MSRI Geometric Representation Theory

workshop 9/2 - 5

Hales

9/2 Talk 1

Introduction to the Langlands program and the fundamental lemma.

Trace formula for finite groups G.

V = class fcts

canonical bases:

- ◦) char. fcts of conj classes C_γ
- ◦) set of irred reps : $\gamma \longmapsto \operatorname{trace} \pi(\gamma)$

  π an irred rep

$\Rightarrow$ h any class fcts

$$\sum_c a_c(h)\, 1_c = \sum_\pi m_\pi(h)\, \operatorname{tr} \pi$$

trace formula

e.g. character of rep of G

left: character formula (geom side)
right: multiplicity of irreds (spectral side)

$\rightsquigarrow$ identity of distributions

$\phi : G \to \mathbb{C}$ get distribution Φ on G, s.t.

$$\Phi(f) : \sum_G \phi(g) f(g)\, dg \quad \forall f \in C_c^\infty(G)$$

$\Rightarrow$ trace formula for distribution

(both sides viewed as fcts on $C_c^\infty(G)$)

e.g. $I_C = \sum_G 1_C(g) f(g)\, dg = \sum_C f(c)\, dg$

$$= \sum_{G_\gamma \backslash G}$$

orbital integral

(see slides)

Arinkin

4/2 Talk 2

Introduction to Geometric Langlands Conjecture

(1)

X = compact Riemann surface

X/k smooth projective curve $/k$

char$(k) = 0$

Automorphic $\longleftrightarrow$ Galois

Automorphic

$$\mathrm{Bun} = \mathrm{Bun}_{GL(N)}(X) = \left\{ \begin{array}{l} \text{rank } N \\ \text{v. bundles on } X \end{array} \right\}$$

classically: $X/\mathbb{F}_q$, $\mathbb{F}_q(X)$

Functions $\mathrm{Bun}(\mathbb{F}_q) \longrightarrow \mathbb{C}$

(automorphic forms)

Geometrically

Category of $\boxed{\text{D-modules on Bun}}$

algebraic systems of PDEs
$\shortparallel$
quasi-coherent sheaves w/ flat connections

has to be viewed as a stack

Rough goal: find a family of "special" D-modules on Bun that form a "basis".

indexed by local systems on X

Galois

Definition: Rank N local systems on X is a (rk N) v. bundle on X together with a flat connection.

$LS = \{\text{loc. system}\}$

Classically: $\pi_1(X) \longrightarrow GL(N)$

(1st talk)

② Example (Geometric class field theory)

$N=1$

$\mathrm{Bun} = J =$ Jacobian of X

Starting from a rank 1 local system ℓ on $X \longmapsto \mathrm{Aut}_\ell$: D-module on J.

ℓ is given by $\mathrm{Mon}(\ell) : \pi_1(X) \longrightarrow \mathbb{C}^*$

$$\begin{tikzcd} \pi_1(X) \arrow[r] \arrow[d] & \mathbb{C}^* \\ \pi_1(X)^{ab} = H_1(X) \arrow[ur] & \\ \pi_1(J) \arrow[uur, bend right] \arrow[u, phantom, "\shortparallel"] & \end{tikzcd}$$

Define Aut to be rk 1 local system on J such that $\mathrm{Mon}(\mathrm{Aut}_\ell)$ is the composition $\pi_1(J) \to \mathbb{C}^*$ above

③ How are $\mathrm{Aut}_\mathcal{L}$ special

$\mathcal{L} \in \mathcal{LS}$

$\mathrm{Aut}_\mathcal{L}$ is a D-module on Bun

Hecke eigenproperty:

Ex: $N=1$

Fix $x \in X$; It gives $\begin{matrix} E \longmapsto E(x) \\ \mathcal{H}_x : \mathrm{Bun} \longrightarrow \mathrm{Bun} \end{matrix}$

Classically: $H = \mathrm{Bun}(\mathbb{F}_q)$ $\circlearrowleft$ $\mathcal{H}_x$ operators $\mathcal{H}_x^*$ act on {fcts $H \to \mathbb{C}$}

find eigenvectors

In example:

$$t : X \times \mathrm{Bun} \longrightarrow \mathrm{Bun}$$
$$(x, E) \longmapsto E(x)$$

Hecke eigenproperty:

$$\boxed{\ell \boxtimes \mathrm{Aut}_\ell = t^*(\mathrm{Aut}_\ell)}$$

$$\rightsquigarrow \quad \ell_x \otimes \mathrm{Aut}_\ell = t_x^*(\mathrm{Aut}_\ell)$$

What if $N > 1$

Universal Hecke correspondence

$$\mathrm{Hecke} = \left\{ \left(\underset{\in \mathrm{Bun}}{E \longrightarrow E_1} \text{ st } 0 \to E \to E_1 \to E_1/E \to 0 \right) \right\}$$

E_1/E length 1 sheaf at $x \in X$

$$X \times \mathrm{Bun} \xleftarrow{\ p\ } \mathrm{Hecke} \xrightarrow{\ p_1\ } \mathrm{Bun}$$
$$(x, E) \longleftarrow \quad \longrightarrow E_1$$

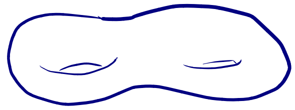
Hecke eigenproperty: $\quad \mathcal{L} \in LS$

$$\boxed{\mathcal{L} \boxtimes \mathrm{Aut}_\mathcal{L} = p_* p_1^! (\mathrm{Aut}_\mathcal{L})}$$

eigenvalues

Arinkin II

9/3 Talk 2

$X =$ 

geometric unramified global Langlands correspondence (function field)

$\mathrm{Bun} = \{\text{rk } N \text{ v. bundles on } X\}$

$LS = \{\text{rk } N \text{ local systems on } X\}$

$$\begin{array}{ccc} \mathcal{L} & \longmapsto & \mathrm{Aut}_{\mathcal{L}} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \mathcal{L}S & & D\text{-mod}(\mathrm{Bun}) \end{array}$$

- Hecke eigenproperty
- $\{\mathrm{Aut}_{\mathcal{L}}\}$ form a basis

(• Parabolic induction)

$\mathrm{Aut} = \{\mathrm{Aut}_{\mathcal{L}}\}_{\mathcal{L} \in \mathcal{L}S}$ should form an $\mathcal{L}S$-family of D_{Bun}-modules,

i.e. Aut on $LS \times \mathrm{Bun}$ is an $\mathcal{O}_{\mathcal{L}S}$-$\mathcal{D}_{\mathrm{Bun}}$-module.

Get a functor

$$QCoh(\mathcal{L}S) \longrightarrow D\text{-mod}(Bun)$$

$$\begin{tikzcd} & \mathcal{L}S \times Bun \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathcal{L}S & & Bun \end{tikzcd} \qquad \underline{p_{2,*}\left(\mathcal{A}ut \otimes p_1^*(-)\right)}$$

<u>(Naive) GLC</u>: This is an equivalence.

<u>Rmk</u>: Derive everything!

<u>Other groups</u>

G = reductive group / $\mathbb{C}$

$G^\vee$ = Langlands dual of G

$Bun_G = \{G\text{-bundles on } X\}$

$\mathcal{L}S_{G^\vee} = \{G^\vee\text{- local systems on } X\}$

$\mathcal{A}ut$ is $\mathcal{L}S_{G^\vee}$-family of D_{Bun_G}-modules giving an equivalence

$$QCoh(\mathcal{L}S_{G^\vee}) \longrightarrow D\text{-mod}(Bun_G)$$

How to modify G-bundle at $x \in X$?

Modification : $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Bun}_G$ and $\mathcal{F}_1|_{X \setminus \{x\}} \simeq \mathcal{F}_2|_{X \setminus \{x\}}$

Classified by : $G(\mathcal{O}) \backslash G(K) / G(\mathcal{O})$

$K = \mathbb{C}((z))$

$\mathcal{O} = \mathbb{C}[[z]]$

z is coord around x

<u>Ex</u> : $GL(N)$; minimal mod $\begin{pmatrix} z & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$.

