习题 1 设 $\boldsymbol{\alpha}(t)$ 是向量值函数，证明：
(1) $|\boldsymbol{\alpha}| =$ 常数当且仅当 $<\boldsymbol{\alpha}(t),\boldsymbol{\alpha}'(t)>=0$
(2) $\boldsymbol{\alpha}(t)$ 的方向不变当且仅当 $\boldsymbol{\alpha}(t)\wedge\boldsymbol{\alpha}'(t)=0$

证明 (1) 因为 $|\boldsymbol{\alpha}|^2=<\boldsymbol{\alpha}(t),\boldsymbol{\alpha}(t)>$，且 $\frac{\mathrm{d}}{\mathrm{d}t}<\boldsymbol{\alpha}(t),\boldsymbol{\alpha}(t)>=2<\boldsymbol{\alpha}(t),\boldsymbol{\alpha}'(t)>$，所以

$$|\boldsymbol{\alpha}|\text{为常数} \iff \text{上式两边求导均为零} \iff <\boldsymbol{\alpha}(t),\boldsymbol{\alpha}'(t)>=0$$

(2) $(\Longrightarrow)$: $\boldsymbol{\alpha}(t)$ 方向不变，则存在常向量 $\boldsymbol{v}$ 和函数 $f(t)$，使得 $\boldsymbol{\alpha}(t)=f(t)\boldsymbol{v}$，两边同时求导得

$$\boldsymbol{\alpha}'(t)=f'(t)\boldsymbol{v}$$

这就说明 $\boldsymbol{\alpha}(t)//\boldsymbol{\alpha}'(t)$，即 $\boldsymbol{\alpha}(t)\wedge\boldsymbol{\alpha}'(t)=0$
$(\Longleftarrow)$: 以下我们讨论向量的方向，故可设 $\boldsymbol{\alpha}(t)\neq 0$，设 $\boldsymbol{e}(t)=\frac{\boldsymbol{\alpha}(t)}{|\boldsymbol{\alpha}(t)|}$，即为 $\boldsymbol{\alpha}(t)$ 方向的单位向量，则由 (1) 知 $<\boldsymbol{e}(t),\boldsymbol{e}'(t)>=0$，设 $\boldsymbol{f}(t)=|\boldsymbol{\alpha}(t)|$，则 $\boldsymbol{\alpha}(t)=f(t)\boldsymbol{e}(t)$，两边同时求导得

$$\boldsymbol{\alpha}'(t)=f'(t)\boldsymbol{e}(t)+f(t)\boldsymbol{e}'(t)$$

所以

$$\boldsymbol{\alpha}(t)\wedge\boldsymbol{\alpha}'(t)=(f(t)\boldsymbol{e}(t)\wedge f'(t)\boldsymbol{e}(t)+f(t)\boldsymbol{e}'(t))=0$$

即 $|f(t)|^2\boldsymbol{e}(t)\wedge\boldsymbol{e}'(t)=0$，由 $\boldsymbol{\alpha}(t)\neq 0$ 知，$|f(t)|^2>0$，即 $\boldsymbol{e}(t)\wedge\boldsymbol{e}'(t)=0$，则 $\boldsymbol{e}(t)//\boldsymbol{e}'(t)$，又因为 $\boldsymbol{e}(t)\perp\boldsymbol{e}'(t)$，即垂直又正交，只能是 $\boldsymbol{e}'(t)=0$，即 $\boldsymbol{e}(t)=\mathrm{const}$，即 $\boldsymbol{\alpha}(t)$ 的方向不变

