BETA AND GAMMA FUNCTIONS

Swiss mathematician and physicist Leonhard Euler (1707-1783) developed many concepts that are integral part of modern mathematics. There are many special functions which have more or less established names and notations due to their importance in mathematical analysis, functional analysis, physics or other applications. In 1729 and 1730, Euler investigated Gamma and Beta functions.

Beta Functions

If $m > 0, n > 0,$ then the integral

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

is called the **Beta function** or the **first Eulerian integral** and is denoted by $\beta(m,n)$.

Examples

(i) $\int_0^1 x^3(1-x)^5\,dx = \beta(4,6)$

(ii) $\int_0^1 \sqrt{x}\,(1-x)^3\,dx = \beta(\frac{3}{2},4)$

Properties of beta function

1. Beta function is a symmetric function, that is, $\beta(m,n) = \beta(n,m)$.

Proof: Let $1-x = u$. Then $dx = -du$. Also, if $x = 0$ then $u = 1$ and if $x = 1$ then $u = 0$.

So that $\beta(m,n) = -\int_1^0 (1-u)^{m-1}\,u^{n-1}du = \int_0^1 u^{n-1}(1-u)^{m-1}\,du = \beta(n,m)$.

2. Trigonometric form: $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$.

Proof: Let $x = \sin^2\theta$. Then $dx = 2\sin\theta\cos\theta\,d\theta$.

If $x = 0$ then $\theta = 0$ and if $x = 1$ then $\theta = \frac{\pi}{2}$.

So that $\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\,2\sin\theta\cos\theta\,d\theta = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$.

3. Prove that $\beta(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$.

Proof: Let $x = \frac{u}{1+u}$. Then $dx = \frac{du}{(1+u)^2}$.

Also, if $x = 0$ then $u = 0$ and if $x = 1$ then $u = \infty$.

So that $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^{\infty}\left(\frac{u}{1+u}\right)^{m-1}\left(1-\frac{u}{1+u}\right)^{n-1}\frac{du}{(1+u)^2}$

$$= \int_0^\infty \frac{u^{m-1}}{(1+u)^{m+n}}\, du = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

Thus, $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$.

4. Prove that $\beta(m,1-m) = \int_0^1 \frac{x^{m-1}+x^{-m}}{1+x}\, dx$.

Proof: We have $\beta(m,1-m) = \int_0^\infty \frac{x^{m-1}}{1+x}\, dx = \int_0^1 \frac{x^{m-1}}{1+x}\, dx + \int_1^\infty \frac{x^{m-1}}{1+x}\, dx.$

Putting $x = \frac{1}{u}$ in the second integral, we get $dx = -\frac{1}{u^2}\, du$.
Also, if $x = 1$ then $u = 1$ and if $x = \infty$ then $u = 0$.

So that $\int_1^\infty \frac{x^{m-1}}{1+x}\, dx = -\int_1^0 \frac{(1/u)^{m-1}}{1+\frac{1}{u}} \frac{du}{u^2} = \int_0^1 \frac{u^{-m}}{1+u}\, du = \int_0^1 \frac{x^{-m}}{1+x}\, dx.$

Thus, $\beta(m,1-m) = \int_0^1 \frac{x^{m-1}+x^{-m}}{1+x}\, dx.$

Worked out problems

1. Evaluate $\int_0^1 x^3 (1-x)^{\frac{4}{3}}\, dx$

Solution: We can write $\int_0^1 x^3 (1-x)^{\frac{4}{3}}\, dx = \beta(4, \frac{7}{3}) = \beta(\frac{7}{3}, 4) = \int_0^1 x^{\frac{4}{3}} (1-x)^3\, dx$

$$= \int_0^1 x^{\frac{4}{3}} (1 - 3x + 3x^2 - x^3) dx$$

$$= \int_0^1 \left(x^{\frac{4}{3}} - 3x^{\frac{7}{3}} + 3x^{\frac{10}{3}} - x^{\frac{13}{3}} \right) dx$$

$$= \frac{243}{7280}.$$

2. Show that $\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta\, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right);\ m > -1,\ n > -1.$

Proof: Let $u = \sin^2\theta$. Then $du = 2\sin\theta\cos\theta\, d\theta$.

Also, if $\theta = 0$ then $u = 0$ and if $\theta = \frac{\pi}{2}$ then $u = 1$.

