Talk 1:

Michigan DAG workshop
May 2012

Motivation

Main analogy:

Non-reduced structure is to algebraic set
as
derived structure is to scheme.

Fix two conics:

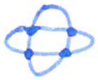
$C = \{q = 0\}$

$C' = \{q' = 0\}$

$q, q' \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$

(over alg. closed field)

Case 1: Transverse intersection



$\rightsquigarrow$ 4 points $\{q = q' = 0\}$, stable under perturbation

Case 2: Tangent intersection

$\rightsquigarrow$ 3 points in $\{q = q' = 0\}$, not stable under perturbation

non-reduced structure sheaf $\rightsquigarrow$ $\dim H^0(\mathbb{P}^2, \underbrace{\mathcal{O}_C \otimes \mathcal{O}_{C'}}_{\text{alg. object}}) = 4$

geom. object: $\operatorname{Spec}(\mathcal{O}_C \otimes \mathcal{O}_{C'})$

Case 3: Degenerate intersection $q = q'$

$C \cap C \rightsquigarrow \mathcal{O}_C \overset{L}{\otimes} \mathcal{O}_C$

resolution: $\mathcal{O}_C \longleftarrow \left[\mathcal{O}_{\mathbb{P}^2} \xleftarrow{\cdot q} \mathcal{O}_{\mathbb{P}^2}(-2) \longleftarrow 0\right]$

$\rightarrow \mathcal{O}_C \overset{L}{\otimes} \mathcal{O}_C = \left(\mathcal{O}_{\mathbb{P}^2} \xleftarrow{(q,q')} \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 2} \xleftarrow{\binom{q'}{-q}} \mathcal{O}_{\mathbb{P}^2}(-4)\right)$

For all C, C' we have:

1) The locus where $\mathcal{O}_C \otimes \mathcal{O}_{C'}$ fails ~~to~~ to be exact equals $C \cap C'$.

2) $\dim \mathbb{H}^0(\mathbb{P}^2, \mathcal{O}_C \overset{L}{\otimes} \mathcal{O}_{C'}) = 4$

non-degenerate case: $\mathbb{H}^0(\mathbb{P}^2, \mathcal{O}_C \overset{L}{\otimes} \mathcal{O}_{C'}) = H^0(\mathbb{P}^2, \mathcal{O}_C \otimes \mathcal{O}_{C'})$

degenerate case:

$$\mathbb{H}^0(\mathbb{P}^2, \mathcal{O}_C \overset{L}{\otimes} \mathcal{O}_C) \underset{\substack{\uparrow \\ \text{spectral} \\ \text{sequence}}}{=} \begin{array}{c} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \\ \oplus \\ H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 2}) \\ \oplus \\ H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-4)) \end{array} = \begin{array}{c} 1 \\ + \\ 0 \\ + \\ 3 \end{array} = 4$$

Q: Is there any geometric object associated to $\mathcal{O}_C \overset{L}{\otimes} \mathcal{O}_{C'}$?

Serre's Tor-Formula: R reg. local ring, $I, J \subseteq R$ with ~~$\exists$~~ $R/I+J$ 0-dimensional

$$\mu(R/I, R/J) = \sum_{i \geq 0} (-1)^i \text{ length } \operatorname{Tor}_i^R(R/I, R/J) = \text{“length } R/I \overset{L}{\otimes} R/J\text{”}$$

$R = k[x,y]$ $\qquad$ $R = (\text{Set}, +, \cdot, 1)$

Categorical ring: $\mathcal{C} = (\text{Category}, \hat{+}, \hat{\cdot}, \hat{1})$

given $R \rightsquigarrow \mathcal{C}_R$: objects $= \{r \in R\}$

morphisms $= \operatorname{Mor}(r, r') = \begin{cases} \{*\} & , r = r' \\ \emptyset & , r \neq r' \end{cases}$

$\hat{+} : \mathcal{C}_R \times \mathcal{C}_R \longrightarrow \mathcal{C}_R$

$(r, r') \longmapsto r + r'$, etc.

Impose $q = 0 \rightsquigarrow$ new category with  , i.e. $\operatorname{Mor}(r, r') = \begin{cases} \{*\} & r - r' \in (q) \\ \emptyset & \text{else} \end{cases}$

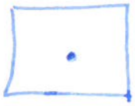
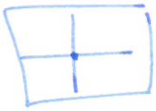
$\Longrightarrow$ iso classes are exactly $R/(q)$

Impose $q = 0$ twice $\rightarrow$ get  (and everything that is forced by axiom)

iso classes ~~of~~ are still R/q

$\rightsquigarrow$ this gives not the complete picture, need higher ~~gen~~ structures.

$$\begin{tikzcd}
\underline{\text{Homological}} & & \underline{\text{Categorical / Simplicial}} \arrow[ll, "\text{Dold - Kan Thm}"'] \\
\text{complexes} & & \text{simplicial algebra} \arrow[ll] \\
H_i^{?}(\text{complex}) & & \pi_i \arrow[ll] \\
\text{Tor}_i \arrow[u, phantom, "\shortparallel"] & &
\end{tikzcd}$$

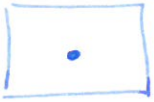
 can arise from, e.g  or 

$\longrightarrow$ different scheme structures.

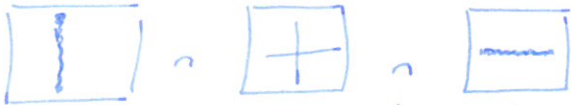
Same for derived structures

Ex: $R = k[x,y]$, $Q = R/(x^2, xy, y^2)$

•) Spec $Q \in \mathbb{A}^2$

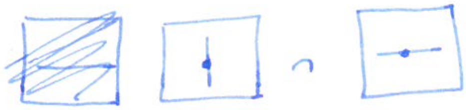


$$Q \leftarrow [R \leftarrow R^3 \leftarrow R^2 \leftarrow 0] \qquad \pi_i = \begin{cases} Q & i=0 \\ 0 & \text{otherwise} \end{cases}$$

•) 

$$R/(x^2) \overset{L}{\otimes} R/(xy) \overset{L}{\otimes} R/(y^2)$$

$$R^1 \leftarrow R^3 \leftarrow R^1 \leftarrow 0 \qquad \pi_i = \begin{cases} Q & , i=0 \\ (x,y)/(x^2,y^2) & , i=1 \\ 0 & , i \geq 2 \end{cases}$$

•) 

$R/(x^2,xy) \overset{L}{\otimes} R/(y^2,xy)$ gives 3 homotopy groups.