□

习题 2 设 $\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3,\boldsymbol{v}_4$ 是 $\mathbb{R}^3$ 的四个向量，验证
(1) $\boldsymbol{v}_1\wedge(\boldsymbol{v}_2\wedge\boldsymbol{v}_3)=<\boldsymbol{v}_1,\boldsymbol{v}_3>\boldsymbol{v}_2-<\boldsymbol{v}_1,\boldsymbol{v}_2>\boldsymbol{v}_3$
(2) Lagrange 恒等式 $<\boldsymbol{v}_1\wedge\boldsymbol{v}_2,\boldsymbol{v}_3\wedge\boldsymbol{v}_4>=<\boldsymbol{v}_1,\boldsymbol{v}_3><\boldsymbol{v}_2,\boldsymbol{v}_4>-<\boldsymbol{v}_1,\boldsymbol{v}_4><\boldsymbol{v}_2,\boldsymbol{v}_3>$
(3) $(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3)=(\boldsymbol{v}_2,\boldsymbol{v}_3,\boldsymbol{v}_1)=(\boldsymbol{v}_3,\boldsymbol{v}_1,\boldsymbol{v}_2)$
(4) $\nabla\wedge(\nabla f)=\mathrm{rot}(\mathrm{grad}\ f)=0$
(5) $<\nabla,\nabla\wedge\boldsymbol{F}>=\mathrm{div}(\mathrm{rot}\boldsymbol{F})=0$

证明 (1) 由线性性知，只需对标准正交基 $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$ 验证即可，再由反对称性，只需验证 $(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3)=(\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3)$ 或 $(\boldsymbol{e}_1,\boldsymbol{e}_1,\boldsymbol{e}_2)$ 或 $(\boldsymbol{e}_1,\boldsymbol{e}_1,\boldsymbol{e}_1)$ 的情况即可

- $(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3)=(\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3)$ 时

$$LHS=\boldsymbol{e}_1\wedge(\boldsymbol{v}_2\wedge\boldsymbol{v}_3)=\boldsymbol{e}_1\wedge\boldsymbol{e}_1=\mathbf{0}=<\boldsymbol{e}_1,\boldsymbol{e}_3>\boldsymbol{e}_2-<\boldsymbol{e}_1,\boldsymbol{e}_2>\boldsymbol{e}_3=RHS$$

- $(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3)=(\boldsymbol{e}_1,\boldsymbol{e}_1,\boldsymbol{e}_2)$ 时

$$LHS=\boldsymbol{e}_1\wedge(\boldsymbol{e}_1\wedge\boldsymbol{e}_2)=\boldsymbol{e}_1\wedge\boldsymbol{e}_3=-\boldsymbol{e}_2=RHS$$

- $(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = (\boldsymbol{e}_1, \boldsymbol{e}_1, \boldsymbol{e}_1)$ 时

$$LHS = \boldsymbol{e}_1 \wedge (\boldsymbol{e}_1 \wedge \boldsymbol{e}_1) = \boldsymbol{0} = RHS$$

(2) 由混合积的轮换对称性（在下一问中证明）和 (1) 知

$$\begin{aligned} <\boldsymbol{v}_1 \wedge \boldsymbol{v}_2, \boldsymbol{v}_3 \wedge \boldsymbol{v}_4> = (\boldsymbol{v}_1 \wedge \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4) = (\boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_1 \wedge \boldsymbol{v}_2) &= <\boldsymbol{v}_3, \boldsymbol{v}_4 \wedge (\boldsymbol{v}_1 \wedge \boldsymbol{v}_2)> \\ &= <\boldsymbol{v}_3, (<\boldsymbol{v}_4, \boldsymbol{v}_2>\boldsymbol{v}_1 - <\boldsymbol{v}_4, \boldsymbol{v}_1>\boldsymbol{v}_3)> \\ &= <\boldsymbol{v}_1, \boldsymbol{v}_3><\boldsymbol{v}_2, \boldsymbol{v}_4> - <\boldsymbol{v}_1, \boldsymbol{v}_4><\boldsymbol{v}_2, \boldsymbol{v}_3> \end{aligned}$$

(3) 直接计算，设 $\boldsymbol{v}_i = (x^1, y^1, z^1), i = 1, 2, 3$

$$\begin{aligned} (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) &= <(x^1, y^1, z^1), (y^2 z^3 - y^3 z^2, x^3 z^2 - x^2 z^3, x^2 y^3 - x^3 y^2)> \\ &= x^1 y^2 z^3 - x^1 y^3 z^2 + y^1 x^3 z^2 - y^1 x^2 z^3 + z^1 x^2 y^3 - z^1 x^3 y^2 \\ &= <(x^2, y^2, z^2), (y^3 z^1 - y^1 z^3, -x^3 z^1 + x^1 z^3, x^3 y^1 - x^1 y^3)> \\ &= (\boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_1) \end{aligned}$$

