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Simplicial Sets and Higher Categories

05/17/12

Introduction

Homological Algebra works for $\mathcal{A}$ an abelian category
Chain complexes in $\mathcal{A}$
Look at Projective / Injective resolution

Want to do Homotopical Algebra

$\mathcal{C}$ a category   look at simplicial objects in $\mathcal{C}$
Look at Cofibrant / fibrant resolutions

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\deg 0"] \arrow[d] & \text{Chain complex in } \mathcal{A} \arrow[d, "\text{Dold-Kan correspondence}"] \\
\mathcal{C} \arrow[r, "\text{constant}"] & \text{simplicial objects in } \mathcal{C}
\end{tikzcd}$$

Simplicial Objects

Definition Simplicial set is a functor $X : \Delta^{op} \longrightarrow Set$
where
Objects $= [n] = \{0, \ldots, n\}$
Morph $=$ non-decreasing maps

$X_k = X([k])$ gives rise to

$$\begin{tikzcd}
X_0 \arrow[d, phantom, "\parallel"] & X_1 \arrow[l, shift right=2, "d_0"'] \arrow[l, "d_1"] \arrow[d, phantom, "\parallel"] & X_2 \arrow[l, shift right=3, "d_0"'] \arrow[l, "d_1"] \arrow[l, shift left=3, "d_2"] \arrow[d, phantom, "\parallel"] & \cdots \\
\{v_0, \ldots, v_l\} \arrow[u, phantom] & \{e_0, \ldots, e_m\} & \{f_0, \ldots, f_n\}
\end{tikzcd}$$

with $s_0 : X_0 \to X_1$ and $s_0, s_1 : X_1 \to X_2$.

$d_i = X(d^i)$ $\qquad [n] \xrightarrow{d^i} [n+1]$

$s_j = X(s^j)$ $\qquad \{0, \dots, i, \dots n\} \mapsto \{0, \dots, \hat{i}, \dots, n\}$

$$[n+1] \xrightarrow{s^j} [n]$$

$$\{0, \dots, j, j+1, \dots n\} \mapsto \{0, \dots, j, j, \dots, n\}$$

To find "Shape" of X look at

Geometric Realization $|\cdot| : sSet \longrightarrow Top$

$$|X.| = \underbrace{\coprod X_k \times |\Delta^k|}_{\sim} \qquad |\Delta^k| \subseteq \mathbb{R}^{k+1}$$

(note $\exists$ $\Delta^k \in sSet$ with $|\Delta^k| = |\Delta^k|$ in $\mathbb{R}^{k+1}$

$\qquad \overset{\shortparallel}{\mathrm{Hom}_\Delta}(-, \emptyset[n])$

Example $\quad Top \xrightarrow{S} sSet$

$$X \longmapsto SX_n = \mathrm{Hom}_{Top}(|\Delta^n|, X)$$

Singular simplices

$$\boxed{\text{Fact} \quad sSet \underset{S}{\overset{|\cdot|}{\rightleftarrows}} Top \quad \text{are adjoint}}$$

Definition Simplicial object is $X : \Delta^{op} \longrightarrow \mathcal{C}$

Examples ① sSet has simplicial sets

② sAb has simplicial abelian groups (This is where Dold-Kan comes from)

③ sRing has simplicial rings

Dold-Kan **Theorem** $\mathcal{A}$ an abelian category gives an equiv. of categories

$$\left\{\begin{matrix}\text{Simplicial Objects}\\ \text{in } \mathcal{A}\end{matrix}\right\} \xrightarrow{N} \left\{\begin{matrix}\text{non-negative}\\ \text{Chain Complexes}\\ \text{in } \mathcal{A}\end{matrix}\right\}$$

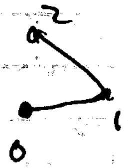
$$A \in \mathcal{A} \quad \pi_i(A) = H_i(NA)$$

"Sketch of Proof"

$$\mathrm{Hom}_{sSet_*}(\Delta^n/\partial\Delta^n, A) = \bigcap_{i=0}^{n} \ker(d_i : A_n \to A_{n-1})$$

$$\Delta^n/\partial\Delta^n \xrightarrow{d^0} \Delta^{n+1}/\Lambda^{n+1}_0 \longrightarrow A$$