<u>Classically</u>: $K = \mathbb{F}_q((z))$

Hecke algebra : $\{$functions on $G(\mathcal{O}) \backslash G(K) / G(\mathcal{O}) \to \mathbb{C}\}$

algebra under convolution

<u>Hecke category</u>

$Gr = G(K)/G(\mathcal{O})$ ... affine Grassmannian

$\mathcal{H} := \text{D-mod}(Gr)^{G(\mathcal{O})}$

"$\text{D-mod}\left(G(\mathcal{O}) \backslash G(K) / G(\mathcal{O})\right)$"

$\mathcal{H}$ is a monoidal category under convolution

$\ast : \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H}$

$\mathcal{H}$ acts on $\text{D-mod}(\mathrm{Bun}_G)$

Informally:
$$\begin{tikzcd}[column sep=small]
 & \left\{ \left(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_1|_{X \setminus \{x\}} \simeq \mathcal{F}_2|_{X \setminus \{x\}} \right) \right\} \arrow[dl, "p_1"'] \arrow[d, "p_2"] \arrow[dr, "\tau"] & \\
\underset{\mathrm{Bun}_G}{\overset{\mathcal{F}_1}{\cap}} & \underset{\mathrm{Bun}_G}{\overset{\mathcal{F}_2}{\cap}} & \text{type of modification } G(\mathcal{O}) \backslash G(\mathcal{K}) / G(\mathcal{O})
\end{tikzcd}$$

$M \in \mathcal{H}$ acts as $p_{1,*}\left(p_2^*(-) \otimes \tau^*(M)\right)$

Geometric Satake equiv

<u>Thm</u>: $\mathcal{H} \simeq \mathrm{Rep}(G^\vee)$

└ Representations of $G^\vee$.

iso as monoidal categories

<u>Rem</u>: for abelian categories (derived version more subtle)

used in <u>Hecke eigenproperty</u>:

$\mathcal{L} \in LS_{G^\vee} \longmapsto \mathrm{Aut}_{\mathcal{L}} \in D\text{-mod}(\mathrm{Bun}_G)$

Given $M \in \mathcal{H}$,

eigenvalue of M on $\mathrm{Aut}_{\mathcal{L}}$

$$M * \mathrm{Aut}_{\mathcal{L}} = \lambda_M \otimes \mathrm{Aut}_M \quad \text{for some } \lambda_M \in \mathrm{Vect}.$$

As x varies, λ_M becomes a local system (of v. spaces) on X

$\mu \in \mathcal{H}$

$\downarrow$

$\rho \in \mathrm{Rep}(G^\vee)$ so $\boxed{\lambda_\mu = \rho(\mathcal{L})}$

Examples: $G = GL(N) = G^\vee$

minimal modification $\in G(\mathcal{O}) \backslash G(K) / G(\mathcal{O})$

$\rightsquigarrow S_{m.m.} \in \mathcal{H}$

$\updownarrow$

standard rep $\in \mathrm{Rep}(G^\vee)$

Frenkel

9/12 Talk 3

Gauge theory and Langlands program

Number theory	$\longleftrightarrow$	Alg curves $/\mathbb{F}_q$	$\longleftrightarrow$	Riemann surfaces
number fields F		$F = \mathbb{F}_q(X)$ funct field		alg. curves $/\mathbb{C}$
symmetries: $Gal(F/\mathbb{Q})$		$Gal(\bar{F}/F)$		
$Gal(\bar{\mathbb{Q}}/\mathbb{Q})$				

Langlands program 1967

$$\left\{ \begin{array}{c} n\text{-dim'l Representations} \\ \text{of } Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \end{array} \right\} \dashleftarrow\dashrightarrow \left\{ \begin{array}{c} \text{automorphic} \\ \text{reps of } GL_n(\mathbb{A}_{\mathbb{Q}}) \end{array} \right\}$$

Quantum physics

S-duality

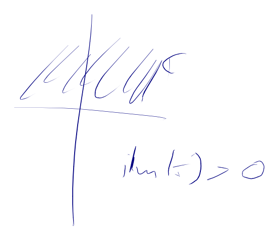
mirror symmetry

Simple ex:

elliptic curves $/\mathbb{Q}$ $\longleftrightarrow$ modular forms

$n=2$: 2 dim rep of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ (on cohomology)

automorphic reps of $GL_2(\mathbb{A})$



$$y^2 + y = x^3 - x \mod p$$

$a_p = p + 1 - \#\text{solutions}$ (projective)

$$q(1-q)^2(1-q^{11})^2(1-q^2)^2(1-q^{22})^2 \cdots$$

$$= \sum_{n \geq 1} b_n q^n$$

$q = e^{2\pi i \tau}$ $\rightsquigarrow$ modular form

Shimura - Taniyama Weil (Wiles thm)

$$\Rightarrow \quad a_p = b_p \quad \forall p \text{ prime}$$

middle column: $\mathbb{Q} \rightsquigarrow F$ function field

$F(X)$,
X smooth projective (geom. connected) curve $/\mathbb{F}_q$

$$\mathrm{Gal}(\bar{F}/F) \supset W(F)$$

(Drinfeld, Lafforgue)

$$\{W(F) \to GL_n\} \overset{\text{Thm}}{\longleftrightarrow} \left\{\begin{array}{l}\text{Automorphic reps of} \\ GL_n(\mathbb{A}_F)\end{array}\right\}$$

$$\mathbb{A}_F = \prod_{x \in X}{}' F_x$$

$\uparrow \mathbb{F}_{q_x}((t_x))$

$GL_n \rightsquigarrow G$ reductive alg. group $/\mathbb{F}_q$

X — curve $/ \mathbb{C}$ $\qquad F = \mathbb{C}(X)$

Analogue of $\mathrm{Gal}(\bar{F}/F)$ is $\pi_1(X, x)$

$$\{\pi_1(X,x) \longrightarrow {}^LG\} \longleftrightarrow \{\mathcal{D}\text{-modules on } \mathrm{Bun}_G\}$$

$$\begin{aligned} &\Vert \\ &\{\text{flat } {}^LG\text{-bundles on } X\} \\ &\Vert \\ &(E^\infty, \nabla = \nabla^{(1,0)} + \nabla^{(0,1)}) \\ &= (E, \nabla_{hol}) \end{aligned}$$

<u>categorically</u>

$$D^b\ \boxed{\mathcal{O}\text{-modules on } \mathrm{Loc}_{{}^LG}} \ \underset{\text{corresp.}}{\overset{\text{cat.}}{\longleftrightarrow}} \ D^b\ \boxed{\mathcal{D}\text{-modules on } \mathrm{Bun}_G}$$

Hecke functors

$G = GL_1$: Theorem: Laumon & Rothstein

G non-abelian: Arinkin–Gaitsgory (formulation)

Frenkel, Part II

4/4 Talk 3

Number theory — Curves over finite fields — Riemann surfaces$/\mathbb{C}$ — Quantum physics

$\underbrace{\text{Number theory} \quad \text{Curves over finite fields}}_{\text{classical}}$

$\underbrace{\text{Curves over finite fields} \quad \text{Riemann surfaces}/\mathbb{C}}_{\text{geometric}}$

Categorical Langlands Correspondence

$$F_{V,x} \circlearrowright \boxed{D^b \ \mathcal{O}\text{-modules on } \mathrm{Loc}_{{}^LG}} \longleftrightarrow \boxed{D^b \ \mathcal{D}\text{-modules on } \mathrm{Bun}_G} \circlearrowleft H_{V,x}$$

$x \in X$
$V \in \mathrm{Rep}\,{}^LG$

$\mathcal{T}$ universal ${}^LG$-bundle

$$\begin{array}{c} \mathcal{T} \\ \downarrow \\ X \times \mathrm{Loc}_{{}^LG} \\ \cup\!\!\!\,\mid \\ \mathcal{E} = (E, \nabla) \\ \downarrow \\ X \end{array}$$

$x \in X$, $V \in \mathrm{Rep}\,{}^LG$ $\qquad V_{\mathcal{T}} = \mathcal{T} \underset{{}^LG}{\times} V$ ... assoc vect. bundle

$$\begin{array}{c} V_{\mathcal{T}} \Big|_{x \times \mathrm{Loc}_{{}^LG}} \\ \downarrow \\ \mathrm{Loc}_{{}^LG} \end{array}$$

$\mathcal{V}_{\tau,x}$ - $\mathcal{O}$-module on $Loc_{{}^L G}$ of sections of $V_\tau|_{x \times Loc_{{}^L G}}$

Frobenius (or Wilson) functor

$$F_{V,x}(\mathcal{F}) = \mathcal{V}_{\tau,x} \otimes \mathcal{F}$$

Let $\mathcal{F} = \mathcal{O}_\mathcal{E}$ $\rightsquigarrow$ $F_{V,x}(\mathcal{O}_\mathcal{E}) \simeq \underline{V} \otimes \mathcal{O}_\mathcal{E}$

$\uparrow$

$\mathcal{E}$ smooth pt $\qquad \forall x, V$

$\mathcal{O}_\mathcal{E}$ skyscraper $\qquad \Rightarrow \mathcal{O}_\mathcal{E}$ is eigensheaf.

