Homework 2: Linear Regression

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Problem 2.1

To minimize the MSE of θ, we computer our estimator $\hat{\theta}$:

$$\hat{\theta} = \arg\min_{\theta} \|\mathbf{y} - \mathbf{X}\theta\|_2^2$$

Expanding that out, we find:

$$\begin{aligned}
\|\mathbf{y} - \mathbf{X}\theta\|_2^2 &= (\mathbf{y} - \mathbf{X}\theta)^\mathsf{T}(\mathbf{y} - \mathbf{X}\theta) \\
&= \mathbf{y}^\mathsf{T}\mathbf{y} - \mathbf{y}^\mathsf{T}\mathbf{X}\theta - \theta^\mathsf{T}\mathbf{X}^\mathsf{T}\mathbf{y} + \theta^\mathsf{T}\mathbf{X}^\mathsf{T}\mathbf{X}\theta \\
&= \mathbf{y}^\mathsf{T}\mathbf{y} - 2\theta^\mathsf{T}\mathbf{X}^\mathsf{T}\mathbf{y} + \theta^\mathsf{T}\mathbf{X}^\mathsf{T}\mathbf{X}\theta
\end{aligned}$$

From there, we take the derivative w.r.t. θ:

$$\frac{d}{d\theta} = 2\mathbf{X}^\mathsf{T}\mathbf{X}\theta - 2\mathbf{X}^\mathsf{T}\mathbf{y}$$

After setting this derivative to zero and solving for θ, we find that our solution is:

$$\hat{\theta} = (\mathbf{X}^\mathsf{T}\mathbf{X})^{-1}\mathbf{X}^\mathsf{T}\mathbf{y}$$

Problem 2.2

Since know that ϵ is distributed normally, we can use the normal distribution formula to derive our expression for the MLE of θ^*:

$$\mathbb{P}(\mathbf{y}, \mathbf{X}|\theta, \mathbf{\Sigma}^*) = \frac{1}{\mathbf{\Sigma}^*\sqrt{2\pi}}\mathbf{e}^{-\frac{1}{2\mathbf{\Sigma}^*}(\mathbf{y}-\theta\mathbf{X})^\mathsf{T}(\mathbf{y}-\theta\mathbf{X})}$$

Maximizing for θ, we find that the MLE is notably the same as the minimizer of the MSE from Problem 2.1 above.

$$\hat{\theta} = \arg\min_{\theta}\|\mathbf{y} - \mathbf{X}\theta\|_{\mathbf{2}}^{\mathbf{2}}$$
$$\hat{\theta} = (\mathbf{X}^{\mathsf{T}}\mathbf{X})^{-\mathbf{1}}\mathbf{X}^{\mathsf{T}}\mathbf{y}$$

Problem 2.3

Problem 2.4

Problem 2.5

Problem 2.6

Problem 2.7

(a)