Motivation

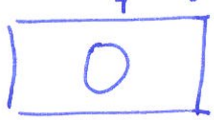
Main analogy:

Non-reduced structure is to algebraic set
as
derived structure is to scheme.

Fix two conics:

$C = \{q = 0\}$

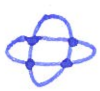
$C' = \{q' = 0\}$



$q, q' \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$

(over alg. closed field)

<u>Case 1</u>: Transverse intersection



$\rightsquigarrow$ 4 points $\{q = q' = 0\}$, stable under perturbation

<u>Case 2</u>: Tangent intersection

$\rightsquigarrow$ 3 points in $\{q = q' = 0\}$, not stable under perturbation

non-reduced structure sheaf $\rightsquigarrow$ $\dim H^0(\mathbb{P}^2, \underbrace{\mathcal{O}_C \otimes \mathcal{O}_{C'}}_{\text{alg. object}}) = 4$

geom. object: $\operatorname{Spec}(\mathcal{O}_C \otimes \mathcal{O}_{C'})$

<u>Case 3</u>: Degenerate intersection $q = q'$

$C \cap C \rightsquigarrow \mathcal{O}_C \overset{L}{\otimes} \mathcal{O}_C$

resolution: $\mathcal{O}_C \leftarrow \left[\mathcal{O}_{\mathbb{P}^2} \xleftarrow{\cdot q} \mathcal{O}_{\mathbb{P}^2}(-2) \leftarrow 0\right]$

$\rightarrow \mathcal{O}_C \overset{L}{\otimes} \mathcal{O}_C = \left(\mathcal{O}_{\mathbb{P}^2} \xleftarrow{(q, q')} \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 2} \xleftarrow{\binom{q'}{-q}} \mathcal{O}_{\mathbb{P}^2}(-4)\right)$

For all C, C' we have:

1) The locus where $\mathcal{O}_C \overset{L}{\otimes} \mathcal{O}_{C'}$ fails ~~to~~ to be exact equals $C \cap C'$.

2) $\dim \mathbb{H}^0(\mathbb{P}^2, \mathcal{O}_C \overset{L}{\otimes} \mathcal{O}_{C'}) = 4$

non-degenerate case: $\mathbb{H}^0(\mathbb{P}^2, \mathcal{O}_C \overset{L}{\otimes} \mathcal{O}_{C'}) = H^0(\mathbb{P}^2, \mathcal{O}_C \otimes \mathcal{O}_{C'})$

degenerate case: $\mathbb{H}^0(\mathbb{P}^2, \mathcal{O}_C \overset{L}{\otimes} \mathcal{O}_C) \underset{\substack{\uparrow \\ \text{spectral} \\ \text{sequence}}}{=}$

$$\begin{array}{ccc} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) & = & 1 \\ \oplus & & + \\ H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 2}) & & 0 \\ \oplus & & + \\ H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-4)) & & 3 \end{array} = 4$$

<u>Q</u>: Is there any geometric object associated to $\mathcal{O}_C \overset{L}{\otimes} \mathcal{O}_{C'}$?

<u>Serre's Tor-Formula</u>: R reg. local ring, $I, J \subseteq R$ with ~~$\exists$~~ $R/_{I+J}$ 0-dimensional

$$\mu(R/I, R/J) = \sum_{i \geq 0} (-1)^i \operatorname{length} \operatorname{Tor}_i^R(R/_I, R/_J) = \text{"}\operatorname{length} R/_I \overset{L}{\otimes} R/_J\text{"}$$

$R = k[x,y]$ $\qquad R = (\mathrm{Set}, +, \cdot, 1)$

Categorical ring: $\mathcal{C} = (\mathrm{Category}, \hat{+}, \hat{\cdot}, \hat{1})$

given $R \rightsquigarrow \mathcal{C}_R$: objects $= \{r \in R\}$

morphisms $= \mathrm{Mor}(r, r') = \begin{cases} \{*\} & , r = r' \\ \emptyset & , r \neq r' \end{cases}$

$\hat{+}: \mathcal{C}_R \times \mathcal{C}_R \longrightarrow \mathcal{C}_R$

$(r, r') \longmapsto r + r'$, etc.

Impose $q = 0 \rightsquigarrow$ new category with $\begin{tikzcd} 0 \arrow[r, bend left, no head] & q \end{tikzcd}$, i.e. $\mathrm{Mor}(r, r') = \begin{cases} \{*\} & r - r' \in (q) \\ \emptyset & \text{else} \end{cases}$

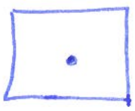
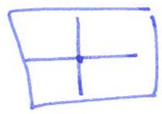
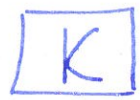
$\Longrightarrow$ iso classes are exactly $R/(q)$

Impose $q = 0$ twice $\rightsquigarrow$ get $\begin{tikzcd} 0 \arrow[r, bend left, no head] \arrow[r, bend right, no head] & q \end{tikzcd}$ (and everything that is forced by axiom)

iso classes ~~of~~ are still R/q

$\rightsquigarrow$ this gives not the complete picture, need higher ~~q~~ structures.

Homological	$\xleftarrow{\text{Dold-Kan Thm}}$	Categorical / Simplicial
complexes	$\longleftarrow$	simplicial algebra
$H_i(\text{complex})$	$\longleftarrow$	π_i
$\shortparallel$		
Tor_i		

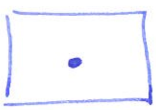
 can arise from, e.g  or 

$\longrightarrow$ different scheme structures.

