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Theory of Computation

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Outline

1. Closure Properties of Regular Languages
2. Elementary Questions about Regular Languages
3. Identifying Nonregular Languages

For regular languages L_1 and L_2, we will prove that:

Union: $L_1 \cup L_2$

Concatenation: $L_1 L_2$

Star: $L_1 *$

Reversal: L_1^R

Complement: $\overline{L_1}$

Intersection: $L_1 \cap L_2$

Difference: $L_1 - L_2$

Are regular Languages

We say: Regular languages are **closed under**

Union: $L_1 \cup L_2$

Concatenation: $L_1 L_2$

Star: $L_1{}^*$

Reversal: $L_1{}^R$

Complement: $\overline{L_1}$

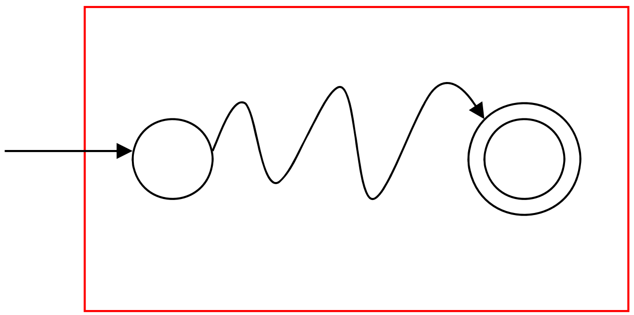
Intersection: $L_1 \cap L_2$

Difference: $L_1 - L_2$

Regular language L_1

$$L(M_1) = L_1$$

NFA M_1

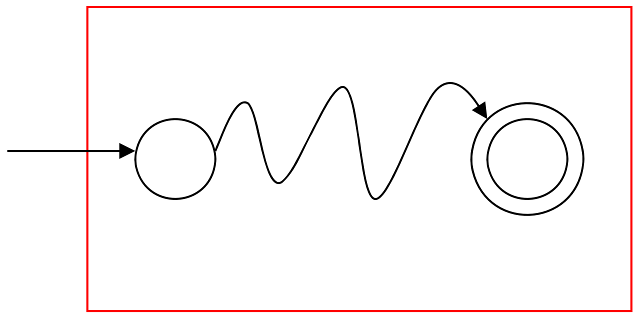


Single final state

Regular language L_2

$$L(M_2) = L_2$$

NFA M_2



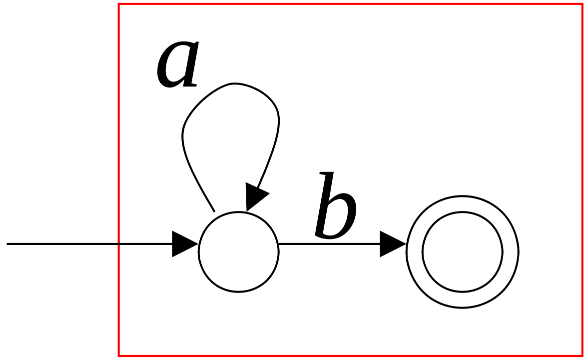
Single final state

Example

$$n \geq 0$$

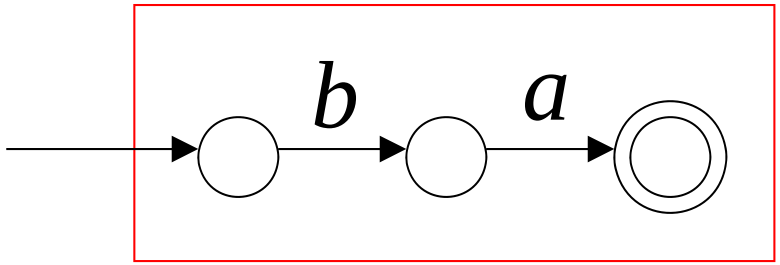
$$L_1 = \{a^n b\}$$

M_1



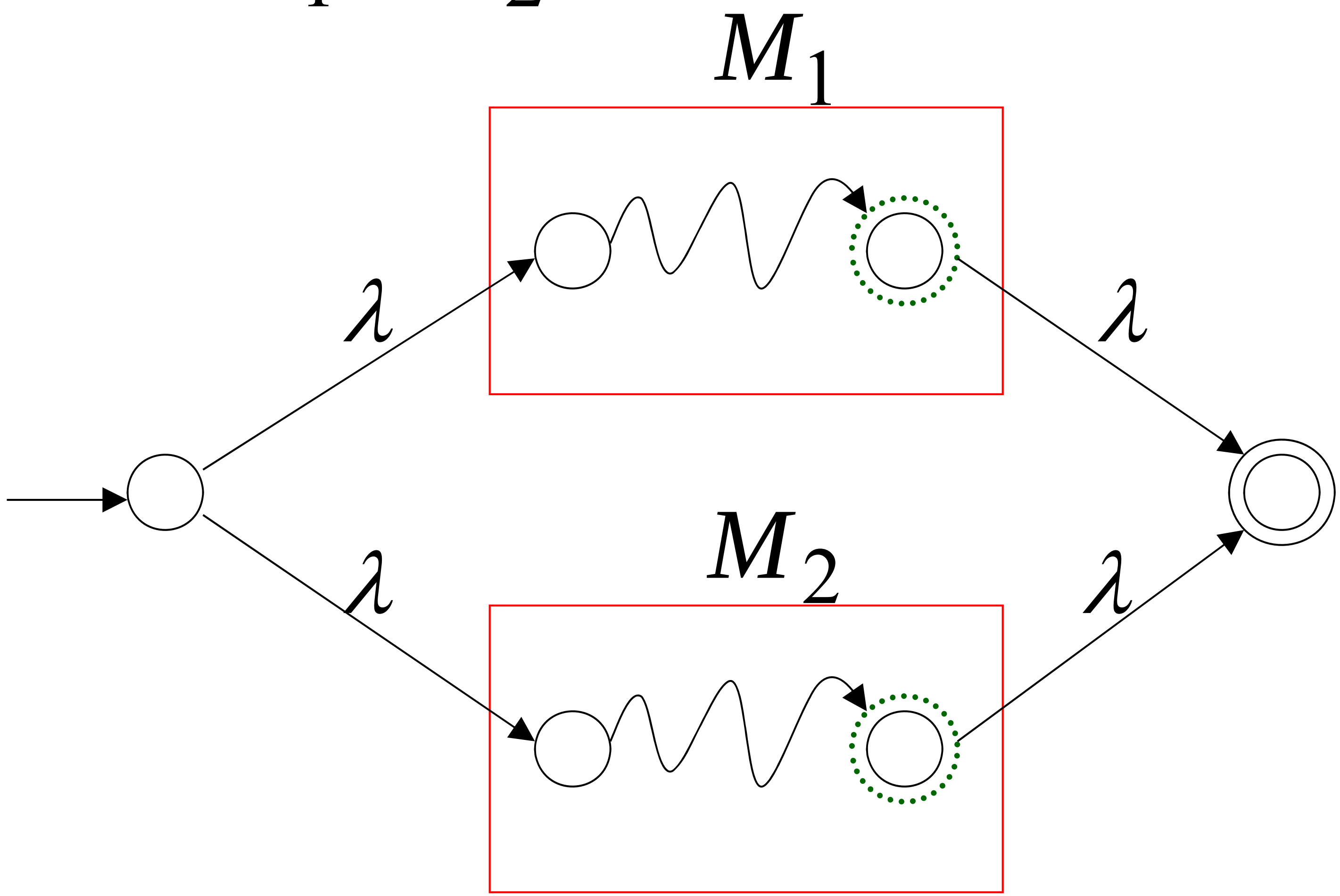
$$L_2 = \{ba\}$$

M_2



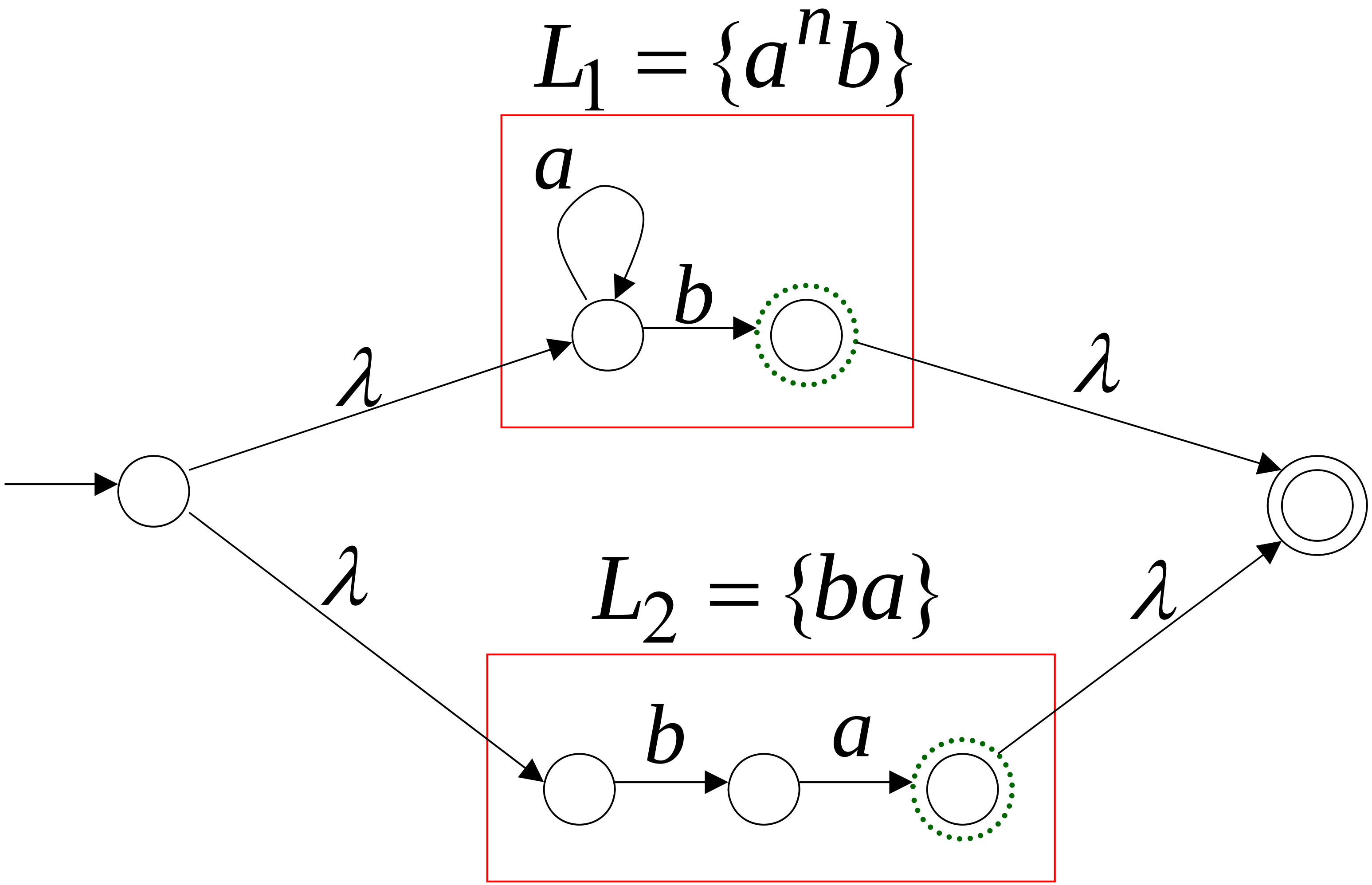
Union

NFA for $L_1 \cup L_2$



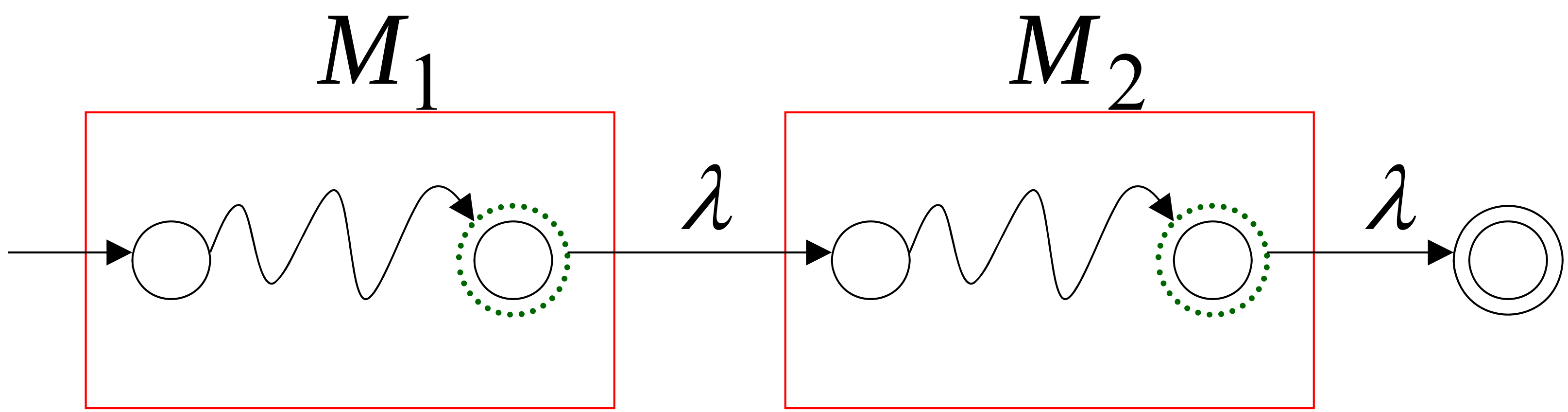
Example

NFA for $L_1 \cup L_2 = \{a^n b\} \cup \{ba\}$



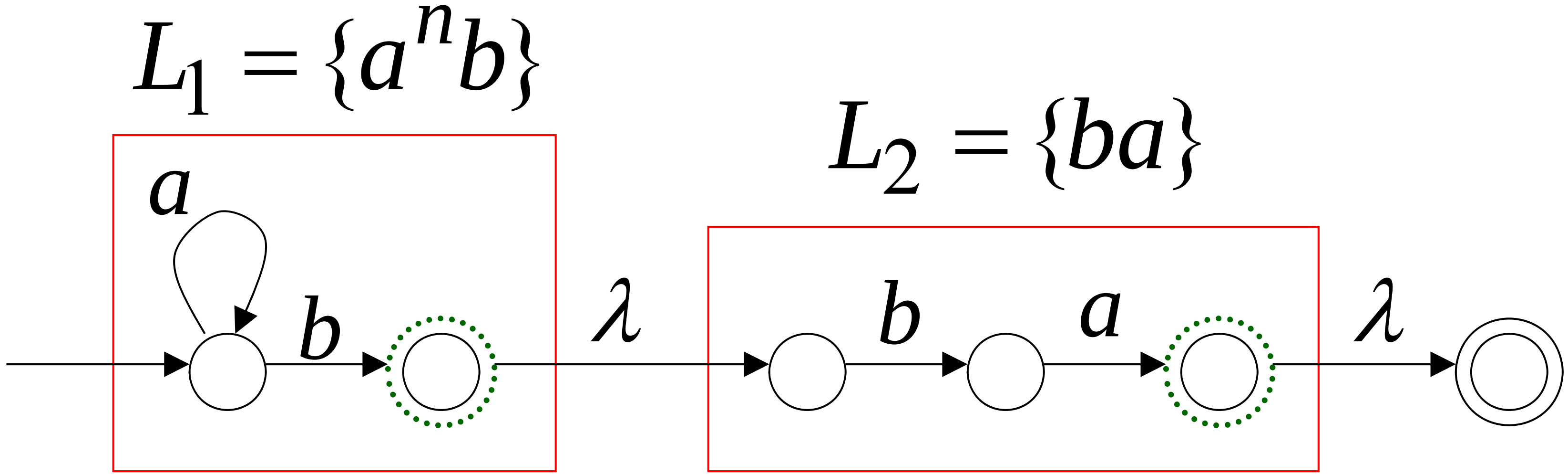
Concatenation

NFA for $L_1 L_2$



