Jonathan Wise

Intersection Theory
App's to Obstruction Theory

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$\tilde{M}$ a derived scheme (expected dim $= 0$, $L_{\tilde{M}}$ perfect, rank 0)

$\chi(\pi_* \mathcal{O}_{\tilde{M}})$ = virt # of points

Example conics Q, Q' in $\mathbb{P}^2$

$\tilde{M} = Q \cap Q'$

$$\chi(\pi_* \mathcal{O}_{\tilde{M}}) = \sum (-1)^i \operatorname{Tor}_i(\mathcal{O}_Q, \mathcal{O}_{Q'}) = 4$$

(Fulton's) Intersection Theory

If $Q = Q'$, $M = Q \cap Q'$ (underived)

$$\int_M c_{top}\left(N_{Q/\mathbb{P}^2}\big|_M\right) = 4$$

Want to do this with derived schemes

$$N_{Q/\mathbb{P}^2}\big|_M = L_{\tilde{M}/Q}[-1]$$

$\boxed{\text{Moduli space of stable maps}}$

Fix X a smooth scheme

$$M(X)(A) = \left\{ \begin{array}{l} C \xrightarrow{\pi} \operatorname{Spec}(A) \text{ is stable curve of genus} \geq 2 \\ \text{and map } C \longrightarrow X \end{array} \right\}$$

a) π is flat, qcqs, almost f.p.

b) ~~$X \times_A \pi_0 A \to \operatorname{Spec}(\pi_0 A)$ is stable curve~~ $\left(\operatorname{Aut}\begin{pmatrix} C \to X \\ \downarrow \\ \operatorname{Spec} A \end{pmatrix} \right)$

Theorem $\overline{M}(X) : SCR \longrightarrow Spaces$

$$A \longmapsto \left\{ \begin{matrix} \text{all stable curves} \\ C \longrightarrow Spec(A) \\ \text{of genus } g \geq 2 \end{matrix} \right\}$$

is represented

Compute $L_{M(X)}$ by evaluating $M(X)$ on $A+J$

$$\begin{tikzcd}
C \arrow[r, dashed] \arrow[d] \arrow[rr, bend left, "f"] & C' \arrow[r, dashed] \arrow[d, dashed] & X \\
Spec\, A \arrow[r] & Spec(A+J) &
\end{tikzcd}$$

$$\begin{tikzcd}
L_{C/A} \arrow[r] & J[1] \\
f^* L_X \arrow[u] \arrow[ur, "0"'] &
\end{tikzcd} \iff \text{def. of } C$$

$Hom(Spec(A+J), M(X))$

$= Hom(L_{M(X)}, J)$

$= Hom(\underbrace{cofib(f^* L_X \longrightarrow L_{C/S})}_{F}, J[1])$

$L_{M(X)} = R\pi_*(F[-1]^\vee)^\vee$

$\Uparrow$

(concentrated in degrees) $[-1, 0]$

$\boxed{\text{Setting}}$ M an ordinary scheme

and map of complexes $E \to L_M \to K$ exact

which is 1-connective (i.e. $K = \tau_{\leq -2} K$)

called obstruction theory (Behrend-Fantechi)

We get maps $Ext^0(L_M, J) \xrightarrow{\sim} Ext^0(E, J)$

and

$$Ext^1(L_M, J) \hookrightarrow Ext^1(E, J)$$

we

$$\begin{tikzcd} S \arrow[r] \arrow[d] & M \\ S' \arrow[ur, dashed] & \end{tikzcd}$$

$I_{S/S'} = J$

Let $\tilde{M}$ = derived scheme

$\pi_0(\tilde{M}) = M$

$\pi_0(\mathcal{O}_{\tilde{M}}) = \mathcal{O}_M$

$\Rightarrow$ the map $\mathcal{O}_{\tilde{M}} \to \mathcal{O}_M$ is 1-connective

$\Rightarrow$ the map $\underbrace{\mathcal{O}_M \otimes_{\mathcal{O}_{\tilde{M}}} L_{\tilde{M}}}_{E} \to L_M$ is also 1-connective

$$L_M \to L_{M/\tilde{M}}$$

gives obstruction theory

Theorem (Schürg) Any perfect (i.e. E is perfect complex) obstruction theory comes from a derived enhancement.

