar{X} \arrow[dr, "\bar{f}"] & \\ X \arrow[ur, "j"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

Basic property: Base change

$$\begin{tikzcd} S \arrow[r, "\psi"] \arrow[d, "\varphi"'] & X \arrow[d, "f"] \\ T \arrow[r, "g"'] & Y \end{tikzcd}$$ Cartesian in dg case

$$\Longrightarrow \quad g^! f_*^{\mathrm{IndCoh}} = \varphi_*^{\mathrm{IndCoh}} \psi^!$$

Application to the theory of D-modules

Recall: Idea of Grothendieck is that D-modules

$\Leftrightarrow$ QCoh sheaves w/ infinitesimal parallel transport

X scheme

$$\cdots \; \substack{\rightarrow\\\rightarrow\\\rightarrow} (X \times X)^{\wedge}_X \rightrightarrows X$$

idea: $D\text{-mod}(X) = \lim\limits_{[n] \in \Delta} \mathrm{QCoh}\big(\underbrace{(X \times \cdots \times X)}_{n \text{ times}}{}^{\wedge}_X\big)$

Idea $X_{dR} = X / {\scriptstyle x = x' \atop x \text{ is infinitesimally close to } x'}$

<u>Def</u>: $\mathrm{PreStk} = \mathrm{Hom}(\mathrm{AffSch}^{op}, \infty\text{-Grp})$

$\quad\cup\!\text{I}$ Yoneda embedding

AffSch

<u>Examples</u>:

1) $\mathrm{Bun}_G^{B\text{-gen}}$

2) X scheme (or prestack)

$$X_{dR}(S) := X(S^{red}) = \{ S^{red} \to X \}$$

If Y is a prestack $\mathrm{QCoh}(Y) = \lim\limits_{\substack{S \to Y \\ \uparrow \\ \mathrm{AffSch}}} \mathrm{QCoh}(S)$

Allows me to define $D\text{-mod}(X) := \mathrm{QCoh}(X_{dR})$

<u>Remember</u>: in D-module world have $f^!$, $f_{*,dR}$

Pullback for q-coh recovers $f^!$

Y prestack "locally almost of finite type"

$\Longrightarrow$ $\mathrm{IndCoh}(Y)$ makes sense.

General structure: $\mathrm{IndCoh}(Y)$ is cocomplete dg cat

nice upper-! functoriality

$\mathrm{QCoh}(Y) \circlearrowright \mathrm{IndCoh}(Y)$

$$\rightsquigarrow \quad \mathrm{QCoh}(Y) \longrightarrow \mathrm{IndCoh}(Y)$$
$$\mathcal{F} \longmapsto \mathcal{F} \underset{\mathcal{O}_Y}{\otimes} \omega_Y$$

<u>Fact</u> (G–R)

$\mathrm{QCoh}(X_{dR}) \longrightarrow \mathrm{IndCoh}(X_{dR})$ is an equivalence

<u>Idea</u> (G–R)

f_*^{IndCoh} is defined an well-behaved for a much larger class of morphisms than in the quasi-coherent setting.

Roughly: fibers of look like

- schemes
- formal completion $Y \hookrightarrow X$ $\quad (X^\wedge_Y)$
- X_{dR}
- $X^\wedge_Y$ / infinitesimal groupoid

<u>Sample statements:</u>

1) Base change holds in this generality

2) If $f|_{\text{reduced schemes}}$ has fibers that are proper:

$$\left(f_*^{\text{IndCoh}}, f^!\right)$$

Toen II

algebraic derived stacks : $\exists n$: n-geometric

Def: $F \in dSt_k$

1) F is 0-geometric if F is affine
2) F is n-geometric if $\exists X_i \in dAff_k$ + $\coprod X_i \xrightarrow{\pi} F$ with π being $(n-1)$-geometric, smooth & projective
3) $f: F \longrightarrow G$ in dSt_k
   - f is n-geometric if $\forall X \in dAff_k$, $\overset{X \to G}{\forall}$, $F \underset{G}{\times} X$ is n-geometric
   - f is moreover smooth (& surjective) if $\forall X \to G$ exists smooth atlas $\coprod X_i \longrightarrow X \underset{G}{\times} F$ (as in 2) s.t. $X_i \longrightarrow X$ is smooth (smooth cov.)

Terminology: $F \in dSt_k \subseteq Fun^{\infty}(cdga, Top)$

$t_0 F = F^{cl}$: $cAlg \longrightarrow Top$

$t_0 : dSt_k$ ... non-dg, cdga, left Kan ext

$\exists$ an adjunction map $i: t_0 F \xrightarrow{i} F$ (think of as $X_{red} \hookrightarrow X$, X scheme)

·) F is an alg. der. stack if it is geometric for some n

·) F is a <u>derived scheme</u> if F is algebraic and t_0F is a scheme

·) F is a derive Deligne-Mumford stack / alg. space if it is geometric and t_0F is so.

·) If F is 1-geometric then F^{cl} is always a "classical" artin stack with affine diagonal.

·) dSt_k is Cartesian closed

$F, G \in dSt_k \rightsquigarrow \exists\ \mathbb{R}Map(F,G)$

$$\mathrm{Hom}_{dSt_k}(S, \mathbb{R}Map(F,G)) = \mathrm{Hom}_{dSt_k}(S \times F, G)$$

<u>Thm</u>: · Y is a "very presentable stack" (e.g. $Y = BG$)

· X be either

- a smooth proper scheme $/k$
- $X = Z_{DR}$, Z smooth proper scheme
- $X = K$, K a finite homotopy type
  $\cap dSt_k$ (constant functor, stackified)

Then $\mathbb{R}Map(X, Y)$ is a derived algebraic stack.

$\rightsquigarrow$ • $\mathbb{R}\mathrm{Bun}_G(X)$ = derived moduli of G-bundles on X

• $\mathbb{R}\mathrm{LocSys}_G(X)$ = derived moduli of flat G-bundles on Z

• $\mathbb{R}\mathrm{Loc}_G(K)$ = derived moduli of G-local systems on K

$(\cdot)^{cl}$ ⇝ usual moduli

ex $K = S^d$, $d > 1$

$(\mathbb{R}\mathrm{Loc}(S^d))^{cl} = BG$ ... no local systems on S^d

$\mathbb{R}\mathrm{Loc}(S^d) = [\mathrm{Spec}\, A_d / G]$

$A_d = \mathrm{Sym}(\mathfrak{g}^*[d-1]) \rightsquigarrow (\mathrm{Spec}\, A_d)^{cl} = \mathrm{Spec}\, \pi_0(A_d) = \mathrm{Spec}\, k$

(co)tangent complexes

Def • $F \in dSt_k$

$$(QCoh(F) =)\ Lqcoh(F) := \underbrace{\lim_{\substack{\mathrm{Spec}\, A \to F \\ dAff/F}}}_{\text{in the } \infty\text{-category of } \infty\text{-categories}} L(A)$$

$L(A) = \infty$-cat of A-dg modules

When F is algebraic then $\exists$ a natural t-structure on $Lqcoh(F)$, whose heart is $QCoh(F^{cl})$.

q-coh sheaves on the algebraic (underived) stack F^{cl}.

(*) $\mathcal{O}_F \in Lqcoh(F)$, $\underline{H}^*(F)$ .. sheaf of graded algebras on F^{cl}

"forgetful" derived alg stacks $\to$ graded underived alg. stacks

<u>Def</u>: F an alg stack, lafp.

$E \in Lqcoh(F)$ is called -) <u>perfect</u> if $\forall$ Spec $A \xrightarrow{u} F$, u^*E is perfect (compact)

-) <u>coherent</u> if $H^0(E)$ is of finite type as an $H^0(\mathcal{O}_F)$-module

F alg derived stack

$M \in Lqcoh(F)^{\leq 0}$

$\mathcal{O}_F \oplus M$ = trivial square-zero extension of $\mathcal{O}_F$ by M

qcoh $\mathcal{O}_F$-algebra (≤ 0)

$$\mathrm{Spec}(\mathcal{O}_F \oplus M) \underset{0 \text{ section}}{\rightleftarrows} \mathrm{Spec}(\mathcal{O}_F) = F$$

$\|$

$F[M]$ = split sq. ~~zero~~ ext. of F by M

$M = \mathcal{O}_F \rightsquigarrow F[M] = F[\varepsilon]$

<u>Fact</u>: $\exists$ an object $\mathbb{L}_F \in Lqcoh(F)^-$

s.t. $\underline{\mathrm{Hom}}_{Lqcoh(F)}(\mathbb{L}_F, M) \simeq \mathrm{Hom}_{F/dSt_k}(F[M], F)$

<u>Def</u>: $\mathbb{L}_F$ is called the <u>cotangent complex</u> of F

(derived analogue of "Ω^1")

e.g. $U \xrightarrow{\pi} F$ smooth atlas

$\pi^* \mathbb{L}_F \longrightarrow \mathbb{L}_U \longrightarrow \mathbb{L}_\pi$ (not eq.)

- $F = \mathrm{Spec}\, A$

$\mathbb{L}_F = \mathbb{L}_A =$ Quillen cotangent complex
André

$\cong$

$\Omega^1_{A^\cdot}$ $A^\cdot$ is a q-free resolution of A

<u>Tangent complex</u> : F is dg & $\mathbb{L}_F$ is perfect

then the tangent complex is $\mathbb{T}_F = \mathbb{L}_F^\vee = \underline{\mathrm{Hom}}(\mathbb{L}_F, \mathcal{O}_F)$

$\in$ $L_{qcoh}(F)$

<u>Fact</u> : X, Y as in the theorem

$$\begin{array}{ccc} \mathbb{R}\mathrm{Map}(X,Y) \times X & \xrightarrow{ev} & Y \\ \downarrow p & & \\ \mathbb{R}\mathrm{Map}(X,Y) & & \end{array}$$

then $\mathbb{T}_{\mathbb{R}\mathrm{Hom}(X,Y)} \simeq p_* ev^* (\mathbb{T}_Y)$ [is perfect]

This is a sheaf version of $\mathbb{T}_{\mathbb{R}\mathrm{Hom}(X,Y), f} = \mathbb{R}\Gamma(X, f^*(\mathbb{T}_Y))$

$f : X \to Y$

<u>Rem</u> : This is wrong without "$\mathbb{R}$".

Atiyah classes & singular support

F derived alg. stack, loc. of f.p. ($\Rightarrow \mathbb{L}_F$ is perfect)

Thm (Kapranov / Hennion / G-R)

* $\mathbb{T}_F[-1]$ is a $\mathcal{O}_F$-linear dg Lie algebra

** $\exists$ a natural ∞-functor

$$\begin{array}{ccc} Lqcoh(F) & \longrightarrow & \mathbb{T}_F[-1]-\text{dg modules on } F \\ & \overset{id}{\searrow} & \downarrow \text{forget} \\ & & Lqcoh(F) \end{array}$$

*** $\mathcal{E} \in Lqcoh(F)$

$\mathbb{T}_F[-1] \otimes \mathcal{E} \longrightarrow \mathcal{E}$ by **

$\rightsquigarrow \mathcal{E} \longrightarrow \mathcal{E} \otimes \mathbb{L}_F[1]$

$\rightsquigarrow at_{\mathcal{E}} \in Ext^1(\mathcal{E}, \mathcal{E} \otimes \mathbb{L}_F[1])$

is the Atiyah class (eg F smooth scheme)

idea: $\mathcal{L}F = Map(S^1, F)$ — $S^1 = B\mathbb{Z}$ = const. stack w/ values S^1

... derived loop stack of F

$\mathcal{L}F \simeq F \times_{F \times F} F \in dSt_k$ (derived version of "inertia stack")

$\mathcal{L}F \xrightarrow{\pi} F$ evaluation at the base point $* \in S^1$

is naturally endowed with a mult. structure / F

given by composition of loops:

$S^1 \longrightarrow S^1 \vee S^1$

$\rightsquigarrow Maps(S^1 \vee S^1, F) \longrightarrow Maps(S^1, F)$

$\overset{\shortparallel}{Maps(S^1, F) \times Maps(S^1, F)}$ $\nearrow$ mult

$\rightsquigarrow$ $\mathcal{L}F$ is derived group stack $/F$

it's "Lie algebra" $e^*(\mathbb{T}_{\mathcal{L}F/F}) \simeq \mathbb{T}_F[-1]$

$\Rightarrow$ $(*)$ of the theorem

$\rightsquigarrow$ the gr $\mathcal{L}F$ acts on any qcoh sheaf

tangent action $\Rightarrow$ $\mathbb{T}_F[-1]$-action.

