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Theory of Computation

Kun-Ta Chuang
Department of Computer Science and Information Engineering
National Cheng Kung University

Outline

1. Regular Expressions (RE)
2. Connection Between REs and Regular Languages
3. Regular Grammars

Specifying Language

How do we specify languages?

- If language is finite, you can list all of its strings.
  - $L = \{a, aa, aba, aca\}$
- Descriptive:
  - $L = \{x \mid n_a(x) = n_b(x)\}$
- Using basic Language operations
  - $L = \{aa, ab\}^* \cup \{b\}\{bb\}^*$
- Regular languages are described using the last method

Regular Expressions

Regular expressions describe regular languages and the notation involves a combination of:

- Strings of symbols from some alphabet Σ
- Parentheses ()
- Operators $+, \cdot, *$

Regular Expressions

Important thing to remember

- A regular expression is not a language
- A regular expression is used to describe a language.
- It is incorrect to say that for a language L, L = (a + b + c)*
- But it's okay to say that L is described by (a + b + c)*

Regular Expressions

All finite languages can be described by regular expressions

Example: $(a+b\cdot c)^*$ ⟺ {{a} U {bc}}*

describes the language

$$\{a,bc\}^* = \{\lambda, a, bc, aa, abc, bca, ...\}$$

Definition 3.1

Let Σ be a given alphabet. Then

1. ϕ, λ, and $a \in \Sigma$ are all regular expressions. These are called primitive regular expressions.
2. If r_1 and r_2 are regular expressions, so are r_1+r_2, $r_1 \cdot r_2$, r_1^* and (r_1).
3. A string is a regular expression ***iff*** it can be derived from the primitive regular expressions by a finite number of applications of the rules in (2).

Example 3.1

A regular expression: $(a+b \cdot c)^* \cdot (c+\varnothing)$

Not a regular expression: (a + b +)

a^n

a^+

Languages of Regular Expressions

$L(r)$: language of regular expression r

Example

$$L((a+b\cdot c)^*)=\{\lambda,a,bc,aa,abc,bca,...\}$$

Definition 3.2

- For primitive regular expressions:

$$L(\varnothing) = \varnothing \qquad (1)$$

$$L(\lambda) = \{\lambda\} \qquad (2)$$

$$L(a) = \{a\} \qquad (3)$$

Definition (continued)

For regular expressions r_1 and r_2

$$L(r_1 + r_2) = L(r_1) \cup L(r_2) \qquad (4)$$

$$L(r_1 \cdot r_2) = L(r_1)\, L(r_2) \qquad (5)$$

$$L(r_1 *) = (L(r_1))* \qquad (6)$$

$$L((r_1)) = L(r_1) \qquad (7)$$

Example 3.2

Regular expression: $(a+b)\cdot a*$

$$
\begin{aligned}
L((a+b)\cdot a*) &= L((a+b))\,L(a*) \\
&= L(a+b)\,L(a*) \\
&= (L(a)\cup L(b))\,(L(a))* \\
&= (\{a\}\cup\{b\})\,(\{a\})* \\
&= \{a,b\}\,\{\lambda,a,aa,aaa,\ldots\} \\
&= \{a,aa,aaa,\ldots,b,ba,baa,\ldots\}
\end{aligned}
$$

Priority of Operators

- Regular expression: $r = a \cdot b + c$

  $r_1 = a \cdot b \quad r_2 = c$ or $r_1 = a \quad r_2 = b + c$

  $L(r) = \{ab, c\} \neq \{ab, ac\}$

- Star closure (*) precedes concatenation (·) precedes union (+)

Example 3.3

$\Sigma = \{a,b\}$

- Regular expression $r = \underbrace{(a+b)^*}_{\text{Stands for any string of a's and b's}}(a+bb)$

$$L(r) = \{a, bb, aa, abb, ba, bbb, \ldots\}$$

L(r) is the set of all strings on {a,b}, terminated by either an a or a bb

Example 3.4

- Regular expression $r = (aa)^*(bb)^*b$

$$L(r) = \{a^{2n}b^{2m+1} : \quad n, m \geq 0\}$$

L(r) is the set of all strings with an even number of a's followed by an odd number of b's

Example 3.5

- For $\Sigma = \{0, 1\}$, give a regular expression r such that

$L(r) = \{ w \in \Sigma^* :$ w has at least one pair of consecutive 0 $\}$

00

Example 3.6

L(r) = { all strings with no pairs of consecutive 0s }

- Regular expression $\boxed{r = (1 + 01)^*(0 + \lambda)}$

$$r = (1^* 011^*)^* \underline{\qquad} \underline{\qquad} \underline{\qquad}$$

Add 1 immediately after a 0

String ending in 0

String with all 1's

There are an unlimited number of REs for any given language!

Equivalent Regular Expressions

Definition:

Regular expressions r_1 and r_2

are **equivalent** if $L(r_1) = L(r_2)$

Example

L = { all strings without two consecutive 0 }

$$r_1 = (1 + 01)^*(0 + \lambda)$$

$$r_2 = (1^* 0 1 1^*)^*(0 + \lambda) + 1^*(0 + \lambda)$$

$L(r_1) = L(r_2) = L$ $\Rightarrow$ r_1 and r_2 are equivalent regular expressions

More Examples

- $L_1 = \{a, aa, aba, aca\}$
- $L_1 = \{a\} \cup \{aa\} \cup \{aba\} \cup \{aca\}$
- Regular expression describing L_1:

  $(a + aa + aba + aca)$

More Examples

- $L_2 = \{x \in \{0,1\}^* \mid |x| \text{ is even}\}$
- $L_2 = \{00, 01, 10, 11\}^*$
- Regular expressions describing L_2:

$(00 + 01 + 10 + 11)^*$

$((0 + 1)(0 + 1))^*$

More Examples

- $L_3 = \{x \in \{0,1\}^* \mid x$ does not end in 01 $\}$

  If x does not end in 01, then either

  x ends in 00, 10, or 11
- A regular expression that describes L_3 is:

  $(0 + 1)^*(00 + 10 + 11)$

More Examples

- $L_4 = \{x \in \{0,1\}^* \mid x$ contains an odd number of 0s $\}$

  Express $x = yz$

  y is a string of the form $y=1^i01^j$

  In z, there must be an even number of 0's

  $z = (01^k01^m)^*$

- A regular expression that describes L_4 is:

  $(1^*01^*)(01^*01^*)^*$

Short Quiz

- Give regular expressions for the following language on Σ= {a, b, c}.
  - All strings containing exactly one a

r = (b+c)*a(b+c)*

Outline

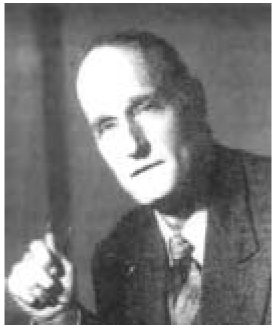
1. Regular Expressions (RE)
2. Connection Between REs and Regular Languages
3. Regular Grammars

Theorem

$$\left\{\begin{array}{l}\text{Languages} \\ \text{Described by} \\ \text{Regular Expressions}\end{array}\right\} = \left\{\begin{array}{l}\text{Regular} \\ \text{Languages}\end{array}\right\}$$

For every regular language there is a regular expression
For every regular expression there is a regular language

Kleene Theorem:
Regular expressions and Finite Automata are equivalent (w.r.t. the languages they describe/accept)



Theorem - Part 1

$$\left\{\begin{array}{l}\text{Languages}\\ \text{Described by}\\ \text{Regular Expressions}\end{array}\right\} \subseteq \left\{\begin{array}{l}\text{Regular}\\ \text{Languages}\end{array}\right\}$$

1. For any regular expression r the language $L(r)$ is regular

■ Theorem 3.1

Theorem - Part 2

$$\left\{\begin{array}{l}\text{Languages} \\ \text{Described by} \\ \text{Regular Expressions}\end{array}\right\} \supseteq \left\{\begin{array}{l}\text{Regular} \\ \text{Languages}\end{array}\right\}$$

2. For any regular language L there is a regular expression r with $L(r) = L$

■Theorem 3.2

Proof - Part 1 $\left\{ \begin{array}{l} \text{Languages} \\ \text{Described by} \\ \text{Regular Expressions} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$

1. For any regular expression r
   the language $L(r)$ is regular

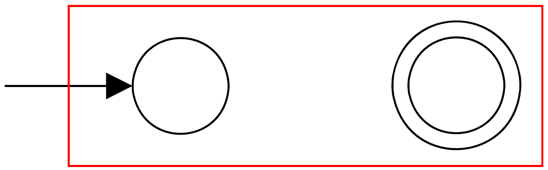
If we have any regular expression r,
we can construct an NFA(DFA) that accepts L(r)

Proof by induction on the size of r

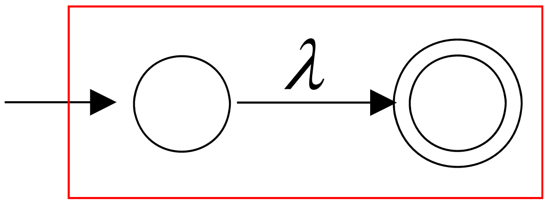
Induction Basis

- Primitive Regular Expressions: $\emptyset, \quad \lambda, \quad \alpha$

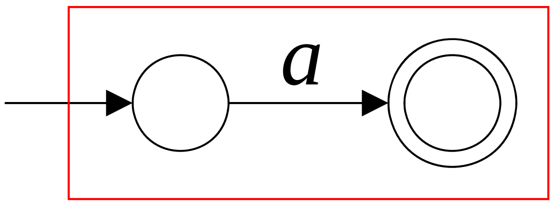
NFAs



$$L(M_1) = \emptyset = L(\emptyset)$$



$$L(M_2) = \{\lambda\} = L(\lambda)$$



$$L(M_3) = \{a\} = L(a)$$

regular languages

Inductive Hypothesis

Assume

for regular expressions r_1 and r_2

that

$L(r_1)$ and $L(r_2)$ are regular languages

Inductive Step

∵ REs are derived from these four rules:

We will prove:

$$\left.\begin{array}{l} L(r_1 + r_2) \\ \\ L(r_1 \cdot r_2) \\ \\ L(r_1 *) \\ \\ L((r_1)) \end{array}\right\} \text{Are regular Languages}$$

- By definition of regular expressions:

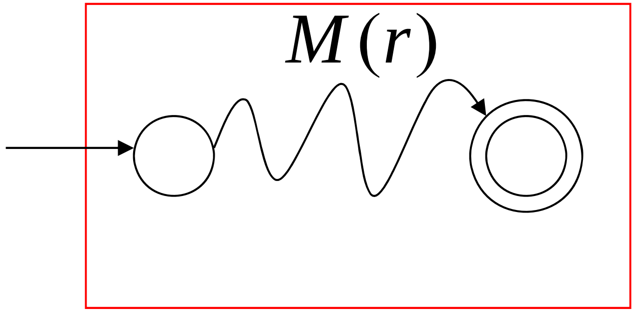
$$L(r_1 + r_2) = L(r_1) \cup L(r_2)$$

$$L(r_1 \cdot r_2) = L(r_1)\, L(r_2)$$

$$L(r_1 *) = (L(r_1))*$$

$$L((r_1)) = L(r_1)$$

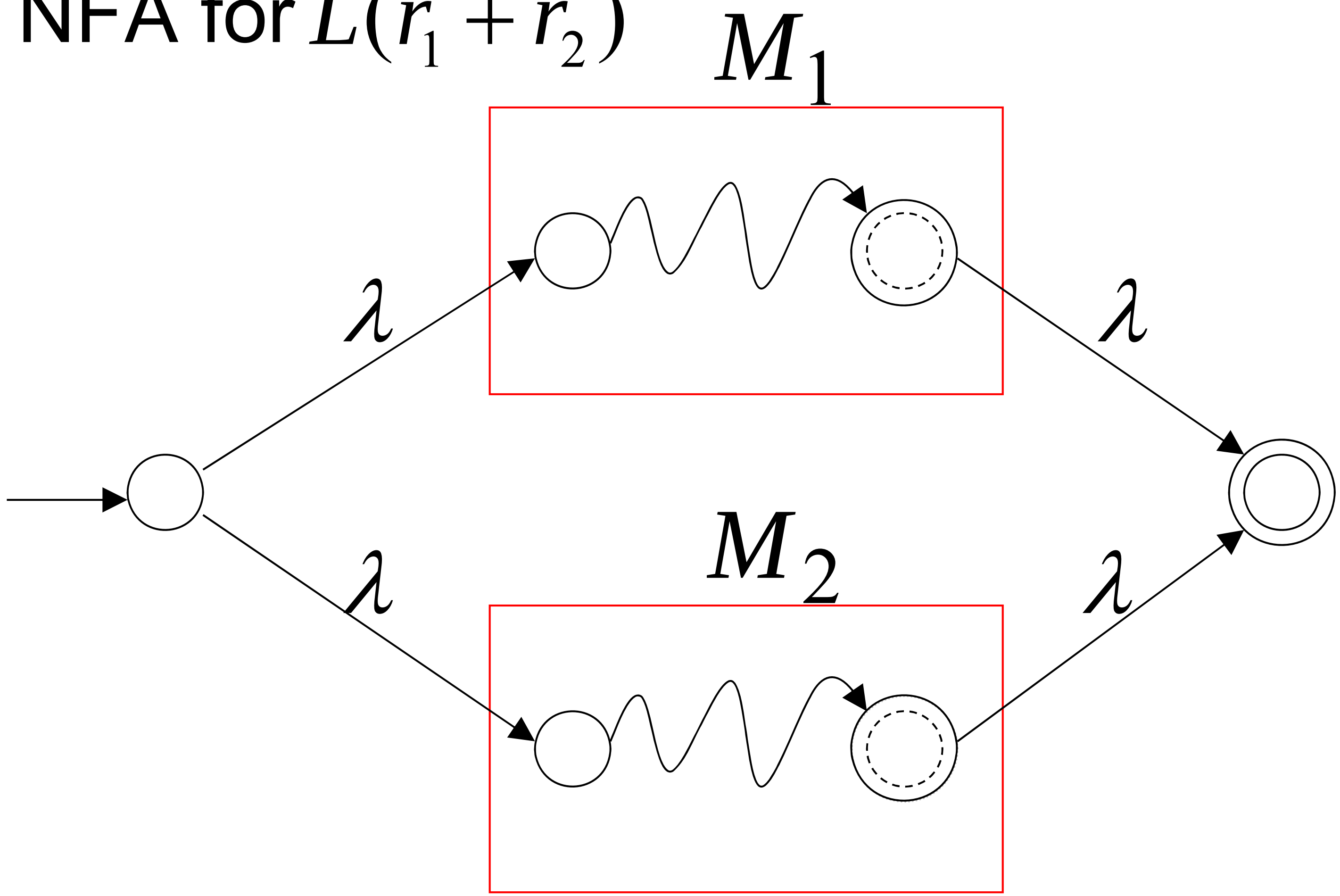
Schematic representation of an NFA (M(r)) accepting L(r)



