III. (Akhil Mathew) Simplicial Commutative Rings (1)

classical: $\mathrm{Aff}_{k\text{-var}} = (\mathrm{Alg}_{k,red})^{op}$

modern: $\mathrm{Aff} = (\mathrm{CRing})^{op}$. Need more topological data.

Candidate: Topological / Simplicial Commutative Rings

Recall:

Quillen Equivalence: Simplicial Sets $\simeq$ Topological spaces

Def A simplicial comm. ring is an elt of $\mathrm{Fun}(\Delta^{op}, \mathrm{CRing}) = SCR$

examples: ① $\mathrm{CRing} \hookrightarrow SCR$

$R \longmapsto$ constant functor

② $X_\cdot \in sSet$ $\quad \mathbb{Z}[X_\cdot])_n := \mathbb{Z}[X_n]$

③ 1-truncated homotopy types $\simeq$ Groupoids

A categorical ring is a ring object in Gpd. Its nerve is then a SCR.

Homotopy groups

$R_\cdot \in SCR$ $\qquad \pi_i R_\cdot := [(S^i_\cdot, *), (R_\cdot, 0)] = H_i(NR_\cdot)$

$S^i_\cdot$: $\Delta^i/\partial\Delta^i$ $\qquad (R_\cdot, 0)$: this is fibrant

Claim: If $R_\cdot \in SCR$, then $\pi_* R_\cdot$ is graded-commutative

$f: S^i \to R_\cdot$

$g: S^j \to R_\cdot$ $\qquad f \wedge g : S^{i+j} \longrightarrow R \wedge R \longrightarrow R$

where $A_\cdot \wedge B_\cdot := \dfrac{A \times B}{* \times B \cup A \times *}$

example: $\pi_0 R = R_0 / (d_1 - d_0) R_1$

$R_\cdot \longmapsto \pi_0 R_\cdot$ is analagous to taking the reduced ring $R \longrightarrow R_{red}$

Next, we want to construct a space (actually a simplicial set) of maps between two commutative rings

$$\underline{Hom}(R_\cdot, S_\cdot)$$

for this, we define $K_\cdot \otimes R_\cdot$ where $K \in sSet$, $R \in sCR$

then $Map_{sSet}(T_\cdot, \underline{Hom}(R_\cdot, S_\cdot)) = Map_{SCR}(T_\cdot \otimes R_\cdot, S_\cdot)$ gives us the desired mapping space by Yoneda.

Let $\mathcal{C}$ be a category with coproducts

$s\mathcal{C} = Fun(\Delta^{op}, \mathcal{C})$ let $X \in s\mathcal{C}$, $K \in sSet$

$$(K \otimes X)_n := \coprod_{K_n} X_n$$

Problems: - this may not be fibrant (Kan complex)
- homotopy compatibility conditions, e.g. $Map(R_\cdot, S_\cdot) \simeq Map(R'_\cdot, S_\cdot)$ for $R' \xrightarrow{\sim}$
- possibly incorrect homotopy type.

Model Structure on SCR

1. weak equivalences are isomorphisms on π_*
2. fibrations are Kan fibrations (of the underlying sSet)
3. uniquely determined.

recall: $sSet \underset{\text{forget}}{\overset{\mathbb{Z}[\cdot]}{\rightleftarrows}} SCR$ adjunction

more generally, imagine $\mathcal{A} \rightleftarrows \mathcal{B}$, with $\mathcal{A}$ a model category

Idea: f is a w.e./fib iff $G(f)$

under reasonable assumptions on $\mathcal{A}$ and $\mathcal{B}$, this induces a model structure on $\mathcal{B}$ (Quillen).

If G preserves fibrations, then F preserves cofibrations.

Thus, if I is a generating set of cofibrations on $\mathcal{A}$, then FI generates the cofib
in $\mathcal{B}$. This allows us to describe cofibrations in SCR.

$R_\cdot \to S_\cdot$ is a cofib if $\exists\ A_n \subset S_n$ s.t.

① $S_n = R_n[A_n]$

② A_n is preserved under degeneracies.

Generating cofibs in sSet are $\partial\Delta^n \hookrightarrow \Delta^n$.