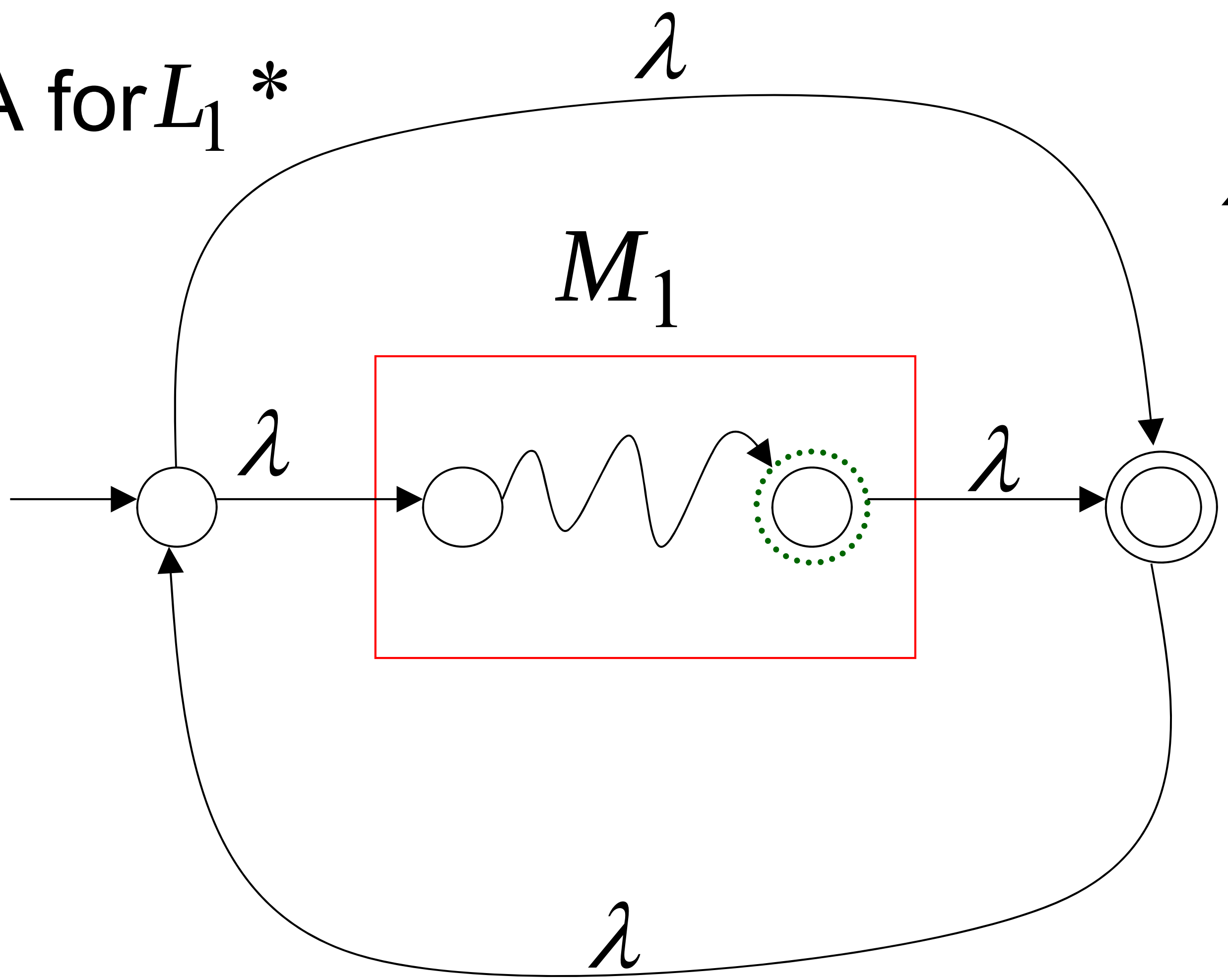
Example

NFA for $L_1 L_2 = \{a^n b\}\{ba\} = \{a^n bba\}$



Star Operation

NFA for $L_1{}^*$



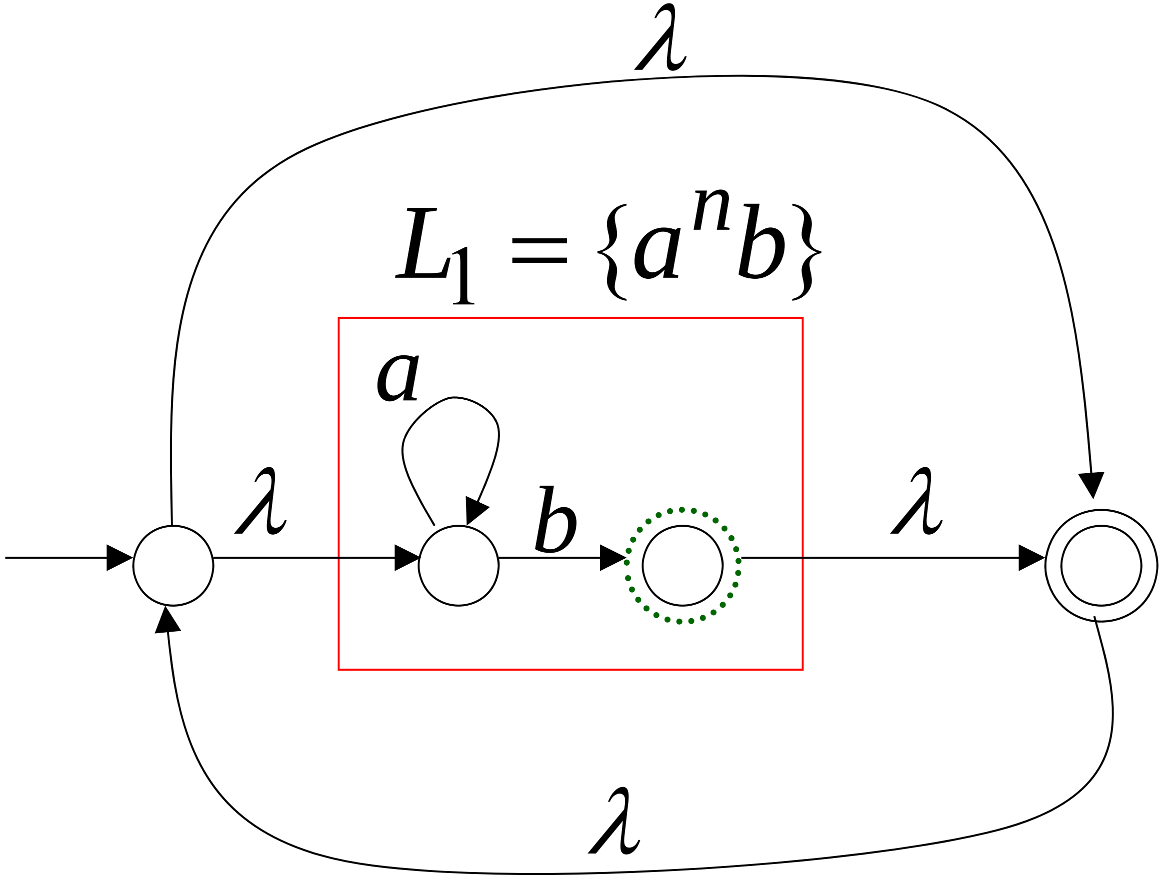
$\lambda \in L_1{}^*$

Example

NFA for $L_1{}^* = \{a^n b\}^*$

$w = w_1 w_2 \cdots w_k$

$w_i \in L_1$



Reverse

NFA for L_1^R

L_1 M_1 M_1'

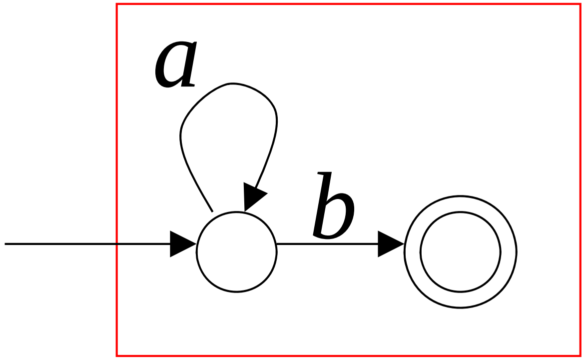
1. Reverse all transitions

2. Make initial state final state
   and vice versa

Example

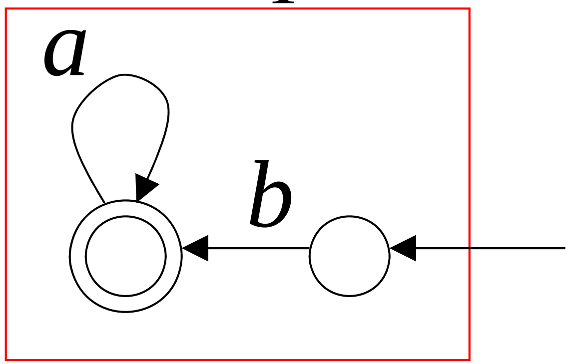
M_1

$L_1 = \{a^n b\}$

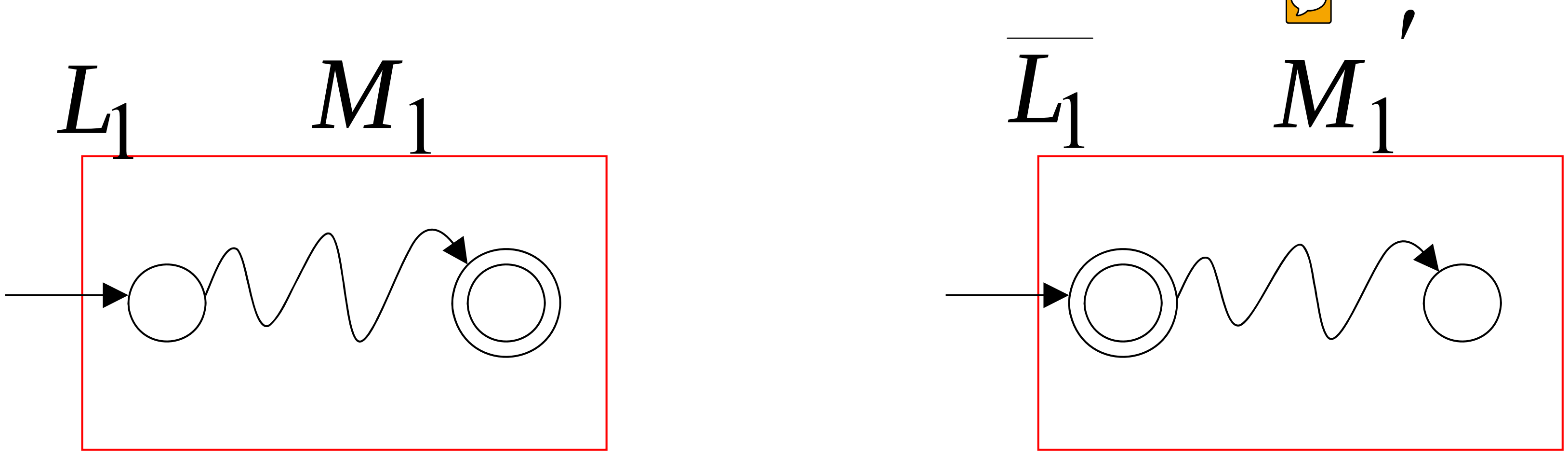


M_1'

$L_1^R = \{ba^n\}$



Complement



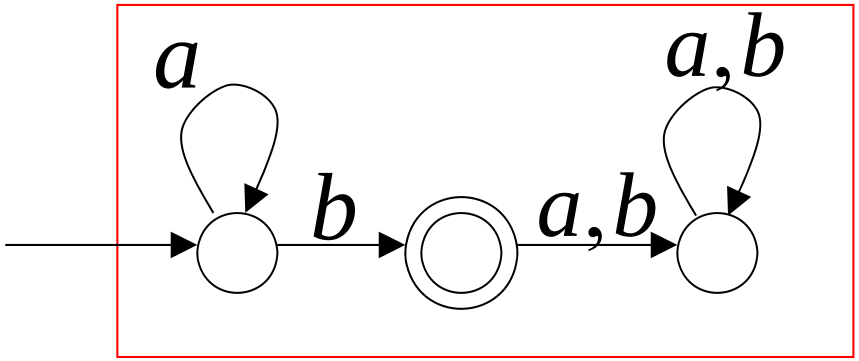
1. Take the **DFA** that accepts L_1

2. Make final states non-final, and vice-versa

Example

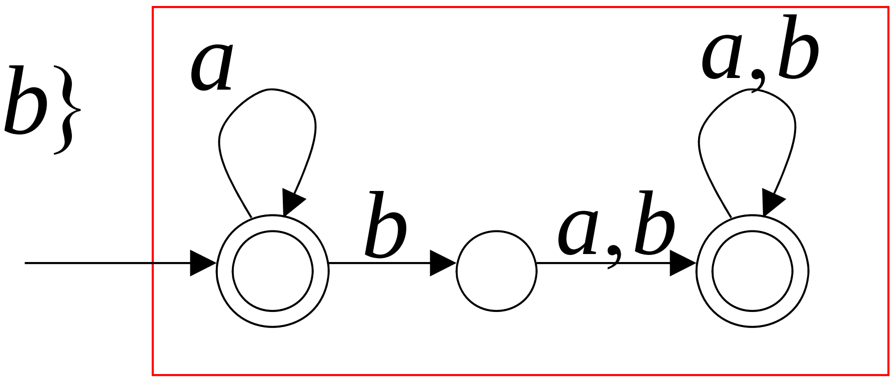
M_1

$$L_1 = \{a^n b\}$$



M_1'

$$\overline{L_1} = \{a,b\}^* - \{a^n b\}$$

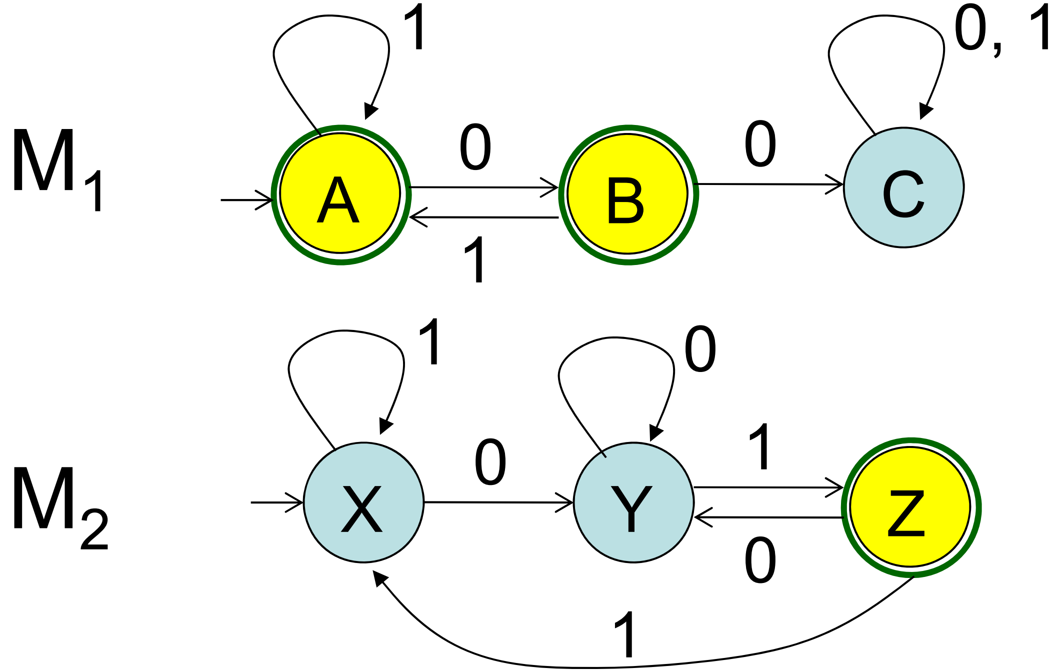


Intersection and Difference

- Let's step through an example

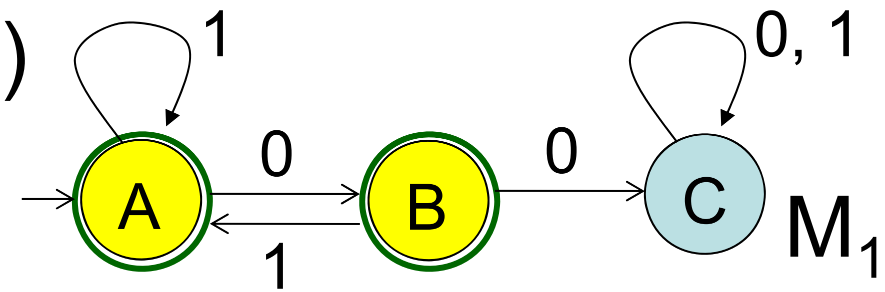
  $L_1 = \{ x \mid 00 \text{ is not a substring of } x\}$

  $L_2 = \{ x \mid x \text{ ends in } 01\}$

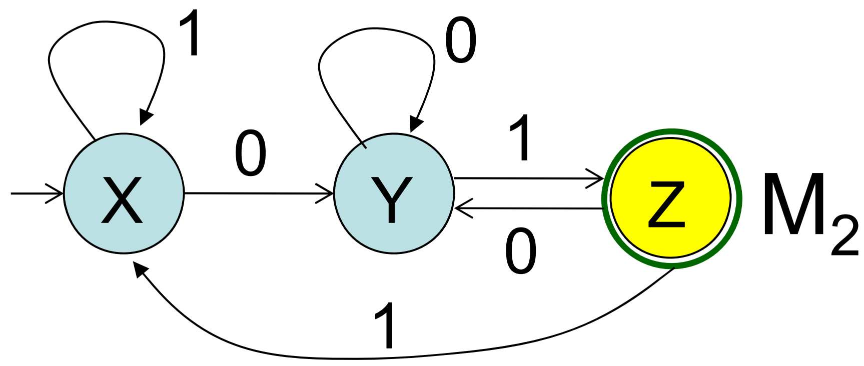


Intersection and Difference

- $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$
  - $Q_1 = \{A, B, C\}$
  - $q_1 = A$
  - $F_1 = \{A, B\}$



- $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$
  - $Q_2 = \{X, Y, Z\}$
  - $q_2 = X$
  - $F_2 = \{Z\}$

