Geom. Langlands Summer School 2012 Freiburg, Germany

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G ... cx reductive alg group $/\mathbb{C}$

Aside: (from topology)

$G \rightsquigarrow BG$ classifying space

$\Omega_* X \longleftarrow X$ connected space

$G \supset B \supset T$ (Borel, torus)
$\supset N$ unipotent radical

$T \cong H = B/N$ universal Cartan $\quad W = N(T)/T$ ... Weil group

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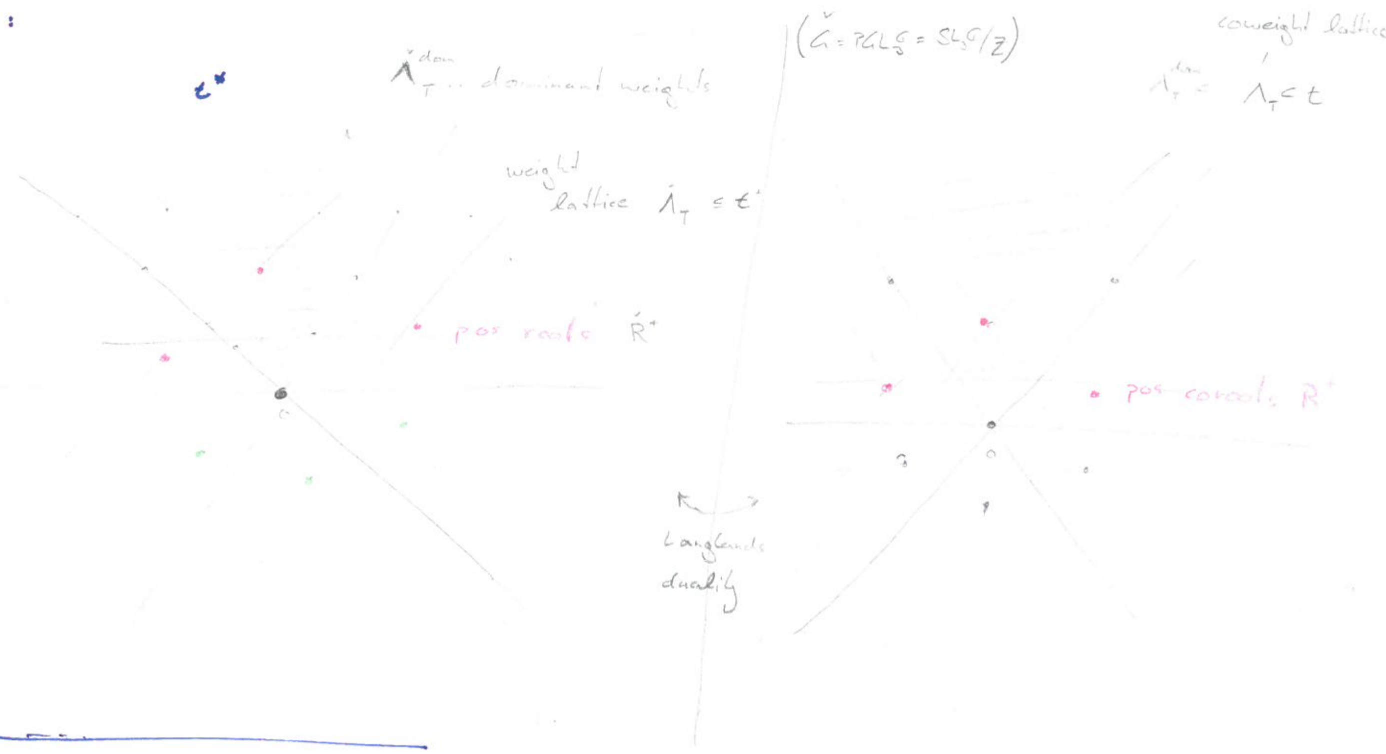
$G = SL_3$:



Towards Hecke-algebras : for G, LG

Want to understand geometry of "coset multiplication" ${}_B\backslash G/_B \times {}_B\backslash G/_B \to {}_B\backslash G/_B$"

What is ${}_B\backslash G/_B$?

G/B ... flag variety.

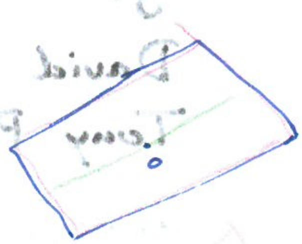
${}_B\backslash(G/B)$ .. B-equivariant geometry of G/B

Ex: Show that ${}_B\backslash G/B = BB \times_{BG} BB$ ($\rightsquigarrow$ $X \times_Y X$ with $X \to Y$ proper)

ictures of ${}_B\backslash G/B$

$G = SL_3$:

G/B

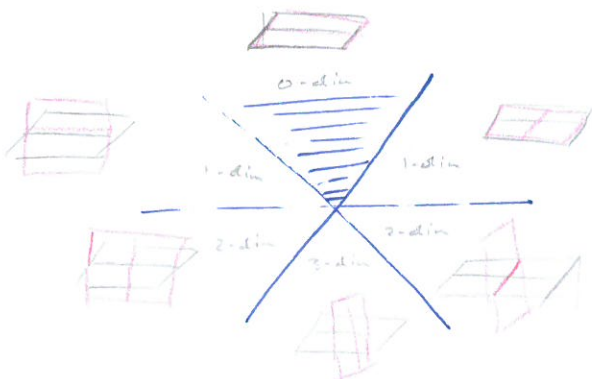


any flag

${}_B\backslash G/B$

std reference flag

B-equiv ~ up to rel pos. w/ std reference flag



$\rightsquigarrow \overset{|W|}{6}$ points

Two questions:

1) How are the w points of ${}_B\backslash G/B$ put together?

points of ~~G~~ ${}_B\backslash G/B$ $\leftrightsquigarrow$ B-orbits in G/B
Schubert cells

Cells interaction: Kazhdan-Lusztig theory

Ex: Show for SL_3 all ~~Sch~~ Schubert cell closures (= Schubert varieties) are smooth.
What are they?

2) what does group mult. look like in terms of ${}_B\backslash G/B$.

Loop groups:

LG = "maps circle $\to G$"

"circle" $D^* = \mathrm{Spec}\, K$, $\quad K = \mathbb{C}((t))$

Def: $LG = G(K)$

Two natural parabolics

max: $L_+G = G(\mathcal{O})$ $\qquad \mathcal{O} = \mathbb{C}[[t]]$ $\qquad D = \mathrm{Spec}\,\mathcal{O}$ "contractible circle"

min: I. (Iwahori)

$$\begin{array}{ccc} L_+G & \xrightarrow{ev_0} & G \\ \cup & & \cup \\ I & \longrightarrow & B \end{array}$$

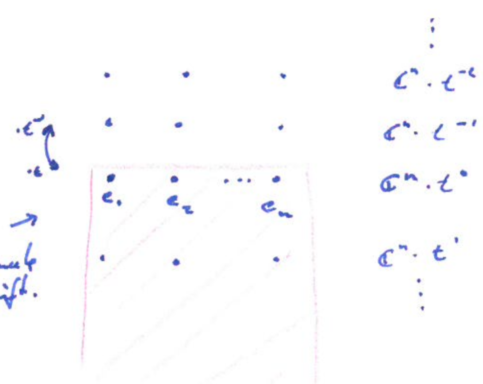
Def: affine flag variety for G: $Fl_G = LG/I$

affine Grassmannian for G. $Gr_G = LG/L_+G$

Note: $Fl_G \to Gr_G$ is a fibration with fiber $G/B = L_+G/I$

How to think about Gr_G? $\qquad G = GL_n$, $LG = GL_n(K)$

$GL_n(K) \circlearrowright K^n$

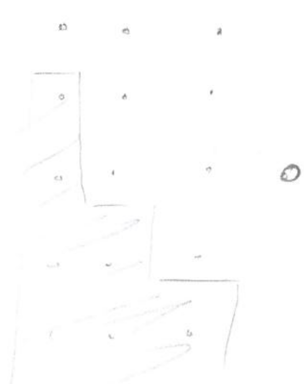


$GL_n(K)$ = auto of $\to$ that commute with shift.

$GL_n(\mathcal{O})$ is precisely what fixes $\mathcal{O}^n$

Prop: $Gr_{GL_n} = \left\{ W \subseteq K^n \ (\mathbb{C}\text{-subsp.}) : \begin{array}{l} 1)\ t \cdot W \subseteq W \\ 2)\ \exists N \gg 0 \text{ s.t. } t^N\mathcal{O}^n \subseteq W \subseteq t^{-N}\mathcal{O}^n \end{array} \right\}$

Ex:

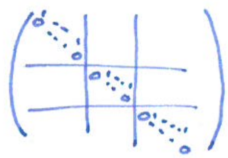


Prop: $Gr_{GL_n} = \bigcup_{k=0}^{\infty} Gr^k_{GL_n}$ with $Gr^k_{GL_n} = \{ W \in Gr_{GL_n} \text{ s.t. } N = k \}$

Prop: $Gr^k_{GL_n}$ is a proj variety

$Gr^2_{GL_n}$

$\swarrow$ subspaces in $t^{-k}\mathcal{O}^n / t^k\mathcal{O}^n = \mathbb{C}^{n(2k)}$

$t =$ 

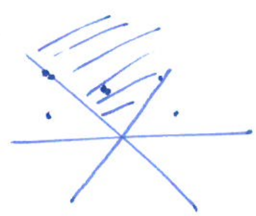
<u>Upshot</u> : Springer fibers!

<u>Double cosets</u> $L_+G \backslash LG / L_+G$, i.e. L_+G-equiv. geometry of Gr_G.

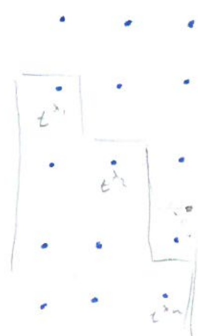
Cartan decomposition:

$$L_+G \backslash LG / L_+G = \Lambda_G^{dom} \text{ dom. coweights}$$
$$= \Lambda_G / W$$

$G = SL_3$

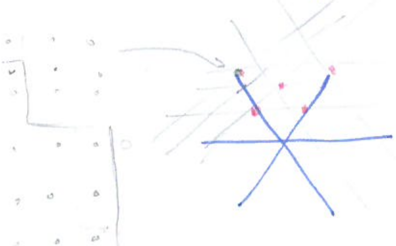


$G = GL_3$:



$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$

<u>Ex</u>: $\lambda \in \Lambda_G^{dom} \rightsquigarrow Gr_G^\lambda \subset Gr_G$ L_+G-orbit through t^λ.
Show Gr_G^λ is a vector bundle over a partial flag variety.
What is the partial flag var? What dim is the vector bundle?



$\rightsquigarrow$ 4-dim vb over $\mathbb{P}^2$

Ex: What is $\cancel{\pi_0(Gr_G)}$ $\pi_0 Gr_G$?

Ex: $Gr_T \simeq \mathbb{Z}^{\dim T}$

Ex: Consider Gr_{GL_2}. Show $\overline{Gr^{\lambda}_{GL_2}}$ is rationally smooth.

One last point of view:

Let $K \subset G$ max compact subgroup.

$$LG \underset{\text{homeo.}}{\simeq} L_+G \times \Omega^{\text{poly}}_{\text{based}} K$$

$$\rightsquigarrow Gr_G \simeq \Omega^{\text{poly}}_{\text{based}} K$$

Ex: $G = PGL_2 \simeq SO_3$

$K \simeq SO_3\mathbb{R}$

$$\Omega^{\text{poly}}_{\text{based}} K \simeq \underbrace{F(S^2)}_{\text{free top gp}} \Big/ \underbrace{(x \cdot a(x) = 1)}_{\text{antipode}}$$

interpret the lecture in terms of this

Lecture II

Categorical linearization of topology:

Geometry:

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & X' \end{tikzcd}$$

Linearization: See "quantum" symmetries

Sheaves: linearize & see full structure of spaces w/ symmetries

Ex: G red. gp $/\mathbb{C}$

$QCoh(BG) = Rep(G)$

Show: You can recover G from $Rep(G)$ (Tanakian reconstruction)

Constructible Sheaves

$Sh(pd) = Sh(pd, \mathbb{C})$

Differential graded bounded derived category

$$C^\bullet = \quad \cdots \to C^{-1} \to C^0 \xrightarrow{d} C^1 \to C^2 \to \cdots \qquad d^2 = 0$$

<u>Obj</u>: Complexes of vector spaces
w/ coh bounded in finite degrees and finite dimensional

<u>Mor</u>: $\mathrm{Hom}(C^\bullet, D^\bullet)$ is hom complex

<u>Derived</u>: Quis become equivalences

X alg variety, S ... finite stratification

0) Strata are smooth
1) $\mathrm{Homeo}(X, S)$ act transitively on each stratum
2) Normal slice to any point in a stratum is a cone over (X', S') of smaller dimension

<u>Examples</u>: 1) Whitney stratifications
2) Alg var w/ group action w/ finitely many orbits.