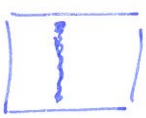
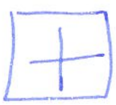
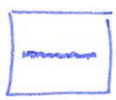
Same for derived structures

<u>Ex</u>: $R = k[x,y]$, $Q = R/(x^2, xy, y^2)$

•) $\operatorname{Spec} Q \subseteq \mathbb{A}^2$

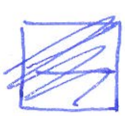


$$Q \leftarrow [R \longleftarrow R^3 \longleftarrow R^2 \longleftarrow 0] \qquad \pi_i = \begin{cases} Q & i=0 \\ 0 & \text{otherwise} \end{cases}$$

•)  $\cap$  $\cap$ 

$$R/(x^2) \overset{L}{\otimes} R/(xy) \overset{L}{\otimes} R/(y^2)$$

$$R^1 \longleftarrow R^3 \longleftarrow R^1 \longleftarrow 0 \qquad \pi_i = \begin{cases} Q & , i=0 \\ (x,y)/(x^2,y^2) & , i=1 \\ 0 & , i\geq 2 \end{cases}$$

•)   $\cap$ 

$$R/(x^2,xy) \overset{L}{\otimes} R/(y^2,xy)$$ gives 3 homotopy groups.

Talk 2:

Simplical sets & higher categories:

I. Introduction

Homologicale algebra	works for Ab. cat $A \overset{\deg 0}{\hookrightarrow}$ via chain cx in A		inj/proj resolutions
		$\updownarrow$ DK-corresp.	
Homotopical	category $\mathcal{C} \overset{\text{const.}}{\hookrightarrow}$ simplicial obj in $\mathcal{C}$		cofib/fib resolution

II. Simplicial objects and DK-corresp.

Def: A simplicial set is a functor $X: \Delta^{op} \longrightarrow$ Sets

etc.

Talk 3

Simplicial Commutative Rings I

$$\mathrm{Aff}_{\text{h-var}} = (\mathrm{Alg}_{\mathfrak{g}k,\mathrm{red}})^{op}$$
$$\downarrow$$
$$\mathrm{Aff} = (\mathrm{CRing})^{op}$$
$$\downarrow$$
$$?$$

Candidate: top~~ological~~ comm. rings
→ simplicial

Equivalence: simplicial sets $\simeq$ top. spaces

Def: A simplicial comm. ring is an element of $\mathrm{Fun}(\Delta^{op}, \mathrm{CRing}) = \mathrm{SCR}$

Ex: $\mathrm{CRing} \hookrightarrow s\mathrm{CR}$
$R \longmapsto$ constant

Ex: $X_\bullet \in s\mathrm{Set} \rightsquigarrow (\mathbb{Z}[X_\bullet])_n = \mathbb{Z}[X_n]$

~~Ex: 1-truncated homotopy-types $\simeq$ groupoids~~
~~A categorical ring =~~

Homotopy groups:

Def: $R_\bullet \in s\mathrm{CR}$, $\pi_i R = [(S^i, *), (R_\bullet, 0)] = H_i(NR_\bullet)$

$S^i = \Delta^i/\partial\Delta^i$; this is fibrant

Claim: $R_\bullet \in s\mathrm{CR}$. Then $\pi_* R$ is graded-commutative.

$f: S^i \to R_\bullet$
$g: S^j \to R_\bullet$ $\quad f\wedge g: S^{i+j} \to R_\bullet \wedge R_\bullet \to R_\bullet$ $\quad A_\bullet, B_\bullet \rightsquigarrow A_\bullet \wedge B_\bullet = \dfrac{A\times B}{*\times B \cup A\times *}$

Ex: $\pi_0 R = R_0 / (d_1 - d_0) R_1$

Ex: If $R \in$ CRing, there is a map $R \to R_{red}$

If $R_\bullet \in$ sCR, there is a map $R_0 \to \pi_0 R$.

Goal: Given $R, S \in$ sCR, want $\underline{\mathrm{Hom}}(R_\bullet, S_\bullet)$

To get this we'll define $K_\bullet \otimes R_\bullet$ for $K_\bullet \in$ sSet, $R_\bullet \in$ sCR

Then $\mathrm{Map}_{sSet}(T_\bullet, \underline{\mathrm{Hom}}(R_\bullet, S_\bullet)) = \mathrm{Map}_{sCR}(T_\bullet \otimes R_\bullet, S_\bullet)$ can be used to define $\underline{\mathrm{Hom}}$

Let $\mathcal{C}$ be a category w/coprods. Set $s\mathcal{C} = \mathrm{Fun}(\Delta^{op}, \mathcal{C})$.

Let $X_\bullet \in s\mathcal{C}$, $K_\bullet \in$ sSet, then $(K \otimes X)_\bullet := \coprod_{K_n} X_n$

The model structure:

1) A weak equivalence $R_\bullet \to S_\bullet$ is an iso on π_*

2) A fibration is a Kan fibration (so everything is fibrant)

3) The cofibrations are determined by this.

$$\mathrm{sSet} \underset{\text{forget}}{\overset{\mathbb{Z}[\cdot]}{\rightleftarrows}} \mathrm{sCR} \quad \text{adjoint}$$

Use this to lift the model structure

Ex: $R_\bullet \to S_\bullet$

If there exist sets $A_n \subseteq S_n$ s.t.
1) $S_n = R_n[A_n]$
2) A_n is preserved under degeneracies

Then $R_\bullet \to S_\bullet$ is a cofibration.

Claim: If $R_\bullet$ is a cofibrant simplicial ring, then for any $S_\bullet$, $\underline{\mathrm{Hom}}(R_\bullet, S_\bullet)$ is the correct homotopy type.

sCR is a simplicial model category.

Let $R_\bullet \in sCR$

Want $Mod(R_\bullet)$, simplicial modules.

$$Mod(R_\bullet) = \left\{ \begin{array}{l} \text{simplicial sets } M_\bullet \text{ w/ a map} \\ R_\bullet \times M_\bullet \longrightarrow M_\bullet \text{ satisfying the usual module axioms.} \end{array} \right\}$$

So: $\pi_* R \times \pi_* M \longrightarrow \pi_* M$ makes $\pi_* M$ a graded module.

