<u>Goal</u>: Spectral geometry of Hecke alg. & traces

function fields/k, char = 0

<u>Overview</u>: I. Derived alg. geometry

II. Microlocal alg. geometry

III. Hecke categories

IV. Geom. traces

I. Derived algebraic geometry

Motivation: Algebraic geometry studies systems of poly. equations.

Ex: Eq's over sol's : importance of nilpotents

matrix

M_n char. polys $= 0$

$\cup$

D_n

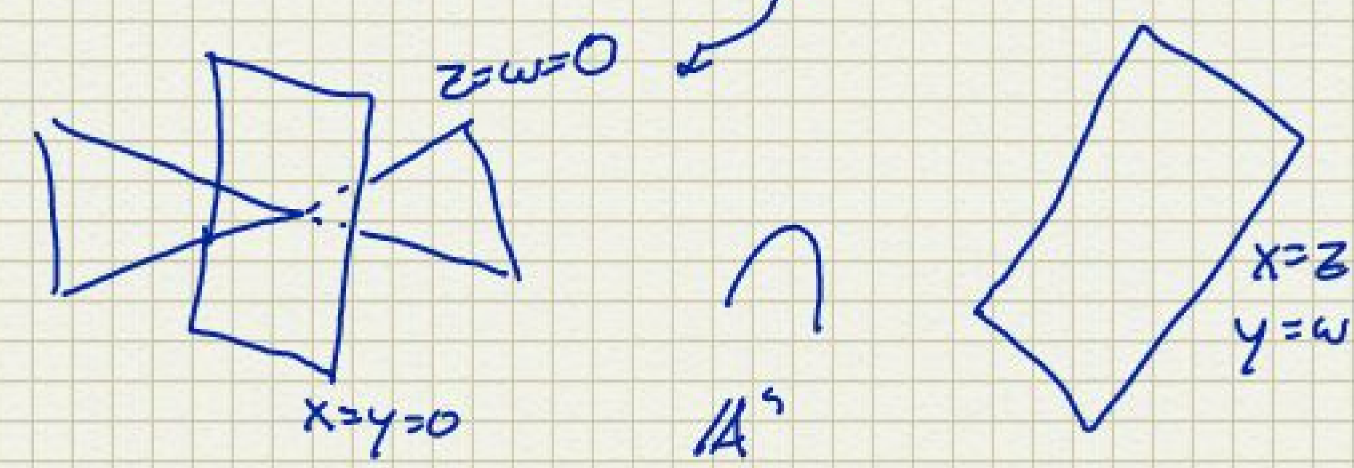
diagonal

$\boxed{\text{Ref: Toen – Vezzosi, Lurie}}$

1. Three examples of "derived equations"

1) Intersection multiplicities

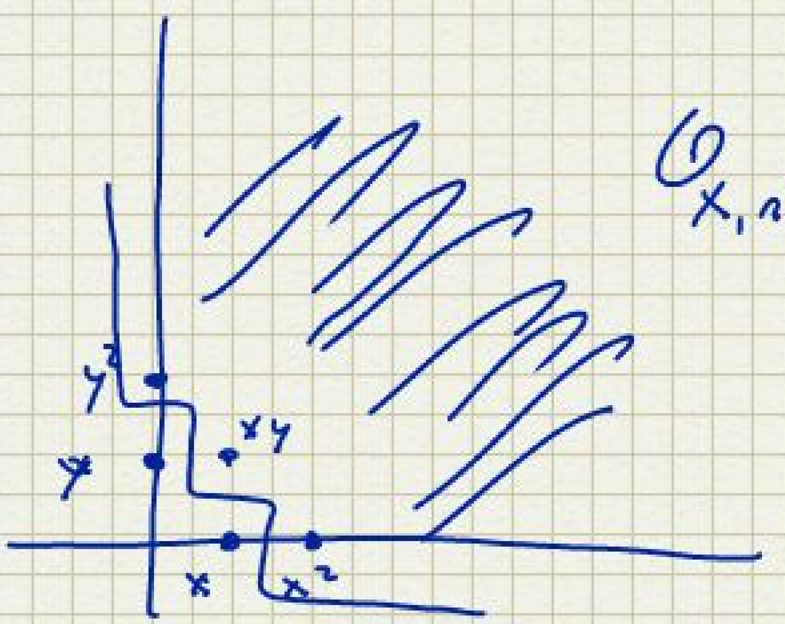
$\mathbb{A}^4 = \operatorname{Spec} k[x,y,z,w]$ $xz = xw = yz = yw = 0$