$Sh_S(X)$ = dg bounded constructible derived category

$\mathcal{F}: \mathrm{Top}(X)^{op} \longrightarrow Sh(p)$

satisfying appropriate form of gluing

on any stratum the cohomology sheaves are locally constant

<u>Naive way</u> to <u>think about</u> $\mathcal{F} \in Sh_S(\underline{X})$.

Dévissage: $\underset{open}{U} \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} V = X \setminus U$

$j_! j^! \mathcal{F} \to \mathcal{F} \longrightarrow i_* i^* \mathcal{F}$ is a d.t.

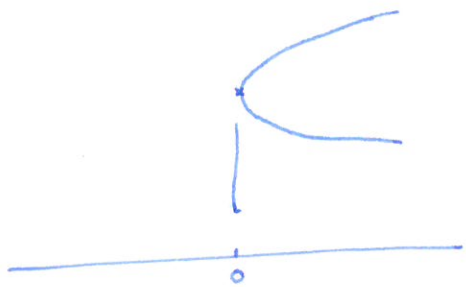
<u>Upshot</u> (applied to S): $\mathcal{F}$ is glued together from complexes of the form $i_{\alpha *} \mathcal{F}_\alpha$,
where $i_\alpha : S_\alpha \hookrightarrow X$, $\mathcal{F}_\alpha \in Sh_{S_\alpha}(S_\alpha)$.

Nearby & vanishing cycles

X alg var $/\mathbb{C}$

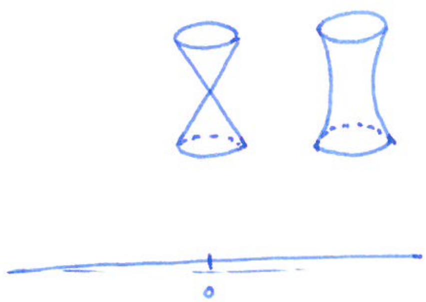
$X \xrightarrow{f} \mathbb{A}^1$

Ex: 0)



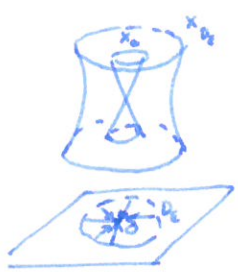
$\mathbb{C} \xrightarrow{f(x) = x^2} \mathbb{C}$

1)



$\mathbb{C}^2 \xrightarrow{f(x,y) = x^2+y^2} \mathbb{C}$

Collapse map: There exists an "appropriately unique retraction" $r : X_{D_\varepsilon} \longrightarrow X_0$



Assume $\mathcal{F} \in Sh(X)$ satisfies $f_*\mathcal{F} \in Sh_S(\mathbb{C})$ where $S = \{\{0\}, \mathbb{C}\setminus 0\}$

~~Def: Given $\mathcal{F} \in Sh(X \setminus X_0)$~~

Def: Nearby cycles of $\mathcal{F}$: $\psi\mathcal{F} = r_*\mathcal{F}|_{X_\delta}$ $\quad \delta \in D_\varepsilon \setminus 0$

monodromy on $\circlearrowright \psi\mathcal{F}$ by moving δ around the circle.

Def: Vanishing cycles of $\mathcal{F}$:

s.t. $\mathcal{F}|_{X_0} \to \psi\mathcal{F} \longrightarrow \underbrace{\varphi\mathcal{F}}_{\text{vanishing cycle}}$

monodromy on $\circlearrowright \varphi\mathcal{F}$ by moving δ around.

$\underline{\text{Ex}}$: 0) $\mathcal{F} = k_{\mathbb{C}} \simeq C^{\bullet}_{\mathbb{C}} \simeq \Omega^{\bullet}_{\mathbb{C}} \cdots$

$$\begin{tikzcd} \mathcal{F}|_{\{0\}} \arrow[r] \arrow[d, equal] & \psi\mathcal{F} \arrow[r] \arrow[d, equal] & \varphi\mathcal{F} \arrow[d, equal] \\ \mathbb{C}_{\{0\}} \arrow[r, "\Delta"] & \mathbb{C}^2_{\{0\}} \arrow[r] \arrow[loop below, "\sigma"] & \mathbb{C}_{\{0\}} \arrow[loop below, "-1"] \end{tikzcd}$$

1) $\mathcal{F} = C^{\bullet}_{\mathbb{C}^2}$

$$\begin{tikzcd} \mathcal{F}|_{X_0} \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \psi\mathcal{F} \arrow[r] \arrow[loop below, "m"] & \varphi\mathcal{F} \arrow[d, phantom, "\cong" rotate=90] \\ C^{\bullet}_{X_0} & & \mathbb{C}_{\{0\}}[-1] \end{tikzcd}$$

$\underline{\text{Perverse Sheaves}}$: $P_{\mathcal{S}}(X) \subseteq Sh_{\mathcal{S}}(X)$

Data of the measurement : $x \in X$, $f : X \to \mathbb{C}$,
$f(x) = 0$, $df|_x$ generic

measurement: $\varphi\mathcal{F}|_x = M_{x,\xi}(\mathcal{F})$ $\qquad \xi = df|_x$

$P_{\mathcal{S}}(X)$ = full subcat of obj s.t. $M_{x,\xi}(\mathcal{F})$ are concentrated in degree 0.

$\underline{\text{Ex}}$: Describe $Sh_{\mathcal{S}}(\mathbb{C})$ and $P(\mathcal{S})$, $\mathcal{S} = \{\{0\}, \mathbb{C}\setminus\{0\}\}$
in terms of measurements

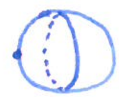
Lecture III

Schubert stratifications : $\quad {}_B\circlearrowright G/B \quad , \quad L_+G \circlearrowright Gr_G \quad , \quad I \circlearrowright Fl_G$

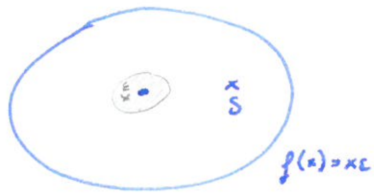
Orbit poset : $\quad W_G \quad , \quad \Lambda_G^{dom} \quad , \quad W_G^{aff}$

Ex : $G = SL_2$, $B \circlearrowright SL_2/B = \mathbb{P}^1$

$P_S(\mathbb{P}^1) = ?$



First : $P_S(\mathbb{A}^1) \ni \mathcal{F}$



Two measurements:

$$M_{0,\varepsilon}(\mathcal{F}) \underset{w}{\overset{\partial}{\rightleftarrows}} \mathcal{F}|_s[-1]$$

generic microlocal stalk

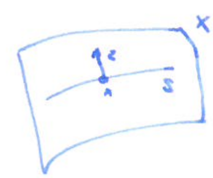
$$\xleftarrow{\ \partial[1]\ } \Gamma(\{\varepsilon\},\mathcal{F}) \longleftarrow \Gamma(D,\mathcal{F}) \longleftarrow M_{0,\varepsilon}(\mathcal{F}) = \Gamma((D,\varepsilon),\mathcal{F})$$

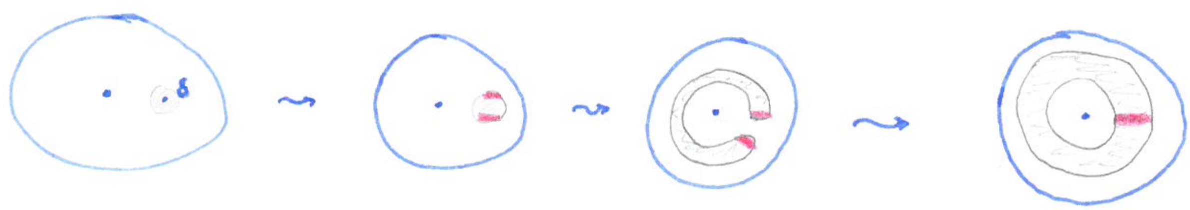
Corrected definition of microlocal stalk

For $x \in S \subseteq X$ stratum

$f : X \to \mathbb{C}$, $f(x) = 0$, $\varepsilon = df|_x$ generic in T_S^*X

quadratic part of $df|_S$ is nonsing at $x \in S$





$\mathcal{F}|_S[-1] \simeq \Gamma((\ ,\ ),\mathcal{F})$

$\simeq \Gamma((\ ,\ -),\mathcal{F})$

w is the corresponding restriction map

Relations

Ex : $w \circ \partial : \mathcal{F}|_S[-1] \longrightarrow \mathcal{F}|_S[-1]$, $w \circ \partial = 1 - \text{monodromy}$

in particular, $1 - w\circ\partial$ is invertible.

$\partial \circ w : M_{0,\delta}(\mathcal{F}) \longrightarrow M_{0,\delta}(\mathcal{F})$, $\partial \circ w = 1 - \text{monodromy}$

$\to 1 - \partial\circ w$ is invertible $\qquad$ of vanishing cycles

Ex : $P_S(\mathbb{A}^1) = \mathrm{Rep}\left(\bullet \underset{w}{\overset{\partial}{\rightleftarrows}} \bullet ,\ 1-\partial\circ w,\ 1-w\circ\partial \text{ are invertible} \right)$

Next: $P_S(\mathbb{P}^1) = \mathrm{Rep}\left(\bullet \underset{w}{\overset{\partial}{\rightleftarrows}} \bullet \,,\ \begin{matrix} 1-\partial w \\ 1-w\partial \end{matrix} \text{ invertible}, \ldots ? \ldots \right)$

monodromy has to be trivial $\Rightarrow$ $1-w\partial = 1$ in the additional condition

Indecomposable objects

2 irreducibles :

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend right, "0"'] & 0 \arrow[l, bend right, "0"'] \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, bend right, "0"'] & \mathbb{C} \arrow[l, bend right, "0"'] \end{tikzcd}$$

$$\updownarrow \qquad\qquad\qquad \updownarrow$$

$$\mathbb{C}_{\{0\}} \qquad\qquad \mathbb{C}_{\mathbb{P}^1}[1]$$

3 others :

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "\sim"] & \mathbb{C} \arrow[l, bend left, "0"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C} \arrow[r, bend left, "0"] & \mathbb{C} \arrow[l, bend left, "\sim"] \end{tikzcd}$$

$$T = \begin{tikzcd} \mathbb{C} \arrow[r, bend right, "i_2"'] & \mathbb{C} \oplus \mathbb{C} \arrow[l, bend right, "p_1"'] \end{tikzcd}$$

Next: Indecomposable B-equivariant

All except T

Def: $\mathcal{H}_G = Sh_B(G/B)$

$\mathcal{H}_{LG} = Sh_I(LG/I)$

$\mathcal{H}^{sph}_{LG} = Sh_{L_+G}(LG/L_+G)$

Aside: $H \circlearrowright X$ $\qquad$ "$_H\backslash^X$" $\leftarrow X \leftleftarrows H \times X \leftleftarrows\!\!\!\leftarrow \cdots$

$Sh_H(X) \longrightarrow Sh(X) \rightrightarrows Sh(H\times X) \rightrightarrows\!\!\!\rightarrow \cdots$

$\quad \parallel$

$\lim$ [cosimplicial diagram $\uparrow$] $\quad \lrcorner$

Why are we able to calculate convolution?

