$X = \operatorname{Spec} R/(f_1, \ldots, f_n)$ $\qquad U = \operatorname{Spec} R$ smooth $\qquad /\mathbb{C}$

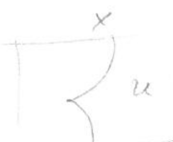
either: •) $f_1, \ldots, f_n$ reg. sequence (codim $X = n$)

•) X dg

Key observation:

For any $F \in A$-mod, there are natural coh. operation $\xi_1, \ldots, \xi_n \in Ext^2(F, F)$

Ex: $A = R/(f)$, F a q-coh sheaf on X, $i: X \hookrightarrow U$



$$L_k i^* i_* F = \begin{cases} F & k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

$F[1] \longrightarrow L i^* i_* F \longrightarrow F \longrightarrow F[2]$ $\qquad$ triangle.

This gives $\xi: F \to F[2]$ $\qquad$, $\xi \in Ext^2(F, F)$

e.g. $X = pt \subseteq \mathbb{A}^1$, Then ξ is zero

Suppose $F \in D(X)$ — qcoh der. cat.

$gr\, End(F) = \bigoplus Ext^k(F, F) \longleftarrow A[\xi_1, \ldots, \xi_n]$, $\deg \xi_i = 2$

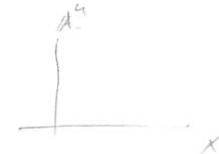
Thm (Gulliksen): If $F \in D^b_{coh}(X)$, then $gr\, End(F)$ is a f.g. module over $A[\xi_1, \ldots, \xi_n]$.

Def: $\operatorname{Singsupp}(F) = \operatorname{Supp}_{A[\xi_1, \ldots, \xi_n]} gr\, End(E). \subseteq \operatorname{Spec} A[\xi_1, \ldots, \xi_n] \subseteq X \times \mathbb{A}^n$

Simple properties:

1) $\mathrm{supp}_A \, \mathrm{gr}\, \mathrm{End}(F) = \mathrm{supp}\, F$

i.e. projection of $\mathrm{singSupp}(F)$ onto X is $\mathrm{supp}\, F$



2) $\mathrm{singSupp}(F)$ is conical, because $\mathrm{gr}\,\mathrm{End}(F)$ is a graded module

3) If F is perfect, then $\mathrm{Ext}^k(F,F)=0$ for $k \gg 0$

$\Rightarrow$ ξ_i's are nilpotent

$\Rightarrow$ $\mathrm{SingSupp}(F) = \mathrm{Supp}\, F \times \{0\}$ (actually iff)

The graded center of $D(X)$ is $\mathrm{gr}\, Z_{D(X)} = \bigoplus_k \mathrm{Hom}(\mathrm{Id}_{D(X)}, \mathrm{Id}_{D(X)}[k])$

$$\begin{tikzcd} A[\xi_1, \dots, \xi_n] \arrow[r] \arrow[dr] & \mathrm{gr}\, Z_{D(X)} \arrow[r] & \mathrm{gr}\,\mathrm{End}(F,F) \\ & HC(X) \arrow[u] & \end{tikzcd}$$

By this naturality: $i_x : x \hookrightarrow X$

$$i_x^! \mathcal{F} = \mathrm{Hom}(\mathbb{C}_x, F)$$

$A[\xi_1, \dots, \xi_n]$ acts on $H^\bullet(i_x^! F) = \mathrm{Ext}^\bullet(\mathbb{C}_x, F)$.

Claim: $\mathrm{supp}\, H^\bullet(i_x^! \mathcal{F}) = (\{x\} \times \mathbb{C}^n) \cap \mathrm{SingSupp}(\mathcal{F})$

Ex: X is a dg point, $U = pt$, $f_1 = \dots = f_n = 0$

$X = \mathrm{Spec}\, A$, $A = \mathbb{C}[\eta_1, \dots, \eta_m]$, $\deg(\eta_i) = -1$, $d\eta_i = 0$

$\deg \xi_i = 2$

Consider $i_x^!(F)$ $\quad D^b_{coh}(X) \xrightarrow[\sim]{i_x^!} D_{coh}(\mathbb{C}[\xi_1, \dots, \xi_n])$

Koszul transform

Where does SingSupp lie?