perturb $\rightsquigarrow$ should get 2

Impose all equations:

$$x^2 = xy = xy = y^2 = 0 \qquad (\text{in } \mathbb{A}^2)$$



$\mathcal{O}_{X_1 \cap X_2}$ 3 dim

Serre: $2 = 3 - 1$

↑ dim of Tor

2) Base change

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & X_2 \arrow[d, "f_2"] \\
X_1 \arrow[r, "f_1"'] & Y
\end{tikzcd}$$

classically fails, even in simple situations

ex
$$\begin{tikzcd}
\underbrace{\{0\} \times_{\mathbb{A}^1} \{0\}}_{\{0\} \text{ classically}} \arrow[r] \arrow[d] & \{0\} \arrow[d] \\
\{0\} \arrow[r] & \mathbb{A}^1
\end{tikzcd}$$

$\{x = 0\}$

$$\begin{tikzcd}
\mathrm{Coh}(X_1) \arrow[r, bend left, "p_{2*}p_1^*"] \arrow[r, "f_2^* f_{1*}"'] & \mathrm{Coh}(X_2)
\end{tikzcd}$$

$$\begin{tikzcd}
k\text{-mod} \arrow[r, bend left, "id"] \arrow[r, "\otimes\left(\underset{-1}{k} \xrightarrow{0} \underset{0}{k}\right)"'] & k\text{-mod}
\end{tikzcd}$$

Derived alg. geometry changes this functor by changing the fiberproduct.

Exercise: $Q^3 = \{ A \in M_2 : \det A = 0 \}$

$$\begin{tikzcd} \tilde{Q} \arrow[r] \arrow[d] & Q^3_- \arrow[d] \\ Q^3_+ \arrow[r] & Q_3 \end{tikzcd}$$

Find correct (derived) fiber product so that base change holds.

3) Moduli problems

classically strange ambiguities; specifically for deformations

Ex: $\mathcal{M}_n = \{$two commuting operators on $k^n\}$

$$\begin{tikzcd} \mathcal{M}_n \arrow[r] \arrow[d] & M_n^2 \arrow[d, "{[\,,\,]}"] \\ \{0\} \arrow[r] & M_n \end{tikzcd}$$

$n_k = 1$:

$$\begin{tikzcd} \mathcal{M}_1 \arrow[r] \arrow[d] & \mathbb{A}^1 \times \mathbb{A}^1 \arrow[d, "{[\,,\,]}"] \\ \{0\} \arrow[r] & M_1 \end{tikzcd}$$

Deform: $[\,,\,] = \lambda \neq 0 \rightsquigarrow$ empty!

2. Local = affine DAG

Equations $\rightsquigarrow$ Tensors $\rightsquigarrow$ Keep track of Tors

Def: A cdga$_k$ $A^\bullet$ is a commutative, assoc. alg in k chain complexes.

Rmk: Up to quasi-isom.

⚠ $A^\bullet$ knows much more than $H^\bullet(A^\bullet)$

(Exer: give examples ... Massey products)

Def: A derived ring ($\overset{\text{opp to}}{\Leftrightarrow}$ affine derived scheme)

$A^\bullet$ is a cdga which is connective, i.e. $H^i A^\bullet = 0$ for $i > 0$.