Decomposition Thm: $f: X \to Y$ proper map of cx varieties

$$f_* IC_{X^\circ, \mathcal{L}^\circ} \cong \bigoplus_{Y_i, \mathcal{L}_i, n_i} IC_{Y_i, \mathcal{L}_i}[n_i]$$

$X^\circ \subset X$ smooth subvariety
$\mathcal{L}^\circ$ trivial local system.

$P(X)$ Abelian category

Def: simple objects $IC_{Y,\mathcal{L}}$

$Y \subseteq X$, $\mathcal{L}$ ... irred local system on Y
smooth subvariety

$P_S(X) \rightsquigarrow IC_{S_\alpha, \mathcal{L}_\alpha}$

$S_\alpha \subset X$ stratum

Convolution:

Generalities

$$\begin{tikzcd} & X \times_Y X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[d] \arrow[dr] & & X \arrow[dl] \\ Y & Y & \end{tikzcd}$$

"Endomorphisms of X over Y"

Sheaves:

$$\begin{array}{ccc} & Sh(X \times_Y X) & \\ & \Psi & \\ Sh(X) & K & Sh(X) \\ \Psi & & \\ \mathcal{F} & & \end{array}$$

$\Psi_K(\mathcal{F}) = p_{2!}(p_1^* \mathcal{F} \otimes K)$ $\rightsquigarrow$ $Sh(X \times_Y X) \longrightarrow \mathrm{End}_{Sh(Y)}(Sh(X))$

Hecke categories $X = BB$, $Y = BG$

$X \times_Y X = {}_B\backslash^G/_B$

$\mathcal{H}_G$

Def: Convolution on $Sh({}_B\backslash^G/_B)$ is the product given by composition of kernels.

We think about $\mathcal{H}_G$ as a monoidal category.

Lecture IV

Full dg category of sheaves $\supseteq$ Abelian category of perverse sheaves $\supseteq$ Basis of simple perverse sheaves

	$\mathcal{H}_G$	$\mathcal{H}_{LG}$	$\mathcal{H}_{LG}^{sph}$
Index for basis	W fin. Weyl gp	W_{aff} aff. Weyl gp	$\Lambda_G^{dom} = \Lambda_G/W$ dom. coroots
	IC_w	IC_w	IC_λ

Characterization of $IC_{S,\mathcal{L}}$, $S \subset X$ stratum , $\mathcal{L}$ – S

for all strata $S' \subset \bar{S}$, $S' \neq S$, $x \in S'$

$i_x^* IC_{S,\mathcal{L}}$ concentrated in degrees $< -\dim S'$

$i_x^! IC_{S,\mathcal{L}}$ —— " —— $> \dim S'$

and $IC_{S,\mathcal{L}}|_S \simeq \mathcal{L}[\dim S]$

Exc: $P_S(\mathbb{P}^1)$ $S =$ Schubert strat. $\{0\}$, $\mathbb{P}^1 \setminus \{0\}$

$$\simeq \operatorname{Rep}\left(\begin{tikzcd} \bullet \arrow[r, bend right, "q"'] & \bullet \arrow[l, bend right, "p"'] \end{tikzcd},\ 1-pq,\ 1-qp \text{ invertible},\ 1-pq = 1 \right)$$

$IC_{\{0\}} = \mathbb{C}_{\{0\}}$, $IC_{\mathbb{P}^1} = \mathbb{C}_{\mathbb{P}^1}[1]$

$$\bullet \quad \overset{\mathbb{C}}{\bullet} \qquad , \qquad \overset{\mathbb{C}}{\bullet} \quad \bullet \qquad , \qquad \begin{tikzcd} \mathbb{C} \bullet \arrow[r, bend left, "0"] & \bullet \mathbb{C} \arrow[l, bend left, "\sim"] \end{tikzcd} \quad , \quad \begin{tikzcd} \mathbb{C} \bullet \arrow[r, bend left, "\sim"] & \bullet \mathbb{C} \arrow[l, bend left, "0"] \end{tikzcd}$$

What are stalks and costalks?

Convolution

$$(X \times_Y X) \times (X \times_Y X) \xleftarrow{p_{12} \times p_{23}} X \times_Y X \times_Y X \xrightarrow{p_{13}} X \times_Y X$$

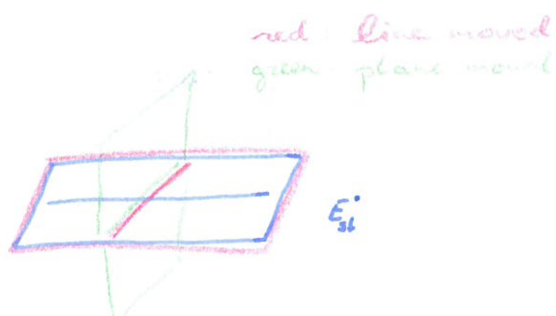
$X = BB$
$Y = BB$

$$B\backslash G/B \times B\backslash G/B \xleftarrow{p} B\backslash G \times_B G/B \xrightarrow{q} B\backslash G/B$$

Monoidal structure: $\mathcal{H}_G \times \mathcal{H}_G \to \mathcal{H}_G$

$$(\mathcal{F}, \mathcal{G}) \longmapsto \mathcal{F} * \mathcal{G} = q_! p^* (\mathcal{F} \boxtimes \mathcal{G})$$

Warmup: Cartoon of "multiplying" double cosets.



Ni

1) $\mathcal{H}_G = Sh({}_B\backslash G/B) \simeq Sh_B(G/B)$, $G = SL_2$

$IC_e \simeq \mathbb{C}_{\{0\}}$ ... unit in $\mathcal{H}_G$

$IC_s = \mathbb{C}_{\mathbb{P}^1}[1]$

$IC_s * IC_s$

$$IC_s * IC_s = IC_s^{[1]} \otimes H^*(\mathbb{P}^1)$$
$$= IC_s^{[1]} \oplus IC_s[-1]$$

Note: good news: "trivial" case of the decomp. theorem

simple basis $\rightsquigarrow$ shifts of simple basis

bad news: there are shifts

2) $\mathcal{H}_{SL_3}$ $IC_{s_1} * IC_{s_2}$ vs $IC_{s_2} * IC_{s_1}$

s_1... move line

s_2... move plane

for each green a unique red

$\rightsquigarrow = IC_{s_1 s_2}$ $= IC_{s_2 s_1}$

bad news: non-commutative

Thm (Geometric Satake)

Miracle 1: $\mathcal{H}_{LG}^{sph}$ convolution preserves perverse sheaves

$\rightsquigarrow \exists\ \mathcal{P}_{LG}^{sph}$ Abelian monoidal category

Miracle 2: $\mathcal{P}_{LG}^{sph}$ is a tensor category (i.e. commutative monoidal cat.)

Thm: $\mathcal{P}_{LG}^{sph} \simeq Rep_{fd}(\check{G})$

Ex: $G = GL_1$

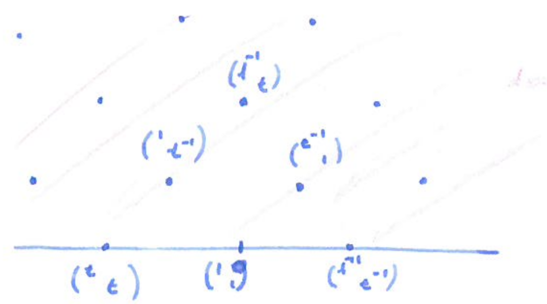
$Gr_{GL_1} = \mathbb{Z} \simeq \Lambda_{GL_1}$

$P^{sph}_{LGL_1} \simeq Vect_{\mathbb{Z}} \simeq Rep(GL_1)$

convolution given by group addition in $\mathbb{Z}$ $\longleftrightarrow$ tensor in $Rep(GL_1)$

Ex: $G = GL_2$

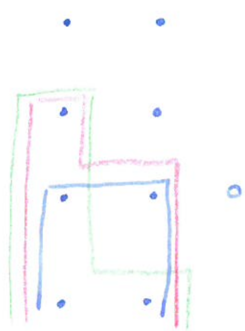
spherical double cosets $\Lambda^{dom}_{GL_2} \subset \Lambda_{GL_2} = Hom(\mathbb{C}^*, T)$



Calculate: $IC_\lambda * IC_\mu \underset{Satake}{\cong}$

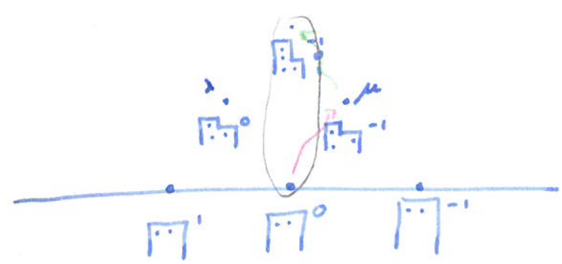
$\left(\text{Satake } V_\lambda \otimes V_\mu \simeq V_{\lambda+\mu} \oplus V_0 \right)$

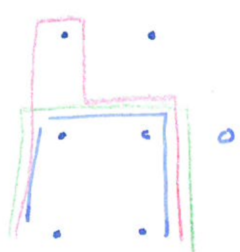
$\rightsquigarrow \; = IC_{\lambda+\mu} \oplus IC_0$



$S_\lambda = \mathbb{P}^1$

$S_\mu = \mathbb{P}^1$

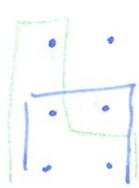




$\underset{S_\lambda}{\mathbb{P}^1} \tilde{\times} \underset{S_\mu}{\mathbb{P}^1} \longrightarrow \overline{S}_{\lambda+\mu} = S_{\lambda+\mu} \sqcup S_0$

$S_{\lambda+\mu}$ = line bundle over $\mathbb{P}^1$

$\underset{Ex}{\simeq} T^*\mathbb{P}^1$

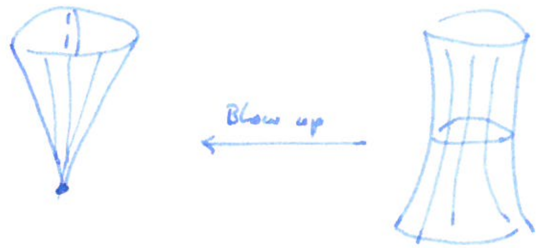


$S_0 = pt$

$\overline{S}_{\lambda+\mu}$ = singular quadric $\subset \mathbb{P}^3$ $\quad x_0^2 + x_1^2 + x_2^2 = 0$

$\cancel{x_0^2 + y^2 + z^2 = 0}$

$S_\lambda \tilde{\times} S_\mu \cong \mathbb{P}^1 \tilde{\times} \mathbb{P}^1$ (compactified) Springer resolution for SL_2



What complexes: $P^{sph}_{LG} \simeq \mathrm{Rep}_{fd}(\check{G})$

$T \subset G$ max torus $\quad \downarrow ? \qquad \downarrow \mathrm{res}^G_T$

$P^{sph}_{LT} \simeq \mathrm{Rep}_{fd}(\check{T})$

$N(K) \circlearrowright Gr_G \qquad \frac{\infty}{2}$-orbits

Iwasawa: $N(K)\backslash LG/L_+G \simeq \Lambda_G$

Notation: $N_\lambda \subseteq Gr_G$ ~~orbit~~

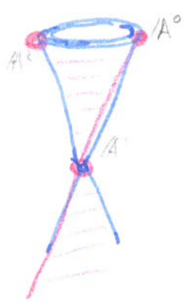
$\quad \shortparallel$

$N(K)\cdot\lambda$

$$Gr_G \xleftarrow{p} \coprod_{\lambda \in \Lambda_G} N_\lambda \xrightarrow[q]{N_\lambda \mapsto \lambda} \cancel{\{\lambda\}} \in Gr_T$$

Prop: res^G_T corresponds to $q_! p^*$ [up to shifts]

Ex: $\overline{S_{\lambda+\mu}} \subset Gr_{GL_2}$



Lecture V

Bundle interpretation

Warmup: $B\backslash G/B \simeq BB \times_{BG} BB$

BG = G-bundles on a pt
$= [pt/G]$

$G = pt \times_{BG} pt$ = G-bundle on point + 2 triv.

