电动力学第二周作业

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1.6

证明 首先

$$
\begin{aligned}
\nabla \cdot \left(\frac{\overrightarrow{R}}{R^3}\right) &= \left(\nabla \frac{1}{R^3}\right) \cdot \overrightarrow{R} + \frac{1}{R^3}\left(\nabla \times \overrightarrow{R}\right) = -3\frac{1}{R^4}\nabla R \cdot \overrightarrow{R} + \frac{1}{R^3} \cdot 3 \\
&= -\frac{3\overrightarrow{e_r}}{R^4} \cdot \overrightarrow{R} + \frac{3}{R^3} = 0
\end{aligned}
$$

$$
\begin{aligned}
\nabla \times \left(\frac{\overrightarrow{R}}{R^3}\right) &= \nabla \left(\frac{\overrightarrow{R}}{R^3}\right) \times \overrightarrow{R} + \frac{1}{R^3}(\nabla \times \overrightarrow{R}) \\
&= -\frac{3\overrightarrow{e_r}}{R^4} \times \overrightarrow{R} + \frac{1}{R^3}(\nabla \times \overrightarrow{R}) = 0 + 0 = 0
\end{aligned}
$$

使用公式

$$
\nabla \times (\overrightarrow{f} \times \overrightarrow{g}) = (\overrightarrow{g} \cdot \nabla)\overrightarrow{f} + (\nabla \cdot \overrightarrow{g})\overrightarrow{f} - (\overrightarrow{f} \cdot \nabla)\overrightarrow{g} - (\nabla \cdot \overrightarrow{f})\overrightarrow{g}
$$

则我们有

$$
\begin{aligned}
\nabla \times \left(\overrightarrow{m} \times \frac{\overrightarrow{R}}{R^3}\right) &= \left(\frac{\overrightarrow{R}}{R^3} \cdot \nabla\right)\overrightarrow{m} + \left(\nabla \cdot \frac{\overrightarrow{R}}{R^3}\right)\overrightarrow{m} - (\overrightarrow{m} \cdot \nabla)\frac{\overrightarrow{R}}{R^3} - (\nabla \cdot \overrightarrow{m})\frac{\overrightarrow{R}}{R^3} \\
&= -(\overrightarrow{m} \cdot \nabla)\frac{\overrightarrow{R}}{R^3}
\end{aligned}
$$

使用公式

$$
\nabla(\overrightarrow{f} \cdot \overrightarrow{g}) = \overrightarrow{f} \times (\nabla \times \overrightarrow{g}) + (\overrightarrow{f} \cdot \nabla)\overrightarrow{g} + \overrightarrow{g} \times (\nabla \times \overrightarrow{f}) + (\overrightarrow{g} \cdot \nabla)\overrightarrow{f}
$$

$$
\begin{aligned}
\nabla \left(\frac{\overrightarrow{m} \cdot \overrightarrow{R}}{R^3}\right) &= \overrightarrow{m} \times \left(\nabla \times \frac{\overrightarrow{R}}{R^3}\right) + (\overrightarrow{m} \cdot \nabla)\frac{\overrightarrow{R}}{R^3} + \frac{\overrightarrow{R}}{R^3} \times (\nabla \times \overrightarrow{m}) + \left(\frac{\overrightarrow{R}}{R^3} \cdot \nabla\right)\overrightarrow{m} \\
&= (\overrightarrow{m} \cdot \nabla)\frac{\overrightarrow{R}}{R^3}
\end{aligned}
$$

所以

$$
\nabla \times \left(\overrightarrow{m} \times \frac{\overrightarrow{R}}{R^3}\right) = -\nabla \left(\frac{\overrightarrow{m} \cdot \overrightarrow{R}}{R^3}\right)
$$

□

1.10

证明 首先，电流圈 L_1 在电流圈 L_2 上的电流元 $I_2\mathrm{d}\boldsymbol{l}_2$ 上产生的磁场为（记 $\boldsymbol{r}_{12}$ 是从 $I_1\mathrm{d}\boldsymbol{l}_1$ 指向 $I_2\mathrm{d}\boldsymbol{l}_2$ 的向量）

$$
\boldsymbol{B}_1 = \frac{\mu_0}{4\pi}\oint_{L_1} \frac{I_1\mathrm{d}\boldsymbol{l}_1 \times \boldsymbol{r}_{12}}{r_{12}^3}
$$