Notation: $\pi_i = H^{-i}$

Rmk:

$$\begin{array}{ccc} A^\bullet & \longrightarrow & \pi_0 A^\bullet \\ \nearrow & & \uparrow \\ \text{derived ring} & & \text{classical ring} \end{array}$$

"derived thickening"

$$\operatorname{Spec} A^\bullet \hookleftarrow \operatorname{Spec} \pi_0 A^\bullet$$

Source of all examples: iterated fibered products, where we keep track of tor complexes.

Structure of cdga: ∞-categories ...

simplicial localization of a model category

main structure: simplicial set of Homs

$$\mathrm{Hom}^n(A, B) = \mathrm{Hom}^0(A, B \otimes \Omega^\bullet(\Delta^n))$$

To calculate derived operations:

1) all objects are fibrant

2) cell cdga are cofibrant

$$k = A_0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \dots A_n \xrightarrow{\simeq} A$$

$$A_{i+1} = A_i[x_i]$$

$$\deg x_i \in \mathbb{Z}$$

$$dx_i \in A_i.$$

<u>Ex</u>: $X = \{f = 0\} \subset \mathbb{A}^n$

Traditional: $\mathcal{O}(X) = k[x_1, \dots, x_n]/(f)$ $\Rightarrow \varepsilon^2 = 0$

Cofibrant replacement: $k[x_1, \dots, x_n][\varepsilon]$ $\deg \varepsilon = -1$

$d\varepsilon = f$

<u>Linearization</u>: A ... derived ring

1. <u>Modules</u>: A-mod : derived dg modules (QCoh)

co-completion $\nearrow$ $\cup$ } compacts

"functions"

A-perf : summands of finite complexes of shifts of A (Perf)

Ind-coh. modules

"distributions" co-completion $\nearrow$ $\cup$ } compacts

A-coh bounded coherent cohomology

2. <u>Cotangent complex</u>:

$\mathbb{L}_A \in A$-mod : connective cotangent bundle

$\pi_0 \mathbb{L}_A \simeq \Omega_{\pi_0 A}$

Univ. char: $\mathrm{Der}(A, M) = \mathrm{Hom}(\mathbb{L}_A, M)$

in particular $d: A \to \mathbb{L}_A$ "Kähler diff."

To calculate: $D = B \otimes^{\mathbb{L}}_A C$

pullback

$\Rightarrow \mathbb{L}_D = \mathrm{Cone}\left(\mathbb{L}_A \otimes_A D \to (\mathbb{L}_B \otimes_B D) \oplus (\mathbb{L}_C \otimes_C D)\right)$

3. Global DAG

Functor of points

$$\begin{tikzcd}
\mathrm{Aff}^{op} \arrow[rr, "\mathrm{Sch}"] \arrow[d, hook] \arrow[drr, "\mathrm{Stacks}"] & & \mathrm{Sets} \arrow[d, hook] \\
\mathrm{DAff}^{op} \arrow[rr, "\mathrm{DStacks}"'] & & \mathrm{Spaces} = \mathrm{SSet}
\end{tikzcd}$$

Colimits in Sets are simple: = coproducts + coequalisers

$$X \leftarrow \Big[\coprod_\alpha S_\alpha \leftleftarrows \coprod_\beta R_\beta \Big]$$

Artin derived stacks:

smooth maps

$$X \leftarrow X_0 \leftleftarrows X_1$$

derived schemes

<u>Ex</u>: 1) $BG \leftarrow pt \leftleftarrows G$

group

2) $\mathcal{M}_{1,1} \leftarrow (? \leftleftarrows ?)$

3) $S^1 \leftarrow (pt \leftleftarrows \mathbb{Z})$

Today: Microlocal A.G.

