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Simplicial Commutative Rings II

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I. Examples
- attaching cells
- Symmetric algebras
- $k \oplus k[t]$

II. Properties of Rings & Modules

Notation $X, Y \in sCR,\ Mod_A,\ K \in sSet$

$$(K \otimes X)_n = \coprod_{K_n} X_n$$

$$(\underline{Hom}(X,Y))_n = Hom(\Delta^n \otimes X, Y)$$

$$\mathbb{R}\underline{Hom}(X,Y) = \underline{Hom}(Q(X), Y)$$

$Q(X) \longrightarrow X$ cofibrant replacement

$M, N \in Mod_A$, $* \in K$ a basepoint

Define $K \wedge M$ by pushout

$$\begin{tikzcd} * \otimes M \arrow[r] \arrow[d] & K \otimes M \arrow[d] \\ 0 \arrow[r] & K \wedge M \end{tikzcd}$$

$$M[n] = \Sigma^n M = S^n \wedge M \quad \text{where } S^n = \Delta^n / \partial\Delta^n$$

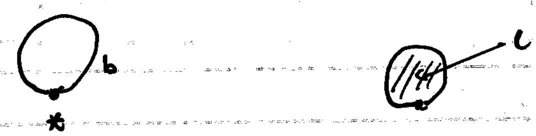
$$(M \otimes N)_n = M_n \otimes_{A_n} N_n \rightsquigarrow M \overset{L}{\otimes} N$$

Attaching Cells

Question - How we construct simplicial resolutions for modules $R^J \longrightarrow R^I \longrightarrow M$

We look at
$$\begin{array}{ccc} S^{n-1} & \hookrightarrow & D^n \\ \| & & \| \\ \Delta^{n-1}/\partial\Delta^{n-1} & & \Delta^n/\Lambda^n_0 \end{array}$$

$$|S^{n-1}| \longrightarrow |D^n|$$



$$d_0(b) = \cdots = d_{n-1}(b) = *$$

$$d_1(c) = \cdots = d_n(c) = * \text{ and } d_0(c) = b$$

<u>Notation</u> $\mathbb{Z}[K] = K \otimes \mathbb{Z}[x]$

i.e. $(\mathbb{Z}[K])_n = \mathbb{Z}[K_n]$, $\mathbb{Z}[(K,*)]_n = \mathbb{Z}[K_n]/(*)$

$$\begin{tikzcd}
A \arrow[r] & A' \\
\mathbb{Z}[(S^{n-1},*)] \arrow[u] \arrow[r] \arrow[ur, phantom, "\urcorner"] & \mathbb{Z}[(D^n,*)] \arrow[u]
\end{tikzcd}$$

$$\mathrm{Hom}_{sCR}(\mathbb{Z}[(S^{n-1},*)], A) = \mathrm{Hom}_{sSet_*}((S^{n-1},*),(A,0))$$

$$[\mathbb{Z}[(S^{n-1},*)], A]_{sCR} = [(S^{n-1},*),(A,0)]_{sSet_*}$$
$$= \pi_{n-1}(A)$$

Fix $(S^{n-1},*) \xrightarrow{f} (A,0)$
representing $\omega \in \pi_{n-1}(A)$

$$(A')_j = A_j[(D^n)_j] \Big/ (x = f(x))_{x \in (S^{n-1})_j}$$

$A'_j = A_j$ for $j < n$

$A'_n = A_n[c]$

$\rightsquigarrow \pi_j(A') = \pi_j(A)$

$$0 \to A_{n-1} \cdot w \to \pi_{n-1}(A) \to \pi_{n-1}(A') \to 0$$

$C \in N(A')_n \quad d(C) = w$

$A' = A[x \mid \partial(x) = w]$

Example $R \to R/(r)$

$R[x, \partial(x) = r]$

$\boxed{\text{Symmetric Algebras}}$

For $A \in sCR$, define $\mathrm{Sym}_A : \mathrm{Mod}_A \rightleftarrows sCR_A$: Forget (right adjoint) (Forgetful Functor)

by $(\mathrm{Sym}_A(M))_n = \mathrm{Sym}_{A_n}(M_n)$

gives rise to right derived functor $\mathbb{L}\mathrm{Sym}_A$

Example
- $\mathbb{L}\mathrm{Sym}^n_A(M[1]) \simeq (\mathbb{L}\Lambda^n_A(M))[n]$
- $\mathbb{L}\mathrm{Sym}^n_A(M[2]) \simeq (\mathbb{L}\Gamma^n_A(M))[2n]$

If A is a $\mathbb{Q}$-algebra then $\mathrm{Sym}^n = \Gamma^n$

Example What is $\mathrm{Sym}\,\mathbb{Z}[n-1]$?

