IX. (Anatoly Preygel) Applications to Loop Spaces

First, talk about <u>based</u> loop spaces. In <u>topology</u>: $pt \in X$

$$\begin{tikzcd} pt \overset{h}{\underset{X}{\times}} pt \simeq \Omega_{pt} X \arrow[r] \arrow[d] & \mathrm{Path}\, X = \left\{ \begin{matrix} \gamma : I \to X \\ \gamma(0) = pt \end{matrix} \right\} \arrow[d, "\gamma(1)"] \\ pt \arrow[r] & X \end{tikzcd}$$

(Path X is contractible)

$\Omega_{pt} X$ "group"

$B\Omega_{pt} X$ classifying space

htopy Čech nerve $\left| pt \leftleftarrows pt \overset{h}{\underset{X}{\times}} pt \,\substack{\leftarrow\\\leftarrow\\\leftarrow}\, pt \overset{h}{\underset{X}{\times}} pt \overset{h}{\underset{X}{\times}} pt \right.$

$$\begin{tikzcd} B\Omega_{pt} X = \left| pt \leftleftarrows \Omega_{pt} X \,\substack{\leftarrow\\\leftarrow\\\leftarrow}\, (\Omega_{pt} X)^2 \cdots \right| \arrow[d] \\ X \end{tikzcd}$$

$B\Omega_{pt} X \simeq$ base pt component of X

<u>Variant</u> $f : Y \to X$ Čech nerve

$$\begin{tikzcd} \left| Y \leftleftarrows Y \underset{X}{\times} Y \,\substack{\leftarrow\\\leftarrow\\\leftarrow}\, Y \underset{X}{\times} Y \underset{X}{\times} Y \cdots \right| \arrow[d] \\ X \end{tikzcd}$$

<u>Algebra</u> $Z \xrightarrow{i} X$ closed immersion (for now) of derived schemes.

$Z = \mathrm{Spec}\, B$, $X = \mathrm{Spec}\, A$

i.e. i is affine, $H_j(i_* \mathcal{O}_Z)$ coherent over $H_0 \mathcal{O}_X$, and $H_0 \mathcal{O}_X \to H_0 \mathcal{O}_Z$ surjective.

Consider $Z \leftleftarrows Z \underset{X}{\times} Z \,\substack{\leftarrow\\\leftarrow\\\leftarrow}\, Z \underset{X}{\times} Z \underset{X}{\times} Z$

$= \mathrm{Spec}\, B \qquad = \mathrm{Spec}\, B \overset{L}{\underset{A}{\otimes}} B \qquad = \mathrm{Spec}\, B \overset{L}{\underset{A}{\otimes}} B \overset{L}{\underset{A}{\otimes}} B$

Q: How badly does closed descent fail?

A: $\left| Z \leftleftarrows Z \underset{X}{\times} Z \,\substack{\leftarrow\\\leftarrow\\\leftarrow}\, \cdots \right| (R) = \left| Z(R) \leftleftarrows Z(R) \underset{X(R)}{\times} Z(R) \,\substack{\leftarrow\\\leftarrow\\\leftarrow}\, \cdots \right|$

realization lives in $\mathrm{Fun}(sCR, \mathrm{Spaces})$

$= \left\{ \begin{matrix} \text{the union of components} \\ \text{of } X(R) \text{ that are hit by } Z(R) \end{matrix} \right\}$

$= \left\{ \begin{array}{l} \text{those pts } \eta \in X(R) \text{ that admit a scheme-theoretic} \\ \text{factorization through } Z \end{array} \right\}$

$\cap|$

$\left\{ \quad " \qquad\qquad " \qquad \text{set-theoretic} \right\} = \hat{X}_Z(R)$

(completion of X along Z)

<u>Prop</u>: $\mathcal{O}_{\hat{X}_Z} \xrightarrow{\text{res}} \mathrm{Tot}\left\{ \mathcal{O}_Z \rightrightarrows \mathcal{O}_{Z \times_X Z} \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots \right\}$

is an equivalence

Concrete Example / Exercise:

$\bullet \longrightarrow$ ——

$\mathrm{Spec}\, k \longrightarrow \mathrm{Spec}\, k[x]$

$$k \overset{L}{\underset{k[x]}{\otimes}} k \simeq k[\underset{\text{degree } +1}{B}] \quad \text{as dg-Hopf alg.} \quad \left(\Delta(B) = B \otimes 1 + 1 \otimes B \right)$$

and k is a comodule for it.

(because the totalization computes $R\mathrm{Hom}_{k[\varepsilon_{-1}]}(k, k)$.)