G/B = G-bundle + triv + red to B
$= pt \times_{BG} BB$

= G-bundle on pt + 2 reductions to B
= 2 B-bundles on pt + ident. of induced G-bundles

Now to LG:

LG = G-bundle on $D^\times$ with 2 trivs.

$Gr_G = LG/L_+G$ = "G-bundle on $D^\times$ + triv + triv that extends over D"
= G-bundle on D + triv over $D^\times$
= G-bundle on $D^\times$ + triv over $D^\times$ + ~~bundle extends over D~~ extension over D

$L_+G\backslash LG/L_+G$ = 2 G-bundles on D + identification over $D^\times$

Exc: Gr_G = G-bundle on C + triv away from $c \in C$.

Exc: $Bun_G^{G\text{ s.s.}}(C) \simeq \Gamma_c\backslash LG/L_+G$, $c \in C$, $\Gamma_c = Maps(C \setminus c, G)$

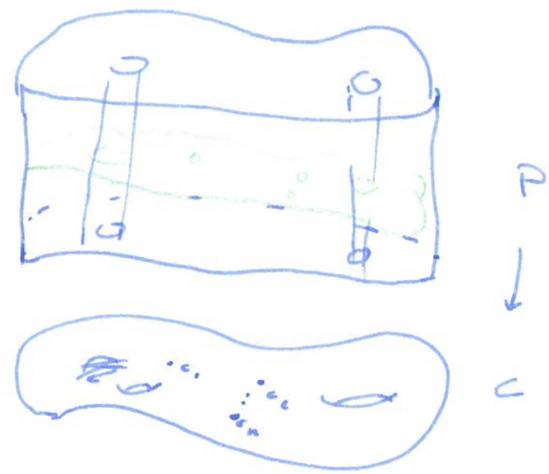
$\circlearrowleft$

$L_+G\backslash LG/L_+G$

Use bundle interpretation to reformulate convolution.

Beilinson – Drinfeld Grassmannian: Fix curve C, $n \geq 0$

$Gr_G^{(n)}$ = G-bundle on C + n pt $c_1, \dots, c_n \in C$
+ triv of bundle on $C \setminus \{c_1, \dots, c_n\}$



$$\begin{array}{ccc} Gr_G^{(n)} & \longrightarrow & C^n \\ \cup & & \cup \\ Gr_G^{(n)}\big|_{\underline{c}} & \longrightarrow & \underline{c} = (c_1, \dots, c_n) \end{array}$$

$|\underline{c}| = \#$ of distinct points

Exc: is

$Gr_G^{|c|} = |c|$ - copies of Gr_G

Grassmannian description of $Gr_G^{(n)}$

$C = \mathbb{A}^1, \quad n = 2, \quad G = GL_n$

$(\mathbb{A}^1)^2 = \mathbb{A}^2$

$\mathbb{C}$-subspaces ^{W} closed under t, s & contain V_c

Exc: $\varepsilon \neq 0 \rightsquigarrow Gr \times Gr$

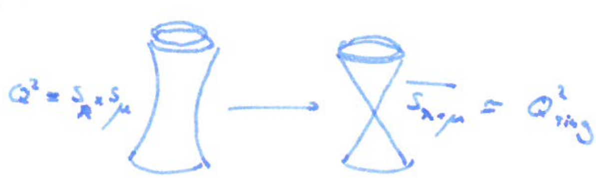
$\varepsilon = 0 \rightsquigarrow Gr$

Ex: $G = GL_2$, $S_\mu = \mathbb{P}^1 \subset Gr$, $S_\lambda = \mathbb{P}^1 \subset Gr$

$S_\mu \times S_\lambda \times (\mathbb{A}^1 \setminus 0) \subseteq Gr^{(2)}\big|_{\mathbb{A}^1 \setminus 0}$

what is family as $\varepsilon \to 0$

Exc: gives $\overline{S_{\lambda+\mu}}$



Thm: Convolution in $\mathcal{P}^{sph}_{LG}$ $\xleftrightarrow{\text{equiv.}}$ nearby cycles along collapse maps in B-D grassmannian

$$IC_\lambda * IC_\mu \simeq \bigoplus_\nu IC_\nu \qquad \psi\left(IC_\lambda \boxtimes IC_\mu \times \mathbb{C}_{\mathbb{A}^1\setminus 0}\right)$$

Rmk: 1) ψ preserves perverse sheaves in general
$\rightsquigarrow$ convolution does so to

2) points on $\mathbb{C}$ can move around each other
$\rightsquigarrow$ convolution is commutative

Finally: Intro to affine Hecke category $\mathcal{H}_{LG} = Sh\left({}_I\backslash^{G}/_I\right)$.

$\mathcal{H}_{LG}$ are kernels of modifications of G-bundles over C + reduction to B at pt $c \in C$

Structure 1: $\mathcal{Z}: \mathcal{P}^{sph}_{LG} \longrightarrow \mathcal{H}_{LG}$
central fun

Consider the moduli: $\widetilde{Fl}_G$ = G-bundle over C
point $c \in C$
triv. away from c
flag (red. to B) at c_0

$\widetilde{Fl}_G \longrightarrow C$

$Gr_G \times G/B \longrightarrow c \neq c_0$

$Fl_G \longrightarrow c = c_0$

$$\mathcal{Z}(IC_\lambda) \simeq \psi\big(IC_\lambda \times \underset{\substack{\uparrow \\ pt \in G/B}}{\mathbb{C}_{B/B}} \times \mathbb{C}_{C\setminus\{c_0\}}\big).$$

Structure 2: Spectral interpretation of $\mathcal{H}_{LG}$

$$\begin{tikzcd}
\mathcal{H}_{LG} \arrow[r, "\simeq", "\text{monoidal equiv}"'] & Coh\left(St_{\check{G}}/\check{G}\right) \\
\mathcal{P}^{sph}_{LG} \arrow[u, bend left, "m \circlearrowright Z \text{ monodromy}"] \arrow[r, "\text{Satake}", "\sim"'] & Coh(B\check{G}) \arrow[u, bend right, "\circlearrowright \text{ universal endo. of } \mathcal{V}"']
\end{tikzcd}$$

$St_{\check{G}} = \tilde{\mathcal{N}} \underset{\check{\mathcal{N}}}{\times} \tilde{\mathcal{N}} \circlearrowleft \check{G}$

Formulation of the GLC

Background data:

C — smooth proj. curve$/\mathbb{C}$, genus $g>1$

G — cx reductive gp.

$T \subset B \subset G$ fixed

$\mathrm{char}(T) = \mathrm{Hom}(T, \mathbb{C}^*)$

$\mathrm{cochar}(T) = \mathrm{Hom}(\mathbb{C}^*, T)$

$\mathrm{roots} \in \mathrm{char}(T) \subset \mathrm{weights}$

$\mathrm{coweights} \subset \mathrm{cochar}(T) \subset \mathrm{coroots}$ $(\mathrm{co}\ldots = \mathrm{Hom}(\ldots, \mathbb{Z}))$

Two reductive gps $G, {}^LG$ are Langlands dual if $\mathrm{char}(G) = \mathrm{cochar}({}^LG)$

Moduli objects:

$\mathcal{B}un$ = moduli stack of principal alg G-~~bundle~~ bundles on C

$\mathcal{L}oc$ = moduli stack of principal alg ~~connections~~ G-bundles on C equipped w/ alg. connections

Aside: If $V \xrightarrow{p} C$ a principal G-bundle on C

we can look at

$$0 \to T_p \to T_{\mathrm{tot}(V)} \xrightarrow{dp} p^* T_C \to 0$$

and the direct image

$$0 \to p_* T_p \to p_* T_{\mathrm{tot}(V)} \to p_* p^* T_C \to 0$$ because ~~ff~~ $V \to C$ is affine fibration

pass to G-invariants

$$0 \to \underset{\substack{\| \\ V \times_{\mathrm{ad}} \mathfrak{g}}}{(p_* T_p)^G} \to \underset{\substack{\| \\ \mathcal{A}(V)}}{\underbrace{(p_* T_{\mathrm{tot}(V)})^G}} \xrightarrow{dp} \underset{\substack{\| \\ T_C}}{(p_* p^* T_C)^G} \to 0$$

Atiyah alg. of V

This is the Atiyah seq. of V: $0 \to \mathrm{ad}\, V \to \mathcal{A}(V) \to T_C \to 0$

An alg connection on V is a splitting of this sequence

$$0 \to \mathrm{ad}\, V \to \mathcal{A}(V) \to T_C \to 0$$

∇

∇ is flat if ∇ is a map of Lie algebras. (automatic over curves)

Geometric properties of these moduli

(1) Bun is a smooth alg. stack.

If G - semisimple : ~~dim~~ $\dim \mathrm{Bun} = (\dim G)(g-1)$

(2) Bun is not quasi-compact and is not of finite type.

(3) Bun^{ss} = substack of semi-stable G-bundles $\subseteq$ Bun - open and dense

is quasi-compact, if we fix c_1.

Note: $\pi_0 \mathrm{Bun}$ is coarsely representable by a pt.

$\pi_0 \mathrm{Bun}^{ss}_{c_1 = a}$ —— " —— by a proj variety

(when G semisimple $\dim \pi_0 \mathrm{Bun}^{ss}_{c_1=a} = (\dim G)(g-1)$)

(4) Loc is an alg. stack of finite type & quasi-compact.

Note: $(V, \Delta) \in \mathrm{Loc}$, then the Atiyah seq for V is split.

$$(0 \to \mathrm{ad} V \to A(V) \to T_C) \in \mathrm{Ext}^1_C(T_C, \mathrm{ad}(V))$$

$$\| $$

$$H^1(C, \mathrm{ad}(V) \otimes \Omega^1_C)$$

$$\swarrow \mathrm{tr}$$

$$c_1(V) = \mathrm{tr}(A(V)) \qquad H^1(\Omega^1_C)$$

$$\Rightarrow c_1(V) = 0$$

Hint for ex : G reductive

$\dim \mathrm{Bun} = \dim G (g-1) + \dim Z(G)$

$= \dim G_{ad} (g-1) + \dim Z(G) \cdot g.$

If G is semisimple, $\dim \mathrm{Loc} = 2 \dim G (g-1)$

(5) Loc is not smooth in general.

However Loc is a l.c.i.

Rem: (V, Δ) is the same thing as a homomorphism $\pi_1(C, x) \to G$

RH - corresp. $\exists$ analytic iso $\mathrm{Loc}^{an} \xrightarrow{\sim}$ (stack of reps of $\pi_1(C,x) \to G$) $=: \mathcal{M}_B(C, G)$

$(V, \Delta) \longrightarrow \mathrm{mon}_x(\nabla)$

Loc and $\mathcal{M}_B$ are both alg. stacks of finite type, but not iso ~~as~~ as alg. stacks

Note: $\mathcal{M}_B$ does not depend on C as an alg. stack (only on genus)

Loc does depend on C as an alg. stack $\qquad \mathcal{M}_g \to$ (moduli of Loc) is injective.

First iteration on the Geometric Langlands Conjecture (Drinfeld's best hope)

There is a canonical equivalence of categories

$$D_{qcoh}(\mathcal{L}oc, \mathcal{O}) \xrightarrow[\cong]{c} D({}^L\mathcal{B}un, \mathcal{D})$$

so that c intertwines the natural symmetries on both sides.

symmetries of $D_{qcoh}(\mathcal{L}oc, \mathcal{O})$:

Tensorization (Wilson) operators.

These are labeled by $x \in C$, $\rho \in \mathrm{Hom}(G, GL(E)) \leftrightarrow \rho^{\mu}$, $\mu \in \mathrm{char}^+(G)$

Note: We have a universal bundle with connection $(\mathcal{V}, \Delta) \longrightarrow \mathcal{L}oc \times C$

Then $\mathcal{V}$ defines $\mathcal{V}_x = \mathcal{V}|_{\mathcal{L}oc \times \{x\}}$ — principal G-bundle on $\mathcal{L}oc$.

