MONDAY EXERCISES

1. Hecke operators in Geometric Langlands: Lecture 1

Topology of Lie groups and loop groups. Basic Lie theory with a focus on flag varieties, Schubert stratifications, relative position correspondences.

1.1. **Exercise.** Draw the weights, coweight, roots, and coroots for $SL_3(\mathbb{C})$ and $PGL_3(\mathbb{C})$ and compare them. Do the same for $SO_5(\mathbb{C})$ and $Sp_4(\mathbb{C})$, and for $SO_7(\mathbb{C})$ and $Sp_6(\mathbb{C})$.

1.2. **Exercise.** (Borel) Calculate the rational cohomology ring of the flag variety G/B in terms of functions $\mathcal{O}(0_{th})$ on the scheme-theoretic fiber

$$\begin{tikzcd} 0_{th} \arrow[r] \arrow[d] & 0 \arrow[d] \\ \mathfrak{t} \arrow[r] & \mathfrak{t}//W \end{tikzcd}$$

1.3. **Exercise.** (Bott-Samelson) Resolve any Schubert variety by an iterated $\mathbb{P}^1$-bundle.

1.4. **Exercise.** Let $Gr_2^4(\mathbb{C})$ be the Grassmannnian of 2-planes in 4-space.

Let $e_1, e_2, e_3, e_4 \in \mathbb{C}^4$ be the standard basis, and $E_2 = \{e_1, e_2\} \subset \mathbb{C}^4$ the standard 2-plane. Describe the Schubert variety

$$X = \{P \in Gr_2^4(\mathbb{C}) | \dim(P \cap E_2) \geq 1\}$$

its singularity, and find a minimal resolution.

1.5. **Exercise.** Compare the presentations of an affine Weyl group in terms of reflections (Coxeter) and as a semi-direct product of the finite Weyl group with a lattice (Bernstein).

1.6. **Exercise.** Let $\mathfrak{p}$ be a partition of n with p parts, and $n_{\mathfrak{p}} \in M_n(\mathbb{C})$ the corresponding nilpotent matrix. Identify the Springer fiber

$$X_{\mathfrak{p}} = \{P \in \textstyle\coprod_{k=0}^{n} Gr_k^n(\mathbb{C}) | n_{\mathfrak{p}}(P) \subset P\}$$

with a subvariety of the affine Grassmannian Gr_{GL_p}.

1.7. **Exercise.** Find as many descriptions as you can for the rational homology of a based loop group ΩK of a compact Lie group K.

1.8. **Exercise.** Let G be a complex reductive group. Describe the fixed points of the natural loop rotation action on the affine Grassmannian Gr_G.

1.9. **Exercise.** Show that the Schubert singularities of the affine Grassmannian of $SO_3(\mathbb{C})$ are all rationally smooth.

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1.10. **Exercise.** Let $a : S^2 \to S^2$ be the antipodal map, $\mathcal{F}r(S^2)$ the free topological group on S^2, and $F = \mathcal{F}r(S^2)/(xa(x))$ its quotient by the relation $xa(x) = 1$ for any $x \in S^2$. Construct an equivalence of topological groups

$$F \xrightarrow{\sim} \Omega_{poly}(SO_3(\mathbb{R}))$$

to the polynomial based loop space of $SO_3(\mathbb{R}) \simeq \mathbb{RP}^3$.

Under the resulting identification of F with the affine Grassmannian of $SO_3(\mathbb{C})$, describe the components and Schubert cells as subspaces of F.

2. Geometric Langlands Duality and its Classical Limit: Lecture 1

Moduli stacks of bundles and local systems. Hecke and tensorization operators. A naive formulation of the Geometric Langlands conjecture.

Notation:

G: a complex reductive group;
C: a smooth complex projective curve of genus $g > 1$;
$\boldsymbol{\mathcal{B}un}$: the moduli stack of principal G bundles on C;
$\boldsymbol{\mathcal{L}oc}$: the moduli stack of G-local systems on C.

2.1. **Exercise.** Use deformation theory to describe the tangent complexes $\mathbb{T}_{\boldsymbol{\mathcal{B}un}}$ and $\mathbb{T}_{\boldsymbol{\mathcal{L}oc}}$.