<u>Thm</u> (Quillen): $\exists$ a model structure on $Mod(R_\bullet)$.

<u>Ex</u>:

$$\begin{tikzcd} C_\bullet & & B_\bullet \\ & A_\bullet \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$B_\bullet \otimes_{A_\bullet} C_\bullet$ only correct if cofibrant.

Define: $B_\bullet \overset{L}{\otimes}_{A_\bullet} C_\bullet = \tilde{B}_\bullet \otimes_{A_\bullet} \tilde{C}_\bullet$ where

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \\ & \tilde{B} \arrow[ur, "\simeq"'] & \end{tikzcd}$$

this is homotopy invariant.

<u>Ex</u>: $R[y] \longrightarrow R$, $y \longmapsto 0$ Want $R \overset{L}{\otimes}_{R[y]} R$

- - -

Let $\mathcal{C}$ be a cat $T: \mathcal{C} \to \mathcal{C}$, X a T-alg (i.e. $TX \to X$)

We will find a simplicial T-alg $B(T,X)_\bullet \longrightarrow X$

$\uparrow$ Bar construction

$$TX \rightleftarrows T^2X \quad T^3X \cdots$$

This is a simplicial homotopy equivalence.

- - -

$\mathcal{C} = Set$ T-alg $= CRing$

$T = \mathbb{Z}[\ ]$

If $X \in CRing$ get cofibrant $B(T,X)_\bullet \longrightarrow X$

not very efficient... $\mathbb{Z}[X] \rightleftarrows \mathbb{Z}[\mathbb{Z}[X]] \cdots$

$B(R)_n = R[y]^{\otimes(n+1)}$

$= \{ g[f_1|\dots|f_n] \quad , g, f_i \in R[y] \quad (R\text{-linear}) \}$

$d_i(g[f_1|\dots|f_n]) = \begin{cases} gf_1[f_2|\dots|f_n] & i=0 \\ g[f_1|\dots|f_if_{i+1}|\dots|f_n] & 0<i<n \\ g\varphi(f_n)[f_1|\dots|f_{n-1}] & i=n \end{cases}$ $\qquad \varphi: R[y] \to R$

$\vdots$

$R[y]\otimes R[y]$

$R[y]$

<u>Thm</u>: $R[y] \hookrightarrow B(R) \xrightarrow{\sim} R$

$R \overset{L}{\underset{R[y]}{\otimes}} R = B(R) \underset{R[y]}{\otimes} R$ gives the same as ↑ except for $i=0$: $g\varphi(f_1)[f_2|\dots|f_n]$

$\pi_*(B(R) \underset{R[y]}{\otimes} R) = \begin{Bmatrix} R & \text{in dim } 0,1 \\ 0 & \text{otherwise} \end{Bmatrix} = \mathrm{Tor}_i^{R[y]}(R,R).$

<u>Ex</u>: If R is any ring $y \in R$ a non-zerodivisor

$R \longrightarrow R/(y)$

$$\begin{array}{ccc} R[x] & \xrightarrow{x\mapsto 0} & R \\ \downarrow{\scriptstyle x\mapsto y} & & \downarrow \\ R & \longrightarrow & \cancel{R/(y)}\ B(R)\underset{R[y]}{\otimes} R \to R/(y) \end{array}$$

Talk 4:

SIMPLICIAL COMMUTATIVE RINGS II

Notation:

$X, Y \in sCR$ or Mod_A , $K \in sSet$

$(K \otimes X)_n = \coprod_{K_n} X_n$

$(\underline{Hom}(X,Y))_n = Hom(\Delta^n \otimes X, Y)$

$\rightsquigarrow \mathbb{R}\underline{Hom}(X,Y) = Hom(Q(X), Y)$ where $Q(X)$ cofib replacement

$M, N \in Mod_A$, $* \in K$ base pt

define $K \wedge M$ by

$$\begin{tikzcd} * \otimes M \arrow[r] \arrow[d] & K \otimes M \arrow[d] \\ 0 \arrow[r] & K \wedge M \end{tikzcd}$$ pushout

$M[n] = \Sigma^n M = S^n \wedge M$, $S^n = \Delta^n / \partial \Delta^n$

$(M \otimes N)_n = M_n \otimes_{A_n} N_n \rightsquigarrow M \otimes^{\mathbb{L}}_A N$

Attaching cells

Q: How do we construct simplicial res for modules $R^J \to R^I \to M$.

$S^{n-1} \hookrightarrow D^n$

$\Delta^{n-1}/\partial\Delta^{n-1} = S^{n-1}$, $D^n = \Delta^n/\Lambda^n_0$

$|S^{n-1}| \longrightarrow |D^n|$

$d_0(b) = \dots = d_{n-1}(b) = *$

$d_1(c) = \dots = d_n(c) = *$

$d_0(c) = b$

notation $\mathbb{Z}[K] = K \otimes \mathbb{Z}[x]$, $(\mathbb{Z}[K])_n = \mathbb{Z}[K_n]$, $\mathbb{Z}[(K,*)]_n = \mathbb{Z}[K_n]/(*)$

ness, étaleness, smoothness

<u>Prop</u>: $A \in sCR$, $M \in Mod_A$ is <u>flat</u> if the following equivalent definitions hold.

1) $\pi_0(M)$ is a flat $\pi_0(A)$-module

$\pi_n(R) \otimes_{\pi_0(R)} \pi_0(M) \xrightarrow{\sim} \pi_n(M) \quad \forall n$

2) $M \otimes^{\mathbb{L}} -$ commutes with finite homotopy limits.

3) M is a filtered colimit of finite free A-modules

1') $M \otimes^{\mathbb{L}} -$ commutes w/ Ω
$$\left(\begin{tikzcd} \Omega M \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & 0 \arrow[d] \\ 0 \arrow[r] & M \end{tikzcd} \right)$$

1'') N a discrete A-module, then $M \otimes^{\mathbb{L}} N$ is discrete.

