电动力学第二周作业

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1.1

证明 因为

$$\begin{aligned}\nabla(\boldsymbol{A}\cdot\boldsymbol{B}) &= \nabla(\boldsymbol{A}\cdot\boldsymbol{B}_c)+\nabla(\boldsymbol{A}_c\cdot\boldsymbol{B})\\ &= (\boldsymbol{B}_c\cdot\nabla)\boldsymbol{A}+(\boldsymbol{A}\times\nabla)\times\boldsymbol{B}_c+(\nabla\cdot\boldsymbol{A}_c)\boldsymbol{B}+\boldsymbol{A}_c\times(\nabla\times\boldsymbol{B})\\ &= (\boldsymbol{B}_c\cdot\nabla)\boldsymbol{A}+\boldsymbol{B}_c\times(\nabla\times\boldsymbol{A})+(\boldsymbol{A}_c\cdot\nabla)\boldsymbol{B}+\boldsymbol{A}_c\times(\nabla\times\boldsymbol{B})\\ &= (\boldsymbol{B}\cdot\nabla)\boldsymbol{A}+\boldsymbol{B}\times(\nabla\times\boldsymbol{A})+(\boldsymbol{A}\cdot\nabla)\boldsymbol{B}+\boldsymbol{A}\times(\nabla\times\boldsymbol{B})\end{aligned}$$

令 $\boldsymbol{A}=\boldsymbol{B}$，则 $\nabla(\boldsymbol{A}\cdot\boldsymbol{A})=2(\boldsymbol{A}\cdot\nabla)\boldsymbol{A}+2\boldsymbol{A}\times(\nabla\times\boldsymbol{A})$，因此

$$\boldsymbol{A}\times(\nabla\times\boldsymbol{A})=\frac{1}{2}\nabla\boldsymbol{A}^2-(\boldsymbol{A}\cdot\nabla)\boldsymbol{A}$$

□

1.2

证明 因为 $f(u)=f(u(x,y,z)),\boldsymbol{A}(u)=(A_x(u(x,y,z)),A_y(u(x,y,z)),A_z(u(x,y,z)))$，所以

$$\begin{aligned}\nabla f(u) &= \overrightarrow{e}_x\frac{\partial}{\partial x}f(u(x,y,z))+\overrightarrow{e}_y\frac{\partial}{\partial y}f(u(x,y,z))+\overrightarrow{e}_z\frac{\partial}{\partial z}f(u(x,y,z))\\ &= \overrightarrow{e}_x f'(u)\frac{\partial u}{\partial x}+\overrightarrow{e}_y f'(u)\frac{\partial u}{\partial y}+\overrightarrow{e}_z f'(u)\frac{\partial u}{\partial z}\\ &= \frac{\mathrm{d}f}{\mathrm{d}u}\nabla u\end{aligned}$$

$$\begin{aligned}\nabla\cdot\boldsymbol{A}(u) &= \frac{\partial}{\partial x}A_x(u(x,y,z))+\frac{\partial}{\partial y}A_y(u(x,y,z))+\frac{\partial}{\partial z}A_z(u(x,y,z))\\ &= A_x'(u)\frac{\partial u}{\partial x}+A_y'(u)\frac{\partial u}{\partial y}+A_z'(u)\frac{\partial u}{\partial z}\\ &= \nabla u\cdot\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}u}\end{aligned}$$

$$\begin{aligned}\nabla\times\boldsymbol{A}(u) &= \begin{vmatrix}\overrightarrow{e}_x & \overrightarrow{e}_y & \overrightarrow{e}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x(u) & A_y(u) & A_z(u)\end{vmatrix}\\ &= \overrightarrow{e}_x\left(\frac{\partial}{\partial y}A_z(u)-\frac{\partial}{\partial z}A_y(u)\right)-\overrightarrow{e}_y\left(\frac{\partial}{\partial x}A_z(u)-\frac{\partial}{\partial z}A_x(u)\right)+\overrightarrow{e}_z\left(\frac{\partial}{\partial x}A_y(u)-\frac{\partial}{\partial y}A_x(u)\right)\\ &= \overrightarrow{e}_x\left(A_z'(u)\frac{\partial u}{\partial y}-A_y'(u)\frac{\partial u}{\partial z}\right)-\overrightarrow{e}_y\left(A_z'(u)\frac{\partial u}{\partial x}-A_x'(u)\frac{\partial u}{\partial z}\right)+\overrightarrow{e}_z\left(A_y'(u)\frac{\partial u}{\partial x}-A_x'(u)\frac{\partial u}{\partial y}\right)\\ &= \begin{vmatrix}\overrightarrow{e}_x & \overrightarrow{e}_y & \overrightarrow{e}_z\\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ A_x'(u) & A_y'(u) & A_z'(u)\end{vmatrix}=\nabla u\times\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}u}\end{aligned}$$