Goal: Geometric Arthur parameters

Setup: work locally $F: \mathbb{A}^n \longrightarrow \mathbb{A}^k$

$X = \{F = 0\}$

Understand: $Coh(X)$ ← derived

Philosophy: Two themes

1) Ex: A-alg , A-mod

$\cup$

$Z(A)$ center

Spec $Z(A)$ — localize A to a sheaf of algebras on this spectrum

& therefore localize A-mod

More generally, if $\mathcal{C}$ a category, we can sheafify $\mathcal{C}$ over $Z(\mathcal{C})$ = endomorphisms of $id_{\mathcal{C}}$

2) Geometric measurements $<$ static $\rightsquigarrow X$; dynamic $\rightsquigarrow T^*X$

<u>Ex</u>: Fourier transform

$$\underset{\substack{\text{dynamic}\\\text{measurements}}}{L^2(S^1) \xrightarrow{\sim} L^2(\mathbb{Z})}$$

<u>Koszul duality</u>

$V = \operatorname{Spec} k[x_1, \dots, x_n]$ $\quad \deg x_i = -1$

$\operatorname{Coh}(V)$? $\qquad V_{cl} = \cancel{\text{pt}} = \operatorname{Spec} \pi_0 V = \text{pt}$

<u>Codirections</u>: introduce $V^*[1] = \operatorname{Spec} k[\xi_1, \dots, \xi_n]$ $\quad \deg \xi_i = 2$

<u>Thm</u> (K-D): $\operatorname{Coh}(V) \simeq \operatorname{Coh}(V^*[1])$

$$\mathcal{O} \longleftrightarrow k$$
$$k \longleftrightarrow \mathcal{O}$$

Can localize coherent sheaves on V on $V^*[1]$

Supports will be closed conic subsets.

Exer: Under KD, $\mathcal{F} \in Coh(V)$ is perfect

$$\iff \operatorname{supp} \mathcal{F} \subset \{0\} \subset V^*[1]$$

Rmk: $Coh(V) \supset Perf(V)$ "dists" "fn" KD measures codirection on which "smoothness" fails.

Exer: $V = \{0\} \times_{\mathbb{A}^n} \{0\}$ $\rightsquigarrow$ V abelian groupoid

$\rightsquigarrow$ $Coh(V)$ is tensor category

$* =$ convolution

Under KD $\quad * \longleftrightarrow \otimes$

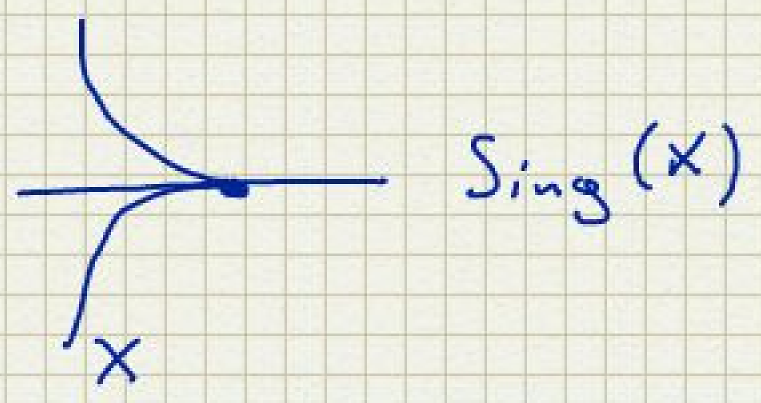
Back to $X = \{F = 0\} \subseteq \mathbb{A}^n$

$$\begin{array}{ll} \mathbb{L}_X = \Omega_{\mathbb{A}^n} & 0 \\ \uparrow dF^* & \\ \Omega_{\mathbb{A}^k} & -1 \end{array}$$

Def: $Sing(X) \to X$

Fiber at $x \in X = H^{-1} \mathbb{L}_X$

<u>Ex</u>: $X = \{x^2 = y^3\}$



<u>Rmk</u>: $\mathrm{Sing}\, X \subseteq X \times \mathbb{A}^k$

Localize $\mathrm{Coh}(X)$ over $\mathrm{Sing}(X)$

<u>Rmk</u>: This will be the ultimate localization.