2.2. **Exercise.** Show that $\boldsymbol{\mathcal{B}un}$ is a smooth algebraic stack and compute its dimension. Compute the dimension of the coarse moduli space of stable principal G bundles on C.

2.3. **Exercise.** Let $G = \mathrm{SL}(2,\mathbb{C})$, then $\boldsymbol{\mathcal{B}un}$ is the moduli stack of rank two vector bundles with trivial determinant on C. For every $n \in \mathbb{Z}_{>0}$ consider the open substack $\boldsymbol{\mathcal{B}un}^{\leq(n)} \subset \boldsymbol{\mathcal{B}un}$ parametrizing rank two vector bundles that do not admit line subbundles of degree $> n$. Show that $\boldsymbol{\mathcal{B}un}^{\leq(n)}$ is quasi-compact. Use the fact that $\boldsymbol{\mathcal{B}un}$ is the union of all $\boldsymbol{\mathcal{B}un}^{\leq(n)}$ to argue that $\boldsymbol{\mathcal{B}un}$ is not quasi-compact.

2.4. **Exercise.** Compute the dimension of the stack $\boldsymbol{\mathcal{L}oc}$. Show that if $G = \mathrm{SL}(2,\mathbb{C})$, then $\boldsymbol{\mathcal{L}oc}$ is not smooth.

2.5. **Exercise.** Write $\boldsymbol{\mathcal{P}ic}$ for the stack of principal $GL(1,\mathbb{C})$ bundles on C. Fix a point $x \in C$ and let $\boldsymbol{\mathcal{P}ic}^{\mathrm{fr},x}$ be the moduli stack of line bundles equipped with a framing at x. By definition the groupoid of $\boldsymbol{\mathcal{P}ic}^{\mathrm{fr},x}$ over a test scheme S is the groupoid of pairs (L, f), where L is a line bundle on $S \times C$, and $f : L_{|S\times\{x\}} \to \mathcal{O}_C$ is an isomorphism. Show that $\boldsymbol{\mathcal{P}ic}^{\mathrm{fr},x}$ is a space.

2.6. **Exercise.** Show that the natural map $\boldsymbol{\mathcal{P}ic}^{\mathrm{fr},x} \to \boldsymbol{\mathcal{P}ic} \to \mathbf{Pic}$ from $\boldsymbol{\mathcal{P}ic}^{\mathrm{fr},x}$ to the coarse moduli space $\mathbf{Pic}$ of $\boldsymbol{\mathcal{P}ic}$ is an isomorphism.

2.7. **Exercise.** Describe all Hecke correspondences on $\boldsymbol{\mathcal{P}ic}$ and $\mathbf{Pic}$. Describe the Hecke functors on $\mathbf{Pic}$ and the tensorization functors on the moduli stack of rank one local systems on C.

2.8. **Exercise.** Let $G = GL(n,\mathbb{C})$. Viewing $\boldsymbol{\mathcal{B}un}$ as the stack of all rank n vecor bundles on C, describe the Hecke correspondences on $\boldsymbol{\mathcal{B}un}$ corresponding to the fundamental weights ε_i of $GL(n,\mathbb{C})$. Fix a point $x \in C$ and a rank n vector bundle E on C. Compute the fibers of $p^{\varepsilon_i,x}, q^{\varepsilon_i,x} : \boldsymbol{\mathcal{H}ecke}^{\varepsilon_i,x} \to \boldsymbol{\mathcal{B}un}$ over the point E.

2.9. **Exercise.** Let $G = GL(n,\mathbb{C})$. Then $\boldsymbol{\mathcal{L}oc}$ is the stack of vector bundles equipped with flat connections. Let $\mathbb{F} = (F, \nabla)$ be a point of $\boldsymbol{\mathcal{L}oc}$ and let $\mathcal{O}_{\mathbb{F}}$ be the corresponding sky-scraper sheaf on $\boldsymbol{\mathcal{L}oc}$. Compute the action of the tensorization operator $W^{\varepsilon_i,x}$ on $\mathcal{O}_{\mathbb{F}}$.