□

1.3

证明 $(1a)$. 因为 $r=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}, \boldsymbol{r}=(x-x',y-y',z-z')$，所以

$$\frac{\partial r}{\partial x}=\frac{2(x-x')}{2\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}=\frac{x-x'}{r},\quad \frac{\partial r}{\partial x'}=\frac{-2(x-x')}{2\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}=-\frac{x-x'}{r}$$

对 y,z 求偏导也类似，因此

$$\begin{aligned}\nabla r&=\vec{e}_x\frac{\partial r}{\partial x}+\vec{e}_y\frac{\partial r}{\partial y}+\vec{e}_z\frac{\partial r}{\partial z}\\&=\left(\frac{x-x'}{r},\frac{y-y'}{r},\frac{z-z'}{r}\right)\\&=\frac{\boldsymbol{r}}{r}=\vec{e}_r\end{aligned}$$

$$\begin{aligned}\nabla' r&=\vec{e}_x\frac{\partial r}{\partial x'}+\vec{e}_y\frac{\partial r}{\partial y'}+\vec{e}_z\frac{\partial r}{\partial z'}\\&=\left(-\frac{x-x'}{r},-\frac{y-y'}{r},-\frac{z-z'}{r}\right)\\&=-\frac{\boldsymbol{r}}{r}=-\vec{e}_r\end{aligned}$$

所以

$$\nabla r=-\nabla' r=\frac{\boldsymbol{r}}{r}$$

$(1b)$. 因为

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right)=\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\right)\cdot\frac{\partial r}{\partial x}=-\frac{x-x'}{r^3},\quad \frac{\partial}{\partial x'}\left(\frac{1}{r}\right)=\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\right)\cdot\frac{\partial r}{\partial x'}=\frac{x-x'}{r^3}$$

对 y,z 求偏导也类似，因此

$$\begin{aligned}\nabla\frac{1}{r}&=\left(-\frac{x-x'}{r^3},-\frac{y-y'}{r^3},-\frac{z-z'}{r^3}\right)\\&=-\frac{\boldsymbol{r}}{r^3}\end{aligned}$$

$$\begin{aligned}\nabla'\frac{1}{r}&=\left(\frac{x-x'}{r^3},\frac{y-y'}{r^3},\frac{z-z'}{r^3}\right)\\&=\frac{\boldsymbol{r}}{r^3}\end{aligned}$$

所以

$$\nabla\frac{1}{r}=-\nabla'\frac{1}{r}=-\frac{\boldsymbol{r}}{r^3}$$

$(1c)$. 因为

$$\nabla\times\frac{\boldsymbol{r}}{r^3}=\begin{vmatrix}\vec{e}_x & \vec{e}_y & \vec{e}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{x-x'}{r^3} & \frac{y-y'}{r^3} & \frac{z-z'}{r^3}\end{vmatrix}$$

考虑 $\vec{e}_x$ 项，展开有

$$\vec{e}_x\left[\frac{\partial}{\partial y}\left(\frac{z-z'}{r^3}\right)-\frac{\partial}{\partial z}\left(\frac{y-y'}{r^3}\right)\right]=\vec{e}_x\left[(z-z')\cdot\frac{-3}{r^4}\cdot\frac{y-y'}{r}-(y-y')\cdot\frac{-3}{r^4}\cdot\frac{z-z'}{r}\right]=0$$