$A \to B$ in sCR is <u>étale</u> (resp. <u>smooth</u>) if it is flat and $\pi_0(A) \to \pi_0(B)$ is étale (resp. smooth).

ess Conditions

$A \to B$ in sCR. Say B is a

) <u>finitely presented</u> A-alg if it can be obtained by attaching finitely many cells;

) <u>locally finitely presented</u> A-alg if it is a retract of a f.p. A-alg.

) <u>almost finitely presented</u> A-alg if $\forall n \ \exists B_n$ f.p. A-alg and $f_n : B_n \to B$ s.t.

$\pi_i(B_n) \xrightarrow{\sim} \pi_i(B) \quad i \leq n.$

logous conditions for modules:
- finitely presented
- perfect
- almost perfect

R discrete, $M \in Mod_R$ f.p.

M perfect $\Rightarrow pd_R M < \infty$

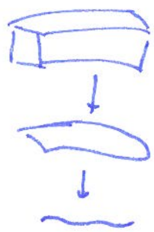
lly f.p. is "compact", i.e. $R\underline{Hom}(B, -)$ commutes will filtered colimits

The Cotangent Complex I: The "Classical Theory"

Motivation

$B/A \rightsquigarrow B \xrightarrow{d} \Omega^1_{B/A}$ well behaved if B/A is smooth

$$\begin{matrix} C & & X \\ \uparrow & & \downarrow \\ B & & Y \\ \uparrow & & \downarrow \\ A & & Z \end{matrix}$$



$$0 \to T_{X/Y} \to T_{X/Z} \to T_{Y/Z} \to 0$$

$$\rightsquigarrow \underset{\substack{\uparrow \\ X/Y \text{ smooth}}}{(0 \to)} f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

$$\underset{0 \text{ if smooth}}{?} \to I/I^2 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to 0$$

Motivation II: Deformation Theory.

$L_{X/Y} \in D(X)$

Thm: (i)

$$\begin{tikzcd} X_0 \arrow[r, "g"] \arrow[d, "j \text{ sq.0}"'] & Y \arrow[d] \\ X \arrow[r] \arrow[ur, dashed, "h"] & S \end{tikzcd}$$

h exists iff obstruction $o(g,j) \in \mathrm{Ext}^1(g^*L_{X/S}, J)$ vanishes.
If $o(g,j)=0$ then the set of lifts is ? $\mathrm{Hom}(g^*L_{X/S}, J)$

(ii)

$$\begin{tikzcd} X_0 \arrow[r, dashed] \arrow[d, "f_0", "\text{flat}"'] & X \arrow[d, dotted, "i"] \\ S_0 \arrow[r, "i"' , "\text{sq.0}"'] & S \end{tikzcd}$$

X exists if $o(X_0, i) = 0$ in $\mathrm{Ext}^2(L_{X_0/S_0}, f_0^*(I))$
Such X ? for $\mathrm{Ext}^1(L_{X_0/S_0}, f_0^*(I))$

Application 1:

A perfect $\mathbb{F}_p$-algebra, e.g. $\mathrm{Frob}_p : A \to A$ is an?

$$\begin{array}{l} \mathrm{Spec}\, A \\ \downarrow \\ \mathrm{Spec}\, \mathbb{F}_p \longrightarrow \mathrm{Spec}\, \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathrm{Spec}\, \mathbb{Z}/p^3\mathbb{Z} \longrightarrow \cdots \end{array}$$

obstruction: $\mathrm{Ext}^2(\underset{\overset{\shortparallel}{0}}{L_{A/\mathbb{F}_p}}, f^\circ(\underset{\overset{\shortparallel}{\mathbb{F}_p}}{\langle p^{k-1}\rangle})) = 0$.

as $L_{A/\mathbb{F}_p} \otimes_A A \xrightarrow[0]{\sim} L_{A/\mathbb{F}_p}$

$$d(x \to x^p) = p\, x^{p-1} = 0$$

Thm: Let A be a complete local Noetherian ring with residue field k.

$X_{0/k}$ is a proper curve, lci, ?

Then $\exists\ X/A$ projective + flat with $X_k \simeq X_0$

$\boxed{\text{etc.}}$

3. Definition / Construction:

Def: A projective A-alg resolution of cB is a factorization
constant simpl. ring

$$\begin{array}{ccc} cA & \longrightarrow & cB \\ & \searrow_{\text{cof}} \ \ P \nearrow_{\text{trivial fibration}} & \end{array}$$

Def (Cotangent Complex):

$$(L_{B/A})_n = (\Omega_{P_n/A} \otimes_{P_n} \cancel{A}B)$$

Def (AQ (co)homology)

$$D_q(B/A, M) = H_q(L_{B/A} \otimes_B M)$$

$$D^q(B/A, M) = H^q(\mathrm{Hom}_B(L_{B/A}, M))$$

4. Properties:

1) $D_0(B/A, B) = D_0(B/A) = \Omega^1_{B/A}$

2) $L_{B/A}$ is a ~~proja~~ complex of projectives

3) An extension of B by M is a s.e.s. $0 \to \underset{\text{sq. } 0}{M} \to X \to B \to 0$

Let $\mathrm{Exalcom}(B/A, M)$ be the set of isoclasses of such extensions.

Then $D^1(B/A, M) = \mathrm{Exalcom}(B/A, M)$

4) Suppose $B = A/I$. Then $D_0 = 0$, $D_1 = I/I^2$

5) Suppose

$$\begin{tikzcd} R' \arrow[r] & S' = S \otimes_R R' \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd} \quad , \quad \mathrm{Tor}_q^R(R', S) = 0 \quad \forall q > 0$$

Then $L_{S/R} \otimes_R R' = L_{S'/R'}$

$$L_{S'/R} \simeq L_{S/R} \otimes_R R' \oplus L_{R'/R} \otimes_R S$$

6) $L_{S^{-1}A/A} = 0$

7) If A is Noetherian and B is a finite type A-alg.