<u>Sing support</u>:

- $\mathcal{E} \in \mathrm{Coh}(F)$
- F is q-smooth: $\mathbb{L}_F$ has amplitude $[-1, \infty[$.
- $\mathrm{Sing}(F) := \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_F}(\mathbb{T}_F[1]))^{cl} \longrightarrow F^{cl}$

$$\parallel$$

$$\mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_{F^{cl}}}(H^1(j^*\mathbb{T}_F))) \qquad j: F^{cl} \hookrightarrow F$$

- $i_x: \mathrm{Spec}\, k \longrightarrow F$ a point

$i_x^! \mathcal{E} \in \mathrm{Lqcoh}(k)$

$\circlearrowleft$ dg module

$\mathbb{T}_{F,x}[-1] = i_x^*(\mathbb{T}_F[-1])$

$H^*(i_x^! \mathcal{E})$ is naturally a $\mathrm{Sym}_k(H^1(\mathbb{T}_{F,x}))$-module

$Y \subseteq \mathrm{Sing}(F)$ closed

<u>Def</u>: $\mathrm{Singsupp}(\mathcal{E}) \subseteq Y$

$\Leftrightarrow$ any pt $i_x: \mathrm{Spec}\, k \to Y$ the support of $H^*(i_x^! \mathcal{E})$ in $\mathrm{Spec}(\mathrm{Sym}_k(H^1(\mathbb{T}_{F,x}))) = \mathrm{Sing}(F)_x$ is contained in Y_x.

Thm. $\mathcal{E}$ is perfect $\iff$ $\mathrm{SingSupp}(\mathcal{E}) \subseteq \underset{\{0\}}{\overset{\shortparallel}{F^{cl}}} \subseteq \mathrm{Sing}(F)$

Ben-Zvi I

Topological Field Theory & Geometric Langlands

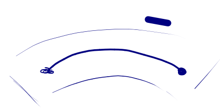
2006: Kapustin-Witten:
See structure of GLC from $N=4$ SYM in $d=4$

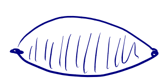
Claim: Can see [de Rham] GLC out of physics $\rightsquigarrow$ 6 dimensions
Not going there.

TFT: representation of a higher category of bordisms of manifolds

$\mathrm{Bord}_n^{\text{framed/oriented}}$ = symmetric monoidal (∞,n)-category

objects: points  w/ tangential structure on n-dim'l tangent bundle

1-morphisms:  e.g. $\mathrm{End}(\phi_0)$ = closed 1-manifolds

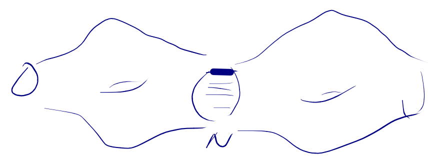
2-morphisms: 

$\vdots$

n-morphisms: classifying spaces of bordisms
$\Longleftrightarrow$ their fundamental ∞-groupoids

e.g. $\mathrm{Bord}_2^{or}(\phi_1,\phi_1) = \coprod_g B\mathrm{Diff}\,\Sigma_g$

Composition = gluing



$M_1 \amalg_N M_2$

symmetric monoidal = $\amalg$

n-dim TFT: $\mathrm{Bord}_n^{?} \xrightarrow{Z} \mathcal{C}$ symmetric monoidal

into some s.m. (∞, n)

Cobordism Hypothesis (Baez-Dolan, Lurie)

- Framed n-dim'l TFTs $\Longleftrightarrow$ n-dualizable objects of $\mathcal{C}$

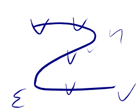
$$Z \longmapsto Z(\bullet)$$

$$\left(M \mapsto \int_M A\right) \longleftarrow\!\shortmid \; A$$

Dualizable object V of sym. monoidal $(\infty,1)$-cat:

$\exists\, V^\vee$,

$1 \xrightarrow{\eta} V \otimes V^\vee$, $V \otimes V^\vee \xrightarrow{\varepsilon} 1$



$= V \xrightarrow{\;id\;} V$

e.g. V vector space is dualizable iff fin. dim'l.

Can ask for η & ε to also have duals (adjoints)

$\eta \dashv \eta^R$ $\quad 1 \to \eta^R \eta$, $\eta \eta^R \to 1$

2-dualizable: dualizable using only dualizable η, ε

⋮

Thm $\Rightarrow$ action of group $O(n)$ on $\mathcal{C}^{n\text{-dualizable}}$ $\leftarrow$ ∞-groupoid
(by acting on framing)

given $G \to O(n) = GL_n$

Corollary: $\mathrm{Bord}_n^G \xrightarrow{Z} \mathcal{C} \iff (\mathcal{C}^{n\text{-dualizable}})^G$

Can think of (this as) increasingly tough finiteness conditions on object $V \in \mathcal{C}$ for convergence of $\int_M V$ on higher dim M

Target of TFT:

Desideratum: $Z(M^n) \in \mathbb{C}$

i.e. $\mathcal{C}(1_{n-1}, 1_{n-1}) = \mathbb{C}$

i.e. $\mathcal{C}$ "n-fold delooping of $\mathbb{C}$"

$n=1$: $\mathrm{Vect}_{\mathbb{C}}$ or $\mathrm{dgVect}_{\mathbb{C}}$ [small or large versions]

$n=2$: $\mathrm{Cat}_{\mathbb{C}}$ or $\mathrm{dgCat}_{\mathbb{C}}$

$\cup$

$\mathrm{Alg}_{\mathbb{C}}$ .. categories w/ one object

or categories generated by one object

$A \rightsquigarrow A\text{-mod} \ni A$ $\quad \mathrm{End}_{A\text{-mod}}(A) = A^{op}$

$n=3$: $2-\mathrm{Cat}_{\mathbb{C}}$ or $\mathrm{Alg}(\mathrm{Cat}_{\mathbb{C}})$ or $\mathrm{Alg}(\mathrm{Alg}(\mathrm{Vect}_{\mathbb{C}}))$

$\mathrm{Alg}(\mathrm{Cat}_{\mathbb{C}})$ = monoidal cats = 2-cat w/ 1 obj

$\mathrm{Alg}(\mathrm{Alg}(\mathrm{Vect}_{\mathbb{C}}))$ = E_2-algebras = 2-cat w/ 1 obj & 1 morphism

$\iff$ algebra over little 2-disks operad

$n=4$ 3-cat$_{\mathbb{C}}$ or monoidal 2-Cat$_{\mathbb{C}}$ or $\boxed{\text{braided } \otimes\text{-cat} \quad Alg(Alg(Cat_{\mathbb{C}}))}$ or E_3-algs
$Alg(2Cat_{\mathbb{C}})$

e.g. $\mathcal{C}$ monoidal cat ($n=3$)
say it's 2+1-dualizable
(2-dualizable in $(\infty, 3)$)

Compactification

Z n-dim TFT, N k-manifold

$\rightsquigarrow Z_N$ a $(n-k)$-dim TFT: $Z_N(M) = Z(N \times M)$

Dimensional reduction

$Z(N) \circlearrowleft \mathrm{Diff}\, N$

$Z_N(M) = Z(N \times M) \circlearrowleft \mathrm{Diff}\, N$

would like to take $\underbrace{\overline{Z}_N(M) = Z(N \times M)^{\mathrm{Diff}\, N}}_{\text{not a TFT!}}$

instead define $\overline{Z}_N(M) = \int_M \left(Z(N) \right)^{\mathrm{Diff}\, N}$

Cobordism hypothesis w/ singularities

gives universal properties of bordism categories w/ singularities.

<u>Example</u>: $\mathcal{C}, \mathcal{D}$ n-dualizable $\mathcal{C} \xrightarrow{\varphi} \mathcal{D}$

φ k-dualizable for some $k < n$

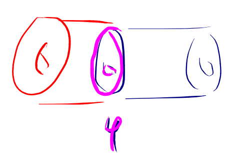
e.g. "k-dualizable object of $\mathcal{D}$"

$$\underset{\mathrm{Vect}_{\mathbb{C}}}{1} \xrightarrow{V} \underset{\mathrm{Cat}}{\mathcal{C}}$$

$1 \xrightarrow{V} \mathcal{C} \iff$ [local] boundary condition for $Z_{\mathcal{C}}$:
can extend $Z_{\mathcal{C}}$ to manifolds w/ boundaries labeled by V

– meaning of $Z(\cdot) = \mathcal{C}$

$\mathcal{C} \xrightarrow{\varphi} \mathcal{D}$ k-dualizable $\iff$ domain wall between $Z_{\mathcal{C}}, Z_{\mathcal{D}}$



$Z_{\mathcal{C}}(N) \longrightarrow Z_{\mathcal{D}}(N)$,
Diff N invariant

$\underline{n=1}$ $V \in \mathrm{Vect}_{\mathbb{C}}$ dualizable $\Leftrightarrow$ f.d.

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\mathrm{Id}=\eta"] \arrow[dr, bend right, "\dim V = \mathrm{Tr}(\mathrm{Id}_V)"'] & \mathrm{End}(V) \arrow[d, phantom, "\wr"] & \\ & V \otimes V^* \arrow[r, "ev=\varepsilon"] & \mathbb{C} \end{tikzcd}$$

$$Z(\bigcirc) = Z\left(\overset{V}{\underset{\eta \quad V^* \quad \varepsilon}{C\,\supset}}\right) = \dim V$$

$\underline{n=2}$ A .. algebra always 1+1-dualizable

$$\begin{tikzcd} \mathbb{C} \arrow[r, "A"] \arrow[rr, bend right] & A \otimes A^{op} \arrow[r, "A"] & \mathbb{C} \end{tikzcd}$$

$$\mathrm{Vect} \xrightarrow{A} A\text{-mod-}A^{op}$$

$$\dim A = \varepsilon(\eta) = Z_A(S^1)$$

$$= A \overset{\mathbb{L}}{\underset{A \otimes A^{op}}{\otimes}} A = HH_\bullet(A)$$

$\mathcal{C}$ compactly generated also 1+1 dualizable

$\mathcal{C}$ dg-cat is 2-dualizable $\Leftrightarrow$ $\mathcal{C}$ is smooth & proper

(looks like $QCoh(X)$, X smooth & proper)

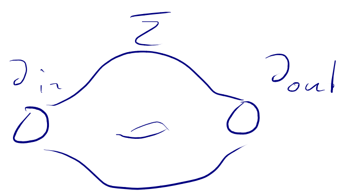
finiteness for diagonal bimodule

X smooth proper : $\mathcal{C} = QCoh\, X$ $\Rightarrow$ <u>B-model</u> $Z_{\mathcal{C}}$

G finite group $\mathcal{C} = Rep\, G$: 2d YM

G finite , $\mathcal{C} = Rep\, G$

$$Z_{\mathcal{C}}(\Sigma) = \# Loc_G(\Sigma) = Map(\Sigma, BG)$$



$$\begin{tikzcd} & Loc_G \Sigma \arrow[dl, "\pi_{in}"'] \arrow[dr, "\pi_{out}"] & \\ Loc_G \partial_{in} & & Loc_G \partial_{out} \end{tikzcd}$$

$$\pi_{out,*}\, \pi_{in}^* : Z_{\mathcal{C}}(\partial_{in}) \longrightarrow Z_{\mathcal{C}}(\partial_{out})$$

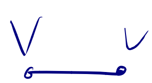
$$\mathbb{C}[\tfrac{G}{G}] = \mathbb{C}[Loc_G \partial_{in}] \qquad \mathbb{C}[Loc_G \partial_{out}]$$

$$(Loc_G \partial_{in} = Loc_G S^1 = \tfrac{G}{G})$$

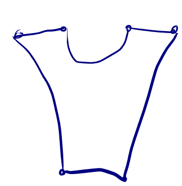
<u>boundary conditions</u> : Objects of $\mathcal{C}$ = A-mod

1-dualizable $\Rightarrow$ compact obj of $\mathcal{C}$

$\mathcal{C} = Rep\, G \ni V$ f. dim. reps of G



End V, $\otimes$



V

$\chi(V) \in Z(S^1) = HH_*(A) = \mathbb{C}\,\frac{G}{G}$

character of V

$\mathcal{C} = Q\,Coh(X)$; $\quad Z(S^1) = HH_*(\mathcal{C})$

$$\underset{HKR}{=} R\Gamma(\Omega_X^{-\bullet})$$

$B \subset G$ subgroup (e.g. $\underline{B}(\mathbb{F}_q) \subseteq \underline{G}(\mathbb{F}_q)$)

Consider G-local systems w/ reductions to B along boundaries

$$Z(\bullet\!\!-\!\!\bullet) \overset{\mathcal{H}=}{=} \mathbb{C}[B\backslash G/B] = End(\mathbb{C}[G/B])$$

$\mathcal{H}$-mod $= \langle \mathbb{C}[G/B] \rangle \subset Rep\, G$ "principal series"