Electromagnetic Duality

$\vec{E}, \vec{B}$ $\qquad$ $\vec{E} \longrightarrow \vec{B}$, $\vec{B} \longrightarrow -\vec{E}$: equations stay the same

Quantum Level : maximal ($N=4$) supersymmetric electromagnetism

e - "electric charge"

has electromagnetic duality with $e \longrightarrow \frac{1}{e}$

perturbative vs non-perturbative

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n \quad \text{vs.} \quad \exp(\pi i) = -1$$

$$\phi(q) = \sum_{n=0}^{\infty} \underset{\substack{\uparrow \\ \text{partitions} \\ \text{of } n}}{p(n)} q^n = \prod_{i=1}^{\infty} (1-q^i)^{-1} \quad \text{vs} \quad \eta(q) = q^{\frac{1}{24}} \phi(q)^{-1}$$

is modular

duality indicates that there is a non-perturbative electromagnetic theory

Gauge (Y-M) theory with gauge group $U(1)$

$N=4$ SUSY ✓ general 4D gauge theory with gauge group G_c, g τ > parameter

$$({}^L G_c, -\tfrac{1}{\tau}) \longleftrightarrow (G_c, \tau)$$

Kapustin - Witten (2006)

$M_4 = X \times \Sigma$ Dimensional reductions $|X| \ll |\Sigma|$

effective 2D-theory on Σ: supersymmetric sigma model with target manifold $\mathcal{M}_H(G, X)$

$$\Sigma \longrightarrow \mathcal{M}_H(G, X)$$

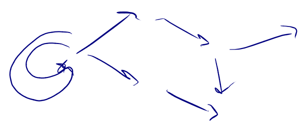
Hitchin moduli space

⋮

Ginzburg

9/2 Talk 4

Geometry of Quiver Varieties

Quiver: 

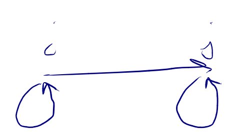
I = vertex set
oriented arrows $\Big\}$ finite

A rep of a quiver Q is an assignment of a f.dim vector space V_i $\forall i \in I$ and a linear map $\hat{e} : V_i \to V_j$ to each arrow $e : i \to j$.

$\dim \rho = \{\dim V_i\} \in \mathbb{Z}^I_{\geq 0}$

$\mathrm{Rep}_\alpha Q$ = set of all reps of $\dim = \alpha \in \mathbb{Z}^I_{\geq 0}$ with $V_i = \mathbb{C}^{\alpha_i}$.

$\uparrow$ vector space with action of $G = \prod_i GL(V_i)$



$\mathrm{Rep}_\alpha Q / G_\alpha$ stack

$\Downarrow$

moduli stack of the category of objects fin dim reps of a fixed quiver Q

Fact: Stack of objects of a category $\mathcal{C}$ is smooth iff $\mathrm{h.dim}(\mathcal{C}) \leq 1$, i.e. $\mathrm{Ext}^i_{\mathcal{C}}(M,N) = 0$ $\forall i \geq 2$ $\forall M, N$

Thm: $\mathcal{C}$ has h. dim 1 only in the following cases:

1) $\mathcal{C}$ = reps of a quiver !
2) Coherent sheaves on a smooth curve
3) lots of exceptions

k a field $\quad kI := k^I \quad \ldots$ k-basis 1_i

$E = \bigoplus_{e \text{ edges}} k e_i$ is a kI bimodule:

$$1_i e 1_j = \begin{cases} e & \text{if source}(e) = i,\ \text{target}(e) = j \\ 0 & \text{else} \end{cases}$$

Path algebra of Q: $\underbrace{T_{kI}}_{\text{tensor algebra}} E = P$

Observation: rep of Q = P-module

Lemma: $\forall$ P-modules M, N we have $\mathrm{Ext}^j_P(M,N) = 0 \ \forall j > 1$

proj resolution

$$0 \to P \otimes_{kI} E \otimes_{kI} P \to P \otimes_{kI} P \xrightarrow{\text{mult}} P \to 0$$

$$p \otimes e \otimes p' \longmapsto pe \otimes p' - p \otimes ep'$$

$- \otimes_P M$:

$$0 \to P \otimes_{kI} E \otimes_{kI} M \to P \otimes_{kI} M \to M \to 0$$

$\mathrm{Hom}(-, N)$:

$$0 \to \mathrm{Hom}_P(M,N) \to \mathrm{Hom}_{kI}(M,N) \to \mathrm{Hom}_{kI}(E \otimes M, N) \to \mathrm{Ext}^1(M,N) \to 0$$

Euler form: $\mathbb{Z}^I \times \mathbb{Z}^I \longrightarrow \mathbb{Z}$

$\alpha, \beta \in \mathbb{Z}^I$: $\langle \alpha, \beta \rangle = \sum \alpha_i \beta_i - \sum \alpha_i \dim E_{ij} \beta_j$

$E_{ij} = \#\{i \longrightarrow j\}$

exact sequence:

$$\dim \mathrm{Hom}(M,N) - \dim \mathrm{Ext}^1(M,N) = \dim M \cdot \dim N - \sum \dim M_i \dim E_{ij} \dim N_j$$

$$\Rightarrow \chi(\mathrm{Ext}^i) = \langle \dim M, \dim N \rangle$$

DT invariants (after Kontsevich-Soibelman)

Q a quiver, P = path alg of Q

W cyclic potential $= P/[P,P]$

Fix $\alpha \in \mathbb{Z}^I_{\geq 0}$

$$W_\alpha : \mathrm{Rep}_\alpha Q \longrightarrow k$$
$$\rho \longmapsto \mathrm{Tr}\, \rho(W_\alpha)$$

$G_\alpha = \prod G_{\alpha_i}$ invariant fn

descends to $\mathrm{Rep}_\alpha Q / G_\alpha \longrightarrow k$

There is a sheaf of vanishing cycles $\phi(W_\alpha)$ on $W_\alpha^{-1}(0)$,

$$\chi\left(H^\circ\left(w_\alpha^{-1}(0),\ \phi_\alpha(w_\alpha)\right)\right) = \widetilde{DT}_\alpha$$

$$\mathrm{Log}\left(\sum_\alpha q^\alpha \widetilde{DT}_\alpha\right) = DT(Q,w)$$

$$H^\circ\left(w_\alpha^{-1}(0),\ \phi(w_\alpha)\right) = H^\circ\left(\Omega_{\mathrm{Rep}_\alpha Q},\ d_{dR} + dw_\alpha\right)$$

"Fact": $\forall$ CY 3-category

$\rightsquigarrow (Q,w) \rightsquigarrow DT(Q,w)$

Reminder on symplectic geometry:

$$\begin{tikzcd} T^*X \arrow[r, "p"] \arrow[d, "\mu"] & X \\ \mathfrak{g}^* & \end{tikzcd}$$

$G \curvearrowright X$ action of alg group on X

$\mathfrak{g} = \mathrm{Lie}\, G$

$\varphi \in T^*_x X$

$\downarrow$

$[\mu(\varphi) : \underset{\in \mathfrak{g}}{a} \longmapsto \langle \varphi, \underset{\in T_x X}{a_x} \rangle$

action of a at x

Hamiltonian reduction

$G \curvearrowright X \qquad G \curvearrowright T^*X$

$\mathrm{Ad}\, G \curvearrowright \mathfrak{g} \qquad \mu$ is G-equivariant

$\forall h \in \mathfrak{g}^* \quad \mu^{-1}(h)$ is a G_h-stable subvariety of T^*X

Hamilt. reduction: $\mu^{-1}(h)/G_h$

Concrete case: $X = \mathrm{Rep}_\alpha Q$

$G = G_\alpha$ acts on $\mathrm{Rep}_\alpha Q$

$\mu : T^*\mathrm{Rep}_\alpha Q \longrightarrow \mathfrak{g}_\alpha^* \quad , \mathfrak{g}_\alpha = \mathrm{Lie}(G_\alpha)$

A "quiver variety" is $\mu^{-1}(h)/G_h$

Relation to Kac conjecture (1981, proved 2013 by (HLRV))

ground field $\mathbb{F}_q$

$\#$ absolutely indecomposable reps in $\mathrm{Rep}_\alpha(Q, \mathbb{F}_q) =: a_\alpha(q)$

Thm: $a_\alpha(q) = \mathrm{Tr}_{\mathrm{Fr}}\, H^i(\mu^{-1}(h)/G_h, \bar{\mathbb{Q}}_\ell)$

for h sufficiently generic.

An analogue of DT invariance over $\mathbb{F}_q$

Fix $\psi : (\mathbb{F}_q, +) \longrightarrow \overline{\mathbb{Q}}_\ell^*$ a multiplicative char.

$$DT(Q, W, \mathbb{F}_q)_\alpha = \sum_{\substack{\rho \in \text{abs. indep} \\ \text{reps of } Q \text{ of} \\ \dim \alpha}} \psi(w_\alpha(\rho))$$

Schiffmann

9/2 Talk 5

Position in categorical Langlands correspondence:

X smooth alg. curve $/\mathbb{C}$

$G, G^\vee$ Langlands dual groups

Rough form of cat. Langlands corresp.

$$\mathcal{O}\text{-mod}(\mathrm{Loc}_{G^\vee} X) \text{ "}\simeq\text{" } D\text{-mod}(\mathrm{Bun}_G X)$$

E irred. local system:

$$\mathcal{O}_E \rightsquigarrow \mathrm{Aut}_E$$

What about more general torsion sheaves on $\mathrm{Loc}_{G^\vee} X$?