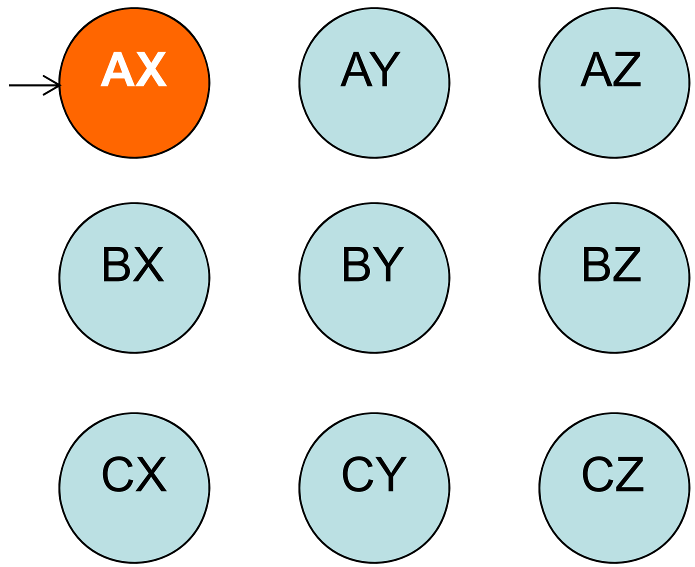


Intersection and Difference

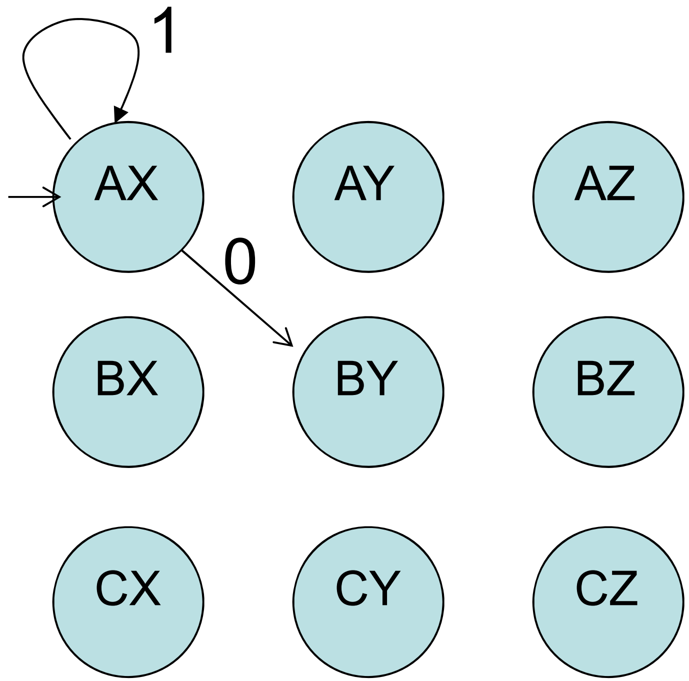
- $M = (Q, \Sigma, \delta, q_0, F)$
  - $Q = \{AX, AY, AZ, BX, BY, BZ, CX, CY, CZ\}$
  - $q_0 = AX$

$M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$
- $Q_1 = \{A, B, C\}$
- $q_1 = A$
- $F_1 = \{A, B\}$

$M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$
- $Q_2 = \{X, Y, Z\}$
- $q_2 = X$
- $F_2 = \{Z\}$

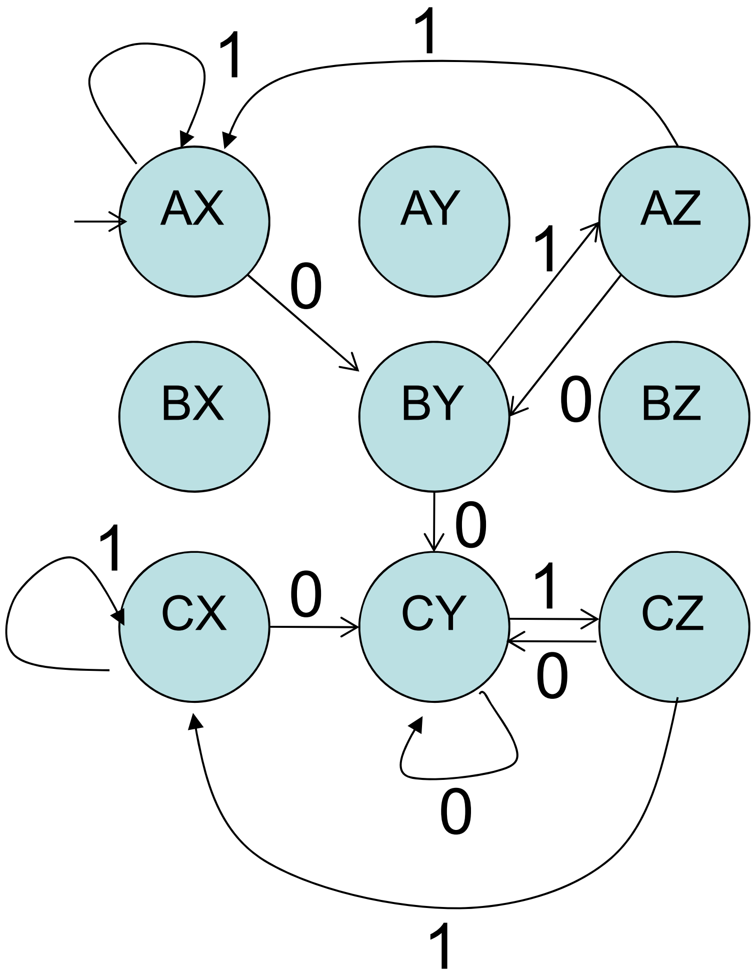


Intersection and Difference



$\delta((A,X), 1) = (\delta1\ (A,1), \delta2\ (X,1)) = (A, X)$
$\delta((A,X), 0) = (\delta1\ (A,0), \delta2\ (X,0)) = (B, Y)$

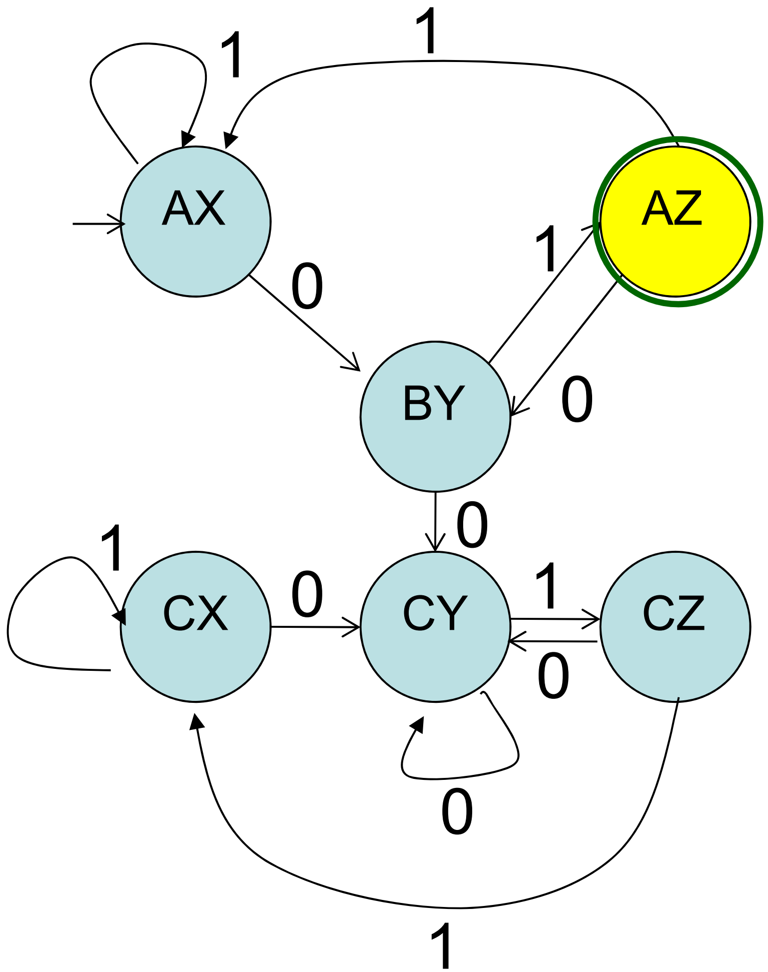
Intersection and Difference

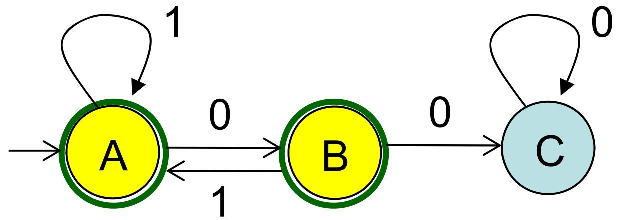


Intersection and Difference

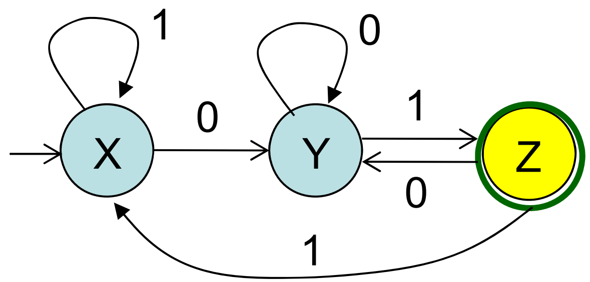
- Finally we can define F, the set of accepting states in M
- Intersection ($L_1 \cap L_2$)
  - $F = \{(p,q) \mid p \in F_1 \text{ and } q \in F_2\}$
- Difference ($L_1 - L_2$)
  - $F = \{(p,q) \mid p \in F_1 \text{ and } q \notin F_2\}$

Intersection ($L_1 \cap L_2$)



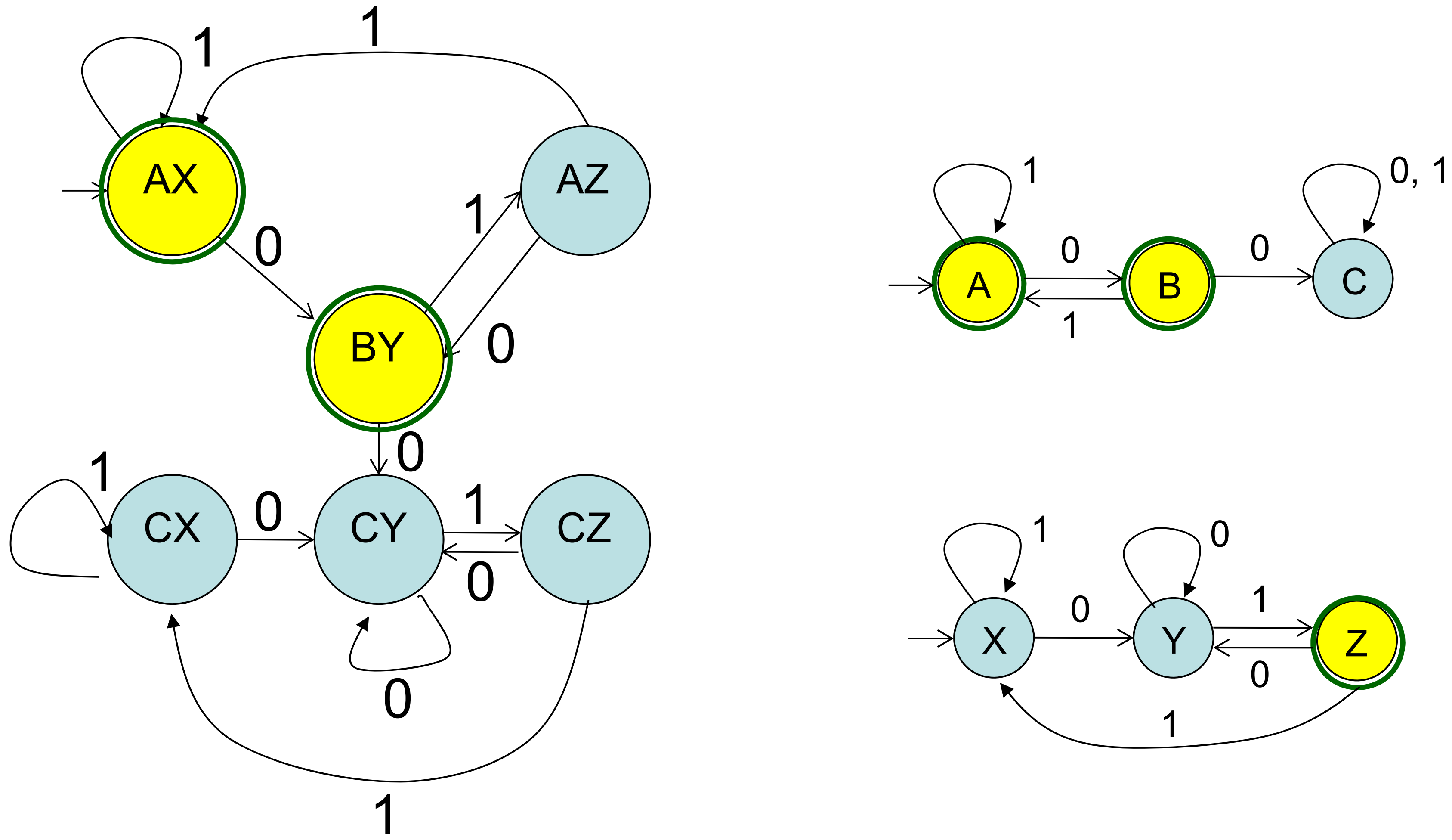


M_1

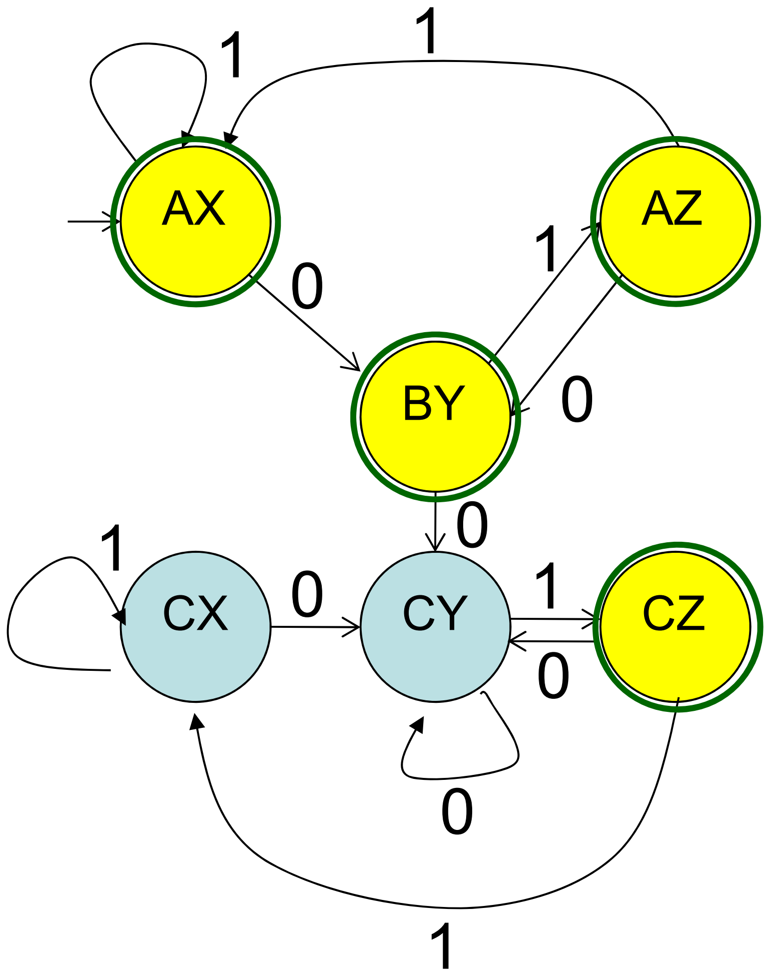


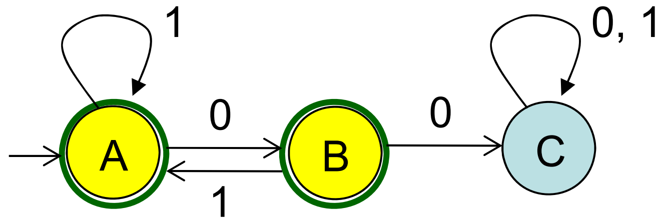
M_2

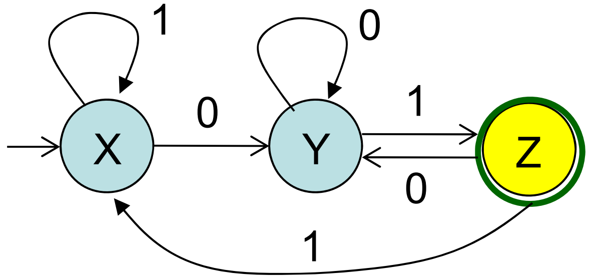
Difference (L_1 - L_2)



Union (L_1 U L_2)







Intersection

DeMorgan's Law: $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$

$L_1 \, , \; L_2$ regular

⇒ $\overline{L_1} \, , \; \overline{L_2}$ regular

⇒ $\overline{L_1} \cup \overline{L_2}$ regular

⇒ $\overline{\overline{L_1} \cup \overline{L_2}}$ regular

⇒ $L_1 \cap L_2$ regular

Example

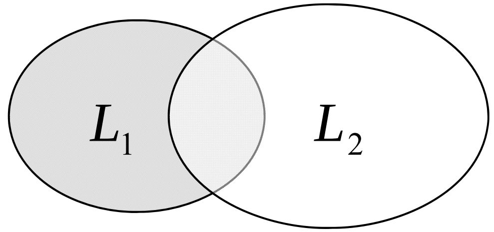
$L_1 = \{a^n b\}$ regular

$L_2 = \{ab, ba\}$ regular

$\Rightarrow$ $L_1 \cap L_2 = \{ab\}$

regular

Difference



Venn diagrams

$$L_1 - L_2 = L_1 \cap \overline{L_2}$$

L_1 , L_2 regular

$\overline{L_2}$ regular

$L_1 \cap \overline{L_2}$ regular

Closure under Other Operations

- Definition 4.1:
  – Suppose Σ and Γ are alphabets. Then a function

$$h: \Sigma \rightarrow \Gamma^*$$

is called a homomorphism. In other words, a homomorphism is a substitution in which a single letter is replaced with a string.