Then B étale $\iff L_{B/A} = 0$

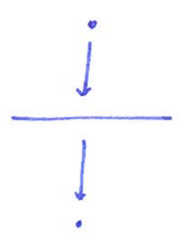
B smooth$/A \iff L_{B/A} \simeq c(\Omega^1_{B/A})$, $\Omega^1_{B/A}$ projective

8) (Transitivity Triangle)

If $A \to B \to C$ are maps of rings, then there is a d.t. in $D(C)$

$$\begin{tikzcd} L_{B/A} \otimes_B C \arrow[rr] & & L_{C/A} \arrow[dl] \\ & L_{C/B} \arrow[ul, dashed, "+1"] & \end{tikzcd}$$

<u>Ex</u>:



$$\begin{tikzcd} k \\ k[\epsilon] \arrow[u, "t \mapsto 0"'] \\ k \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} \overset{k[\epsilon]}{\overset{\parallel}{L_{k[\epsilon]/k} \otimes_{k[\epsilon]} k}} \arrow[rr] & & \overset{0}{\overset{\parallel}{L_{k/k}}} \arrow[dl] \\ & L_{k/k[\epsilon]} \arrow[ul, dashed] & \end{tikzcd} \quad \Rightarrow k[1]$$

Talk 6

Cotangent Complex II

I. Review

$f: A \to B$, $A, B \in sCR$

Quillen adjoint pair

$$\left.\begin{array}{ll} \text{left}: & F: sCR_{A//B} \longrightarrow Mod_B \\ & \quad X \longmapsto \Omega_{X/A} \otimes B \\ \text{right} & G: Mod_B \longrightarrow sCR_{A//B} \\ & \quad M \longmapsto B \oplus M \end{array}\right\} \Rightarrow \text{Total derived functors exist and are adjoint.}$$

$\rightsquigarrow L_{B/A} := LF(B)$

$$\underset{\substack{\uparrow \\ h(Mod_B)}}{Hom}(L_{B/A}, M) = \underset{\substack{\uparrow \\ h(sCR_{A//B})}}{Hom}(B, \overset{B\oplus M}{\overset{\|}{RG(M)}})$$

II. Connectivity Results

Thm: $f: A \to B$ in sCR. If $cofib(f)$ is n-connected, then the natural map
$\varepsilon: cofib(f) \otimes_A B \to L_{A/B}$ is $(2n+1)$-connected

Cor: If $cofib(f)$ is n-connected then $L_{B/A}$ is n-connected.

The converse is true if $\pi_0(A) \simeq \pi_0(B)$.

Cor: $f: A \to B$ is an equivalence iff

1) $\pi_0(A) \simeq \pi_0(B)$

2) $L_{B/A} \simeq 0$

Cor: $A \longrightarrow \pi_0(A)$ is an equiv iff $L_{\pi_0(A)/A} \simeq 0$.

III. Deformation Theory

A, B, M ordinary rings/module

$\mathrm{Ext}^i(L_{B/A}, M)$
$\|$
$\mathrm{Ext}^0(L_{B/A}, M[i]) = \mathrm{Hom}(B, B \oplus M[i])$

"$\mathrm{Spec}(B \oplus M[i]) \to \mathrm{Spec}\, B$"

$$\begin{tikzcd} X \arrow[d, "\text{schemes}"'] \\ Y \end{tikzcd} \rightsquigarrow L_{X/Y} \qquad \begin{tikzcd} X \arrow[r, dashed] \arrow[d, "f"'] \arrow[dr, phantom, "?"] & X' \\ Y \arrow[r, hook, "j", "\text{sq. } 0"'] \arrow[dr] & Y' \arrow[u, dashed] \arrow[d] \\ & S \end{tikzcd}$$

M

Thm: (i) There is an obstruction $ob(f) \in \mathrm{Ext}^2(L_{X/Y}, f^*M)$, vanishing iff deformation exists

(ii) if $ob(f) = 0$, then $\{\text{deformations}\}/_{\simeq} = \mathrm{Ext}^1(L_{X/Y}, f^*M)$.

Rmks: 1) $\mathrm{Ext}^1(L_{Y/S}, M)$ classifies sq. 0. ext's of Y by M over S.

$$\begin{tikzcd} X \arrow[d] \\ Y \arrow[r] & Y' \end{tikzcd}$$

2) $\underset{\underset{\mathrm{Ext}^2(L_{X/Y}, f^*M)}{\cap}}{ob(f)} = \text{composition of } \underset{\underset{\mathrm{Ext}^1(f^*L_{Y/S}, f^*M)}{\cap}}{f^*[j]} \text{ with } KS(X \to \overset{\overset{\mathrm{Ext}^1(L_{X/Y}, f^*L_{Y/S})}{\cup}}{Y} \to S)$

Construction: $A \in sCR$, $M \in \mathrm{Mod}_A$

$$\begin{tikzcd} A \oplus M[i] \arrow[d] \\ A \arrow[u, bend left, "s"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} A' \arrow[r, "\varphi_s"] \arrow[d] & A \arrow[d, "s"] \\ A \arrow[r, "0"'] & A \oplus M[i] \end{tikzcd} \quad \text{pullback.}$$

Def: φ_s is called a <u>square zero extension</u> of A by M.

Ex: ·) $A \oplus M$ (for $s = 0$)

·) $k \oplus k[i]$

·) square-zero extensions of ordinary rings.

Prop: Given $A \in sCR$, then every map in

$$\cdots \to \tau_{\leq 2} A \to \tau_{\leq 1} A \to \tau_{\leq 0} A = \pi_0 A$$

is a square zero extension.