$Z_{\mathcal{H}}$ (superselection sector)

picture above with reduction on boundary

$$\rightsquigarrow \frac{B}{B} = \{g \in G,\ x \in (G/B)^g\}/G = \tilde{G}/G$$

$$\downarrow \pi$$

$$\frac{G}{G}$$

version of Grothendieck-Springer

domain wall

T B G

$$\begin{array}{ccc} & {}^{\bullet}/B & \\ \swarrow & & \searrow \\ {}^{\bullet}/T & & {}^{\bullet}/G \end{array} \rightsquigarrow \text{parabolic induction}$$

$$\mathrm{Rep}\, T \longrightarrow \mathrm{Rep}\, G$$

$$\mathbb{C}\left[\tfrac{T}{T}\right] \longrightarrow \mathbb{C}\left[\tfrac{G}{G}\right]$$

Dimensional reduction of B-model $\overline{Z}_{S^1}$

$$Z(S^1) = HH_*(A) = R\Gamma(\Omega^{-\bullet})$$

Diff $S^1 \curvearrowright S^1$

$$C_*(S^1) = \mathbb{C}[B]/B^2=0 \qquad |B| = 1$$

Connes B-differential

$\rightsquigarrow$ de Rham differential

$$\overline{Z}_{S^1} = \bullet \longmapsto (HH_*(A))^{S^1} =: HC_-(A)$$

Note $\overline{Z}_N$ is linear over $C^*(B\mathrm{Diff}\, N)$

$\rightsquigarrow$ $\overline{Z}_{S^1}$ is linear over $C^*(BS^1) = \mathbb{C}[\varepsilon]$ $\quad |\varepsilon| = 2$

[Nekrasov Ω-background]

$\overline{Z}_{S^1}$ is family over $\mathbb{C}[\varepsilon]$, $|\varepsilon| = 2$

Would like to take $\varepsilon = 1$ $\quad R\Gamma(\Omega^{-\bullet}), d \rightsquigarrow H_{dR}(X)$

does not make sense, but can invert ε

$$\rightsquigarrow \overline{Z}_{S^1}(\ ) \otimes_{\mathbb{C}[\varepsilon]} \mathbb{C}[\varepsilon, \varepsilon^{-1}] = H^{\bullet}_{dR}(X) \otimes \mathbb{C}[\varepsilon, \varepsilon^{-1}]$$

Nadler

Betti Langlands in genus 1

I. Overview

setting: geometric Langlands

$$\text{D-mod}(\mathrm{Bun}_G C) \simeq \mathrm{IndCoh}_N(\mathrm{Loc}_{G^\vee} C)$$

Ansatz: Find a version that is topological in C
(loc. constant in moduli of curves)

Ex: $G = T$

$$\mathrm{Pic}_T C \simeq \mathrm{Jac}_T C \times BT \times \Lambda_T$$

$$\mathrm{BettiLoc}_{T^\vee} C = \mathrm{Hom}(\pi_1 C, T^\vee) \times BT^\vee \times \mathrm{Spec}\,\mathrm{Sym}(t[1])$$

$$\underset{\underset{\mathrm{Loc}}{\shortparallel}}{\mathrm{Sh}_N(\mathrm{Pic}_T)} \simeq \underset{\underset{\mathrm{QCoh}}{\shortparallel}}{\mathrm{IndCoh}_N}(\mathrm{BettiLoc}_{T^\vee} C)$$

Λ_T-graded modules over $k[\pi_1 C \otimes \Lambda_T] \otimes \mathrm{Sym}(t[1])$

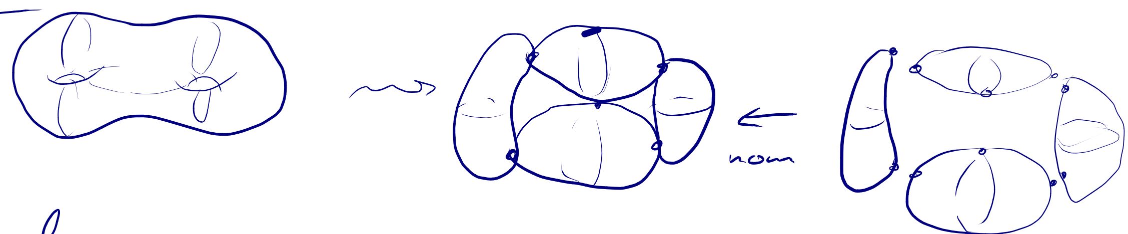
Betti Langlands conjecture

$$Sh_N(Bun_G C) \simeq IndCoh_N(BettiLoc_{G^\vee} C)$$

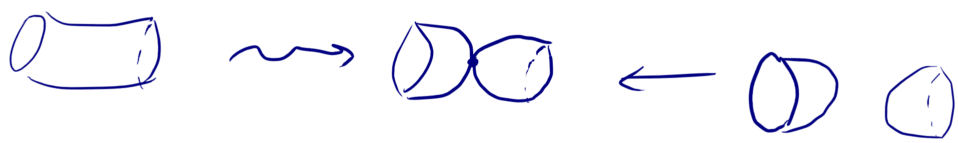
Today: C genus 1

Strategy "Verlinde Formula" – take curve C to the boundary of moduli

global



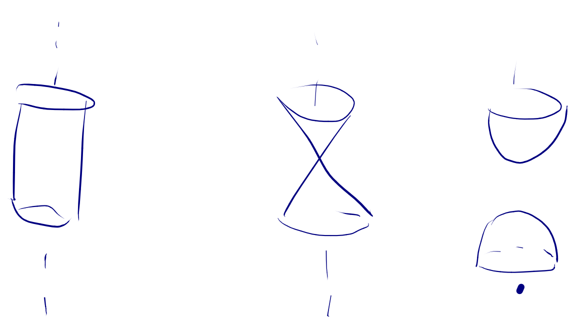
local



Gauge symmetries – wonderful compactification

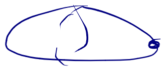
$$LG \rightsquigarrow LG/I_0^u \underset{1}{\times} I_\infty^u \backslash LG$$

TFT interpretation



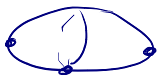
0-cobordism w/ single Morse singularity type $1, -1, -1$

Atomic building blocks

 $\rightsquigarrow$ "Lattic" module for aff Hecke category

 $\rightsquigarrow$ "regular bimodule" over affine Hecke category

} Kazhdan–Lusztig, Bezrukavnikov

 "regular trimodule" — Yun's talk

$\mathbb{RP}^2$

"nonoriented module" — real groups

Betti Langlands Verlinde Conj

$Sh_N(Bun_G C)$ results from gluing

$Sh_N(Bun_G(\text{atoms}))$ wrt aff. Hecke cat.

Gluing (colimits): A alg, M right module, N left module

$$M \otimes_A N = \operatorname{colim} \left[M \otimes N \leftleftarrows M \otimes A \otimes N \;\overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}}\; \cdots \right]$$

<u>Thm</u> $\mathrm{Coh}_N(\mathrm{Betti\,loc}_{G^\vee} C)$ results from such gluing.

<u>Motivation</u> genus 1

pursuing "affine character sheaves"

Lusztig's <u>character sheaves</u> $Ch_G \subset Sh(\frac{G}{G})$

image of horocycle corresp.

$$\frac{N\backslash G/N}{T} \xleftarrow{p} \frac{G}{B} \xrightarrow{q} \frac{G}{G}$$

H_G = Hecke cat. of bim. sheaves $\xrightarrow{q_! p^*} Ch_G$

adjoint $p_* q^!$ = nearby cycles at boundary of wonderful compactification

$$G \rightsquigarrow G/N \underset{T}{\times} N\backslash G$$

<u>Mirković–Vilonen</u>: $Ch_G = Sh_N(\frac{G}{G})$

gluing characterization: Ch_G = Drinfeld center of H_G

$$= H_G \underset{H_G \otimes H_G^{op}}{\otimes} H_G$$

Back to affine setting

$\mathcal{H}_G^{aff}$ affine Hecke cat of Lin. sheaves on

$$\frac{I_0^u \backslash LG / I_\infty^u}{T}$$

Def: aff character sheaves

Ch_G^{aff} = Drinfeld center of $\mathcal{H}_G^{aff}$

Verlinde Conj (genus 1) $\Longrightarrow$ $Ch_G^{aff} = Sh_N(Bun_G E)$

genus 1

Consequence Bez's local Langlands duality

Genus 1 Verlinde conj $\Rightarrow$ Genus 1 Betti Langlands conj

only atoms are bivalent not trivalent

II. Coherent sheaves in DAG

Point of view: categorified functional analysis

$$X \rightsquigarrow \mathcal{O}(X) \rightsquigarrow \mathrm{Perf}(X)$$

reasonable derived alg. stack — comm alg — sym. monoidal

$$\mathrm{Dist}(X) \rightsquigarrow \mathrm{Coh}(X)$$

module (not nec. compactly supported) — module category

$$\begin{array}{l} X \\ \downarrow \text{ proper} \\ S \text{ smooth} \end{array}$$

Thm
$$\mathrm{Fun}^{ex}_{\mathrm{Perf}(S)}(\mathrm{Perf}(X), \mathrm{Perf}(S)) = \mathrm{Coh}(X)$$

$$\mathrm{Fun}^{ex}_{\mathrm{Perf}(S)}(\mathrm{Coh}(X), \mathrm{Perf}(S)) = \mathrm{Perf}(X)$$

Many variation on theme (Toen, ...)

Thm: X, X' over S: $X \to S$ proper, $X' \to S$ smooth, S smooth

$$\mathrm{Fun}^{ex}_{\mathrm{Perf}(S)}(\mathrm{Perf}(X), \mathrm{Perf}(X')) = \mathrm{Coh}(X \overset{\mathbb{R}}{\underset{S}{\times}} X')$$

Ex: $X = X' = \frac{B^\vee}{B^\vee} = \frac{\tilde{G}^\vee}{G^\vee}$

$$\begin{array}{c} X \\ {\scriptstyle \text{Gr-Spr}}\big\downarrow{\scriptstyle \mu} \\ Y = \frac{G^\vee}{G^\vee} \end{array}$$

$$\mathcal{H}^{\mathrm{aff}}_{G^\vee} = \mathrm{Coh}\left(\frac{B^\vee}{B^\vee} \underset{\frac{G^\vee}{G^\vee}}{\times} \frac{B^\vee}{B^\vee} \right)$$

$$\simeq \mathrm{Fun}^{\mathrm{ex}}_{\mathrm{Perf}\left(\frac{G^\vee}{G^\vee}\right)}\left(\mathrm{Perf}\left(\frac{\tilde{G}^\vee}{G^\vee}\right), \mathrm{Perf}\left(\frac{\tilde{G}^\vee}{G^\vee}\right) \right)$$

<u>Gluing Ideas</u>

all smooth

$$\begin{array}{c} X \\ {\scriptstyle \text{proper}}\big\downarrow \\ S \end{array}$$

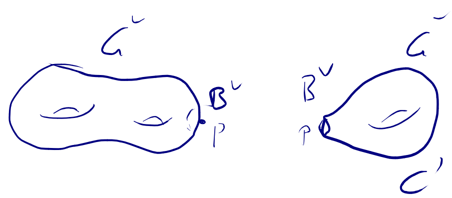
$$Z = X \times_S W$$

$$Z' = X \times_S W'$$

$$\mathrm{Coh}_{\substack{\text{sing} \\ \text{supp}}}\left(W \times_S W' \right) = \mathrm{Coh}_{\substack{\text{sing} \\ \text{supp}}}(Z) \underset{\mathrm{Coh}_{\substack{\text{sing} \\ \text{supp}}}\left(X \times_S X \right)}{\otimes} \mathrm{Coh}_{\substack{\text{sing} \\ \text{supp}}}(Z')$$

explicit conditions

<u>Ex</u>



$$Z = \mathrm{BettiLoc}_{G^\vee}(C \setminus p)$$

$$Z' = \ldots$$

<u>Spectral Verlinde formula:</u>

$$\mathrm{Coh}_N\left(\mathrm{BettiLoc}_{G^\vee}(C \setminus p) \right) \underset{\mathcal{H}^{\mathrm{aff}}_{G}}{\otimes} \mathrm{Coh}_N\left(\mathrm{BettiLoc}_{G^\vee}(C' \setminus p) \right)$$

$$= \mathrm{Coh}_N\left(\mathrm{BettiLoc}_{G^\vee}(C \#_p C') \right)$$

Special case

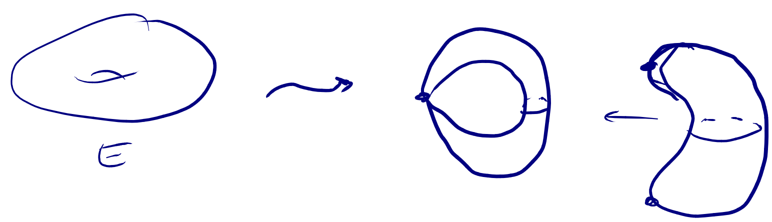
$$\begin{matrix}\text{Drinfeld center} \\ \text{of } \mathcal{H}_G^{\mathrm{aff}}\end{matrix} = \mathrm{Coh}_N(\mathrm{Bellloc}_{G^\vee}(E))$$

genus 1

III Genus 1 tools: uniformization

Goal: Gluing result $Sh_N(\mathrm{Bun}_G(E)) = Ch_G^{\mathrm{aff}}$

Drinfeld center of $\mathcal{H}_G^{\mathrm{aff}}$



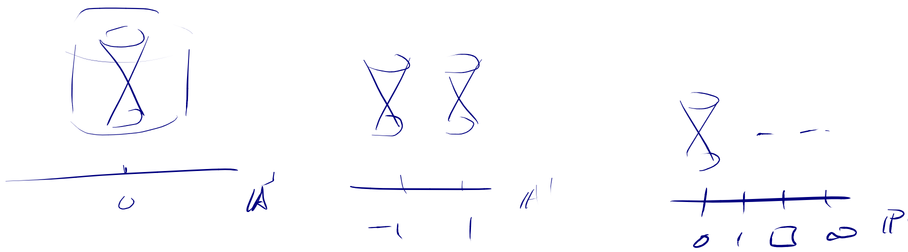
Confirm ansatz: need $Sh_N(\mathrm{Bun}_G E)$ is topology analytic étale