We can claim that for every NFA there is only one final state (by exercise 7, section 2.3)

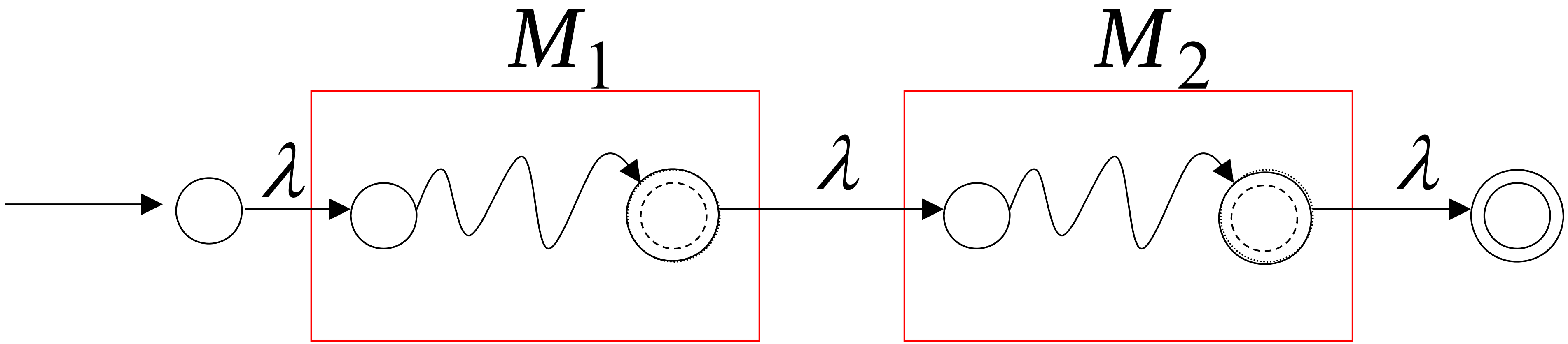
Union

- NFA for $L(r_1 + r_2)$



Concatenation

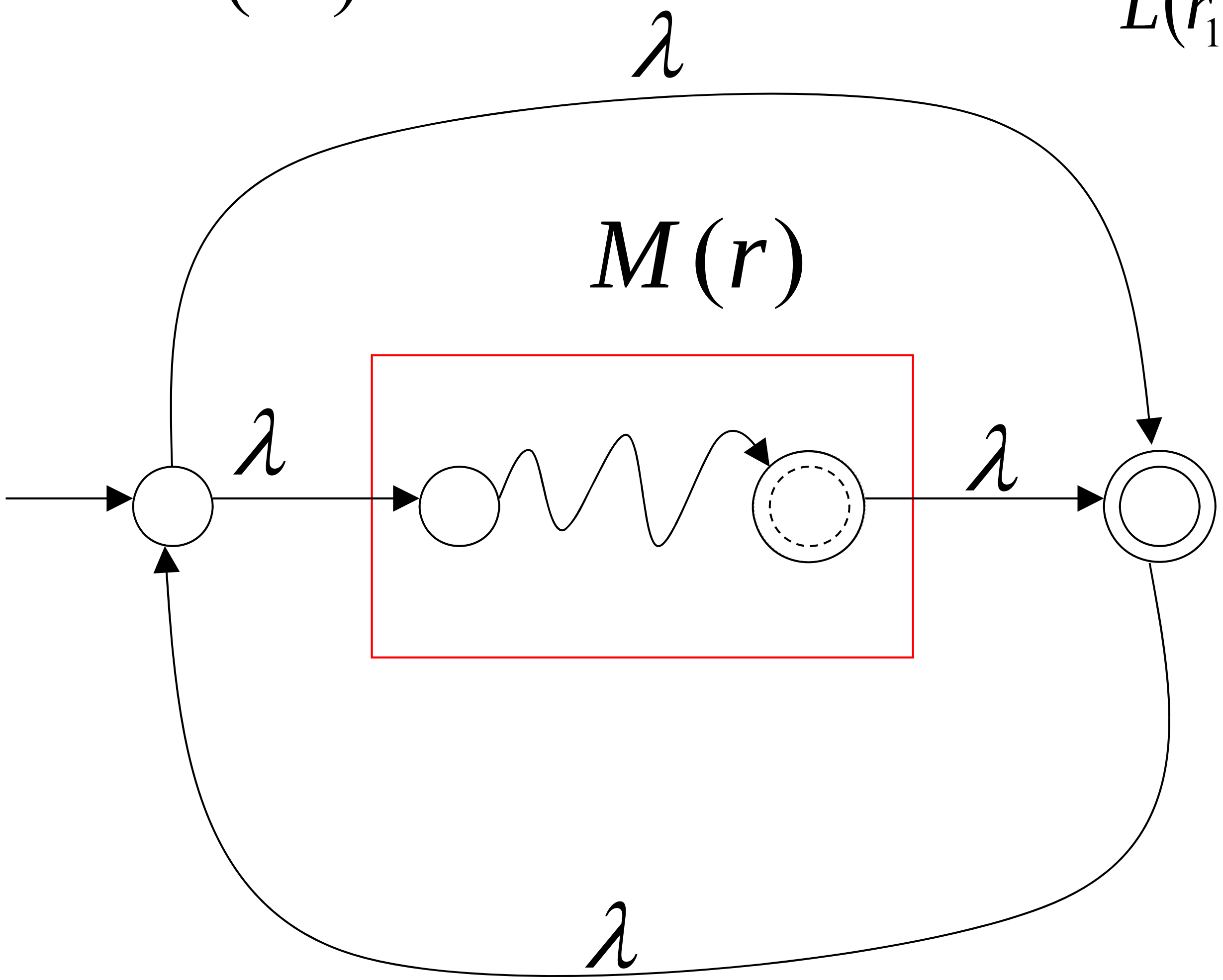
- NFA for $L(r_1 r_2)$



Star Operation

- NFA for $L(r^*)$

$$L(r_1{}^*) = (L(r_1))^*$$



By inductive hypothesis we know:

$L(r_1)$ and $L(r_2)$ are regular languages

We also know:

Regular languages are closed under:

Union	$L(r_1) \cup L(r_2)$
Concatenation	$L(r_1)\,L(r_2)$
Star	$(L(r_1))^*$

Therefore:

$$L(r_1 + r_2) = L(r_1) \cup L(r_2)$$

$$L(r_1 \cdot r_2) = L(r_1)\, L(r_2)$$

$$L(r_1 *) = (L(r_1))*$$

Are regular languages

And trivially:

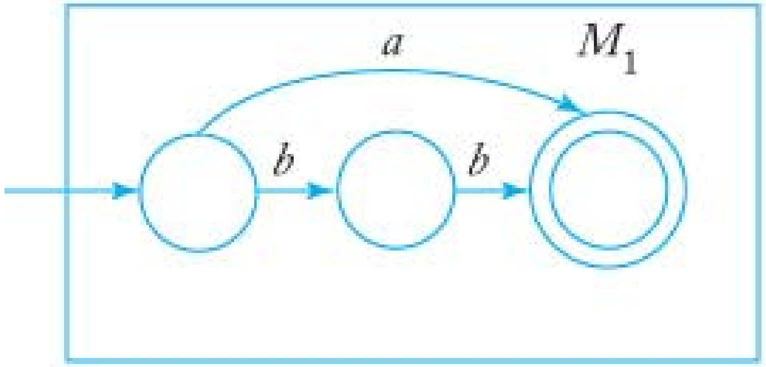
$L((r_1))$ is a regular language

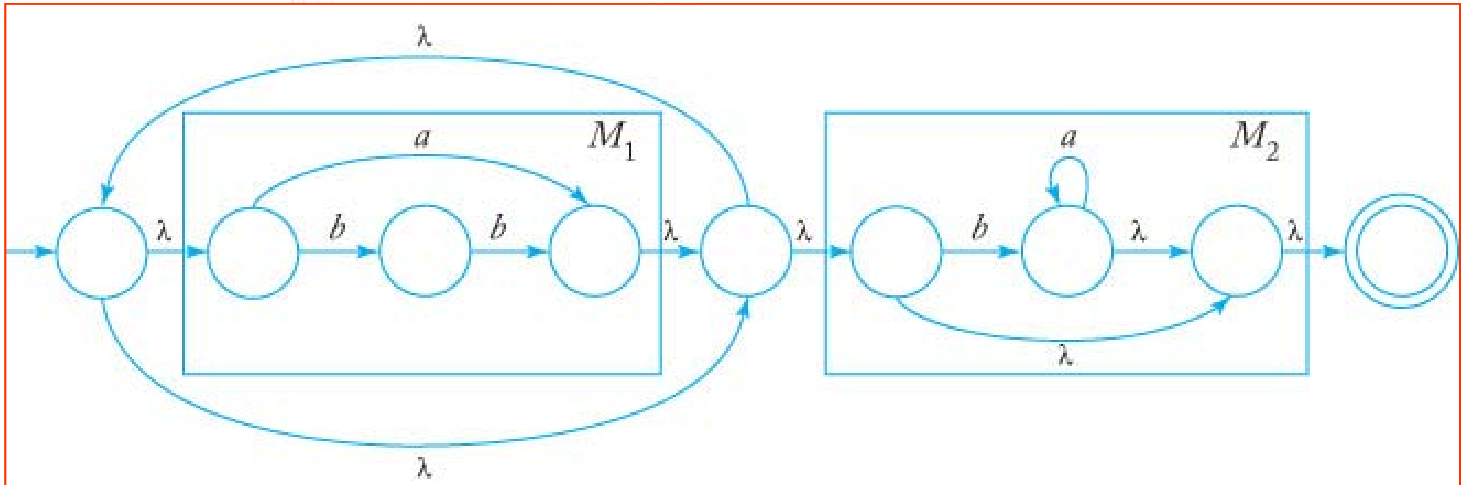
$\therefore$ For any regular expression r
the language $L(r)$ is regular

Example 3.7

- Find an NFA that accepts L(r), where

$$r = (a + bb)^*(ba^* + \lambda)$$





Proof – Part 2 $\left\{\begin{array}{l}\text{Languages}\\ \text{Described by}\\ \text{Regular Expressions}\end{array}\right\} \supseteq \left\{\begin{array}{l}\text{Regular}\\ \text{Languages}\end{array}\right\}$

2. For any regular language L there is a regular expression r with $L(r) = L$

⇩

Since any regular language has an associated NFA and hence a transition graph,
all we need to do is to find a regular expression capable of generating the labels of all the walks from q_0 to any final state.

⇩

Proof by construction of regular expression

Generalized Transition Graphs (GTG)

