复分析第二周作业

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习题 2.1

T4

证明 由 $f(z)$ 在 D 上全纯知，$\forall z_0 \in D$，

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \stackrel{\text{def}}{=} f'(z_0)$$

所以

$$\begin{aligned}\lim_{\Delta z \to 0} \frac{\overline{f(\overline{z_0 + \Delta z})} - \overline{f(\overline{z_0})}}{\Delta z} &= \lim_{\Delta z \to 0} \overline{\left(\frac{f(\overline{z_0 + \Delta z}) - f(\overline{z_0})}{\overline{\Delta z}} \right)} \\ &= \overline{f'(\overline{z_0})}\end{aligned}$$

因为 $z_0 \in D \iff \overline{z_0} \in G$，这就说明了 $\forall z_0 \in D$，有 $\overline{z_0} \in G$，$\overline{f(\overline{z_0})}$ 可微，且 $\left(\overline{f(\overline{z_0})}\right)' = \overline{f'(\overline{z_0})}$，故 $\overline{f(\overline{z})}$ 是 G 上的全纯函数 □

习题 2.2

T2

证明 设 $f(z) = f(x+iy) = u(x,y) + iv(x,y)$，由 $f \in H(D)$ 知，f 满足 *Cauchy-Riemann* 方程，即

$$\begin{cases} u_x(x,y) = v_y(x,y) \\ u_y(x,y) = -v_x(x,y) \end{cases}$$

所以

(*i*) $\text{Re} f(z) = u(x,y) = C$，则 $u_x = u_y \equiv 0$，因此 $v_x = -u_y \equiv 0, v_y = u_x \equiv 0$，故 f 是常数

(*ii*) $\text{Im} f(z) = v(x,y) = C$，则 $v_x = v_y \equiv 0$，因此 $u_x = v_y \equiv 0, u_y = -v_x \equiv 0$，故 f 是常数

(*iii*) $|f(z)| = f(z)\overline{f(z)} = u^2(x,y) + v^2(x,y) = c$，若 $c = 0$，则 f 显然为常数；若 $c > 0$，则

$$\begin{cases} uu_x + vv_x = 0 \cdots ① \\ uu_y + vv_y = 0 \cdots ② \end{cases}$$

所以 $① \times u + ② \times v, ① \times v - ② \times u$ 得

$$\begin{cases} (u^2+v^2)u_x = (u^2+v^2)v_y = 0 \\ (u^2+v^2)v_x = (u^2+v^2)u_y = 0 \end{cases}$$

因此 $u_x = u_y = v_x = v_y = 0$，故 f 是常数

(iv) 因为 $\arg f(z) = \arctan\left(\frac{v(x,y)}{u(x,y)}\right)$，所以

$$\begin{cases} \frac{\partial}{\partial x}\arg f(z) = \frac{2\cdot\frac{v}{u}\cdot\left(\frac{v_x u - u_x v}{u^2}\right)}{1+\left(\frac{v}{u}\right)^2} = 0 \\ \frac{\partial}{\partial x}\arg f(z) = \frac{2\cdot\frac{v}{u}\cdot\left(\frac{v_y u - u_y v}{u^2}\right)}{1+\left(\frac{v}{u}\right)^2} = 0 \end{cases}$$

若 $v(x,y)\equiv 0$，则 $v_x, v_y \equiv 0$，由 *Cauchy-Riemann* 方程知 $u_x, u_y \equiv 0$，故 f 是常数；若 $v(x,y)\not\equiv 0$，则

$$\begin{cases} v_x u - u_x v = 0 \cdots ① \\ v_y u - u_y v = 0 \cdots ② \end{cases}$$

所以 $① \times u + ② \times v, ① \times v - ② \times u$ 得

$$\begin{cases} v_x(u^2+v^2) = u_y(u^2+v^2) = 0 \\ u_x(u^2+v^2) = v_y(u^2+v^2) = 0 \end{cases}$$