Most singular pt: $E =$ trivial local system

$\mathcal{O}\text{-mod}_{triv}(\mathrm{Loc}_{G^\vee} X) \rightsquigarrow$ a certain category of D-modules (perverse sheaves?)

$\exists$ parabolic induction functors on both sides

Levi factor, parabolic subgroup

$$L \leftarrow P \rightarrow G$$

$$L^\vee \leftarrow P' \rightarrow G^\vee$$

induce maps

$$\mathrm{Loc}_{L^\vee} \leftarrow \mathrm{Loc}_{P^\vee} \longrightarrow \mathrm{Loc}_{G^\vee}$$

$$\mathrm{Bun}_L \leftarrow \mathrm{Bun}_P \longrightarrow \mathrm{Bun}_G$$

"inducing" functors

$$\mathcal{O}\text{-mod}(\mathrm{Loc}_{L^\vee} X) \rightleftarrows \mathcal{O}\text{-mod}(\mathrm{Loc}_{G^\vee} X)$$

$$D\text{-mod}(\mathrm{Bun}_L X) \rightleftarrows D\text{-mod}(\mathrm{Bun}_G X)$$

More manageable categories:
subcategories generated by $\mathcal{O}_{\mathrm{triv}}(\mathrm{Loc}_{T^\vee} X)$ & $\mathbb{C}_{\mathrm{Bun}_T}$

$G = GL_r$

$$\begin{array}{ccc} \mathrm{Bun}_B X & \xhookrightarrow[\text{open}]{} & \mathrm{Coh}_B X \\ \pi \downarrow & & \downarrow \pi' \\ \mathrm{Bun}_G X & \xhookrightarrow[\text{open}]{} & \mathrm{Coh}_G X \end{array}$$

π not proper, bad π' is.

$\rightsquigarrow \mathrm{Eis}_G X$ = subcategory generated by $\pi_!(\mathbb{C}_{\mathrm{Coh}_B X})$

Here functors

$$\mathrm{Eis}_{GL_{n+m}} \rightleftarrows \mathrm{Eis}_{GL_n} \times \mathrm{Eis}_{GL_m}$$

induce the structure of a Hopf algebra on

$$\bigoplus_n K_0(\mathrm{Eis}_{GL_n})$$

Hope / observation: These categories $\mathrm{Eis}_{GL_n} X$ & their K_0 have a rich & rigid structure.

$\rightsquigarrow$ ∞-dim quantum groups, DAHA,
combinatorics
Knot theory
Rep. theory

Analogy with quivers

$\vec{Q}$ quiver
$\underline{d}$ dimension vector

$\mathrm{Bun}_{GL_n} X \rightsquigarrow \mathrm{Rep}_{\underline{d}} \vec{Q}$

$\mathrm{Eis}_{GL_n} X \rightsquigarrow \mathcal{Q}_{\underline{d}}$ category of perverse sheaves introduced by Lusztig

$\rightsquigarrow$ canonical bases

$$\bigoplus_n K_0(\mathrm{Eis}_{GL_n}(X)) \sim \bigoplus_{\underline{d}} K_0(\mathcal{Q}_{\underline{d}}) \underset{\text{Lusztig Ringel}}{\simeq} U_q^+(\mathfrak{g}_{\vec{Q}})$$

$\uparrow$
Kac-Moody alg associated to $\vec{Q}$

Thm(?)

Fix $\vec{Q}, \underline{d}$

$$\mathrm{Ext}^\bullet\Big(\bigoplus_{\mathbb{P}} \mathbb{P}, \bigoplus_{\mathbb{P}} \mathbb{P}\Big)$$

where $\mathbb{P}$ runs through simple obj. in $Q_{\underline{d}}$

is isom. to K-L-R algebra $A_{\vec{Q}, \underline{d}}$

(fundamental tool in categorification)

Aim: use "combinatorics" of $\bigoplus_r K_0(\mathcal{E}is_{Gl_r} X)$ to compute:

- cohomology of moduli space of stable Higgs bundle $\mathrm{Higgs}^{st} X$ for $X/\mathbb{C}$
- $\#(\mathrm{Higgs}^{st} X)$ for $X/\mathbb{F}_q$

---

§1 Facts about vector bundles & Higgs bundles on curves

X smooth proj. curve $/k$, $k \in \{\mathbb{C}, \mathbb{F}_q, \bar{\mathbb{F}}_q\}$

$\mathrm{Coh}(X)$ — abelian cat of gl. dim 1

For $\mathcal{F} \in \mathrm{Coh}(X)$ $\quad ch(\mathcal{F}) = (rk(\mathcal{F}), \deg(\mathcal{F})) \in \mathbb{Z}^2$

$$\langle \mathcal{F}, \mathcal{G} \rangle = \hom(\mathcal{F}, \mathcal{G}) - \mathrm{ext}(\mathcal{F}, \mathcal{G})$$
$$\underset{R-R}{=} (1-g) r_{\mathcal{F}} r_{\mathcal{G}} + (r_{\mathcal{F}} d_{\mathcal{G}} - r_{\mathcal{G}} d_{\mathcal{F}})$$

$\mathrm{Bun}_{r,d}$ = moduli stack of vector bundles on X of
$\alpha = (r, d)$

smooth stack, locally of finite type.

of $\dim = -\langle (r,d), (r,d) \rangle = (g-1) r^2$

* Cohomology of $\mathrm{Bun}_{r,d}$

Let $\mathcal{E}$ be the universal bundle on $X \times \mathrm{Bun}_{r,d}$,

i.e. $\mathcal{E}_{|X \times \{E\}} = E$

$c_i(\mathcal{E}) \in H^*(X \times \mathrm{Bun}_{r,d}) = H^*(X) \otimes H^*(\mathrm{Bun}_{r,d})$

set $c_i(\mathcal{E}) = [X] \otimes a_i + \sum_j^{2g} \gamma_j \otimes b_i^j + [pt] \otimes f_i$

$a_i, b_i^j, f_i \in H^*(\mathrm{Bun}_{r,d})$

Thm (Atiyah-Bott, Harder-Narasimhan)

$$H^*(\mathrm{Bun}_{r,d}) = \mathbb{C}\langle a_i, b_i^j, f_i \rangle_{i=1,\dots,r}$$

* Point count

$k = \mathbb{F}_q$

$\mathrm{Bun}_{r,d}(\mathbb{F}_q)$ is a groupoid

$$\mathrm{Vol}(\mathrm{Bun}_{r,d}(\mathbb{F}_q)) = \sum_{V \in \mathrm{Bun}_{r,d}/\sim} \frac{1}{\# \mathrm{Aut}(V)}$$

Let $\zeta_X(z) = \dfrac{\prod_{i=1}^{2g}(1 - z\alpha_i)}{(1-z)(1-qz)}$

$\alpha_1, \dots, \alpha_{2g}$ Frob. eigenvalues on $H^1(\bar{X}, \mathbb{Q}_\ell)$

"Weil numbers of X"

<u>Thm</u> (Harder, Langlands, Siegel)

$$\mathrm{Vol}(\mathrm{Bun}_{r,d}) = \frac{q^{(g-1)(r^2-1)}}{q-1} |\mathrm{Jac}(X(\mathbb{Q}))| \prod_{i=2}^{r} \zeta_X(q^{-i})$$

* <u>Relation between H^* & vol</u>

<u>Thm</u> : $X/\mathbb{F}_q$

$H^*(\mathrm{Bun}_{r,d})$ is pure.