Geometry of the proof: "non-abelian uniformization of $\mathrm{Bun}_G^{ss,0}(E) = \mathfrak{g}(E)$

Analogue: $G = \mathbb{G}_m$, $\mathrm{Pic}^0(E) = \mathbb{C}^\times / q^{\mathbb{Z}}$

Ex : $G = SL_2$

$$\begin{array}{ccc} \mathfrak{g} & G & \mathfrak{g}_E \\ \downarrow & \downarrow & \downarrow \\ \mathfrak{h}/\!/W & H/\!/W & (E \otimes \Lambda_H)/W \end{array}$$



Uniformization (via two adj. quotients of LG)

$$\mathfrak{g}_E = \left(\frac{G_0^{q\text{-reg}}}{G_0} \sqcup \frac{G_1^{q\text{-reg}}}{G_1} \right) \Big/ \sim$$

$$G_0, G_1 \subset LG$$

$$\begin{array}{cc} \cup & \cup \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \begin{pmatrix} a & bt \\ t^{-1}c & d \end{pmatrix} \end{array}$$

Gaitsgory II

$P \rightsquigarrow \mathrm{Whit}^P$

$P_1 \hookrightarrow P_2 \quad \rightsquigarrow \mathrm{Whit}_{P_2} \xrightarrow{j_! \circ j_{P_2}^!} \mathrm{Whit}_{P_1}$

$\mathrm{Glue}(\mathrm{Whit}^P) \simeq \mathrm{Whit}^{ext}$

$$\begin{array}{l} \quad \Big\uparrow \mathrm{coeff}^{ext} \rightsquigarrow \text{conjecturally fully faithful} \\ \mathrm{Dmod}(\mathrm{Bun}_G) \end{array}$$

•) $\mathrm{LocSys}_{\check{G}} = \mathrm{Maps}(X_{dR}, B\check{G})$

– Y prestack: $\mathrm{Hom}(S, Y_{dR}) = \mathrm{Hom}(S_{red}, Y)$

– $\mathrm{Hom}(S, \mathrm{Maps}(Z_1, Z_2)) = \mathrm{Hom}(S \times Z_1, Z_2)$

<u>Observation</u> $\mathrm{LocSys}_{\check{G}}$ is quasi-smooth (derived l.c.i.)

$\sigma \in \mathrm{LocSys}_{\check{G}}$

$H^0(T_\sigma \mathrm{LocSys}_{\check{G}}) = H^1_{dR}(X, \check{g}_\sigma)$ $\quad \Big\lceil\, T_y \mathcal{Y}[-1] = \mathrm{Lie}(pt \underset{\mathcal{Y}}{\times} pt)$

$H^{-1}(T_\sigma \mathrm{LocSys}_{\check{G}}) = H^0_{dR}(X, \check{g}_\sigma)$

$H^1(T_\sigma \mathrm{LocSys}_{\check{G}}) = H^2_{dR}(X, \check{g}_\sigma)$

$\mathcal{Y}$ – quasi-smooth alg. stack

$\rightsquigarrow$

$$\mathrm{Sing}(\mathcal{Y}) = \{(y \in \mathcal{Y}, \xi \in \underbrace{H^{-1}(T_y^*\mathcal{Y})}_{= H^1(T_y\mathcal{Y})^*})\}$$

Y – quasi-smooth scheme, $\mathcal{F} \in \mathrm{Coh}(Y)$

$$pt \xrightarrow{i_y} Y$$

$$i_y^!(\mathcal{F}) \curvearrowleft T_y Y[-1]$$

in degrees 1 and 2

The support of $H^*(i_y^! \mathcal{F})$ over $\mathrm{Sym}(H^2(T_y Y[-1]))$ will be exactly a subset of $H^{-1}(T_y^* Y)$.

$$N \subseteq \mathrm{Sing}(Y) \rightsquigarrow \mathrm{Coh}_N(Y) \subseteq \mathrm{Coh}(Y)$$

$$\mathrm{IndCoh}_N(Y) \subseteq \mathrm{IndCoh}(Y)$$

For quasi-smooth alg. stacks glue from charts.

For any derived scheme Y there is a canonically defined functor

$$\mathrm{QCoh}(Y) \longrightarrow \mathrm{IndCoh}(Y)$$

$$\mathcal{F} \longmapsto \mathcal{F} \otimes \omega_Y$$

<u>Thm</u>: If $N = \{0\}$ then $\quad \mathrm{QCoh}(Y) = \mathrm{IndCoh}_{\{0\}}(Y)$

$$\xrightarrow{\otimes \omega_Y} \quad \cap\!\!\mathrm{I} \; \mathrm{IndCoh}(Y)$$

$$\mathrm{Sing}(\mathrm{LocSys}_{\check{G}}) = (\sigma, \ A \in H^0_{dR}(X, \check{\mathfrak{g}}_\sigma))$$

$\mathrm{Nilp} \subseteq \mathrm{Sing}(\mathrm{LocSys}_{\check{G}})$, A be nilpotent.

$$\text{D-mod}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$$

For a map $f: Y_1 \to Y_2$ between derived schemes we have a functor $f^!: \mathrm{IndCoh}(Y_2) \to \mathrm{IndCoh}(Y_1)$

If $\mathcal{Y}$ is a prestack, $\mathrm{IndCoh}(Y) = \lim\limits_{S \to \mathcal{Y}} \mathrm{IndCoh}(S)$

$\mathrm{Crys}(\mathcal{Y}) = \mathrm{IndCoh}(\mathcal{Y}_{dR})$

Lem: Y a smooth alg. variety
$\mathrm{Crys}(Y) \simeq \text{D-mod}(Y)$

Lem: $- \otimes \omega_{Y_{dR}} : \mathrm{QCoh}(Y_{dR}) \to \mathrm{IndCoh}(Y_{dR})$
is an equivalence.

$$\begin{tikzcd} Z \arrow[d] \\ Y \end{tikzcd}$$

$\mathrm{IndCoh}(Z_{dR} \underset{Y_{dR}}{\times} Y)$ .. relative D-modules

$Z \to Z_{dR} \underset{Y_{dR}}{\times} Y$

Z & Y quasi-smooth

$$\begin{tikzcd} \mathrm{QCoh}(Z_{dR} \underset{Y_{dR}}{\times} Y) \arrow[r, Rightarrow] \arrow[rr, bend left, "\otimes\omega"] & \boxed{\mathrm{IndCoh}_{\{0\}}(Z_{dR} \underset{Y_{dR}}{\times} Z)} \arrow[r] & \mathrm{IndCoh}(Z_{dR} \underset{Y_{dR}}{\times} Y) \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{IndCoh}(Z_{dR} \underset{Y_{dR}}{\times} Z) \arrow[r] & \mathrm{IndCoh}(Z) \\ \cup\!\shortmid & \cup\!\shortmid \\ \mathrm{IndCoh}_{\{0\}}(Z_{dR} \underset{Y_{dR}}{\times} Z) \arrow[r] & \mathrm{IndCoh}_{\{0\}}(Z) \end{tikzcd}$$

this is the definition of this

$$\begin{tikzcd} \mathrm{LocSys}_{P^\vee} \arrow[d] \\ \mathrm{LocSys}_{G^\vee} \end{tikzcd}$$

Consider

$$\begin{tikzcd} \mathrm{IndCoh}_{\{0\}}\left((\mathrm{LocSys}_{P^\vee})_{dR} \underset{(\mathrm{LocSys}_{G^\vee})_{dR}}{\times} (\mathrm{LocSys}_{P^\vee})_{dR}\right) \arrow[d, "\text{Raskin's work-in-progress}"'] \\ \mathrm{Whit}^P(G) \end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Glue}\left((\mathrm{LocSys}_{\check{P}})_{dR} \underset{(\mathrm{LocSys}_{\check{G}})_{dR}}{\times} (\mathrm{LocSys}_{\check{P}})_{dR}\right) \arrow[r, hook] & \mathrm{Glue}(\mathrm{Whit}^?) = \mathrm{Whit}^{ext} \\
\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}}) \arrow[u, "\text{now } (*)"] & \text{D-mod}(\mathrm{Bun}_G) \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}}) \arrow[r, hook] & \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}}) \arrow[d, "! - \text{pullback}"] \\
& \mathrm{IndCoh}\left((\mathrm{LocSys}_{\check{P}})_{dR} \underset{(\mathrm{LocSys}_{\check{G}})_{dR}}{\times} (\mathrm{LocSys}_{\check{P}})_{dR}\right) \arrow[d, shift left, "\text{loses information!}"] \\
& \mathrm{IndCoh}_{\{0\}}\left((\mathrm{LocSys}_{\check{P}})_{dR} \underset{(\mathrm{LocSys}_{\check{G}})_{dR}}{\times} (\mathrm{LocSys}_{\check{P}})_{dR}\right) \arrow[u, hook, shift left]
\end{tikzcd}$$

<u>Thm</u> (Arinkin) The combined functor $(*)$ is fully faithful.

Y – quasi-smooth derived scheme

$$\mathrm{Ind}\mathring{\mathrm{Coh}}(Y) = \mathrm{IndCoh}(Y) / \mathrm{IndCoh}_{\{0\}}(Y)$$

<u>Thm:</u> (a) There exists a canonically defined monoidal action of $\mathrm{D\text{-}mod}(\mathbb{P}(\mathrm{Sing}\,Y))$ on $\mathrm{Ind}\mathring{\mathrm{Coh}}(Y)$,

(b)

$$\mathrm{Ind}\mathring{\mathrm{Coh}}_{\{0\}}(Z_{dR} \underset{Y_{dR}}{\times} Y) = \mathrm{Dmod}(\mathbb{P}(\mathrm{Sing}(f)^{-1}(\{0\}))) \underset{\mathrm{Dmod}(\mathbb{P}(\mathrm{Sing}\,Y))}{\otimes} \mathrm{Ind}\mathring{\mathrm{Coh}}(Y)$$

$$\begin{tikzcd} Z \arrow[d, "f"] \\ Y \end{tikzcd} \qquad \begin{tikzcd} Z \underset{Y}{\times} \mathrm{Sing}(Y) \arrow[r, "\mathrm{Sing}(f)"] \arrow[d] & \mathrm{Sing}\,Z \\ \mathrm{Sing}(Y) \end{tikzcd}$$

$$\leadsto \quad \mathrm{Sing}(f)^{-1}(\{0\}) \subseteq Z \underset{Y}{\times} \mathrm{Sing}(Y)$$

<u>Ex</u>: $Z = pt$, $Y = pt \underset{\mathbb{A}^1}{\times} pt$

Y – affine derived scheme

C – category

$\text{D-mod}(Y) \curvearrowright C$

" Y_{dR} is 1-affine "

Thm: An action of $\text{Dmod}(Y)$ on C is equivalent to the following data:

- $\forall S,\ S_{red} \to Y \rightsquigarrow C_S \curvearrowleft QCoh(S)$
- $S_1 \xrightarrow{f} S_2$; $\quad C_{S_1} \simeq QCoh(S_1) \underset{QCoh(S_2)}{\otimes} C_{S_2}$ + compatibilities for composition
- $\lim\limits_{S;\, S_{red} \to Y} C_S \simeq C$

$S_{red} \xrightarrow{f} \mathbb{P}(\mathrm{Sing}\, Y) \underset{\text{need to define}}{\rightsquigarrow} \mathrm{Ind}\mathring{C}oh(Y)_S$

$\left(QCoh(S) \otimes \mathrm{Ind}\mathring{C}oh(Y)\right)_M \qquad M \subseteq S \times \mathbb{P}\mathrm{Sing}\, Y$

Define $\mathrm{Ind}\mathring{C}oh(Y)_S := \left(QCoh(S) \otimes \mathrm{Ind}\mathring{C}oh(Y)\right)_{\mathrm{Graph}\, f}$.