Consequence:

$$\begin{tikzcd}
\tau_{\leq n} A \arrow[r] \arrow[d] & \tau_{\leq n-1} A \arrow[d, "k_n \in \mathrm{Hom}(L_{\tau_{\leq n-1} A},\, \pi_n(A)[n+1])"] \\
\tau_{\leq n-1} A \arrow[r, "0"] & \tau_{\leq n-1} A \oplus \pi_n(A)[n+1]
\end{tikzcd}$$

Let

$$\begin{tikzcd}
B & B' \arrow[l, dashed] \\
A \arrow[u, "f"] & A' \arrow[l, "\text{s.z.e by } M"'] \arrow[u, dashed]
\end{tikzcd}$$

f flat

Thm: (i) There exists an obstruction $ob(f) \in Ext^2(L_{B/A}, B \otimes_A M)$ which vanishes iff deformation exists.

(ii) When $ob(f) = 0$, then $\{\text{deformations}\}/_{\cong} = Ext^1(L_{B/A}, B \otimes_A M)$.

explicitly, $ob(f)$ is $L_{B/A}[-1] \to B \otimes L_A \to B \otimes M[1]$.

$$\begin{tikzcd}
X = X_0 \arrow[r] \arrow[d, "f"', "\text{smooth proj}"] & X_1 \arrow[r, dashed] \arrow[d, "f_1"] & X_2 \arrow[d, dashed] \\
\mathrm{Spec}\, k \arrow[r] & \mathrm{Spec}(k[\varepsilon]/\varepsilon^2) \arrow[r] & \mathrm{Spec}(k[\varepsilon]/\varepsilon^3)
\end{tikzcd}$$

$$ob(f_1) \in H^2(T_X) = Ext^2(L_X, k) = Ext^1(L_X, k[1])$$

classifies

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X' \arrow[d] \\
\mathrm{Spec}\, k \arrow[r] & \mathrm{Spec}(k \oplus k[1])
\end{tikzcd}$$

$$\begin{tikzcd}
k[\varepsilon]/\varepsilon^3 \arrow[r] \arrow[d] & k[\varepsilon]/\varepsilon^2 \arrow[d] \\
k \arrow[r] & k \oplus k[1]
\end{tikzcd}$$

pullback

$$\begin{tikzcd}
& X_1^{ob} \arrow[rr] \arrow[d, "ob(f_1)"] & & X_1 \arrow[d, "f_1"] \\
X^{triv} \arrow[r] \arrow[d] & \mathrm{Spec}(k \oplus k[1]) \arrow[rr] \arrow[d] & & \mathrm{Spec}(k) \arrow[d] \\
X \arrow[r] & \mathrm{Spec}(k[\varepsilon]/\varepsilon^2) \arrow[rr] & & \mathrm{Spec}(k[\varepsilon]/\varepsilon^3)
\end{tikzcd}$$

pushout

DERIVED SCHEMES : representability theorems

I. Artin's theorem

R an excellent noetherian ring

$F: Alg_{R/-} \longrightarrow (Grpd)$

F is representable by an Artin stack locally of f.p. $/R$ iff

a) $F(\operatorname{colim}_i A_i) = \operatorname{colim}_i F(A_i)$ for filtered diagrams $i \mapsto A_i$

b) F is a sheaf for the étale topology:

If $\{U_i \to X\}$ is an étale cover of affine, then $F(X) \Rightarrow \operatorname{holim}\left(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_X U_j) \Rrightarrow \prod_{i,j,k} F(U_i \times_X U_j \times_X U_k) \cdots\right)$

c) $F(B) \xrightarrow{\sim} \operatorname{holim}_n (B/m^n)$ for every complete Noeth. local R-alg (B, m).

d) F admits a def/obs theory satisfying Schlessinger's condition

e) $\Delta: F \to F \times F$ is representable by algebraic spaces.

Examples:

1) $F(A) = \left\{ \begin{matrix} \text{smooth proper curves } /A \\ \text{of genus } g \end{matrix} \right\} = \mathcal{M}_g(A)$

2) Fix $X/_{\operatorname{Spec}(R)}$ proper & flat.

$F(A) = \left\{ \begin{matrix} \text{vector bundles} \\ \text{on } X \times_R A \end{matrix} \right\} = \operatorname{Vect}(X/_R)(A)$

II. Lurie's rep. thm (Lurie's thesis, DAG XIV, Pridham's paper)

Thm: R a derived G-ring

$F: sCR_{R/-} \longrightarrow$ Spaces a functor

Then F is representable by a derived DM-stack almost of f.p. $/R$, iff

a) F commutes with filtered colimits on $sCR_{R/-, \leq k}$ for all $k \in \mathbb{Z}_{\geq 0}$.

b) F is an étale sheaf

c) $F(B) \simeq \operatorname{holim}_n F(B/m^n)$ for discrete complete noetherian local R-alg.

d₁) F has a connective cotangent complex

d₂) F is infinitesimally cohesive.

e) F is nilcomplete

Explanation

<u>Setup</u>: $F \in \mathrm{Fun}(sCR_{R/-}, \text{Spaces})$

a) if $I \xrightarrow{A} sCR_{R/-, \leq k}$ filtered, then

$$F(\operatorname{colim}_i A_i) \xleftarrow{\sim} \operatorname{colim}_i F(A_i) \qquad \text{(homotopy colimit)}$$

<u>Ex</u>: 1) If $B \in sCR_{R/-}$, $F = \underline{\mathrm{Hom}}_{sCR_{R/-}}(B, -)$ satisfies (a) for all k if B is almost f.p./R.

