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05/17/12

Motivation

Ref
Luries
Thesis

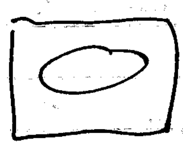
Two entry points: Intersection and Deformations

Nonreduced Structure is to Algebraic Set

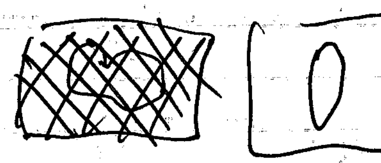
as

Derived Structure is to Scheme

Fix two conics in the plane



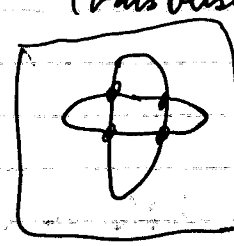
$C = \{q=0\}$



$C' = \{q'=0\}$

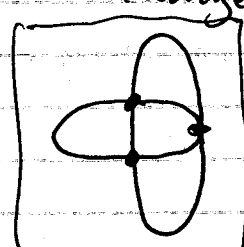
$$q, q' \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$$

$\boxed{\text{Case 1}}$ - Transverse Intersection



$\{q = q' = 0\}$ ← 4 points

$\boxed{\text{Case 2}}$ - Tangent Intersection



$\{q = q' = 0\}$ ← 3 points
not stable under perturbation

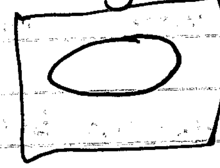
Endow with scheme structure

Structure sheaf of intersection $\mathcal{O}_C \otimes \mathcal{O}_{C'}$

$$\dim H^0(\mathbb{P}^2, \mathcal{O}_c \otimes \mathcal{O}_{c'}) = 4$$

Spec$(\mathcal{O}_C \otimes \mathcal{O}_{C'})$ scheme theoretic intersection

$\boxed{\text{Case 3}}$ - Degenerate Intersection



$\xi = \xi'$

Think of $C \cap C'$ $\longrightarrow$ $\mathcal{O}_C \overset{L}{\otimes} \mathcal{O}_{C'}$ Derived Tensor Product

$$\mathcal{O}_C \leftarrow \mathcal{O}_{\mathbb{P}^2} \xleftarrow{\xi} \mathcal{O}_{\mathbb{P}^2}(-2) \leftarrow 0$$

$$\mathcal{O}_{C'} \leftarrow \mathcal{O}_{\mathbb{P}^2} \xleftarrow{\xi'} \mathcal{O}_{\mathbb{P}^2}(-2) \leftarrow 0$$

$\boxed{\text{Motivation: Want to do Alg. Geom. on this Complex}}$

Think of $\mathcal{O}_C \overset{L}{\otimes} \mathcal{O}_{C'} = \left[\mathcal{O}_{\mathbb{P}} \xleftarrow{(\xi + \xi')} \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}(-2) \xleftarrow{\binom{\xi'}{-\xi}} \mathcal{O}_{\mathbb{P}^2}(-4) \leftarrow 0\right]$

Koszul Complex on 2 elements

$\forall C, C'$ we have:

1. The locus where $\mathcal{O}_C \overset{L}{\otimes} \mathcal{O}_{C'}$ fails to be exact equals $C \cap C'$.
2. Hypercohomology (cohomology for complexes of sheaves)

$$\cancel{\dim \mathbb{H}^0(\mathbb{P}^2, \mathcal{O}_C \overset{L}{\otimes} \mathcal{O}_{C'}) = 4}$$

$$\dim \mathbb{H}^0(\mathbb{P}^2, \mathcal{O}_C \overset{L}{\otimes} \mathcal{O}_{C'}) = 4$$

$\boxed{\text{Non degenerate Case}}$

$$\mathbb{H}^0(\mathbb{P}^2, \mathcal{O}_C \otimes \mathcal{O}_{C'}) = H^0(\mathbb{P}^2, \mathcal{O}_C \otimes \mathcal{O}_{C'})$$

$\boxed{\text{Degenerate Case}}$ $\mathbb{H}^0(\mathbb{P}^2, \mathcal{O}_C \overset{L}{\otimes} \mathcal{O}_{C'}) = \begin{array}{ll} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) & 1 \\ \oplus & + \\ H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^2(-2)) & 0 \quad = 4 \\ \oplus & + \\ H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-4)) & 3 \end{array}$

spectral sequence

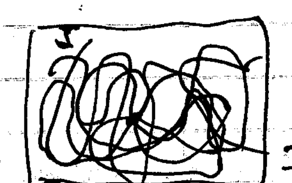
(Geometric way to think of this decomposition ?)