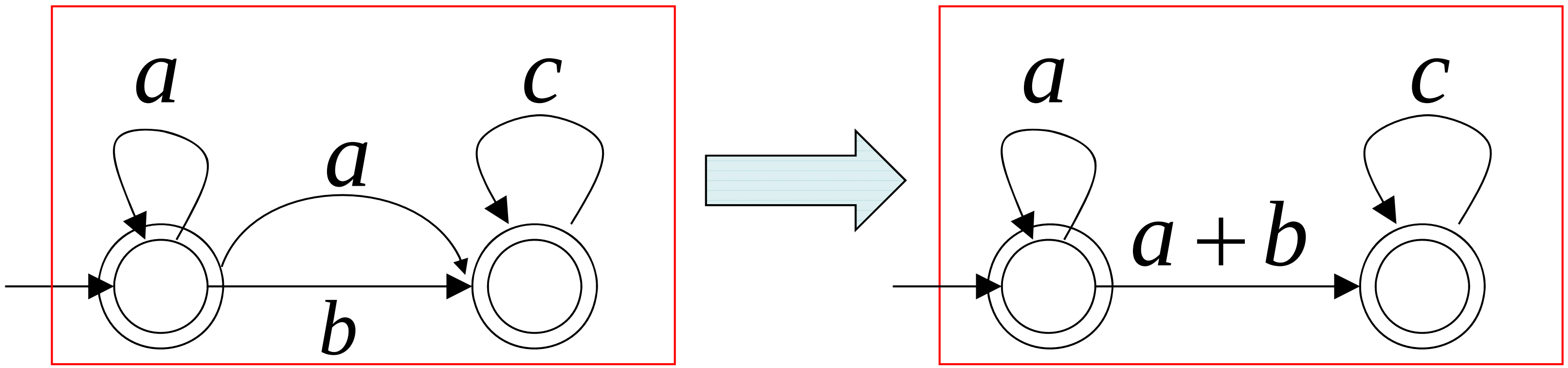
From M construct the equivalent

Generalized Transition Graph

in which transition labels are regular expressions

Example: M

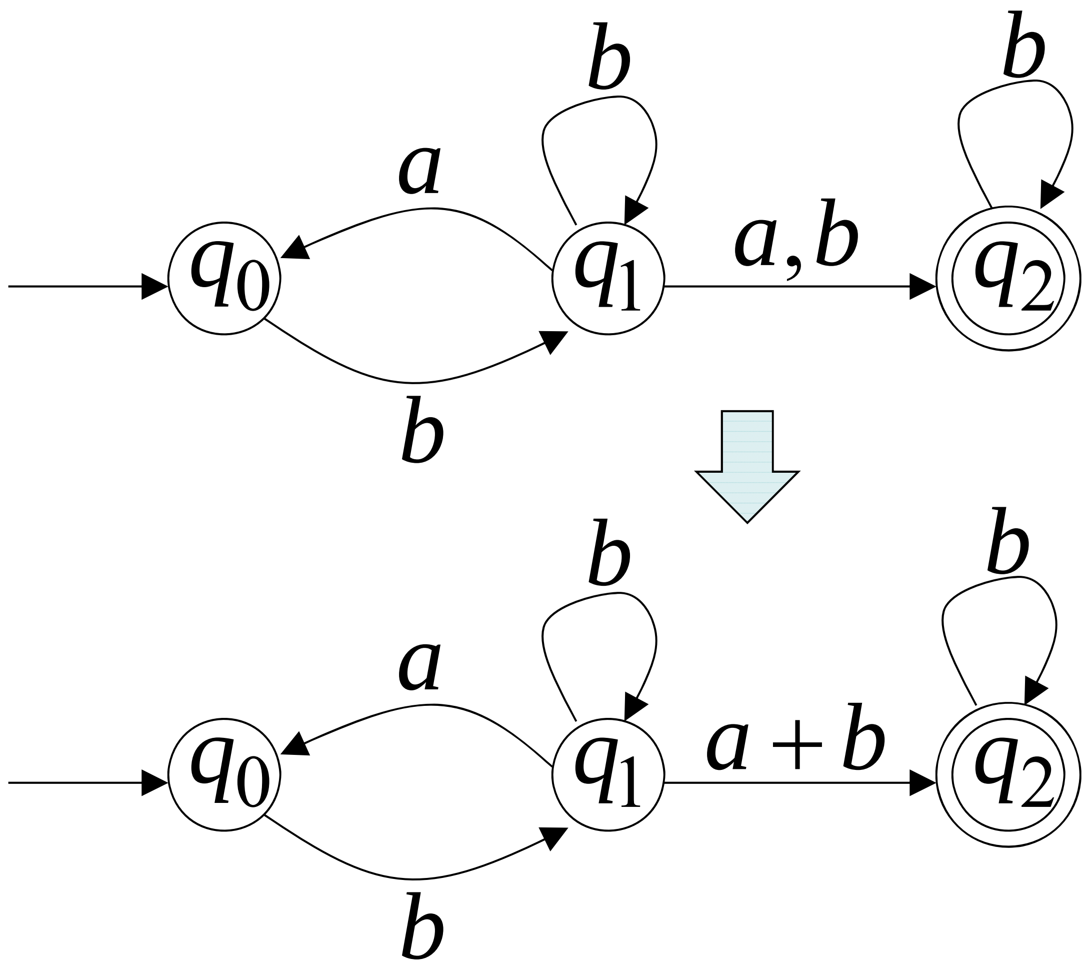
L(a* + a*(a + b)c*)



Example 3.8

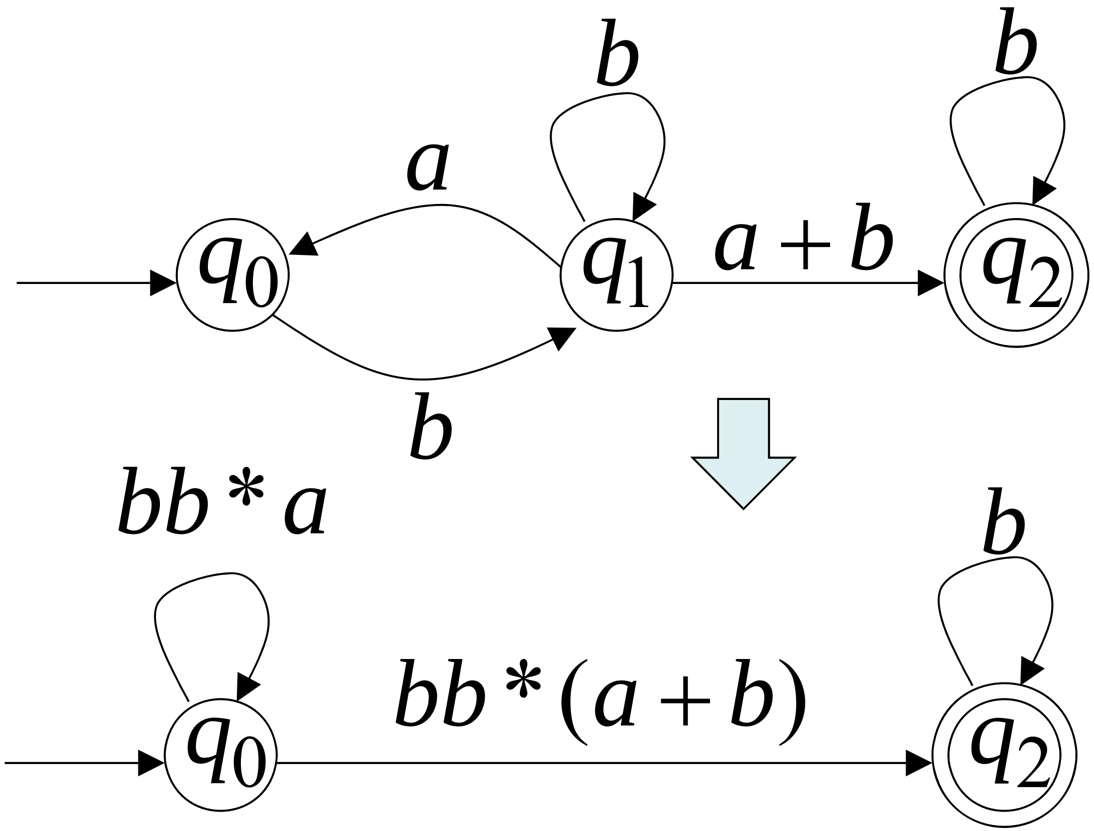
GTG may have many states

Enumerating all walks is time-consuming



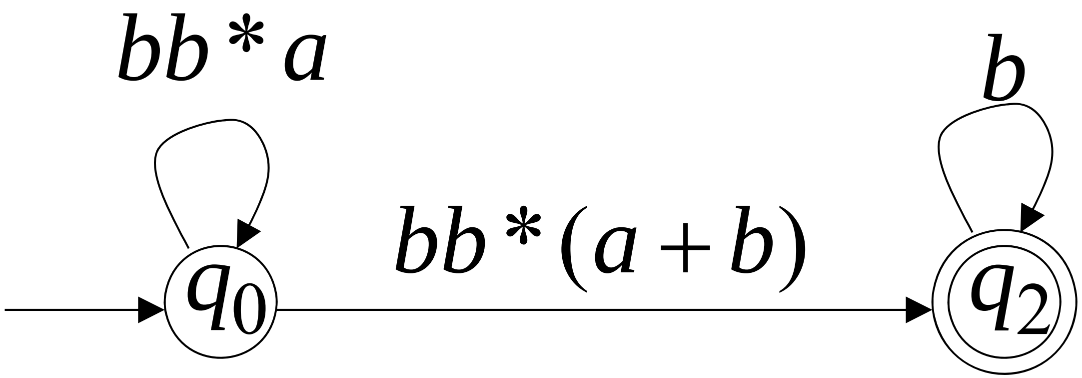
Reducing the states:

Ex. reduce q_1



Simple two-state GTG

Resulting Regular Expression:

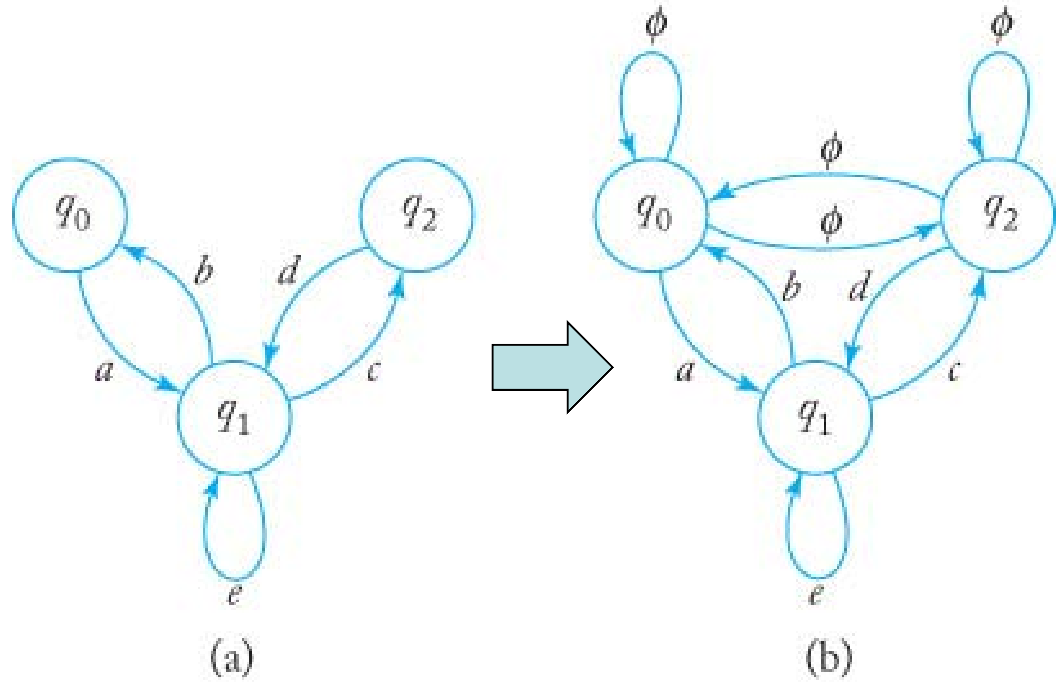


$$r = (bb^*a)^*bb^*(a+b)b^*$$

$$L(r) = L(M) = L$$

Complete GTG

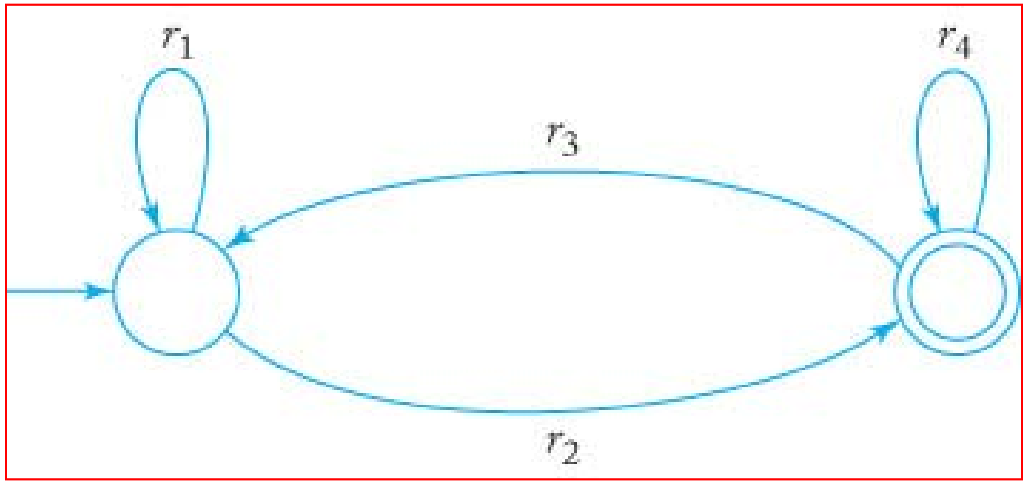
GTG



Complete GTG

- If a GTG, after conversion from an NFA, has some edges missing, we put them in and label them with ϕ
- A complete GTG with |V| vertices has exactly $|V|^2$ edges

Example 3.9



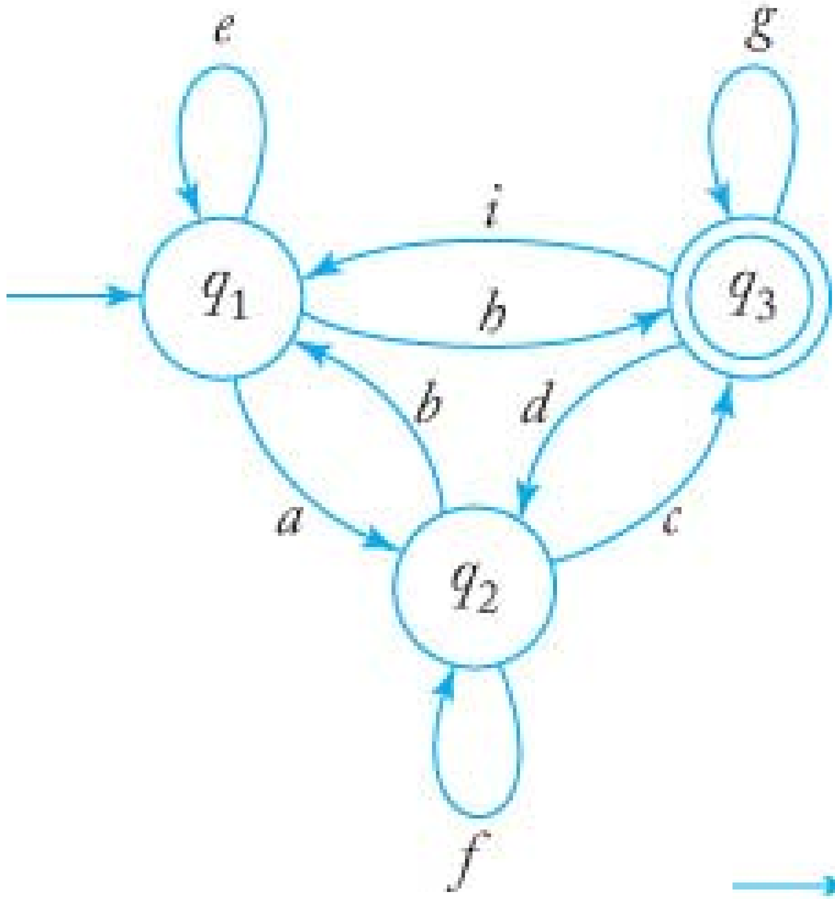
RE?

$r = r_1{}^*r_2(r_4 + r_3r_1{}^*r_2)^*$

How about a GTG with more than two states?