因此 $u_x = u_y = v_x = v_y = 0$，故 f 是常数

(v) $u = v^2$，则

$$\begin{cases} 2vv_x = u_x = v_y \\ 2vv_y = u_y = -v_x \end{cases}$$

所以 $v_y = 2vv_x = -4v^2 v_y \Rightarrow (1+4v^2)v_y = 0$，故 $v_y \equiv 0$；若 $v \equiv 0$，则 $u = v^2 \equiv 0$，则 f 显然为常数；若 $v \not\equiv 0$，则由 $2vv_x = v_y$ 知，$v_x \equiv 0$，这就说明 $u_x = u_y = v_x = v_y = 0$，故 f 是常数 □

T3

证明 因为此时 $u(x,y) = \sqrt{xy}, v(x,y) = 0$，且

$$\frac{\partial u}{\partial x}(0,0) = \lim_{|\Delta x|\to 0}\frac{\sqrt{\Delta x\cdot 0} - 0}{\Delta x} = 0, \quad \frac{\partial u}{\partial y}(0,0) = \lim_{\Delta y\to 0}\frac{\sqrt{0\cdot\Delta y} - 0}{\Delta y} = 0, \quad v_x = v_y = 0$$

因此 $u_x(0,0) = v_y(0,0) = 0, u_y(0,0) = -v_x(0,0) = 0$，故 $f(z) = \sqrt{xy}$ 在 $z=0$ 处满足 *Riemann* 方程

任取 $\Delta z = \Delta x + i\Delta y, \Delta x = k\Delta y, k \geq 0$，则

$$\lim_{\substack{\Delta z\to 0\\ \Delta x = k\Delta y}}\frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\substack{\Delta z\to 0\\ \Delta x = k\Delta y}}\frac{\sqrt{k(\Delta y)^2}}{(k+i)\Delta y}$$

当 $\Delta y > 0$ 时，上述极限为 $\frac{\sqrt{k}}{k+i}$；当 $\Delta y < 0$ 时，上述极限为 $\frac{\sqrt{k}}{k+i}$，故 $\lim\limits_{\Delta z\to 0}\frac{f(\Delta z)-f(0)}{\Delta z}$ 的极限不存在！ □

T4

证明 因为 $x = r\cos\theta, y = r\sin\theta$，所以

$$\begin{cases} \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \cdots ① \\ \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial u}{\partial x}r\sin\theta + \frac{\partial u}{\partial y}r\cos\theta \cdots ② \\ \frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta \cdots ③ \\ \frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial v}{\partial x}r\sin\theta + \frac{\partial v}{\partial y}r\cos\theta \cdots ④ \end{cases}$$

将 $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ 代入③,④两式得

$$\begin{cases}\frac{\partial v}{\partial r}=-\frac{\partial u}{\partial y}\cos\theta+\frac{\partial u}{\partial x}\cos\theta\cdots⑤\\ \frac{\partial v}{\partial \theta}=\frac{\partial u}{\partial y}r\sin\theta+\frac{\partial u}{\partial x}r\cos\theta\cdots⑥\end{cases}$$

再将④,⑤与①,②对比，即得

$$\begin{cases}\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}\\ \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}\end{cases}$$

□

T5

证明 因为 $r=\sqrt{x^2+y^2},\theta=\arctan\left(\frac{y}{x}\right)$，所以

$$\frac{\partial r}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}=\cos\theta,\quad \frac{\partial r}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}=\sin\theta,\quad \frac{\partial \theta}{\partial x}=\frac{-y}{x^2+y^2}=-\frac{\sin\theta}{r},\quad \frac{\partial \theta}{\partial y}=\frac{x}{x^2+y^2}=\frac{\cos\theta}{r}$$

进而

$$\begin{cases}\frac{\partial f}{\partial x}=\frac{\partial f}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}=\frac{\partial f}{\partial r}\cos\theta-\frac{\partial f}{\partial \theta}\frac{\sin\theta}{r}\\ \frac{\partial f}{\partial y}=\frac{\partial f}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}=\frac{\partial f}{\partial r}\sin\theta+\frac{\partial f}{\partial \theta}\frac{\cos\theta}{r}\end{cases}$$