If L is a language on Σ, then its homomorphic image is defined as

$$h(L) = \{h(w): w \in L\}$$

Example 4.2

- $\Sigma = \{a, b\}$ and $\Gamma = \{a, b, c\}$ and define h by

$$h(a) = ab, h(b) = bbc.$$

$h(aba) = abbbcab$

The homomorphic image of $L = \{aa, aba\}$ is
$h(L) = \{abab, abbbcab\}$

Example 4.3

- $\Sigma = \{a, b\}$ and $\Gamma = \{b, c, d\}$ and define h by

$$h(a) = dbcc,\ h(b) = bdc.$$

If L is the regular language denoted by

$$r = (a + b^*)(aa)^*$$

then

$$r_1 = (dbcc + (bdc)^*)(dbccdbcc)^*$$

Theorem 4.3

- Let h be a homomorphism. If L is a regular language, then its homomorphic image $h(L)$ is also regular.

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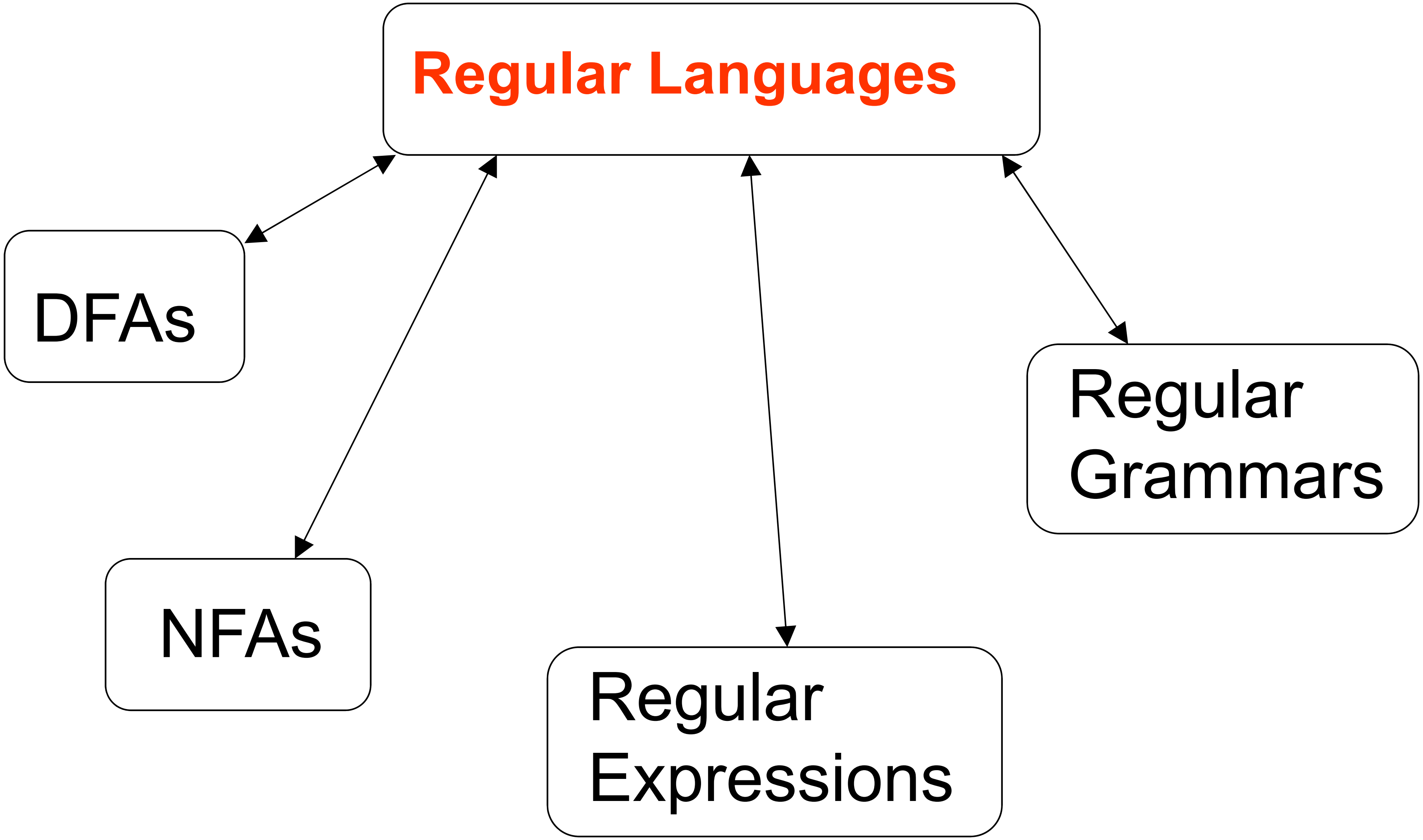
1 Closure Properties of Regular Languages

2 Elementary Questions about Regular Languages

3 Identifying Nonregular Languages

Standard Representations of Regular Languages



When we say: We are given a Regular Language L

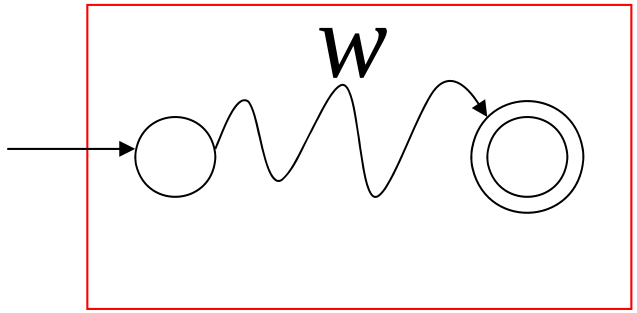
We mean: Language L is in a standard representation

Membership Question

Question: Given regular language L
and string w
how can we check if $w \in L$ **?**

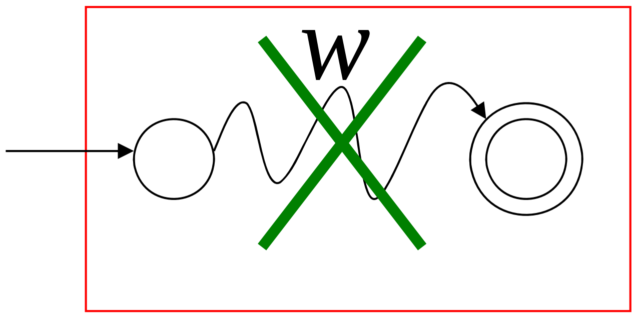
Answer: Take the DFA that accepts L
and check if w is accepted

DFA



$$w \in L$$

DFA



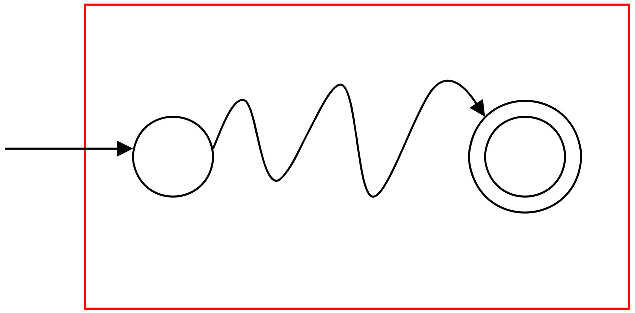
$$w \notin L$$

Question: Given regular language L how can we check if L is empty: $(L = \varnothing)$?

Answer: Take the DFA that accepts L

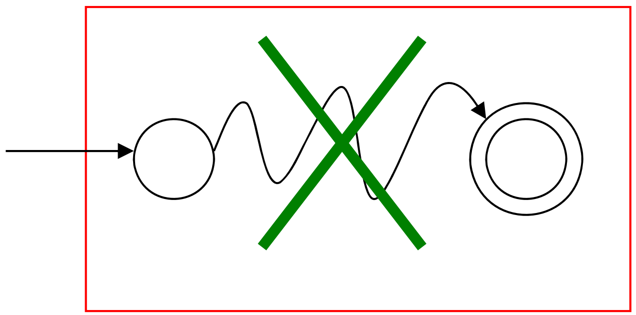
Check if there is any path from the initial state to a final state

DFA



$$L \neq \emptyset$$

DFA



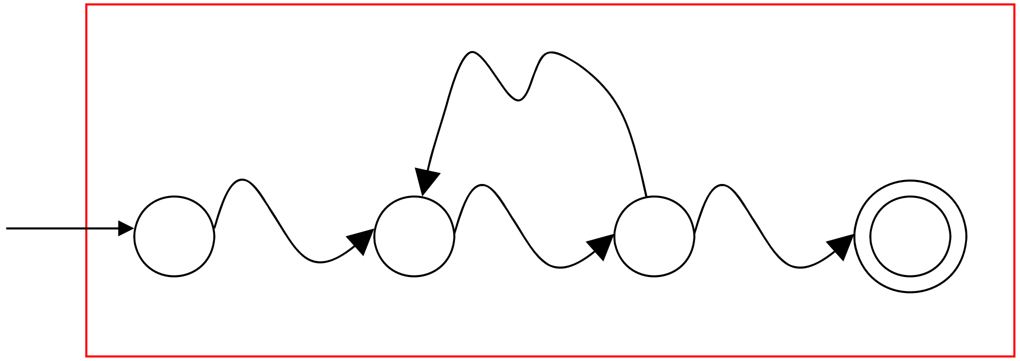
$$L = \emptyset$$

Question: Given regular language L how can we check if L is finite?

Answer: Take the DFA that accepts L

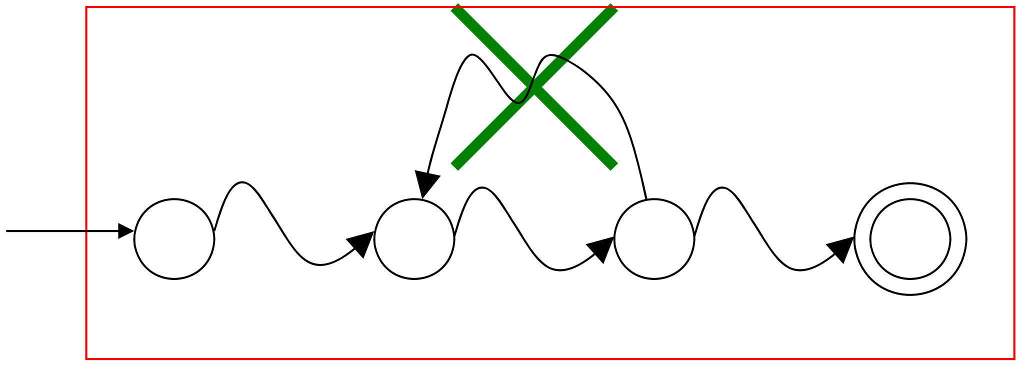
Check if there is a walk with cycle from the initial state to a final state

DFA



L is infinite

DFA



L is finite

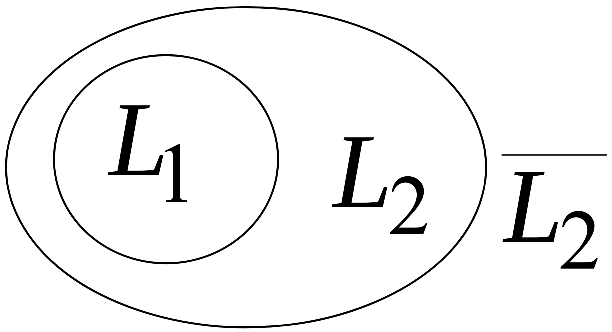
Question: Given regular languages L_1 and L_2 how can we check if $L_1 = L_2$ **?**

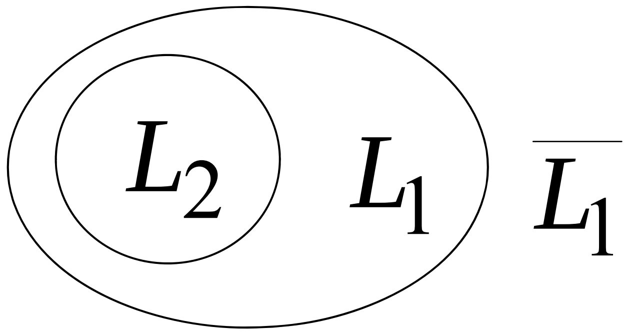
Answer: Find if $(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) = \emptyset$

$$(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) = \varnothing$$

$$\Downarrow$$

$L_1 \cap \overline{L_2} = \varnothing$ and $\overline{L_1} \cap L_2 = \varnothing$





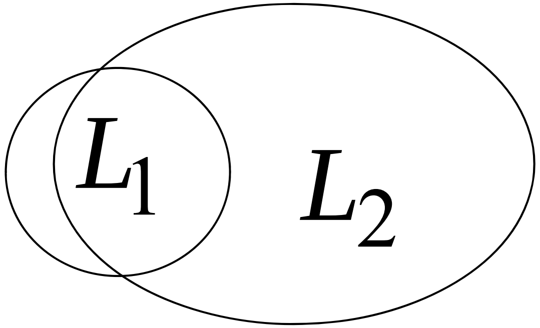
$L_1 \subseteq L_2$ $L_2 \subseteq L_1$

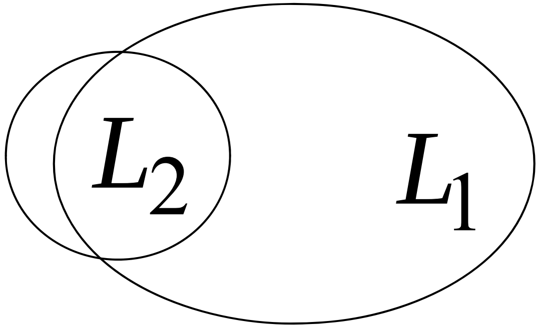
$$\Downarrow$$

$$L_1 = L_2$$

$$(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) \neq \emptyset$$

$L_1 \cap \overline{L_2} \neq \emptyset$ or $\overline{L_1} \cap L_2 \neq \emptyset$





$L_1 \not\subset L_2$ $L_2 \not\subset L_1$

$$L_1 \neq L_2$$

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Non-regular languages

$$\{a^n b^n : \quad n \geq 0\}$$

$$\{vv^R : \quad v \in \{a,b\}^*\}$$

Regular languages

$$a^*b \qquad b^*c + a$$

$$b + c(a+b)^* \qquad etc...$$

Finite languages

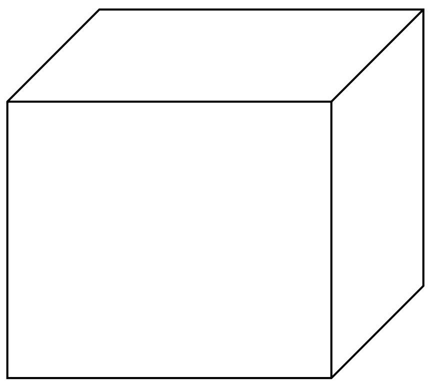
How can we prove that a language L
is not regular?

Prove that there is no DFA that accepts L

Problem: this is not easy to prove

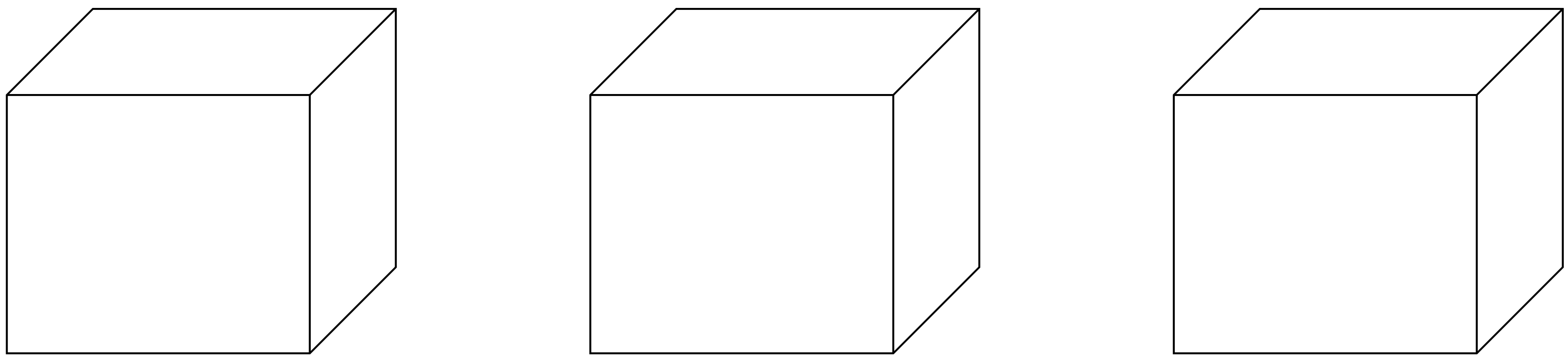
Solution: the Pumping Lemma !!!