* <u>semistable bundles</u>

<u>Def</u>: $\mu(\mathcal{F}) = \dfrac{\deg(\mathcal{F})}{\mathrm{rk}(\mathcal{F})} \in \mathbb{Q}$

$\mathcal{F}$ is semistable if $\forall\, \mathcal{G} \subseteq \mathcal{F}$ $\mu(\mathcal{G}) \leq \mu(\mathcal{F})$

$$\begin{array}{ccc} \mathrm{Bun}^{\frac{1}{2}st}_{r,d} & \underset{\text{open}}{\hookrightarrow} & \mathrm{Bun}_{r,d} \\ \downarrow \text{ coarse moduli space} & & \\ M_{r,d} & & \end{array}$$

$M_{r,d} \longleftarrow$ moduli space of $\frac{1}{2}$stable v. bundles (projective)

Harder–Narasimhan filtration

Fact: $\forall\ \mathcal{V} \in \mathrm{Bun}_{r,d}\ \exists!$ filtration

$$\mathcal{V}_s \subseteq \mathcal{V}_{s-1} \subseteq \dots \subseteq \mathcal{V}_1 = \mathcal{V}$$

st $\begin{cases} \mathcal{V}_i/\mathcal{V}_{i+1} \text{ is semistable} \\ \mu(\mathcal{V}_i/\mathcal{V}_{i+1}) < \mu(\mathcal{V}_{i+1}/\mathcal{V}_{i+2}) \end{cases}$

Def: type of $\mathcal{V} = \left(d(\mathcal{V}/\mathcal{V}_2), \dots, d(\mathcal{V}_s) \right)$

$\rightsquigarrow$ partition of $\mathrm{Bun}_{r,d} = \coprod_{\underline{\alpha}} HN_{\underline{\alpha}}$

$$HN_{\underline{\alpha}} \longrightarrow \mathrm{Bun}^{\frac{1}{2}sl}_{\alpha_1} \times \dots \times \mathrm{Bun}^{\frac{1}{2}st}_{\alpha_s}$$

$\uparrow$
stack v. bundle
"affine fibration"

$\rightsquigarrow$ allows you to recursively compute
$\mathrm{vol}(\mathrm{Bun}^{\frac{1}{2}sl}_{r,d})$
$H^*(\mathrm{Bun}^{\frac{1}{2}sl}_{r,d})$

ref: J. Heinloth, lectures on "Modulistacks of vector bundles on curves"

Higgs bundles

$\mathcal{H}iggs_{r,d} = T^* Bun_{r,d}$ singular stack

$T_E Bun_{r,d} = Ext^1(E,E)$

$\rightsquigarrow T^*_E Bun_{r,d} = Ext^1(E,E)^* \underset{\text{Serre}}{\simeq} Hom(E, E\otimes\Omega)$

$\rightsquigarrow \mathcal{H}iggs_{r,d} = \{(\mathcal{F},\vartheta) : \mathcal{F}\in Bun_{r,d},\ \vartheta\in Hom(\mathcal{F},\mathcal{F}\otimes\Omega)\}$

$\gcd(r,d)=1$

$\exists\ Higgs^{st}_{r,d}$ moduli space, smooth, symplectic

$\mu : \mathcal{H}iggs^{st}_{r,d} \longrightarrow \mathbb{A}$ Hitchin map, Lagrangian fibration

Compute: $H^\bullet(\mathcal{H}iggs^{st}_{r,d})$

$\#(\mathcal{H}iggs^{st}_{r,d}(\mathbb{F}_q))$

Fix $g\geq 0$

$T_g = \{(z_1,\dots,z_{2g})\in \mathbb{G}_m^{2g} : z_{2i-1}z_{2i} = z_{2j-1}z_{2j}\ \forall i,j\}$

$\vdots$

Achar

9/3 Talk 3

Springer Correspondence

G – connected reductive gp. / $\mathbb{C}$

$\cup$

B – Borel subgroup

$\mathcal{B} = G/B$ = flag variety

$\mathcal{B}_g$ $(g \in G)$ – Springer fiber

$\shortparallel$

$\{xB \in G/B \;:\; g \in xBx^{-1}\}$

$\mathcal{U}$ = set of unipotent elts in G

Observation (1900)

$$\left\{\begin{matrix}\text{unipotent classes}\\ \text{of } GL_n\end{matrix}\right\} \xleftrightarrow{\text{bij}} \left\{\begin{matrix}\text{partitions}\\ \text{of } n\end{matrix}\right\} \longleftrightarrow \mathrm{Irr}(W)$$

$C \longmapsto$ sizes of Jordan blocks $\qquad W \shortparallel \mathfrak{S}_n$

In 1976 Springer discovered

$$W \circlearrowright H^*(\mathcal{B}_u, \mathbb{C}) \qquad u \in \mathcal{U}$$

Moreover, among the $H^{top}(\mathcal{B}_u, \mathbb{C})$ each irred. W-rep occurs for a unique u up to conj, i.e.

get $\mathrm{Irr}(W) \longrightarrow \left\{\begin{matrix}\text{unip.}\\ \text{classes}\end{matrix}\right\} \longleftarrow$ Later: enrich this side with info $A_G(u) = C_G(u)/C^\circ_G(u)$

Warmup: Special case $\mathcal{B}_e = \mathcal{B}$
$\uparrow$ id elt.

Trick:

1) $G/T \to G/B$ induces an isom.

$$H^\bullet(G/B) \xrightarrow{\sim} H^\bullet(G/T)$$

2) $W = N(T)/T$ acts on G/T by mult. on the right

What is the action?

Let V = refl. rep. of W

$S(V)$ = symmetric algebra on V

$$\mathrm{Coinv}(W) = S(V) / \left(\underset{\substack{\uparrow \\ \text{homog } W\text{-inv. of pos. degree}}}{S(V)^W_+} \right)$$

Thm (Borel): $H^*(\mathcal{B}) = \mathrm{Coinv}(W)$

Other Springer fibers

$\tilde{G} := \{ (g, xB) \in G \times \mathcal{B} : x^{-1} g x \in B \}$

$$\begin{array}{c} \tilde{G} \\ \downarrow \pi \\ G \end{array} \qquad \text{by def: } \pi^{-1}(g) = \mathcal{B}_g$$

$\tilde{\mathcal{U}} := \pi^{-1}(\mathcal{U})$

Main diagram

$$\begin{tikzcd}
\mathcal{B}_u \arrow[r, hook] \arrow[d] & \tilde{\mathcal{U}} \arrow[r, hook] \arrow[d] & \tilde{G} \arrow[d, "\pi"] & \tilde{G}_{rs} \arrow[l, hook'] \arrow[d] \\
\{u\} \arrow[r, hook] & \mathcal{U} \arrow[r, hook] & G & G_{rs} \arrow[l, hook', "j"'] & \mathcal{L} \arrow[l, squiggly]
\end{tikzcd}$$

Step 0: Note : $R\pi_* \mathbb{C}|_{\{u\}} = H^*(\mathcal{B}_u)$

Plan: 1) Get W to act on $R\pi_*\mathbb{C}$

2) Decompose $R\pi_*\mathbb{C}|_{\mathcal{U}}$

Lemma ①: Let $G_{rs} = \{\text{reg. semisimple elts in } G\}$

$\tilde{G}_{rs} = \pi^{-1}(G_{rs})$

Then $\tilde{G}_{rs} = \{(g, xT) \in G \times G/T \;:\; x^{-1} g x \in T\}$

$\rightsquigarrow W \circlearrowright \tilde{G}_{rs}$

Lemma ②: $\pi|_{\tilde{G}_{rs}} : \tilde{G}_{rs} \to G_{rs}$ is a Galois covering map w/ covering group W

Cor ③: Let $\mathcal{L} = (R\pi_*\mathbb{C})|_{G_{rs}}$, then $\mathcal{L}$ is the local system corresp. to the reg. rep. of W.

In particular $\mathrm{End}(\mathcal{L}) = \mathbb{C}[W]$.

Thm ④: a) $R\pi_* \mathbb{C}[\dim G]$ is perverse.

In fact it is $\cong$ to $j_{!*}(\mathcal{L}[\dim G])$

b) $R\pi_* \mathbb{C}[\dim \mathcal{U}]|_{\mathcal{U}}$ is a perverse sheaf on $\mathcal{U}$.

Pf sketch: perverse sheaves are characterized by some coh-vanishing bounds. Prove these bounds by bounding $\dim \mathcal{B}_u$.

Cor of 4a + 3: $\mathrm{End}(\underbrace{R\pi_* \underline{\mathbb{C}}}_{\text{lives on } G}) = \mathbb{C}[W]$

Restrict to $\mathcal{U}$: get a map

$$(*) \quad \mathbb{C}[W] \longrightarrow \mathrm{End}(R\pi_* \underline{\mathbb{C}}|_{\mathcal{U}})$$

Let $A = R\pi_* \underline{\mathbb{C}}|_{\mathcal{U}}[\dim \mathcal{U}]$ -- the <u>Springer sheaf</u>

Thm ⑤ (Borho - MacPherson)

$(*)$ is an isomorphism, so

$$\mathrm{End}(A) = \mathbb{C}[W]$$

(pf: Restrict further to $\mathcal{B}_e$, use lattice-ulp.)