We can find an equivalent graph by removing one state at a time

Example 3.10



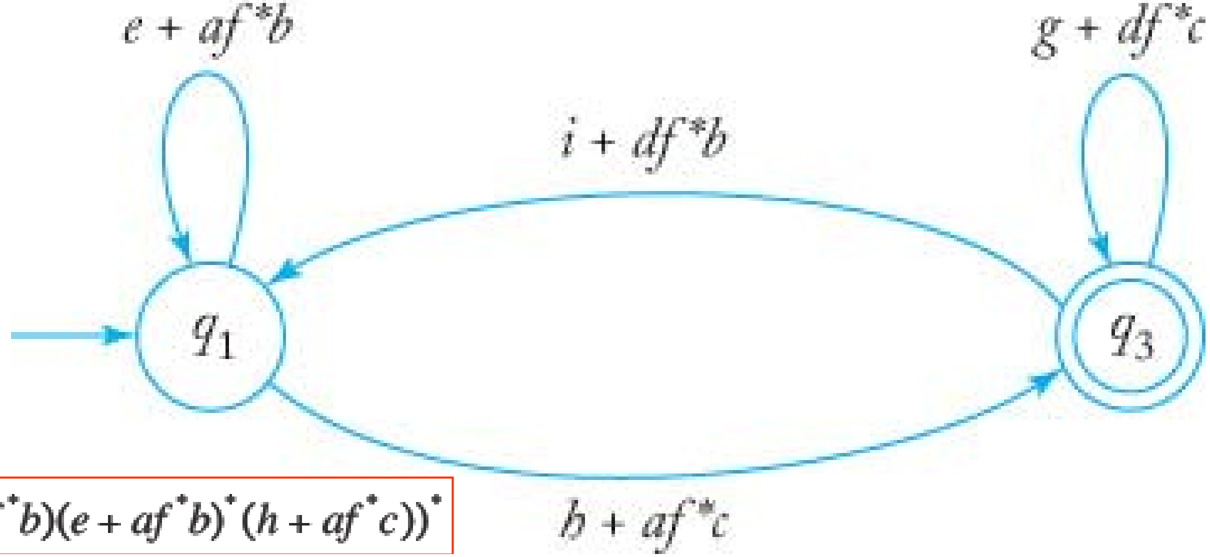
To remove q_2, we create edges as follows:

$\overrightarrow{q_1q_1} \rightarrow$

$\overrightarrow{q_1q_3} \rightarrow$

$\overrightarrow{q_3q_1} \rightarrow$

$\overrightarrow{q_3q_3} \rightarrow$



$r = r_1{}^*r_2(r_4 + r_3r_1{}^*r_2)^*$

$$\boxed{(e+af^*b)^*(h+af^*c)((g+df^*c)+(i+df^*b)(e+af^*b)^*(h+af^*c))^*}$$

NFA → RE

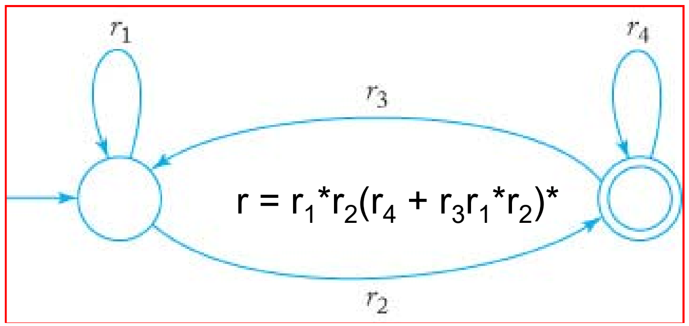
1. Convert the NFA (with single final state) into a complete GTG. Let r_{ij} stand for the label of the edge from q_i to q_j.
2. If the GTG has only two states with $q_i \in q_0$ and $q_j \in F$, as its associated RE is:

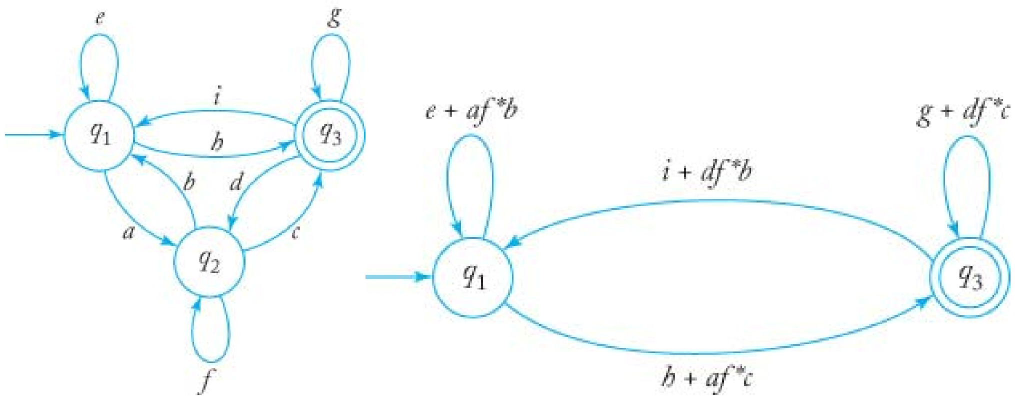
$$r = r_{ii}^{*} r_{ij} (r_{jj} + r_{ji} r_{ii}^{*} r_{ij})^{*}$$

3. If the GTG has three states with $q_i \in q_0$, $q_j \in F$, and $q_k \in Q$, introduce new edges, labeled:

$$r_{pq} + r_{pk} r_{kk}^{*} r_{kq}$$

for p = i, j, q = i, j. When this is done, remove vertex q_k and its associated edges.





NFA → RE

4. If the GTG has four or more states, pick a state q_k to be removed. Apply rule 3 for all pairs of states (q_i, q_j), $i \neq k$, $j \neq k$. At each step apply the simplifying rules

   $$r + \phi = r, \quad r \cdot \phi = \phi, \quad \phi^* = \lambda$$

   wherever possible. When this is done, remove state q_k.

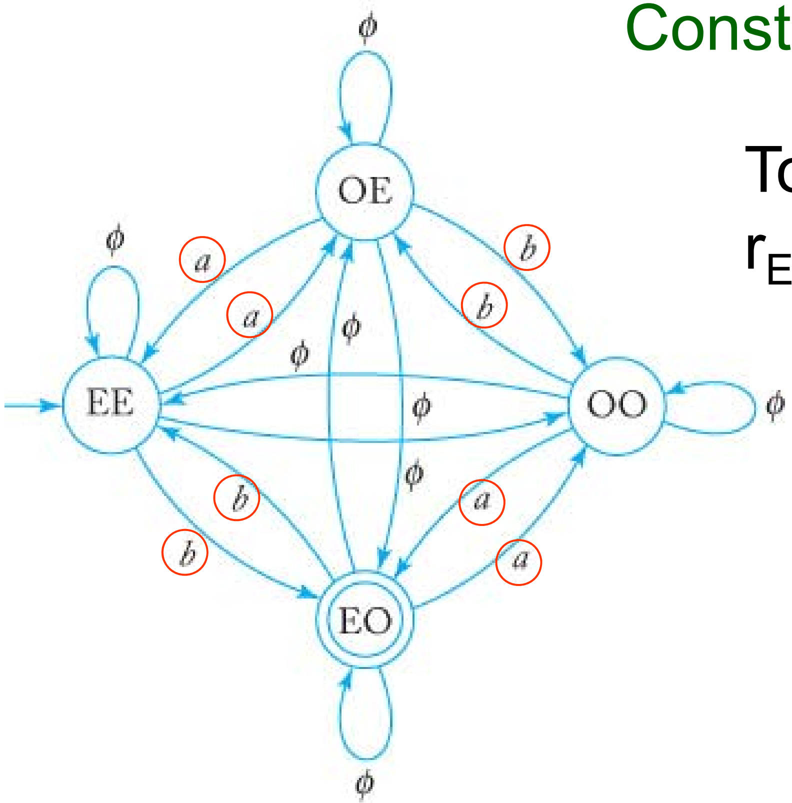
5. Repeat step 2 to 4 until the correct RE is obtained.

Example 3.11

- Find a RE for the language

$$L = \{w \in \{a,b\}^* : n_a(w) \text{ is even and } n_b(w) \text{ is odd}\}.$$

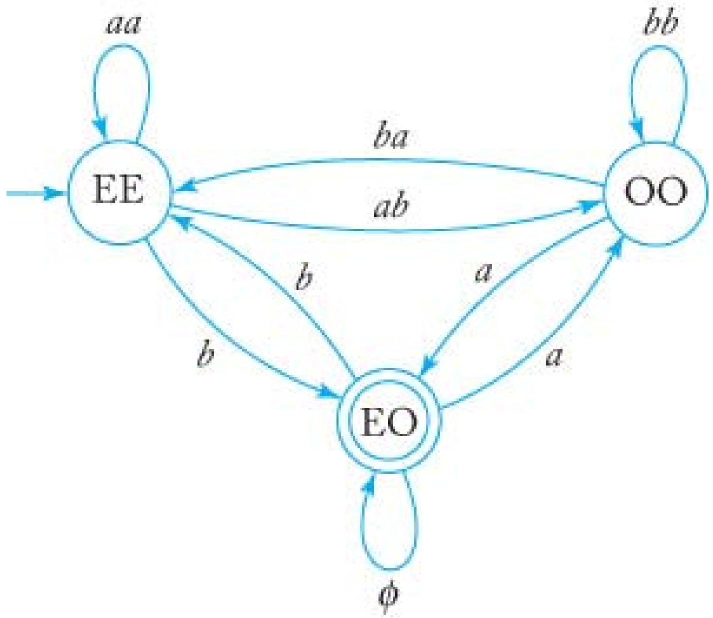
Construct NFA first!



To remove OE

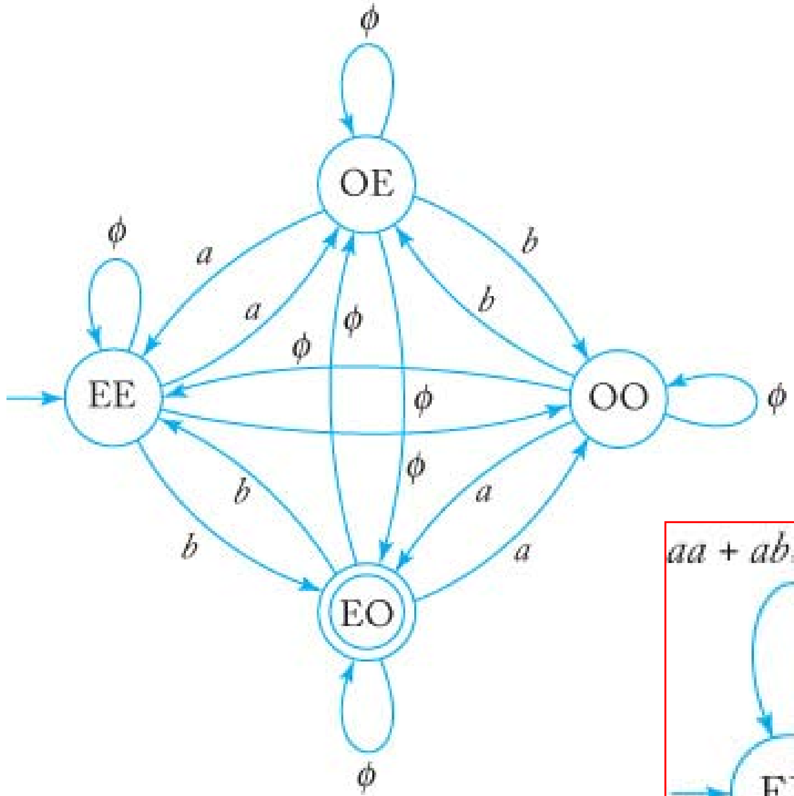
$r_{EE} = \phi + a\,\phi^* a$

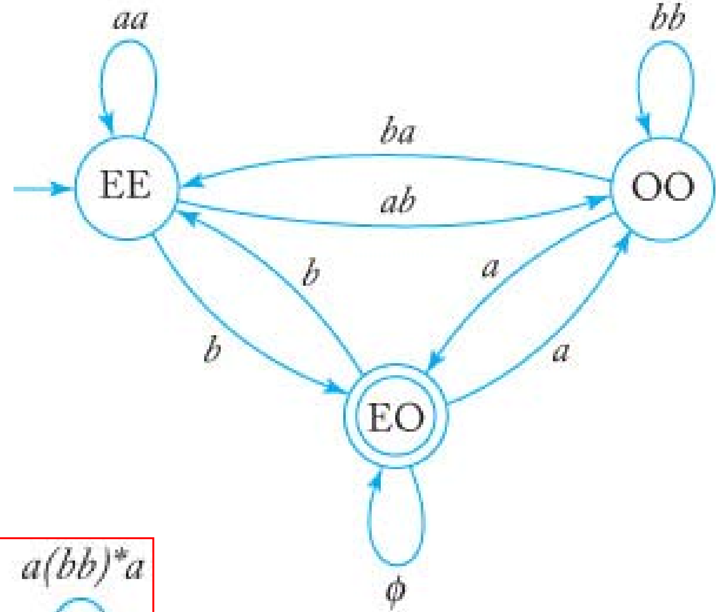
$= aa$

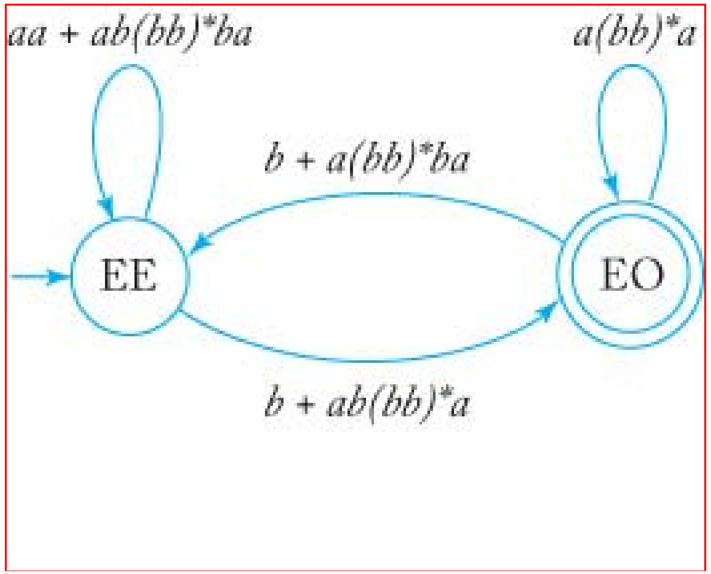


Example 3.11

$$L = \{w \in \{a,b\}^* : n_a(w) \text{ is even and } n_b(w) \text{ is odd}\}.$$







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Grammar Recap

- A grammar G is defined as a 4-tuple:

$$G = (V, T, S, P)$$

where

- V is a finite set of variables
- T is a finite set of terminals
- $S \in V$, called start variable
- P is a finite set of production rules

Grammar Recap

- Let $G = (V, T, S, P)$ be a grammar. Then the set

  $$L(G) = \{w \in T^*: S \overset{*}{\Rightarrow} w\}$$

  is the language generated by G

- If $w \in L(G)$, then the sequence

  $$S \Rightarrow w_1 \Rightarrow w_2 \Rightarrow \ldots \Rightarrow w_n \Rightarrow w$$

  is a derivation of the sentence w.