The Pigeonhole Principle

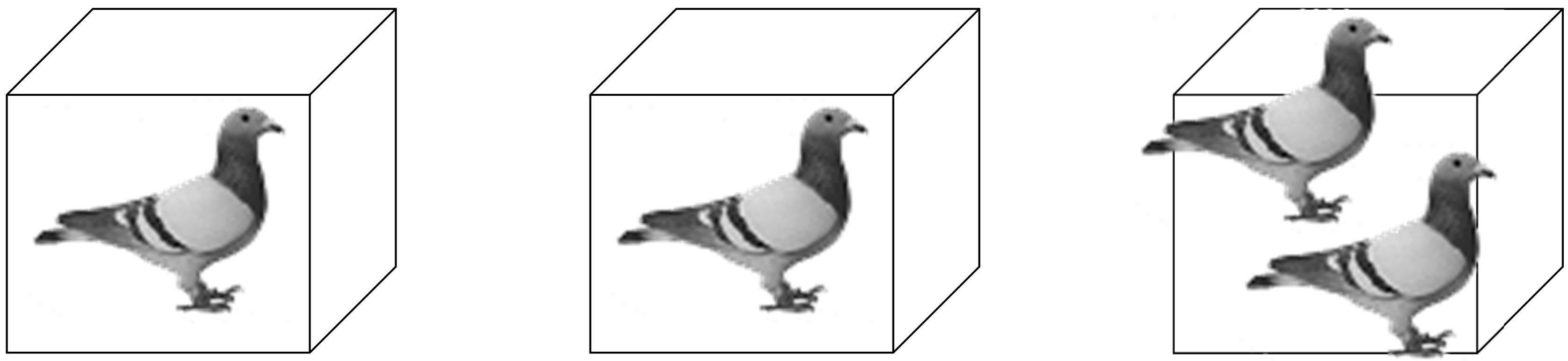


4 pigeons

3 pigeonholes

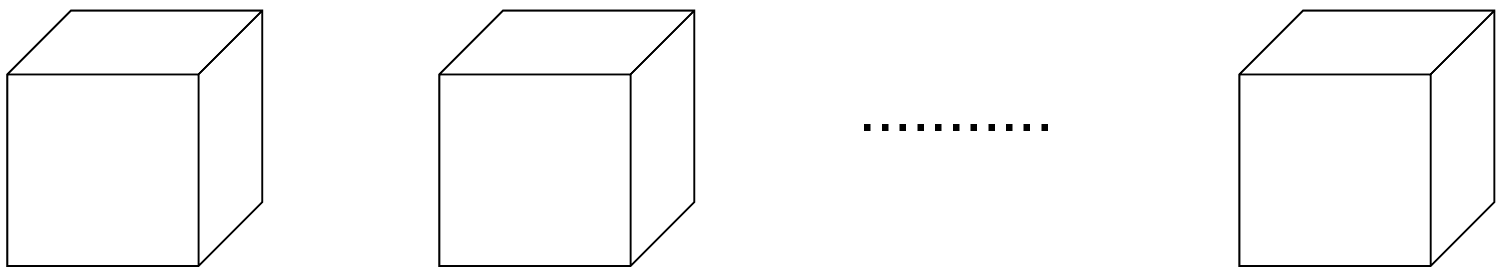


A pigeonhole must
contain at least two pigeons



n pigeons

m pigeonholes $\qquad n > m$



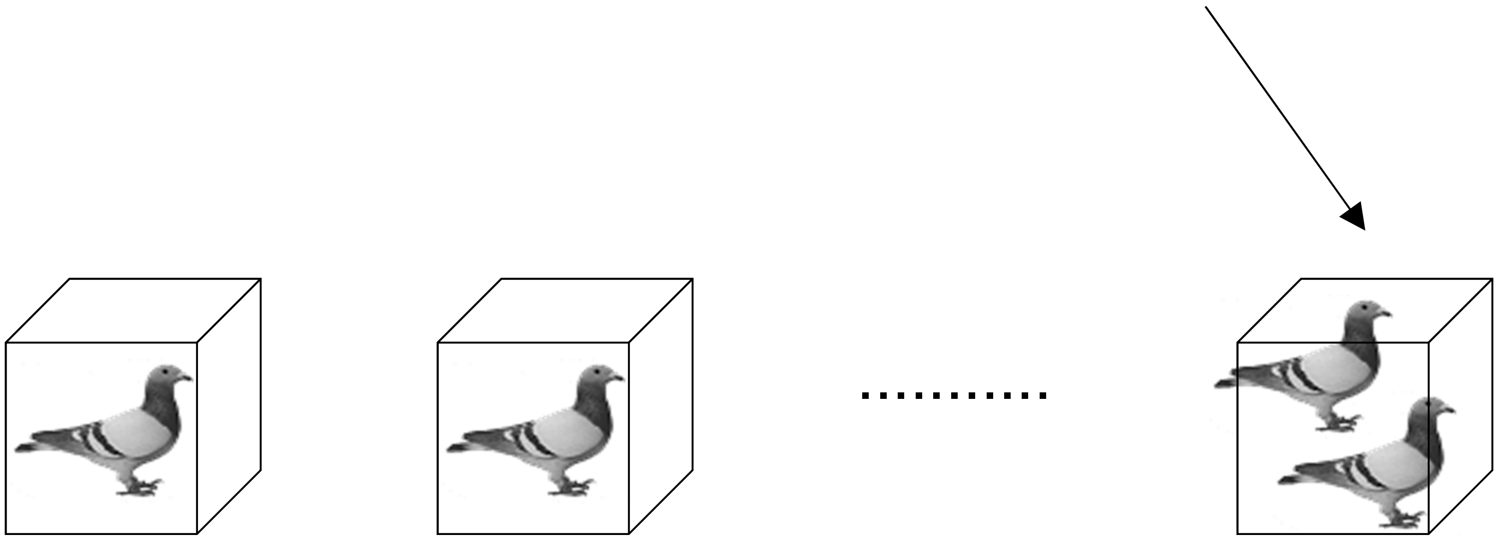
The Pigeonhole Principle

n pigeons

m pigeonholes

$$n > m$$

There is a pigeonhole with at least 2 pigeons

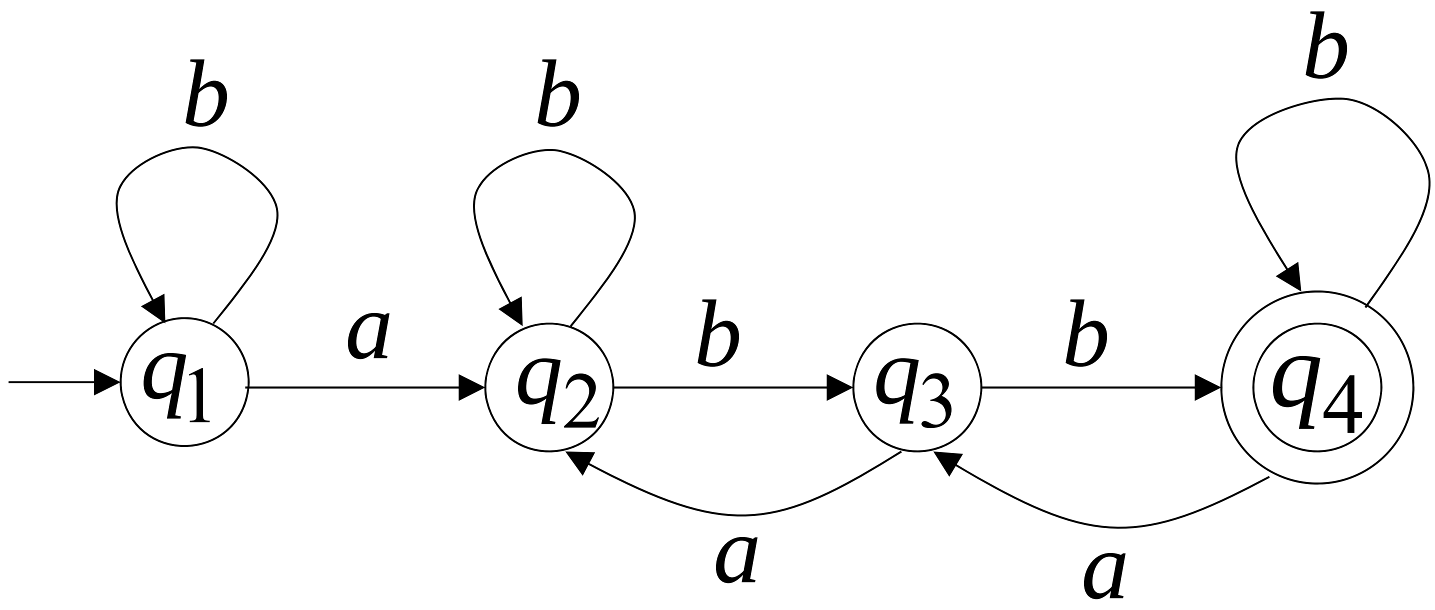


The Pigeonhole Principle

and

DFAs

DFA with 4 states

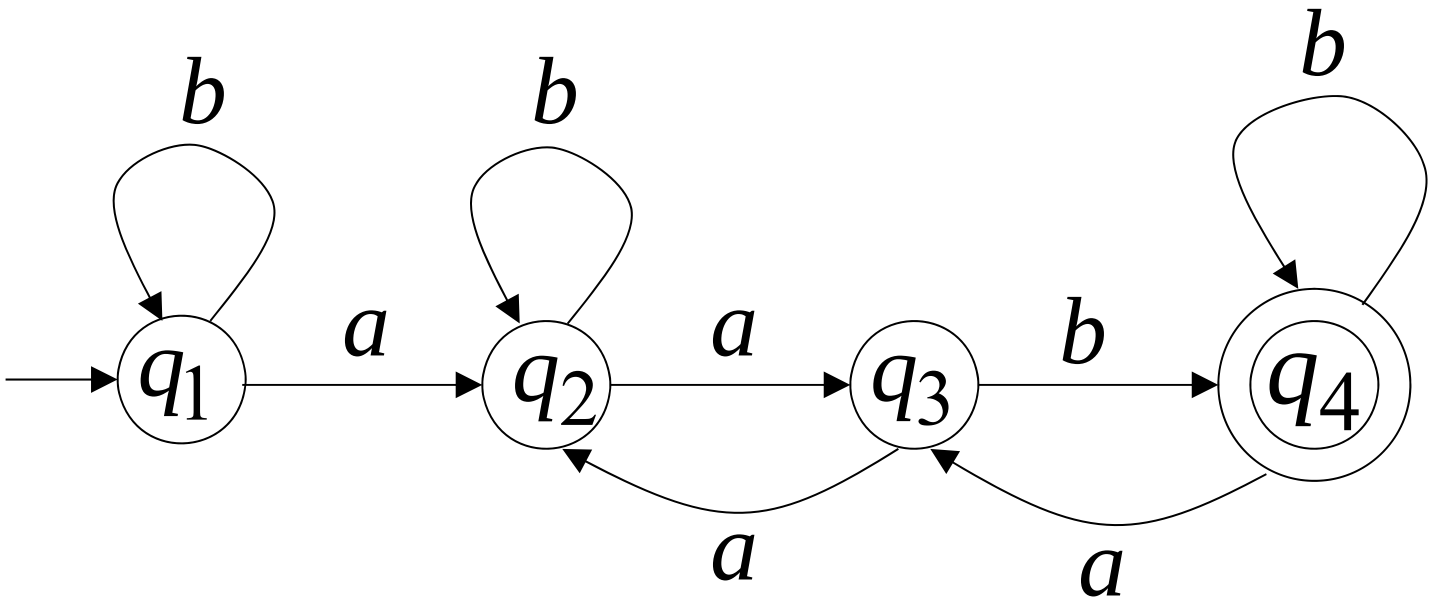


In walks of strings: a

aa

aab

no state
is repeated



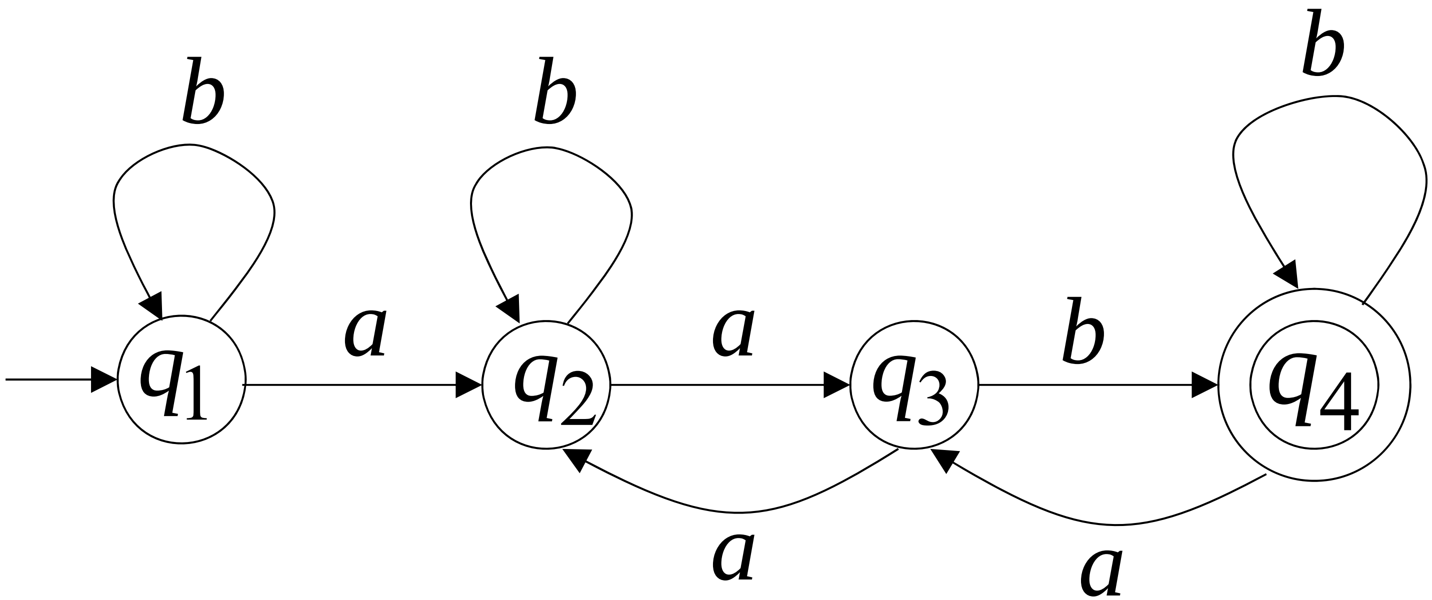
In walks of strings: *aabb*

bbaa

abbabb

abbbabbabb...