Talk 2:

Simplical sets & higher categories:

I. Introduction

$$\begin{tikzcd} \text{Homological algebra} & \text{works for Ab. cat } \mathcal{A} \arrow[r, hook, "\deg 0"] & \text{via chain cx in } \mathcal{A} \arrow[d, leftrightarrow, "\text{DK-corresp.}"] & \text{inj/proj resolutions} \\ \text{Homotopical} & \text{category } \mathcal{C} \arrow[r, hook, "\text{const.}"] & \text{simplicial obj in } \mathcal{C} & \text{cofib/fib resolution} \end{tikzcd}$$

II. Simplicial objects and DK-corresp.

Def.: A simplicial set is a functor $X : \Delta^{op} \longrightarrow \text{Sets}$

etc.

Talk 5

Simplicial Commutative Rings I

$$\begin{tikzcd}
\mathrm{Aff}_{n\text{-var}} \arrow[d, rightsquigarrow] & = (\mathrm{Alg}_{/k,red})^{op} \\
\mathrm{Aff} \arrow[d, rightsquigarrow] & = (\mathrm{CRing})^{op} \\
?
\end{tikzcd}$$

Candidate: ~~topological~~ comm. rings

$\rightarrow$ simplicial

Equivalence: simplicial sets $\simeq$ top. spaces

Def: A simplicial comm. ring is an element of $\mathrm{Fun}(\Delta^{op}, \mathrm{CRing}) = \mathrm{SCR}$

Ex: $\mathrm{CRing} \hookrightarrow s\mathrm{CR}$

$R \longmapsto$ constant

Ex: $X_\bullet \in s\mathrm{Set} \rightsquigarrow (\mathbb{Z}[X_\bullet])_n = \mathbb{Z}[X_n]$

~~Ex: 1-truncated homotopy-types $\simeq$ groupoids~~

~~A categorical ring =~~

Homotopy groups:

Def: $R_\bullet \in s\mathrm{CR}$, $\pi_i R = [(\underset{\Delta^i/\partial\Delta^i}{\underset{\|}{S^i}}, *), (R_\bullet, 0)] = H_i(NR_\bullet)$

$\uparrow$ this is fibrant

Claim: $R_\bullet \in s\mathrm{CR}$. Then $\pi_* R$ is graded-commutative.

$f: S^i \to R_\bullet$

$g: S^j \to R_\bullet$ $\quad f \wedge g : S^{i+j} \longrightarrow R_\bullet \wedge R_\bullet \longrightarrow R_\bullet$

$\ulcorner A_\bullet, B_\bullet \rightsquigarrow A_\bullet \wedge B_\bullet = \dfrac{A \times B}{* \times B \cup A \times *}$

Ex: $\pi_0 R = R_0 / (d_1 - d_0) R_1$

Ex: If $R \in \mathrm{CRing}$, there is a map $R \to R_{red}$

If $R_\bullet \in \mathrm{sCR}$, there is a map $R_\bullet \to \pi_0 R.$

Goal: Given $R, S \in \mathrm{sCR}$, want $\underline{\mathrm{Hom}}(R_\bullet, S_\bullet)$

To get this we'll define $K_\bullet \otimes R_\bullet$ for $K_\bullet \in \mathrm{sSet}$, $R_\bullet \in \mathrm{sCR}$

Then $\mathrm{Map}_{\mathrm{sSet}}(T_\bullet, \underline{\mathrm{Hom}}(R_\bullet, S_\bullet)) = \mathrm{Map}_{\mathrm{sCR}}(T_\bullet \otimes R_\bullet, S_\bullet)$ can be used to define $\underline{\mathrm{Hom}}$

Let $\mathcal{C}$ be a category w/coprods. Set $s\mathcal{C} = \mathrm{Fun}(\Delta^{op}, \mathcal{C})$.

Let $X_\bullet \in s\mathcal{C}$, $K_\bullet \in \mathrm{sSet}$, then $(K \otimes X)_\bullet := \coprod_{K_n} X_n$

The model structure:

1) A weak equivalence $R_\bullet \to S_\bullet$ is an iso on π_*

2) A fibration is a Kan fibration (so everything is fibrant)

3) The cofibrations are determined by this.

$$\begin{tikzcd} \mathrm{sSet} \arrow[r, shift left, "\mathbb{Z}[\cdot]"] & \mathrm{sCR} \arrow[l, shift left, "\text{forget}"] \end{tikzcd}$$ adjoint

Use this to lift the model structure

Ex: $R_\bullet \to S_\bullet$

If there exist sets $A_n \subseteq S_n$ s.t.

1) $S_n = R_n[A_n]$

2) A_n is preserved under degeneracies

Then $R_\bullet \to S_\bullet$ is a cofibration.

Claim: If $R_\bullet$ is a cofibrant simplicial ring, then for any $S_\bullet$, $\underline{\mathrm{Hom}}(R_\bullet, S_\bullet)$ is the correct homotopy type.

sCR is a simplicial model category.

Let $R_\bullet \in sCR$

Want $\mathrm{Mod}(R_\bullet)$, simplicial modules.

$$\mathrm{Mod}(R_\bullet) = \left\{ \begin{array}{l} \text{simplicial sets } M_\bullet \text{ w/ a map} \\ R_\bullet \times M_\bullet \longrightarrow M_\bullet \text{ satisfying the usual module axioms.} \end{array} \right\}$$

So: $\pi_* R \times \pi_* M \longrightarrow \pi_* M$ makes $\pi_* M$ a graded module.

Thm (Quillen): $\exists$ a model structure on $\mathrm{Mod}(R_\bullet)$.

Ex: $$\begin{tikzcd} C_\bullet & & B_\bullet \\ & A_\bullet \arrow[ul] \arrow[ur] & \end{tikzcd}$$ $B_\bullet \otimes_{A_\bullet} C_\bullet$ only correct if cofibrant.

Define: $B_\bullet \overset{L}{\otimes}_{A_\bullet} C_\bullet = \tilde{B}_\bullet \otimes_{A_\bullet} \tilde{C}_\bullet$ where $$\begin{tikzcd} A \arrow[rr] \arrow[dr, hook] & & B \\ & \tilde{B} \arrow[ur, "\sim"'] & \end{tikzcd}$$

this is homotopy invariant.

Ex: $R[y] \longrightarrow R$, $y \longmapsto 0$ want $R \overset{L}{\otimes}_{R[y]} R$

- - -

Let $\mathcal{C}$ be a cat $T: \mathcal{C} \longrightarrow \mathcal{C}$, X a T-alg (i.e. $TX \to X$)

We ~~f~~ will find a simplicial T-alg $B(T,X)_\bullet \longrightarrow X$

$\uparrow$ Bar construction

$$\begin{tikzcd} TX \arrow[r, leftarrow, shift left=2] \arrow[r, shift right] \arrow[r, leftarrow, shift right=3] & T^2X \arrow[r, shift left=3] \arrow[r, leftarrow, shift left=1] \arrow[r, leftarrow, shift right=1] \arrow[r, leftarrow, shift right=3] & T^3X \cdots \end{tikzcd}$$

This is a simplicial homotopy equivalence.