对 $\vec{e}_y,\vec{e}_z$ 项展开也同样得零，因此

$$\nabla\times\frac{\boldsymbol{r}}{r^3}=0$$

(1d). 因为

$$
\begin{aligned}
\nabla \cdot \frac{\boldsymbol{r}}{r^3} &= \frac{\partial}{\partial x}\left(\frac{x-x'}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y-y'}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z-z'}{r^3}\right) \\
&= \frac{r^2-3(x-x')^2}{r^5} + \frac{r^2-3(y-y')^2}{r^5} + \frac{r^2-3(z-z')^2}{r^5} \\
&= \frac{3r^2-3[(x-x')^2+(y-y')^2+(z-z')^2]}{r^5} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\nabla' \cdot \frac{\boldsymbol{r}}{r^3} &= \frac{\partial}{\partial x'}\left(\frac{x-x'}{r^3}\right) + \frac{\partial}{\partial y'}\left(\frac{y-y'}{r^3}\right) + \frac{\partial}{\partial z'}\left(\frac{z-z'}{r^3}\right) \\
&= \frac{-r^2+3(x-x')^2}{r^5} + \frac{-r^2+3(y-y')^2}{r^5} + \frac{-r^2+3(z-z')^2}{r^5} \\
&= \frac{-3r^2+3[(x-x')^2+(y-y')^2+(z-z')^2]}{r^5} \\
&= 0
\end{aligned}
$$

所以

$$
\nabla \cdot \frac{\boldsymbol{r}}{r^3} = -\nabla' \cdot \frac{\boldsymbol{r}}{r^3} = 0, \quad r \neq 0
$$

(2).

$$
\begin{aligned}
\nabla \cdot \boldsymbol{r} &= \frac{\partial(x-x')}{\partial x} + \frac{\partial(y-y')}{\partial y} + \frac{\partial(z-z')}{\partial z} \\
&= 1+1+1 = 3
\end{aligned}
$$

$$
\begin{aligned}
\nabla \times \boldsymbol{r} &= \begin{vmatrix} \overrightarrow{e}_x & \overrightarrow{e}_y & \overrightarrow{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-x' & y-y' & z-z' \end{vmatrix} \\
&= \overrightarrow{e}_x\left[\frac{\partial}{\partial y}(z-z') - \frac{\partial}{\partial z}(y-y')\right] - \overrightarrow{e}_y\left[\frac{\partial}{\partial x}(z-z') - \frac{\partial}{\partial z}(x-x')\right] + \overrightarrow{e}_z\left[\frac{\partial}{\partial x}(y-y') - \frac{\partial}{\partial y}(x-x')\right] \\
&= \boldsymbol{0}
\end{aligned}
$$

$$
\begin{aligned}
(\boldsymbol{a}\cdot\nabla)\boldsymbol{r} &= \left(a_x\frac{\partial}{\partial x} + a_y\frac{\partial}{\partial y} + a_z\frac{\partial}{\partial z}\right)(x-x', y-y', z-z') \\
&= (a_x, a_y, a_z) = \boldsymbol{a}
\end{aligned}
$$

$$
\begin{aligned}
\nabla(\boldsymbol{a}\cdot\boldsymbol{r}) &= \nabla(a_x(x-x') + a_y(y-y') + a_z(z-z')) \\
&= (a_x, a_y, a_z) = \boldsymbol{a}
\end{aligned}
$$

因为 $\frac{\partial}{\partial x}\sin(\boldsymbol{k}\cdot\boldsymbol{r}) = \frac{\partial}{\partial x}\sin(k_x(x-x')+k_y(y-y')+k_z(z-z')) = k_x\cos(k_x(x-x')+k_y(y-y')+k_z(z-z')) = k_x\cos(\boldsymbol{k}\cdot\boldsymbol{r})$，对 y, z 求偏导也类似，所以

$$
\begin{aligned}
\nabla\cdot[\boldsymbol{E}_0\sin(\boldsymbol{k}\cdot\boldsymbol{r})] &= \frac{\partial}{\partial x}(E_{0x}\sin(\boldsymbol{k}\cdot\boldsymbol{r})) + \frac{\partial}{\partial y}(E_{0y}\sin(\boldsymbol{k}\cdot\boldsymbol{r})) + \frac{\partial}{\partial z}(E_{0z}\sin(\boldsymbol{k}\cdot\boldsymbol{r})) \\
&= E_{0x}k_x\cos(\boldsymbol{k}\cdot\boldsymbol{r}) + E_{0y}k_y\cos(\boldsymbol{k}\cdot\boldsymbol{r}) + E_{0z}k_z\cos(\boldsymbol{k}\cdot\boldsymbol{r}) \\
&= (\boldsymbol{k}\cdot\boldsymbol{E}_0)\cos(\boldsymbol{k}\cdot\boldsymbol{r})
\end{aligned}
$$