$\rho(\mathcal{V}_x) = \mathcal{V}_x \times_G E$ = assoc. v. bundle on $\mathcal{L}oc$

Def: the tensorization operator assoc. to (x, ρ) is

$$\begin{aligned} W^{x,\rho} : D_{qcoh}(\mathcal{L}oc, \mathcal{O}) &\longrightarrow D_{qcoh}(\mathcal{L}oc, \mathcal{O}) \\ \mathcal{F} &\longmapsto \mathcal{F} \otimes \rho(\mathcal{V}_x) \end{aligned}$$

Symmetries on $D(\mathcal{B}un, \mathcal{D})$: Hecke operators

Labeled by $x \in C$, $\rho \in \mathrm{Hom}(G, GL(E))$

There is a natural geometric correspondence

$$\begin{tikzcd} & {}^L\mathcal{H}eck \arrow[dl, "{}^Lp"'] \arrow[dr, "{}^Lq"] & \\ {}^L\mathcal{B}un & & {}^L\mathcal{B}un \times C \end{tikzcd}$$

${}^L\mathcal{H}eck$ = moduli stack of $(V, V', x, \beta : V|_{C \setminus \{x\}} \xrightarrow{\cong} V'|_{C \setminus \{x\}})$

${}^Lp(V, V', x, \beta) = V$

${}^Lq(V, V', x, \beta) = (V', x)$

Properties of ${}^L\mathcal{H}eck$: ${}^L\mathcal{H}eck$ is not an alg. stack, but it is ind-alg.

Fix $x \in C$, get substack

$$\begin{tikzcd} & {}^L\mathcal{H}eck \arrow[dl, "{}^Lp^x"'] \arrow[dr, "{}^Lq^x"] & \\ {}^L\mathcal{B}un & & {}^L\mathcal{B}un \times \{x\} \end{tikzcd}$$

${}^L\mathcal{H}eck^x$ is a locally trivial fibration on ${}^L\mathcal{B}un$ (via either ${}^Lp^x$ or ${}^Lq^x$). The fibers are Gr_G.

$\hookrightarrow$ ind-scheme

Rmk: "Hecke" is formally smooth but not smooth as an ind-stack.

In fact Gr_G is formally smooth (but not smooth) as an ind-scheme.

This follows from $G((t))$ being not smooth as an ind-scheme.

Aside: An ind-scheme is $\mathfrak{X} = \varinjlim_{a \in A} X_a$, where X_a is (quasi-compact) scheme.

A .. poset

$X_a \overset{\text{closed}}{\hookrightarrow} X_b$ for $a \le b$.

$h_{\mathfrak{X}} : (Sch)^{op} \to Sets$: $h_{\mathfrak{X}} = \varinjlim_{a \in A} h_{X_a}$.

$\mathfrak{X}$ is called smooth at a point if locally near that pt, $\mathfrak{X} = \varinjlim_{a \in A} U_a$, U_a smooth.

— " — formally smooth if $h_{\mathfrak{X}}$ satisfies $h_{\mathfrak{X}}(T) \twoheadrightarrow h_{\mathfrak{X}}(S)$ for $S \subset T$ infinitesimal thickening.

Thm (Simpson - Teleman)

$G((t))$ is not smooth.

Thm (Simpson - Teleman)

If $\mathfrak{X}$ is a proj ind-scheme which is smooth at each pt, then the Hodge-to-deRham spectral sequence on $\mathfrak{X}$ degenerates at E_1

Thm (Fishel - Grojnowski - Teleman)

$H^2_{dR}(G((t)), \mathbb{C}) = \mathbb{C}$

$H^1_{dR}(G((t)), \Omega^1_{G((t))}) = H^1(\mathfrak{g}((t)), \mathfrak{g}; \mathfrak{g}[[t]]\, dt) \cong \mathfrak{g}$.

Lecture II

We had

$$\begin{tikzcd} & {}^L\mathcal{H}ecke \arrow[dl] \arrow[dr] & \\ {}^L\mathrm{Bun} & & {}^L\mathrm{Bun} \times C \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & {}^L\mathcal{H}ecke_x \arrow[dl] \arrow[dr] & \\ {}^L\mathrm{Bun}_x & & {}^L\mathrm{Bun} \end{tikzcd}$$

<u>Note</u>: There is a composition

$${}^L\mathcal{H}ecke \underset{\mathrm{Bun}}{\times} {}^L\mathcal{H}ecke \longrightarrow {}^L\mathcal{H}ecke$$

$$(V, V', \beta),\ (V', V'', \beta') \longmapsto (V, V'', \beta' \circ \beta)$$

Given $x \in C$, $\rho \in \mathrm{Rep}(G) = \mathrm{dom}^+(G) = \cancel{\mathrm{do}}\ \mathrm{codom}^+({}^LG)$, $\rho \longleftrightarrow \mu$, we get a substack ${}^L\mathcal{H}ecke_x^\rho \subseteq {}^L\mathcal{H}ecke_x$ $\quad {}^L\mathcal{H}ecke_x = \{(V, V', \beta)\}$

such that poles of β are bounded by ρ.

<u>Explicitly</u>: $(V, V', \beta) \in {}^L\mathcal{H}ecke_x^\rho$ iff for every $\lambda \in \mathrm{dom}^+({}^LG)$ the induced map of locally free sheaves $\rho^\lambda(V) \xrightarrow{\rho^\lambda(\beta)} \rho^\lambda(V)(\underline{\langle\lambda,\mu\rangle}\, x)$ for all λ.

map exists for some number here.
condition is that this number is less than $\langle\lambda,\mu\rangle$.

${}^L\mathcal{H}ecke_x^\rho$ is an alg. stack and ${}^L\mathcal{H}ecke_x = \varinjlim_\rho {}^L\mathcal{H}ecke_x^\rho$

Again, ${}^Lp_x^\rho,\ {}^Lq_x^\rho : {}^L\mathcal{H}ecke_x^\rho \longrightarrow {}^L\mathrm{Bun}$ — locally triv fibrations

the fibers are closures of affine Schubert varieties.

<u>Note</u>: ${}^L\mathcal{H}ecke_x^\rho$ is smooth if and only if μ is miniscule.

The Hecke operator given by (x, ρ) is the functor ${}^LH^{x,\rho} : D({}^L\mathrm{Bun}, \mathcal{D}) \circlearrowleft$

$${}^LH^{x,\rho}(\mathcal{F}) = ({}^Lq_x^\rho)_!\left(({}^Lp_x^\rho)^* \mathcal{F} \otimes IC_{{}^L\mathcal{H}ecke_x^\rho}\right)$$

Best hope conjecture requires $c \circ W^{x,\rho} = {}^L H^{x,\rho} \circ c \quad \forall x, \rho$.

$D_{qcoh}(\mathcal{L}oc, \mathcal{O})$ has orthogonal collection of spanning objects : structure sheaves of points.
This is a basis of eigensheaves for $W^{x,\rho}$
If ~~$\mathbb{V}$~~ $\mathbb{V} = (V, \Delta)$ is a G-local system, then $c(\mathcal{O}_{\mathbb{V}})$ will have to be an eigen D-module for the action of ${}^L H^{x,\rho}$.

$${}^L H^{x,\rho}(c(\mathcal{O}_{\mathbb{V}})) = c(\mathcal{O}_{\mathbb{V}}) \otimes \rho(V_x)$$

If we don't fix x, then we still have "Hecke" and ${}^L H^{\rho}: D({}^L Bun, \mathcal{D}) \to D({}^L Bun \times C, \mathcal{D})$
and $\Sigma\, {}^L H^{\rho}(c(\mathcal{O}_{\mathbb{V}})) = c(\mathcal{O}_{\mathbb{V}}) \boxtimes \rho(\mathbb{V})$

<u>Problems with the naive conjecture</u>:

(1) The categories $D_{qcoh}(\mathcal{L}oc, \mathcal{O})$, $D({}^L Bun, \mathcal{D})$ have different complexity, so can not be equivalent.

$$\pi_0(\mathcal{L}oc) = H^2(C, \pi_1(G)_{tor}) \simeq \pi_1(G)_{tor}$$

so $\mathcal{L}oc = \coprod_{\chi \in \pi_1(G)_{tor}} \mathcal{L}oc_\chi$

$\mathcal{L}oc$ is a stack s.t. for each object $\mathbb{V} = (V, \Delta)$, $\mathrm{Aut}(\mathbb{V}) \supseteq Z(G)$
There is a rigidification of $\mathcal{L}oc$: $\underline{\mathcal{L}oc} = \mathcal{L}oc /\!\!/ Z_{\rho}(G)$
↑ alg. stack equipped w/ a map $\mathcal{L}oc \to \underline{\mathcal{L}oc}$ so that $\mathcal{L}oc$ is $Z_0(G)$ gerbe over $\underline{\mathcal{L}oc}$

$D_{qcoh}(\underline{\mathcal{L}oc}_\chi, \mathcal{O})$ is irreducible.

<u>Rmk</u>: ~~If~~ If $\mathfrak{X}$ is an alg. stack, it comes equipped with a group scheme $I_{\mathfrak{X}} \to \mathfrak{X}$
$\quad\quad\quad\quad \| \quad \mathfrak{X} \times_{\mathfrak{X} \times \mathfrak{X}} \mathfrak{X}$

Suppose we have a flat subgroupscheme $H < I_{\mathfrak{X}}$. Then we can rigidify $\mathfrak{X}$ by H.
($\Rightarrow H \triangleleft I_{\mathfrak{X}}$)
i.e. construct a new stack $\mathfrak{X} /\!\!/ H$ s.t. (1) $\mathfrak{X} \xrightarrow{p} \mathfrak{X} /\!\!/ H$
(2) $p^* I_{\mathfrak{X} /\!\!/ H} = I_{\mathfrak{X}} / H$
(3) p is a flat H-gerbe.

<u>Example</u>: $\mathfrak{X}$ = stack of rank n vector bundles on C
$Z(Gl_n(\mathbb{C})) = \mathbb{C}^\times \subset I_{\mathfrak{X}}$
We can rigidify $\mathfrak{X} /\!\!/ \mathbb{C}^\times$
$\mathfrak{X} /\!\!/ \mathbb{C}^\times$ is the moduli stack s.t. $(\mathfrak{X} /\!\!/ \mathbb{C}^\times)(S)$: ob: $E \to S \times C$ flat families of rank n v.b.
mor: (L, φ) : L: S-line bundle
$\varphi: E \xrightarrow{\sim} E' \otimes p_S^* L$.

<u>Rmk</u>: If $\mathcal{Y}$ is a stack, $H \to \mathcal{Y}$ a sheaf of ab. gps, $\mathcal{X} \to \mathcal{Y}$ an H-gerbe then

$$D_{qcoh}(\mathcal{X}, \mathcal{O}) = \coprod_{\alpha \in \mathrm{Chars}(H)} \underbrace{D_{qcoh}(\mathcal{X}, \mathcal{O})^{\alpha}}_{\shortparallel \atop D_{qcoh}(\mathcal{Y}, \mathcal{O}; \alpha)}$$

<u>Conclusion:</u>

$$D_{qcoh}(\mathcal{L}oc, \mathcal{O}) = \coprod_{(\gamma, \alpha) \in \pi_1(G)_{tor} \times Z(G)^{\wedge}} D_{qcoh}(\underline{\mathcal{L}oc}_{\gamma}, \mathcal{O}; \alpha)$$

What about $D({}^L\mathcal{B}un, \mathcal{D})$?

Again ${}^L\mathcal{B}un$ can be rigidified by $\underline{{}^L\mathcal{B}un} = {}^L\mathcal{B}un /\!/ Z({}^LG)$ and we have connected components ${}^L\mathcal{B}un = \coprod_{\alpha \in \pi_0 {}^L\mathcal{B}un} {}^L\mathcal{B}un_{\alpha}$.

$$\pi_0({}^L\mathcal{B}un) = H^2(C, \pi_1({}^LG))$$

${}^L\mathcal{B}un \longrightarrow \underline{{}^L\mathcal{B}un}$ is a $Z({}^LG)$-gerbe.