同理可得 $(\boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_1) = (\boldsymbol{v}_3, \boldsymbol{v}_1, \boldsymbol{v}_2)$

(4)

$$\begin{aligned} \nabla \wedge (\nabla f) &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\ &= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) = \boldsymbol{0} \end{aligned}$$

(5) 设 $\boldsymbol{F} = (P, Q, R)$

$$<\nabla, \nabla \wedge \boldsymbol{F}> = \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 P}{\partial y \partial z} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} = 0$$

□

习题 3 设 $\mathcal{T}$ 是 $\mathbf{E}^3$ 的一个合同变换，$\boldsymbol{v}$ 和 $\boldsymbol{w}$ 是 $\mathbf{E}^3$ 的两个向量，求 $(\mathcal{T}\boldsymbol{v}) \wedge (\mathcal{T}\boldsymbol{w})$ 与 $\mathcal{T}(\boldsymbol{v} \wedge \boldsymbol{w})$ 的关系

解 设 $\mathcal{T} = \boldsymbol{X}T + P$，其中 $T \in \mathrm{O}(3), P$ 是常向量

当 $P = \boldsymbol{0}$ 时，若 $\boldsymbol{v} // \boldsymbol{w}$，则二者均为零，不妨设 $\boldsymbol{v}$ 与 $\boldsymbol{w}$ 不共线，则存在 $\boldsymbol{z}$ 使得矩阵 $A \overset{\text{def}}{=} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix}$ 可逆，此时考虑 $A^{-1} = \frac{1}{\det(A)} A^*$，由伴随矩阵的定义知

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} (\boldsymbol{v} \wedge \boldsymbol{w})^T & * & * \end{pmatrix}$$

另一方面

$$(AT)^{-1}=\frac{1}{\det(A)\det(T)}\begin{pmatrix}(\boldsymbol{v}T\wedge\boldsymbol{w}T)^T & * & *\end{pmatrix}$$

因此

$$\frac{T^{-1}}{\det(A)}\begin{pmatrix}(\boldsymbol{v}\wedge\boldsymbol{w})^T & * & *\end{pmatrix}=T^{-1}A^{-1}=(AT)^{-1}=\frac{1}{\det(A)\det(T)}\begin{pmatrix}(\boldsymbol{v}T\wedge\boldsymbol{w}T)^T & * & *\end{pmatrix}$$

两边同时转置得

$$(\boldsymbol{v}\wedge\boldsymbol{w})T=\frac{\boldsymbol{v}T\wedge\boldsymbol{w}T}{\det(T)}$$

即此时有 $\det(T)\mathcal{T}(\boldsymbol{v}\wedge\boldsymbol{w})=\mathcal{T}\boldsymbol{v}\wedge\mathcal{T}\boldsymbol{w}$

当 $P\neq\boldsymbol{0}$ 时，此时有

$$\begin{cases}(\mathcal{T}\boldsymbol{v})\wedge(\mathcal{T}\boldsymbol{w})=(\boldsymbol{v}T\wedge\boldsymbol{w}T)+\boldsymbol{v}T\wedge P+P\wedge\boldsymbol{w}T\\ \mathcal{T}(\boldsymbol{v}\wedge\boldsymbol{w})=(\boldsymbol{v}\wedge\boldsymbol{w})T+P\end{cases}$$

即

$$(\mathcal{T}\boldsymbol{v})\wedge(\mathcal{T}\boldsymbol{w})=\det(T)(\mathcal{T}(\boldsymbol{v}\wedge\boldsymbol{w})-P)+(\boldsymbol{v}T-\boldsymbol{w}T)\wedge P$$