TUESDAY EXERCISES

1. Hecke operators in Geometric Langlands: Lecture 2

Categorical tools. Differential graded categories of $\mathcal{D}$-modules, constructible sheaves, Lagrangian A-branes.

1.1. Exercise. Let T be a complex torus $T \simeq (\mathbb{C}^\times)^n$ for some n. Give spectral descriptions for local systems on T and for $\mathcal{D}$-modules on T in terms of the dual torus T^*, the dual to the Lie algebra $\mathfrak{t}^*$, and the weight lattice $\Lambda^* = \mathrm{Hom}(T, \mathbb{C}^\times)$.

1.2. Exercise. Classify regular holonomic $\mathcal{D}$-modules on $\mathbb{A}^1$ with singular support in

$$T^*_{\mathbb{A}^1}\mathbb{A}^1 \cup T^*_{\{0\}}\mathbb{A}^1 \subset T^*\mathbb{A}^1$$

Describe the corresponding constructible complexes.

1.3. Exercise. Let $E = \mathcal{D}_{\mathbb{A}^1}/\mathcal{D}_{\mathbb{A}^1}(\partial_x - 1)$ be the "exponential" $\mathcal{D}$-module on $\mathbb{A}^1$. Calculate its stalks and global sections.

1.4. Exercise. Let $f : Y \hookrightarrow X$ be the inclusion of a locally closed submanifold of a submanifold. Describe the following constructible complexes (all functors are derived):

$$i) f^*\mathbb{C}_X \qquad ii) f_*\mathbb{C}_Y \qquad iii) f_!\mathbb{C}_Y \qquad iv) f^!\mathbb{C}_X$$

1.5. Exercise. Describe natural triangles of constructible complexes which underly the standard long exact sequences of cohomology groups of pairs.

1.6. Exercise. Explain how Verdier duality and sheaf operations realize Poincaré duality. Construct Lefschetz duality for a manifold with boundary from sheaf operations.

1.7. Exercise. Given a stratification $\mathcal{S} = \{S_\alpha\}$ of a manifold X, show that an $\mathcal{S}$-constructible complex or $\mathcal{D}$-module on X will have singular support in $T^*_{\mathcal{S}}X = \coprod_\alpha T^*_{S_\alpha}X$. Show the converse holds as well.

Show that if the strata of $\mathcal{S}$ are the orbits of a contractible group N acting on X, then it is equivalent to consider N-equivariant constructible complexes or $\mathcal{D}$-modules.

1.8. Exercise. Let $f : X \to Y$ be a proper, finite map of complex varieties. Decompose $f_*\mathbb{C}_X$ as a direct sum of complexes and describe the indecomposable summands.

1.9. Exercise. Let $f : S^3 \to S^2$ be the Hopf fibration. Compare the constructible complex $f_*\mathbb{C}_{S^3}$ to the sum of its cohomology sheaves.

1.10. Exercise. Let $f : \tilde{X} \to X$ be a resolution of the Schubert variety

$$X = \{P \in Gr_2^4(\mathbb{C}) |\, \dim(P \cap E_2) \geq 1\}$$

Calculate the pushforward $f_*\mathbb{C}_{\tilde{X}}$.

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2. Geometric Langlands Duality and its Classical Limit: Lecture 2

Connected components and gerbes. Singular supports for coherent sheaves. The precise formulation of the Geometric Langlands conjecture.

Notation:

G: a complex reductive group;
C: a smooth complex projective curve of genus $g > 1$;
$\mathcal{B}un$: the moduli stack of principal G bundles on C;
$\mathcal{L}oc$: the moduli stack of G-local systems on C.

2.1. **Exercise.** Let G and ${}^LG$ be Langlands dual complex semisimple groups. Check that $\pi_1(G) \cong Z({}^LG)^\wedge$ and $Z(G) \cong \pi_1({}^LG)^\wedge$.

2.2. **Exercise.** Let $\mathbb{L} = (L, \nabla)$ be a rank one local system on C. For any $d > 0$ let $g_d : C^{\times d} \to C^{(d)} = C^{\times d}/\mathsf{S}_d$ be the natural projection, and let $\mathsf{aj}^d : C^{(d)} \to \mathbf{Pic}^d$ be the Abel-Jacobi map.