Ben-Zvi II

Last time:

$\mathcal{C}$ dg category: 2-dualizable $\Longleftrightarrow$ smooth & proper

- "2d YM": G finite, $\mathcal{C} = \mathrm{Rep}\, G$
- B-model: X smooth proper variety, $\mathcal{C} = \mathrm{QCoh}(X)$

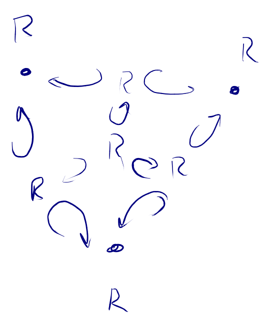
- Any commutative dga $(/\mathbb{C})$ R is $n+1$ dualizable $\forall n$

Consider R as n-category for any n:

$$(((R\text{-mod})\text{-mod})\text{-mod})\cdots$$

M manifold, in fact M any homotopy type: $(\mathrm{Spec}\, R)^M = \mathrm{Map}(M, \mathrm{Spec}\, R)$

$$\mathcal{O}(\mathrm{Spec}\, R)^M = R \otimes M$$



$M^k \longrightarrow R \otimes M$ $(n-k)$-category

$$R \otimes S^1 = R \otimes_{R \otimes R} R = R \otimes_{R\otimes R} R = HH_* R$$

$$= \mathcal{O}(\underbrace{\mathrm{Spec}\, R^{S^1}}_{= \mathcal{L}\,\mathrm{Spec}\, R})$$

$$\int_M R = M \otimes R \quad \text{considered as } k\text{-category}$$

$\mathcal{C},\otimes$ symmetric monoidal category is $n+2$ dualizable $\forall n$

$M \otimes \mathcal{C}$ same construction

perfect
X stack $\rightsquigarrow$ $\mathrm{QCoh}\, X = \mathcal{C}$

$ \;\; \| $

[R-mod if $X = \mathrm{Spec}\, R$]

Fact: $(\mathrm{QCoh}(X)) \otimes M \simeq \mathrm{QCoh}\, X^M$

perfect stack: $\mathrm{QCoh}\, X$ is compactly generated by dualizable objects

$(\mathrm{QCoh}\, X)_c$ = dualizable = $\mathrm{Perf}\, X$

$\Rightarrow$ $\mathrm{QCoh}\, X, \otimes$ is $2+1$ dualizable

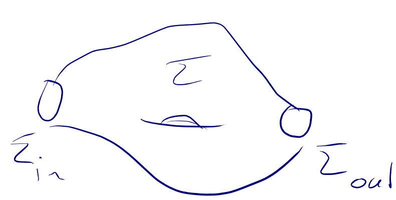
$\rightsquigarrow$ 3d TFT: "Rozansky-Witten theory"

$\bullet \longmapsto \mathrm{QCoh}(X), \otimes \iff \mathrm{QC}(X)$-mod

is
quasicoherent sheaves of categories $/X$

$S^1 \longmapsto \mathrm{QCoh}(\mathcal{L}X) = \mathrm{QCoh}(X) \otimes S^1$

$\Sigma \longmapsto \mathcal{O}(X^\Sigma)$



$$\begin{array}{ccc} & X^{\Sigma} & \\ \swarrow & & \searrow \\ X^{\Sigma_{in}} & & X^{\Sigma_{out}} \\ \mathrm{QCoh} & \longrightarrow & \mathrm{QCoh} \end{array}$$

e.g. $X = BG$, G affine

$\bullet \longmapsto \mathrm{Rep}\, G\text{-mod}$

$\simeq$ algebraic G-categories

$\|$

$(QC(G), *)\text{-mod}$

$G \times G \to G$

$S^1 \longmapsto QC(\cancel{\mathcal{L}BG}) = QC(\frac{G}{G})$

$\Sigma \longmapsto \mathcal{O}(BG^{\Sigma}) = \mathcal{O}(\mathrm{Loc}_G \Sigma)$

X smooth scheme : "RW" $\leftrightsquigarrow$ RW theory of T^*X

compactify on S^1

B-model of T^*X

2-periodically

$QCoh(T^*X) \otimes \mathbb{C}[\varepsilon, \varepsilon^{-1}]$

$|\varepsilon| = 2$

Koszul duality

$S^1 \longmapsto QCoh(\mathcal{L}X)$

$\|$

$T_X[-1]$

$\Omega_X^\bullet\text{-mod}$

IndCoh $\mathcal{L}X$

$QCoh(T_X^*[2])$

$\mathrm{Sym}\, T[-2]\text{-mod}$

only see some formal completion

Character theory

G reductive $\rightsquigarrow$ $(\mathcal{D}(G), *)$-mod

$= $ sheaves of categories over $(BG)_{dR}$

only 1+2 dualizable

$$Z(S^1) = \mathcal{D}\left(\tfrac{G}{G}\right)$$

$\circledcirc$

expect "$Z(\Sigma) = H^*_{dR}(\mathrm{Loc}_G \Sigma)$", but theory doesn't make it to dimension 2!

Let's fix action of $Z(U\mathfrak{g}) = \mathbb{C}[\mathfrak{h}^*/W]$

Examples of $\mathcal{D}(G)$-mod

- $\mathcal{D}(X)$ for $G \circlearrowright X$

$$G \times G \times X \rightrightarrows G \times X \to X$$

- $U\mathfrak{g}$-mod

$\circlearrowleft_G$ adjoint action

Let $\mathcal{O}_0 = U\mathfrak{g}\text{-mod}_0 \overset{\text{Beilinson-Bernstein}}{\simeq} \mathcal{D}(G/B)$

"unipotent principal series" $\quad \mathcal{D}_\lambda(G/N)$

$\mathcal{P}_0 \quad \mathcal{P}_0$

$$\operatorname{End}_{\mathcal{D}(G)} \mathcal{P}_0 \overset{\text{Fact}}{=} \mathcal{D}(B\backslash G/B), * = \mathcal{H}_0, *$$

$$\circlearrowright$$

$$\mathcal{D}(G/B)$$

Thm: $\mathcal{H}_0$ is 2+1 dualizable.

$$Z = \mathcal{D}^{G\times G}$$

$$S^1 \longmapsto \int_{S^1} (\mathcal{H}_0, *) = HH_*(\mathcal{H}_0, *) = Ch_0 G \subseteq \mathcal{D}_{\mathcal{N}}\left(\frac{G}{G}\right)$$

Z acts as on HC_0: $\{ z \cdot f = \underbrace{\chi_0(z)}_{\text{augmented}} f \ : \ z \in Z \}$

any algebra:

$$A \xrightarrow{Tr} HH_* A$$

$$\begin{tikzcd}
 & \mathcal{D}\left(\frac{G}{B}\right) \arrow[dr] & \\
\mathcal{H}_0 = \mathcal{D}(B\backslash G/B) \arrow[ur] \arrow[rr, "Tr"'] \arrow[rr, bend right, "Ch_0"'] & & \mathcal{D}\left(\frac{G}{G}\right)
\end{tikzcd}$$

$Tr(1_{\mathcal{H}})$ = character of $\mathcal{H}$ as $\mathcal{H}$-module

$= \quad$ —— " —— $\mathcal{P}_0$ as $\mathcal{D}(G)$

Grothendieck-Springer sheaf

$= \mathcal{S} = \left(\begin{array}{c} \frac{B}{B} = \frac{\tilde{G}}{G} \\ \downarrow \\ \frac{G}{G} \end{array} \right)_* \mathcal{O}$

$= HC_0$

$\wr\!\!\uparrow$

Hotta-Kashiwara

4d : • Betti Geometric Langlands theory
• Betti Quantum Geometric Langlands theory

$\mathrm{Rep}\, G = QC(BG)$ symmetric monoidal category
($\rightsquigarrow$ 3-cat)

Fact (Lurie, Walker): $\mathrm{Rep}\, G$ is 3+1 dualizable

$$\bullet \mapsto ((\mathrm{Rep}\, G)\text{-mod})\text{-mod}$$

$$S^1 \mapsto (QC(\mathcal{L}BG = \tfrac{G}{G}), \otimes)\text{-mod}$$

$$\Sigma \mapsto QC(\mathrm{Loc}_G \Sigma)$$

(am really want IndCoh $\mathrm{Loc}_G \Sigma$)

Frenkel-Gaitsgory
$\mathrm{Conn}_G D^*$

X alg curve
Σ top surface

$\mathrm{Conn}_G X \underset{\text{symplectomorphic}}{\overset{\text{analytically}}{\sim}} \mathrm{Loc}_G \Sigma$

$\rightsquigarrow$ 0-dimensional sheaves are the same

quantize

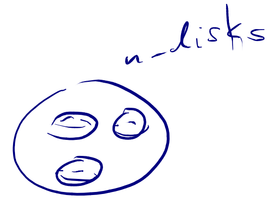
quantize

$\mathcal{D}_{\det^{\frac{1}{k}}}(\mathrm{Bun}_G X)$

R comm. ring $\rightsquigarrow$ $R \otimes M \longleftarrow$ any simplicial set

$\quad \shortparallel \qquad\qquad\qquad\qquad \shortparallel$

$\quad E_\infty \qquad\qquad\qquad\quad \int_M R$

R an E_n-algebra = very degenerate n-category = 

$= \mathrm{Alg}(\mathrm{Alg}(\mathrm{Alg}(\ldots(\mathrm{dgVect}))))$

$\Leftarrow$ $n+1$ dualizable

$R \otimes M$ makes sense for M any n-manifold [framed]

— factorization homology $\quad \shortparallel \; \int_M R$

R an E_n-algebra $\rightsquigarrow$ n-category

$\overbrace{(\underbrace{R\text{-mod}}_{E_{n-1}\text{-category}})\text{-mod}}^{E_{n-2}\ 2\text{-category}} \ldots$

E_2-category = braided $\otimes$-category $\ni \mathrm{Rep}_q G$ quantum group

$\wr$

f.d. $U_q\mathfrak{g}$-modules

$\mathrm{Rep}_q G$ is $3+1$ dualizable

→ Betti quantum Geom. Langland theory

Boundary conditions $(Z(\cdot))$ – algebras $\mathcal{C}, *$ over $\mathrm{Rep}_q G$

geom. Eisenstein series $\longleftarrow$ domain wall $\quad$ e.g. $\mathrm{Rep}_q B$ $\circlearrowleft \mathrm{Rep}_q G$, $\circlearrowleft \mathrm{Rep}_q$ $\quad$ algebra object in bimodules

$$
\begin{tikzcd}
\mathrm{Conn}_G X \arrow[d, squiggly] & \mathrm{Loc}_G \Sigma \arrow[d, squiggly] \\
\mathcal{D}_{1/k}(\mathrm{Bun}_G X) & \int_\Sigma \mathrm{Rep}_q G \\
(\hat{\mathfrak{g}}_{1/k}, G(\mathcal{O}))\text{-mod} \arrow[u, "BB"] \arrow[r, "\simeq"] & \mathrm{Rep}_q G \arrow[u]
\end{tikzcd}
$$

Kazhdan-Lusztig

$$R \to \int_M R$$

dooon a point

$$x \hookrightarrow \Sigma \qquad
\begin{tikzcd}
\mathrm{Loc}_G \Sigma \arrow[d] \\
BG
\end{tikzcd}
\qquad
\begin{tikzcd}
QC(\mathrm{Loc}_G \Sigma) \arrow[r, "\simeq"] & (A\text{-mod})_{\mathrm{Rep}\, G} \\
\mathrm{Rep}\, G \arrow[u]
\end{tikzcd}$$

$\circlearrowleft$ Rep G

Σ punctured surface:

BZ - Brochier - Jordan

$$\int_\Sigma \mathrm{Rep}_q G \simeq (A\text{-mod})_{\mathrm{Rep}_q G}$$

$$T^2 \setminus x \rightsquigarrow A = \mathcal{D}_q(G)$$

Langlands duality:

$$Z_{q,G} \simeq Z_{q^\vee, G^\vee} \qquad\qquad \int_{\underline{\ }} \mathrm{Rep}_q G$$

Conjecture says that $Z_{q,G}$ is q-expansion of modular TFT

$$\rightsquigarrow \quad Z_{q,G} = Z_{E,G}$$

Beraldo

Goal: Decompose $D(\mathrm{Bun}_G)$ into simple pieces parametrized by $\{P \supseteq B\}$

$$D(\mathrm{Bun}_G) \xrightarrow{\text{coeff}_{\text{ext}}} Wh_{\text{ext}} = \underset{P \supseteq B}{\mathrm{Glue}}(Wh_P)$$

Thm: for $G = GL_n$ this functor is fully faithful.