So that $\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta\, d\theta = \int_0^{\frac{\pi}{2}} (\sin^{m-1}\theta \cos^{n-1}\theta) \sin\theta \cos\theta\ d\theta$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}(\sin^2\theta)^{\frac{m-1}{2}}(\cos^2\theta)^{\frac{n-1}{2}}\,2\sin\theta\,\cos\theta\,d\theta=\frac{1}{2}\int_0^1 u^{\frac{m-1}{2}}(1-u)^{\frac{n-1}{2}}\,du=\frac{1}{2}\beta\left(\frac{m+1}{2},\frac{n+1}{2}\right).$$

3. Evaluate $\int_0^1 x^m(1-x^2)^n\,dx; m>-1,\ n>-1$

Solution: Let $x^2=u$. Then $dx=\frac{1}{2}u^{-\frac{1}{2}}\,du$.
Also, if $x=0$ then $u=0$ and if $x=1$ then $u=1$.
So that $\int_0^1 x^m(1-x^2)^n\,dx=\int_0^1 u^{\frac{m}{2}}(1-u)^n\frac{1}{2}u^{\frac{-1}{2}}\,du=\frac{1}{2}\int_0^1 u^{\frac{m-1}{2}}(1-u)^n\,du=\frac{1}{2}\beta\left(\frac{m+1}{2},n+1\right).$

4. Evaluate $\int_0^2 x^4(8-x^3)^{-\frac{1}{3}}\,dx$

Solution: Let $x^3=8u$. Then $dx=\frac{2}{3}u^{-\frac{2}{3}}\,du$.
Also, if $x=0$ then $u=0$ and if $x=2$ then $u=1$.
So that $\int_0^2 x^4(8-x^3)^{-\frac{1}{3}}\,dx=\int_0^1 16u^{\frac{4}{3}}(8-8u)^{\frac{-1}{3}}\frac{2}{3}u^{\frac{-2}{3}}\,du=\frac{16}{3}\int_0^1 u^{\frac{2}{3}}(1-u)^{\frac{-1}{3}}\,du=\frac{16}{3}\beta\left(\frac{5}{3},\frac{2}{3}\right).$

5. Evaluate $\int_0^2\sqrt{x}\,(4-x^2)^{-\frac{1}{4}}\,dx.$

Solution: Let $x^2=4u$. Then $dx=u^{-\frac{1}{2}}\,du$.
Also, if $x=0$ then $u=0$ and if $x=2$ then $u=1$.
So that $\int_0^2\sqrt{x}\,(4-x^2)^{-\frac{1}{4}}\,dx=\int_0^1 2^{\frac{1}{2}}u^{\frac{1}{4}}(4-4u)^{\frac{-1}{4}}u^{\frac{-1}{2}}\,du=\int_0^1 u^{\frac{-1}{4}}(1-u)^{\frac{-1}{4}}\,du=\beta\left(\frac{3}{4},\frac{3}{4}\right).$

Gamma Functions

If $n>0$, then the integral

$$\int_0^\infty x^{n-1}e^{-x}\,dx$$

is called the **Gamma function** or the **second Eulerian integral** and is denoted by $\Gamma(n)$. The Gamma function is convergent for $n>0$ (see page 322, Advanced Calculus 2nd edition, Schaum Series Textbook). This integral has important applications in solving different type of problems that arise in science and engineering.

Properties of gamma function

1. $\Gamma(n+1)=n\Gamma(n)$ for $n\geq 1$.

Proof: We have $\Gamma(n+1)=\int_0^\infty x^n\, e^{-x}dx=[-x^n\, e^{-x}]_0^\infty+n\int_0^\infty x^{n-1}e^{-x}dx=0+n\Gamma n$.

Thus $\Gamma(n+1)=n\Gamma(n)$.

2. If $n\geq 0$ is an integer, then $\Gamma(n+1)=n!$.

Proof: We have $\Gamma n=\int_0^\infty x^{n-1}\, e^{-x}dx$ (1)

So that $\Gamma(1)=\int_0^\infty e^{-x}dx=1=0!$, $\Gamma(2)=\int_0^\infty x\,e^{-x}dx=[-x\,e^{-x}]_0^\infty+\int_0^\infty e^{-x}dx=1=1!$

Similarly, we can show that $\Gamma(3)=2\Gamma(2)=2!$, $\Gamma(4)=3\Gamma(3)=3!$

In general, $\Gamma(n+1)=n!$.