Decomposition Thm $\Rightarrow$ A is a semi-simple perverse sheaves

$$A = \bigoplus_{(C,\mathcal{E})} IC(C,\mathcal{E}) \otimes V_{C,\mathcal{E}}$$

nilpotent class

equiv irred loc system $\leftarrow$ irr. rep of $A_G(u)$

"multiplicity vector space"

Thm 5 $\Rightarrow$ $A = \bigoplus_{\mathrm{Virr}(W)} \left(\begin{smallmatrix}\text{simple} \\ \text{obj}\end{smallmatrix}\right) \otimes V$

distinct

Math terms

set $\boxed{\mathrm{Irr}(W) \xhookrightarrow[\text{inj. map}]{} \{(C,\mathcal{E})\}}$

Springer correspondence

or

$\boxed{\mathrm{Irr}(W) \hookrightarrow \mathrm{Irr}(\mathrm{Perv}_G(\mathcal{U}))}$

Examples

1. GL_n: All $A_G(u)$'s are triv
   - So no non-triv loc. sys
     $\{(C,\mathcal{E})\} = \{(C,\underline{\mathbb{C}})\}$

   $\Rightarrow$ Springer corresp. is a bij.

2. SL_2: $u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad A_G(u) = \mathbb{Z}/2$

   (class of u, non-triv loc system) is <u>NOT</u> in the image of the Springer corresp.

Problem: explain the missing pairs.

Setup for Thursday:

Fix a parabolic subgroup $P = LV$

$$i_P : P \hookrightarrow G \quad , \quad q_P : P \twoheadrightarrow L$$

Two functors:

$$\mathrm{res}^G_{L\subset P} : D^b_G(G) \longrightarrow D^b_L(L)$$

$$\mathrm{res}^G_{L\subset P} = Rq_{P!}\, i_P^*$$

Rmk: If $H \circlearrowright X$, $H \subset G$, then $D^b_G(G \times^H X) = D^b_H(X)$

$$\nearrow$$

$$G \times X / (gh, x) \sim (g, hx)$$

$$G \times^H (-) : D^b_H(X) \to D^b_G(G \times^H X)$$

$$\mathrm{ind}^G_{L\subset P} : D^b_L(L) \longrightarrow D^b_G(G)$$

$$R\pi_*\big(\underbrace{G \times^P (q_P^!(-))}_{\text{sheaf on } G\times^P P \xrightarrow{\pi} G}\big)$$

Exercise: $A = \mathrm{ind}^G_{T\subset B}\, \mathbb{C}_{\{e\}}$ $\leftarrow$ skyscraper at $e \in T$.

Springer Corresp., Part II

9/4 Talk 2

Let $X_G = \{ (C, \mathcal{E}) \} = \mathrm{Irr}(\mathrm{Perv}_G(\mathcal{U}))$

unip. class

irred G-equi. loc. sys.

$P = LV$ parabolic.

Say $(C_1, \mathcal{E}_1) \in X_L$

$IC(C_1, \mathcal{E}_1)$ supported on $\bar{C}_1 \subset \mathcal{U}_L \subset L$

$q_P^!($ — " — $)$ supported on $\bar{C}_1 V \subset P$

$$\begin{array}{c} G \times^P \bar{C}_1 V \\ \big\downarrow \pi \\ \mathcal{U}_G \end{array}$$

$\rightsquigarrow \mathrm{ind}_{L \subset P}^G (IC(C_1, \mathcal{E}_1))$ is supported on $\mathcal{U}_G$

Yesterday: $\mathrm{ind}_{T \subset B}^G IC(C_0, \underline{\mathbb{C}}) = \underbrace{A}_{\text{Springer sheaf}} = \bigoplus_{(C,\mathcal{E}) \in X_G} IC(C, \mathcal{E}) \otimes V_{C,\mathcal{E}}$

→ either 0 or an irred W-rep

→ every irred W-rep occurs once!

Get $\mathrm{Irr}(W) \longleftrightarrow \{$ simple summands of $A \} \subset X_G$

Problem: What about the missing pairs?

Ex: $G = SL_4$: $|X_G| = 9$, $|\mathrm{Irr}(S_4)| = 5$

$\longrightarrow$ 4 missing pairs.

Idea: Develop a "Harish-Chandra theory" for $\mathrm{Perv}_G(\mathcal{U})$

Thm 6:

- (res, ind) is an adjoint pair
- They restrict to exact functors

$$\mathrm{Perv}_L(\mathcal{U}_L) \rightleftarrows \mathrm{Perv}_G(\mathcal{U}_G)$$

- They take semisimple perverse sheaves to semisimple perverse sheaves.

Pf sketch:

Exactness:

- Can get 1-sided exactness statement by dim calculations in Thm 4(b)
- Braden's theory of hyperbolic localization

Semisimplicity:

- ind: Decomposition Thm
- Braden's theorem

Side note: $\mathrm{Perv}_G(\mathcal{U})$ is semisimple!

<u>Def</u>: $(C, \mathcal{E})$ is <u>cuspidal</u> if $\mathrm{res}^G_{L\subset P}\, IC(C,\mathcal{E}) = 0 \;\; \forall P \subsetneq G$

<u>Thm 7</u> ("Induction series")

$\forall (C,\mathcal{E}) \in X_G \quad \exists (L, C_0, \mathcal{E}_0)$ where unique up to G-conjugacy

L Levi, $(C_0, \mathcal{E}_0) \in X_L$ cuspidal

s.t. $IC(C, \mathcal{E})$ is a summand of $\mathrm{ind}^G_{L\subset P}(C_0, \mathcal{E}_0)$

<u>Pf sketch</u>:

Existence: obvious

uniqueness: use "Mackey formula" that relates res o ind to ind o res.

So:

$$X_G = \coprod_{(L, C_0, \mathcal{E}_0)} X_G^{(L, C_0, \mathcal{E}_0)} \qquad (*)$$

$$X_G^{(L, C_0, \mathcal{E}_0)} = \text{simple summands of } \mathrm{Ind}^G_{L\subset P}(IC(C_0, \mathcal{E}_0))$$

One piece of $(*)$:

$$X_G^{(T, \{e\}, \underline{\mathbb{C}})} = \text{simple summands of } A = \mathrm{Irr}(W)$$

Thm ⑧: $\forall (L, C_0, \mathcal{E}_0)$

$$X_G^{(L, C_0, \mathcal{E}_0)} \longleftrightarrow \mathrm{Irr}\big(\overbrace{N_G(L)/L}^{\text{"relative Weyl" group}}\big)$$

pf sketch: $P = LV$ $\qquad Z^\circ(L) =$ connected component of centralizer of L

$$Z^\circ(L)_{reg} = \{x \in Z^\circ(L) : C_G^\circ(x) \subset L\}$$

main diagram: $\quad \Sigma = Z^\circ(L) C_0, \quad \Sigma_{reg} = Z^\circ_{reg}(L) C_0$

$$\begin{tikzcd}
\underset{IC(C_0,\mathcal{E}_0)}{G \times^P \overline{C_0} V} \arrow[r, hook] \arrow[d] & \underset{IC(\Sigma, \mathbb{C} \boxtimes \mathcal{E}_0)}{G \times^P \overline{\Sigma} V} \arrow[d, "\pi"] & G \times^P \Sigma_{reg} V \arrow[l, hook'] \arrow[d, "\pi_{reg}"] \\
\mathcal{U}_G \arrow[r, hook] & G & G \cdot \Sigma_{reg} \,(= G \cdot \Sigma_{reg} V) \arrow[l, hook']
\end{tikzcd}$$

Steps:

1) $G \times^P \Sigma_{reg} V \simeq \{(g, xL) \in G \times G/L : x^{-1} g x \in \Sigma_{reg}\}$

$N_G(L)/L$ acts on this.

2) π_{reg} is a Galois covering w/ gp $N_G(L)/L$

Let $\mathcal{L} = \pi_{reg,*}(G \times^P q_P^*(\mathbb{C} \boxtimes \mathcal{E}_0))$

3) $\mathrm{End}(\mathcal{L}) = \mathbb{C}[N_G(L)/L]$

most delicate step ... need to understand what $(C_0, \mathcal{E}_0)$ can be

4) $R\pi_* \, G \times^P (q_P^! \, IC(\Sigma, \mathbb{C} \boxtimes \mathcal{E}_0))$

$\quad \rightsquigarrow$ restrict to $\mathcal{U}_G$ $\qquad \| \qquad j_{!*} \mathcal{L}[\ldots]$

$\mathrm{ind}_{L \subset P}^G \, IC(C_0, \mathcal{E}_0)$

5) The map

$$\mathbb{C}[N_G(L)/L] = \mathrm{End}(\mathcal{L}) \longrightarrow \mathrm{End}(\mathrm{ind}\, IC(C_0, \mathcal{E}_0))$$

is an isomorphism.