$$\begin{tikzcd} * \otimes \mathbb{Z} \arrow[r] \arrow[d] & S^{n-1} \otimes \mathbb{Z}^n \arrow[d] \\ 0 \arrow[r] & \mathbb{Z}[n-1] \end{tikzcd} \quad \overset{\mathrm{Sym}_{\mathbb{Z}}}{\rightsquigarrow} \quad \begin{tikzcd} \mathbb{Z}[*] \arrow[r] \arrow[d] & \mathbb{Z}[S^{n-1}] \arrow[d] \\ \mathbb{Z} \arrow[r] & \mathrm{Sym}\,\mathbb{Z}[n-1] \arrow[ul, phantom, "\lrcorner"] \end{tikzcd}$$

$\rightsquigarrow \mathrm{Sym}\,\mathbb{Z}[n-1] \simeq \mathbb{Z}[(S^{n-1}, *)]$

We can also discretize attaching cells by

$$A \overset{\mathbb{L}}{\otimes}_{\mathrm{Sym}\,\mathbb{Z}[n-1]} \mathbb{Z}$$

$$A[X, \partial(X)=0] = \mathrm{Sym}_A A[n]$$

$$\boxed{k \oplus k[i]}$$

For $A \in sCR$, $M \in \mathrm{Mod}_A$.

Form "trivial extension" $A \oplus M$ by

$$(A\oplus M)_n = A_n \oplus M_n$$

$$(a,m)(a',m') = (aa', am' + a'm)$$

For k a field, $k \oplus k[i]$ "higher dual numbers"

$$k \oplus k[0] = k[\varepsilon]/(\varepsilon^2)$$

$\exists$ homotopy pullback

$$(*) \quad \begin{tikzcd} k[\varepsilon]/(\varepsilon^{n+1}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & k[\varepsilon]/(\varepsilon^n) \arrow[d] \\ k \arrow[r] & k \oplus k[1] \end{tikzcd}$$

$$\pi_0\left(\text{"deformation over } (-)\text{"}\right)$$

$$\begin{tikzcd} \left(\begin{matrix}\text{deformations over}\\ k[\varepsilon]/\varepsilon^{n+1}\end{matrix}\right)/\sim \arrow[r] & \left(\begin{matrix}\text{deformations}\\ \text{over } k[\varepsilon]/\varepsilon^{n}\end{matrix}\right)/\sim \arrow[d, "\text{obstruction map}"] \\ \text{Obstruction classes} \arrow[r, phantom, "="] & \left(\begin{matrix}\text{deformations over}\\ k\oplus k[1]\end{matrix}\right)/\sim \end{tikzcd}$$

How do we produce $(*)$?

- Char $K = 0 \rightsquigarrow$ we can produce it using CDGAs
- In general, replace $K[\varepsilon]/\varepsilon^n$ by $K[\varepsilon][x \mid \partial(x) = \varepsilon^n]$

to get diagram

$$\begin{tikzcd} & K[\varepsilon][x \mid \partial(x) = \varepsilon^n] \arrow[d] \\ K \arrow[r] & K \oplus K[1] \end{tikzcd} \qquad \varepsilon \mapsto 0 \quad x \mapsto 1 \in K[1]$$

$\boxed{\text{Properties}}$ Flatness, Étaleness, Smoothness

Flatness

Prop./Def. If $A \in sCR$, $M \in Mod_A$ is flat if the following equiv. conditions hold:

(1) $\pi_0(M)$ is flat $\pi_0(A)$-module

$\pi_n(R) \otimes_{\pi_0(R)} \pi_0(M) \xrightarrow{\sim} \pi_n(M)$ for all n

(2) $M \overset{\mathbb{L}}{\otimes} \_$ commutes with finite homotopy limits

(3) M is a filtered colimit of finite free A-modules

(2') $M \overset{\mathbb{L}}{\otimes} \_$ commutes with Ω $\left(\text{i.e. } \begin{tikzcd} \Omega M \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & 0 \arrow[d] \\ 0 \arrow[r] & M \end{tikzcd} \right)$

(2'') For N discrete A-module,

$M \overset{\mathbb{L}}{\otimes} N$ is discrete étale/smooth

Definition $A \to B$ in sCR is étale (resp. smooth)

if it is flat and $\pi_0(A) \longrightarrow \pi_0(B)$

is étale (resp. smooth)

Finiteness Conditions

Definition $A \to B$ in sCR. We say B is a

(1) • finitely presented A-alg if it can be obtained by attaching finitely many cells

(2) • locally finitely presented A-alg if it is a retract of a fin. presented A-alg.

(3) • almost fin. presented if $\forall n$, $\exists B_n$ a fin. presented A-alg and $f_n : B_n \to B$ such that

$$\pi_i(B_n) \xrightarrow{\sim} \pi_i(B) \text{ for } i \leq n$$

Analogous conditions for Modules

(1) finitely presented
(2) Perfect
(3) almost perfect

Example R a discrete ring, $M \in \mathrm{Mod}_R$ fin. presented (in ordinary commut alg. sense)

• M perfect $\Rightarrow$ $\mathrm{Pd}_R M < \infty$

• locally fin. presented is "Compact"
(i.e. $\mathbb{R}\underline{\mathrm{Hom}}(B, -)$ commutes with filtered Colims