I) Classical Theory $k = \mathbb{F}_q$

$$\mathrm{Fun}\left(G(K) \backslash^{G(\mathbb{A})} / G(\mathcal{O})\right) \hookrightarrow \mathrm{Fun}\left(\cancel{{}_{N(K)}\backslash} Gr_G\right)^{B(K)} \to \mathrm{Fun}(Gr_G \times ch)^{T(\mathcal{O}) \ltimes N(\mathbb{A}), ev}$$

$$f \mapsto Wf(g,\chi) = \int_{N(\mathbb{A})/N(K)} f(n^{-1}g)\, e^{\chi(n)}\, dn$$

$K =$ function field on X

$Gr_G = G(\mathbb{A})/G(\mathcal{O})$

$\circlearrowleft$
$N(\mathbb{A})$

Do F.T. wrt $N(\mathbb{A})$

$ch :=$ additive chars of $N(\mathbb{A})/N(K)$

Thm (strong approx): $N(\mathbb{A})/N(K)$ is cpt
$\Longrightarrow \int$ converges

analogy in geometry: Av function is cts.

$$\mathrm{Fun}(G_r \times \mathfrak{h})^{T(K)\ltimes N(A),\, ev} = \left\{ \varphi \;\middle|\; \begin{array}{l} \varphi(ng, x) = e^{\chi(n)} \varphi(g,x) \ \forall n \in N(A) \\ \varphi(tg, \mathrm{Ad}_t x) = \varphi(g,x) \ \forall t \in T(K) \end{array} \right\}$$

Assume center Z_G is connected.

$T(K) \circlearrowright \mathfrak{h}$

$$\left\{ \begin{array}{l} T(K)\text{-orbits} \\ \text{of } \mathfrak{h} \end{array} \right\} \xleftrightarrow{\ \simeq\ } \{ P \supseteq B \}$$

$$\text{orbit thru } x_P \longleftarrow\!\shortmid\ P$$

$$\mathfrak{h} \cong (\Omega^1(K))^{\times r} \simeq K^r \qquad r = \mathrm{rank}(G)$$

$$ev : N(A) \times \mathfrak{h} \longrightarrow k$$

$$(n; \xi_1, \dots, \xi_r) \longmapsto \sum_{i=1}^{r} \mathrm{Res}(\alpha_i(n)\, \xi_i)$$

$$\mathrm{Stab}_{T(K)}(x_P) = Z_M(K) \qquad [M = \text{Levi of } P]$$

<u>Obvious</u>: $\mathrm{Whit}_{ext} = \prod_{P \supseteq B} \mathrm{Fun}(G_r)^{Z_M(K) \ltimes N(A),\, \chi_P}$

easy: for $G = GL_n, PGL_n$ coeff_{ext} is injective

(Idea: use $P_{n-1,1} = \begin{pmatrix} * & * \\ 0 \cdots 0 & * \end{pmatrix}$)

$$\underset{M \ltimes V}{\overset{\text{Levi}}{}} \quad P_{n-1,1} = M \ltimes V$$

II) Geometry

Ran: $S \mapsto \left\{ \underline{x} : \text{finite nonempty subsets of } X(S) \right\}$

$G(A) : S \mapsto \left\{ \underline{x},\ \mathring{D}_{\underline{x}} \xrightarrow{f} G \right\}$

$G(O) : S \mapsto \left\{ \underline{x},\ D_{\underline{x}} \to G \right\}$

$G(K) : S \mapsto \left\{ \underline{x},\ f : X_S \setminus \Gamma_{\underline{x}} \to G \right\}$ $\qquad X_S := S \times X$

$Gr_G : S \mapsto \left\{ \underline{x},\ P_G \text{ G-bundle on } X_S,\ \text{trivialization defined on } X_S \setminus \Gamma_{\underline{x}} \right\}$

$ch : S \mapsto \left\{ \underline{x},\ \xi_1, \dots, \xi_r \in \Omega_X \boxtimes O_S |_{X_S \setminus \Gamma_{\underline{x}}} \right\}$

Naive def of Wh_{ext}

$$Wh_{ext}[\mathcal{D}(Gr_G)] = \mathcal{D}(Gr_G \times_{Ran} ch)^{T(K) \ltimes N(A), ev}$$

$$= \left(\mathcal{D}(Gr_G) \otimes_{\mathcal{D}(Ran)} \mathcal{D}(ch) \right)^{T(K) \ltimes N(A), \omega}$$

invariant category

Paradigm of rep theory

$(Sch, \times)^{op} \xrightarrow{\mathcal{D}} (DGcat, \otimes)$

is symmetric monoidal

$$\mathcal{D}(Y \times Z) = \mathcal{D}(Y) \otimes \mathcal{D}(Z)$$

group $(G, m) \longrightarrow$ get a comonoidal cat $(\mathcal{D}(G), m^!)$

Saying $G \curvearrowright \mathcal{C}$ <u>means</u> $\mathcal{C}$ is given a coaction of $(D(G), m^!)$

e.g. $G \curvearrowright Vect$

<u>Def</u>: invariant cat of $\mathcal{C}$

$$\mathcal{C}^G := \lim \left(\mathcal{C} \underset{\text{triv}}{\overset{\text{coact}}{\rightrightarrows}} D(G) \otimes \mathcal{C} \rightrightarrows \cdots \right)$$

$\hookrightarrow$ objects $\{ (c, \ \text{coact}(c) \simeq \omega_G \otimes c) \}$

e.g. $G \circlearrowright X \Rightarrow G \circlearrowright D(X)$

$$D(X)^G \cong D(X/G)$$

Toën III

Today: Deformation quantization of the derived stacks $Map(?, BG)$.

i.e. deform, more-or-less canonically, $L_{qcoh}(Map(?, BG))$ into non-commutative versions

ex: $? = \Sigma$ cpt surface

$Map(\Sigma, BG)$

$$L_{qcoh}(Map(\Sigma, BG)) = \int_{\Sigma} Rep(G)$$
$$\rightsquigarrow \int_{\Sigma} Rep_q(G)$$

Here: General setting to construct these kinds of quantizations.

⚠ Still in progress + incomplete

---

1) De Rham theory

$A \in cdga_k^{\leq 0}$

$\mathbb{L}_A \in L(A) \simeq A\text{-mod}$

The de Rham algebra $DR(A) := Sym_A(\mathbb{L}_A[-1])$

- graded comm. dg algebra
- $\exists$ a mixed structure $\epsilon = d_{DR}$ differential

} graded mixed cdga

There is a relative story : $A \longrightarrow B$

$$DR(B/A) = \mathrm{Sym}_B(\mathbb{L}_{B/A}[-1])$$

is a graded mixed cdga ; A-linear

<u>Def</u> : $X \to Y$ of der. alg. stacks

$$DR(X/Y) := \lim_{\substack{\operatorname{Spec} B \to X \\ \downarrow \qquad \downarrow \\ \operatorname{Spec} A \to Y}} DR(B/A) \in \text{graded mixed cdga}$$

<u>Fact</u> $DR(X/Y) \simeq \bigoplus_p \Gamma(X, \mathrm{Sym}^p(\mathbb{L}_{X/Y}[-1]))$

as a graded cdga

$$\| \\ \bigwedge_p \Gamma(X, \Lambda^p \mathbb{L}_{X/Y})[-p]$$

$\Rightarrow$ is a graded mixed cdga

(*) $A^{\cdot}_{dR}(X/Y) := \underline{\mathrm{Hom}}_{\varepsilon\text{-}dg^{gr}}(k, DR(X,Y)) \simeq \prod_{p \geq 0} \Gamma(X, \Lambda^p \mathbb{L}_{X/Y})[-p]$

"$D = d + dR$"

$\varepsilon\text{-}dg^{gr}$ = graded mixed modules

derived de Rham complex of $X \to Y$

$F^i := \prod_{p \geq i}$

(**) $F^i A^{\cdot}_{dR}(X/Y) \subseteq A_{dR}(X/Y)$ filtration

(***) $X \to *$ $\quad H^*(A_{dR}(X/Y)) \xrightarrow{\sim} H^*_{dR}(X^d_{/k})$

shifted symplectic structures

X alg der. stack$/k$ (laft)

- A closed 2-form on X of degree (shift) $n \in \mathbb{Z}$ is
  $\omega \in H^n(F^2 A_{dR}(X/k)[2])$

Note: $X/_k$ smooth scheme: closed 2-forms of deg 0 on X
$$\simeq \Gamma(X, \Omega_X^{2,cl})$$

$\rightsquigarrow$ "being closed up to homology" as $\omega = \omega_0 + \omega_1 + \dots$
$$\omega_i \in \Gamma(X, \Lambda^{2+i} \mathbb{L}_X)^{n-i}$$

$$dR(\omega_0) = -d(\omega_1)$$
$$dR(\omega_1) = -d(\omega_2)$$
$$\vdots$$

Def: An <u>n-shifted symplectic structure</u> on X is a closed 2-form of degree n such that ω_0 is non-degenerate: $\mathbb{T}_X \overset{\nu_{\omega_0}}{\simeq} \mathbb{L}_X[n]$ in $Lqcoh(X)$

Examples

- G reductive

  2-shifted symplectic structure on $BG \simeq Sym^2(\mathfrak{g}^*)^G_{\text{ind}}$

- $X \rightsquigarrow T^*X[n] = \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}}(\mathbb{T}_X[-n]))$
  carries a canonical n-shifted symplectic structure.

– Perf = derived stack of perfect complexes

Perf : cdga $\longrightarrow$ Top

$A \longmapsto \mathrm{Perf}(A) = \infty$-gp of perfect A-modules

Fact : Perf is alg (∞-) stack, laft and it carries a canonical 2-shifted symplectic structure ω.

$$\mathbb{T}_X \xrightarrow{\vartheta_{\omega_0}} \mathbb{L}_X[2]$$

$$\mathbb{T}_X = \underline{\mathcal{E}nd}(\mathcal{E})[1] \qquad \mathcal{E} \in \mathrm{Perf}(\mathrm{Perf}) \text{ universal}$$

Thm : Let X ~~be a~~ derived alg stack with n-shifted symplectic structure

- Z be an oriented derived stack of dim. d ("CY of dim d")

Then $\mathrm{Map}(Z, X)$ has a canonical $(n-d)$-shifted symplectic structure.

Ex : $X = BG$

- $\mathrm{Map}(M, BG)$ $\quad$ M ~~..~~ manifold of dim d
  $(2 - \dim M)$-symplectic

- $\mathrm{Map}(Y_{dR}, BG)$ $\quad$ Y sm. proper$/k$
  $(2 - 2\dim Y)$-symplectic

⋮

Pb: Want deformation quantization of these n-shifted symplectic structures.

2) Shifted Poisson structures

X der. alg stack loc. of f. pr $/k$
$n \in \mathbb{Z}$

- n-shifted polyvectors on X

$$\mathrm{Pol}(X,n) := \bigoplus_{p \geq 0} \Gamma\left(X, \mathrm{Sym}^p_{\mathcal{O}_X}\left(\mathbb{T}_X[-1-n]\right)\right)$$

-- graded alg.

Fact: $\mathrm{Pol}(X,n)$ has a natural structure of a graded $\mathbb{P}_{n+2}$-alg. ($\mathbb{P}_{n+2} = H_*(E_{n+2})$, $n \geq 0$)

We are interested in the underlying graded Lie algebra
$$\mathrm{Pol}(X,n)[n+1]$$
This Lie bracket is a version of Schouten bracket.