<u>Claim</u> If $R_\cdot$ is cofibrant, $\underline{\mathrm{Hom}}(R_\cdot, S_\cdot)$ is the correct homotopy type.
SCR is a simplicial model category. (recall that $S_\cdot$ is a fib

Let $R_\cdot \in SCR$. want $\underset{\substack{\text{simplicial} \\ \text{modules}}}{\mathrm{Mod}(R_\cdot)} = \left\{ \begin{array}{l} \text{simplicial sets } M_\cdot \text{ with a map} \\ R_\cdot \times M_\cdot \to M_\cdot \text{ satisfying the usual axioms} \end{array} \right.$

$\pi_* R \times \pi_* M \to \pi_* M$ so $\pi_* M$ is a graded module.

<u>Thm</u> (Quillen): $\exists$ model structure on $\mathrm{Mod}(R_\cdot)$

Daniel: If $R_\cdot$ is constant, then the model structure induced on Ch_R via Dold-Kan agrees with the usual one.

Define $B_\cdot \overset{L}{\underset{A_\cdot}{\otimes}} C_\cdot := \tilde{B}_\cdot \underset{A_\cdot}{\otimes} \tilde{C}_\cdot$ where
$$\begin{tikzcd} A_\cdot \arrow[rr] \arrow[dr] & & B_\cdot \\ & \tilde{B} \arrow[ur, "\sim"'] & \end{tikzcd} , \quad \begin{tikzcd} A_\cdot \arrow[rr] \arrow[dr] & & C_\cdot \\ & \tilde{C} \arrow[ur, "\sim"'] & \end{tikzcd}$$
cofibr replac

this is analogous to taking projective resolutions.

Remark: In fact, it suffices to replace $B_\cdot$ with $\tilde{B}_\cdot$ and leave $C_\cdot$ as is (or vice versa).

Consider

$$R[y] \longrightarrow R \qquad \text{Let's compute } R \overset{L}{\otimes}_{R[y]} R \ \ldots$$
$$y \longmapsto 0$$

Let $\mathcal{C}$ be a category, $T : \mathcal{C} \to \mathcal{C}$ X a T-algebra (i.e. $TX \to X$ satisfies monad
we will find a simplicial T-algebra $B(T,X) \longrightarrow X$

by taking $T^3 X \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \; T^2 X \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \; TX$ this is a simplicial htopy equ

For example, $\mathcal{C} = \mathrm{Set}$ $T\text{-alg} = CR$
$T = \mathbb{Z}[\ ]$ If $X \in CR$
then replace $B(T,X) \longrightarrow X$

$$B(R)_n = R[y]^{\otimes n+1}$$
$$= \left\{ g[f_1 | f_2 | \ldots | f_n] \quad g, f_i \in R[y] \right.$$

$$d_i(g[f_1|\ldots|f_n]) = \begin{cases} i=0 & g f_1 [f_2|\ldots|f_n] \\ 0<i<n & g[f_1|\ldots|f_i f_{i+1}|\ldots f_n] \\ i=n & g f_n [f_1|\ldots|f_{n-1}] \end{cases}$$

$R[y] \hookrightarrow B(R) \xrightarrow{\sim} R$ is the desired cofibrant replacement.

$$R \overset{L}{\otimes}_{R[y]} R = B(R) \otimes_{R[y]} R$$

$$\mathrm{Tor}^{R[y]}_{*}(R,R) = \pi_*\left(B(R) \otimes_{R[y]} R\right) = \begin{cases} R & \text{in dim } 0 \\ R & \text{in dim } 1 \\ 0 & \text{else} \end{cases}$$

If R is any ring, $y \in R$ is a non-zero divisor

To find a cofib replacement for $R \longrightarrow R/(y)$,
consider the pushout square:

$$\begin{array}{ccc} R[x] & \longrightarrow & B(R) \\ \downarrow{\scriptstyle x \mapsto y} & & \downarrow \\ R & \longrightarrow & B(R) \otimes_{R[x]} R \ \underset{\text{htopy}}{\simeq} \ R/(y) \end{array}$$