This rigidification factors as

$$\begin{tikzcd}
{}^L\mathcal{B}un \arrow[r] \arrow[dr] & {}^L\mathcal{B}un /\!/ Z_0({}^LG) = {}^L\mathrm{Bun} \arrow[d, "/\!/ \pi_1(Z({}^LG))"] \\
& \underline{{}^L\mathcal{B}un}
\end{tikzcd}$$

<u>Note:</u> $\pi_1({}^LG) = Z(G)^{\wedge}$

$Z_0({}^LG) = (\pi_1(G)_{free})^{\wedge}$

$\pi_0(Z({}^LG)) = (\pi_1(G)_{tor})^{\wedge}$

This indicates that on the RHS we must take $\overset{{}^L\mathrm{Bun}}{\cancel{{}^L\underline{\underline{\mathrm{Bun}}}}} = {}^L\mathcal{B}un /\!/ Z_0({}^LG)$

Then $D(\overset{{}^L\mathrm{Bun}}{\cancel{{}^L\underline{\mathrm{Bun}}}}, \mathcal{D}) = \coprod_{(\alpha, \gamma) \in \pi_1({}^LG) \times \pi_0(Z({}^LG))} D({}^L\underline{\mathrm{Bun}}_{\alpha}, \mathcal{D}; \gamma)$.

<u>Example</u>: $G = GL_1$

$\mathcal{L}oc_1 = \underline{\mathcal{L}oc}_1 \times B\mathbb{C}^* = T^{\vee}_{\mathcal{O}} J^0(C) \times B\mathbb{C}^*$

$\mathcal{B}un_1 = \mathcal{P}ic(C) = J^0(C) \times \mathbb{Z} \times B\mathbb{C}^*$

$R\mathcal{L}oc_1 = (T^{\vee}_{\mathcal{O}} J^0(C), \mathcal{O} \otimes A) \times B\mathbb{C}^*$, where A = freely generated dga$/\mathbb{C}$ by one generator in degree -1

$= T^{\vee}_{\mathcal{O}} J^0(C) \times B\mathbb{C}^* \times R\mathrm{Spec}(A)$

c: $D_{qcoh}(T^{\vee}_{\mathcal{O}} J^0(C), \mathcal{O}) \xrightarrow{\sim} D(J^0(C), \mathcal{D})$ Laumon–Fourier transform

$D_{qcoh}(\overset{B\mathbb{C}^*}{\cancel{B\mathbb{Z}}}, \mathcal{O}) \xrightarrow{\sim} D(\mathbb{Z}, \mathcal{D})$

$D_{qcoh}(\mathrm{Spec}\, A, \mathcal{O}) \xrightarrow{\sim} D(B\mathbb{C}^*, \mathcal{D})$

Lecture III

Yesterday:

$$D_{qcoh}(\mathcal{L}oc, \mathcal{O}) = \coprod_{(\gamma,\alpha) \in \pi_1(G)_{tors} \times Z(G)^\wedge} D_{qcoh}(\underline{\mathcal{L}oc}_\gamma, \mathcal{O}; \alpha)$$

$$D({}^L\mathcal{B}un, \mathcal{D}) = \coprod_{(\tilde{\jmath}, \gamma) \in \pi_1({}^LG) \times \pi_0(Z({}^LG))^\wedge} D(\underline{\mathcal{B}un}_\alpha, \mathcal{D}; \gamma)$$ ← these categories are too big to match (when G not semi-simple)

Version: $D({}^L\mathrm{Bun}, \mathcal{D}) = \coprod_{\alpha, \gamma} D(\mathrm{Bun}_\alpha, \mathcal{D}; \gamma)$

↑

${}^L\mathcal{B}un /\!\!/ Z_0({}^LG)$

Computation for GL_1 suggests:

The conjectural ~~&~~ geometric Langlands ~~conjecture~~ correspondance should identify

$$D_{qcoh}(\mathcal{L}oc, \mathcal{O}) \xrightarrow{\sim} D({}^L\mathrm{Bun}, \mathcal{D})$$

or

$$D_{qcoh}(R\mathcal{L}oc, \mathcal{O}) \xrightarrow{\sim} D({}^L\mathcal{B}un, \mathcal{D})$$

and should identify pieces labeled by the same data.

This modification is not enough, since $D({}^L\mathcal{B}un, \mathcal{D})$ behaves as the derived category of qcoh on a smooth space.

Aside: Smoothness is a categorical notion.

Def: If $\mathcal{C}$ a cocomplete, $\mathbb{C}$-linear dg category, an object $x \in ob(\mathcal{C})$ is compact if

$$\mathrm{Hom}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathrm{Compl}(\mathbb{C})$$

commutes with colimits.

The full subcategory of compact objects in $\mathcal{C}$ is denoted by $\mathrm{Perf}(\mathcal{C})$.

We say $\mathcal{C}$ is compactly generated if $\mathcal{C} \simeq \widehat{\mathrm{Perf}(\mathcal{C})} = \mathrm{Fun}(\mathrm{Perf}(\mathcal{C}), \mathrm{Compl}(\mathbb{C}))$

Thm (Neeman, TT): If X a quasi-compact, quasi-sep. scheme, then

$$\mathrm{Perf}(D_{qcoh}(X)) = \mathrm{Perf}(X)$$

↑ complexes locally quis to complexes of vector bundles

and $D_{qcoh}(X) = \widehat{\mathrm{Perf}(X)}$.

<u>Fact</u>: X is smooth iff $\mathrm{Perf}(X) = D_{coh}(X)$.

X is singular iff $\mathrm{Perf}(X) \subsetneq D_{coh}(X)$

<u>Def</u>: If $\mathcal{C}$ is complete & cocomplete, we say that $\mathcal{C}$ is <u>nc-smooth</u> if $\mathcal{C}$ is compact as an object in the category $\widehat{\mathcal{C}\otimes\mathcal{C}^{op}}$.

<u>Note</u>: X is smooth iff $\widehat{\mathrm{Perf}}(X)$ is nc-smooth.

<u>Note</u>: if X is singular it can happen that $\widehat{D_{coh}}(X)$ is nc-smooth.

<u>Exercise</u>: $X = \mathrm{Spec}\,(\mathbb{C}[\varepsilon]/_{\varepsilon^2})$

$\underset{\text{not nc-smooth}}{\mathrm{Perf}(X)} \neq \underset{\text{check this is nc-smooth}}{D_{coh}(X)} \longleftarrow = \mathrm{Perf}(A\text{-mod})$ for some dg-~~category~~ algebra A

<u>Def</u>: If X is a scheme or (derived) stack, then

$\hat{D}_{coh}(X)$ = stable derived category of X

$=: \mathrm{Ind\,Coh}(X)$ = category of ind-coherent sheaves.

Explicit model: unbounded complexes of injectives with homotopy classes of maps

We have a localization functor $\mathrm{Ind\,Coh}(X) \longrightarrow D_{qcoh}(X)$, which has a right adjoint $\Xi : D_{qcoh}(X) \hookrightarrow \mathrm{Ind\,Coh}(X)$.

$$\mathrm{Perf}(X) \subseteq D_{coh}(X) \subseteq \underset{\underset{\widehat{\mathrm{Perf}}(X)}{\|}}{D_{qcoh}(X)} \subseteq \underset{\underset{\widehat{D_{qcoh}}(X)}{\|}}{\mathrm{Ind\,Coh}(X)}$$

Derived category of singularities: $D_{sing}(X) := D_{coh}(X)/_{\mathrm{Perf}(X)}$

$$\hat{D}_{sing}(X) = \mathrm{Ind\,Coh}(X)/_{D_{qcoh}(X)}$$

Facts: •) If U is a quasi-compact scheme or stack, then $D(U,\mathcal{D})$ is compactly generated.

•) (Arkhipov–Gaitsgory): ${}^L\mathrm{Bun} = \bigcup_\alpha U_\alpha$, U_α quasi-compact

s.t. $j_{\alpha\beta}: U_\alpha \hookrightarrow U_\beta$ satisfy $j_{\alpha\beta *}: D(U_\alpha,\mathcal{D}) \to D(U_\beta,\mathcal{D})$ preserve compactness.

•) (Drinfeld–Gaitsgory) If U is a q-compact stack, then

$$\mathrm{Perf}(D(U,\mathcal{D})) \xrightarrow{\sim} \cancel{\mathrm{Perf}}\, D_{coh}(U,\mathcal{D})$$

and moreover

$$\mathrm{Perf}(D({}^L\mathrm{Bun},\mathcal{D})) = \text{!-extension of compact objects in } D(U_\alpha,\mathcal{D})$$

$\Longrightarrow$ $D({}^L\mathrm{Bun},\mathcal{D})$ is compactly generated.

Rmk: Because of DG statement, we should ~~probably extend~~ probably extend $D_{coh}(\mathcal{L}oc,\mathcal{O})$ to $\mathrm{IndCoh}(\mathcal{L}oc,\mathcal{O})$.

However, $\mathrm{IndCoh}(\mathcal{L}oc,\mathcal{O})$ is too big.

Reason: $c: \mathrm{IndCoh}(\mathcal{L}oc,\mathcal{O}) \xrightarrow{\sim} D({}^L\mathrm{Bun},\mathcal{D})$

should intertwine the tensorization & Hecke operators, but should also be functorial in G.

Functoriality in G:

If $P \subset G$ parabolic, $M = \mathrm{Levi}(P)$ $\quad (\longleftrightarrow {}^LP \subset {}^LG,\ {}^LM = \mathrm{Levi}({}^LP))$, get

$$\begin{tikzcd} & \mathcal{L}oc_P \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathcal{L}oc_M & & \mathcal{L}oc_G \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{Bun}_{{}^LP} \arrow[dl, "\varphi"'] \arrow[dr, "\gamma"] & \\ \mathrm{Bun}_{{}^LM} & & \mathrm{Bun}_{{}^LG} \end{tikzcd}$$

We get integral transforms $\Phi_P = g_! f^*: \mathrm{IndCoh}(\mathcal{L}oc_M,\mathcal{O}) \to \mathrm{IndCoh}(\mathcal{L}oc_G,\mathcal{O})$

$\mathrm{Eis}_{{}^LP} = \gamma_! \varphi^*: D(\mathrm{Bun}_{{}^LM},\mathcal{D}) \to D(\mathrm{Bun}_{{}^LG},\mathcal{D})$

and we should have $\mathrm{Eis}_{{}^LP} \circ c_G = c_M \circ \Phi_P$.

Rmk: •) Φ_P, $\mathrm{Eis}_{{}^LP}$ preserve coherence

•) $\mathrm{Eis}_{{}^LP}$ preserves perfection, but Φ_P does not preserve perfection.

Guess: On the LHS of GLC we want the subcategory of $\mathrm{IndCoh}(\mathcal{L}oc,\mathcal{O})$ generated by

$\Phi_P(D_{coh}(\mathcal{L}oc_M,\mathcal{O}))$ for all P.

How do we understand this subcategory?

Preview : This subcategory will be the subcategory of $\mathrm{IndCoh}(\mathrm{Loc}_G, \mathcal{O})$ consisting of all ind-coh sheaves with singular support in the nilpotent cone.

Singular supports for IndCoh sheaves on a derived stack.

A derived stack X is defined as functor $h_X : (\mathbb{C}\text{-dga}^{\leq 0}) \to (\mathrm{SSet})$.