- - - -

$\mathcal{C} = s\mathrm{Set}$ T-alg $= s\mathrm{CRing}$

$T = \mathbb{Z}[\;]$

If $X \in s\mathrm{CRing}$ get cofibrant $B(T,X)_\bullet \longrightarrow X$

not very efficient... $\mathbb{Z}[X] \leftrightarrows \mathbb{Z}[\mathbb{Z}[X]] \equiv \cdots$

$$B(R)_n = R[y]^{\otimes(n+1)}$$

$$= \{\, g[f_1|\dots|f_n] \quad , g, f_i \in R[y] \qquad (R\text{-linear}) \,\}$$

$$d_i\,(g[f_1|\dots|f_n]) = \begin{cases} gf_1[f_2|\dots|f_n] & i=0 \\ g[f_1|\dots|f_if_{i+1}|\dots|f_n] & 0<i<n \\ g\varphi(f_n)[f_1|\dots|f_{n-1}] & i=n \end{cases} \qquad \varphi : R[y] \to R$$

$$\begin{tikzcd} \vdots \arrow[d, shift right=4] \arrow[d, shift right=2, leftarrow] \arrow[d] \arrow[d, shift left=2, leftarrow] \arrow[d, shift left=4] \\ R[y]\otimes R[y] \arrow[d, shift right=2] \arrow[d, leftarrow] \arrow[d, shift left=2] \\ R[y] \end{tikzcd}$$

Thm: $R[y] \hookrightarrow B(R) \xrightarrow{\sim} R$

$R \overset{L}{\underset{R[y]}{\otimes}} R = B(R) \underset{R[y]}{\otimes} R$ gives the same as ↗ except for $i=0$: $g\varphi(f_1)[f_2|\dots|f_n]$

$$\pi_*\Big(B(R)\underset{R[y]}{\otimes} R\Big) = \left\{ \begin{matrix} R & \text{in dim } 0,1 \\ 0 & \text{otherwise} \end{matrix} \right\} = \mathrm{Tor}_i^{R[y]}(R,R).$$

Ex: If R is any ring $y \in R$ a non-zerodivisor

$$R \longrightarrow R/(y)$$

$$\begin{tikzcd} R[x] \arrow[r, "x\mapsto 0"] \arrow[d, "x\mapsto y"] & R \arrow[d] \\ R \arrow[r] & \cancel{R/(y)}\ B(R)\underset{R[y]}{\otimes} R \arrow[r] & R/(y) \end{tikzcd}$$

Talk 4:

SIMPLICIAL COMMUTATIVE RINGS II

Notation:

$X, Y \in sCR$ or Mod_A , $K \in sSet$

$(K \otimes X)_n = \coprod_{K_n} X_n$

$(\underline{Hom}(X, Y))_n = Hom(\Delta^n \otimes X, Y)$

$\rightsquigarrow$ $\mathbb{R}\underline{Hom}(X, Y) = Hom(Q(X), Y)$ where $Q(X)$ cofib replacement

$M, N \in Mod_A$, $* \in K$ base pt

define $K \wedge M$ by

$$\begin{tikzcd} * \otimes M \arrow[r] \arrow[d] & K \otimes M \arrow[d] \\ 0 \arrow[r] & K \wedge M \end{tikzcd}$$ pushout

$M[n] = \Sigma^n M = S^n \wedge M$, $S^n = \Delta^n / \partial \cancel{\Delta}\Delta^n$

$(M \otimes N)_n = M_n \otimes_{A_n} N_n$ $\rightsquigarrow$ $M \overset{\mathbb{L}}{\otimes}_A N$

Attaching cells

Q: How do we construct simplicial res for modules $R^J \to R^I \to M$.

$$\begin{tikzcd} S^{n-1} \arrow[r, hook] \arrow[d, equal] & D^n \arrow[d, equal] \\ \Delta^{n-1}/\partial\Delta^{n-1} & \Delta^n/\Lambda^n_0 \end{tikzcd}$$

$|S^{n-1}| \longrightarrow |D^n|$

$d_0(b) = \cdots = d_{n-1}(b) = x$

$d_1(c) = \ldots = d_n(c) = *$

$d_0(c) = b$

notation $\mathbb{Z}[K] = K \otimes \mathbb{Z}[x]$, $(\mathbb{Z}[K])_n = \mathbb{Z}[K_n]$, $\mathbb{Z}[(K,*)]_n = \mathbb{Z}[K_n]/(*)$

$$\begin{tikzcd} A \arrow[r] & A' \\ \mathbb{Z}[(S^{n-1},*)] \arrow[u] \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathbb{Z}[(D^n,*)] \arrow[u] \end{tikzcd}$$

$\mathrm{Hom}_{sCR}(\mathbb{Z}[(S^{n-1},*)],$ ~~$\mathbb{Z}\mathbb{Z}$~~ $A) = \mathrm{Hom}_{sSet_*}((S^{n-1},*),(A,0))$

$[\mathbb{Z}[(S^{n-1},*)], A]_{sCR} = [(S^{n-1},*),(A,0)]_{sCR_*} = \pi_{n-1}(A)$

fix $f : (S^{n-1},*) \longrightarrow (A,0)$ representing $w \in \pi_{n-1}(A)$.

$(A'_{\cancel{j}})_j = A_j[(D^n)_j] \,/\, (x = f(x)) \quad x \in (S^{n-1})_j$

$\rightarrow$ $A'_j = A_j$ for $j < n$

$A'_{\cancel{j}n} = A_n[c]$

$\rightsquigarrow$ $\pi_j(A') = \pi_j(A)$

$0 \rightarrow A_{n-1} w \rightarrow \pi_{n-1}(A) \rightarrow \pi_{n-1}(A') \rightarrow 0 \qquad c \in N(A')_n, \; d(c) = w$

write $A' = A[X \mid \partial(X) = w]$

~~Ex.~~

Symmetric Algebras

$A \in sCR$

Define $\mathrm{Sym}_A : \mathrm{Mod}_A \longrightarrow sCR_{A/}$ by $\mathrm{Sym}_A(M)_n = \mathrm{Sym}_{A_n}(M_n)$

has right adjoint the forgetful functor

$\rightsquigarrow$ $\mathbb{L}\mathrm{Sym}_A$.