Rem: $\Gamma(X, \mathbb{T}_X) \subseteq \mathrm{Pol}(X,n)[n+1]$
sub dg-Lie
Lie bracket $= \Gamma(X,\mathbb{T}_X) =$ tangent Lie of $\mathrm{Aut}(X)$

Def An n-shifted poisson structure on X is a morphism in the ∞-category of graded dg Lie,

$$k[-1](2) \xrightarrow{p} \mathrm{Pol}(X,n)[n+1].$$

* p determines a first piece of data "underlying bimodule"

$$p_0 \in H^{-n}(X, \mathrm{Sym}^2(\mathbb{T}_X[-1-n])[n])$$

** higher structures $p = p_0 + p_1 + \cdots$

$$[p_0, p_0] = d p_1$$

$$\vdots$$

Thm (V. Melani): $X = \mathrm{Spec}\, A$ $\quad A \in$ cdga loc f. pres

$\exists$ an equivalence of spaces

$$\left\{\begin{matrix} n\text{-shifted Poisson} \\ \text{structure on } X \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \mathbb{P}_{n+1} \text{ structures} \\ \text{on } A \end{matrix}\right\}$$

$$= \left\{\begin{matrix} B \ \mathbb{P}_{n+1}\text{-alg} \\ + B \simeq A \text{ as cdga} \end{matrix}\right\}$$

Pre Fact: There is a natural map

$$\left\{\begin{matrix} n\text{-shifted sympl.} \\ \text{structure on } X \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} n\text{-shifted Poisson} \\ \text{structure on } X \end{matrix}\right\}$$

3) Quantization of Poisson structures $(n>0)$

general strategy: to quantize each formal completion $\hat{X}_x$

+ integrate over $x \in X_{dR}$

$$\int_{X_{dR}} \hat{X}_x = X$$

$$\begin{tikzcd} \hat{X}_x X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\ x \arrow[r] & X_{dR} \end{tikzcd}$$

X – derived stack $\quad X_{DR}$: cdga $\longrightarrow$ Top

$A \longrightarrow X(A_{red})$

$X \xrightarrow{\pi} X_{dR}$

$L_{qcoh}(X) = \Gamma(X_{dR}, \pi_* L_{qcoh})$

We introduce a sheaf of graded mixed $\mathcal{O}_{X_{dR}}$-cdga.

$$\begin{tikzcd} & \overset{S}{\mathrm{Spec}\, A} \arrow[r] & X_{dR} \\ & \mathrm{Spec}\, A_{red} \arrow[u] \arrow[r, "u"] \arrow[ul] \arrow[r, bend right, "\hat{\Gamma}_u"'] & X \arrow[u] \end{tikzcd}$$

For

Γ_u : $\mathrm{Spec}\, A_{red} \longrightarrow X \times \mathrm{Spec}\, A$

$DR(S_{red}/X \times S)$ graded mixed cdga, A-linear

This defines the wanted sheaf $\mathcal{B}_X \in$ graded mixed $\mathcal{O}_{X_{dR}}$-cdga

Theorem: The sheaf $\mathcal{B}_X$ is an algebraic model for $X \longrightarrow X_{dR}$

Cor: $\exists$ a natural functor

$$\begin{array}{ccc} L_{qcoh}(X) & \longrightarrow & \mathcal{B}_X\text{-graded mixed modules} \\ \cup & & \cup \\ \boxed{\mathrm{Perf}(X) \xrightarrow{\sim} \mathrm{Perf}(\mathcal{B}_X)} & & \end{array}$$

Cor: To quantize $(X, \omega) \rightsquigarrow (X, p)$ p poisson

$\Rightarrow$ $\mathbb{P}_{n+1}$ structure on $\mathcal{B}_X$

$\Rightarrow$ deform to E_{n+1}-structure by formality

$$\boxed{\mathrm{Perf}\, X \simeq \mathrm{Perf}(\mathcal{B}_X) \overset{def}{\rightsquigarrow} \mathrm{Perf}_q(X) \quad E_n\text{-}\otimes\text{-}\infty\text{cat}}$$

Gaitsgory III

$$\begin{array}{l} Z \\ \downarrow f \\ Y \end{array} \quad \text{quasismooth derived scheme}$$

$$\begin{tikzcd} \mathrm{Sing}\,Y \underset{Y}{\times} Z \arrow[r, "\mathrm{Sing}\,f"] \arrow[d] & \mathrm{Sing}\,Z \\ \mathrm{Sing}\,Y & \end{tikzcd}$$

$$Z_{dR} \underset{Y_{dR}}{\times} Y$$

$$\begin{tikzcd} \mathrm{QCoh}(Z_{dR} \underset{Y_{dR}}{\times} Y) \arrow[r] \arrow[rr, bend right, "\otimes \omega_{Z_{dR} \underset{Y_{dR}}{\times} Y}"'] & \mathrm{IndCoh}_{\{0\}}(Z_{dR} \underset{Y_{dR}}{\times} Y) \arrow[r] & \mathrm{IndCoh}(Z_{dR} \underset{Y_{dR}}{\times} Y) \end{tikzcd}$$

$i \in I$

$$\begin{array}{l} Z_i \\ \downarrow \\ Y \end{array}$$

(our case $Y = \mathrm{LocSys}_{\check{G}}$, $Z_i = \mathrm{LocSys}_{\check{P}}$

$I =$ conj. classes of parabolics

$=$ subsets of Dynkin diagram)

$$\mathrm{IndCoh}_N(Y) \longrightarrow \mathrm{Glue}\left(\mathrm{IndCoh}_{\{0\}}(Z_{i,dR} \underset{Y_{dR}}{\times} Y)\right)$$

$N \subseteq \mathrm{Sing}\,Y$

$$\mathrm{IndCoh}_N(Y) \hookrightarrow \mathrm{IndCoh}(Y) \longrightarrow \mathrm{IndCoh}(Z_{i,dR} \underset{Y_{dR}}{\times} Y) \hookleftarrow \mathrm{IndCoh}_{\{0\}}(\ldots)$$

Thm 0: In our situation the above functor is fully faithful.

$$\mathring{\mathrm{IndCoh}}_N(Y) = \mathrm{IndCoh}_N(Y) / \mathrm{QCoh}(Y)$$

$$\mathring{\mathrm{IndCoh}}_{\{0\}}(Z_{dR} \underset{Y_{dR}}{\times} Y) = \mathrm{IndCoh}_{\{0\}}(Z_{dR} \underset{Y_{dR}}{\times} Y) \Big/ \mathrm{QCoh}(Z_{dR} \underset{Y_{dR}}{\times} Y)$$

$$\mathring{\mathrm{IndCoh}}_{\{0\}}(Y) \longrightarrow \mathrm{Glue}\left(\mathring{\mathrm{IndCoh}}_{\{0\}}(Z_{dR} \underset{Y_{dR}}{\times} Y)\right)$$

Lemma: It's enough to show that this functor is fully faithful.

Thm (Recap)

a) $\mathring{\mathrm{IndCoh}}(Y) \circlearrowleft \mathrm{Dmod}(\mathbb{P}\mathrm{Sing}(Y))$

b) $\mathring{\mathrm{IndCoh}}_{\{0\}}(Z_{dR} \underset{Y_{dR}}{\times} Y) = \mathrm{Dmod}(\mathbb{P}(\underbrace{\mathrm{Sing}(f)^{-1}(\{0\})}_{\text{in } \mathrm{Sing}\, Y \underset{Y}{\times} Z})) \otimes \mathring{\mathrm{IndCoh}}(Y)$

$\mathrm{Dmod}(\mathbb{P}\mathrm{Sing}(Y))$

c) $\mathring{\mathrm{IndCoh}}_N(Y) = \mathrm{Dmod}(\mathbb{P}N) \underset{\mathrm{Dmod}(\mathbb{P}\mathrm{Sing}(Y))}{\otimes} \mathring{\mathrm{IndCoh}}(Y)$

$$\begin{array}{c} Z_i' \\ \downarrow f_i \\ Y' \end{array} \qquad \mathcal{M}_i \subseteq Z_i'$$

$$i \xrightarrow{\alpha} j \rightsquigarrow f_\alpha : Z_i' \to Z_j'$$

require: $f_\alpha^{-1}(\mathcal{M}_j) \subseteq \mathcal{M}_i$

$$\begin{array}{c} f_\alpha^{-1}(\mathcal{M}_j) \hookrightarrow \mathcal{M}_i \\ \downarrow \\ \mathcal{M}_j \end{array}$$

$$\mathrm{Dmod}(\mathcal{M}_j) \longrightarrow \mathrm{Dmod}(\mathcal{M}_i)$$

$$\mathrm{Dmod}(Y) \longrightarrow \mathrm{Glue}(\mathrm{Dmod}(\mathcal{M}_i))$$

$$\mathrm{Dmod}(Y') \xrightarrow{f_i^!} \mathrm{Dmod}(Z_i') \leftrightarrows \mathrm{Dmod}(\mathcal{M}_i)$$

Set $Y' = \mathbb{P}\mathrm{Sing}(Y)$

$Z_i' = \mathbb{P}\mathrm{Sing}(Y) \underset{Y}{\times} Z_i$

$\mathcal{M} = \mathbb{P}(N)$

$\mathcal{M}_i = \mathbb{P}(\mathrm{Sing}\, f_i^{-1}(\{0\}))$

Thm $\Rightarrow$

<u>Lem</u>: If $\mathrm{Dmod}(\mathcal{M}) \longrightarrow \mathrm{Glue}(\mathrm{Dmod}(\mathcal{M}_i))$ is fully faithful then

$$\mathrm{IndCoh}_N(Y) \longrightarrow \mathrm{Glue}(\mathrm{IndCoh}_{\{0\}}(Z_{dR} \underset{Y_{dR}}{\times} Y))$$

is also fully faithful

The Lemma reduces Thm 0 to

Thm 2: In our situation

$$\mathrm{Dmod}(M) \longrightarrow \mathrm{Glue}(\mathrm{Dmod}(M_i))$$

is fully faithful.

Assume that $\begin{matrix} Z_i' \\ \downarrow \\ Y' \end{matrix}$ are proper.

$m \in M$.

Consider $Z_i' \underset{Y'}{\times} \{m\} \cong M_i \underset{Y'}{\times} \{m\}$

Lem: The functor $\mathrm{Dmod}(M) \to \mathrm{Glue}(\mathrm{Dmod}(M_i))$ is fully faithfull iff $\forall m \in M$

$$\mathrm{Vect} \longrightarrow \mathrm{Glue}\left(\mathrm{Dmod}\left(M_i \underset{Y'}{\times} \{m\}\right)\right)$$

is fully faithful.

Suppose

$$i \longrightarrow W_i$$

$$i \xrightarrow{\alpha} j \rightsquigarrow W_i \xrightarrow{f_\alpha} W_j$$

$W = \operatorname{colim} W_i$

$$\mathrm{Dmod}(W) = \lim \mathrm{Dmod}(W_i)$$

require strict compatibilities

$$\begin{tikzcd} \mathrm{Dmod}(W) \arrow[d] \\ \mathrm{Glue}(\mathrm{Dmod}(W_i)) \end{tikzcd}$$

require isos

$W \xrightarrow{p} pt$

$W_i \xrightarrow{p_i} pt$

$$\begin{tikzcd} \mathrm{Vect} \arrow[r, "p^!"'] \arrow[dr] & \mathrm{Dmod}(W) \arrow[l, bend right, "p_!"'] \arrow[d] \\ & \mathrm{Glue}(\mathrm{Dmod}(W_i)) \end{tikzcd}$$

$p^!$ is fully faithful if $p_! \circ p^!(k) \to k$ is an isomorphism.

$$p_! p^!(k) = \operatorname{colim}_i p_{i,!} \circ p_i^!(k) = \operatorname{colim}_i H_*(W_i)$$

Drop $!$ and bare decorations from notation

$Z_i \supseteq M_i$ $\qquad \alpha : i \to j$, $\quad f_\alpha : Z_i \to Z_j$ $\qquad f_\alpha^{-1}(M_j) \subseteq M_i$

$$\begin{tikzcd} \mathrm{Vect} \arrow[r, "\phi"'] & \mathrm{Glue}(\mathrm{Dmod}(M_i)) \arrow[l, bend right, "\phi^L"'] \end{tikzcd}$$

M_j

Claim: ϕ^L exists

fully faithfulness $\Leftrightarrow \phi^L \circ \phi(k) \to k$ is an iso

Lem:

$$\phi^L \circ \phi(k) = \operatorname*{colim}_{i_0 \to \dots \to i_n} H_0(M_n \times_{Z_n} Z_0)$$

morphism from $[n] \to [m]$

$$\mathrm{Glue}(M_i) = \operatorname*{colim}_{i_0 \to \dots \to i_n} M_n \times_{Z_n} Z_0$$

want to check that $H_*(\mathrm{Glue}(M_i)) \to k$ is an isomorphism.

Remember our situation..

σ – local system

A – nilpotent endomorphism of σ, $H_{dR}(X, g_\sigma)$

$Z_i = \mathrm{Spr}_P^\sigma$ = scheme of reductions of σ to P

$M_i = \mathrm{Spr}_{P,unip}^{\sigma,A}$ = those reductions that A is contained in the unipotent radical

$$\operatorname*{colim}_{P_1 \subseteq \dots \subseteq P_n} \mathrm{Spr}_{P_0}^\sigma \times_{\mathrm{Spr}_{P_n}^\sigma} \mathrm{Spr}_{P_n, unip}^{\sigma,A} =: \mathrm{Glue}(\mathrm{Spr}_{unip}^{\sigma,A})$$

want: $H_*(\mathrm{Glue}(\mathrm{Spr}_{unip}^{\sigma,A})) \to k$

Thm 3: If $A \neq 0$ and we assume the result for proper Levi subgroups, there exists a canonical isomorphism between

$$H_*\left(\mathrm{Glue}\left(\mathrm{Spr}^{\sigma,A}_{unip}\right)\right) = H_*\left(\mathrm{Glue}\left(\mathrm{Spr}^{\sigma,A}\right)\right)$$

where $\mathrm{Glue}(\mathrm{Spr}^{\sigma,A}) = \underset{P\text{ - proper parabolic}}{\mathrm{colim}}\ \mathrm{Spr}^{\sigma,A}_P$

$\mathrm{Spr}^{\sigma,A}_P$ = reductions of σ to P such that is contained in $H_{dR}(X, p_\sigma)$.