□

习题 4 求下列曲线的弧长与曲率

(1) $y=ax^2$

(2) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(3) $\boldsymbol{r}(t)=(a\cosh t,b\sinh t)$

(4) $\boldsymbol{r}(t)=(t,a\cosh\frac{t}{a}),a>0$

解 (1) 曲线的参数表示为 $\boldsymbol{r}(t)=(t,at^2),t\in\mathbb{R}$，所以

$$\begin{aligned}s(t)&=\int_0^t|\boldsymbol{r}'(u)|\mathrm{d}u=\int_0^t\sqrt{1+4a^2u^2}\mathrm{d}u\\&=\frac{1}{2}t\sqrt{1+4a^2t^2}+\frac{1}{4|a|}\log\left|2|a|t+\sqrt{1+4a^2t^2}\right|\end{aligned}$$

利用下一题的公式

$$\kappa(t)=\frac{2a}{(1+4a^2t^2)^{\frac{3}{2}}}$$

(2) 曲线的参数表示为 $\boldsymbol{r}(t)=(a\cos t,b\sin t),t\in[0,2\pi)$，所以

$$s(t)=\int_0^t|\boldsymbol{r}'(t)|\mathrm{d}t=\int_0^t\sqrt{a^2\sin^2u+b^2\cos^2u}\mathrm{d}u$$

特别地，若 $a=b$，则 $s(t)=at$；曲率如下

$$\kappa(t)=\frac{(-a\sin t)\cdot(-b\sin t)-(-a\cos t)\cdot b\cos t}{(a^2\sin^2t+b^2\cos^2t)^{\frac{3}{2}}}=\frac{ab}{(a^2\sin^2t+b^2\cos^2t)^{\frac{3}{2}}}$$

(3) 弧长

$$s(t)=\int_0^t|\boldsymbol{r}'(u)|\mathrm{d}u=\int_0^t\sqrt{a^2\sinh^2u+b^2\cosh^2u}\mathrm{d}u$$

曲率

$$\kappa(t)=-\frac{ab}{(a^2\sinh^2t+b^2\cosh^2t)^{\frac{3}{2}}}$$

(4) 弧长

$$\begin{aligned}s(t)&=\int_0^t|\boldsymbol{r}'(u)|\mathrm{d}u=\int_0^t\sqrt{1+\sinh^2\frac{u}{a}}\mathrm{d}u\\&=\int_0^t\cosh\frac{u}{a}\mathrm{d}u=a\sinh\frac{t}{a}\end{aligned}$$

曲率

$$\kappa(t)=\frac{\cosh\frac{t}{a}}{a(1+\sinh^2\frac{t}{a})^{\frac{3}{2}}}=\frac{1}{a\cosh^2\frac{t}{a}}$$

□

习题 5 设曲线 $\boldsymbol{r}(t)=(x(t),y(t))$，证明：它的曲率为

$$\kappa(t)=\frac{x'(t)y''(t)-y'(t)x''(t)}{[(x')^2+(y')^2]^{\frac{3}{2}}}$$

证明 对于正则曲线，$\frac{\mathrm{d}}{\mathrm{d}t}s(t)=|\boldsymbol{r}'(t)|>0$，由反函数定理知，$t$ 可以表示为 s 的函数，记为 $t(s)$，且

$$\frac{\mathrm{d}t}{\mathrm{d}s}=\frac{1}{\frac{\mathrm{d}s}{\mathrm{d}t}}=\frac{1}{|\boldsymbol{r}'(t)|}=\frac{1}{\sqrt{(x')^2+(y')^2}}$$

$$\begin{aligned}\frac{\mathrm{d}^2t}{\mathrm{d}s^2}&=\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\frac{\mathrm{d}s}{\mathrm{d}t}}\right)=\frac{\mathrm{d}\left(\frac{1}{\frac{\mathrm{d}s}{\mathrm{d}t}}\right)}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s}\\&=\frac{-\frac{\mathrm{d}^2s}{\mathrm{d}t^2}}{\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2}\frac{1}{\frac{\mathrm{d}s}{\mathrm{d}t}}=-\frac{1}{|\boldsymbol{r}'(t)|^3}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{r}'(t)|\\&=-\frac{x'(t)x''(t)+y'(t)y''(t)}{[(x')^2+(y')^2]^2}\end{aligned}$$