2) $F(A) = \mathrm{Mod}_A^{f.p.} \subset \mathrm{Mod}_A$

b) F is an étale sheaf if

-) F commutes with finite products

-) if $A \to B$ is an étale cover in $sCR_{R/-}$, then

$$F(A) \xrightarrow{\sim} \operatorname{holim}_n F(B^{\otimes(n+1)})$$

<u>Ex</u>: $F = \underline{\mathrm{Hom}}_{sCR_{R/-}}(C, -)$ is an étale sheaf

key pt: $A \xrightarrow{\sim} \operatorname{holim}_n (B^{\otimes(n+1)})$

d) <u>cotangent complex on a functor</u>:

Say $F: sCR \longrightarrow \text{Spaces}$.

-) $QC_F^{\geq 0} = \operatorname{holim}_{\substack{A \in sCR \\ \eta \in F(A)}} \mathrm{Mod}_A$ $\rightarrow$ For each (A, η) get $M(\eta) \in \mathrm{Mod}_A$. For $\phi: A \to B$

$$M(\eta) \otimes_A B \xrightarrow{\sim} M(\phi_* \eta)$$

<u>Ex</u>: $F = \underline{\mathrm{Hom}}(A, -)$ then $QC_F^{\geq 0} = \mathrm{Mod}_A$

-) F has a connective cotangent complex ~~if~~ $L_F \in QC_F^{\geq 0}$ if there exists an equivalence

$\mathrm{Hom}_{\mathrm{Mod}_A}(L_F(\eta), N) =$ fibre of $F(A \oplus N) \to F(A)$ over η.

+ functoriality in A, η, N.

d_2) ~~if~~ inf. cohesive: —

e) nilcomplete: $F(B) \xrightarrow{\sim} \operatorname{holim}_n F(\tau_{\leq n} B)$.

Example : $\mathcal{M}_g$

Def: A map $f: X \to \mathrm{Spec}(A)$ is a stable curve of genus $g \geq 2$ if

a) f is flat, qcqs, almost f.p.

b) $X \times_A \pi_0(A) \longrightarrow \mathrm{Spec}(\pi_0 A)$ is a stable curve

Thm: $\overline{\mathcal{M}}_g : \mathrm{sCR} \longrightarrow \mathrm{Spaces}$

$$A \longmapsto \{\text{all stable curves } X \to \mathrm{Spec}(A) \text{ of genus } g \geq 2\}$$

is representable by $\overline{\mathcal{M}}_g^{cl}$.

Prop: If $f: X \to \mathrm{Spec}(A)$ is a derived scheme and $M \in \mathrm{Mod}_A$, then

$$\mathrm{Hom}_{QC_X}(L_{X/A}, f^*M[1]) = \{\text{deformations of } X \text{ over } \mathrm{Spec}(A \oplus M)\}$$

$\overline{\mathcal{M}}_g$ has a connective cotangent complex

$$\begin{array}{ccc} \overline{\mathcal{M}}_g(A \oplus M) & \longrightarrow & \overline{\mathcal{M}}_g(A) \\ \Psi & & \Psi \\ \mathrm{Hom}_X(L_{X/A}, f^*M[1]) & \to & \eta : (X \xrightarrow{f} \mathrm{Spec}(A)) \\ \| ?? & & \\ \mathrm{Hom}_A(\eta^* L_{\overline{\mathcal{M}}_g/\mathbb{Z}}, M) & & \end{array}$$

$$\eta^*(L_{\overline{\mathcal{M}}_g/\mathbb{Z}}) = \left(Rf_* L_{X/A}^\vee\right)^\vee[-1]$$

$$\uparrow$$

$$\overline{\mathcal{M}}_g(A) \ni \eta : X \to \mathrm{Spec}\, A$$

Talk #9

Application of Based Loop Spaces

In topology: $\quad pt \subset X$

$$\begin{tikzcd} pt \times^h_X pt \simeq \Omega_{pt} X \arrow[r] \arrow[d] & \mathrm{Path}\, X \arrow[d] \\ pt \arrow[r] & X \end{tikzcd}$$

paths starting at pt

$\Omega_{pt} X$ "group"

$B\Omega_{pt} X$ classifying space

$$\left| pt \leftleftarrows \Omega_{pt} X \Lleftarrow (\Omega_{pt} X)^2 \Lleftarrow \cdots \right|$$

homotopy ~~Cts~~ Čech nerve

$$\left| pt \leftleftarrows pt \times^h_X pt \Lleftarrow pt \times^h_X pt \times^h_X pt \Lleftarrow \cdots \right|$$

$\downarrow$ realises $B\Omega_{pt} X \cong$ base point component of X

X

Variant: $f : Y \to X$

Čech nerve

$$\left| Y \leftleftarrows Y \times^h_X Y \Lleftarrow Y \times^h_X Y \times^h_X Y \Lleftarrow \cdots \right| \simeq \text{unions of components of } X \text{ hit by } Y$$

$\downarrow$

X

Algebra:

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ \| & & \| \\ \mathrm{Spec}\, B & & \mathrm{Spec}\, A \end{array}$$

closed immersion of derived schemes

i.e. i is affine, $H_j(i_* \mathcal{O}_Z)$ coherent $/ H_0 \mathcal{O}_X$ $\forall j$

and $H_0(\mathcal{O}_X) \to H_0 \mathcal{O}_Z$ surjective

Construction

$$\begin{array}{ccccccc} Z & \leftleftarrows & Z \times_X Z & \Lleftarrow & Z \times_X Z \times_X Z & \Lleftarrow & \cdots \\ \mathrm{Spec}\, B & & \mathrm{Spec}\, B \otimes^L_A B & & \mathrm{Spec}\, B \otimes^L_A B \otimes^L_A B & & \end{array}$$

Q: How badly does closed descent fail?