- $S, w_1, w_2, \ldots, w_n$ are called sentential forms

Linear Grammars

Grammars with

at most one variable at the right side

of a production

Examples:

$$S \to aSb$$
$$S \to \lambda$$

$$S \to Ab$$
$$A \to aAb$$
$$A \to \lambda$$

Another Linear Grammar

Grammar G :

$$S \to A$$

$$A \to aB \mid \lambda$$

$$B \to Ab$$

$$L(G) = \{a^n b^n : n \geq 0\}$$

A Non-Linear Grammar

Grammar G:

$$S \to SS$$

$$S \to \lambda$$

$$S \to aSb$$

$$S \to bSa$$

$$L(G) = \{w: \quad n_a(w) = n_b(w)\}$$

Number of a in string w

Right-Linear Grammars

- All productions have form: $A \to xB$

  or

  $A \to x$

  string of terminals

- Example: $S \to abS$

  $S \to a$

Left-Linear Grammars

- All productions have form: $A \to Bx$

  or

  $A \to x$

  string of terminals

- Example: $S \to Aab$

  $A \to Aab \mid B$

  $B \to a$

Regular Grammars

A regular grammar is either
right-linear or left-linear grammar

Examples:

G_1

$$S \to abS$$

$$S \to a$$

G_2

$$S \to Aab$$

$$A \to Aab \mid B$$

$$B \to a$$

G_3

$$S \to A$$

$$A \to aB \mid \lambda$$

$$B \to Ab$$

Observation

Regular grammars generate regular languages

A regular grammar is always linear, but
not all linear grammars are regular.

$$G_3$$

G_3 is linear grammar
but not regular grammar

$$S \to A$$

$$A \to aB \mid \lambda$$

$$B \to Ab$$

Example 3.13

Regular grammars generate regular languages

$$G_1$$

$$S \to abS$$

$$S \to a$$

$$G_2$$

$$S \to Aab$$

$$A \to Aab \mid B$$

$$B \to a$$

$$L(G_1) = (ab)^* a$$

$$L(G_2) = aab(ab)^*$$

Theorem

$$\left\{\begin{array}{l}\text{Languages}\\ \text{Generated by}\\ \text{Regular Grammars}\end{array}\right\} = \left\{\begin{array}{l}\text{Regular}\\ \text{Languages}\end{array}\right\}$$

Theorem - Part 1

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Any regular grammar generates a regular language

- Theorem 3.3

Theorem - Part 2

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Any regular language is generated by a regular grammar

■ Theorem 3.4

Proof – Part 1

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

The language $L(G)$ generated by any regular grammar G is regular

The case of Right-Linear Grammars

Let G be a right-linear grammar

We will prove: $L(G)$ is regular

Proof idea: We will construct NFA M with $L(M) = L(G)$

- Grammar G is right-linear

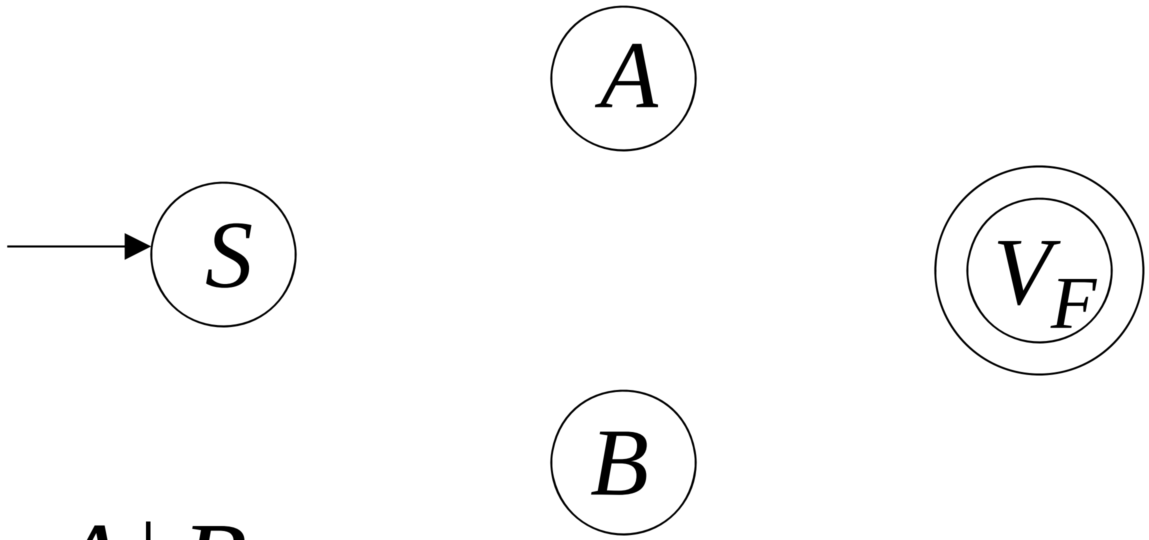
Example:

$$S \rightarrow aA \mid B$$

$$A \rightarrow aa\ B$$

$$B \rightarrow b\ B \mid a$$

Construct NFA M such that every state is a grammar variable:

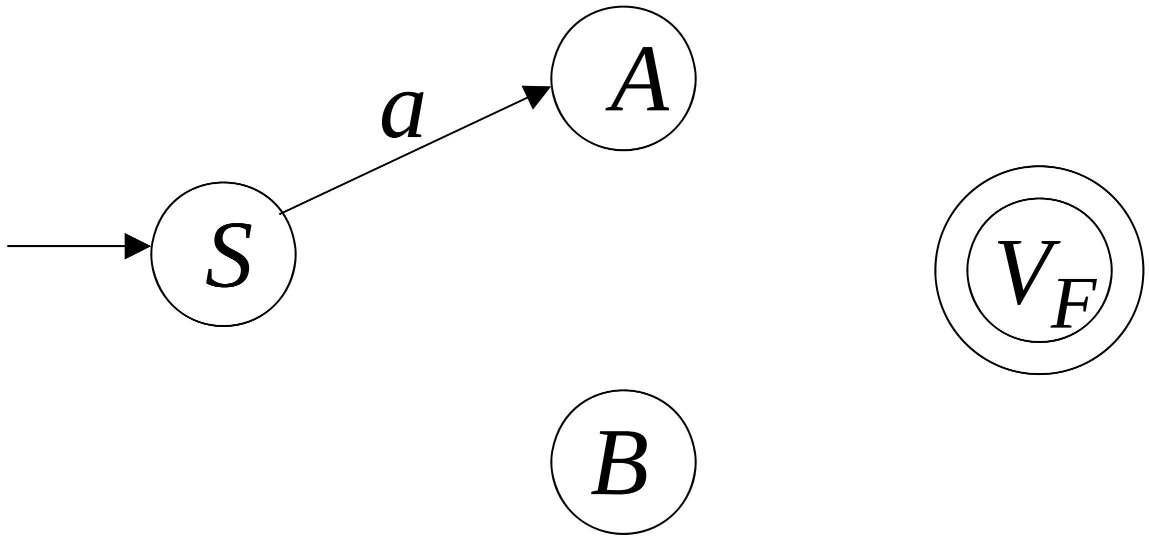


$$S \to aA \mid B$$

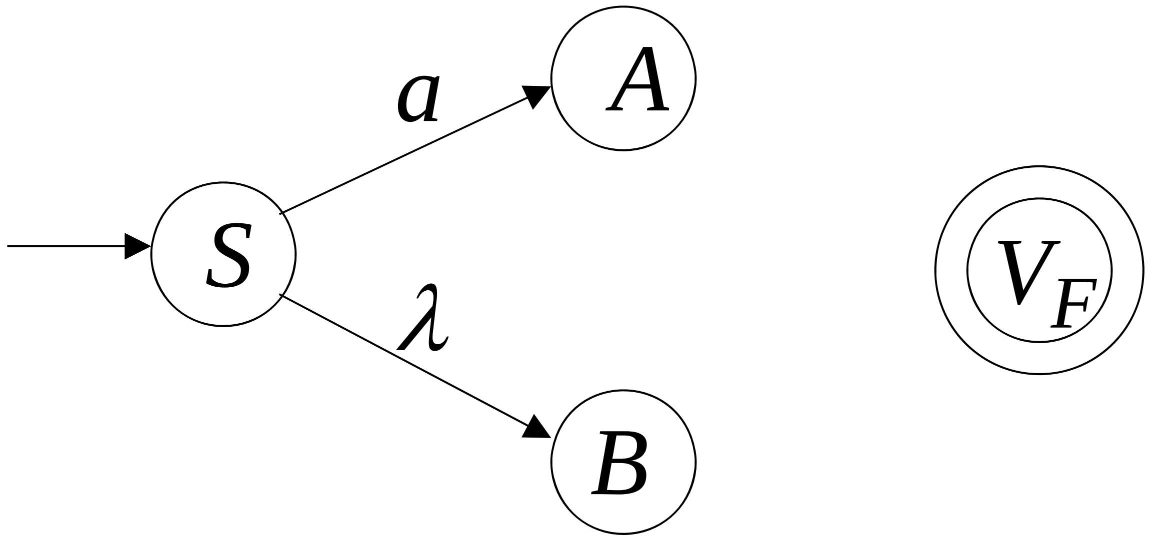
$$A \to aa\, B$$

$$B \to b\, B \mid a$$

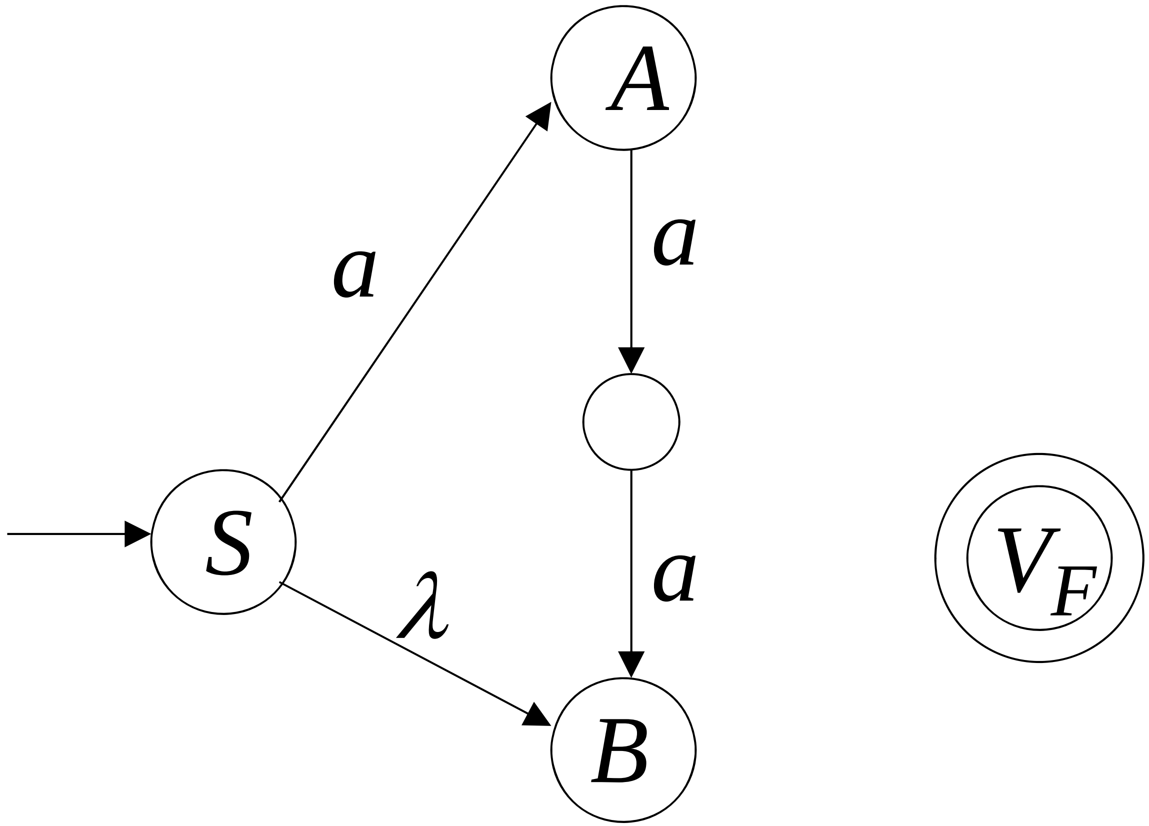
- Add edges for each production:



$$S \to aA$$

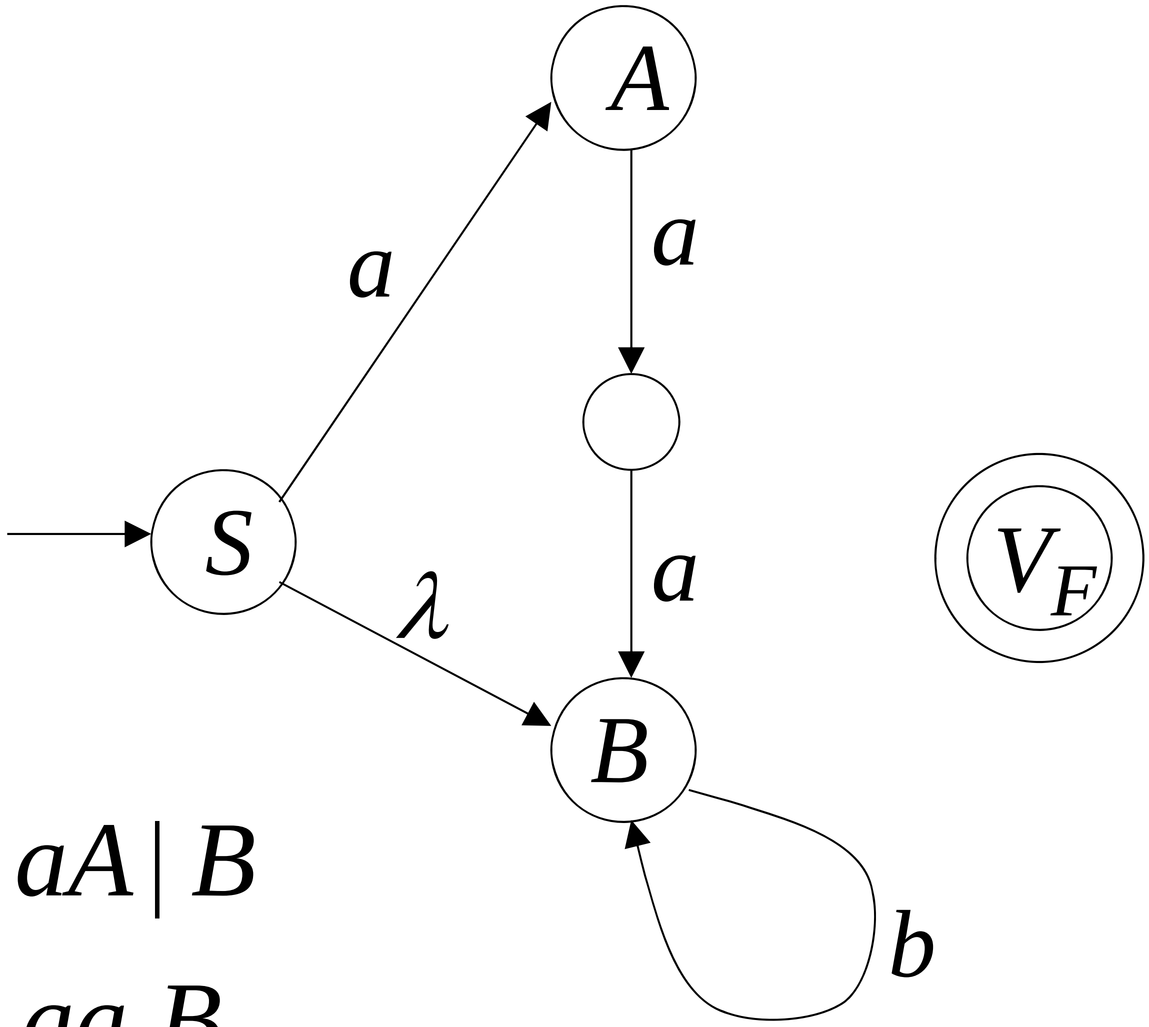


$$S \to aA \mid B$$



$$S \to aA \mid B$$

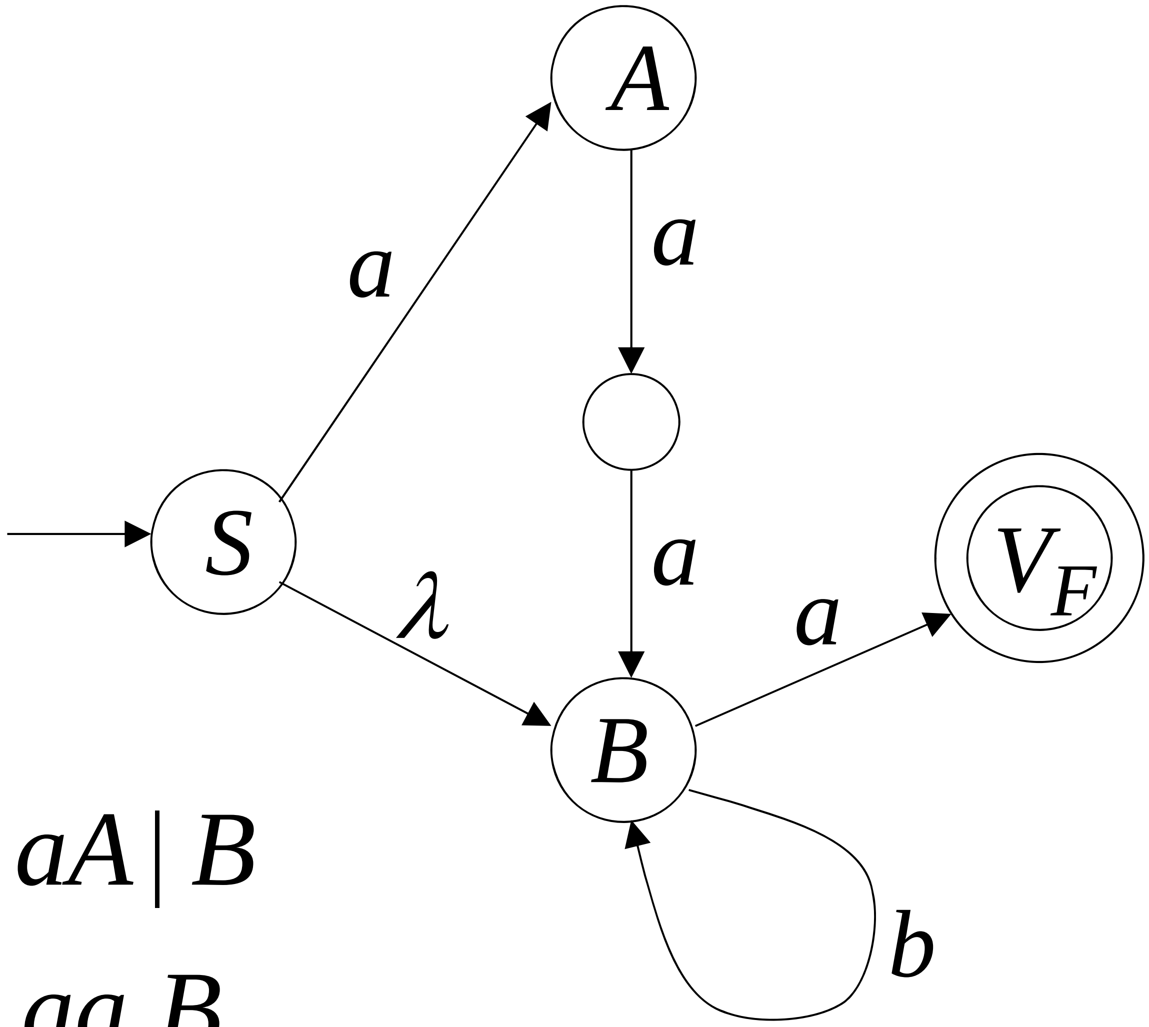
$$A \to aa\,B$$



$$S \to aA \mid B$$

$$A \to aa\ B$$

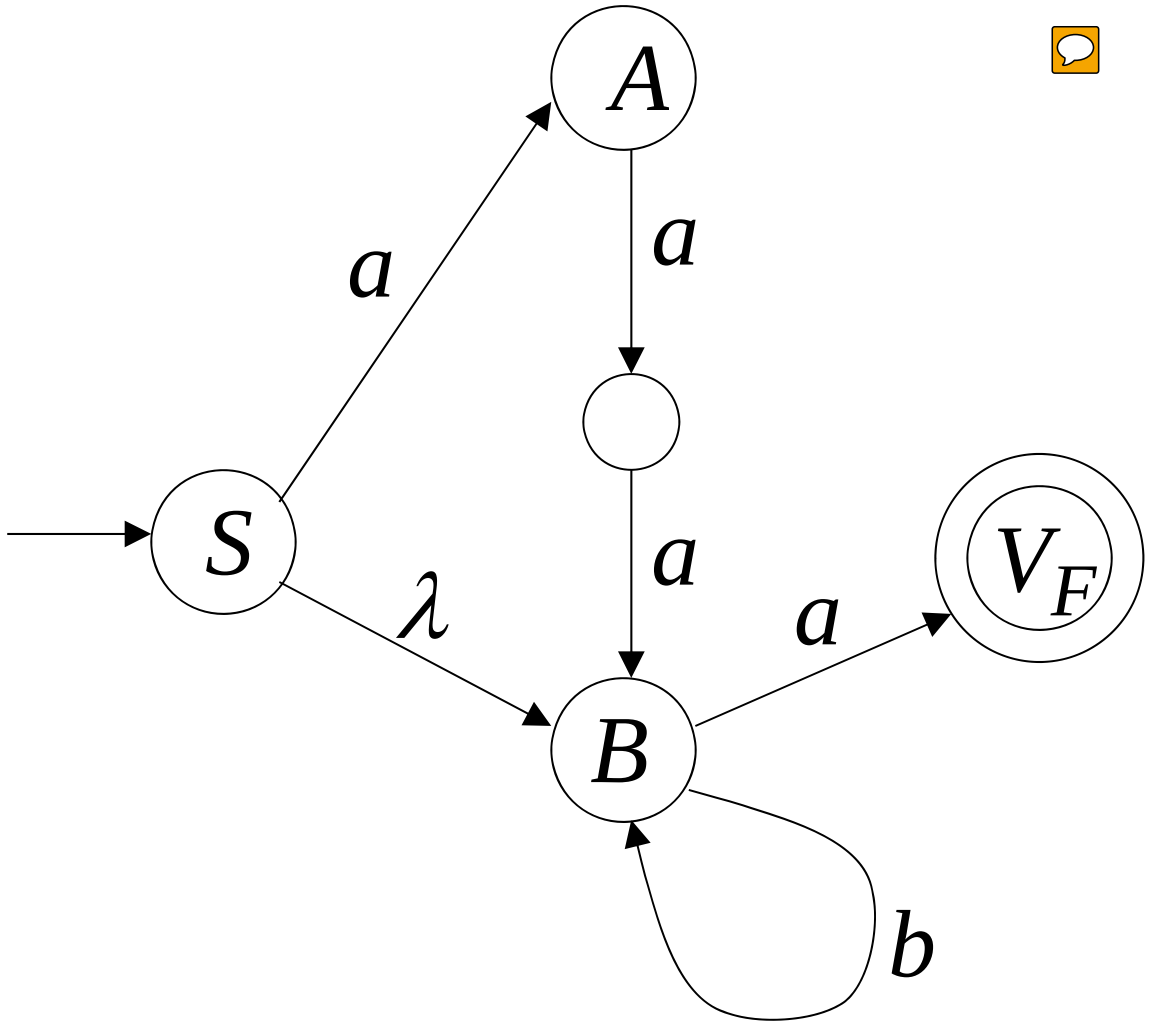
$$B \to bB$$



$$S \to aA \mid B$$

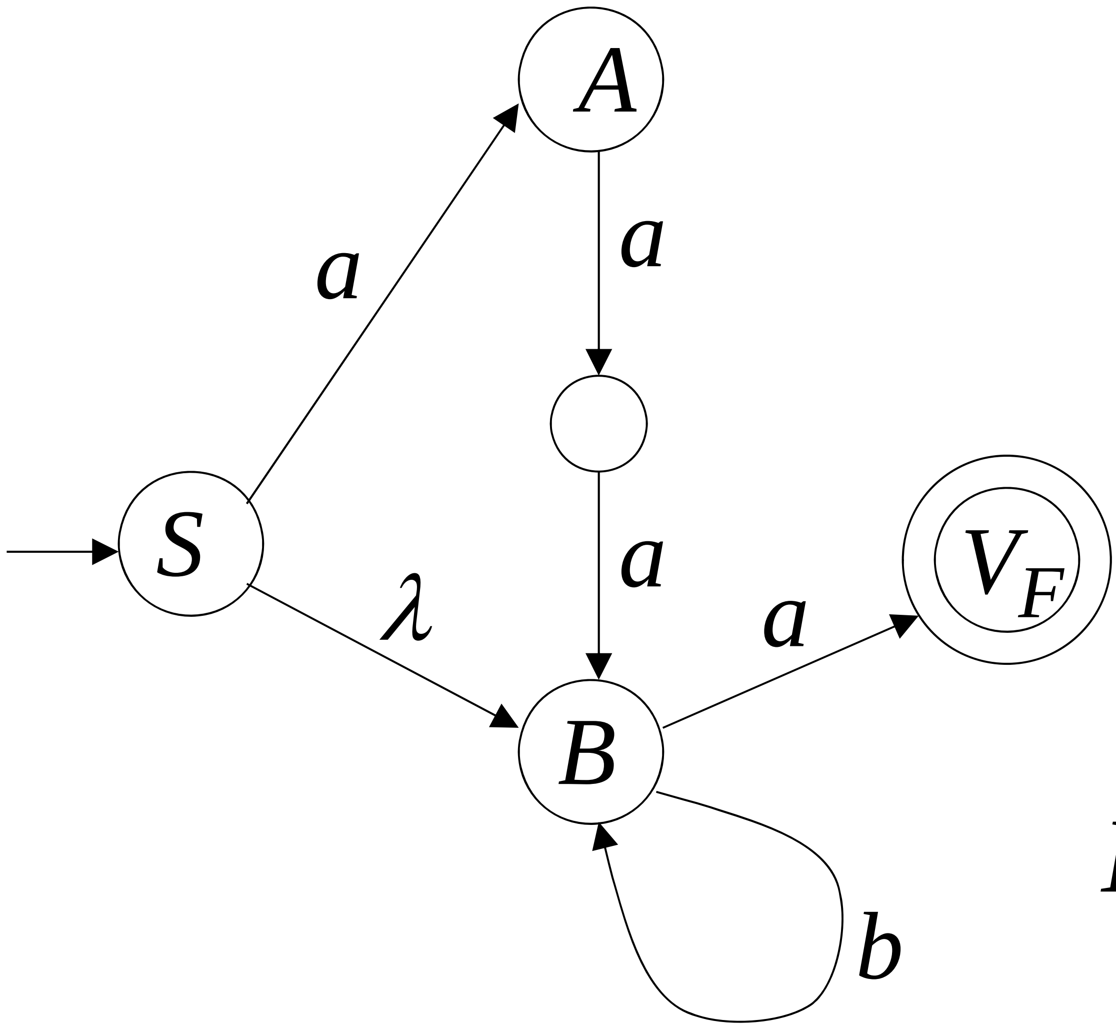
$$A \to aa\ B$$

$$B \to bB \mid a$$



$$S \Rightarrow aA \Rightarrow aaaB \Rightarrow aaabB \Rightarrow aaaba$$

NFA M



Grammar G

$$S \to aA \mid B$$

$$A \to aa\,B$$

$$B \to bB \mid a$$

$$L(M) = L(G) =$$

In General

A right-linear grammar G

has variables: $V_0, V_1, V_2, \dots$

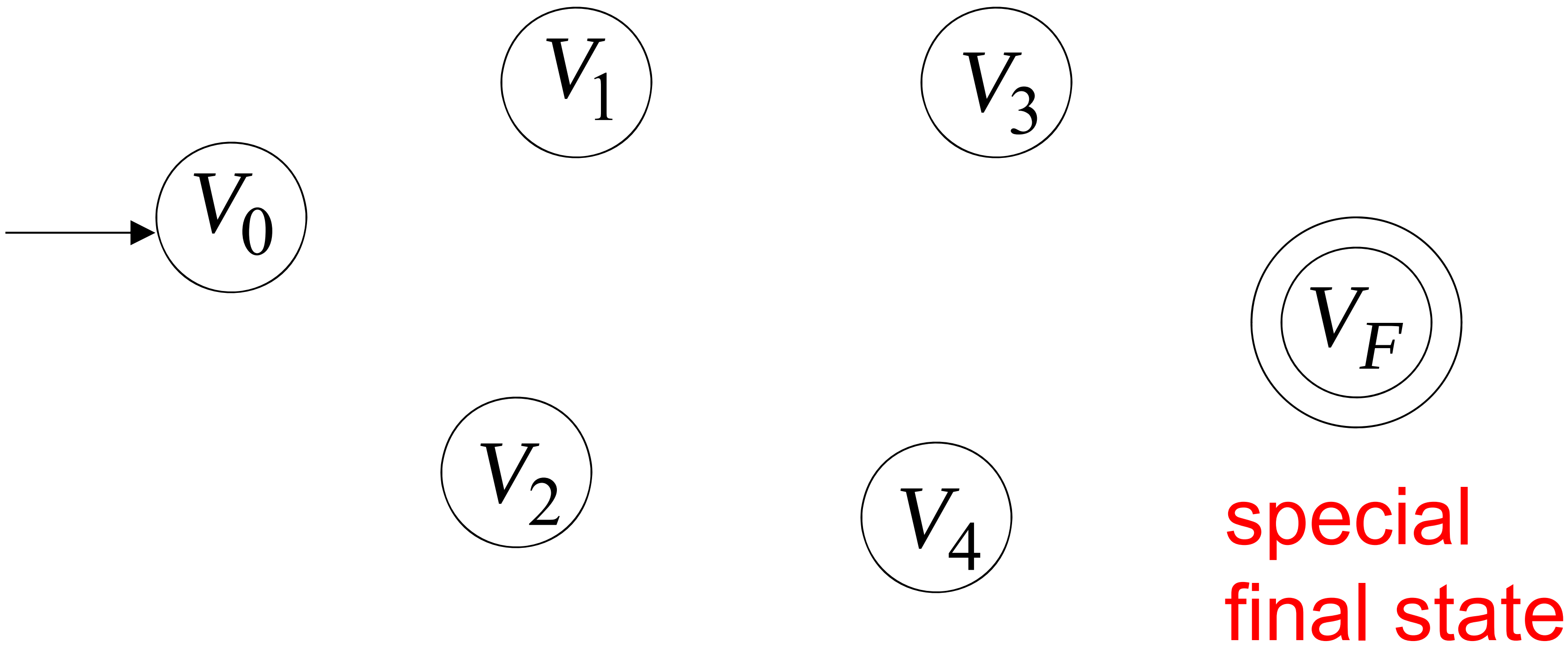
and productions: $V_i \to a_1 a_2 \cdots a_m V_j$

or

$$V_i \to a_1 a_2 \cdots a_m$$

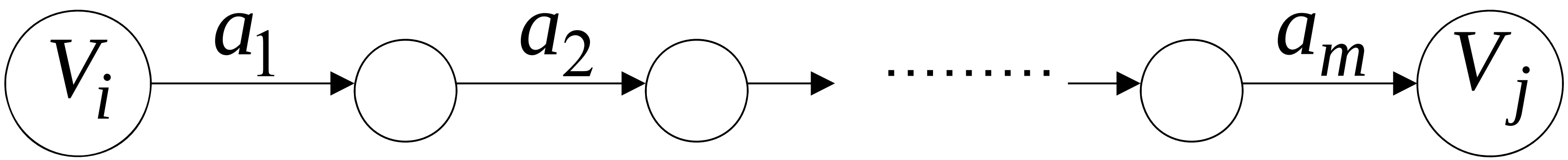
We construct the NFA M such that:

each variable V_i corresponds to a node:



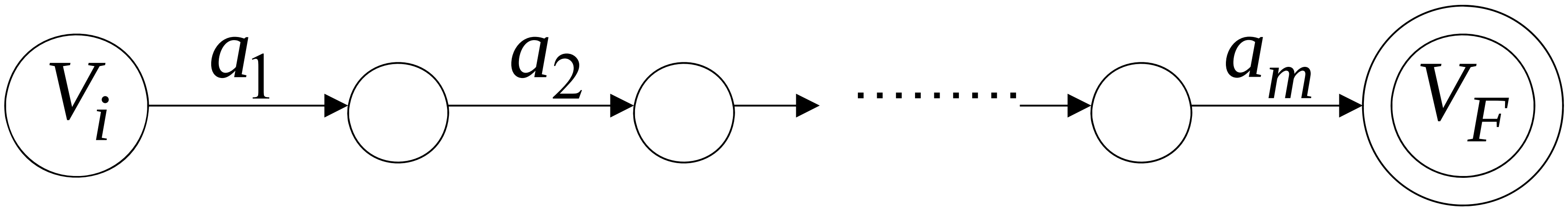
For each production: $V_i \to a_1 a_2 \cdots a_m V_j$

we add transitions and intermediate nodes



For each production: $V_i \to a_1 a_2 \cdots a_m$

we add transitions and intermediate nodes

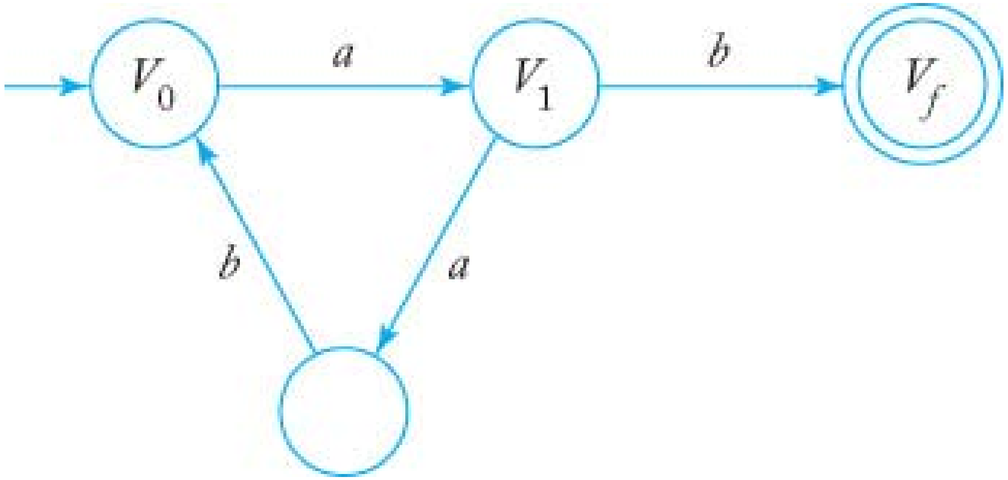


Example 3.15

- Construct a FA that accepts the language generated by the grammar

$V_0 \rightarrow aV_1,$

$V_1 \rightarrow abV_0|b$



L(G)=