6) $$\mathrm{ind}\, IC^L(C_0, \mathcal{E}_0) = \bigoplus_{(C,\mathcal{E})} IC^G(C, \mathcal{E}) \otimes V_{C,\mathcal{E}}$$

$$= \bigoplus_{V \in \mathrm{Irr}(N_G(L)/L)} \begin{pmatrix} \text{simple} \\ \text{obj.} \end{pmatrix} \otimes V$$

match terms. $\square$

Combine Thms 7 & 8:

$$\boxed{\bigsqcup_{(L, C_0, \mathcal{E}_0)} \mathrm{Irr}(N_G(L)/L) \xrightarrow{\sim} X_G = \{(C, \mathcal{E})\}}$$

generalized Springer correspondence

Ex SL_4:

$$|\mathcal{X}_G^{(T,1,\mathbb{C})}| = 5$$

$$|\mathcal{X}_G^{(A_1\times A_1, \cdot, \cdot)}| = 2$$

& 2 cuspidal pairs

Context: character sheaves (~1980s)

- Certain G-equiv. perverse sheaves on G
- Goal: for $G/\mathbb{F}_q$

$\rightsquigarrow$ compute char. values of $G(\mathbb{F}_q)$

- $\mathrm{Perv}_G(\mathcal{U}_G)$ - usually not char. sheaves ($j_{!*}\mathcal{L}$ is a char. sheaf)

Lübeck

More or less reduce to compute stalks of IC's on $\mathcal{U}_G$

- Stalks of IC's on $\mathcal{U}_G$:

"Lusztig-Shoji algorithm":

- explicit knowledge of gen Springer
- Basic rep theory of $N_G(L)/L$
- How to manipulate matrices

Lie algebra version

$\mathcal{N}$ = nilpotent cone $\subset \mathfrak{g}$

Yesterdays main diagram

$$\begin{tikzcd} \tilde{\mathcal{N}} \arrow[r, hook] \arrow[d] & \tilde{\mathfrak{g}} \arrow[d] & \tilde{\mathfrak{g}}_{rs} \arrow[l, hook'] \arrow[d] \\ \mathcal{N} \arrow[r, hook] & \mathfrak{g} & \mathfrak{g}_{rs} \arrow[l, hook'] \end{tikzcd}$$

$\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ parabolic $\quad \Sigma = C_0 + \mathfrak{z}_L$

$$\begin{tikzcd} G \times^P (C_0 + \mathfrak{n}) \arrow[r, hook] \arrow[d] & G \times^P (\overline{\Sigma} + \mathfrak{n}) \arrow[d] & G \times^P (\Sigma_{reg} + \mathfrak{n}) \arrow[l, hook'] \arrow[d] \\ \mathcal{N} \arrow[r, hook] & \mathfrak{g} & G \cdot \Sigma_{reg} \arrow[l, hook'] \end{tikzcd}$$

transfer info

$$(j_{!*} \mathcal{L})|_{\mathcal{N}} \simeq \operatorname{ind} IC(C_0, \mathcal{E}_0)$$

or: FOURIER TRANSFORM

~ Deligne
Sato

$\mathbb{F}: \operatorname{Perv}_G(\mathfrak{g}) \longrightarrow \operatorname{Perv}_G(\mathfrak{g})$

Prop : $\mathbb{T}(j_{!*}\mathcal{L}) = \mathrm{ind}\, IC(C_0, \mathcal{E}_0)$

$$
\begin{tikzcd}
 & \mathbb{C}[N_G(L)/L] \cong \mathrm{End}(j_{!*}\mathcal{L}) \arrow[dl, "(\ldots)|_N"', "\sim"] \arrow[dr, "\mathbb{T}", "\sim"'] & \\
\mathrm{End}(\mathrm{ind}\, IC(C_0,\mathcal{E}_0)) & & \mathrm{End}(\mathrm{ind}\, IC(C_0,\mathcal{E}_0))
\end{tikzcd}
$$

these differ by the sign character

Proudfoot

9/4 Talk 5

Quantizations of symplectic resolutions

G simple alg. group $/\mathbb{C}$

$B \subseteq G$ Borel

$X = G/B$

$Z = M \times_{M_0} M$ ... Steinberg variety

$M = T^*X = \{(gB, a) \in X \times \mathrm{nil}(\mathfrak{g}) : g^{-1} x g \in \mathfrak{b}\}$

$\downarrow$

$M_0 = \mathrm{nil}(\mathfrak{g})$ ... Springer resolution

Ex: $G = SL_2$

$M = T^*\mathbb{P}^1$

$\downarrow$

$M_0 = \mathbb{C}^2/\{\pm 1\}$

$Z = T^*\mathbb{P}^1 \times_{\mathbb{C}^2/\{\pm 1\}} T^*\mathbb{P}^1$

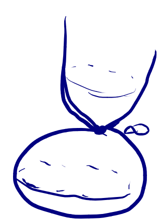
$= T^*\mathbb{P}^1_\Delta \cup_{\mathbb{P}^1} (\mathbb{P}^1 \times \mathbb{P}^1)$

Fact: All components of Z have dimension $d = \dim M$

$M^+ = \bigcup \left\{\begin{matrix}\text{conormal bundles} \\ \text{to } B\text{-orbits in } X\end{matrix}\right\}$

Ex: $X = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$

$M^+ = \mathbb{C} \cup T^*_\infty \mathbb{CP}^1$



Consider $H^{BM}_{2d}(Z) = \mathbb{C}^{\#\text{ of components of } Z}$

This is an algebra:

$$\begin{tikzcd}
& \mathcal{M} \times_{\mathcal{M}_0} \mathcal{M} \times_{\mathcal{M}_0} \mathcal{M} \arrow[dl, "p_1"'] \arrow[d, "p_2"] \arrow[dr, "p_3"] & \\
Z & Z & Z
\end{tikzcd}$$

$$\alpha * \beta = p_{2,*}(p_1^* \alpha \cap p_2^* \beta)$$

$H^{BM}_{d}(\mathcal{M}^+) = \mathbb{C}^{\#\text{ components of } \mathcal{M}^+}$

is module of $H^{BM}_{2d}(Z)$

<u>Thm</u> (Ginzburg): $H^{BM}_{2d}(Z) \cong \mathbb{C}[W]$

$H^{BM}_{d}(\mathcal{M}^+) \simeq \mathbb{C}[W]$

$G \circlearrowright X \rightsquigarrow \mathfrak{g} \longrightarrow \mathrm{vf}(X)$

$\rightsquigarrow U(\mathfrak{g}) \longrightarrow \mathrm{Diff}(X) = \Gamma(X, D_X)$

<u>Fact</u>: This map is surjective.

Let $U(\mathfrak{g})_0 = U(\mathfrak{g})/_{\ker} \simeq \mathrm{Diff}(X)$

<u>Thm</u> (Beilinson – Bernstein)

$$U(\mathfrak{g})_0\text{-mod} \underset{\Gamma}{\overset{\mathrm{Loc}}{\underset{\sim}{\rightleftarrows}}} D_X\text{-mod}$$

$$U(\mathfrak{g})_0\text{-mod} \xrightarrow{\text{Loc}} D_X\text{-mod} \overset{\text{"microlocal support"}}{\rightsquigarrow} \text{cycles on } \mathcal{M} = T^*X$$

$$\cup\!\!\mid \qquad\qquad\qquad\qquad\qquad\qquad \cup\!\!\mid$$

$\mathcal{O}_0 \longrightarrow$ weakly B-equiv D_X-modules $\rightsquigarrow$ cycles on $\mathcal{M}^+$

$\mathcal{O}_0 :=$ f.g. $U(\mathfrak{g})_0$-modules that are locally finite for the action of $U(\mathfrak{b})$

$$K(\mathcal{O}_0)_{\mathbb{C}} \xrightarrow{\ \approx\ } H_d^{BM}(\mathcal{M}^+)$$

$U(\mathfrak{g})_0$-bimod $\xrightarrow{\text{Loc}}$ $D_X \boxtimes D_X^{op}$-mod $\rightsquigarrow$ cycles on $\mathcal{M} \times \mathcal{M}$

$\cup\!\!\mid \qquad\qquad\qquad\qquad\qquad \cup\!\!\mid$

$HC_0 \longrightarrow$ G-equiv D-modul on $X \times X$ $\rightsquigarrow$ cycles on Z

$HC_0 :=$ f.g. $U(\mathfrak{g})_0$-bimodules that are locally finite for the adjoint action

Thm: (1) HC_0 is a tensor category acting on $\mathcal{O}_0$

(2) Support intertwines $\overset{L}{\otimes}$ with $*$.

(3) $\exists$ bimodules^{HC} $\{H_w : w \in W\}$ s.t.