Ex : •) $\mathbb{L}\mathrm{Sym}^n_A(M[1]) \simeq (\mathbb{L}\Lambda^n_A(M))[n]$

•) $\mathbb{L}\mathrm{Sym}^n_A(M[2]) \simeq (\mathbb{L}\Gamma^n_A(M))[2n]$

If A is a $\mathbb{Q}$-algebra $\mathrm{Sym}^n = \Gamma^n$

Ex: what is Sym $\mathbb{Z}[n-1]$?

$$\begin{tikzcd}
* \otimes \mathbb{Z} \arrow[r] \arrow[d] & S^{n-1} \otimes \mathbb{Z}^{n} \arrow[d] & & \mathbb{Z}[*] \arrow[r] \arrow[d] & \mathbb{Z}[S^{n-1}] \arrow[d] \\
0 \arrow[r] & \mathbb{Z}[n-1] \arrow[r, squiggly, "\mathrm{Sym}_{\mathbb{Z}}"] & {} & \mathbb{Z} \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & \mathrm{Sym}\, \mathbb{Z}[n-1]
\end{tikzcd}$$

$\to$ $\mathrm{Sym}\, \mathbb{Z}[n-1] \simeq \mathbb{Z}[(S^{n-1}, *)]$

$\rightsquigarrow$ Can also describe attaching cells by $A \overset{\mathbb{L}}{\underset{\mathrm{Sym}\,\mathbb{Z}[n-1]}{\otimes}} \mathbb{Z}$

- $A[X, \partial(X) = 0] = \mathrm{Sym}_A A[n]$

$K \oplus K[i]$

$A \in sCR$, $M \in \mathrm{Mod}_A$

$\rightsquigarrow$ "trivial extension" $A \oplus M$: $(A \oplus M)_n = A_n \oplus M_n$

$(a, m)(a', m') = (aa', am' + a'm)$

K a field. $K \oplus K[i]$ "higher dual numbers"

ex: $K \oplus K[0] = K[\varepsilon]/\varepsilon^2$

$\exists$ a homotopy pullback

$$\begin{tikzcd}
(*) & k[\varepsilon]/\varepsilon^{n+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & k[\varepsilon]/\varepsilon^{n} \arrow[d] \arrow[r, squiggly, "\pi_0 \text{ ``deformations'' over } (-)"] & \begin{pmatrix} \text{defos over} \\ k[\varepsilon]/\varepsilon^{n+1} \end{pmatrix}/\sim \arrow[r] & \begin{pmatrix} \text{defos over} \\ k[\varepsilon]/\varepsilon^{n} \end{pmatrix}/\sim \arrow[d] \\
& k \arrow[r] & K \oplus K[1] & & \begin{pmatrix} \text{``defos over} \\ k \oplus k[1]\text{''} \end{pmatrix}/\sim
\end{tikzcd}$$

$\qquad$ ‖

obstruction classes

How do we produce $(*)$?

- char 0 $\rightsquigarrow$ can see it using cdgas
- In general replace $k[\varepsilon]/\varepsilon^n$ by $K[\varepsilon][X \mid \partial(X) = \varepsilon^n]$

$$\begin{tikzcd}
& k[\varepsilon][X \mid \partial(X) = \varepsilon^n] \arrow[d] & \varepsilon \arrow[d, mapsto] & X \arrow[d, mapsto] \\
k \arrow[r] & k \oplus k[1] & 0 & 1 \in k[1]
\end{tikzcd}$$

Flatness, étaleness, smoothness

<u>Def / Prop</u>: $A \in sCR$, $M \in Mod_A$ is <u>flat</u> if the following equivalent definitions hold.

1) $\pi_0(M)$ is a flat $\pi_0(A)$-module

   $\pi_n(R) \otimes_{\pi_0(R)} \pi_0(M) \xrightarrow{\sim} \pi_n(M) \quad \forall n$

2) $M \overset{L}{\otimes} -$ commutes with finite homotopy limits.

3) M is a filtered colimit of finite free A-modules

2') $M \overset{L}{\otimes} -$ commutes w/ Ω $\left(\begin{tikzcd} \Omega M \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & 0 \arrow[d] \\ 0 \arrow[r] & M \end{tikzcd} \right)$

2'') N a discrete A-module, then $M \overset{L}{\otimes} N$ is discrete.

<u>Def</u>: $A \to B$ in sCR is <u>étale</u> (resp. <u>smooth</u>) if it is flat and $\pi_0(A) \to \pi_0(B)$ is étale (resp. smooth).

Finiteness Conditions

<u>Def</u>: $A \to B$ in sCR. Say B is a

- <u>finitely-presented</u> A-alg if it can be obtained by attaching finitely many cells;
- <u>locally-finitely-presented</u> A-alg if it is a retract of a f.p. A-alg.
- <u>almost finitely presented</u> A-alg if $\forall n \; \exists B_n$ f.p. A-alg and $f_n : B_n \to B$ s.t. $\pi_i(B_n) \xrightarrow{\sim} \pi_i(B)$ $i \leq n$.

analoguous conditions for modules:
- finitely presented
- perfect
- almost perfect

<u>Ex</u>: R discrete, $M \in Mod_R$ f.p.

M perfect $\Rightarrow pd_R M < \infty$

locally f.p. is "compact", i.e. $\mathbb{R}\underline{Hom}(B, -)$ commutes will filtered colimits

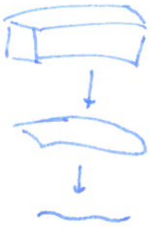
Talk 5

THE COTANGENT COMPLEX I: THE "CLASSICAL THEORY"

Motivation

$B/A \rightsquigarrow B \xrightarrow{d} \Omega^1_{B/A}$ well behaved if B/A is smooth

$$\begin{tikzcd} C & X \arrow[d] \\ B \arrow[u] & Y \arrow[d] \\ A \arrow[u] & Z \end{tikzcd}$$



$0 \to T_{X/Y} \to T_{X/Z} \to T_{Y/Z} \to 0$

$\rightsquigarrow \underset{\substack{\uparrow \\ X/Y \text{ smooth}}}{(0 \to)} f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$

$\underset{\text{0 if smooth}}{?} \to I/I^2 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to 0$

Motivation II: Deformation Theory.

$L_{X/Y} \in D(X)$

Thm: (i)

$$\begin{tikzcd} X_0 \arrow[r, "g"] \arrow[d, "j \text{ sq.0}"'] & Y \arrow[d] \\ X \arrow[r] \arrow[ur, dashed, "h"] & S \end{tikzcd}$$

h exists iff obstruction $o(g,j) \in \mathrm{Ext}^1(g^* L_{X/S}, J)$ vanishes.
If $o(g,j)=0$ then the set of lifts is ? $\mathrm{Hom}(g^* L_{X/S}, J)$

(ii)

$$\begin{tikzcd} X_0 \arrow[r, dashed] \arrow[d, "\text{flat}"', "f_0"] & X \arrow[d, dashed, "i"] \\ S_0 \arrow[r, "i \text{ sq.0}"'] & S \end{tikzcd}$$

X exists if $o(X_0, i) = 0$ in $\mathrm{Ext}^2(L_{X_0/S_0}, f_0^*(I))$

Such X ? for $\mathrm{Ext}^1(L_{X_0/S_0}, f_0^*(I))$

Application 1:

A perfect $\mathbb{F}_p$-algebra, e.g. $\mathrm{Frob}_p : A \to A$ is an?

$$\begin{tikzcd}
\mathrm{Spec}\, A \arrow[d] & & \\
\mathrm{Spec}\, \mathbb{F}_p \arrow[r] & \mathrm{Spec}\, \mathbb{Z}/p^2\mathbb{Z} \arrow[r] & \mathrm{Spec}\, \mathbb{Z}/p^3\mathbb{Z} \arrow[r] & \dots
\end{tikzcd}$$

obstruction: $\mathrm{Ext}^2(\underset{\substack{\shortparallel \\ 0}}{L_{A/\mathbb{F}_p}}, \underset{\substack{\shortparallel \\ \mathbb{F}_p}}{f^\circ(\langle p^{k-1}\rangle)}) = 0$.

as $L_{A/\mathbb{F}_p} \otimes_A A \xrightarrow[0]{\sim} L_{A/\mathbb{F}_p}$

$$d(x \to x^p) = p\, x^{p-1} = 0$$

Thm: Let A be a complete local Noetherian ring with residue field k.