Example

If $\Delta^2 =$  then $\Lambda^2_1 =$ 

im d_0 is null homotopic

$$N(X)_n = \bigcap_{i=1}^{n} \ker(d_i : A_n \to A_{n-1})$$

$$N(X)_n \xrightarrow{d_n} N(X)_{n-1}$$

Example G a abelian group

$$G[-n] = \cdots \to 0 \to \overset{n^{th}\text{ degree}}{G} \to 0 \to \cdots$$

gives rise to $K(G, n)$ simplicial object (Eilenberg-MacLane space)

Exercise $K(G,1) = BG$

Model Categories

$\mathcal{C}$ a category

We want to invert some morphisms

so we can regard some morphisms as isomorphisms

(aka Weak Equivs)

Example $\mathcal{C} = Top$

Weak Equivalence = Weak Homotopy Equiv's

(i.e. f is w.e if $\pi_n(X) \xrightarrow{f_*} \pi_n(Y)$ $\forall n$)

Good Objects: CW cmplx $\subset Top$

Thm Whitehead Theorem: weak equiv. = homotopy equiv.

Theorem (CW approximation)

For $X \in Top$ $\exists$ CW cmplx QX with

$$QX \underset{w.e.}{\simeq} X$$

We want

$$\begin{tikzcd} QX \arrow[r, "Qf"] \arrow[d] & QY \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

unique up to homotopy

We need homotopy category

$$Hom_{Ho}(Top)(X,Y) = \overset{\text{Homotopy classes of}}{Hom}(QX,QY)$$

Model Categories have: Weak Equivs W

Cofibrations

fibrations

These give ① good objects

② homotopy classes between good objects

③ lots of well-defined actions on objects

Want to invert W to get

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[dr, bend right] & \mathcal{C}[W^{-1}] \arrow[d, dashed] \\
& D
\end{tikzcd}$$

$\mathrm{Hom}(X, Y)$ = homotopy classes of $\mathrm{Hom}(QX, QY)$

Example $\mathcal{C} = \mathrm{Top}$

Weak equiv = weak homotopy equiv.
fibration = Serre fibration
cofibration = cofibration.

QX = cofibrant replacement

Facts $\exists$ initial object $\emptyset$
terminal object $*$

X is ~~[illegible]~~ cofibrant if $\emptyset \longrightarrow X$ is a cofibration

X is fibrant if $X \longrightarrow *$ is a fibration

Cofibrant replacement - For any X, $\exists\, \hat{X}$ with

$$\begin{tikzcd}
\emptyset \arrow[r, hook, "" , "\text{cofib.}"'] & \hat{X} \arrow[r, "\sim", "\text{w.e.}"'] & X
\end{tikzcd}$$

In: Top Fibrant Objects = Every object

Cofibrant Objects = CW complexes

<u>Example</u> $C = sSet$ Weak equiv. = weak equiv of $|\cdot|$

fibration =

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists \text{lift}"] & Y \end{tikzcd}$$

If lift exists the f is a fibration

Cofibration = $X \xrightarrow{f} Y$

~~such that~~ $X_n \xrightarrow{f_n} Y_n$ is injective for all n.

$$\begin{tikzcd} Set \arrow[r] & sSet \\ \emptyset \arrow[r, mapsto] & \emptyset \\ * \arrow[r, mapsto] & * \end{tikzcd}$$

Everything is Cofibrant

fibrant objects are Kan Complexes

Example $Ch_*(R)$

weak equiv. = quasi-isomorphisms

fibrations = $M \xrightarrow{f} N$ if

$M_n \xrightarrow{f_n} N_n$ surjective for $n \geq 1$

Cofibrations = $M \xrightarrow{f} N$

$M_n \xrightarrow{f} N_n$ injective with proj. coker for $n \geq 0$

Everything is fibrant

($\exists$ 0-object = 0-complex)

Cofibrant Objects = Proj. for each degree

Remarks ① $0 \hookrightarrow \hat{M} \xrightarrow{\simeq} M$ cofib. resolution

② $\exists$ other Model Category Structures which emphasize injective resolutions

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∞-Category

local coord : Manifold ⟷ Model Category : ∞-Category

Look at Simplicially enriched model category

① Cof-fib

② Simplicial nerve functor