Assignment: due July the 17th of 2023, 23:59 hrs

An investor has total capital of 1 that she wants to invest in n companies. The run time for the investment is t time units. The market value of company i at time t is given by X_i. Let $x_i \in \mathbb{R}, i = 1, \dots, n$ be given thresholds. Company i will not be able to generate any profit if $X_i < x_i$. In case $X_i \geq x_i$ the return on the investment is given by Y_i. Investing a fraction p_i of the capital yields expected return

$$p_i \mathbb{E}[Y_i 1_{X_i \geq x_i}]$$

for company i. The investor wants to find the optimal investment strategy, that is, the investor wants to solve the problem

$$\max \sum_{i=1}^{n} p_i \mathbb{E}[Y_i 1_{X_i \geq x_i}], \tag{1}$$

such that

$$\sum_{i=1}^{n} p_i = 1 \quad \text{and} \quad 0 \leq p_i \leq 1. \tag{2}$$

(a). Consider the following *small investor problem*. Assume that $n = 3$ and Y_i are independent uniformly distributed on $[0, X_i]$, and we let

$$X_i = \frac{\rho V + \sqrt{1-\rho^2}\,\eta_i}{\max(W, 1)}, \quad 1 \leq i \leq n,$$

with η_i normally distributed with mean 0 and variance i modelling the company's idiosyncratic risk, V standard normally distributed modelling the common factor that affects the economy, and W exponentially distributed with rate $1/0.3$ modelling common market shocks. The variables V, W, and η_i's are all independent. Set the weight factor $\rho = 0.6$, and the thresholds $x_1 = 2, x_2 = 3$, and $x_3 = 1$. Let

$$J(p_1, p_2, p_3) = \sum_{i=1}^{3} p_i \mathbb{E}[Y_i 1_{X_i \geq x_i}].$$

Find the optimal allocation vector (p_1, p_2, p_3) by applying an appropriate SA algorithm. Discuss your choice of algorithm and make an output analysis.

(b). Consider the *big investor problem*. The model is as for the small investor with the difference that investing in company i will affect the return of investment. More specifically, assume that Y_i is uniform on

$$\big[0, i + X_i p_i\big],$$

for $i = 1, 2, 3$. Find the optimal allocation vector (p_1, p_2, p_3) by applying a SA algorithm.

(c). Consider the small investor problem again. Suppose that the realizations of (Y_1, Y_2, Y_3) and (X_1, X_2, X_3) are revealed to you successively. Construct a SA learning algorithm that finds the optimal allocation vector (p_1, p_2, p_3) based on the streaming data. Put differently, construct an algorithm that updates the current guess for (p_1, p_2, p_3) when a new observation of (Y_1, Y_2, Y_3) and (X_1, X_2, X_3) becomes available.

Can you construct an algorithm that is robust against changes in the underlying data stream (i.e., that can adjust if the distribution of the X_i's change)?

Bonus Question: Consider the small investor problem again. Taking the riskiness of a chosen investment policy into account, consider the optimization problem of minimizing the risk given by

$$\tilde{J}(p_1, p_2, p_3) = \mathbb{E}\left[\frac{\sum_{i=1}^{3} p_i Y_i 1_{X_i \geq x_i}}{std\left(\sum_{i=1}^{3} p_i Y_i 1_{X_i \geq x_i}\right)} \right],$$

where $std(X)$ denotes the standard deviation of random variable X. Find the optimal allocation vector (p_1, p_2, p_3) by applying an appropriate SA algorithm. Discuss your choice of algorithm and make an output analysis.

What is the difference between minimizing the risk and maximizing the expected pay-off of the investment strategy?