所以电流元 $I_2\mathrm{d}\boldsymbol{l}_2$ 在该磁场下受到的作用力为

$$
\mathrm{d}\boldsymbol{F}_{12} = I_2\mathrm{d}\boldsymbol{l}_2 \times \boldsymbol{B}_1 = I_2\mathrm{d}\boldsymbol{l}_2 \times \frac{\mu_0}{4\pi}\oint_{L_1} \frac{I_1\mathrm{d}\boldsymbol{l}_1 \times \boldsymbol{r}_{12}}{r_{12}^3}
$$

再在 L_2 上积分，故电流圈 L_1 对 L_2 的作用力为

$$
\begin{aligned}
\boldsymbol{F}_{12} &= \oint_{L_2} \mathrm{d}\boldsymbol{F}_{12} = \oint_{L_2} I_2 \mathrm{d}\boldsymbol{l}_2 \times \frac{\mu_0}{4\pi} \oint_{L_1} \frac{I_1 \mathrm{d}\boldsymbol{l}_1 \times \boldsymbol{r}_{12}}{r_{12}^3} \\
&= \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_2} \mathrm{d}\boldsymbol{l}_2 \times \oint_{L_1} \frac{\mathrm{d}\boldsymbol{l}_1 \times \boldsymbol{r}_{12}}{r_{12}^3} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_2} \oint_{L_1} \mathrm{d}\boldsymbol{l}_2 \times \frac{\mathrm{d}\boldsymbol{l}_1 \times \boldsymbol{r}_{12}}{r_{12}^3} \\
&= \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_2} \oint_{L_1} \left[\frac{\mathrm{d}\boldsymbol{l}_1 (\boldsymbol{r}_{12} \cdot \mathrm{d}\boldsymbol{l}_2)}{r_{12}^3} - \frac{\boldsymbol{r}_{12} (\mathrm{d}\boldsymbol{l}_1 \cdot \mathrm{d}\boldsymbol{l}_2)}{r_{12}^3} \right]
\end{aligned}
$$

由 *Stokes* 定理

$$
\begin{aligned}
\oint_{L_2} \oint_{L_1} \frac{\mathrm{d}\boldsymbol{l}_1 (\boldsymbol{r}_{12} \cdot \mathrm{d}\boldsymbol{l}_2)}{r_{12}^3} &= \oint_{L_1} \mathrm{d}\boldsymbol{l}_1 \oint_{L_2} \frac{\boldsymbol{r}_{12} \cdot \mathrm{d}\boldsymbol{l}_2}{r_{12}^3} \\
&= \oint_{L_1} \mathrm{d}\boldsymbol{l}_1 \iint_{S_2} \nabla \times \left(\frac{\boldsymbol{r}_{12}}{r_{12}^3} \right) \cdot \mathrm{d}\boldsymbol{S} \\
&= \oint_{L_1} 0 \ \mathrm{d}\boldsymbol{l}_1 = 0
\end{aligned}
$$

所以

$$
\boldsymbol{F}_{12} = -\frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_2} \oint_{L_1} \frac{\boldsymbol{r}_{12} (\mathrm{d}\boldsymbol{l}_1 \cdot \mathrm{d}\boldsymbol{l}_2)}{r_{12}^3}
$$

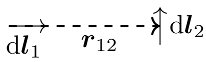
同理，交换下标 1,,2 得电流圈 L_2 对 L_1 的作用力

$$
\boldsymbol{F}_{21} = -\frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{\boldsymbol{r}_{21} (\mathrm{d}\boldsymbol{l}_2 \cdot \mathrm{d}\boldsymbol{l}_1)}{r_{21}^3}
$$