Examples : (1) If A is a $(\mathbb{C}\text{-cdga})^{\leq 0}$ can define $\mathbb{R}\mathrm{Spec}(A)$ which is given by

$$h_{\mathbb{R}\mathrm{Spec}(A)} : (\mathbb{C}\text{-cdga})^{\leq 0} \longrightarrow (\mathrm{SSet})$$

$$B \longmapsto \underline{\mathrm{Hom}}(A, B)$$

(2) If M is a smooth scheme, $E \to M$ a vector bundle, $s \in H^0(M, E)$, then $\mathrm{zero}(s) = X$, a l.c.i, has a natural derived structure : derived scheme RX

$$RX = (M, \mathcal{O}^{\bullet}_{RX})$$

$\uparrow$ sheaf of dg-algebras s.t. $H^0(\mathcal{O}^{\bullet}_{RX}) = \mathcal{O}_X$.

$$\mathcal{O}^{\bullet}_{RX} = (\Lambda^{\bullet} E^{\vee}, \lrcorner s)$$

(3) $L^{\bullet}$ is $\mathbb{Z}_{\geq 0}$-graded complex dg Lie algebra (or L_∞-algebra)

Then $L^{\bullet}$ gives a derived stack

$$RX = \left[\mathbb{R}\mathrm{Spec}\left((\mathrm{Sym}^{\bullet} L_{\geq 1}^{\vee}[1], Q) \right) \Big/ \exp L^0 \right]$$

$$X = \pi_0(RX) = [MC(L_1) / \exp(L^0)]$$

Lecture IV

Singular supports for indcoh sheaves on a derived stack

Idea: If $\mathfrak{X}$ is a nice derived stack, then $\mathbb{T}_{\mathfrak{X}} \in D_{qcoh}(\mathfrak{X}, \mathcal{O})$ and $\mathbb{T}_{\mathfrak{X}}[-1]$ is a Lie alg. obj. in $D_{qcoh}(\mathfrak{X}, \mathcal{O})$. The Lie bracket on $\mathbb{T}_{\mathfrak{X}}[-1]$ is given by the Atiyah class of $\mathfrak{X}$.

~~the~~ Concretely:

$$
\begin{array}{ccccccc}
\mathbb{T}_{\mathfrak{X}}^{\vee} \otimes \mathcal{H}om(\mathcal{O}_{\mathfrak{X}^{(1)}}, \mathcal{O}_{\mathfrak{X}}) & & \xrightarrow{\ \sigma\ } & & \mathrm{End}(\mathbb{T}_{\mathfrak{X}}) \overset{L}{\otimes} \mathbb{T}_{\mathfrak{X}} & & \\
\cup & & & & \cup & & \\
\mathrm{End}(\mathbb{T}_{\mathfrak{X}}) \longrightarrow \mathcal{A}(\mathbb{T}_{\mathfrak{X}}) & & \longrightarrow & & \mathbb{T}_{\mathfrak{X}} & \longrightarrow & \mathrm{End}(\mathbb{T}_{\mathfrak{X}})[1]
\end{array}
$$

$\otimes^{\mathbb{T}_{\mathfrak{X}}}$

Atiyah class ~~$a(\mathbb{T}_{\mathfrak{X}}) \in \mathrm{Hom}^1(\mathbb{T}_{\mathfrak{X}}$~~

$$a(\mathbb{T}_{\mathfrak{X}}) \in \mathrm{Ext}^1(\mathbb{T}_{\mathfrak{X}}, \mathbb{T}_{\mathfrak{X}}^{\vee} \otimes \mathbb{T}_{\mathfrak{X}})$$

$$a(\mathbb{T}_{\mathfrak{X}}) : \mathbb{T}_{\mathfrak{X}} \otimes \mathbb{T}_{\mathfrak{X}} \longrightarrow \mathbb{T}_{\mathfrak{X}}[1]$$

$$\updownarrow$$

$$a(\mathbb{T}_{\mathfrak{X}}) : \mathbb{T}_{\mathfrak{X}}[-1] \otimes \mathbb{T}_{\mathfrak{X}}[-1] \to \mathbb{T}_{\mathfrak{X}}[-1]$$

Also, for every object $\mathcal{F} \in D_{qcoh}(\mathfrak{X}, \mathcal{O})$ we get an Atiyah extension

$$a(\mathcal{F}) : \mathrm{End}(\mathcal{F}) \to \mathcal{A}(\mathcal{F}) \to \mathbb{T}_{\mathfrak{X}} \to \mathrm{End}(\mathcal{F})[1]$$

$$a(\mathcal{F}) : \mathbb{T}_{\mathfrak{X}}[-1] \otimes \mathcal{F} \to \mathcal{F}$$

$(\mathcal{F}, a(\mathcal{F}))$ is a module over $(\mathbb{T}_{\mathfrak{X}}[-1], a(\mathbb{T}_{\mathfrak{X}}))$.

So $D_{qcoh}(\mathfrak{X}, \mathcal{O})$ (or $\mathrm{IndCoh}(\mathfrak{X}, \cancel{\mathcal{O}})$) can be viewed as a category of modules over the filtered ~~category~~ algebra $U(\mathbb{T}_{\mathfrak{X}}[-1])$.

If we can endow $\mathcal{F}$ with a good filtration, then ~~the~~ $\mathrm{gr}\,\mathcal{F}$ will be a module over $\mathrm{Sym}(\mathbb{T}_{\mathfrak{X}}[-1])$, or $\mathcal{F}$ will be a sheaf on a derived stack $\mathrm{RSpec}(\mathrm{Sym}(\mathbb{T}_{\mathfrak{X}}[-1]))$
$= \mathrm{tot}(\mathbb{L}_{\mathfrak{X}}[1])$.

⌈**Ex.** Do this for spectrum of dual numbers.⌋

Suppose that $\mathcal{X}$ is a quasi-smooth. This means that $\Pi_{\mathcal{X}}$ is perfect of amplitude 1

Ex: If Z is an alg. stack which is a l.c.i., then the natural derived enhancement RZ of Z is quasi-smooth.

If $\mathcal{X}$ is quasi-smooth, then locally $\mathcal{X} \simeq RZ$, where Z is a l.c.i. scheme.

$Z = \mathrm{zero}(s)$, $s \in H^0(M, E)$, M... smooth scheme, E... vector bundle

$\underbrace{\mathrm{tot}(\mathcal{H}^{-1}(\mathbb{L}_{RZ}))}_{\substack{\| \\ \mathrm{Sing}(Z)}}$ is a subscheme in $\mathrm{tot}(E^\vee)$.

$\mathrm{Sing}(Z) \leftarrow$ conical subscheme in

Given $\mathcal{F} \in \mathrm{IndCoh}(RZ)$ we will define $\mathrm{SSupp}(\mathcal{F}) \subseteq \mathrm{Sing}(Z)$ as a certain conical subscheme.

Thm (Isik): $D_{qcoh}(RZ) \simeq D_{sing}(\mathrm{tot}(E^\vee), w)^{\mathbb{C}^\times}$, $w : \mathrm{tot}(E^\vee) \to \mathbb{C}$, $w = p^* s \cdot \lambda$

$$\Longrightarrow \mathrm{IndCoh}(RZ) \simeq \widehat{D_{sing}}([\mathrm{tot}(E^\vee)/\mathbb{C}^\times], w) = \widehat{D_{sing}([w^{-1}(0)/\mathbb{C}^\times])} = \frac{\mathrm{IndCoh}([w^{-1}(0)/\mathbb{C}^\times])}{D_{qcoh}([w^{-1}(0)/\mathbb{C}^\times])}.$$

Def: $\mathcal{F} \in \mathrm{IndCoh}(RZ)$

$$\updownarrow$$

$\mathcal{F}' \in \dfrac{\mathrm{IndCoh}([w^{-1}(0)/\mathbb{C}^\times])}{D_{qcoh}([w^{-1}(0)/\mathbb{C}^\times])}$

$\mathrm{ssupp}(\mathcal{F}) = \mathrm{supp}(\mathcal{F}')$.

Remark: $\mathcal{F} \in D_{coh}(RZ, \mathcal{O})$ then $\underset{\text{conical subsch}}{\mathrm{SSupp}(\mathcal{F})} \subset \underset{\text{conical subsch}}{\mathrm{Sing}(Z)} \subset \mathrm{tot}(E^\vee)$, then $\mathcal{F} \in \mathrm{Perf}(D_{qcoh}(RZ, \mathcal{O})) \Leftrightarrow \mathrm{sscupp}(\mathcal{F}) = \mathcal{O}$.

Given any reductive G, the stack $\mathcal{L}oc$ is a l.c.i. so has a natural derived enhancement $R\mathcal{L}oc$.

Explicitly, we can describe $R\mathcal{L}oc$ as follows: Fix $x \in C$ and consider

$\mathcal{L}oc_{\log x}$ = moduli stack of pairs $(V, \overset{\triangledown}{\Delta})$, where V is a principal G-bundle & $\overset{\triangledown}{\Delta}$ is meromorphic connection on V with a log pole at x.

Fact: $\mathcal{L}oc_{\log x}$ is a smooth alg. stack.

__Note:__ $Loc \subset Loc_{\log x}$ and is the zero locus of a section of a vector bundle

$$(\mathcal{V}, \nabla) \longrightarrow Loc_{\log x} \times C$$

↑ universal local system

∇ — relative connection on $\mathcal{V}$ (differentiates in the C-direction) and has 1st order pole at $Loc_{\log x} \times \{x\} =: D$.

$Res_D(\nabla) \in \Gamma(D, ad\,\mathcal{V}|_D)$

on $Loc_{\log x}$ we have $E = ad(\mathcal{V}_x)$, $s = Res_D \nabla \in H^0(Loc_{\log x}, E)$ and $Loc = zero(s)$.

$\rightsquigarrow R\,Loc \cong (Loc_{\log x}, (\Lambda^\bullet E^\vee, \lrcorner s))$

Let $\mathcal{C} \subset tot(ad\,\mathcal{V}_x)$ be a conical closed substack.

__Def:__ $IndCoh_{\mathcal{C}}(R\,Loc, \mathcal{O}) := \{\mathcal{F} : ssupp\, \mathcal{F} \subset \mathcal{C}\}$

From the functoriality constraint of c_G:
we want to be the LHS of GLC to be the subcategory in $IndCoh(RLoc, \mathcal{O})$ generated by $\Phi_P(Perf(D_{coh}(RLoc_M, \mathcal{O})))$ for all parabolics

__Thm__ (Arinkin-Gaitsgory): This ↑ category is equal to $IndCoh_{\mathcal{N}}(R\,Loc, \mathcal{O})$.

($\mathcal{N}$ is nilpotent cone in $ad\,\mathcal{V}_x$)

__GLC__: There exists a functorial equivalence of categories

$$c_G : IndCoh_{\mathcal{N}}(R\,Loc, \mathcal{O}) \xrightarrow{\sim} D({}^L Bun, \mathcal{D})$$

intertwining tornization & Hecke operators.

(__Variant__: Rigidified version.
$c_G : IndCoh_{\mathcal{N}}(RLoc, \mathcal{O}) \xrightarrow{\sim} D({}^L Bun, \mathcal{D})$.)

True for $G = GL_1$, $G = GL_2$.

Classical limit conjecture

$G, {}^L G$ ... semisimple

$\mathrm{IndCoh}(R\mathrm{Loc}, \mathcal{O})$ comes in a natural 1-parameter family of categories
$D({}^L\mathrm{Bun}, \mathcal{D})$ — — "

The first family comes from a 1-parameter deformation of Loc (or RLoc).

There is a moduli stack $\mathcal{H} \to \mathbb{A}^1$.

$\mathcal{H}$ = moduli of triples (V, ∇, t) where
- V a principal G-bundle
- $t \in \mathbb{A}^1$
- ∇ a flat t-connection on V.

$$0 \to \mathrm{ad}\, V \to \mathcal{A}(V) \underset{\nabla}{\overset{dp}{\rightleftarrows}} T_C \to 0$$

∇ a t-connection, if ∇ is an $\mathcal{O}$-linear map such that $dp \circ \nabla = t \cdot \mathrm{id}$.

~~If $t \neq 0$~~
If $z \in \mathbb{C}^\times$, ~~z acts as~~ $z \cdot (V, t, \nabla) = (V, zt, z\nabla)$,
hence $\mathcal{H} \to \mathbb{A}^1$ equivariant

If (V, t, ∇) is a t-connection, then $(V, \frac{1}{t}\nabla)$ is a connection.

So $\mathcal{H}|_{\mathbb{A}^1 \setminus 0} = \mathbb{A}^1 \setminus \{0\} \times \mathrm{Loc}$.

