Daniel Litt

05/18/12

Cotangent Complex I

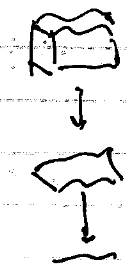
— The "Classical Theory"

Motivation | Kahler differentials

For B/A we have $B \xrightarrow{d} \Omega^1_{B/A}$

(1)

$$\begin{array}{ccc} \text{Rings} & & \text{Spaces} \\ C & & X \\ \uparrow & & \downarrow \\ B & & Y \\ \uparrow & & \downarrow \\ A & & Z \end{array}$$

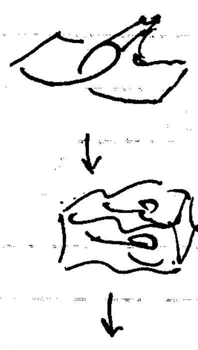


$$0 \to T_{X/Y} \to T_{X/Z} \to T_{Y/Z} \to 0$$

X/Y smooth

$$(0 \to)\ f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

$$\begin{array}{c} X \\ \downarrow \\ Y \\ \downarrow \\ Z \end{array}$$



$$0 \to T_{X/Z} \to f^*T_{Y/Z} \to N_{X/Y} \to 0$$

$$I/I^2 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to 0.$$

(2) Deformation Theory

$L_{X/Y} \in D(X)$

Theorem (1)

$$\begin{tikzcd} X_0 \arrow[r, "g"] \arrow[d, "j"] & Y \arrow[d] \\ X \arrow[r] \arrow[ur, dashed, "h"] & S \end{tikzcd}$$

$j =$ square 0 extension

h exists $\iff$ obstruction $o(g,j) \in Ext(g^* L_{Y/S}, J)$

If $o(g,j) = 0$ then set of lifts is a torsor for $Hom(g^* L_{Y/S}, J)$

(ii)
$$\begin{tikzcd} X_0 \arrow[r, dashed] \arrow[d, "f_0"', "\text{flat}"] & X \arrow[d] \\ S_0 \arrow[r, "i"', "\text{sq } 0"] & S \end{tikzcd}$$

X exists $\iff$ $\underset{\substack{\| \\ 0}}{\mathcal{O}(X,i)} \in \mathcal{E}xt^2(L_{X_0/S_0}, f_0^*(I))$

Such X is a torsor for $Ext^1(L_{X_0/S_0}, L_0^! I)$

$\boxed{\text{Application 1}}$ A a perfect $\mathbb{F}_p$-algebra (e.g. $Frob_p: A \to A$ is isomorphism)

$$\begin{array}{l} \operatorname{Spec} A \\ \downarrow \\ \operatorname{Spec} \mathbb{F}_p \longrightarrow \operatorname{Spec} \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \operatorname{Spec} \mathbb{Z}/p^3\mathbb{Z} \longrightarrow \cdots \end{array}$$

obstruction in $\mathcal{E}xt^2(\underset{\substack{\| \\ 0}}{L_{A/\mathbb{F}_p}}, \underset{\substack{\| \\ \mathbb{F}_p}}{f^0(\langle p^{n+1} \rangle)})$

Frobenius induces $L_{A/\mathbb{F}_p} \otimes_A A \xrightarrow{\cong} L_{A/\mathbb{F}_p}$

$d(x \to x^p)$ $\quad px^{p-1} = 0$

(Application 2) Theorem - Let A be complete local Noetherian ring with residue field k.

X_0/k is a proper curve, lci, smooth complement finitely many points

Then $\exists$ X/A proj. and flat with $X_k \simeq X_0$

We'll need X/k is lci then $L_{X/k}$ is quasi-isomorphic perfect of amplitude $[-1, 0]$

Assume lifted to S_n : $\operatorname{Spec} A/m^n$

then

obstruction $\in \mathcal{E}xt^2(L_{X_0/S_n}, \mathcal{O}_{X_i}) \otimes m^{n+1}/m^{n+2}$

$$\mathcal{E}xt^2(L_{X_0/S_n}, \mathcal{O}_{X_i}) = 0$$

$$H^i(X_0, \mathcal{E}xt^j(L_{X_0/k}, \mathcal{O}_{X_i})) \Rightarrow \mathcal{E}xt^{i+j}(L_{X_0/S}, \mathcal{O}_{X_i})$$

$$\mathcal{E}xt^2(L_{X_0/S_n}, \mathcal{O}_{X_0}) = 0$$

$$H^2(X_n, \mathcal{H}om(L_{X_0/k}, \mathcal{O}_{X_0})) = 0 \quad \text{as } \dim X_n = 1$$

$$H^1(X_i, \mathcal{E}xt^1(L_{X_n/k}, \mathcal{O}_{X_0})) = 0$$

$\boxed{\text{Definition / Construction}}$

Definition A projective A-algebra resolution (i.e. cofibrant replacement) of cB is a factorization

$$\begin{tikzcd} cA \arrow[r] \arrow[dr, "\text{cof.}"'] & cB \\ & P \arrow[u, "\text{triv. fib.}"'] \end{tikzcd}$$