因为 $\boldsymbol{r}_{12} = -\boldsymbol{r}_{21}$，则对比上面两式，显然有 $\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$，因此两个电流圈之间的相互作用力大小相等、方向相反；但对于电流元 $I_1 \mathrm{d}\boldsymbol{l}_1, I_2 \mathrm{d}\boldsymbol{l}_2$ 之间作用力

$$
\mathrm{d}\boldsymbol{F}_{12} = I_2 \mathrm{d}\boldsymbol{l}_2 \times \frac{\mu_0}{4\pi} \frac{I_1 \mathrm{d}\boldsymbol{l}_1 \times \boldsymbol{r}_{12}}{r_{12}^3}, \quad \mathrm{d}\boldsymbol{F}_{12} = I_1 \mathrm{d}\boldsymbol{l}_1 \times \frac{\mu_0}{4\pi} \frac{I_2 \mathrm{d}\boldsymbol{l}_2 \times \boldsymbol{r}_{21}}{r_{21}^3}
$$

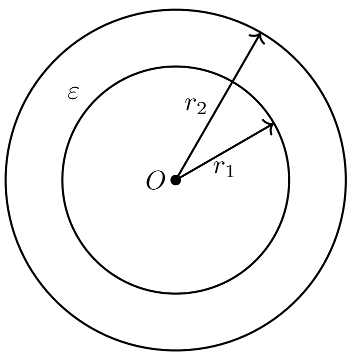
仅考虑方向，$\mathrm{d}\boldsymbol{l}_2 \times (\mathrm{d}\boldsymbol{l}_1 \times \boldsymbol{r}_{12}), \mathrm{d}\boldsymbol{l}_1 \times (\mathrm{d}\boldsymbol{l}_2 \times \boldsymbol{r}_{21})$ 都不一定反向：考虑如下情形



则 $\mathrm{d}\boldsymbol{l}_2 \times (\mathrm{d}\boldsymbol{l}_1 \times \boldsymbol{r}_{12}) = \boldsymbol{0}, \mathrm{d}\boldsymbol{l}_1 \times (\mathrm{d}\boldsymbol{l}_2 \times \boldsymbol{r}_{21}) \neq \boldsymbol{0}$，故显然不服从牛顿第三定律 $\square$

1.7

解 (1).



取半径为 r 的 *Gauss* 面 S，则由均匀分布知

$$
\oiint_S \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = 4\pi r^2 D
$$

由 $Gauss$ 定理知

$$\oiint_S \boldsymbol{D}\cdot \mathrm{d}\boldsymbol{S} = 4\pi r^2 D = \iiint_V \rho_f \mathrm{d}V = \begin{cases} 0, & r<r_1 \\ \frac{4\pi}{3}(r^3-r_1^3)\rho_f, & r_1<r<r_2 \\ \frac{4\pi}{3}(r_2^3-r_1^3)\rho_f, & r>r_2 \end{cases}$$

又因为 $\boldsymbol{D}=\varepsilon\boldsymbol{E}$，所以

$$\boldsymbol{E} = \begin{cases} 0, & r<r_1 \\ \frac{(r^3-r_1^3)\rho_f}{3\varepsilon r^3}\boldsymbol{r}, & r_1<r<r_2 \\ \frac{(r_2^3-r_1^3)\rho_f}{3\varepsilon_0 r^3}\boldsymbol{r}, & r>r_2 \end{cases}$$

(2). 因为在介质内 $\boldsymbol{D}=\varepsilon_0\boldsymbol{E}+\boldsymbol{P}$，所以

$$\boldsymbol{P}=\boldsymbol{D}-\varepsilon_0\boldsymbol{E}=\left(1-\frac{\varepsilon_0}{\varepsilon}\right)\frac{(r^3-r_1^3)\rho_f}{3r^3}\boldsymbol{r}$$

所以

$$\rho_p = -\nabla\cdot\boldsymbol{P} = -\frac{\rho_f}{3}\left(1-\frac{\varepsilon_0}{\varepsilon}\right)\nabla\cdot\left(\frac{r^3-r_1^3}{r^3}\boldsymbol{r}\right) = -\left(1-\frac{\varepsilon_0}{\varepsilon}\right)\rho_f, \quad r_1<r<r_2$$