Remark

Gamma function for even and odd integers.

We have $\Gamma(n)=\dfrac{\Gamma(n+1)}{n}$.

For $n=0$, $\Gamma(0)=\dfrac{\Gamma(0+1)}{0}=+\infty$

For $n=-1$, $\Gamma(-1)=\dfrac{\Gamma(-1+1)}{-1}=-\infty$

For $n=-2$, $\Gamma(-2)=\dfrac{\Gamma(-2+1)}{-2}=+\infty$

For $n=-3$, $\Gamma(-3)=\dfrac{\Gamma(-3+1)}{-3}=-\infty$

Thus, it is seen that for even negative integers n, $\Gamma n=+\infty$ and for odd negative integers n, $\Gamma n=-\infty$.

3. Show that $\Gamma(n)=2\int_0^\infty x^{2n-1}\, e^{-x^2}\, dx$.

Proof: We have $\Gamma(n)=\int_0^\infty x^{n-1}\, e^{-x}dx$ (1)

Let $x=u^2$. Then $dx=2u\,du$.

Also, if $x=0$ then $u=0$ and if $x=\infty$ then $u=\infty$.

So that $\Gamma(n)=\int_0^\infty (u^2)^{n-1}\, e^{-u^2}\, 2u\,du=2\int_0^\infty u^{2n-1}\, e^{-u^2}\, du=2\int_0^\infty x^{2n-1}\, e^{-x^2}\, dx$

4. Prove that $\Gamma(1/2)=\sqrt{\pi}$.

Proof: We have $\Gamma n=2\int_0^\infty x^{2n-1}\, e^{-x^2}\, dx$ (2)

Then $\Gamma(1/2)=2\int_0^\infty x^{2(1/2)-1}\, e^{-x^2}\, dx=2\int_0^\infty e^{-x^2}\, dx=2\int_0^\infty e^{-y^2}\, dy$

So that $\{\Gamma(1/2)\}^2 = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dxdy$

Let $x = r\cos\theta$ and $y = r\sin\theta$. Then $x^2 + y^2 = r^2$ and $\theta = \tan^{-1}\dfrac{y}{x}$.

If $x = 0,\ y = 0$ then $r = 0$ and if $x = \infty,\ y = \infty$ then $\theta = \pi/2$.

Therefore, $\{\Gamma(1/2)\}^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,drd\theta$ because $dxdy = |J|\,drd\theta$, where

$$|J| = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

So, $\{\Gamma(1/2)\}^2 = 2\int_0^{\pi/2}\int_0^\infty e^{-r^2}\,2r\,dr = 2\int_0^{\pi/2}\Big[-e^{-r^2}\Big]_0^{+\infty} d\theta = 2\int_0^{\pi/2} d\theta = 2\left(\dfrac{\pi}{2}\right) = \pi$

Thus, $\Gamma(1/2) = \sqrt{\pi}$.

Remark

$$\Gamma(-\tfrac{1}{2}) = \frac{\Gamma(-\frac{1}{2}+1)}{-\frac{1}{2}} = -2\Gamma(\tfrac{1}{2}) = -2\sqrt{\pi}$$

$$\Gamma(-\tfrac{3}{2}) = \frac{\Gamma(-\frac{3}{2}+1)}{-\frac{3}{2}} = -\frac{2}{3}\Gamma(-\tfrac{1}{2}) = -\frac{2}{3}(-2\sqrt{\pi}) = \frac{4}{3}\sqrt{\pi}$$

Similarly, we can compute $\Gamma(-\dfrac{2n+1}{2})$ for $n = 0,1,2,\ldots$

5. Prove that $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Proof: We have $\Gamma(m) = 2\int_0^\infty x^{2m-1}e^{-x^2}\,dx$ and $\Gamma(n) = 2\int_0^\infty y^{2n-1}e^{-y^2}\,dy$