(a) Show that $\left(g_{d*}\mathbb{L}^{\boxtimes d}\right)^{\mathsf{S}_d}$ is a rank one local system.
(b) Show that for any $d > 2g-2$ the push-forward $\mathsf{aj}^d_*\left[\left(g_{d*}\mathbb{L}^{\boxtimes d}\right)^{\mathsf{S}_d}\right]$ is a rank one local system.

Translation $(\bullet) \otimes \omega_C$ by the canonical line bundle transports the local system in part (b) to components $\mathbf{Pic}^d$ of $\mathbf{Pic}$ with $d \leq 2g-2$. This defines a local system $\mathfrak{c}(\mathbb{L})$ on $\mathbf{Pic}$. Show that $\mathfrak{c}(\mathbb{L})$ is a Hecke eigensheaf with eigenvalue $\mathbb{L}$.

2.3. **Exercise.** Show that the derived category of coherent D-modules on $B\mathbb{G}_m$ is equivalent to the homotopy category of dg modules over the commutative dg algebra A freely generated over $\mathbb{C}$ by a single generator in degree (-1).

2.4. **Exercise.** Show that we have an equivalence of derived categories $D(\mathbb{Z}, \mathcal{D}) \cong D_{\mathsf{coh}}(B\mathbb{G}_m, \mathcal{O})$.

2.5. **Exercise.** Fix a point $x \in C$. Let $\mathscr{M}$ be the moduli stack of rank one meromorphic local systems on C with logarithmic poles at x. Show that $\mathscr{M}$ is a smooth algebraic stack.

2.6. **Exercise.** Let $(\mathscr{L}, \nabla) \to \mathscr{M} \times C$ be the universal logarithmic local system. Write $D := \mathscr{M} \times \{x\} \subset \mathscr{M} \times C$ for the divisor corresponding to $x \in C$. Here ∇ is a relative connection on $\mathscr{L}$ with logarithmic poles along D. By definition ∇ is a $\mathbb{C}$-linear map $\nabla : \mathscr{L} \to \mathscr{L} \otimes p_C^*\Omega^1_C(D)$ satisfying the Leibnitz rule with respect to multiplication by locally defined holomorphic functions. The residue $\mathrm{res}_D(\nabla)$ is a algebraic section of $\mathcal{O}_D$, i.e. can be viewed as a regular function $\boldsymbol{r} : \mathcal{M} \to \mathbb{C}$. Show that the moduli stack $\mathcal{L}oc_{GL(1,\mathbb{C})}$ of rank one local systems on C is naturally isomorphic to the zero locus of the section $\boldsymbol{r}$, i.e. show that the natural map $\mathcal{L}oc_{GL(1,\mathbb{C})} \to \mathcal{M}$ induces an isomorphism

$$\mathcal{L}oc_{GL(1,\mathbb{C})} \cong \mathcal{M} \times_{0,\mathbb{C},\boldsymbol{r}} \mathcal{M}.$$

2.7. **Exercise.** Define the derived stack $\mathbb{R}\,\mathcal{L}oc_{GL(1,\mathbb{C})}$ of rank one local systems as the derived zero locus of $\boldsymbol{r}$, i.e.

$$\mathbb{R}\,\mathcal{L}oc_{GL(1,\mathbb{C})} \cong \mathcal{M} \times^h_{0,\mathbb{C},\boldsymbol{r}} \mathcal{M}.$$

Prove that the derived structure on $\mathbb{R}\,\mathcal{L}oc_{GL(1,\mathbb{C})}$ splits: show that $\mathbb{R}\,\mathcal{L}oc_{GL(1,\mathbb{C})}$ is isomorphic to the dg stack whose underlying stack is $\mathcal{L}oc_{GL(1,\mathbb{C})}$, and whose structure sheaf is $\mathcal{O}_{\mathcal{L}oc_{GL(1,\mathbb{C})}} \otimes A$, where A is the dg algebra from Exercise ??.