$\boxed{\text{Serre's Tor-Formula}}$

(Homological Motivation for Derived Schemes)

R a regular local ring , $I, J \in R$ ideals

$R/(I+J) \leftarrow$ 0-dim



$$\mu(R/I, R/J) = \sum_{i \geq 0} (-1)^i \text{ length } \mathrm{Tor}_i^R(R/I, R/J)$$
$$= \text{"length of } R/I \overset{L}{\otimes} R/J\text{"} \quad \left(\text{length of Complex ?}\right)$$

We are seeking generalized theory of rings that includes derived tensor product (Simplicial Rings)

<u>Example</u> $R = k[x,y]$

Ring Data
$R = (\text{Set}, +, \cdot, 1)$
$+ : R \times R \longrightarrow R$
$\cdot : R \times R \longrightarrow R$ ... etc.

$\boxed{\text{Categorical Ring}}$ $\mathcal{C} = (\text{Category}, \hat{+}, \hat{\cdot}, \hat{1})$ ← functors

C_R = Category with Objects = $\{r \in R\}$

(R is a C_R)

Morphisms of C_R $\quad Mor(r,r') = \begin{cases} \{*\} & r=r' \\ \emptyset & r \neq r' \end{cases}$

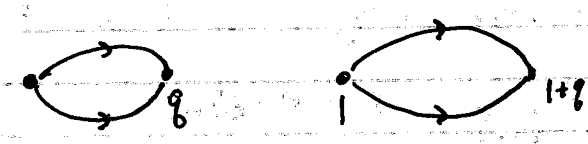
$\hat{+} : C_R \times C_R \longrightarrow C_R$

$(r,r') \longmapsto r+r'$ ... etc.

Impose the equation $q=0$ to get a new category

$$Mor(r,r') = \begin{cases} \{*\} & r-r' \in C_q \\ \emptyset & \text{otherwise} \end{cases} \qquad 0 \longrightarrow q \qquad 1 \longrightarrow 1+q$$

Impose this equation twice to get a third category



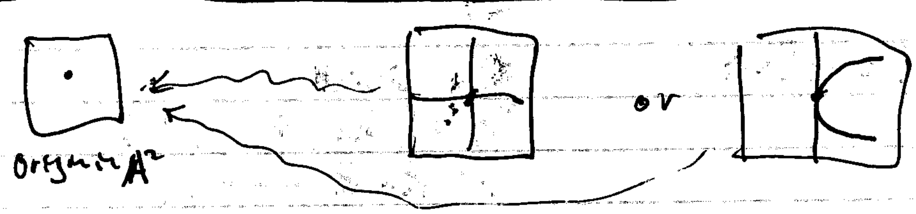
$\left\{\begin{array}{c}\text{Isomorphism Classes} \\ \text{of objects}\end{array}\right\} \cong R/(q)$ (same ~~underlying~~ underlying ring but with extra data)

$\boxed{\text{Homological}} \xleftarrow{\text{Dold-Kan Theorem}} \boxed{\text{Categorical / Simplicial}}$

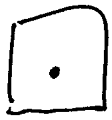
Complexes $\xleftarrow{\text{give rise to}}$ Simplicial Algebra

H_i(complex) $\qquad\qquad \pi_i$

$\quad = Tor_i$



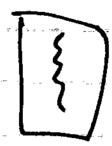
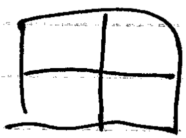
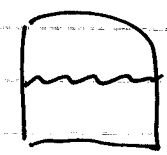
Example $\quad R = k[x,y] \qquad Q = R/(x^2, xy, y^2)$

 $\mathrm{Spec}\,\mathbb{Q} \subseteq \mathbb{A}^2$

$$\mathbb{Q} \leftarrow \left[R^1 \xleftarrow{(x^2,xy,y^2)} R^3 \longleftarrow R^2 \leftarrow 0 \right]$$

$$\pi_i = \begin{cases} \mathbb{Q} & i=0 \\ 0 & i>0 \end{cases}$$

Alternatively (intersection of 3 curves)

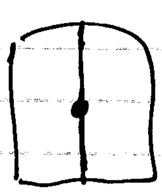
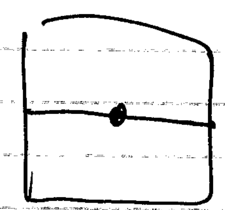
 $\cap$  $\cap$ 

$$R/(x^2) \overset{L}{\otimes} R/(xy) \overset{L}{\otimes} R/(y^2)$$

Koszul Complex on 3 elements $\Big\downarrow$

$$\left[R^1 \longleftarrow R^3 \longleftarrow R^3 \longleftarrow R^1 \leftarrow 0 \right]$$

$$\pi_i = \begin{cases} \mathbb{Q} & i=0 \\ (x,y)/(x^2,y^2)^{\oplus} & i=1 \\ 0 & i>1 \end{cases}$$

Example  $\cap$ 

$$R/(x^2,xy) \overset{L}{\otimes} R/(y^2,xy)$$

has higher homotopy groups

(geometric interpretations of higher π_i ?)