So that $\Gamma(m)\Gamma(n) = 2\int_0^\infty e^{-x^2}x^{2m-1}\,dx \times 2\int_0^\infty e^{-y^2}y^{2n-1}\,dy$

$$= 4\int_0^{\pi/2}\int_0^{+\infty} e^{-r^2}(r^{2m-1}\cos^{2m-1}\theta)(r^{2n-1}\sin^{2n-1}\theta)\,rdrd\theta$$

$$= 2\int_0^{\pi/2}\cos^{2m-1}\theta\,\sin^{2n-1}\theta\,d\theta \times 2\int_0^{+\infty} e^{-r^2}r^{2(m+n)-1}\,dr$$

$$= \beta(m,n)\ \Gamma(m+n)$$

Thus, $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

6. Show that $\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta\, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$; $m > -1,\, n > -1$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)}.$$

Proof: We have $\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta\, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$; $m > -1,\, n > -1$

We also have $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Thus, $\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta\, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$; $m > -1,\, n > -1$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)}.$$

7. Given $\int_0^{\infty} \frac{x^{m-1}}{1+x}dx = \frac{\pi}{\sin(m\pi)}$, show that $\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi}$, $0 < m < 1$.

Solution: We have $\beta(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Therefore, $\beta(m,1-m) = \int_0^{\infty} \frac{x^{m-1}}{1+x}dx = \Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin(m\pi)}$.

Solved Problems

1. Evaluate each of the following:

(i) $\frac{\Gamma(8/3)}{\Gamma(2/3)}$, (ii) $\frac{\Gamma(6)}{2\Gamma(3)}$, (iii) $\frac{\Gamma(5/2)}{\Gamma(1/2)}$, (iv) $\frac{\Gamma(3)\Gamma(2.5)}{\Gamma(5.5)}$, (v) $\frac{6\Gamma(8/3)}{5\Gamma(2/3)}$.

Solution: (i) We can write $\Gamma(8/3) = \Gamma(5/3+1) = \frac{5}{3}\Gamma(5/3) = \frac{5}{3}\Gamma(2/3+1) = \frac{5}{3}\frac{2}{3}\Gamma(2/3)$.

Thus, $\frac{\Gamma(8/3)}{\Gamma(2/3)} = \frac{10}{9}$.

(ii) $\frac{\Gamma(6)}{2\Gamma(3)} = \frac{5!}{2(2!)} = \frac{(5)(4)(3)(2!)}{2(2!)} = 30$.

(iii) $\frac{\Gamma(5/2)}{\Gamma(1/2)} = \frac{(3/2)(1/2)\Gamma(1/2)}{\Gamma(1/2)} = \frac{3}{4}$.

(iv) $\frac{\Gamma(3)\Gamma(2.5)}{\Gamma(5.5)} = \frac{(2!)\Gamma(2.5)}{(4.5)(3.5)(2.5)\Gamma(2.5)} = \frac{(2)(2)(2)(2)}{(9)(7)(5)} = \frac{16}{315}$.

(v) $\dfrac{6\Gamma(8/3)}{5\Gamma(2/3)} = \dfrac{(6)(5/3)(2/3)\Gamma(2/3)}{5\Gamma(2/3)} = \dfrac{4}{3}.$

2. Evaluate each of the following integrals:

(i) $\int_0^\infty x^3 e^{-x}\,dx$, (ii) $\int_0^\infty x^6 e^{-2x}\,dx$, (iii) $\int_0^\infty 3^{-4x^2}\,dx$, (iv) $\int_0^1 \dfrac{dx}{\sqrt{-\ln x}}$.

Solution: (i) By definition of Gamma function, $\int_0^\infty x^3 e^{-x}\,dx = \Gamma(4) = 3! = 6.$

(ii) Let $2x = t$. Then $dx = (1/2)dt$.

Therefore, $\int_0^\infty x^6 e^{-2x}\,dx = \int_0^\infty \left(\dfrac{t}{2}\right)^6 e^{-t}(1/2)dt = \dfrac{1}{2^7}\int_0^\infty t^6 e^{-t}dt = \dfrac{1}{2^7}\Gamma(7) = \dfrac{6!}{2^7} = \dfrac{720}{16\times 8} = \dfrac{45}{8}.$

(iii) We can write $\int_0^\infty 3^{-4x^2}\,dx = \int_0^\infty (e^{\ln 3})^{-4x^2}\,dx = \int_0^\infty e^{-(4\ln 3)x^2}\,dx.$