A: $|Z \leftleftarrows Z \times_X Z \leftleftarrows \cdots|(R) = |Z(R) \leftleftarrows Z(R) \times_{X(R)} Z(R) \leftleftarrows \cdots| = \left\{ \begin{array}{l} \text{the union of components of } X(R) \\ \text{such that they are hit by } Z(R) \end{array} \right\}$

$\uparrow$
$\mathrm{Fun}(sCR^{op}, \mathrm{Spaces})$

$$= \left\{ \begin{array}{l} \text{those points } y \in X(R) \text{ that admit a} \\ \text{(scheme-theoretic) factorization through } Z \end{array} \right\}$$

$$\cap$$

$$X^{\wedge}_Z(R) = \{ \text{same for } \underline{\text{set-theoretic}} \text{ factorization} \}$$

Prop: $\mathcal{O}_{\hat{X}_Z} \xrightarrow[\sim]{\text{restriction}} \mathrm{Tot}\{ \mathcal{O}_Z \rightrightarrows \mathcal{O}_{Z \times Z} \mathrel{\substack{\to\\\to\\\to}} \cdots \}$

$$\mathrm{Tot}\,(B \rightrightarrows B \overset{L}{\underset{A}{\otimes}} B \mathrel{\substack{\to\\\to\\\to}} \cdots)$$

Concrete Ex / Exercise

$\mathrm{Spec}\, k \longrightarrow \mathrm{Spec}\, k[x]$

$k \overset{L}{\underset{k[x]}{\otimes}} k \simeq k[B]$ as dg-Hopf alg $\Delta(B) = B \otimes 1 + 1 \otimes B$
$\uparrow$ deg $+1$

k coalgebra ..

Key The totalization computes $R\mathrm{Hom}_{k[\varepsilon_1]}(k,k) \simeq k[\![x]\!]$

Application of Unbased Loop Spaces

Prop: X der. scheme (or der stack) $[\text{eg } X = \mathrm{Spec}\, A]$
K finite simplicial set
Then $X^K: R \longmapsto \mathrm{Map}_{sSet}(K, X(R)) = X(R)^K$ is (representable by) a derived scheme (/der stack) $[X = \mathrm{Spec}(K \otimes A)]$

pf 1: $\mathrm{Map}_{sCR}(K \otimes A, R)_p = \mathrm{Map}_{CR}(K_p \otimes A_p, R_p) = \mathrm{Map}_{CR}(A_p, R_p)^{K_p}$

$$K_p \otimes A_p = \bigotimes_{K_p} A_p$$

$$= \mathrm{Map}_{sSet}(K, \mathrm{Map}_{sCR}(A,R))$$

pf 2: Write K as a finite sequence of cell attaching (retracts?)
$\Downarrow$
X^K = corresponding sequence of pullbacks
$\uparrow$ corr. rel $\otimes$

$\square$

Ex: 1) K discrete $\simeq \{1,\dots,n\}$

$$X^K = X^n \quad [= \operatorname{Spec} R^{\otimes n}]$$

2)
$$\begin{tikzcd} \Sigma K & * \arrow[l] \\ * \arrow[u] \arrow[ur, phantom, "\text{ho pushout}"] & K \arrow[l] \arrow[u] \end{tikzcd} \qquad \left\{ \begin{tikzcd} \Sigma K & * \arrow[l] \\ \operatorname{cone}(K) \arrow[u] \arrow[ur, phantom, "\text{pushout}"] & K \arrow[l] \arrow[u] \end{tikzcd} \right.$$

$$\begin{tikzcd} X^{\Sigma K} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{ho }\lrcorner"] & X \arrow[d] \\ X \arrow[r] & X^K \end{tikzcd}$$

$\rightarrow \left[X^{\Sigma K} = \operatorname{Spec}\; R \overset{L}{\underset{K\otimes R}{\otimes}} R \right]$

$\simeq \operatorname{Spec}\; (\operatorname{cone}(K)\otimes R) \underset{K\otimes R}{\otimes} R$

Rmk $\nearrow$ Bar construction for $\overset{L}{\otimes}$

3) $K = S^1 = \Sigma S^0$

$$\begin{tikzcd} X^{S^1} \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & X^2 \end{tikzcd}$$

$\mathcal{O}_{X^{S^1}}$ in affine case

Funny cone thing:

$| R\otimes R \leftleftarrows R\otimes R^2 \otimes R \Lleftarrow R\otimes (R^{\otimes 2})^{\otimes 2} \otimes R \Lleftarrow \cdots)$

Cone(S^0): $\Lambda \hookleftarrow \cdot \;\; \cdot$ S^0

$(\Delta^1_\bullet \otimes R) \underset{S^0\otimes R}{\otimes} R$ gives

$| R \leftleftarrows R\otimes R \Lleftarrow R\otimes R\otimes R \Lleftarrow \cdots |$ cyclic bar complex

Prop^n: char 0 HKR-type R, $X = \operatorname{Spec} R$

$$\mathcal{O}_{LX} \xrightarrow{\sim \text{quis}} \operatorname{Sym}_R (\mathbb{L}_R[1])$$

$| R \leftleftarrows R\otimes R \Lleftarrow R\otimes R \otimes R \Lleftarrow \cdots |$ $\qquad \psi$

$r_0 \otimes r_1 \otimes \cdots \otimes r_p \longmapsto r_0\, dr_1\, dr_2 \cdots dr_p / p!$

$\circlearrowleft$ Connes B $\qquad \circlearrowleft d_{dR}$

Regard these as complexes

Cpx w/ S^1-action $\simeq H_* S^1$-mod $= k[B_{+1}]$

The HKR-type map is compatible with it.

$\exists$ relation between $QC(LX)^{S^1}$ and D-modules.

Ex of Loops

1) X classical scheme not ~~[illegible]~~ $\amalg$pt $\Rightarrow LX$ not classical

2)
$$\begin{tikzcd} L(BG) \arrow[r] \arrow[d] & BG \arrow[d] \\ BG \arrow[r] & BG^2 \end{tikzcd}$$
$\rightsquigarrow L(BG) \simeq G^{ad}/G$

3) Y/G affine

when is $L(Y/G)$ classical again?

$\rightsquigarrow$ A: $Y \circlearrowleft$ w/ finitely many orbits.