因为只有在 $r=r_1, r=r_2$ 处，极化强度发生改变，当 $r=r_1$ 时，$\boldsymbol{n}=\boldsymbol{e}_r$，$\boldsymbol{P}^{(2)}=\boldsymbol{P}|_{r=r_1}, \boldsymbol{P}^{(1)}=0$，所以

$$\sigma_P|_{r=r_1} = -\boldsymbol{n}\cdot(\boldsymbol{P}^{(2)}-\boldsymbol{P}^{(1)}) = 0$$

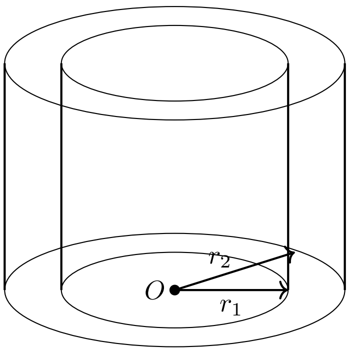
当 $r=r_2$ 时，$\boldsymbol{n}=\boldsymbol{e}_r, \boldsymbol{P}^{(1)}=\boldsymbol{P}|_{r=r_2}, \boldsymbol{P}^{(2)}=0$，所以

$$\sigma_P|_{r=r_2} = -\boldsymbol{n}\cdot(\boldsymbol{P}^{(2)}-\boldsymbol{P}^{(1)}) = \frac{r_2^3-r_1^3}{3r_2^2}\left(1-\frac{\varepsilon_0}{\varepsilon}\right)\rho_f$$

□

1.8

解



取沿圆柱轴线向上为正方向，取圆心在圆柱中轴线上的半径为 r 的圆环 L，由均匀分布知

$$\oint_L \boldsymbol{H}\cdot\mathrm{d}\boldsymbol{l} = H\cdot 2\pi r$$

由安培环路定理知

$$\oint_L \boldsymbol{H}\cdot\mathrm{d}\boldsymbol{l} = 2\pi r H = \iint_S \boldsymbol{J}_f \mathrm{d}\boldsymbol{S} = \begin{cases} 0, & r<r_1 \\ \pi(r^2-r_1^2)\boldsymbol{J}_f, & r_1<r<r_2 \\ \pi(r_2^2-r_1^2)\boldsymbol{J}_f, & r>r_2 \end{cases}$$

又因为 $\boldsymbol{B}=\mu\boldsymbol{H}$，所以

$$\boldsymbol{B}=\begin{cases}0, \quad r<r_1\\ \frac{\mu(r^2-r_1^2)}{2r^2}\boldsymbol{J}_f\times\boldsymbol{r}, \quad r_1<r<r_2\\ \frac{\mu_0(r_2^2-r_1^2)}{2r^2}\boldsymbol{J}_f\times\boldsymbol{r}, \quad r>r_2\end{cases}$$

又因为 $\boldsymbol{H}=\frac{\boldsymbol{B}}{\mu_0}-\boldsymbol{M}$，所以当 $r_1<r<r_2$ 时

$$\boldsymbol{M}=\frac{\boldsymbol{B}}{\mu_0}-\boldsymbol{H}=\left(\frac{\mu}{\mu_0}-1\right)\boldsymbol{H}=\frac{\mu-\mu_0}{\mu_0}\frac{(r^2-r_1^2)}{2r^2}\boldsymbol{J}_f\times\boldsymbol{r}$$

$$\boldsymbol{J}_M=\nabla\times\boldsymbol{M}=\left(\frac{\mu-\mu_0}{\mu_0}\right)\boldsymbol{J}_f$$

因为只有在 $r=r_1,r=r_2$ 时，$\boldsymbol{M}$ 发生改变，让 $r=r_1$ 时，$\boldsymbol{n}=\boldsymbol{e}_r,\boldsymbol{M}^{(2)}=\boldsymbol{M}|_{r=r_1},\boldsymbol{M}^{(1)}=0$，所以

$$\boldsymbol{\alpha}_M|_{r=r_1}=\boldsymbol{n}\times(\boldsymbol{M}^{(2)}-\boldsymbol{M}^{(1)})=0$$