- $H_w \overset{L}{\otimes} - : D^b(\mathcal{O}_0) \underset{\Theta_w}{\xrightarrow{\sim}} D^b(\mathcal{O}_0)$
- $\Theta_w \circ \Theta_{w'} \simeq \Theta_{ww'}$ if $\ell(w) + \ell(w') = \ell(ww')$

Braid group

Thus $B_W \curvearrowright D^b(\mathcal{O}_0)$ categorifying $W \curvearrowright K(\mathcal{O}_0)_{\mathbb{C}}$

Def: A conical symplectic resolution is

- a smooth variety $M/\mathbb{C}$
- a symplectic form $\omega \in \Omega^2_{alg}(M)$
- commuting actions of $S = \mathbb{C}^\times$, $T = \mathbb{C}^\times$

such that

- T preserves ω
- S rotates ω: $s^*\omega = s\omega \quad \forall s \in S$
- $S \circlearrowright \mathbb{C}[M]$ with non-negative weights and $\mathbb{C}[M]^S = \mathbb{C}$
- $|M^T| < \infty$
- The map

$$\begin{array}{c} M \\ \downarrow \\ M_0 = \operatorname{Spec} \mathbb{C}[M_0] \end{array}$$

is a projective resolution

Ex: ① $M = T^*(G/B)$ $\quad$ S scales fibers

$M_0 = \mathrm{nil}(\mathfrak{g})$ $\quad$ $T \hookrightarrow G$ generic

② $M = \mathrm{Hilb}^n(\widetilde{\mathbb{C}^2/\Gamma})$

$\downarrow$ $\quad\quad$ $\Gamma = \mathbb{Z}/k$

$M_0 = \mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$

③ $G \circlearrowright V$ linear rep

$\chi: G \to \mathbb{C}^\times$

$G \circlearrowright T^*V \xrightarrow{\mu} \mathfrak{g}^*$

(+ assumption)

$$\begin{array}{c} M = \mu^{-1}(0) /\!\!/_{\chi} G \\ \downarrow \\ M_0 = \mu^{-1}(0) /\!\!/ G \end{array}$$

e.g. quiver varieties
hypertoric varieties

Let $Z = \mathcal{M} \times_{\mathcal{M}_0} \mathcal{M}$

$\mathcal{M}^+ = \{ p \in \mathcal{M} : \lim_{t \to 0} tp \text{ exists} \}$

$H^{BM}_{2d}(Z) \circlearrowright H^{BM}_{d}(\mathcal{M}^+)$ as before

<u>Def</u>: A quantization of $(\mathcal{M}, \omega)$ is

- a T-equivariant sheaf $\mathcal{A}$ of filtered algebras on $\mathcal{M}$
- a $T \times S$-graded isomorphism $\operatorname{gr} \mathcal{A} \simeq \operatorname{Fun}_{\mathcal{M}}$

+ compatibility of Poisson bracket.

Let $A = \Gamma(\mathcal{M}, \mathcal{A})$

<u>Ex</u>: ① $\mathcal{M} = T^*(G/B)$, A is a quotient of $U(\mathfrak{g})$

② $\mathcal{M} = \overline{\operatorname{Hilb}^n(\mathbb{C}^2/\Gamma)}$, A is a quotient of a spherical rat. Cherednik alg.

③ $\mathcal{M} = T^*V /\!/ G$, A a quotient of the $\operatorname{Diff}(V)^G$.

<u>Thm</u> (BPW):

$$\mathcal{A}\text{-mod} \underset{\mathrm{Loc}}{\overset{\Gamma}{\rightleftarrows}} \mathscr{A}\text{-mod}$$

is an equivalence for "most" quantization.

$$\begin{array}{ccccc} \mathcal{A}\text{-mod} & \xrightarrow{\mathrm{Loc}} & \mathscr{A}\text{-mod} & \xrightarrow{\mathrm{supp}} & \text{cycles on } \mathcal{M} \\ \cup\!\mid & & & & \cup\!\mid \\ \mathcal{O} & & \rightsquigarrow & & \text{cycles on } \mathcal{M}^+ \end{array}$$

$\mathcal{O}$ = f.g. $\mathcal{A}$-modules that are locally finite for $\mathcal{A}^+$

↑ Tachin

<u>Thm</u> (BLPW): $K(\mathcal{O})_{\mathbb{C}} \xrightarrow{\cong} H^{BM}_{d}(\mathcal{M}^+)$

$$\begin{array}{ccccc} \mathcal{A}\text{-bimod} & \xrightarrow{\mathrm{Loc}} & \mathscr{A}\otimes\mathscr{A}^{op}\text{-mod} & \xrightarrow{\mathrm{supp}} & \text{cycles on } \mathcal{M}\times\mathcal{M} \\ \cup\!\mid & & & & \cup\!\mid \\ HC & & \rightsquigarrow & & \text{cycles on } Z \end{array}$$

HC = N s.t. $\mathrm{gr}\, N$ is supported on diagonal of $\mathcal{M}_0 \times \mathcal{M}_0$

Thm (BPW)

1. HC is a tensor category acting on $\mathcal{O}$
2. support intertwines $\overset{L}{\otimes}$ with $*$
3. $\exists$ nice bimodules that fit together into a generalized Braid group action on $D^b(\mathcal{O})$.

Proudfoot, pt II

9/4 Talk 3

Summary

$$M \xrightarrow{\text{conical symplectic resolution}} M_0$$

$$T^*\mathbb{P}^1 \longrightarrow \mathbb{C}^2/\{\pm 1\}$$

$\mathbb{C}^\times = T \circlearrowright M$

$M^+ = \{p \in M \mid \lim_{t\to 0} tp \text{ exists}\}$

$Z = M \times_{M_0} M$

$T^*\mathbb{P}^1 \cup \mathbb{P}^1 \times \mathbb{P}^1$

A a filtered algebra s.t.
$\operatorname{gr} A \cong \mathbb{C}[M]$

$\Gamma(\mathbb{P}^1, D_{\mathbb{P}^1}) = \mathcal{U}(sl_2)_0$

$\mathcal{O}$ = some category of A-modules

$$K(\mathcal{O})_{\mathbb{C}}$$
$$\cong$$
$$H^{BM}_d(M^+) \circlearrowleft H^{BM}_{2d}(Z)$$
$$\cong$$
$$H^d_T(M)$$

"little simples" "big simples"

$$\begin{tikzcd}
\mathbb{C}\{[L] : L \text{ f.d. simple}\} \arrow[d, hook] & \subseteq & K(\mathcal{O})_{\mathbb{C}} \arrow[d, "\cong"] & \supseteq & \mathbb{C}\{[L] : L \text{ simple w/ } \mathrm{Ann}(L) = 0\} \arrow[d] \\
H^d(M) & \subset & H^d_T(M) & \supseteq & H^d_T(M_0)
\end{tikzcd}$$

Conjecture (BLPW)

① $\mathcal{O}$ is Koszul

Ex: i) $T^*(G/B)$ (BGS)

ii) $\lambda \geq \mu$ partitions of r

$\bar{N}_\lambda \supset N_\mu$ $\quad X_{\lambda,\mu}$ resolution of slice to N_μ inside $\bar{N}_\lambda$

$\mathcal{O} \simeq$ singular block of parabolic $\mathfrak{sl}_r$ BGG cat $\mathcal{O}$

$\uparrow$ Koszul by BGS

(Losev, Webster)

iii) $\mathcal{M}(k,r) = \mathrm{Hilb}^r \widetilde{\mathbb{C}^2/\mathbb{Z}_k}$

$\mathcal{M}(k,r) = \left\{ \begin{array}{l} \text{torsion free sheaves on} \\ \mathbb{P}^2, \text{ framed at } \infty, \\ rk = k, \; c_2 = r \end{array} \right\}$

iv) Hypertoric varieties

② $\exists$ $\begin{array}{c} M^! \\ \downarrow \\ M^!_0 \end{array}$ such that $\mathcal{O}^!$ is Koszul dual to $\mathcal{O}$.

Ex: i) $T^*(G/B)$ is dual to $T^*(G^\vee/B^\vee)$ (BGS)

ii) $X_{\lambda, \mu}$ is dual to X_{μ^t, λ^t} (Backelin)

iii) $\mathcal{H}(k,r)$ is dual to $\mathcal{M}(k,r)$ (RSW)

iv) hypertoric varieties are dual to other hypertoric varieties (BLPW)

(3) $\mathcal{O}$ dual to $\mathcal{O}^!$ $\Rightarrow$ $\exists$ bijection

$\{$simples in $\mathcal{O}\}$ $\longleftrightarrow$ $\{$simples in $\mathcal{O}^!\}$

big simple $\longleftrightarrow$ little simple

little $\longmapsto$ big

Upshot: If $\mathcal{M}$ is dual to $\mathcal{M}^!$

$$\begin{array}{ccc} \mathcal{M} & & \mathcal{M}^! \\ \downarrow & & \downarrow \\ \mathcal{M}_0 & & \mathcal{M}_0^! \end{array}$$

the $H^d(\mathcal{M}) \simeq \mathbb{C}\{[L] : L \text{ little simple in } \mathcal{O}\}$

looks like $H^{d^!}_{T^!}(\mathcal{M}_0^!) = \mathbb{C}\{[L^!] : L^! \text{ big simple in } \mathcal{O}^!\}$

examples: see video