Definition $(L_{B/A})_n = (\Omega_{P_n/A} \otimes_{P_n} B)$

((cotangent complex)

Definition (AQ (co)homology)

$$D_q(B/A, M) = H_q(L_{B/A} \otimes_B M)$$

$$D^q(B/A, M) = H^q(\mathrm{Hom}_B(L_{B/A}, M))$$

$\boxed{\text{Properties}}$ ① $D_0(B/A, B) = D_0(B/A)$

$$= \Omega^1_{B/A}$$

② $L_{B/A}$ is a complex of projectives

③ An extension of B by M is an SES

$$0 \to M \xrightarrow{\text{sq}=0} X \to B \to 0$$

Let $\operatorname{Exalcomm}(B/A, M)$ is the set of isom. classes of extensions

Then $D^1(B/A, M) \cong \operatorname{Exalcomm}(B/A, M)$

Proof $P \to B$

$$\begin{tikzcd} & 0 \arrow[d] \\ & M \arrow[d] \\ & X \arrow[d] \\ P_0 \arrow[r] \arrow[ur, dashed, "\theta"] & B \arrow[d] \\ & 0 \end{tikzcd}$$

$i^{-1}(\theta d_0 - \theta d_1)$

$P_1 \to M$

This is class in AQ^1

$D_1\ P^1 \to M$ is a cocycle

Then $X = \operatorname{Coker}\left(P_1 \xrightarrow[(d_1, 0)]{(d_0, D)} P_0 \oplus M\right)$

$$0 \to M \to X \to B \to 0$$

④ Suppose $B = A/I$ then $D_0 = 0$ and $D_1 = I/I^2$

(5) Suppose
$$\begin{tikzcd} R' \arrow[r] & S' = S \otimes_R R' \\ R \arrow[u] \arrow[r, "v"] & S \arrow[u] \end{tikzcd}$$

$\mathrm{Tor}_q^R(R', S) = 0 \quad q > 0$

Then $L_{S/R} \otimes_R R' = L_{S'/R'}$

and $L_{S'/R^*} \cong L_{S/R} \otimes_R R' \oplus L_{R'/R} \otimes_R S$

(6) $L_{S^{-1}A/A} = 0$

(7) If A Noetherian, B is finite type A-alg. then

(i) B étale $\iff L_{B/A} = 0$

(ii) B smooth $/A \iff L_{B/A} \simeq c(\Omega^1_{B/A})$

$\Omega^1_{B/A}$ projective

Proof ($\Leftarrow$) $D^1(B/A, M) = 0$
$\quad \quad \quad \quad ''$
$\mathrm{ExalComm}(B/A, M) = 0 \Rightarrow B/A$ formally smooth
$\Rightarrow B/A$ smooth as well

($\Rightarrow$) B/A étale $\Rightarrow$ diag. map $\Delta : \mathrm{Spec}\, B \to \mathrm{Spec}\, B \otimes_A B$ is open immersion

$p \in B, \quad q = \Delta(p)$

$(B \otimes_A B)_q = B_p$

$(L_{B/A})_p = (L_{B/A} \otimes_A B)_q = (L_{B \otimes B/A})_q = L_{(B \otimes)_p} = 0$

(8) (Transitivity Triangle)

If $A \to B \to C$ are maps of rings, then there is a distinguished triangle in $D(C)$

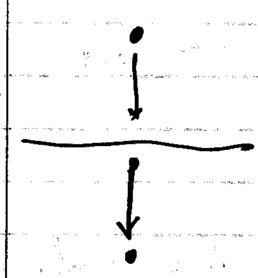
$$\begin{tikzcd} L_{B/A}\otimes_B C \arrow[rr] & & L_{C/A} \arrow[dl] \\ & L_{C/B} \arrow[ul, dashed] & \end{tikzcd}$$

Proof we have

$$\begin{tikzcd} & & & Q \arrow[d, "\cong"] \arrow[ddr, "fib"] & \\ & & P \arrow[ur, "cof"] \arrow[dr, "triv.\ fib."] & B\otimes Q \arrow[d] \arrow[dr, "\simeq"] & \\ & cA \arrow[ur, "cof"] \arrow[rr] & & cB \arrow[r] & cC \end{tikzcd}$$

We have

$$0 \to (\Omega_{P/A}\otimes B)\otimes_B C \to \Omega_{Q/A}\otimes_Q C \to \Omega_{B\otimes_P Q/P}\otimes_{B\otimes_P Q} C \to 0$$

Example 

$$\begin{tikzcd} K & 0 \\ K[t] \arrow[u] & t \arrow[u, mapsto] \\ K \arrow[u] & \end{tikzcd}$$

$$\begin{tikzcd} L_{K[t]/K}\otimes_{K[t]} K \arrow[rr] & & L_{K/K}=0 \arrow[dl] \\ & L_{K/K[t]}=K[1] \arrow[ul, dashed] & \end{tikzcd}$$