$X_{0/k}$ is a proper curve, lci, ?

Then $\exists$ X/A projective + flat with $X_k \simeq X_0$

$\boxed{\text{etc.}}$

3. Definition / Construction:

Def: A projective A-alg resolution of $\overset{\swarrow \text{ constant simpl. ring}}{cB}$ is a factorization

$$\begin{tikzcd}
cA \arrow[r] \arrow[dr, "\text{cof}"'] & cB \\
& P \arrow[u, "\text{trivial fibration}"']
\end{tikzcd}$$

Def (Cotangent Complex):

$$(L_{B/A})_n = (\Omega_{P_n/A} \otimes_{P_n} \overset{B}{\cancel{A}})$$

Def (AQ (co)homology)

$$D_q(B/A, M) = H_q(L_{B/A} \otimes_B M)$$

$$D^q(B/A, M) = H^q(\mathrm{Hom}_B(L_{B/A}, M))$$

4. Properties:

1) $D_0(B/A, B) = D_0(B/A) = \Omega^1_{B/A}$

2) $L_{B/A}$ is a ~~projec~~ complex of projectives

3) An extension of B by M is a s.e.s. $0 \to \underset{\uparrow \text{sq. } 0}{M} \to X \to B \to 0$

Let $\mathrm{Exalcom}(B/A, M)$ be the set of isoclasses of such extensions.

Then $D^1(B/A, M) = \mathrm{Exalcom}(B/A, M)$

4) Suppose $B = A/I$. Then $D_0 = 0$, $D_1 = I/I^2$

5) Suppose

$$\begin{tikzcd} R' \arrow[r] & S' = S \otimes_R R' \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

, $\mathrm{Tor}_q^R(R', S) = 0 \quad \forall q > 0$

Then $L_{S/R} \otimes_R R' = L_{S'/R'}$

$$L_{S'/R} \simeq L_{S/R} \otimes_R R' \oplus L_{R'/R} \otimes_R S$$

6) $L_{S^{-1}A/A} = 0$

7) If A is Noetherian and B is a finite type A-alg.

Then B étale $\iff L_{B/A} = 0$

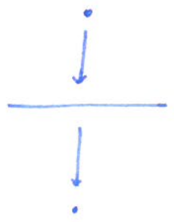
B smooth$/A \iff L_{B/A} \simeq c(\Omega^1_{B/A})$, $\Omega^1_{B/A}$ projective

8) (Transitivity Triangle)

If $A \to B \to C$ are maps of rings, then there is a d.t. in $D(C)$

$$\begin{tikzcd} L_{B/A} \otimes_B C \arrow[rr] & & L_{C/A} \arrow[dl] \\ & L_{C/B} \arrow[ul, dashed, "+1"] & \end{tikzcd}$$

Ex:



$$\begin{tikzcd} k \\ k[\epsilon] \arrow[u, "0 \mapsfrom t"'] \\ k \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} k[\epsilon] \arrow[d, phantom, "\shortparallel"] & & 0 \arrow[d, phantom, "\shortparallel"] \\ L_{k[\epsilon]/k} \otimes_{k[\epsilon]} k \arrow[rr] & & L_{k/k} \arrow[dl] \\ & L_{k/k[\epsilon]} \arrow[ul, dashed] \arrow[r, Rightarrow] & k[1] \end{tikzcd}$$

Talk 6

COTANGENT COMPLEX II

I. Review

$f: A \to B$, $A, B \in sCR$

Quillen adjoint pair

$$\left.\begin{array}{llll} \text{left}: & F: & sCR_{A//B} \longrightarrow & \mathrm{Mod}_B \\ & & X \longmapsto & \Omega_{X/A} \otimes B \\ \text{right} & G: & \mathrm{Mod}_B \longrightarrow & sCR_{A//B} \\ & & M \longmapsto & B \oplus M \end{array}\right\} \Rightarrow \text{Total derived functors exist and are adjoint.}$$

$$\rightsquigarrow L_{B/A} := LF(B)$$

$$\underset{\substack{\uparrow \\ h(\mathrm{Mod}_B)}}{\mathrm{Hom}}(L_{B/A}, M) = \underset{\substack{\uparrow \\ h(sCR_{A//B})}}{\mathrm{Hom}}(B, \overset{\substack{B\oplus M \\ \parallel}}{RG(M)})$$

II. Connectivity Results

Thm: $f: A \to B$ in sCR. If $\mathrm{Cofib}(f)$ is n-connected, then the natural map $\varepsilon: \mathrm{cofib}(f) \otimes_A B \to L_{A/B}$ is $(2n+1)$-connected

Cor: If $\mathrm{cofib}(f)$ is n-connected then $L_{B/A}$ is n-connected.

The converse is true if $\pi_0(A) \simeq \pi_0(B)$.

Cor: $f: A \to B$ is an equivalence iff

1) $\pi_0(A) \simeq \pi_0(B)$

2) $L_{B/A} \simeq 0$

Cor: $A \longrightarrow \pi_0(A)$ is an equiv iff $L_{\pi_0(A)/A} \simeq 0$.

III. Deformation Theory

A, B, M ordinary rings/module

$$\begin{array}{l} \mathrm{Ext}^i(L_{B/A}, M) \\ \quad \| \\ \mathrm{Ext}^0(L_{B/A}, M[i]) = \mathrm{Hom}(B, B\oplus M[i]) \end{array}$$

"$\mathrm{Spec}(B\oplus M[i]) \to \mathrm{Spec}\, B$"

$$\begin{tikzcd} X \arrow[d, "\text{schemes}"'] & & \\ Y & \arrow[r, rightsquigarrow] & L_{X/Y} \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[r, dashed] \arrow[d, "f"'] \arrow[dr, phantom, "?"] & X' \\ M & Y \arrow[r, hook, "j", "\text{sq. 0}"'] \arrow[d] & Y' \arrow[u, dashed] \arrow[dl] \\ & S & \end{tikzcd}$$

Thm: (i) There is an obstruction $ob(f) \in \mathrm{Ext}^2(L_{X/Y}, f^*M)$, vanishing iff deformation exists

(ii) if $ob(f) = 0$, then $\{\text{deformations}\}/\simeq \; = \mathrm{Ext}^1(L_{X/Y}, f^*M)$.