因此

$$\begin{aligned}\frac{\partial f}{\partial z}&=\frac{1}{2}\left(\frac{\partial f}{\partial x}-i\frac{\partial f}{\partial y}\right)\\&=\frac{1}{2}\left[\left(\frac{\partial f}{\partial r}\cos\theta-\frac{\partial f}{\partial \theta}\frac{\sin\theta}{r}\right)-i\left(\frac{\partial f}{\partial r}\sin\theta+\frac{\partial f}{\partial \theta}\frac{\cos\theta}{r}\right)\right]\\&=\frac{1}{2}\left[\frac{\partial f}{\partial r}(\cos\theta-i\sin\theta)-\frac{\partial f}{\partial \theta}\frac{\sin\theta+i\cos\theta}{r}\right]\\&=\frac{1}{2}e^{-i\theta}\left(\frac{\partial f}{\partial r}-\frac{i}{r}\frac{\partial f}{\partial \theta}\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial \overline{z}}&=\frac{1}{2}\left(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\right)\\&=\frac{1}{2}\left[\left(\frac{\partial f}{\partial r}\cos\theta-\frac{\partial f}{\partial \theta}\frac{\sin\theta}{r}\right)+i\left(\frac{\partial f}{\partial r}\sin\theta+\frac{\partial f}{\partial \theta}\frac{\cos\theta}{r}\right)\right]\\&=\frac{1}{2}\left[\frac{\partial f}{\partial r}(\cos\theta+i\sin\theta)+\frac{\partial f}{\partial \theta}\frac{-\sin\theta+i\cos\theta}{r}\right]\\&=\frac{1}{2}e^{i\theta}\left(\frac{\partial f}{\partial r}+\frac{i}{r}\frac{\partial f}{\partial \theta}\right)\end{aligned}$$

□

T9

证明 因为

$$\frac{\partial f}{\partial z}=\frac{1}{2}[(u_x+v_y)+i(v_x-u_y)],\quad \frac{\partial f}{\partial \overline{z}}=\frac{1}{2}[(u_x-v_y)+i(v_x+u_y)]$$

所以

$$\begin{aligned}\left|\frac{\partial f}{\partial z}\right|^2-\left|\frac{\partial f}{\partial \overline{z}}\right|^2&=\frac{1}{4}[(u_x+v_y)^2+(v_x-u_y)^2-(u_x-v_y)^2-(v_x+u_y)^2]\\&=u_xv_y-u_yv_x=\begin{vmatrix}\frac{\partial u}{\partial x}&\frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x}&\frac{\partial v}{\partial y}\end{vmatrix}\end{aligned}$$

特别地，若 f 全纯，则 $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$，且 $\frac{\partial f}{\partial \overline{z}} = 0, f'(z) = u_x + iv_x$，因此

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = u_x^2 + v_x^2 = |f'|^2$$

即 *Jacobi* 行列式表示将区域 D 从复平面变换到 uOv 平面后的面积膨胀率

T12

我们首先证明一个引理

引理 1 若 $f: D \to E, g: E \to F$ 全纯，则 $g \circ f: D \to F$ 也全纯

证明 由 f, g 全纯知，f, g 实可微，且 $\frac{\partial f}{\partial \overline{z}} = \frac{\partial g}{\partial \overline{z}} = 0$，所以 $g \circ f$ 实可微，设 $g = g(\omega, \overline{\omega}), \omega = f(z, \overline{z})$，则

$$\frac{\partial}{\partial \overline{z}}(g \circ f) = \frac{\partial g}{\partial \omega}\frac{\partial f}{\partial \overline{z}} + \frac{\partial g}{\partial \overline{\omega}}\frac{\partial \overline{f}}{\partial \overline{z}}$$

由 f, g 全纯知，$\frac{\partial f}{\partial \overline{z}} = 0, \frac{\partial g}{\partial \overline{\omega}} = 0$，所以 $\frac{\partial}{\partial \overline{z}}(g \circ f) = 0$，这就证明了 $g \circ f$ 也全纯