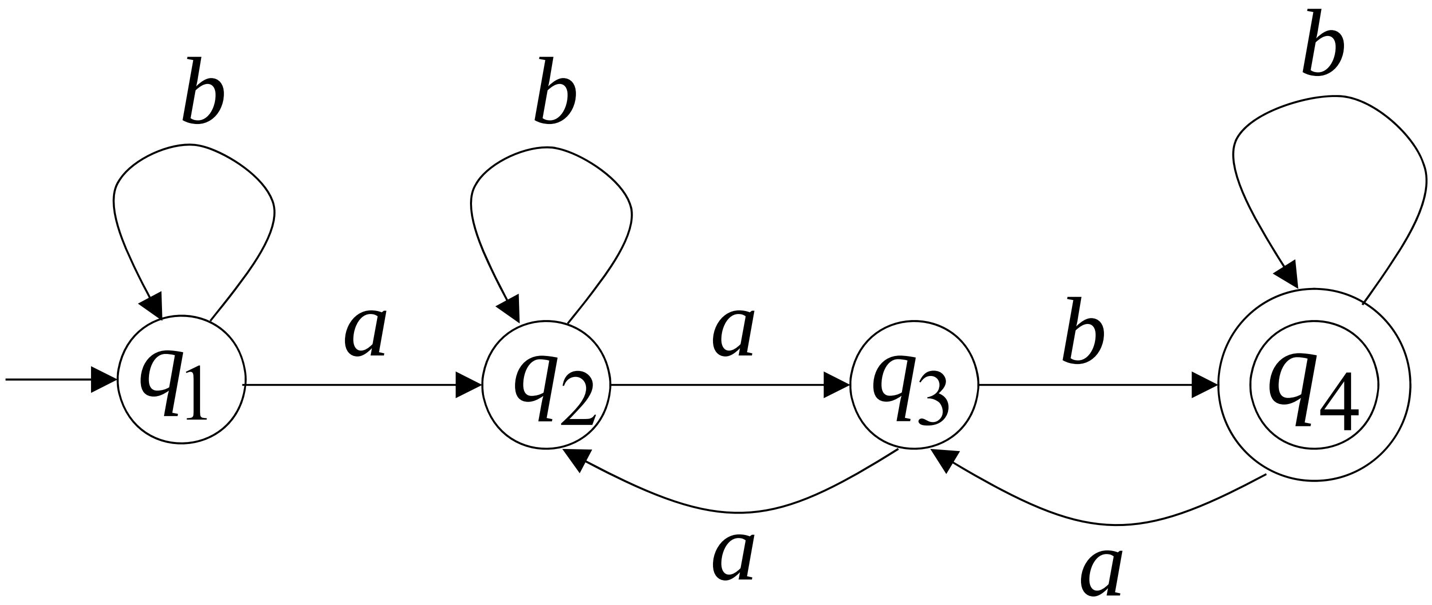
a state
is repeated



If string w has length $|w| \geq 4$:

Then the transitions of string w are more than the states of the DFA

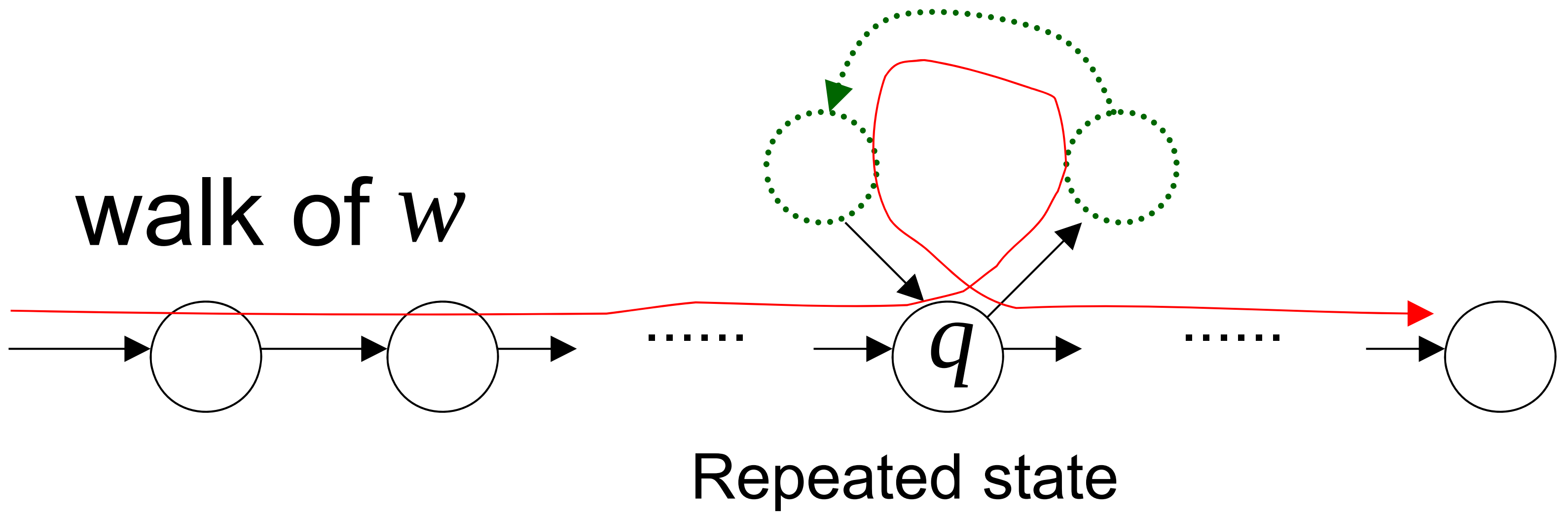
Thus, a state must be repeated



In general, for any DFA:

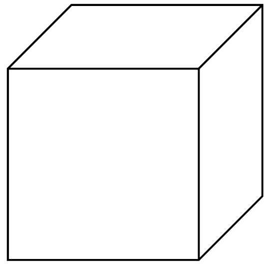
String w has length $\geq$ number of states

A state q must be repeated in the walk of w

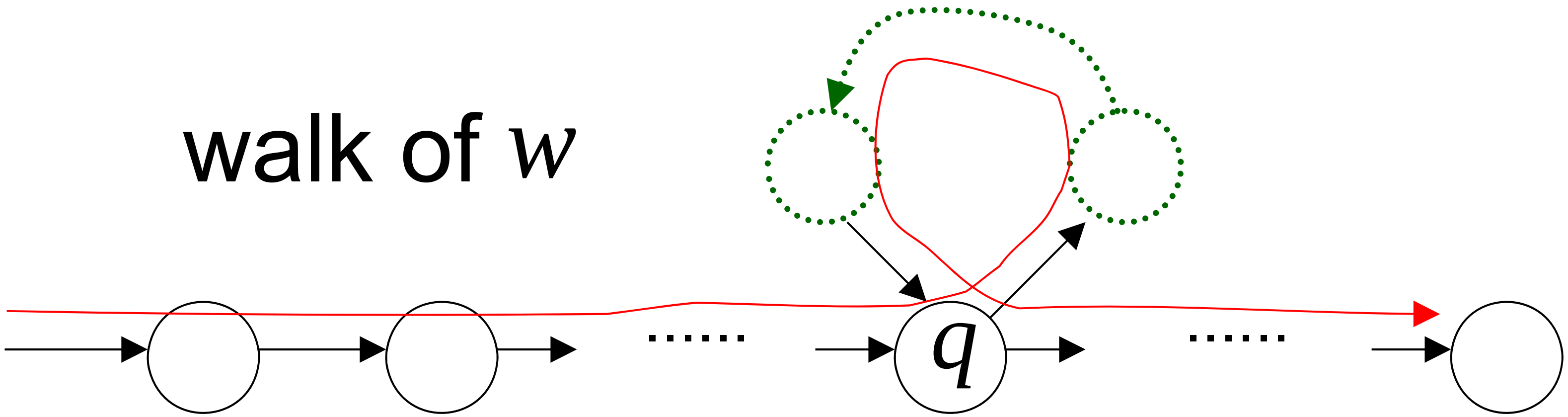


In other words for a string w :

$\xrightarrow{a}$ transitions are pigeons 

q states are pigeonholes 

walk of w



Repeated state

Example 4.6

- (Is $L = \{a^n b^n : n \geq 0\}$ regular?)
- Suppose L is regular → A DFA M exists for it
  - $\delta^*(q_0, a^i)$ for i = 1, 2, 3, … (unlimited)
  - But only a finite number of states in M
  - By pigeonhole principle, there must some state q s.t.
    $\delta^*(q_0, a^n) = q$ and $\delta^*(q_0, a^m) = q$ with $n \neq m$
  - Since M accepts $a^n b^n$ we must have
    $\delta^*(q, b^n) = q_f \in F$
    $\delta^*(q_0, a^m b^n) = q_f \in F$ (contradiction!! $\because n \neq m$)

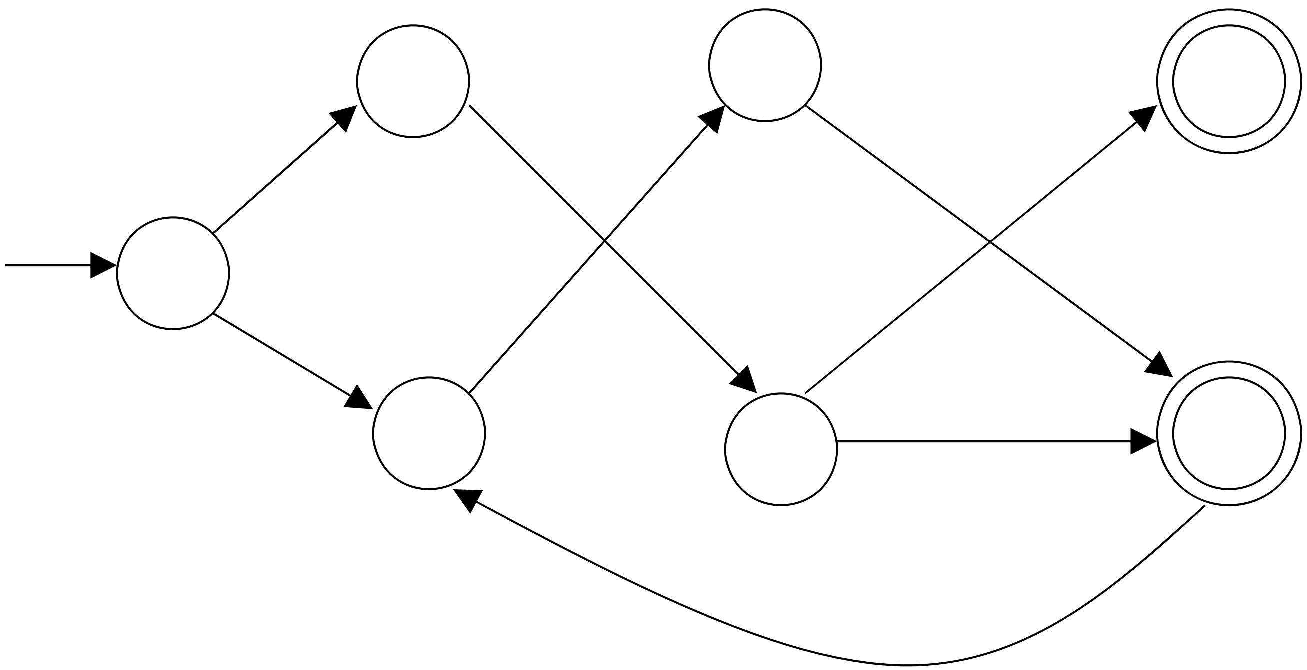
To accept all $a^n b^n$, an automaton would have to differentiate between all prefixes a^n and a^m.

But since there are only a finite number of internal states with which to do this, there are some n and m for which the distinction cannot be made.

The Pumping Lemma

Take an infinite regular language L

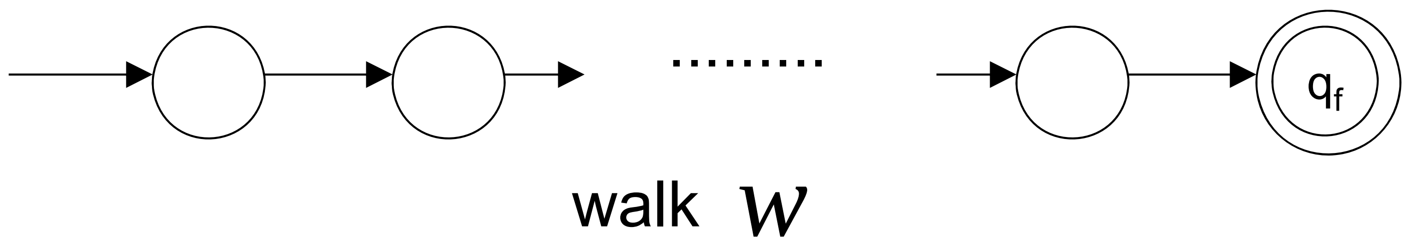
There exists a DFA M that accepts L



m
states

Take a string w with $w \in L$ (drive to q_f)

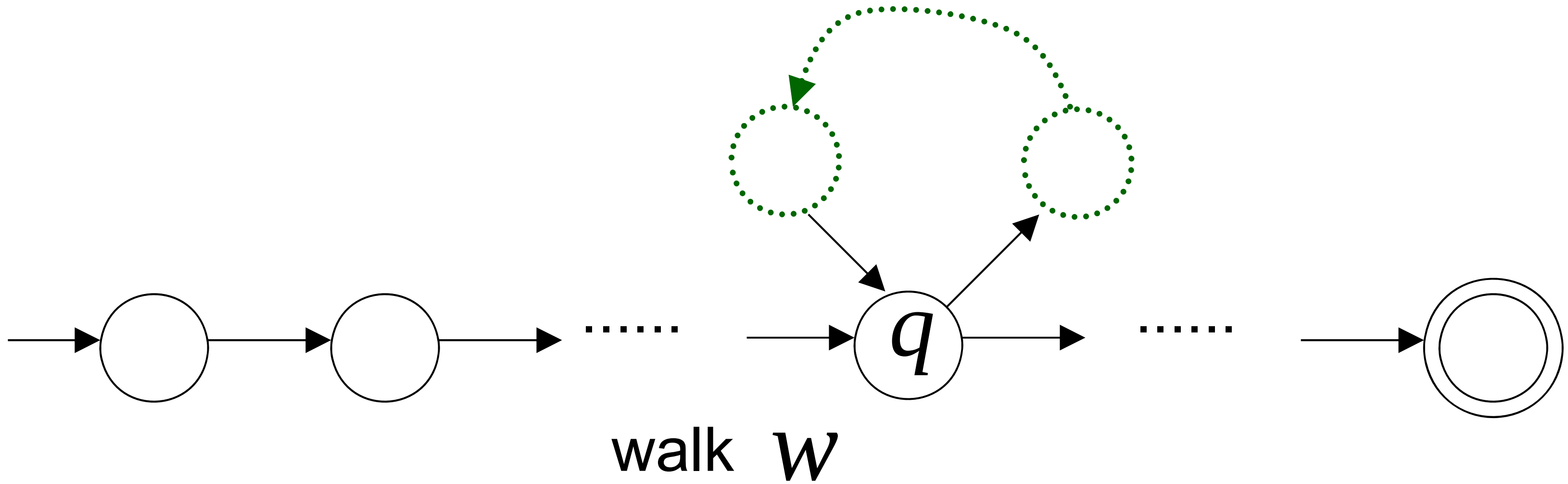
There is a walk with label w :



If string w has length $|w| \geq m$ (number of states of DFA)

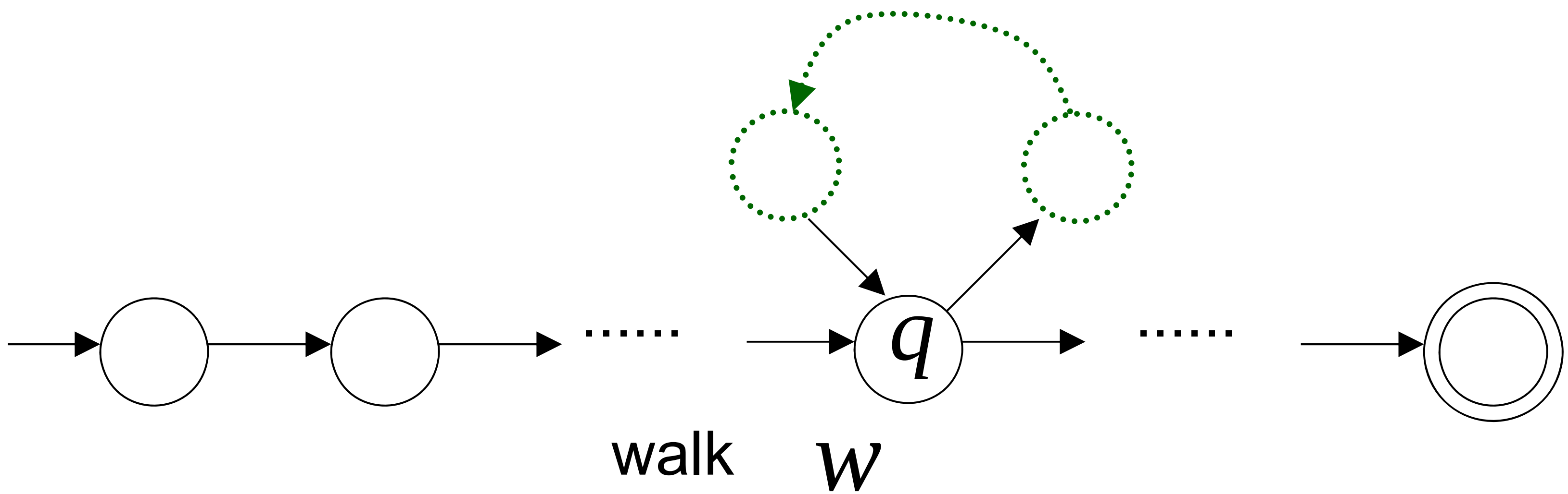
then, from the pigeonhole principle:

a state is repeated in the walk w

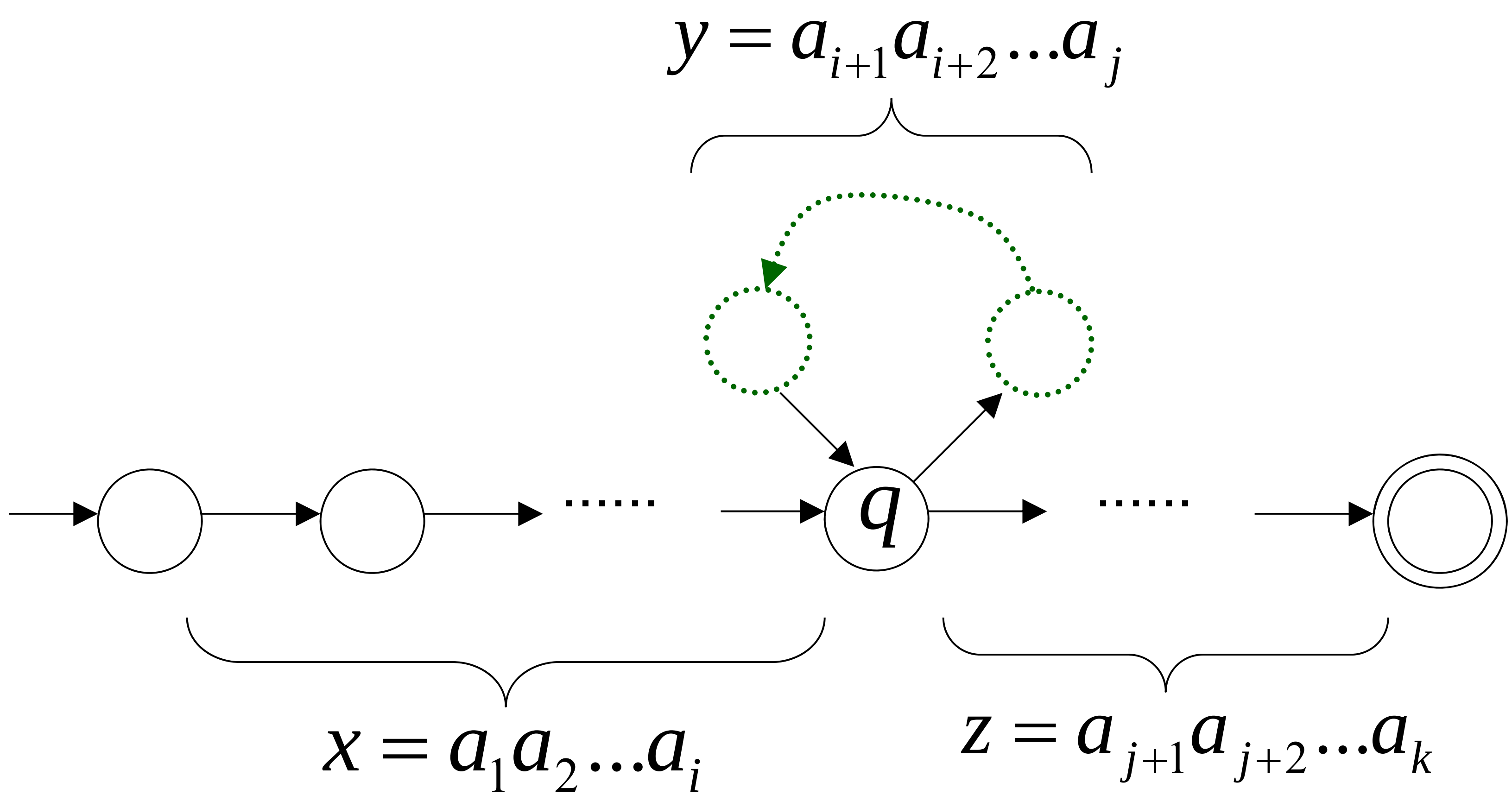


Let q be the first state repeated in the walk of w

(such a repetition must start no later than the m^{th} move)



Write $w = x\ y\ z$



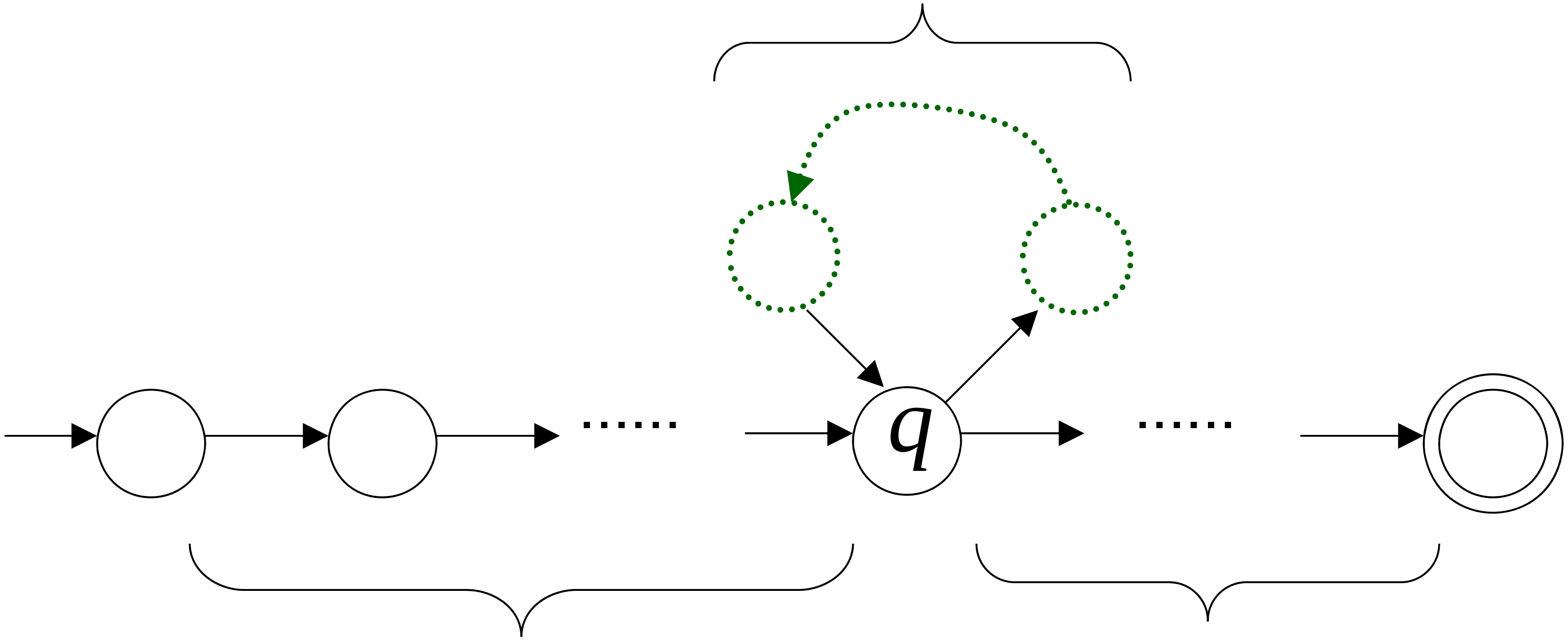
Observations:

length $|x\,y| \leq m$ Number of states of DFA

length $|y| \geq 1$

Remember 'q' is the first repeated state, meaning that a_1, a_2, …, a_i, a_{i+1} …, a_j, are passed through different states

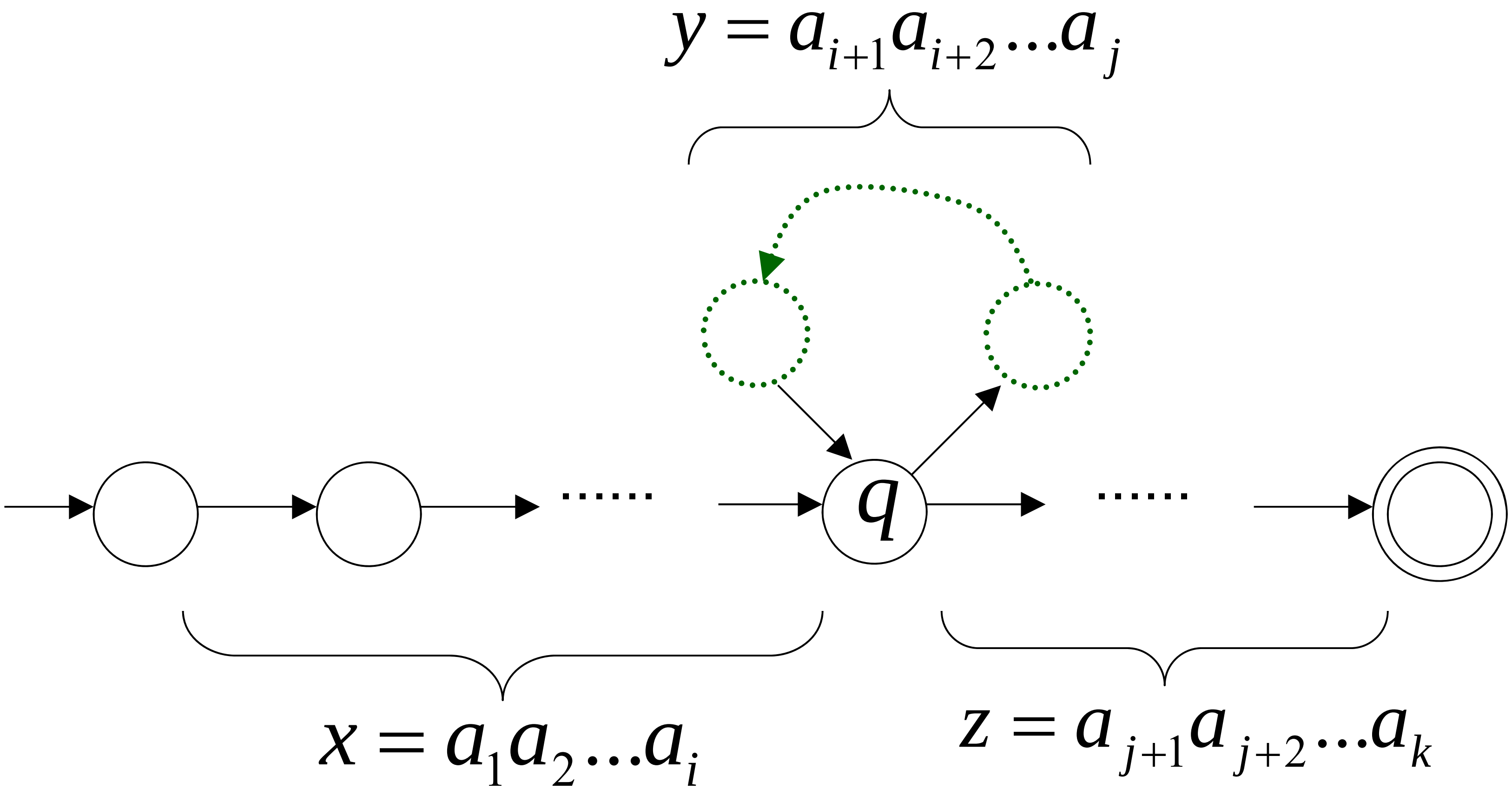
$$y = a_{i+1}a_{i+2}\ldots a_j$$



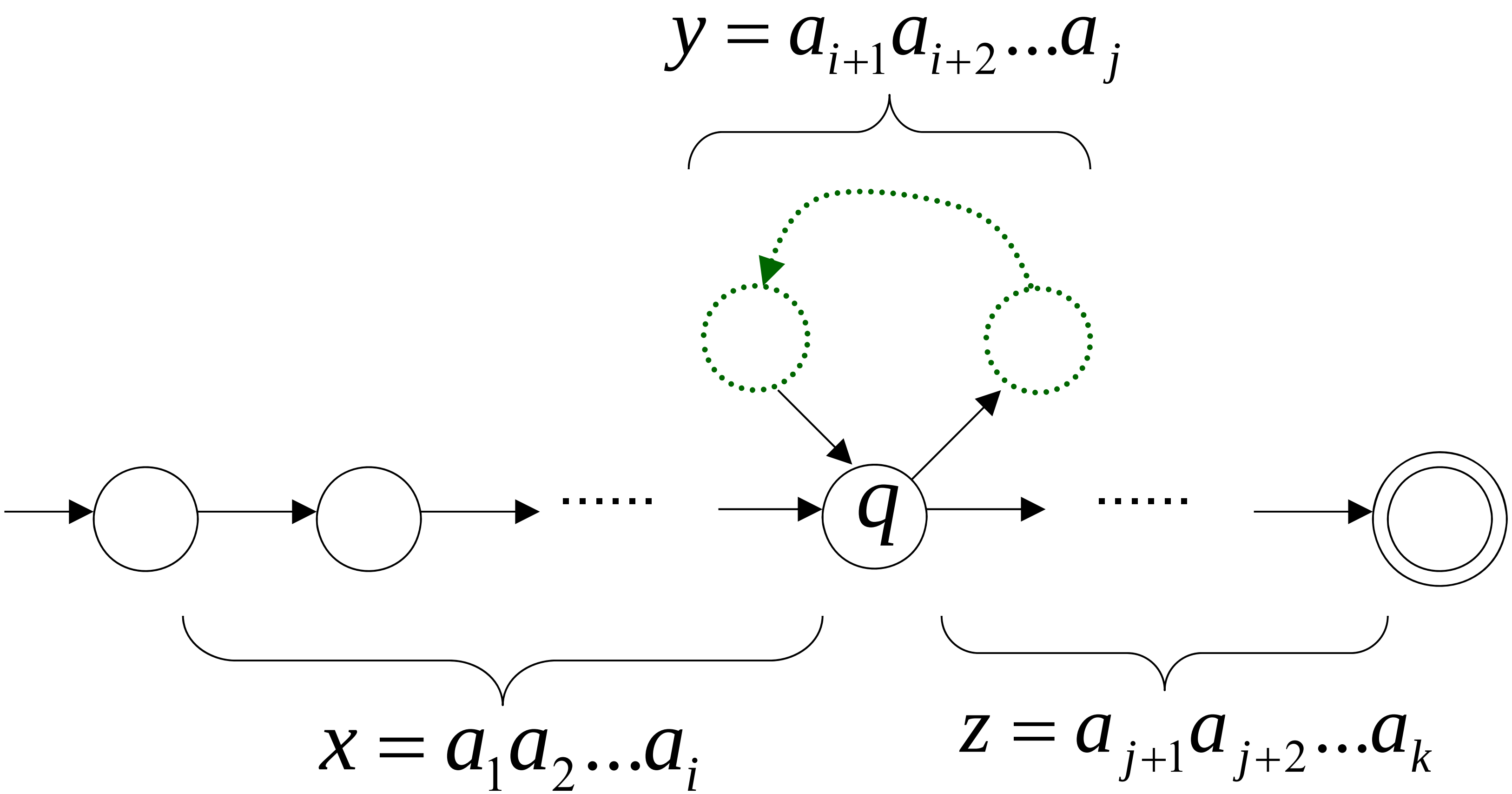
$$x = a_1 a_2 \ldots a_i$$

$$z = a_{j+1}a_{j+2}\ldots a_k$$

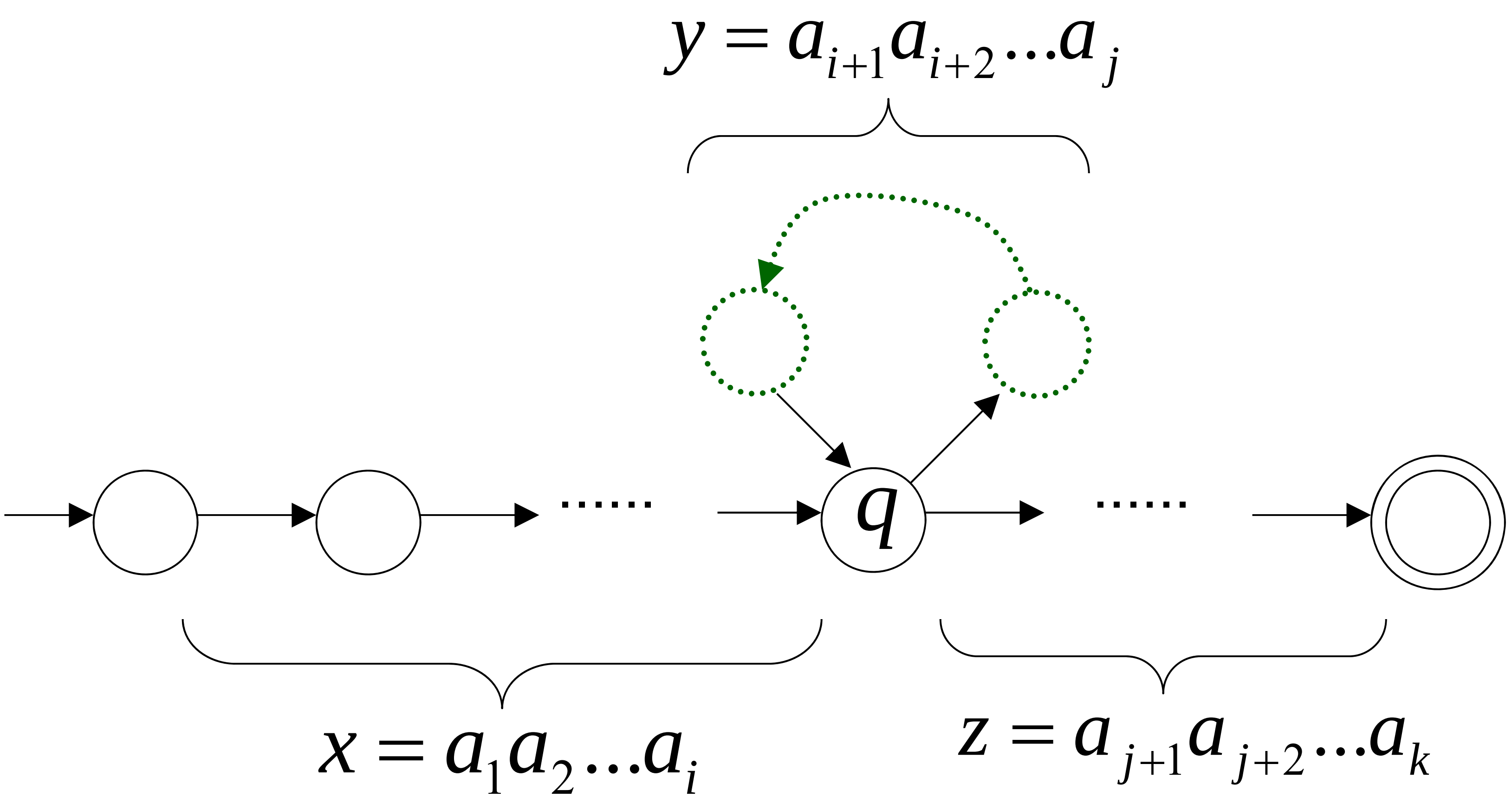
Observation: The string $x\,z$ is accepted