Need to find action of $\mathcal{O}(\mathbb{A}^k) = k[\xi_1, \dots, \xi_k]$

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{A}^n \arrow[d, "F"] \\ \{0\} \arrow[r, hook] & \mathbb{A}^k \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r] \arrow[d] & \{0\} \arrow[d] \\ \{0\} \arrow[r] & \mathbb{A}^k \end{tikzcd}$$

$$\circlearrowright \overset{\{0\}}{\bullet} \longrightarrow F \qquad \circlearrowright^{\{0\}}$$

Upshot: $\underset{*}{\mathrm{Coh}(V)} \circlearrowright \mathrm{Coh}(X)$

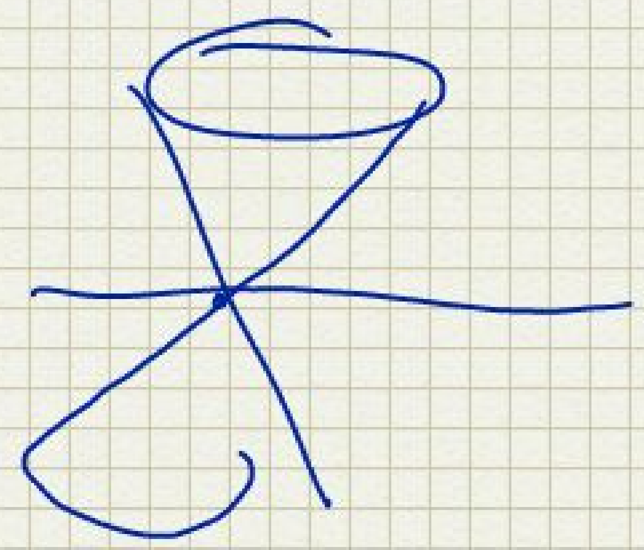
<u>Rmk</u>: Unit of $*$ is k. $\rightsquigarrow$ $\underset{k[\xi_1, \dots, \xi_k]}{\underset{\|}{\mathrm{End}(k)}} \longrightarrow \mathrm{End}(\mathrm{id}_{\mathrm{Coh}(X)})$

Conclusion: $k[\xi_1, \dots, \xi_k]$ acts universally on every object.

$\rightsquigarrow$ $\mathrm{Coh}(X)$ localized on $X \times \mathbb{A}^k$

<u>Exer</u>: supp on $\mathrm{Sing}(X)$

<u>Ex</u>: $Q^3 = \{\det A = 0\}$
$A \in M_2$



$$\begin{tikzcd} Q^3 \arrow[r, hook] \arrow[d] & \mathbb{A}^4 \arrow[d, "\det"] \\ \{0\} \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

$\mathrm{Sing}(X) \subset X \times \mathbb{A}^1$

<u>Exer</u>: $\mathcal{F} \in \mathrm{Coh}(X)$
$\mathrm{supp}\, \mathcal{F} \subseteq \{0\} \subset \mathrm{Sing}(X)$ iff $\mathcal{F} \in \mathrm{Perf}(X)$

In Ex: $k_{\{0\}}$ is not perfect

Geom. Arthur parameters

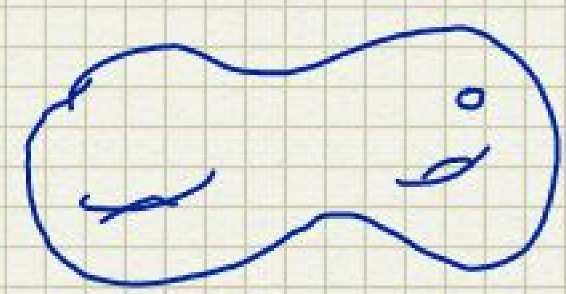
C smooth projective curve $/_k$

$Loc_G(C)$ moduli of local G-systems on C

For today: Betti version: $\{\pi_1 C \to G\}/_G$

$$Loc_G(C) \leftarrow \left[Loc_G(C)' \leftleftarrows G \right]$$

$$\begin{tikzcd}
Loc_G(C)' \arrow[r] \arrow[d] & G^{2g} \arrow[d, "{[\,,\,]}"] \\
\{e\} \arrow[r] & G
\end{tikzcd}$$



$$Sing(Loc_G(C)) \subset (Loc_G(C)' \times \mathfrak{g}^*)/_G$$

$\uparrow$

moduli of geom. Arthur's parameters

Expect automorphic objects to lead to coherent sheaves w/ nilpotent Arthur parameters.