$$
\begin{aligned}
\nabla \times [\boldsymbol{E}_0 \sin(\boldsymbol{k}\cdot\boldsymbol{r})] &= \begin{vmatrix} \overrightarrow{e}_x & \overrightarrow{e}_y & \overrightarrow{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{0x}\sin(\boldsymbol{k}\cdot\boldsymbol{r}) & E_{0y}\sin(\boldsymbol{k}\cdot\boldsymbol{r}) & E_{0z}\sin(\boldsymbol{k}\cdot\boldsymbol{r}) \end{vmatrix} \\
&= \overrightarrow{e}_x \left[\frac{\partial}{\partial y}(E_{0z}\sin(\boldsymbol{k}\cdot\boldsymbol{r})) - \frac{\partial}{\partial z}(E_{0y}\sin(\boldsymbol{k}\cdot\boldsymbol{r}))\right] - \overrightarrow{e}_y \left[\frac{\partial}{\partial x}(E_{0z}\sin(\boldsymbol{k}\cdot\boldsymbol{r})) - \frac{\partial}{\partial z}(E_{0x}\sin(\boldsymbol{k}\cdot\boldsymbol{r}))\right] \\
&\qquad + \overrightarrow{e}_z \left[\frac{\partial}{\partial x}(E_{0y}\sin(\boldsymbol{k}\cdot\boldsymbol{r})) - \frac{\partial}{\partial y}(E_{0x}\sin(\boldsymbol{k}\cdot\boldsymbol{r}))\right] \\
&= \overrightarrow{e}_x[E_{0z}k_y\cos(\boldsymbol{k}\cdot\boldsymbol{r}) - E_{0y}k_z\cos(\boldsymbol{k}\cdot\boldsymbol{r})] - \overrightarrow{e}_y[E_{0z}k_x\cos(\boldsymbol{k}\cdot\boldsymbol{r}) - E_{0x}k_z\cos(\boldsymbol{k}\cdot\boldsymbol{r})] \\
&\qquad + \overrightarrow{e}_z[E_{0y}k_x\cos(\boldsymbol{k}\cdot\boldsymbol{r}) - E_{0x}k_y\cos(\boldsymbol{k}\cdot\boldsymbol{r})] \\
&= \begin{vmatrix} \overrightarrow{e}_x & \overrightarrow{e}_y & \overrightarrow{e}_z \\ k_x & k_y & k_z \\ E_{0x}\cos(\boldsymbol{k}\cdot\boldsymbol{r}) & E_{0y}\cos(\boldsymbol{k}\cdot\boldsymbol{r}) & E_{0z}\cos(\boldsymbol{k}\cdot\boldsymbol{r}) \end{vmatrix} \\
&= (\boldsymbol{k}\times\boldsymbol{E}_0)\cos(\boldsymbol{k}\cdot\boldsymbol{r})
\end{aligned}
$$

1.4

证明 对任意常矢量 $\boldsymbol{c}$，我们有

$$
\begin{aligned}
\nabla\cdot(\boldsymbol{f}\times\boldsymbol{c}) &= \nabla\cdot(\boldsymbol{f}_c\times\boldsymbol{c}) + \nabla\cdot(\boldsymbol{f}\times\boldsymbol{c}_c) \\
&= \boldsymbol{f}_c\cdot(\boldsymbol{c}\times\nabla) + \boldsymbol{c}_c\cdot(\nabla\times\boldsymbol{f}) \\
&= -\boldsymbol{f}_c\cdot(\nabla\times\boldsymbol{c}) + \boldsymbol{c}_c\cdot(\nabla\times\boldsymbol{f}) \\
&= (\nabla\times\boldsymbol{f})\cdot\boldsymbol{c}
\end{aligned}
$$

对上式两边同时在区域 V 上积分得

$$\iiint_V \mathrm{d}V\nabla\cdot(\boldsymbol{f}\times\boldsymbol{c}) = \iiint_V \mathrm{d}V(\nabla\times\boldsymbol{f})\cdot\boldsymbol{c}$$