Thm 4: $H_*\left(\mathrm{Glue}\left(\mathrm{Spr}^{A,\sigma}\right)\right) \xrightarrow{\sim} k$

- $G = SL_2$
- $G = SL_3$

1) A regular

$$\begin{tikzcd} & B \arrow[dl] \arrow[dr] & \\ P_1 & & P_2 \end{tikzcd} \qquad \mathrm{Spr}^{\sigma,A}_{P_i} = pt$$

$$\rightsquigarrow \begin{tikzcd} & pt \arrow[dl] \arrow[dr] & \\ pt & & pt \end{tikzcd}$$

2) A subregular

$$\begin{tikzcd} & \mathbb{P}^1 \sqcup_{pt} \mathbb{P}^1 \arrow[dl] \arrow[dr] & \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{tikzcd} \qquad \begin{tikzcd} & W_1 \sqcup_{pt} W_2 \arrow[dl] \arrow[dr] & \\ W_1 & & W_2 \end{tikzcd}$$

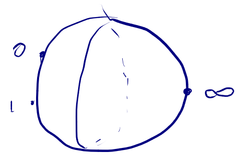
Yun

joint work w/ Nadler

Motivation

"Verlinde" approach to Betti-Langlands:

building blocks



Automorphic side:

$\mathrm{Bun}_G(I_{0,1,\infty})$

$= \{G\text{-bundles on } \mathbb{P}^1 \text{ with flag structure at } 0,1,\infty\}$

Spectral side:

$\mathrm{Loc}_{\check{G}}(\mathrm{unip}_{0,1,\infty}) =$

$= \{(\mathcal{E}, \nabla) : \mathcal{E}^\vee - \check{G}\text{-local system}$

∇ – connection w/ reg sing at $0,1,\infty$ with nilpotent residue at $0,1,\infty$

+ flag structure of $\mathcal{E}$ at $0,1,\infty$ compatible with $\mathrm{Res}\,\nabla$. $\}$

$\boxed{G = PGL_2, \quad \check{G} = SL_2}$

$$\mathrm{Bun}_G(I_{0,1,\infty}) \simeq \left\{ \begin{array}{l} \mathcal{E}: \text{rk } 2 \text{ v.b. on } \mathbb{P}^1 \\ \ell_0, \ell_1, \ell_\infty \text{ lines in } \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_\infty \end{array} \right\} \Big/ \mathrm{Pic}(\mathbb{P}^1)$$

$$= \mathrm{Bun}_G^{\mathrm{even}} \amalg \mathrm{Bun}_G^{\mathrm{odd}}$$

Betti version of spectral side:

$$\left\{ \begin{array}{l} A, B, C \\ \text{unip. matrices in } \check{G} = SL_2 \circlearrowright \mathbb{C}^2 \\ \ell_A, \ell_B, \ell_C \subset \mathbb{C}^2 \end{array} \middle| \begin{array}{l} ABC = 1 \\ A \text{ preserves } \ell_A \\ B \text{ — " — } \ell_B \\ C \text{ — " — } \ell_C \end{array} \right\} \Big/ \check{G} =: S_{\check{G}} / \check{G} \quad \overset{\mathrm{Loc}^{\mathrm{Betti}}_{\check{G}}(0,1,\infty)}{\sim}$$

<u>Thm</u> (Nadler – Y.) $\quad G = PGL_2, \ \check{G} = SL_2$

$$\mathrm{Coh}^{\check{G}}(S_{\check{G}}) \xrightarrow{\sim} D_!(\mathrm{Bun}_G(I_{0,1,\infty}))$$

$D_! = D^b_c$ ... constructible sheaves of the form $j_! \tilde{\mathcal{F}}$,

where $j: \left\{ \begin{array}{l} \text{open substack} \\ \text{of f. t.} \end{array} \right\} \hookrightarrow \mathrm{Bun}_G(I_{0,1,\infty})$

$$\widetilde{\mathcal{N}}^\vee = T^*(G^\vee/B^\vee) = T^*\mathbb{P}^1$$

a priory derived Cartesian product

$$\begin{tikzcd} S_{G^\vee} \arrow[r] \arrow[dd] & \widetilde{\mathcal{N}}^\vee \times \widetilde{\mathcal{N}}^\vee \times \widetilde{\mathcal{N}}^\vee \arrow[d] \\ & \mathcal{N}^\vee \times \mathcal{N}^\vee \times \mathcal{N}^\vee \arrow[d, "\text{mult}"] \\ \{1\} \arrow[r] & G^\vee \end{tikzcd}$$

<u>Fact</u>: •) $S_{G^\vee}$ is an l.c.i. scheme

•) $S_{G^\vee}/G^\vee \simeq \Sigma_{G^\vee}/G^\vee \simeq \mathrm{Loc}^{(2K)}_{G^\vee}(0,1,\infty)$

$$\Sigma_{G^\vee} = \{ \underbrace{A, B, C}_{\text{nilp in } sl_2}, \ell_A, \ell_B, \ell_C : A + B + C = 0 \}$$

Fix $\ell_C = [1:0]$

$\ell_A = [1:x]$, $\ell_B = [1:y]$

$$A = 1 + \begin{pmatrix} -ax & a \\ -ax^2 & ax \end{pmatrix} \qquad B = 1 + \begin{pmatrix} -by & b \\ -by^2 & by \end{pmatrix}$$

$C = \ldots$

a, b, x, y are variables

$$\begin{cases} ax + by = 0 \\ ax^2 + by^2 = 0 \end{cases}$$

5 components of $S_{G^\vee}$

- $a=b=0$
- $a=0,\ y=0$
-
-
- $\begin{cases} x^2=xy=y^2 \\ ax+by=0 \end{cases}$ non-reduced

$A=B=C=1$

$A=1,\ B=C^{-1},\ \ell_B=\ell_a$

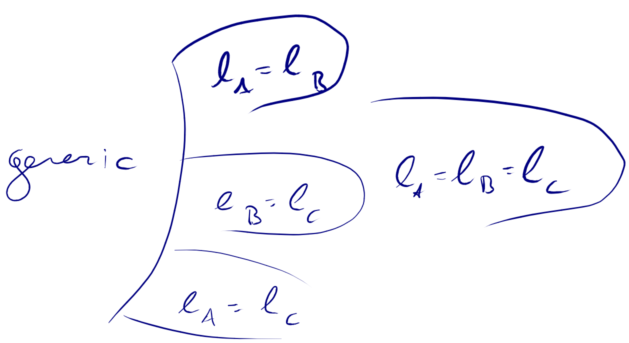
$B=1$ - - -

$C=1$ - - -

reduced part parametrizes $\ell_A=\ell_B=\ell_C$

$$\Sigma_{G^\vee/G^\vee} = \text{Hamiltonian reduction of } G^\vee \circlearrowright \left(T^*G^\vee/B^\vee\right)^3$$

$SL_2 = G^\vee \circlearrowright (\mathbb{P}^1)^3$ has 5 orbits



orbit $Z\subset(\mathbb{P}^1)^3$

$\rightsquigarrow T_Z(\mathbb{P}^1)^3 \subset (T^*\mathbb{P}^1)^3$

match components above.

Steps of proof

- $\mathrm{Perf}^{\check{G}}(S_{G^\vee}) \curvearrowright D_!(\mathrm{Bun}_G(I_{0,1,\infty}))$
- Find the automorphic object corresp. to $\mathcal{O}_{S_{\check{G}}}$

$\Downarrow$

Def functor $\mathrm{Coh} \to D_!$

- show fully faithfulness
- surjectivity

Make $\mathrm{Perf}^{\check{G}}(S_{G^\vee}) \curvearrowright D_!$

use Bezrukavnikov's result (Arkhipov-Bez.):

G general $\rightsquigarrow$ LG loop group

$$\mathcal{H}_{aff} = D_!(I \backslash LG / I) \underset{\otimes}{\simeq} \mathrm{Coh}^{G^\vee}(\tilde{N}^\vee \overset{R}{\underset{\mathfrak{g}^\vee}{\times}} \tilde{N}^\vee) \xleftarrow{\Delta_*} \mathrm{Coh}^{G^\vee}(\tilde{N}^\vee)$$

This can be viewed as geom. Langlands for I_0, I_∞

Radon

$$D_!(\mathrm{Bun}_G(I_{0,\infty})) \simeq \mathrm{Coh}^{G^\vee}(S^{(2)}_{G^\vee})$$

derived

$$\begin{tikzcd} S^{(2)}_{G^\vee} \arrow[r, hook] \arrow[d] & \tilde{N}^\vee \times \tilde{N}^\vee \arrow[d] \\ \{1\} \arrow[r] & G^\vee \end{tikzcd}$$

$x \in \{0, 1, \infty\}$

$$\mathrm{Perf}^{G^\vee}_{\otimes}(\tilde{\mathcal{N}}^\vee) \longrightarrow \mathcal{H}_{\mathrm{aff}} \underset{x}{\curvearrowright} D_!(\mathrm{Bun}_G(I_{0,1,\infty}))$$

$$\Rightarrow \mathrm{Perf}^{G^\vee}(\tilde{\mathcal{N}}^\vee)^{\otimes 3} \curvearrowright D_!$$

$$\downarrow i^*$$

$$\mathrm{Perf}^{G^\vee}(S_{G^\vee})$$

$\Leftarrow$ calculation of monodromy of the $\mathcal{H}_{\mathrm{aff}}$-action using Gaitsgory's nearby cycles construction.

"Automorphic" structure sheaf

Answer: $\mathrm{Bun}^{\mathrm{even}}_{PGL_2} \supset \mathcal{U}_{\mathrm{triv}} = G \backslash (\mathbb{P}^1)^3$

$\mathcal{O}^{\oplus 2}$

$\mathrm{Bun}^{\mathrm{odd}}_{PGL_2} \supset V$

$\mathcal{O} \oplus \mathcal{O}^{-1}$

$$\circledast \xrightarrow{j_0} (\odot \ \circledast) \xrightarrow{j_1} \mathrm{Bun}_G^{\mathrm{odd}}$$

$$\mathcal{A} = j_{1,!}\, j_{0,*}\, \mathbb{Q}$$

this corresponds to $\mathcal{O}_{S_{G^\vee}}$ under geom Langlands

$\wr$

Rozenblyum

Higher differential operators

X variety. Can think of D_X = alg. of differential operators as a quantization of T^*X

shifted symplectic structures

$T^*[n]Y$

X is "n-oriented" then

$Maps(X, T^*[n]Y) \simeq T^* Maps(X, Y)$

want to study D-modules on $Bun_G = Maps(C, BG)$

Q: what is the quantization of $T^*[n]Y$?

Ex: $X = S^1$

① $Maps(X, T^*Y) = HH_\bullet(T^*Y)$

quantized version

$T^*Y \rightsquigarrow D_Y$ $\qquad$ dim $Y = n$

$$\int_{S^1} D_Y = HH_*(D_Y) = H_{DR}^{2n-*}(Y) \simeq H_{dR}^{2n-*}(T^*Y)$$

Higher dimensional version of this story.

Idea: Work locally on X

$$\mathrm{Maps}(X,Y) = \mathrm{Sect}\left(X, \begin{array}{c} X\times Y \\ \downarrow \\ X \end{array}\right) = \mathrm{Sect}_\nabla\left(X, \begin{array}{c} \mathrm{Jets}(Y) \\ \downarrow \\ X_{dR} \end{array}\right)$$

flat sections

$$\begin{array}{c} \mathrm{Jets}(Y) \\ \downarrow \\ X_{dR} \end{array} \quad D\text{-space}$$

fiber at x: $\mathrm{Maps}(D_x, Y)$

Framework

$$\begin{array}{c} Z \\ \downarrow \\ X_{dR} \end{array} \quad D_X\text{-space}$$

Want to study $\mathrm{Sect}_\nabla(X,Z)$

Ansatz: A quantization of $T^*[n]Y$ on X should be a factorization algebra on X

Rough def: A factorization algebra is a D-module on $\mathrm{Ran}(X)$ w/ compatible factorization isos

$$A_{S\amalg S'} \simeq A_S \otimes A_{S'}$$

Fact: (Loc. constant) factorization algebras on $\mathbb{R}$ $\simeq$ associative algebras.

$$\begin{array}{ccc} Z & & Z_{(\mathrm{Ran})} \\ \downarrow & \rightsquigarrow & \downarrow \\ X_{dR} & & \mathrm{Ran}(X)_{dR} \end{array}$$

$$Z_{(\mathrm{Ran})_S} = \mathrm{Sect}_{\nabla}(\hat{X}_S, Z)$$
$S \subset X$

$$\begin{array}{ccc} \mathrm{Sect}_{\nabla}(X, Z) \times \mathrm{Ran}(X)_{dR} & \xrightarrow{ev} & Z_{(\mathrm{Ran})} \\ \downarrow p & & \\ \mathrm{Sect}_{\nabla}(X, Z) & & \end{array}$$

This gives a localization functor

$$\mathrm{Loc}^{\mathcal{O}} := p_* ev^! : \mathrm{IndCoh}^!(Z_{(\mathrm{Ran})}) \to \mathrm{IndCoh}(\mathrm{Sect}_{\nabla}(X, Z))$$

Goal: upgrade this functor to produce D-modules.