则此时

$$\boldsymbol{t}(t)=\boldsymbol{t}(s(t))=\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}=\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s}=\frac{1}{\sqrt{(x')^2+(y')^2}}(x'(t),y'(t))$$

再次求导得

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s}&=\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2+\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\frac{\mathrm{d}^2t}{\mathrm{d}s^2}\\&=\frac{1}{(x')^2+(y')^2}(x''(t),y''(t))-\frac{x'(t)x''(t)+y'(t)y''(t)}{[(x')^2+(y')^2]^2}(x'(t),y'(t))\\&=\left(-\frac{y'(t)[x'(t)y''(t)-x''(t)y'(t)]}{[(x')^2+(y')^2]^2},\frac{x'(t)[x'(t)y''(t)-x''(t)y'(t)]}{[(x')^2+(y')^2]^2}\right)\end{aligned}$$

又因为

$$\boldsymbol{n}(t)=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\boldsymbol{t}(t)=\frac{1}{\sqrt{(x')^2+(y')^2}}(-y'(t),x'(t))$$

所以

$$\kappa(t)=<\frac{\mathrm{d}\boldsymbol{t}(t)}{\mathrm{d}s},\boldsymbol{n}(t)>=\frac{x'(t)y''(t)-y'(t)x''(t)}{[(x')^2+(y')^2]^{\frac{3}{2}}}$$

□

习题 6 设曲线 C 在极坐标 (r,θ) 下的表示为 $r=f(\theta)$，证明：曲线 C 的曲率表达式为

$$\kappa(\theta)=\frac{f^2(\theta)+2\left(\frac{\mathrm{d}f}{\mathrm{d}\theta}\right)^2-f(\theta)\frac{\mathrm{d}^2f}{\mathrm{d}\theta^2}}{\left[f^2(\theta)+\left(\frac{\mathrm{d}f}{\mathrm{d}\theta}\right)^2\right]^{\frac{3}{2}}}$$

证明 因为 $\boldsymbol{r}(\theta)=(f(\theta)\cos\theta,f(\theta)\sin\theta)$，所以

$$\begin{cases}x'(\theta)=f'(\theta)\cos\theta-f(\theta)\sin\theta\\ y'(\theta)=f'(\theta)\sin\theta+f(\theta)\cos\theta\\ x''(\theta)=f''(\theta)\cos\theta-2f'(\theta)\sin\theta-f(\theta)\cos\theta\\ y''(\theta)=f''(\theta)\sin\theta+2f'(\theta)\cos\theta-f(\theta)\sin\theta\end{cases}$$

代入上一题的公式即证 □

习题 7 设 $\boldsymbol{u}(t),\boldsymbol{v}(t)$ 是 $\mathbb{R}^3$ 中的两个向量值函数，如果其导数满足

$$\boldsymbol{u}'(t)=a\boldsymbol{u}(t)+b\boldsymbol{v}(t),\quad \boldsymbol{v}'(t)=c\boldsymbol{u}(t)-a\boldsymbol{v}(t)$$

其中 a,b,c 是常数，证明 $\boldsymbol{u}(t)\wedge\boldsymbol{v}(t)$ 是常向量

证明 因为

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}(t)\wedge\boldsymbol{v}(t))&=\frac{\mathrm{d}\boldsymbol{u}(t)}{\mathrm{d}t}\wedge\boldsymbol{v}(t)+\boldsymbol{u}(t)\wedge\frac{\mathrm{d}\boldsymbol{v}(t)}{\mathrm{d}t}\\&=(a\boldsymbol{u}(t)+b\boldsymbol{v}(t))\wedge\boldsymbol{v}(t)+\boldsymbol{u}(t)\wedge(c\boldsymbol{u}(t)-a\boldsymbol{v}(t))\\&=a\boldsymbol{u}(t)\wedge\boldsymbol{v}(t)-a\boldsymbol{u}(t)\wedge\boldsymbol{v}(t)=0\end{aligned}$$

