# Combinatorics of Schwinger Energy Hamiltonian Coefficients

So we have the following Hamiltonian

$$H_E = \sum_{i=0}^{N-2} \left( \frac{\theta}{2\pi} + \sum_{j=0}^{i} \frac{Z_j + (-1)^j}{2} \right)^2$$

First, we can divide out the common $\left(\frac{1}{2}\right)^2$

$$H_E = \frac{1}{4} \sum_{i=0}^{N-2} \left( \frac{\theta}{\pi} + \sum_{j=0}^{i} Z_j + (-1)^j \right)^2$$

and we want to extract its coefficients. In order to do so, let's leverage some notational abstraction:

$$f(i) = \frac{\theta}{\pi} + \sum_{j=0}^{i} (-1)^j$$

Which we can reduce further using the following fact

$$\sum_{j=0}^{i} (-1)^j = ((i+1) \bmod 2)$$

making

$$f(i) = \frac{\theta}{\pi} + ((i+1) \bmod 2)$$

Plugging this all back into $H_E$ we get

$$H_E = \frac{1}{4} \sum_{i=0}^{N-2} \left( f(i) + \sum_{j=0}^{i} Z_j \right)^2$$

Now, we can multiply the square out!

$$
\begin{aligned}
H_E &= \frac{1}{4}\sum_{i=0}^{N-2}\left(f(i)+\sum_{j=0}^{i} Z_j\right)^2 \\
&= \frac{1}{4}\sum_{i=0}^{N-2}\left(f(i)+\sum_{j=0}^{i} Z_j\right)\left(f(i)+\sum_{k=0}^{i} Z_k\right) \\
&= \frac{1}{4}\sum_{i=0}^{N-2}\left(f(i)^2+\sum_{j=0}^{i}\sum_{k=0}^{i} Z_j Z_k + 2f(i)\sum_{j=0}^{i} Z_j\right)
\end{aligned}
$$

Great! Now we have our term seperated into three more easlily malleable terms.

## Identity

$$
\begin{aligned}
\frac{1}{4}\sum_{i=0}^{N-2} f(i)^2 &= \frac{1}{4}\sum_{i=0}^{N-2}\left(\frac{\theta}{\pi}+((i+1) \bmod 2)\right)^2 \\
&= \frac{1}{4}\sum_{i=0}^{N-2}\left(\frac{\theta}{\pi}\right)^2+((i+1) \bmod 2)^2+2\frac{\theta}{\pi}((i+1) \bmod 2) \\
&\text{Since } ((i+1) \bmod 2) \text{ can only be 0 or 1} \\
&= \frac{1}{4}\sum_{i=0}^{N-2}\left(\frac{\theta}{\pi}\right)^2+((i+1) \bmod 2)+2\frac{\theta}{\pi}((i+1) \bmod 2) \\
&= \frac{1}{4}\sum_{i=0}^{N-2}\left(\frac{\theta}{\pi}\right)^2+\left(1+2\frac{\theta}{\pi}\right)((i+1) \bmod 2) \\
&= \frac{N-1}{4}\left(\frac{\theta}{\pi}\right)^2+\frac{1}{4}\left\lceil\frac{N-1}{2}\right\rceil\left(1+2\frac{\theta}{\pi}\right)
\end{aligned}
$$

## Two-Qubit

For the two-qubit coefficients we we know that if $j = k$ then $Z_j Z_k = I$

$$
\begin{aligned}
\frac{1}{4}\sum_{i=0}^{N-2}\sum_{j=0}^{i}\sum_{k=0}^{i} Z_j Z_k &= \frac{1}{4}\sum_{i=0}^{N-2}\sum_{j=0}^{i}\sum_{k=0}^{i} Z_j Z_k \\
&= \frac{1}{4}\frac{N(N-1)}{2}+\frac{1}{4}\sum_{i=0}^{N-2}\sum_{j=0}^{i}\sum_{k=0}^{j-1} 2\cdot Z_j Z_k \\
&= \frac{N(N-1)}{8}+\frac{1}{2}\sum_{j=0}^{N-2}\sum_{k=0}^{j-1}(N-j-1) Z_j Z_k
\end{aligned}
$$

## Single-Qubit

Finally we have the single-qubit coefficients

$$
\begin{aligned}
\frac{1}{4}\cdot 2\sum_{i=0}^{N-2} f(i)\sum_{j=0}^{i} Z_j &= \frac{1}{2}\sum_{j=0}^{N-2}\sum_{i=j}^{N-2} f(i) Z_j \\
&= \frac{1}{2}\sum_{j=0}^{N-2}\sum_{i=j}^{N-2}\left(\frac{\theta}{\pi} + ((i+1) \bmod 2)\right) Z_j
\end{aligned}
$$

This term itself splits into two parts

$$
\frac{\theta}{2\pi}\sum_{j=0}^{N-2}\sum_{i=j}^{N-2} Z_j = \sum_{j=0}^{N-2}\frac{\theta}{2\pi}(N-j-1)Z_j
$$

and

$$
\frac{1}{2}\sum_{j=0}^{N-2}\sum_{i=j}^{N-2}((i+1) \bmod 2)Z_j = \sum_{j=0}^{N-2}\frac{1}{2}\left(\left\lceil \frac{N-j-1}{2}\right\rceil - ((j\cdot N) \bmod 2)\right)Z_j
$$

Thus

$$
\begin{aligned}
&\frac{1}{2}\sum_{j=0}^{N-2}\sum_{i=j}^{N-2}\left(\frac{\theta}{\pi} + ((i+1) \bmod 2)\right) Z_j \\
=&\frac{1}{2}\sum_{j=0}^{N-2}\left(\frac{\theta}{\pi}(N-j-1) + \left\lceil \frac{N-j-1}{2}\right\rceil + ((j\cdot N) \bmod 2)\right)Z_j
\end{aligned}
$$

With the dust settled we have

### Identity

$$
\frac{1}{4}\left((N-1)\left(\frac{\theta}{\pi}\right)^2 + \left\lceil \frac{N-1}{2}\right\rceil\left(1+2\frac{\theta}{\pi}\right) + \frac{N(N-1)}{2}\right)
$$

### Multi Qubit

$$
\frac{1}{2}\sum_{j=0}^{N-2}\sum_{k=0}^{j-1}(N-j-1)Z_j Z_k
$$

### Single Qubit

$$
\frac{1}{2}\sum_{j=0}^{N-2}\left(\frac{\theta}{\pi}(N-j-1) + \left\lceil \frac{N-j-1}{2}\right\rceil - ((j\cdot N) \bmod 2)\right)Z_j
$$

$$H_E = \frac{g^2 a}{2} \sum_{i=0}^{N-2} \left( \frac{\theta}{2\pi} + \sum_{j=0}^{i} \frac{Z_j + (-1)^j}{2} \right)^2$$

$$H_M = \frac{m}{2} \sum_{i=0}^{N-1} (-1)^i Z_i$$

$$H_I = \frac{1}{4a} \sum_{i=0}^{N-2} X_i X_{i+1} + Y_i Y_{i+1}$$