Geometry of Hecke categories

Satake correspondence

G red gp

$$\mathcal{H}_{sph} = \mathbb{C}_c\left[G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p)\right]$$

$G^\vee$ Langlands dual

G root datum $\quad (\Lambda_G, \Lambda^\vee_G, R_G, R^\vee_G)$

Λ_G: coul. (?) wt. lattice $\quad \Lambda^\vee_G$ $\quad R_G$: coroots $\quad R^\vee_G$: roots

<u>Satake isom</u>:

$$\underset{\text{comm. alg.}}{\mathcal{H}_{sph} \simeq} \mathcal{O}(G^\vee)^{G^\vee} \quad \text{= representation ring}$$

$*$

<u>Goal</u>: geom. version

Affine Grassmannian $\quad k = \mathbb{C}, \ldots$

$K = k((t))$ ... Laurent series

$\cup$

$\mathcal{O} = k[[t]]$ ... power series

$LG = G(K)$ loop group

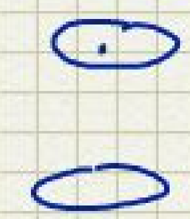
$L_+G = G(\mathcal{O})$ arc group

Ex: $G = GL_n$

$\rightsquigarrow G(K)$ = inv. matrices with K coeff.

Spec K

Spec $\mathcal{O}$



$Gr := LG / L_+G$ aff Gr. / loop Gr.

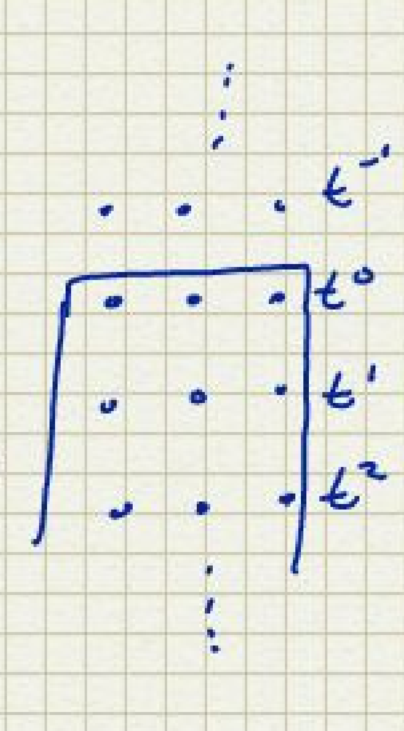
Increasing union of proj. varieties.