当 $r=r_2$ 时，$\boldsymbol{n}=\boldsymbol{e}_r,\boldsymbol{M}^{(2)}=0,\boldsymbol{M}^{(1)}=\boldsymbol{M}|_{r=r_2}$，所以

$$\boldsymbol{\alpha}_M|_{r=r_2}=\boldsymbol{n}\times(\boldsymbol{M}^{(2)}-\boldsymbol{M}^{(1)})=-\left(\frac{\mu-\mu_0}{\mu_0}\right)\frac{r_2^2-r_1^2}{2r_2}\boldsymbol{J}_f$$

□

1.9

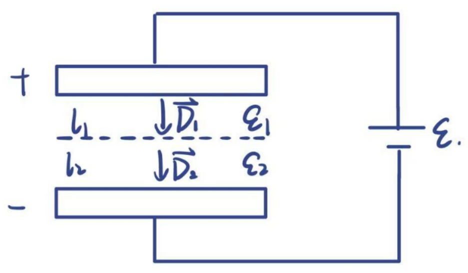
证明 因为各向同性线性均匀介质内有 $\boldsymbol{D}=\varepsilon\boldsymbol{E}$，且 $\boldsymbol{D}=\varepsilon_0\boldsymbol{E}+\boldsymbol{P}$，所以 $\boldsymbol{P}=\left(1-\frac{\varepsilon_0}{\varepsilon}\right)\boldsymbol{D}$

$$\begin{aligned}\rho_P&=-\nabla\cdot\boldsymbol{P}=-\left(1-\frac{\varepsilon_0}{\varepsilon}\right)\nabla\cdot\boldsymbol{D}\\&=-\left(1-\frac{\varepsilon_0}{\varepsilon}\right)\rho_f\end{aligned}$$

□

1.11

解 (2).



因为介质内没有自由电荷，所以在介质分界面上，自由电荷密度 $\omega_f=0$

(1). 因为在介质交界处，自由电荷密度 $\omega_f=0$，取向下为正方向，则

$$\boldsymbol{n}\cdot(\boldsymbol{D}_2-\boldsymbol{D}_1)=\omega_f=0$$

所以 $\boldsymbol{D}_2=\boldsymbol{D}_1$，而 $\boldsymbol{D}=\varepsilon\boldsymbol{E}$，所以 $\varepsilon_1\boldsymbol{E}_1=\varepsilon_2\boldsymbol{E}_2$，又因为 $E_1l_1+E_2l_2=\mathscr{E}$，联立二式解得

$$\begin{cases}E_1=\frac{\varepsilon_2\mathscr{E}}{\varepsilon_1l_2+\varepsilon_2l_1}\\E_2=\frac{\varepsilon_1\mathscr{E}}{\varepsilon_1l_2+\varepsilon_2l_1}\end{cases}\Rightarrow\boldsymbol{D}_2=\boldsymbol{D}_1=\boldsymbol{E}_1=\frac{\varepsilon_1\varepsilon_2\mathscr{E}}{\varepsilon_1l_2+\varepsilon_2l_1}\boldsymbol{n}$$

对于上极板，$\boldsymbol{D}^{(2)}=\boldsymbol{D}_1,\boldsymbol{D}^{(1)}=0$，所以

$$\omega_{up}=\boldsymbol{n}\cdot(\boldsymbol{D}^{(2)}-\boldsymbol{D}^{(1)})=\frac{\varepsilon_1\varepsilon_2\mathscr{E}}{\varepsilon_1l_2+\varepsilon_2l_1}$$

对于下极板，$\boldsymbol{D}^{(2)}=0,\boldsymbol{D}^{(1)}=\boldsymbol{D}_2$，所以

$$\omega_{down}=\boldsymbol{n}\cdot(\boldsymbol{D}^{(2)}-\boldsymbol{D}^{(1)})=-\frac{\varepsilon_1\varepsilon_2\mathscr{E}}{\varepsilon_1l_2+\varepsilon_2l_1}$$