All this stuff so far is an example of non-flat descent

Applications to <u>unbased</u> loop spaces:

Prop: X - der. scheme (or der. stack) $\qquad$ [e.g. $X = \operatorname{Spec} A$]

K - finite simplicial set

then $X^K : R \longmapsto \operatorname{Map}_{sSet}(K, X(R)) = X(R)^K$

is represented by a der. scheme (or der. stack) $\qquad$ [e.g. $\operatorname{Spec}(K \otimes A)$]

Prf 1: $\operatorname{Map}_{sCR}(K \otimes A, R)_p = \operatorname{Map}_{CR}(K_p \otimes A_p, R_p)$

$$\operatorname{Map}_{sSet}(K, \operatorname{Map}_{sCR}(A, R)) = \operatorname{Map}_{CR}(A_p, R_p)^{K_p}$$

Prf 2: Write K as a finite sequence of cell attachments (retracts?)

X^K = corresponding sequence of pullbacks.

Example: 1) $K = \{1, \dots, n\}$

$$X^K = X^n \; [= \operatorname{Spec} R^{\otimes n}]$$

2)
$$\begin{tikzcd} \Sigma K & * \arrow[l] \\ * \arrow[u] & K \arrow[l] \arrow[u] \arrow[ul, phantom, "\text{h.pushout}"] \end{tikzcd} \qquad \begin{tikzcd} \Sigma K & * \arrow[l] \\ \operatorname{cone}(K) \arrow[u] & K \arrow[l] \arrow[u] \arrow[ul, phantom, "\text{pushout}"] \end{tikzcd} \qquad \begin{tikzcd} X^{\Sigma K} \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & X^K \end{tikzcd}$$

$X^{\Sigma K} \simeq \operatorname{Spec} R \overset{L}{\otimes}_{K \otimes R} R \simeq \operatorname{Spec}(\operatorname{cone}(K) \otimes R) \otimes_{K \otimes R} R$

Bar construction for $\overset{L}{\otimes}$.

3) $K = S^1 \simeq \Sigma S^0$

$$\begin{tikzcd} X^{S^1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{h.pus}"] & X \arrow[d] \\ X \arrow[r] & X^2 \end{tikzcd}$$

In affine case: $R \overset{L}{\otimes}_{R^{\otimes 2}} R$

$$\left| R \otimes R \leftleftarrows R \otimes R^{\otimes 2} \otimes R \;\overset{\leftarrow}{\leftleftarrows}\; R \otimes (R^{\otimes 2})^{\otimes 2} \otimes R \cdots \right|$$

$\operatorname{cone}(S^0)$ $\quad \Lambda \leftarrow \underset{S^0}{\cdot\ \cdot}$

$(\Delta^1 \otimes R) \otimes_{S^1 \otimes R} R$ $\quad \left| R \leftleftarrows R \otimes R \;\overset{\leftarrow}{\leftleftarrows}\; R \otimes R \otimes R \cdots \right|$ $\quad$ "cyclic bar complex"

Prop: char 0 (HKR-type)

$X = \operatorname{Spec} R$. $\qquad \mathcal{O}_{LX} \xrightarrow{\text{quasi-iso}} \operatorname{Sym}_R \mathbb{L}_R[+1]$

$| R \leftleftarrows R^{\otimes 2} \leftleftarrows R^{\otimes 3} \cdots |$

$r_0 \otimes \cdots \otimes r_p \longmapsto r_0 \, dr_1 dr_2 \cdots dr_p$

$\circlearrowleft$ Connes B $\qquad \circlearrowleft d_{DR}$

Regard these as complexes

Cpx w/ S^1-action $\simeq H_* S^1$-mod
$\qquad\qquad\qquad\qquad \| $
$\qquad\qquad\qquad\qquad k[B_{+1}]$

(and the HKR-type map is compatible with it.)

Also, $\exists$ relation between $QC(LX)^{S^1}$ and D-modules on X.

Ex of Loops

1) X classical scheme, not $\amalg$ pt.
$\Rightarrow LX$ not classical.

2)
$$\begin{tikzcd} & L(BG) \arrow[r] \arrow[d] \arrow[dl, "\simeq"'] & BG \arrow[d] \\ G^{ad}/G & BG \arrow[r] & BG^2 \end{tikzcd}$$

3) Y/G (affine). when is $L(Y/G)$ classical again?
$\Updownarrow$
finitely many orbits.

this is important from the point of view of geometric representation thy (Nadler - Ben-Zvi)