Ex: $G = GL_n$

$LGL_n \circlearrowright K^n$

basic lattice $\mathcal{O}^n \subset K^n$

preserved by L_+GL_n



$n = 3$ in image

Exc: $Gr_{GL_n} = \{ k$ subspaces $W \subset K^n$ s.t. $\cdot)\ tW \subset W$

$\cdot)\ t^N \mathcal{O}^n \subset W \subset t^{-N} \mathcal{O}$

$N \gg 0 \}$

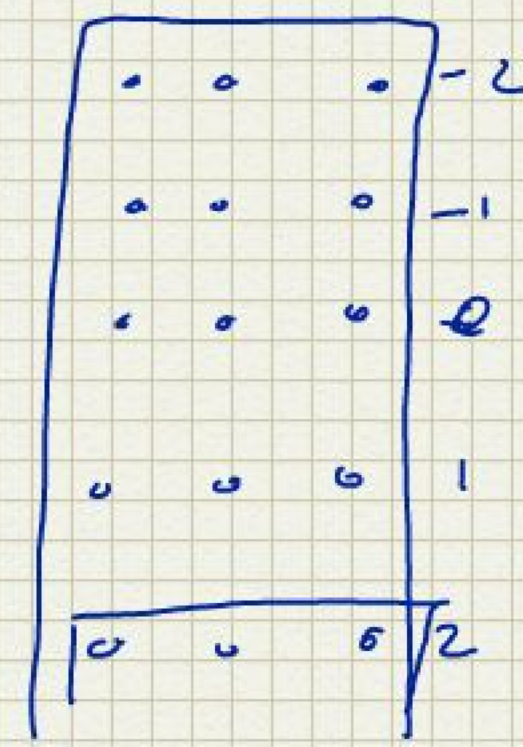
$$\rightsquigarrow Gr = \coprod_{N=0}^{\infty} Gr_N$$

$\nearrow$

condition 2 holds for N

$\uparrow$

maj. variety



$N=2$

<u>Alternative descriptions</u>

$D = \operatorname{Spec} \mathcal{O}$

$D^{\times} = \operatorname{Spec} K$

1) <u>Bundles:</u> $Gr = \{ P$ a G-bundle on D, trivialized on $D^{\times} \}$

$$L_+G \backslash Gr = L_+G \backslash LG / L_+G =$$

$| \ \mathbb{D} = D \amalg_{D^{\times}} D$

$$= \{ G\text{-bundles on } \mathbb{D} \}$$

2) <u>Topological</u> : $K \subset G$ max. compact

ΩK = based poly loop group

as maps $S^1 \to K$ finite Fourier expansion

$Gr \simeq \Omega K$

top. space

in fact $LG \simeq \Omega K \times L_+ G$

<u>Exc:</u> In particular for $G = PGL_2$

$$Gr_{PGL_2} = \mathrm{Free}(S^2) / {x a(x) = 1}$$

free top. group.

$a : S^2 \to S^2$ antipodal

A little structure theory:

Cartan: $L_+G \backslash Gr = L_+G \backslash LG / L_+G \simeq \Lambda_G / W \simeq \Lambda_G^+$

set; Weil gp; dominant coweights

Iwasawa: $N(K) \backslash G = N(K) \backslash LG / L_+G \simeq \Lambda_G$

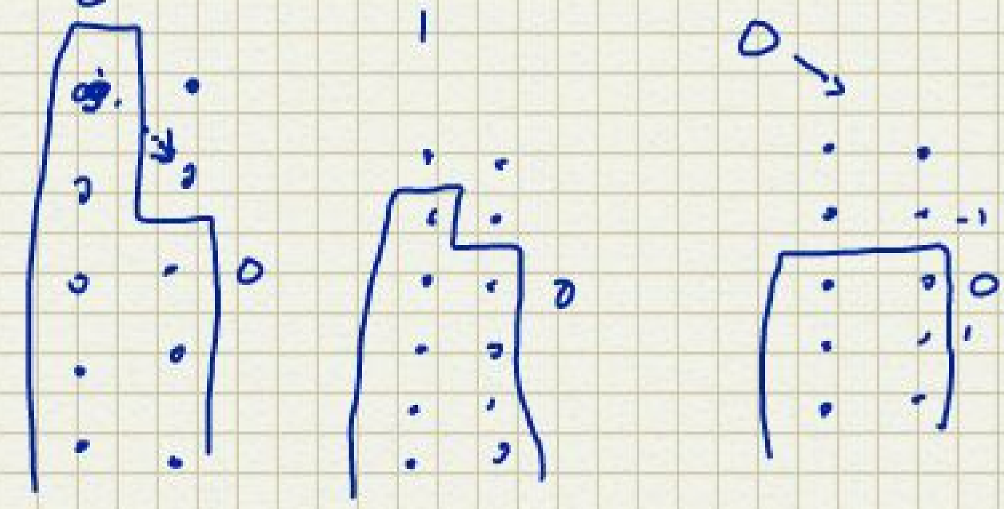
Ex: PGL_2

$+$ $\quad \mathbb{P}^1 \quad T\mathbb{P}^1 (?)$

$0 \quad 2 \quad 4 \quad 6$

$- \quad \mathbb{P}^1$

$1 \quad 3 \quad 5 \quad 7$



Geometric L_+G-invariant "distributions" on Gr

$\rightsquigarrow$ $P_{L_+G}(Gr) = L_+G$ - eq. perv. sheaves

Abelian category

semisimple!