Consider T^*X as a complex,

$$\begin{array}{ccc} \deg -1 & & \deg 0 \\ \mathcal{O}_X^n & \xrightarrow{df_1, \dots, df_n} & (T^*U)|_X \\ a_1, \dots, a_n & \longrightarrow & \sum a_i df_i \end{array}$$

Consider "$H^{-1}(T^*X)$" total space $X \times \mathbb{A}^n$

$$= \{(x, a_1, \dots, a_n) \in X \times \mathbb{A}^n \ : \ \sum a_i df_i(x) = 0\}$$

$$= \mathrm{Spec}\,(\mathrm{Sym}\, TX[1])$$

Claim: ~~SingSupp~~ For any $F \in D^b_{coh}(X)$, $\mathrm{singSupp}(F) \subseteq$ "$H^{-1}(T^*X)$"

Any closed conical ~~support~~ subset appears as SingSupp

Can be made independent of i.

2^{nd} part

Ref: Avramov – Buchweitz,
Benson – Iyengar – Krause.

What is $\mathrm{Singsupp}(F) \setminus$ zero-section.

$\rightarrow$ only depends on $[F] \in D^b_{coh}(X) / \mathrm{Perf}(X) = \cancel{\mathrm{Sing}(X)}\ D_{\mathrm{Sing}}(X)$ = singularity category of X.

Thm (Orlov) Set $Y = \{(u, b_1, \dots, b_n) \in U \times \mathbb{P}^{n-1} : f_1(u) b_1 + \dots + f_n(u) b_n = 0\}$

Then $D_{\mathrm{Sing}}(X) \xrightarrow[\sim]{\Phi} D_{\mathrm{Sing}}(Y)$

via
$$\begin{array}{ccc} X \times \mathbb{P}^{n-1} & \longrightarrow & Y \\ \downarrow & & \\ X & & \end{array}$$

Exercise: Y is a hypersurface, its singular locus is $\left(H^{-1}(T^*X) \setminus \text{zero section}\right) / \mathbb{G}_m$

Claim: $F \in D^b_{coh}(X)$, $\left(\mathrm{Singsupp}(F) \setminus \text{zero section}\right)/\mathbb{G}_m \subset Y$ is the "support" of $\Phi([F])$: the smallest closed set such that $\Phi([F])$ is perfect on the complement.

Singsupp is local in Zariski topology
so it makes sense if X is lci (not nec. affine)
(also smooth-local, so X could be a stack)

So can look at $D^b_{coh}(X)$: makes sense to enlarge it.

$\rightsquigarrow \mathrm{IndCoh}(X) = QC^!(X)$

Def (H Krause): $\mathrm{IndCoh}(X)$ = homotopy category of complexes of q.coh sheaves on X.

•) is triangulated, with infinite $\oplus$, compactly generated by $D^b_{coh}(X)$

$\rightsquigarrow$ $D^b_{coh}(X) \underset{\text{inj. resolutions}}{\hookrightarrow} \mathrm{IndCoh}(X)$

Other defs: •) View $D^b_{coh}(X)$ as a dg category. Then $\mathrm{IndCoh}(X)$ is its Ind-completion.

•) (Positselski) "coderived category"

There is a notion of $\underset{\subseteq H^{-1}T^*X}{\mathrm{SingSupp}(F)}$ for $F \in \mathrm{IndCoh}(X)$.

For any conical $Y \subset H^{-1}T^*X$ can define $\mathrm{IndCoh}_Y(X) = \{F \in \mathrm{IndCoh}\,X : \mathrm{Singsupp}(F) \subset Y\}$

(ex: $QC^* = D(X) = \mathrm{Ind}(\mathrm{Perf}(X)) = \mathrm{IndCoh}_{\text{zero section}}(X)$)

Define SingSupp F for ~~of~~ $F \in \mathrm{IndCoh}$ by sampling $\mathrm{Ext}^\bullet_{\mathrm{IndCoh}(X)}(G, F)$ for all $G \in D^b_{coh}(X)$

ξ_i

Sing Supp has nice functoriality:

$X_1 \xrightarrow{f} X_2$

$$\begin{array}{c} \mathrm{IndCoh}(X_1) \\ f_* \downarrow \uparrow f^! \\ \mathrm{IndCoh}(X_2) \end{array}$$

$$\begin{array}{ccc} H^{-1}T^*X_1 & & H^{-1}T^*X_2 \\ \cup\!\!\!\mid & & \cup\!\!\!\mid \\ Y_1 & & Y_2 \\ & \nwarrow \quad \nearrow & \\ & (H^{-1}T^*X_1) \times_{X_2} X_1 & \end{array}$$

$f_* \mathrm{IndCoh}_{Y_1}(X_1) \subseteq \mathrm{IndCoh}_{Y_2}(X_2)$