Proof Suppose $E \to L_M \to K$ is an obstruction theory

This gives an extension $M^{(1)}$ of M by $K[-1]$

$$\begin{tikzcd}
M \subset M^{(1)} \arrow[r, squiggly] & \mathcal{O}_M \otimes_{\mathcal{O}_{M^{(1)}}} L_{M^{(1)}} \arrow[r] \arrow[d] & L_M \arrow[r] \arrow[d] & L_{M/M^{(1)}} \arrow[d] \\
& K^{(1)} \arrow[r] & K(1) \arrow[r] & L_{M/M^{(1)}}
\end{tikzcd}$$

$L_{M^{(1)}} \to K^{(1)}$ gives an obstruction theory on $M^{(1)}$

We have $M \subset M^{(1)} \subset M^{(2)} \subset \cdots$

$$\mathcal{O}_M \leftarrow \mathcal{O}_{M^{(1)}} \leftarrow \mathcal{O}_{M^{(2)}} \leftarrow \cdots$$

Then $\mathcal{O}_{\tilde{M}} \simeq \underset{\overleftarrow{i}}{\text{holim}}\ \mathcal{O}_{M^{(i)}}$

$\boxed{\text{Construction of fundamental class}}$

Every $\overset{\text{obstruction theory}}{(M, E)}$ is locally a fiber product

$$\underline{E} = \left[N^\vee_{Y/W}\big|_M \to \Omega_Z\big|_M \right] \qquad \begin{tikzcd}
M \arrow[r] \arrow[d] & Z \arrow[d] \\
Y \arrow[r, hook, "i", "lci"'] & W
\end{tikzcd} \quad \text{smooth}$$

Try intersection class

$i^![Z]$ is a cycle class on M = "fundamental class"

$\exists$ map $C_{M/Z} \hookrightarrow N_{Y/W}|_M$

$\exists!$ cycle class $[M]^{vir}$ that pulls back to $[C_{M/Z}]$ in $N_{Y/W}|_M$

Ambiguity: we might have

$$\begin{tikzcd} & Z' \arrow[d] \\ M \arrow[ur] \arrow[r, hook] & Z \end{tikzcd} \quad \text{then } C_{M/Z} = C_{M/Z'} \Big| T_{Z'/Z}$$

Try to kill this ambiguity –

replace $C_{M/Z}$ with quotient $\underbrace{\left[C_{M/Z} / T_Z \right]}_{\text{Stack quotient}} = \mathfrak{C}_M$ Intrinsic Normal Cone

$\mathcal{E} = \left[N_{Y/W}|_Z / T_Z \right]$ ($=$ total space of $E^\vee$)

$\mathfrak{C}_M \subset \mathcal{E}$

$[M]^{virt}$ = unique cycle class on M that pulls back to $[\mathfrak{C}_M]$ in $\mathcal{E}$

<u>Note</u> This applies to $M(X)$

Gromov-Witten invariants of $M(X)$ are $\int_{[M]^{virt}} (\ \ )$.

$\boxed{\text{Properties}}$

- Invariant under deformation $\Rightarrow$ gives computational technique
- If $M' \subset M$ smooth, then $[M']^{virt} = C_{top}$ (obstruction bundle)
- If you understand geometry of $M(X)$, then it can be related to $GW(X)$

Extracting enumerative results:

$X = CY3$ (Calabi-Yau 3-fold)

$E \subset X$ a rational curve $\left(N_{E/X} = \mathcal{O}(-1) + \mathcal{O}(-1) \right)$

We get 1 to $GW(X)$ of class $\beta = [E] \in H_2(X)$

$$\begin{array}{c} \shortparallel \\ \displaystyle\int_{[M(X,\beta)]^{virt}} 1 \end{array}$$

$C \longrightarrow E$ a deg d map contributes $\frac{1}{d^3}$ to $GW(X)$ of class $d\beta$.