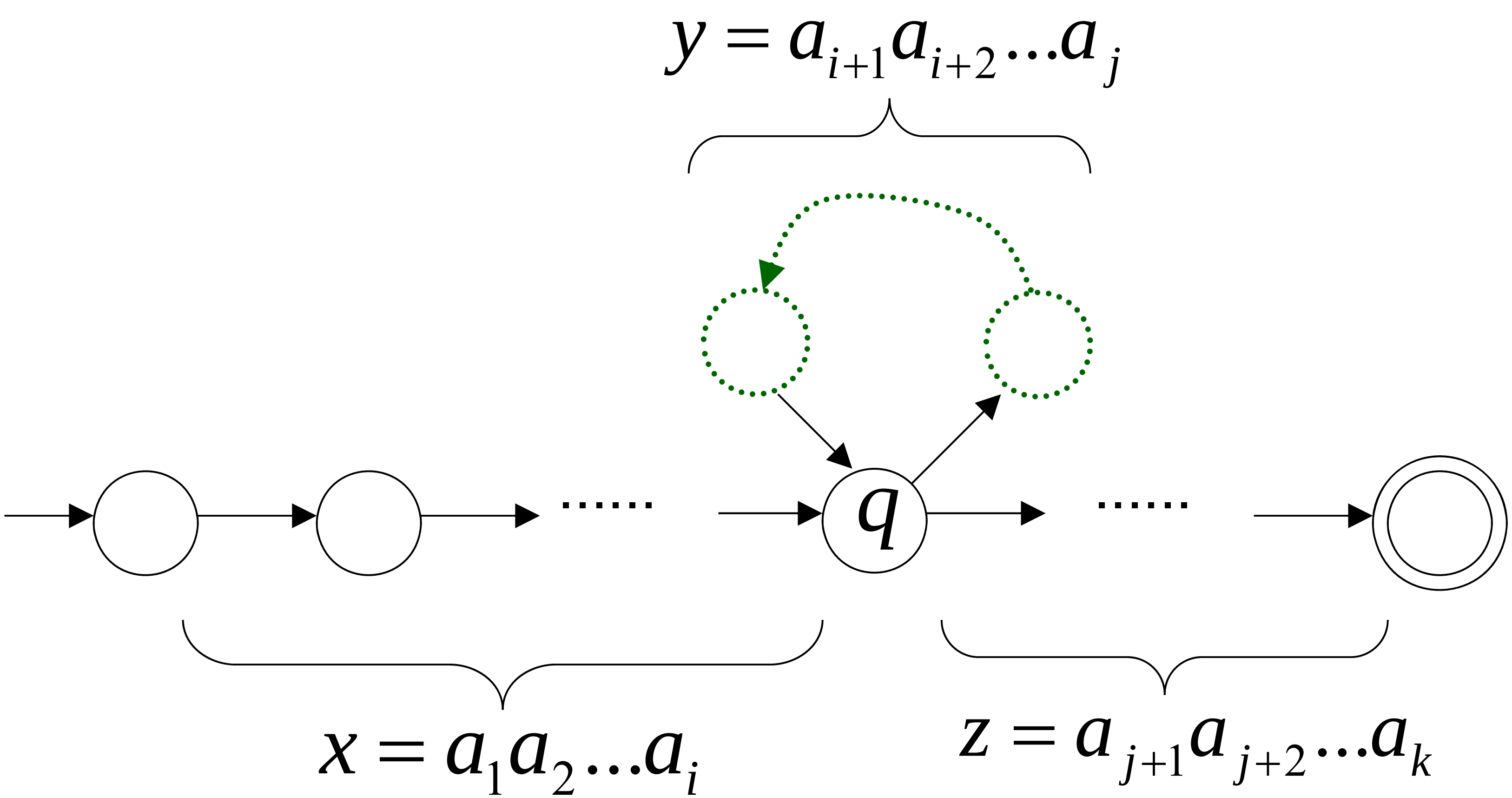
Observation: The string $x\ y\ y\ z$ is accepted



Observation: The string $x\ y\ y\ y\ z$ is accepted



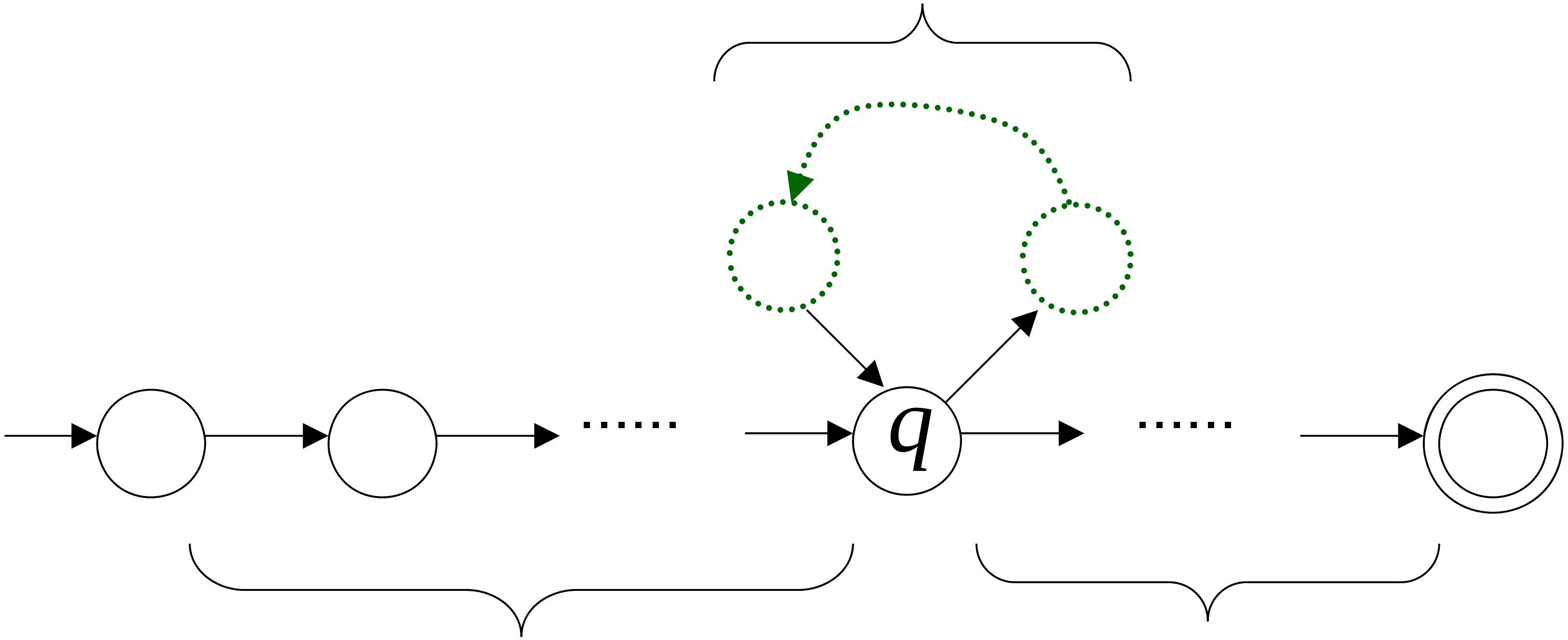
In General: The string $x\,y^i\,z$ is accepted $i = 0, 1, 2, \ldots$



In General: $x\, y^{i}\, z \in L \qquad i = 0, 1, 2, \ldots$

Language accepted by the DFA

$y = a_{i+1} a_{i+2} \ldots a_j$



$x = a_1 a_2 \ldots a_i$

$z = a_{j+1} a_{j+2} \ldots a_k$

In other words, we described:

The Pumping Lemma !!!

The Pumping Lemma:

- Given a infinite regular language L

there exists an integer m

for any string $w \in L$ with length $|w| \geq m$

we can write $w = x\, y\, z$

with $|x\, y| \leq m$ and $|y| \geq 1$

such that: $x\, y^i\, z \in L \quad i = 0, 1, 2, \ldots$

The Pumping Lemma Game

- Goal: Win the game by establishing a contradiction of the pumping lemma

O Picks m

P Picks a string w in L of length equal or greater than m. We are free to choose any w, subject to w ϵ L and |w| ≥ m.

O Chooses the decomposition xyz, subject to |xy| ≤ m, |y| ≥ 1.

P Picks i such that the pumped string w_i is not in L.

Applications

of

the Pumping Lemma

Example 4.7

Theorem: The language $L = \{a^n b^n : n \geq 0\}$

is not regular

Proof: Use the Pumping Lemma

$$L = \{a^n b^n : n \geq 0\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{a^n b^n : n \geq 0\}$$

O

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$

length $|w| \geq m$

P We pick $w = a^m b^m$

- Write: $a^m b^m = x\ y\ z$

From the Pumping Lemma

it must be that length $|x\ y| \leq m, \quad |y| \geq 1$

$$xyz = a^m b^m = \overbrace{a \ldots a a \ldots a a \ldots a}^{m} \overbrace{b \ldots b}^{m}$$

$$\underbrace{a \ldots a}_{x} \underbrace{a \ldots a}_{y} \underbrace{a \ldots a b \ldots b}_{z}$$

Thus: $y = a^k, \quad k \geq 1$

$$x\,y\,z = a^m b^m \qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $x\,y^i\,z \in L$

$i = 0, 1, 2, \ldots$

Thus: $x\,y^2\,z \in L$

$$x\,y\,z = a^m b^m \qquad\qquad y = a^k, \quad k \ge 1$$

P

From the Pumping Lemma: $x\,y^2\,z \in L$

$$\overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}a...a}^{m+k}$$

$$xy^2z = \underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}\underbrace{a...a\overbrace{b...b}^{m}}_{z} \in L$$

Thus: $a^{m+k} b^m \in L$

$$a^{m+k}b^m \in L \qquad k \geq 1$$

BUT: $L = \{a^n b^n : n \geq 0\}$

$$\Downarrow$$

$$a^{m+k}b^m \notin L$$

CONTRADICTION!!!

Example 4.8

- Show that $L = \{ww^R : w \in \Sigma^*\}$ is not regular

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{ww^R : w \in \Sigma^*\}$$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$ and

length $|w| \geq m$

We pick $w = a^m b^m b^m a^m$

Write $a^m b^m b^m a^m = x\ y\ z$

From the Pumping Lemma

it must be that length $|x\ y| \leq m, \quad |y| \geq 1$

$$xyz = \underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a\overbrace{b...b}^{m}\overbrace{b...b}^{m}\overbrace{a...a}^{m}}_{z}$$

(with the first m spanning all the leading a's)

Thus: $y = a^k, \quad k \geq 1$

$$x\ y\ z = a^m b^m b^m a^m \qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $x\ y^i\ z \in L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\ y^2\ z \in L$

$$x\ y\ z = a^m b^m b^m a^m \qquad\qquad y = a^k, \quad k \ge 1$$

From the Pumping Lemma: $x\ y^2\ z \ \in \ L$

$$xy^2z = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}a...a}^{m\ +\ k}\overbrace{b...b}^{m}\overbrace{b...b}^{m}\overbrace{a...a}^{m} \ \in L$$

with z the part $a...ab...bb...ba...a$ after the second y.

Thus: $a^{m+k} b^m b^m a^m \ \in L$

$$a^{m+k}b^m b^m a^m \in L \qquad k \geq 1$$

BUT: $L = \{ww^R : w \in \Sigma^*\}$

$$\Downarrow$$

$$a^{m+k}b^m b^m a^m \notin L$$

CONTRADICTION!!!

$$L = \{ww^R : w \in \Sigma^*\}$$

If we choose $w = a^{2m} \in L$

The opponent picks $y = a^k$?

To apply the pumping lemma, we assume that the opponent will make the best move.
Ex. $y = aa$

Example 4.9

- Let Σ = {a, b}. The language

  $L = \{w \in \Sigma^* : n_a(w) < n_b(w)\}$ is not regular

O Given m

P Picks w = $a^m b^{m+1}$

O $|xy| \leq m$ → picks y with all a's → $y = a^k$, $1 \leq k \leq m$

P Picks i = 2 → $w_2 = a^{m+k} b^{m+1}$ is not in L

Example 4.10

- Let Σ = {a, b}. The language

  $L = \{(ab)^n a^k : n > k, k \geq 0\}$ is not regular

O Given m

P Picks w = $(ab)^{m+1}a^m$

O |xy| ≤ m → picks y = a (or ab)

P Picks i = 0 → $w_0 = (ab)^p b(ab)^q a^m$ is not in L

($w_0 = (ab)^m a^m$ is not in L)

Example 4.11

- Let Σ = {a}. The language
  $L = \{a^n : n \text{ is a perfect square}\}$ is not regular

O Given m

P Picks $w = a^{m^2}$

O $|xy| \leq m$ → picks $y = a^k$, $1 \leq k \leq m$

P Picks $i = 0$ → $w_0 = a^{m^2-k}$ is not in L

∵ $m^2-k > (m-1)^2$

Example 4.12

- Let $\Sigma = \{a, b, c\}$. The language $L = \{a^n b^k c^{n+k} : n \geq 0, k \geq 0\}$ is not regular

O Given m

P Picks $w = a^m b^m c^{2m}$

O $|xy| \leq m \rightarrow$ picks $y = a^k$, $1 \leq k \leq m$

P Picks $i = 0 \rightarrow w_0 = a^{m-k} b^m c^{2m}$ is not in L

Use homomorphism $h(a) = a$, $h(b) = a$, $h(c) = c$

$\rightarrow h(L) = \{a^{n+k} c^{n+k} : n+k \geq 0\}$

Example 4.13

- Let Σ = {a, b}. The language $L = \{a^n b^l : n \neq l\}$ is not regular

Set n = l + 1?

$$L_1 = \overline{L} \cap L(a^* b^*)$$

O Given m

P Picks $w = a^{m!} b^{(m+1)!}$

O |xy| ≤ m → picks $y = a^k$, $1 \leq k \leq m$

P Pumps i times → $w_i = a^{m!+(i-1)k} b^{(m+1)!}$

$$if \ \exists i \quad \text{s.t.} \quad \mathrm{m!+(i-1)k = (m+1)!}$$

$$\mathrm{i = 1 + \frac{mm!}{k}}$$

∵ k ≤ m → i is an integer

Common Pitfalls Using Pumping Lemma

- Use pumping lemma to show that a language is regular
- Start with a string not in L
- Make some assumptions about the decomposition xyz

Common Pitfalls Using Pumping Lemma

- Use pumping lemma to show that a language is regular
  - Even if you can show that no string in a language L can ever be pumped out, you cannot conclude that L is regular.
- Start with a string not in L
- Make some assumptions about the decomposition xyz

Common Pitfalls Using Pumping Lemma

- Use pumping lemma to show that a language is regular
- Start with a string not in L
  - EX. L = {a^n: n is a prime number}
  - Given m, let w = a^m (incorrect)
  - Given m, let w = a^P, where P is a prime number larger than m
- Make some assumptions about the decomposition xyz

Common Pitfalls Using Pumping Lemma

- Use pumping lemma to show that a language is regular
- Start with a string not in L
- Make some assumptions about the decomposition xyz
  - EX. L = {a^n: n is a prime number}
  - y = a^k, with k odd. Then w = xz is an even-length string and thus not in L (incorrect)

More Example

- Let $\Sigma = \{a\}$. The language

  $L = \{a^n$: n is a prime number$\}$ is not regular

O Take p to be the smallest prime # $\geq m$

P Picks $w = a^p$

O $|xy| \leq m \rightarrow$ picks y with all a's $\rightarrow$ $y = a^k, 1 \leq k \leq m$

P Pumps i times$\rightarrow$ $w_i = a^{p+(i-1)k}$

if we take $i-1=p$, then $p+(i-1)k=p(k+1)$ is composite and w_{p+1} is not in L

Short Quiz

Please use the pumping lemma to show that each of these languages is nonregular:

$$A = \{x \in \{0,1\}^* \ : \ \text{the length of } x \text{ is odd, and its middle symbol is } 1\}$$

$$\{a^n b^n a^m \text{ where } n = 0, 1, 2, \ldots \text{ and } m = 0, 1, 2, \ldots\}$$

Questions?