The case of Left-Linear Grammars

Let G be a left-linear grammar

We will prove: $L(G)$ is regular

Proof idea:

We will construct a right-linear grammar G' with $L(G) = L(G')^R$

Since G is left-linear grammar
the productions look like:

$$A \to Ba_1 a_2 \cdots a_k$$

$$A \to a_1 a_2 \cdots a_k$$

- Construct right-linear grammar G'

Left linear G

$$A \to Ba_1 a_2 \cdots a_k$$

$$A \to Bv$$

Right linear G'

$$A \to a_k \cdots a_2 a_1 B$$

$$A \to v^R B$$

- Construct right-linear grammar G'

Left linear G

$$A \to a_1 a_2 \cdots a_k$$

$$A \to v$$

Right linear G'

$$A \to a_k \cdots a_2 a_1$$

$$A \to v^R$$

It is easy to see that: $L(G) = L(G')^R$

Since G' is right-linear, we have:

$L(G') \Rightarrow L(G')^R \Rightarrow L(G)$

$L(G')$		$L(G')^R$		$L(G)$
Regular Language		Regular Language		Regular Language

Proof - Part 2

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Any regular language L is generated by some regular grammar G

Any regular language L is generated by some regular grammar G

Proof idea:

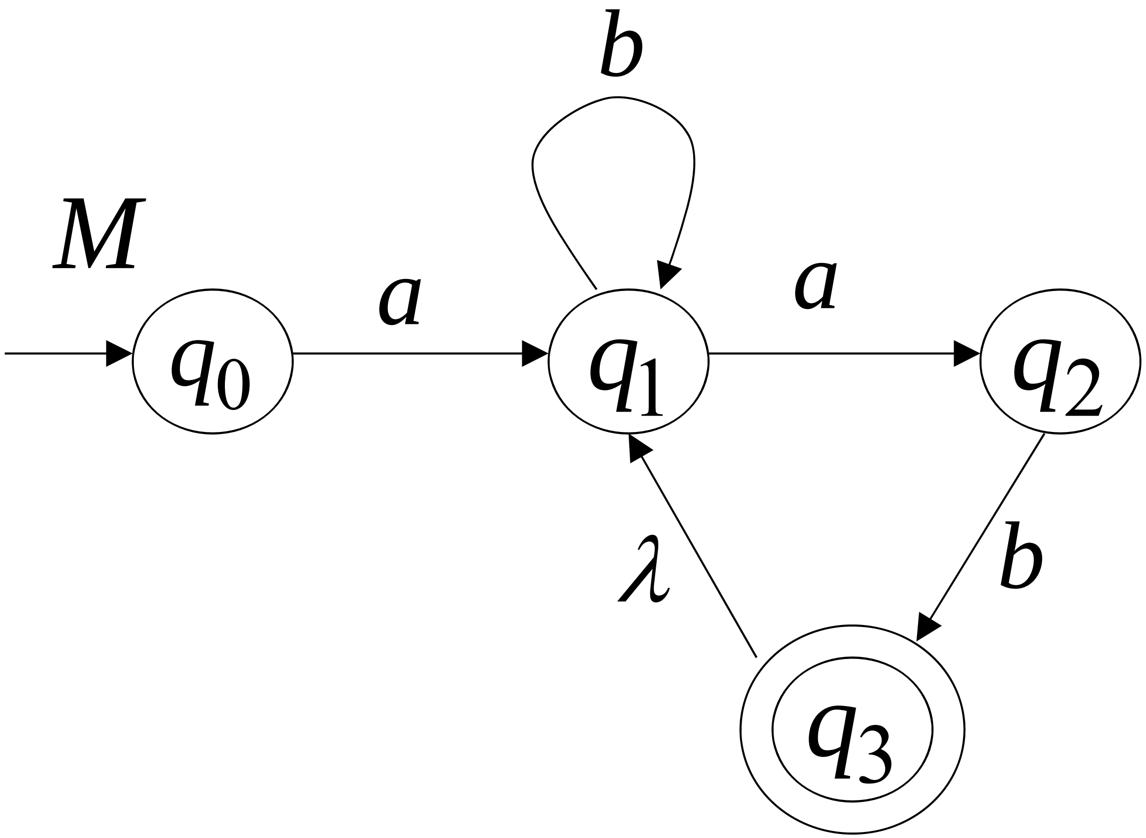
Let M be the NFA with $L = L(M)$.

Construct from M to a regular grammar G such that $L(M) = L(G)$

Since L is regular

there is an NFA M such that $L = L(M)$

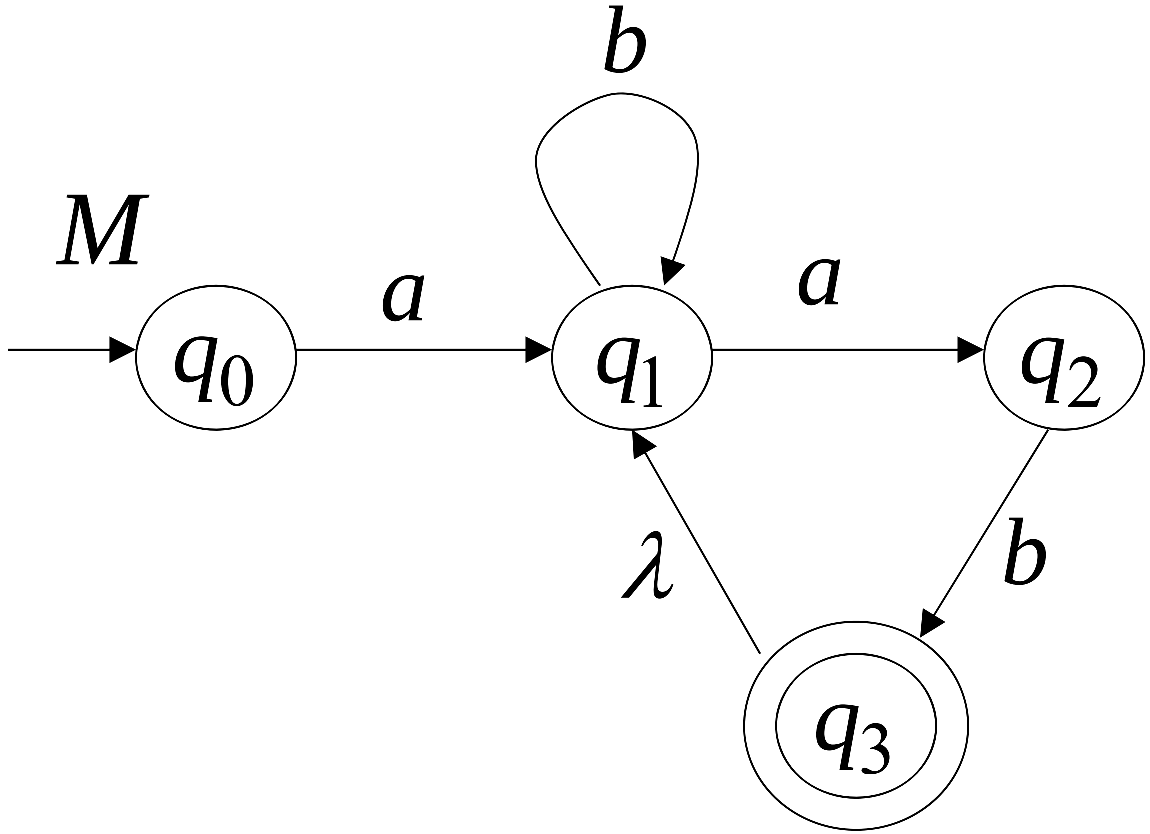
Example:



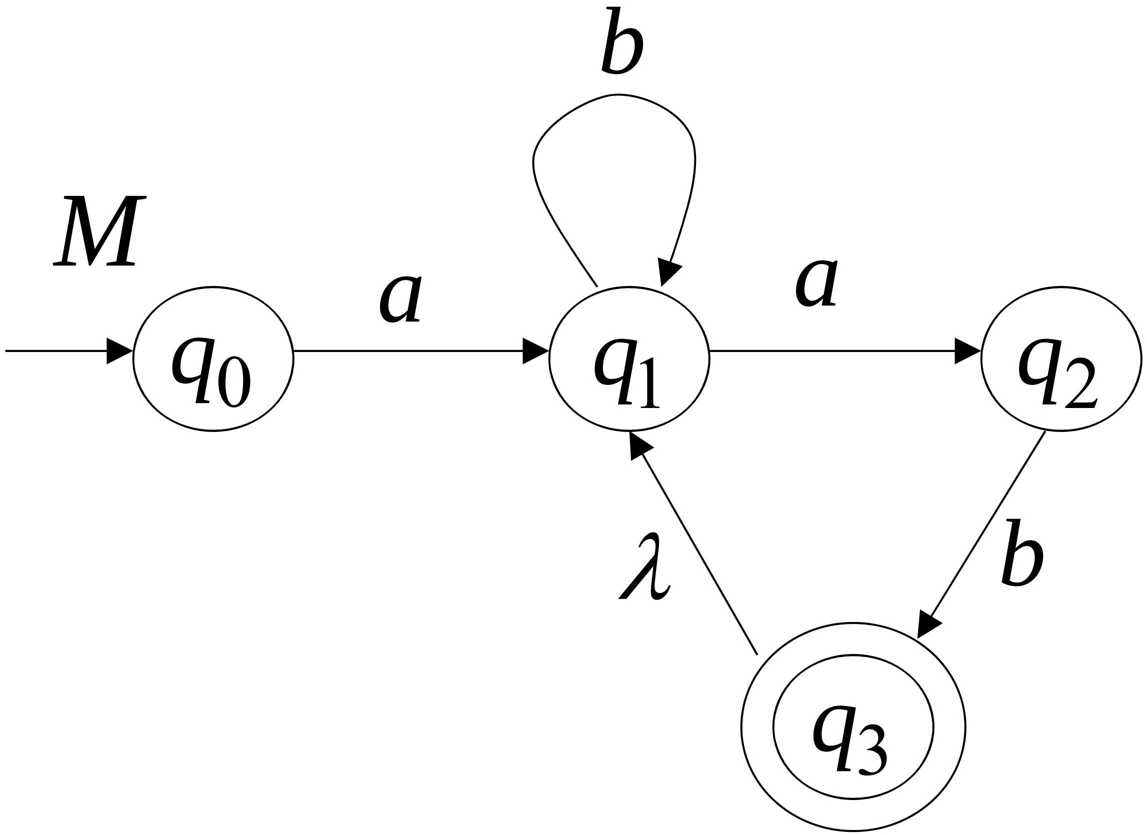
$L =$

$L = L(M)$

Convert M to a right-linear grammar (BFS-like)



$$q_0 \to aq_1$$



$$q_0 \rightarrow aq_1$$

$$q_1 \rightarrow bq_1$$

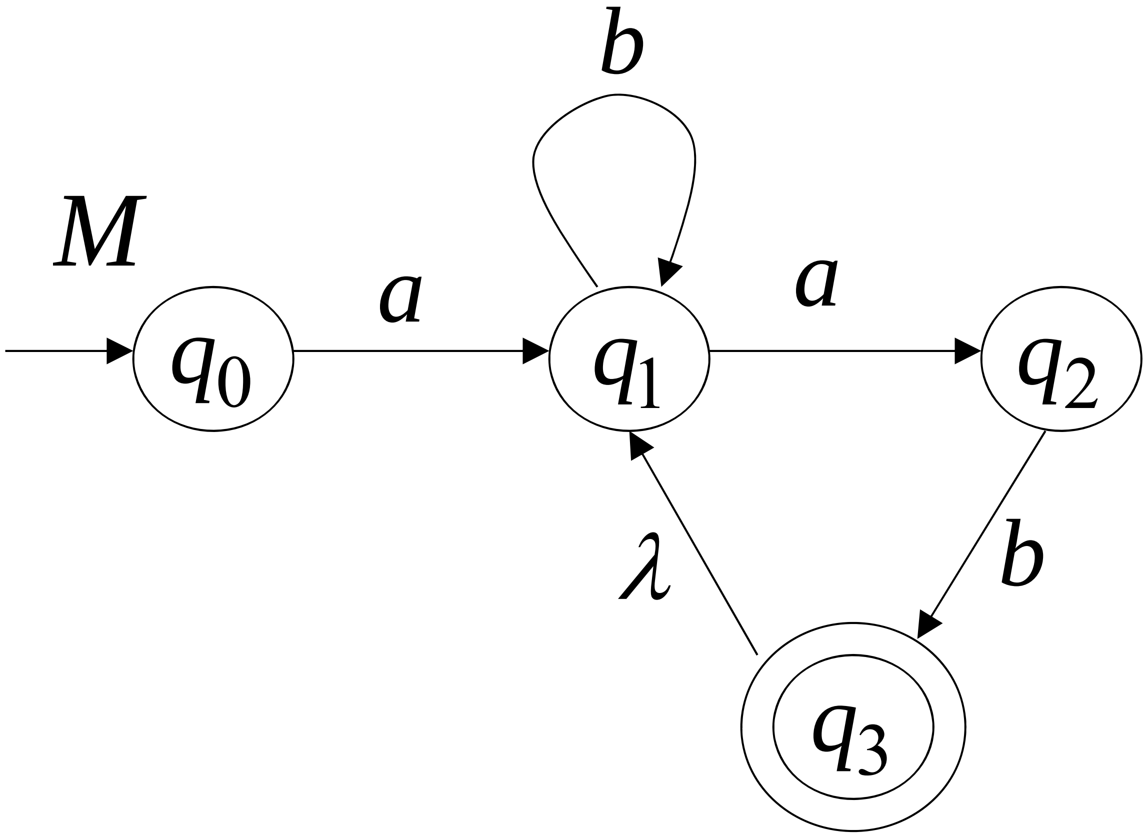
$$q_1 \rightarrow aq_2$$

$$q_0 \to aq_1$$

$$q_1 \to bq_1$$

$$q_1 \to aq_2$$

$$q_2 \to bq_3$$



$$L(G) = L(M) = L$$

G

$$q_0 \to aq_1$$

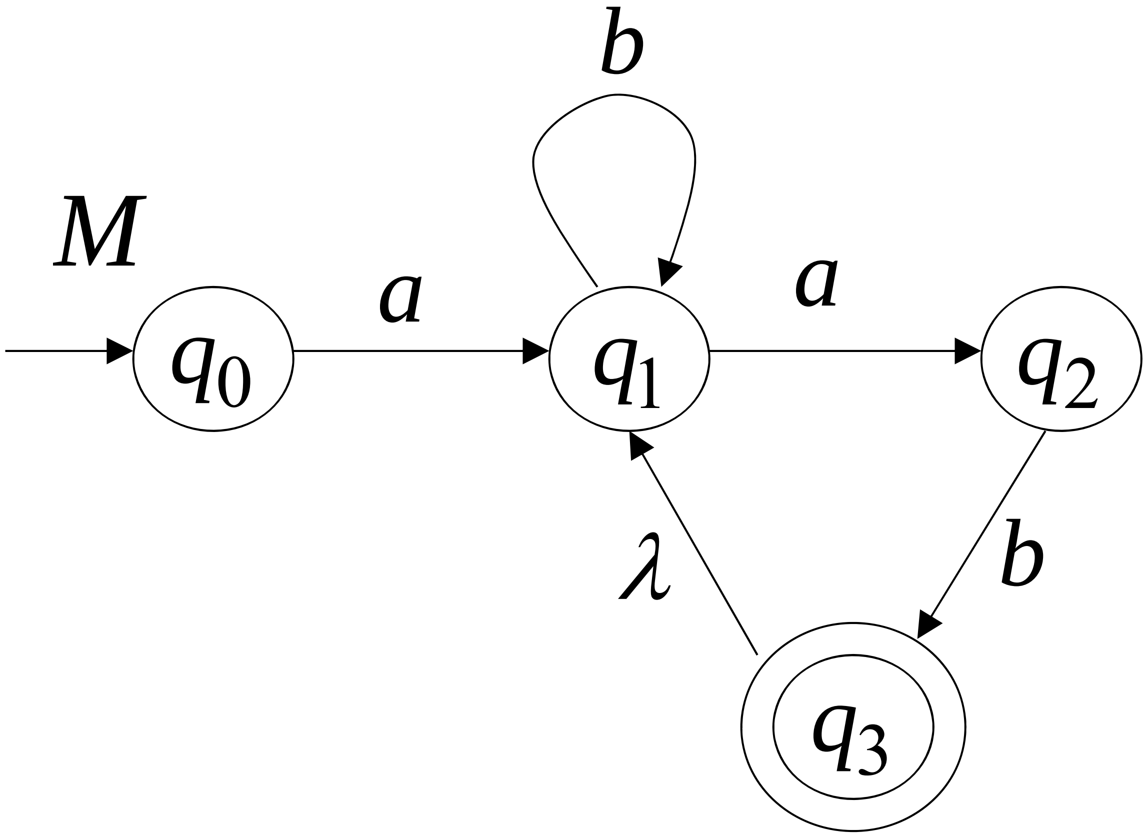
$$q_1 \to bq_1$$

$$q_1 \to aq_2$$

$$q_2 \to bq_3$$

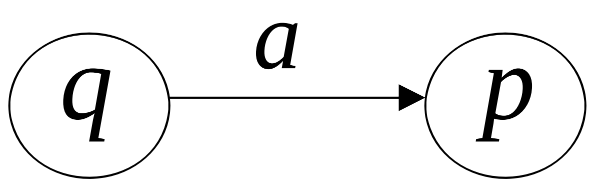
$$q_3 \to q_1$$

$$q_3 \to \lambda$$



In General

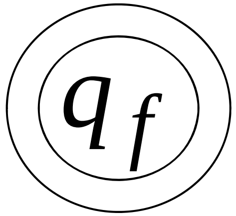
For any transition:



Add production:

$$q \to ap$$

variable terminal variable

For any final state: 

Add production: $q_f \to \lambda$

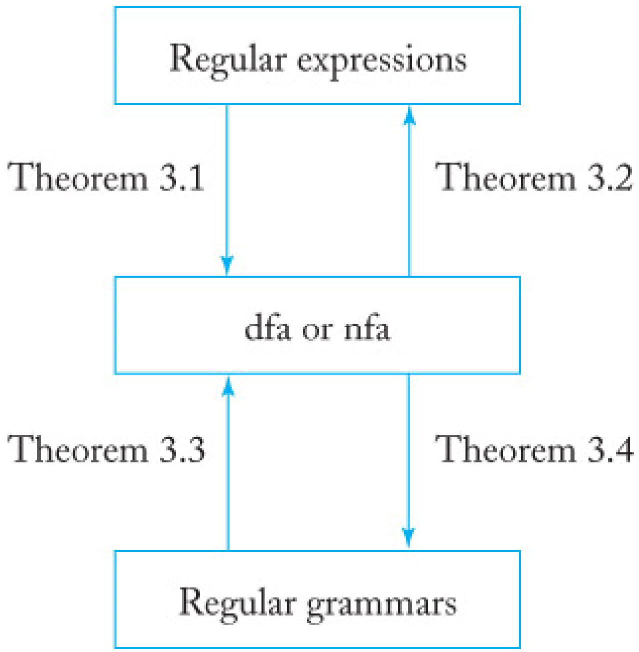
Since G is right-linear grammar

G is also a regular grammar

with $L(G) = L(M) = L$

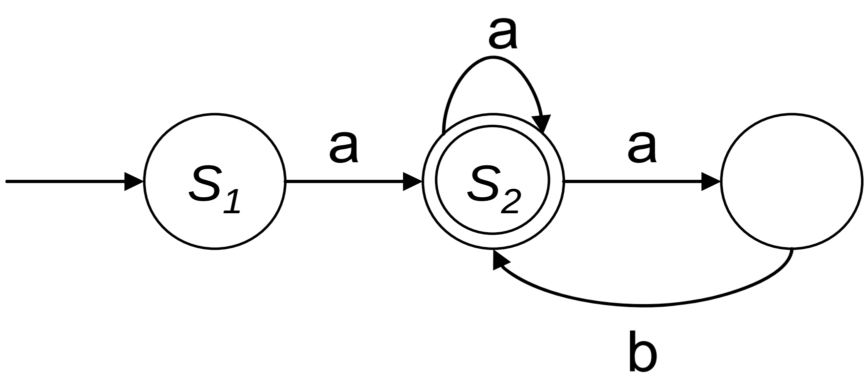
Summary

- We now have several ways of describing regular languages:
  - DFA
  - NFA
  - RE
  - RG



Short Quiz

- Find a regular grammar that generates the language L(aa*(ab+a)*).



$$S_1 \to aS_2$$
$$S_2 \to aS_2 \mid abS_2 \mid \lambda$$

Questions?