即说明 $\boldsymbol{u}(t)\wedge\boldsymbol{v}(t)$ 是常向量 □

习题 8 考虑曲线 $\boldsymbol{r}(t)=(\sin t,\cos t+\log\tan\frac{t}{2}),t\in(0,\pi)$，该曲线称为曳物线，其参数 t 表示切线 $\boldsymbol{r}'(t)$ 与 y 轴的夹角，证明

(1) 当 $t\neq\frac{\pi}{2}$ 时，$\boldsymbol{r}(t)$ 是正则曲线

(2) 设 L 是 $\boldsymbol{r}(t)$ 点的切线，证明 L 从切点 $\boldsymbol{r}(t)$ 到 L 与 y 轴交点之间的线段的长度是常数

证明 (1) 因为

$$\boldsymbol{r}'(t) = (\cos t, -\sin t + \frac{1}{\sin t})$$

所以

$$|\boldsymbol{r}'(t)|^2 = \frac{1}{\sin^2 t} - 1 = \cot^2 t$$

当 $t \neq \frac{\pi}{2}$ 时，$|\boldsymbol{r}'(t)| > 0$，故为正则曲线

(2) 在 $\boldsymbol{r}(t)$ 点的单位切向量 $\boldsymbol{t}(t) = \frac{1}{\tan t}(\cos t, -\sin t + \frac{1}{\sin t})$，故在 $\boldsymbol{r}(t)$ 点的切线方程为

$$\frac{x - \sin t}{\cos t} = \frac{y - (\cos t + \log \tan \frac{t}{2})}{-\sin t + \frac{1}{\sin t}}$$

取 $x = 0$，则切线与 y 轴的交点 $P(t) = (0, \log \tan \frac{t}{2})$，则 $\boldsymbol{r}(t), P(t)$ 之间的长度为

$$l = \sqrt{\sin^2 t + \cos^2 t} = 1$$

□

习题 9 考虑对数螺线 $\boldsymbol{r}(t) = (ae^{bt}\cos t, ae^{bt}\sin t), t \in \mathbb{R}, a > 0, b < 0$ 是常数，证明：当 $t \to +\infty$ 时，$\boldsymbol{r}(t)$ 趋向于原点，$\boldsymbol{r}'(t) \to (0,0)$，且 $\lim\limits_{t\to\infty} \int_{t_0}^{t} |\boldsymbol{r}'(u)| du$ 是有限的，即对数螺线在 $[t_0, +\infty)$ 上有有限的长度

证明 因为

$$|\boldsymbol{r}(t)| = \sqrt{(ae^{bt}\cos t)^2 + (ae^{bt}\sin t)^2} = ae^{bt} \Longrightarrow \lim_{t\to\infty} |\boldsymbol{r}(t)| = 0$$

即 $t \to +\infty$ 时，$\boldsymbol{r}(t)$ 趋向于原点，又因为 $\boldsymbol{r}'(t) = ae^{bt}(b\cos t - \sin t, \cos t + b\sin t)$，所以

$$|\boldsymbol{r}'(t)| \leq a\sqrt{b^2+1}e^{bt} \to 0$$

即 $\boldsymbol{r}'(t) \to (0,0)$，又因为

$$\begin{aligned}\int_{t_0}^{t} |\boldsymbol{r}'(u)| du &= \int_{t_0}^{t} ae^{bt}\sqrt{b^2\cos^2 t + \sin^2 t + \cos^2 t + b^2\sin^2 t}\, dt \\ &= a\sqrt{1+b^2}\int_{t_0}^{t} e^{bt} dt = \frac{a}{b}\sqrt{1+b^2}(e^{bt} - e^{bt_0})\end{aligned}$$

令 $t \to \infty$ 知 $\lim\limits_{t\to\infty} \int_{t_0}^{t} |\boldsymbol{r}'(u)| du = -\frac{ae^{bt_0}}{b}\sqrt{1+b^2}$ □