Rmks: 1) $\mathrm{Ext}^1(L_{Y/S}, M)$ classifies sq. 0. ext's of Y by M over S.

$$\begin{tikzcd} X \arrow[d] & \\ Y \arrow[r] & Y' \end{tikzcd}$$

2) $$\underset{\mathrm{Ext}^2(L_{X/Y}, f^*M)}{\underset{\cap}{ob(f)}} = \text{composition of } \underset{\mathrm{Ext}^1(f^*L_{Y/S}, f^*M)}{\underset{\cap}{f^*[j]}} \text{ with } \overset{\mathrm{Ext}^1(L_{X/Y}, f^*L_{Y/S})}{\overset{\cup}{KS(X \to Y \to S)}}$$

Construction: $A \in sCR$, $M \in \mathrm{Mod}_A$

$$\begin{tikzcd} A\oplus M[1] \arrow[d] \\ A \arrow[u, bend left, "s"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} A' \arrow[r, "\varphi_s"] \arrow[d] & A \arrow[d, "s"] \\ A \arrow[r, "0"'] & A\oplus M[1] \end{tikzcd} \quad \text{pullback.}$$

Def: φ_s is called a <u>square zero extension</u> of A by M.

Ex:
- $A \oplus M$ (for $s = 0$)
- $k \oplus k[i]$
- square-zero extensions of ordinary rings.

Prop: Given $A \in sCR$, then every map in

$$\cdots \to \tau_{\leq 2} A \to \tau_{\leq 1} A \to \tau_{\leq 0} A = \pi_0 A$$

is a square zero extension.

onsequence:

$$\begin{tikzcd}
\tau_{\leq n} A \arrow[r] \arrow[d] & \tau_{\leq n-1} A \arrow[d, "k_n \in \mathrm{Hom}(L_{\tau_{\leq n-1} A}\,,\ \pi_n(A)[n+1])"] \\
\tau_{\leq n-1} A \arrow[r, "0"] & \tau_{\leq n-1} A \oplus \pi_n(A)[n+1]
\end{tikzcd}$$

Let

$$\begin{tikzcd}
B & B' \arrow[l, dashed] \\
A \arrow[u, "f"] & A' \arrow[l, "\substack{\text{s.z.e}\\ \text{by } M}"'] \arrow[u, dashed]
\end{tikzcd}$$

f flat

Thm: (i) There exists an obstruction $ob(f) \in Ext^2(L_{B/A}, B \otimes_A M)$ which vanishes iff deformation exists.

(ii) when $ob(f) = 0$, then $\{\text{deformations}\}/_{\cong} = Ext^1(L_{B/A}, B \otimes_A M)$.

explicitly, $ob(f)$ is $L_{B/A}[-1] \to B \otimes L_A \to B \otimes M[1]$.

$$\begin{tikzcd}
X = X_0 \arrow[r] \arrow[d, "f"', "\substack{\text{smooth}\\ \text{proj}}"] & X_1 \arrow[r, dashed] \arrow[d, "f_1"] & X_2 \arrow[d, dashed] \\
\mathrm{Spec}\, k \arrow[r] & \mathrm{Spec}(k[\varepsilon]/_{\varepsilon^2}) \arrow[r] & \mathrm{Spec}(k[\varepsilon]/_{\varepsilon^3})
\end{tikzcd}$$

$$ob(f_1) \in H^2(T_X) = Ext^2(L_X, k) = Ext^1(L_X, k[1])$$

classifies

$$\begin{tikzcd}
 & & X \arrow[r] \arrow[d] & X' \arrow[d] & \\
 & & \mathrm{Spec}\, k \arrow[r] & \mathrm{Spec}(k \oplus k[1]) & \\
 & X_1^{ob} \arrow[rr] \arrow[d, "ob(f_1)"] & & X_1 \arrow[d, "f_1"] & \\
X^{triv} \arrow[r] \arrow[d] & \mathrm{Spec}(k \oplus k[1]) \arrow[rr] \arrow[d] & & \mathrm{Spec}(k) \arrow[d] & \text{pushout} \\
X \arrow[r] & \mathrm{Spec}(k[\varepsilon]/_{\varepsilon^2}) \arrow[rr] & & \mathrm{Spec}(k[\varepsilon]/_{\varepsilon^3}) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & k[\varepsilon]/_{\varepsilon^3} \arrow[r] \arrow[d] & k[\varepsilon]/_{\varepsilon^2} \arrow[d] \\
\text{llback} & k \arrow[r] & k \oplus k[1]
\end{tikzcd}$$

Talk 7

DERIVED SCHEMES : representability theorems

I. Artin's theorem

R an excellent noetherian ring

F: $\mathrm{Alg}_{R/-} \longrightarrow (\mathrm{Grpd})$

F is representable by an Artin stack locally of f.p. /R iff

a) $F(\operatorname{colim}_i A_i) = \operatorname{colim}_i F(A_i)$ for filtered diagrams $i \mapsto A_i$

b) F is a sheaf for the étale topology:

If $\{U_i \to X\}$ is an étale cover of affine, then $F(X) \xrightarrow{\sim} \operatorname{holim}\Big(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_X U_j) \substack{\to\\\to\\\to} \prod_{i,j,k} F(U_i \times_X U_j \times_X U$

c) $F(B) \xrightarrow{\sim} \operatorname{holim}_n (B/m^n)$ for every complete Noeth. local R-alg (B, m).

d) F admits a def/obs theory satisfying Schlessinger's condition

e) $\Delta : F \to F \times F$ is representable by algebraic spaces.

Examples:

1) $F(A) = \left\{ \begin{matrix} \text{smooth proper curves }/A \\ \text{of genus } g \end{matrix} \right\} = \mathcal{M}_g(A)$

2) Fix $X/_{\mathrm{Spec}(R)}$ proper & flat.

$F(A) = \left\{ \begin{matrix} \text{vector bundles} \\ \text{on } X \times_R A \end{matrix} \right\} = \mathrm{Vect}(X/_R)(A)$

II. Lurie's rep. thm (Lurie's thesis, DAG XIV}, Pridham's paper)

Thm: R a derived G-ring

F: $sCR_{R/-} \longrightarrow$ Spaces a functor

Then F is representable by a derived DM-stack almost of f.p. /R, iff

a) F commutes with filtered colimits on $sCR_{R/-, \le k}$ for all $k \in \mathbb{Z}_{\geq 0}$.

b) F is an étale sheaf

c) $F(B) \simeq \operatorname{holim}_n F(B/m^n)$ for discrete complete noetherian local R-alg.

d₁) F has a connective cotangent complex

d₂) F is infinitesimally cohesive.

e) F is nilcomplete

Explanation

Setup : $F \in \mathrm{Fun}(sCR_{R/-}, \mathrm{Spaces})$

a) if $I \xrightarrow{A} sCR_{R/-, \leq k}$ filtered, then

$$F(\operatorname{colim}_i A_i) \xleftarrow{\sim} \operatorname{colim}_i F(A_i) \qquad \text{(homotopy colimit)}$$

Ex : 1) If $B \in sCR_{R/-}$, $F = \underline{\mathrm{Hom}}_{sCR_{R/-}}(B,-)$ satisfies (a) for all k if B is almost f.p./R.