$\mathcal{H}_0$ = stack of 0-connection = $(V, 0, \nabla)$, $\nabla : T_C \to \mathcal{A}(V)$, $T_C \to \mathrm{ad}(V) \hookrightarrow \mathcal{A}(V)$

$\rightsquigarrow \nabla \in H^0(C, \mathrm{ad}(V) \otimes \Omega^1_C)$ ... Higgs field

$\cdot$ = stack of G-Higgs bundles.

Lecture V

Classical limit conjecture $\quad G, {}^LG$ - semisimple

$$\begin{tikzcd} \mathrm{IndCoh}_{\mathcal{N}}(R\mathrm{Loc}, \mathcal{O}) \arrow[r, "\sim"] \arrow[d, squiggly] & D({}^L\mathrm{Bun}, \mathcal{D}) \arrow[d, squiggly] \\ \mathrm{IndCoh}_{\mathcal{N}}(R\mathrm{Higgs}, \mathcal{O}) \arrow[r, "\sim", "cl"'] & \mathrm{IndCoh}(R\,{}^L\mathrm{Higgs}, \mathcal{O}) \end{tikzcd}$$

Yesterday: The moduli stack of t-connections on G-bundles

$$\begin{tikzcd} \mathrm{Higgs} \arrow[d] & \subset & \mathcal{H} \arrow[d] & \supset & (\mathbb{A}^1\setminus 0)\times \mathrm{Loc} \arrow[d] \\ 0 & \in & \mathbb{A}^1 & \supset & \mathbb{A}^1\setminus 0 \end{tikzcd}$$

Higgs = moduli of pairs (E, ϑ)
E - principal G-bundle
$\vartheta \in H^0(C, \mathrm{ad}\,E \otimes \Omega^1_C)$
$\{\vartheta \wedge \vartheta = 0\}$
$\uparrow$ automatic on curves

Aside: Higgs $= T^\vee \mathrm{Bun}$ and Higgs acts on Loc_1 by translations.

Get a family of derived stacks

$$\begin{tikzcd} R\mathcal{H} \arrow[d] \\ \mathbb{A}^1 \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Sing}(R\mathcal{H}) \arrow[dr] & \supset & \mathcal{N} \arrow[dl] \\ & \mathbb{A}^1 & \end{tikzcd}$$

Thus we get a family of dg cats. $\mathrm{IndCoh}_{\mathcal{N}}(R\mathcal{H}/\mathbb{A}^1)$

$$\begin{tikzcd} \mathrm{IndCoh}_{\mathcal{N}}(R\mathcal{H}/\mathbb{A}^1) \arrow[d] \\ \mathbb{A}^1 \end{tikzcd}$$

gives the LHS vertical specialization.

On the RHS, we have a family of categories which is dequantization.

$\mathcal{D}$ is a filtered sheaf of algebras, so it can be realized as a fiber of a Rees ring.

We have a sheaf of algebras $\mathcal{R} \to {}^L\mathrm{Bun} \times \mathbb{A}^1$ s.t.

$$\mathcal{R}|_{{}^L\mathrm{Bun} \times (\mathbb{A}^1 \setminus \{0\})} = p_1^* \mathcal{D}$$

$$\mathcal{R}|_{{}^L\mathrm{Bun} \times \{0\}} = \mathrm{gr}\, \mathcal{D} = \mathrm{Sym}^\cdot T_{{}^L\mathrm{Bun}}$$

$\mathcal{R} \longleftrightarrow$ module over $\mathcal{O}_{\mathrm{Bun}}[\epsilon]$

$\mathcal{R} \subset \mathcal{D}[\epsilon, \epsilon^{-1}]$

$\mathcal{R} = \{ \sum \epsilon^{-i} p_i \; : \; p_i \in \mathcal{D}^{\leq i} \}$

Get a family of categories $\begin{array}{c} D({}^L\mathrm{Bun}, \mathcal{R}/\mathbb{A}^1) \\ \downarrow \\ \mathbb{A}^1 \end{array}$ which gives a specialization of

$D({}^L\mathrm{Bun}, \mathcal{D})$ to $D({}^L\mathrm{Bun}, \mathrm{Sym}^\cdot T_{{}^L\mathrm{Bun}}) = D(\underbrace{T^\vee {}^L\mathrm{Bun}}_{= {}^L\mathrm{Higgs}}, \mathcal{O})$.

To formulate the classical limit conjecture we need to understand limits of $W^{x,\mu}$ and ${}^L H^{x,\mu}$.

$W^{x,\mu} \rightsquigarrow W^{x,\mu}_{cl}$ given by $W^{x,\mu}_{cl}(\mathcal{F}) = \mathcal{F} \otimes \rho^\mu(\mathcal{E}_x)$

${}^L H^{x,\mu} \rightsquigarrow$?

$$\begin{tikzcd} & {}^L\mathrm{Hecke}^{x,\mu} \arrow[dl, "{}^L p^{x,\mu}"'] \arrow[dr, "{}^L q^{x,\mu}"] & \\ {}^L\mathrm{Bun} & & {}^L\mathrm{Bun} \end{tikzcd}$$

$\longleftarrow$ want to interpret these as correspondences acting on $\mathrm{Sym}^\cdot T_{{}^L\mathrm{Bun}}$-modules.

Aside. If X - smooth space (or stack), then there is an equiv of categories

$$\mathrm{sc}: D_{\mathrm{perf}}(T^\vee X, \mathcal{O}) \xrightarrow{\sim} D(\mathrm{Higgs}(X)).$$

$\uparrow$

pushforward ~~along~~ along $p: T^\vee X \to X$

Def: $IC_{{}^L\mathrm{Hecke}^{x,\mu}}$ is a mixed Hodge module and has a Hodge filtration which is a good filtration.

$$I^{\mu,x}_{cl} := gr_F\left(IC^{x,\mu}_{{}^L\cancel{\mathrm{Hecke}}\ \mathrm{Bun}}\right)$$

This defines ${}^L H^{x,\mu}_{cl}$ = (integral transform with this kernel)

Conjecture: $\exists!\ c_{cl}$ s.t. —, intertwining $W^{x,\mu}_{cl}$ and $H^{x,\mu}_{cl}$ and is functorial for change of groups

Proof of the classical limit

Restrict to open substacks where we have no derived structure on no N.

Higgs^{reg} = stack of regular Higgs bundles
(E,ϑ) where $\vartheta_x \in ad(E_x) \otimes \Omega^1_{C,x}$ is a regular element for all x.

$R\mathrm{Higgs}^{reg} = \mathrm{Higgs}^{reg}$ be $\mathrm{Sing}(\mathrm{Higgs}^{reg}) = 0$

We want to construct

$$c_{cl} : D_{qcoh}(\mathrm{Higgs}^{reg}, \mathcal{O}) \to D_{qcoh}({}^L\mathrm{Higgs}^{reg}, \mathcal{O}).$$

Idea: Abelianize both sides.

Important Fact: Higgs^{reg} is a Abelian group stack.
This is Hitchin's abelianisation (in the form of Donagi-Gaitsgory)
There is a natural morphism $h: \mathrm{Higgs}^{reg} \to \mathcal{B}$ ← smooth variety of dim = dim $G(g-1)$ the Hitchin ~~base~~
the Hitchin map.

If (E,ϑ) is a G-Higgs bundle,

$$\vartheta : C \to ad(E) \otimes \Omega^1_C.$$

Let

$$\begin{array}{ccc} C & \xrightarrow{\vartheta} & ad(G)\otimes\Omega^1_C \\ & \searrow^{v(\vartheta)} & \downarrow v \\ & & \frac{ad(E)}{G}\otimes\Omega^1_C \\ & & \cong (\mathfrak{g}/G)\otimes\Omega^1_C = (\mathfrak{t}\otimes\Omega^1_C)/W \end{array}$$

ϑ gives a section

$$\begin{array}{ccc} v(\vartheta) : C & \longrightarrow & (\mathfrak{t}\otimes\Omega^1_C)/W \\ \uparrow & & \uparrow \\ \tilde{C}^\vartheta & \longrightarrow & (\mathfrak{t}\otimes\Omega^1_C) \end{array}$$

get $\tilde{C} \to C$ which is W-Galois cover of C, the cameral cover associated with ϑ.

$B = \Gamma((\mathfrak{t} \otimes \Omega^1_C)/W)$ = moduli of W-cameral covers of C.

If $p_1, \ldots, p_d$ - basis of G-invariant polys on $\mathfrak{g}$, then

$$B = \bigoplus_{i=1}^{d} H^0(C, (\Omega^1_C)^{\otimes \deg p_i})$$

$h(\vartheta) := \tilde{C} \longleftrightarrow v(\vartheta) \in \Gamma(C, (\mathfrak{t} \otimes \Omega^1_C)/W)$

Given $\tilde{C} \to C$ we get an Abelian group scheme on C

$$J_{\tilde{C}} \text{ " }= (p_*(\mathrm{cochar}(G) \otimes \mathcal{O}^\times))^W \text{ "}$$

and if $\tilde{C} = h((E, \vartheta))$ for (E, ϑ) regular, then

$$J_{\tilde{C}} = \text{centralizer of } \vartheta \text{ in } \mathrm{Aut}(E).$$

Thm: $\mathrm{Higgs}^{reg} \cong \mathbf{B}\mathcal{J}$, where $\mathcal{J} \to B \times C$.

So Higgs^{reg} is a comm. group stack over B.

There is a natural identification $B = {}^L B$.

Hope:

$$\begin{tikzcd}
\mathrm{Higgs}^{reg} \arrow[dr] & & {}^L\mathrm{Higgs}^{reg} \arrow[dl] \\
& B = {}^L B &
\end{tikzcd}$$

are Cartier dual.

There are natural symmetries acting on both sides

$W^{\kappa,\mu}_{ab}$ — abelianized tensorization — tensorization with translation invariant line bundles

${}^L H^{\kappa,\mu}_{ab}$ — abelianized Heckes — translation by sections

$(W^{\kappa,\mu}_{ab}) = (W^{\kappa,\mu}_{cl})$ are the same alg. of endofunctors

Thm (A-B) $({}^L H^{\kappa,\mu}_{ab}) = ({}^L H^{\kappa,\mu}_{cl})$

Thm: If $\mathrm{Discr} \subset B = {}^LB$ is the discriminant of h ($=$ discriminant of ${}^Lh$), then there is a Poincaré ~~sheaf $\mathcal{P}$~~ line bundle

$$\mathcal{P} \longrightarrow \mathcal{H}iggs^{reg} \underset{B \setminus \mathrm{Discr}}{\times} {}^L\mathcal{H}iggs^{reg}$$

~~s.t.~~ which identifies $\mathcal{H}iggs^{reg} \simeq ({}^L\mathcal{H}iggs^{reg})^D = \mathrm{Hom}({}^L\mathcal{H}iggs^{reg}, B\mathbb{G}_m)$ and gives an equiv. of D_{qcoh} intertwining W_{ab}^{sym} and ${}^LH_{ab}^{sym}$.

Remark: Fix $\tilde{C}$

$$h^{-1}(\tilde{C}) \simeq BJ_{\tilde{C}} = \begin{pmatrix} \text{moduli space of} \\ J_{\tilde{C}}\text{-bundles on } C \end{pmatrix} \times BZ(G)$$

$$= \underbrace{\pi_0 \begin{pmatrix} \text{moduli} \\ \text{space} \end{pmatrix}}_{\pi_1(G)} \times \underbrace{\begin{pmatrix} \text{moduli space of} \\ \text{top. trivial} \\ J_{\tilde{C}}\text{-bundle} \end{pmatrix}}_{\substack{\text{abelian group} \\ \mathrm{Prym}^G(\tilde{C}/C)}} \times BZ(G)$$

$${}^Lh^{-1}(\tilde{C}) = \pi_1({}^LG) \times \mathrm{Prym}^{{}^LG}(\tilde{C}/C) \times BZ({}^LG)$$

$(\cdot)^D$:
$$(\pi_1(G))^D = BZ({}^LG)$$
$$(\mathrm{Prym}^G(\tilde{C}/C))^D = \mathrm{Prym}^{{}^LG}(\tilde{C}/C)$$
$$BZ(G)^D = \pi_1({}^LG).$$