若介质是漏电的，由电荷守恒定理

$$\nabla\cdot\boldsymbol{J}+\frac{\partial\rho}{\partial t}=0$$

因为稳恒条件下 $\frac{\partial\rho}{\partial t}=0$，所以我们有 $\nabla\cdot\boldsymbol{J}=0$，故对于介质 *1,2* 的交界面，我们有 $J_{1n}=J_{2n}=J_n$，又因为 $\boldsymbol{J}=\sigma\boldsymbol{E}$，所以 $\sigma_1\boldsymbol{E}_1=\sigma_2\boldsymbol{E}_2=\boldsymbol{J}_n$，联立

$$\begin{cases}E_1l_1+E_2l_2=\mathscr{E}\\\sigma_1E_1=\sigma_2E_2=J_n\end{cases}\Rightarrow J_n=\frac{\sigma_1\sigma_2\mathscr{E}}{\sigma_1l_2+\sigma_2l_1}$$

对于上极板，$\boldsymbol{D}^{(2)}=\boldsymbol{D}_1=\varepsilon_1\boldsymbol{E}_1=\frac{\varepsilon_1\boldsymbol{J}_n}{\sigma_1},\boldsymbol{D}^{(1)}=0$ ，所以

$$\omega_{f_1}=\boldsymbol{n}\cdot(\boldsymbol{D}^{(2)}-\boldsymbol{D}^{(1)})=\frac{\varepsilon_1\sigma_2\mathscr{E}}{\sigma_1l_2+\sigma_2l_1}$$

对于下极板，$\boldsymbol{D}^{(2)}=0,\boldsymbol{D}^{(1)}=\boldsymbol{D}_2=\varepsilon_2\boldsymbol{E}_2=\frac{\varepsilon_2\boldsymbol{J}_n}{\sigma_2}$，所以

$$\omega_{f_2}=\boldsymbol{n}\cdot(\boldsymbol{D}^{(2)}-\boldsymbol{D}^{(1)})=-\frac{\varepsilon_2\sigma_1\mathscr{E}}{\sigma_1l_2+\sigma_2l_1}$$

对于介质交界面，$\boldsymbol{D}^{(2)}=\boldsymbol{D}_2,\boldsymbol{D}^{(1)}=\boldsymbol{D}_1$，所以

$$\omega_{f_3}=\boldsymbol{n}\cdot(\boldsymbol{D}^{(2)}-\boldsymbol{D}^{(1)})=\frac{(\varepsilon_2\sigma_1-\varepsilon_1\sigma_2)\mathscr{E}}{\sigma_1l_2+\sigma_2l_1}$$

□

1.12

证明 (1). 分界面上没有自由电荷，$\sigma_f=0$，由边值关系

$$\begin{cases}\boldsymbol{n}\cdot(\boldsymbol{D}^{(2)}-\boldsymbol{D}^{(1)})=0\\\boldsymbol{n}\times(\boldsymbol{E}^{(2)}-\boldsymbol{E}^{(1)})=0\end{cases}$$

故

$$E_{2\tau}=E_{1\tau},\quad D_{2n}=D_{1n}$$

又因为 $\boldsymbol{D}=\varepsilon\boldsymbol{E}$，所以

$$\begin{cases} E_2\sin\theta_2=E_1\sin\theta_1 \\ \varepsilon_2E_2\cos\theta_2=\varepsilon_1E_1\cos\theta_1 \end{cases}$$

解得

$$\frac{\tan\theta_1}{\tan\theta_2}=\frac{\varepsilon_2}{\varepsilon_1}$$

(2). 由于电流恒定时 $\frac{\partial\rho}{\partial t}=0$，由边值关系

$$\begin{cases} \boldsymbol{n}\cdot(\boldsymbol{J}^{(2)}-\boldsymbol{J}^{(1)})=0 \\ \boldsymbol{n}\times(\boldsymbol{E}^{(2)}-\boldsymbol{E}^{(1)})=0 \end{cases}$$

故

$$E_{2\tau}=E_{1\tau},\quad J_{2n}=J_{1n}$$

又因为 $\boldsymbol{J}=\sigma\boldsymbol{E}$，所以

$$\begin{cases} E_2\sin\theta_2=E_1\sin\theta_1 \\ \sigma_2E_2\cos\theta_2=\sigma_1E_1\cos\theta_1 \end{cases}$$

解得

$$\frac{\tan\theta_2}{\tan\theta_1}=\frac{\sigma_2}{\sigma_1}$$

□