2) $F(A) = \mathrm{Mod}_A^{f.p.} \subset \mathrm{Mod}_A$

b) F is an étale sheaf if

-) F commutes with finite products

-) if $A \to B$ is an étale cover in $sCR_{R/-}$, then

$$F(A) \xrightarrow{\sim} \operatorname{holim}_n F(B^{\otimes(n+1)})$$

Ex : $F = \underline{\mathrm{Hom}}_{sCR_{R/-}}(C,-)$ is an étale sheaf

key pt : $A \xrightarrow{\sim} \operatorname{holim}_n (B^{\otimes(n+1)})$

d) cotangent complex on a functor :

Say $F : sCR \longrightarrow \mathrm{Spaces}$.

-) $QC_F^{\geq 0} = \operatorname{holim}_{\substack{A \in sCR \\ \eta \in F(A)}} \mathrm{Mod}_A$ $\rightarrow$ For each (A, η) get $M(\eta) \in \mathrm{Mod}_A$. For $\phi : A \to B$

$M(\eta) \otimes_A B \xrightarrow{\sim} M(\phi_* \eta)$

Ex : $F = \underline{\mathrm{Hom}}(A, -)$ then $QC_F^{\geq 0} = \mathrm{Mod}_A$

-) F has a connective cotangent complex $L_F \in QC_F^{\geq 0}$ if there exists an equivalence $\mathrm{Hom}_{\mathrm{Mod}_A}(L_F(\eta), N) =$ fibre of $F(A \oplus N) \to F(A)$ over η.

+ functoriallity in A, η, N.

d_2) inf. cohesive : —

e) nilcomplete : $F(B) \xrightarrow{\sim} \operatorname{holim}_n F(\tau_{\leq n} B)$.

Example : M_g

Def : A map $f: X \to \mathrm{Spec}(A)$ is a stable curve of genus $g \geq \overset{2}{\cancel{0}}$ if

a) f is flat, qc qs, almost f.p.

b) $X \times_A \pi_0(A) \longrightarrow \mathrm{Spec}(\pi_0 A)$ is a stable curve

Thm : $\overline{M}_g : sCR \longrightarrow \mathrm{Spaces}$

$A \longmapsto \{ \text{all stable curves } X \to \mathrm{Spec}(A) \text{ of genus } g \geq 2 \}$

is representable by $\overline{M}_g^{cl}$.

Prop : If $f: X \to \mathrm{Spec}(A)$ is a derived scheme and $M \in \mathrm{Mod}_A$, then

$$\mathrm{Hom}_{QC_X}(L_{X/A}, f^*M[1]) = \{ \text{deformations of } X \text{ over } \mathrm{Spec}(A \oplus M) \}$$

$\overline{M}_g$ has a connective cotangent complex

$$\begin{tikzcd}
\overline{M}_g(A \oplus M) \arrow[r] & \overline{M}_g(A) \\
\mathrm{Hom}_X(L_{X/A}, f^*M[1]) \arrow[u, phantom, sloped, "\in"] \arrow[r] & \eta : (X \xrightarrow{f} \mathrm{Spec}(A)) \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

$\| \; ??$

$$\mathrm{Hom}_A(\eta^* L_{\overline{M}_g/\mathbb{Z}}, M)$$

$$\begin{tikzcd}
\eta^*(L_{\overline{M}_g/\mathbb{Z}}) = (Rf_* L_{X/A}^{\vee})^{\vee}[-1] \\
\overline{M}_g(A) \ni \eta : X \to \mathrm{Spec}\, A \arrow[u]
\end{tikzcd}$$

Talk 9

Application of Based Loop Spaces

In topology: $pt \subset X$

paths starting at pt

$$\begin{tikzcd} pt \overset{h}{\underset{X}{\times}} pt \simeq \Omega_{pt} X \arrow[r] \arrow[d] & \mathrm{Path}\, X \arrow[d] \\ pt \arrow[r] & X \end{tikzcd}$$

$\Omega_{pt} X$ "group"

$B\Omega_{pt} X$ classifying space

$$\begin{tikzcd} \left| pt \leftleftarrows \Omega_{pt} X \mathrel{\substack{\leftarrow\\ \leftarrow\\ \leftarrow}} (\Omega_{pt} X)^2 \mathrel{\substack{\leftarrow\\ \leftarrow\\ \leftarrow\\ \leftarrow}} \cdots \right| \arrow[d] \\ X \end{tikzcd}$$

realises $B\Omega_{pt} X$ is ~~b~~ base point component of X

homotopy ~~Cec~~ Čech nerve

$\left| pt \leftleftarrows pt \overset{h}{\underset{X}{\times}} pt \mathrel{\substack{\leftarrow\\ \leftarrow\\ \leftarrow}} pt \overset{h}{\underset{X}{\times}} pt \overset{h}{\underset{X}{\times}} pt \mathrel{\substack{\leftarrow\\ \leftarrow\\ \leftarrow\\ \leftarrow}} \cdots \right|$

Variant: $f: Y \longrightarrow X$

Čech nerve

$$\begin{tikzcd} \left| Y \leftleftarrows Y \overset{h}{\underset{X}{\times}} Y \mathrel{\substack{\leftarrow\\ \leftarrow\\ \leftarrow}} Y \overset{h}{\underset{X}{\times}} Y \overset{h}{\underset{X}{\times}} Y \mathrel{\substack{\leftarrow\\ \leftarrow\\ \leftarrow\\ \leftarrow}} \cdots \right| \arrow[d] \\ X \end{tikzcd}$$

$\simeq$ unions of components of X hit by Y

Algebra:

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, phantom, "\shortparallel"] & X \arrow[d, phantom, "\shortparallel"] \\ \mathrm{Spec}\, B & \mathrm{Spec}\, A \end{tikzcd}$$

closed immersion of derived schemes

i.e. i is affine, $H_j(i_* \mathcal{O}_Z)$ coherent $/ H_0 \mathcal{O}_X$ $\forall j$

and $H_0(\mathcal{O}_X) \to H_0 \mathcal{O}_Z$ surjective

Construction

$$Z \leftleftarrows Z \underset{X}{\times} Z \mathrel{\substack{\leftarrow\\ \leftarrow\\ \leftarrow}} Z \times_X Z \times_X Z \mathrel{\substack{\leftarrow\\ \leftarrow\\ \leftarrow\\ \leftarrow}} \cdots$$

$\mathrm{Spec}\, B \quad \mathrm{Spec}\, B \overset{L}{\underset{A}{\otimes}} B \quad \mathrm{Spec}\, B \overset{L}{\underset{A}{\otimes}} B \overset{L}{\underset{A}{\otimes}} B$

<u>Q</u>: How badly does closed descent fail?