<u>simple objects</u>: $IC^\lambda = IC(Gr^\lambda)$, where $Gr^\lambda \subseteq Gr$ is closure of L_+G-orbit indexed by $\lambda \in \Lambda_G/W$

intersection complex

"best basis of Hecke alg."

<u>Hint of Satake</u>: (Lusztig)

$\dim R\Gamma(Gr, IC^\lambda) = \dim V^\lambda$

V^λ irred rep of $G^\vee$ of hw $\lambda \in \Lambda^\vee_{G^\vee}/W = \Lambda_G/W$

$\dim IC^\lambda|_\mu = \dim wt_\mu(V^\lambda)$

μ point in Gr^μ

Geom. satake:

$$P_{L_+G}(Gr) \underset{\otimes\text{-cat}}{\simeq} Rep(G^\vee)$$

Where is $\otimes$-str?

Monoidal $*$ str. - usual convolution pattern

via. $Gr = \{G\text{-bundles on } \mathbb{D}\}$

Alternative "better" point of view - <u>fusion</u>

<u>Recall</u>: $Gr = \{G\text{-bundles on } D, \text{ triv on } D^\times\}$

C ... smooth curve $\qquad \simeq \{G\text{-bundles on } C,$
$\cup\!\!\!\!\shortmid$
c closed pt $\qquad$ triv. on $C\setminus c\}$

All points of C look similar!

Introduce BD-Grassmanian (Beilinson-Drinfeld)

$S = \{c_i\}_{i=1}^k \subseteq C \qquad Gr^{(k)} = \{G\text{-bundles on } C,$

triv. away from $S\}$

$$Gr^{(k)} = \prod_{i=1}^{k} Gr \qquad \text{(Exercise.)}$$

Allow S to move & points to collide

Evidently commutative tensor product on $\mathcal{P}_{L_+G}(Gr)$ via taking nearby cycles as points collide.

<u>Derived version</u>

$$D_{c,L_+G}(Gr) = ? \qquad \text{spectral interpretation}$$

const. derived cat, L_+G-equiv

$$L_+G \backslash LG / L_+G = \{ G\text{-bundles on } \mathbb{D} \}$$

Spectral expectation: $Loc_{G^\vee}(\mathbb{D}) = \left(\{1\} \underset{G^\vee}{\times} \{1\} \right) / G^\vee =$

$$\mathbb{D} = D \underset{D^*}{\amalg} D \quad \Bigg| \quad = \mathfrak{g}^\vee[1] / G^\vee$$

$$\mathfrak{g}^\vee[1] = \operatorname{Spec} \operatorname{Sym}\left((\mathfrak{g}^\vee)^* [1] \right)$$

ext. alg.

Consider $\mathrm{Coh}(\mathcal{L}oc_{G^\vee}(\mathbb{D}))$

<u>Note</u>: $H^{-1}\,\mathbb{1}_{\mathcal{L}oc_{G^\vee}(\mathbb{D})} \simeq (\mathfrak{g}^\vee)^*$

Consider $\mathrm{Coh}_{\mathcal{N}}(\mathcal{L}oc_{G^\vee}(\mathbb{D}))$

$$\uparrow$$

$\mathcal{N} \subset (\mathfrak{g}^\vee)^*$ nilpotent cone

<u>Thm</u>: $D_{c,\,L_+G}(Gr) \simeq \mathrm{Coh}_{\mathcal{N}}(\mathcal{L}oc_{G^\vee}(\mathbb{D}))$

$$\text{IS}$$

$D_c(\mathrm{Bun}_G(\mathbb{D}))$