接下来回到本题

证明

$\forall z \in D$，记 $\omega = \varphi(z)$，则 $\exists r > 0, \text{s.t. } B(\omega, r) \subseteq G$，因为 u 是单连通区域 $B(\omega, r)$ 上的调和函数，因此存在 u 的共轭调和函数 v，因此 $f \overset{\text{def}}{=} u + iv$ 全纯，又因为 φ 也是全纯函数，由引理得 $f \circ \varphi = u \circ \varphi + i \cdot v \circ \varphi$ 在 $\varphi^{-1}(B(\omega, r))$ 上全纯，故它的实部 $u \circ \varphi$ 是 $\varphi^{-1}(B(\omega, r))$ 上的调和函数，由 z 的任意性知，$u \circ \varphi$ 是 D 上的调和函数 $\square$

T14

证明

上课时已证：$\triangle f = 4\frac{\partial^2 f}{\partial \overline{z} \partial z}$；设 $u = u(\omega, \overline{\omega}), \omega = \varphi(z, \overline{z})$，因为 $\varphi \in H(D)$，所以 $\frac{\partial \omega}{\partial \overline{z}} = \frac{\partial \overline{\omega}}{\partial z} = 0$，设 $\varphi = a + ib$，则 a, b 调和，所以

$$4\frac{\partial^2 \omega}{\partial \overline{z} \partial z} = \triangle \varphi = \triangle a + i \triangle b = 0$$

因此

$$\begin{aligned} \triangle(u \circ \varphi) &= 4\frac{\partial^2}{\partial \overline{z} \partial z}(u \circ \varphi) = 4\frac{\partial}{\partial \overline{z}}\left(\frac{\partial u}{\partial \omega}\frac{\partial \omega}{\partial z} + \frac{\partial u}{\partial \overline{\omega}}\frac{\partial \overline{\omega}}{\partial z}\right) = 4\frac{\partial}{\partial \overline{z}}\left(\frac{\partial u}{\partial \omega}\frac{\partial \omega}{\partial z}\right) \\ &= 4\frac{\partial \omega}{\partial z}\left(\frac{\partial^2 u}{\partial \omega^2}\frac{\partial \omega}{\partial \overline{z}} + \frac{\partial^2 u}{\partial \overline{\omega} \partial \omega}\frac{\partial \overline{\omega}}{\partial \overline{z}}\right) + 4\frac{\partial u}{\partial \omega}\frac{\partial^2 \omega}{\partial \overline{z} \partial z} \\ &= 4\frac{\partial^2 u}{\partial \overline{\omega} \partial \omega}\frac{\partial \omega}{\partial z}\frac{\partial \overline{\omega}}{\partial \overline{z}} = \triangle u \cdot \frac{\partial \omega}{\partial z}\frac{\partial \overline{\omega}}{\partial \overline{z}} \end{aligned}$$

因为 $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$，所以

$$\begin{aligned} \frac{\partial \omega}{\partial z}\frac{\partial \overline{\omega}}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(a + ib) \cdot \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(a - ib) \\ &= \frac{1}{4}\left(a_x + ib_x - ia_y + b_y\right)\left(a_x - ib_x + ia_y + b_y\right) \\ &= \frac{1}{4}(2a_x + 2ib_x))(2a_x - 2ib_x) \\ &= \varphi'\overline{\varphi'} = |\varphi'|^2 \end{aligned}$$

综上我们有 $\triangle(u \circ \varphi) = \triangle u \cdot |\varphi'|^2$ $\square$

习题 2.3

T4

证明 课上已证 *P50.T3*，构造 $g(z)=\frac{f(z_0z)}{f(z_0)}$，因为 $\forall z\in B(0,1), |z_0z|\le|z_0|<1$，且 $g(1)=\frac{f(z_0)}{f(z_0)}=1$；由 f 在 $B(0,1)$ 上全纯，且 $|z_0|<1$ 知，g 在 $B(0,1)\cup\{1\}$ 上全纯，由 *P50.T3* 的结论知

$$g'(1)=\frac{z_0f'(z_0)}{f(z_0)}>0$$

不取等是因为在 *P50.T3* 的证明过程中，若 $f'(1)\neq 0$，则 $f'(1)>0$ $\square$