由 *Gauss* 定理：$\iiint_V \nabla\cdot\boldsymbol{A}\mathrm{d}V = \oiint_S \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S}$，则

$$\iiint_V \mathrm{d}V(\nabla\times\boldsymbol{f})\cdot\boldsymbol{c} = \oiint_S (\boldsymbol{f}\times\boldsymbol{c})\cdot\mathrm{d}\boldsymbol{S} = \oiint_S (\mathrm{d}\boldsymbol{S}\times\boldsymbol{f})\cdot\boldsymbol{c}$$

由 $\boldsymbol{c}$ 的任意性可知

$$\iiint_V \mathrm{d}V\nabla\times\boldsymbol{f} = \oiint_S \mathrm{d}\boldsymbol{S}\times\boldsymbol{f}$$

对任意常矢量 $\boldsymbol{c}$，我们有

$$
\begin{aligned}
\nabla\times(\boldsymbol{c}\varphi) &= \nabla\times(\boldsymbol{c}_c\varphi) + \nabla\times(\boldsymbol{c}\varphi_c) \\
&= \nabla\varphi\times\boldsymbol{c}_c + \varphi_c(\nabla\times\boldsymbol{c}) \\
&= \nabla\varphi\times\boldsymbol{c} + \varphi(\nabla\times\boldsymbol{c}) \\
&= \nabla\varphi\times\boldsymbol{c}
\end{aligned}
$$

两边同时对任意闭合曲面 S 积分得

$$\iint_S \mathrm{d}\boldsymbol{S}\nabla\times(\boldsymbol{c}\varphi) = \iint_S \mathrm{d}\boldsymbol{S}\nabla\varphi\times\boldsymbol{c}$$

由 *Stokes* 定理：$\oint_L \int_L \boldsymbol{f}\cdot\mathrm{d}\boldsymbol{l} = \iint_S(\nabla\times\boldsymbol{f})\cdot\mathrm{d}\boldsymbol{S}$，则

$$\oint_L \boldsymbol{c}\cdot(\varphi\mathrm{d}\boldsymbol{l}) = \oint_L(\boldsymbol{c}\varphi)\cdot\mathrm{d}\boldsymbol{l} = \iint_S \mathrm{d}\boldsymbol{S}\cdot(\nabla\varphi\times\boldsymbol{c}) = \iint_S \boldsymbol{c}\cdot(\mathrm{d}\boldsymbol{S}\times\nabla\varphi)$$

对比上式第一项和第四项，由 $\boldsymbol{c}$ 的任意性可知

$$\oint_L \varphi \mathrm{d}\boldsymbol{l} = \iint_S \mathrm{d}\boldsymbol{S} \times \nabla\varphi$$

□

1.5

证明

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} &= \iiint_V \frac{\partial\rho}{\partial t}(\boldsymbol{x}', t)\boldsymbol{x}'\mathrm{d}V' \\ &= -\iiint_V [\nabla'\cdot\boldsymbol{J}(\boldsymbol{x}', t)]\boldsymbol{x}'\mathrm{d}V'\end{aligned}$$

因为（下面的 $\boldsymbol{J} = \boldsymbol{J}(\boldsymbol{x}', t)$）

$$\nabla'\cdot(\boldsymbol{J}\boldsymbol{x}') = (\nabla'\cdot\boldsymbol{J})\boldsymbol{x}' + \boldsymbol{J}\cdot\nabla'\boldsymbol{x}'$$

所以

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} &= \iiint_V \boldsymbol{J}(\boldsymbol{x}', t)\mathrm{d}V' - \iiint_V \nabla'\cdot(\boldsymbol{J}(\boldsymbol{x}', t)\boldsymbol{x}')\mathrm{d}V' \\ &= \iiint_V \boldsymbol{J}(\boldsymbol{x}', t)\mathrm{d}V' - \oiint_S \mathrm{d}\boldsymbol{S}'\cdot(\boldsymbol{J}(\boldsymbol{x}', t)\boldsymbol{x}')\end{aligned}$$

我们取积分区域 $\tilde{V}$ 远大于电荷分布区域 V，因此在新的 $\tilde{V}$ 的边界 $\tilde{\boldsymbol{S}}$ 上，有 $\boldsymbol{J} = 0$，故上式积分的第二项为零，因此我们有

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \iiint_V \boldsymbol{J}(\boldsymbol{x}', t)\mathrm{d}V'$$

□