<u>A</u>: $| Z \leftleftarrows Z \times_X Z \leftleftarrows \cdots |(R) = | Z(R) \leftleftarrows Z(R) \underset{X(R)}{\times} Z(R) \leftleftarrows \cdots | = \left\{ \begin{array}{l} \text{the union of components of } X(R) \\ \text{such that they are hit by } Z(R) \end{array} \right\}$

$\uparrow$
$\mathrm{Fun}(sCR^{op}, \mathrm{Spaces})$

$= \left\{ \begin{array}{l} \text{those points } y \in X(R) \text{ that admit a} \\ \text{(\underline{scheme-theoretic}) factorization through } Z \end{array} \right\}$

$\cap$

$X^{\wedge}_{Z}(R) = \{ \text{same for \underline{set-theoretic} factorization} \}$

<u>Prop</u>: $\mathcal{O}_{\hat{X}_Z} \xrightarrow[\sim]{\text{restriction}} \mathrm{Tot}\{ \mathcal{O}_Z \rightrightarrows \mathcal{O}_{Z \times Z} \ldots \}$

$\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad \mathrm{Tot}(B \rightrightarrows B \overset{L}{\underset{A}{\otimes}} B \ldots)$

<u>Concrete Ex / Exercise</u>

$\mathrm{Spec}\, k \longrightarrow \mathrm{Spec}\, k[x]$

$k \overset{L}{\underset{k[x]}{\otimes}} k \simeq k[\underset{\uparrow \atop \deg +1}{B}]$ as dg-Hopf alg $\Delta(B) = B \otimes 1 + 1 \otimes B$

k coalgebra ..

<u>Key</u> The totalization computes $R\mathrm{Hom}_{k[\varepsilon_1]}(k, k) \simeq k[[x]]$

<u>Application of Unbased Loop Spaces</u>

<u>Prop</u>: X der. scheme (or der stack) $\qquad [\text{eg } X = \mathrm{Spec}\, A]$

K finite simplicial set

Then $X^K : R \longmapsto \mathrm{Map}_{sSet}(K, X(R)) = X(R)^K$ is (representable by) a derived scheme (/der stack) $\qquad [X = \mathrm{Spec}(K \otimes A)]$

<u>pf 1</u>: $\mathrm{Map}_{sCR}(K \otimes A, R)_p = \mathrm{Map}_{CR}(K_p \otimes A_p, \cancel{A_p} R_p) = \mathrm{Map}_{CR}(A_p, \cancel{A_p} R_p)^{K_p}$

$\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad \bigotimes_{K_p} A_p$

$\qquad\qquad\qquad\qquad\qquad\qquad = \mathrm{Map}_{sSet}(K, \mathrm{Map}_{sCR}(A, R))$

<u>pf 2</u>: Write K as a finite sequence of cell attaching (retracts?)

$\quad \Downarrow$

X^K = corresponding sequence of <u>pull</u>backs

$\qquad\qquad\qquad\qquad\qquad\qquad \uparrow$ corr. rel $\otimes$

Ex: 1) K discrete or $\{1, \ldots, n\}$

$X^K = X^n \quad [= \operatorname{Spec} R^{\otimes n}]$

2)
$$\begin{tikzcd}
\Sigma \cancel{*} K & * \arrow[l] \\
* \arrow[u] \arrow[ur, phantom, "\text{ho pushout}"] & K \arrow[l] \arrow[u]
\end{tikzcd}
\qquad
\left\{
\begin{tikzcd}
\Sigma K & * \arrow[l] \\
\operatorname{cone}(K) \arrow[u] \arrow[ur, phantom, "\text{pushout}"] & K \arrow[l] \arrow[u]
\end{tikzcd}
\right.$$

$$\begin{tikzcd}
X^{\Sigma K} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{ho } \lrcorner"] & X \arrow[d] \\
X \arrow[r] & X^K
\end{tikzcd}$$

$\to \left[X^{\Sigma K} = \operatorname{Spec} R \overset{L}{\underset{K \otimes R}{\otimes}} R\right]$

$= \operatorname{Spec} (\operatorname{cone}(K) \otimes R) \underset{K \otimes R}{\otimes} R$

Rmk ↗ Bar construction for $\overset{L}{\otimes}$

3) $K = S^1 = \Sigma S^0$

$$\begin{tikzcd}
X^{S^1} \arrow[r] \arrow[d] & X \arrow[d] \\
X \arrow[r] & X^2
\end{tikzcd}$$

$\mathcal{O}_{X^{S^1}}$ in affine case

Funny cone thing: $\$$

$| R \otimes R \leftleftarrows R \otimes R^2 \otimes R \substack{\leftarrow \\ \leftarrow \\ \leftarrow} R \otimes (R^{\otimes 2})^{\otimes 2} \otimes R \substack{\leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow} \cdots)$

Cone(S^0):

$(\Delta^1_\bullet \otimes R) \underset{S^0 \otimes R}{\otimes} R$ gives

$| R \leftleftarrows R \otimes R \substack{\leftarrow \\ \leftarrow \\ \leftarrow} R \otimes R \otimes R \substack{\leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow} \cdots |$ cyclic bar complex

Propn: char 0 HKR-type R, $X = \operatorname{Spec} R$

$$\mathcal{O}_{LX} \xrightarrow{\sim \text{ quis}} \operatorname{Sym}_R (\mathbb{L}_R[+1])$$

$| R \leftleftarrows R \otimes R \substack{\leftarrow \\ \leftarrow \\ \leftarrow} R \otimes R \otimes R \substack{\leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow} \cdots |$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \psi$

$r_0 \otimes r_1 \cdots \otimes r_p \longmapsto r_0 \, dr_1 \, dr_2 \cdots dr_p / p!$

$\circlearrowright_{\text{Connes } B} \qquad\qquad \circlearrowright_{d_{dR}}$

Regard these as complexes

Cpx w/ S^1-action $\simeq$ $H_* S^1$-mod

$\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{k[B_{+1}]}$

The HKR-type map is compatible with it.

$\exists$ relation between $QC(LX)^{S^1}$ and D-modules.

Ex of Loops

not ~~étale~~ llpl

1) X classical scheme $^{\vee}$ $\Rightarrow$ LX not classical

2)
$$\begin{tikzcd}
L(BG) \arrow[r] \arrow[d] & BG \arrow[d] \\
BG \arrow[r] & BG^2
\end{tikzcd}$$
$\rightsquigarrow$ $L(BG) \simeq G^{ad}/G$

3) Y/G $\overset{\uparrow}{\text{affine}}$

when is $L(Y/G)$ classical again?

$\rightsquigarrow$ A: $Y \circlearrowleft$ w/ finitely many orbits.