Let $(4\ln 3)x^2 = t$. Then $x = \dfrac{1}{\sqrt{4\ln 3}}t^{1/2}$ and $dx = \dfrac{1}{\sqrt{4\ln 3}}(1/2)t^{-1/2}dt.$

Thus, $\int_0^\infty 3^{-4x^2}\,dx = \int_0^\infty e^{-(4\ln 3)x^2}\,dx = \dfrac{1}{2\sqrt{4\ln 3}}\int_0^\infty t^{-1/2}e^{-t}\,dt = \dfrac{\Gamma(1/2)}{4\sqrt{\ln 3}} = \dfrac{\sqrt{\pi}}{4\sqrt{\ln 3}}.$

(iv) Let $-\ln x = t$. Then $x = e^{-t}$ and $dx = -e^{-t}dt$.

If $x = 0$ then $t = \infty$ and if $x = 1$ then $t = 0$.

Therefore, $\int_0^1 \dfrac{dx}{\sqrt{-\ln x}} = -\int_\infty^0 e^{-t}t^{-1/2}dt = \int_0^\infty e^{-t}t^{-1/2}dt = \Gamma(1/2) = \sqrt{\pi}.$

3. Show that $\int_0^\infty \sqrt{y}\, e^{-y^3}\,dy = \dfrac{\sqrt{\pi}}{3}.$

Solution: Let $y^3 = t$. Then $y = t^{1/3} \Rightarrow dy = \dfrac{1}{3}t^{-2/3}dt.$

Therefore, $\int_0^{+\infty} \sqrt{y}\, e^{-y^3}\,dy = \dfrac{1}{3}\int_0^{+\infty} t^{1/6}\, e^{-t}\, t^{-2/3}dt = \dfrac{1}{3}\int_0^{+\infty} t^{-1/2}e^{-t}\,dt = \dfrac{1}{3}\Gamma(1/2) = \dfrac{\sqrt{\pi}}{3}.$

4. Show that $\int_0^1 x^m(\log x)^n\,dx = \dfrac{(-1)^n\Gamma(n+1)}{(m+1)^{n+1}}.$

Solution: Let $x = e^{-y}$. Then $dx = -e^{-y}dy$.

So that if $x = 0$ then $y = +\infty$ and if $x = 1$ then $y = 0$.

Therefore, $\int_0^1 x^m(\log x)^n\,dx = \int_{+\infty}^0 e^{-my}(-y)^n(-e^{-y})\,dy = (-1)^n\int_0^{+\infty} e^{-(m+1)}y^n dy$

Again, let $(m+1)y = t$. Then $dy = \dfrac{1}{m+1}dt.$

Thus, $\int_0^1 x^m (\log x)^n\, dx = (-1)^n \int_0^{+\infty} e^{-t}\left(\frac{t}{m+1}\right)^n \frac{1}{m+1}\, dt = \frac{(-1)^n}{(m+1)^{n+1}} \int_0^{+\infty} e^{-t}\, t^n\, dt = \frac{(-1)^n \Gamma(n+1)}{(m+1)^{n+1}}.$

5. Show that $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n} \frac{\Gamma(\frac{1}{n})}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}.$

Solution: Let $x^n = \sin^2\theta$. Then $dx = \frac{2}{n}\sin^{\frac{2}{n}-1}\theta \cos\theta\, d\theta$.
So that if $x=0$ then $\theta = 0$ and if $x=1$ then $\theta = \pi/2$.

Therefore, $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \int_0^{\pi/2} \frac{1}{\cos\theta}\frac{2}{n}\sin^{\frac{2}{n}-1}\theta\cos\theta\, d\theta = \frac{2}{n}\int_0^{\pi/2} \sin^{\frac{2}{n}-1}\theta \cos^0\theta\, d\theta$

$$= \frac{2}{n} \frac{\Gamma\left(\frac{\frac{2}{n}-1+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{\frac{2}{n}-1+2}{2}\right)} = \frac{\sqrt{\pi}}{n}\frac{\Gamma(\frac{1}{n})}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}.$$

6. Show that $\int_0^1 x^m (1-x^n)^p\, dx = \frac{\Gamma(p+1)\Gamma\left(\frac{m+1}{n}\right)}{n\Gamma\left(p+1+\frac{m+1}{n}\right)}.$

Solution: Let $x^n = t$. Then $dx = \frac{1}{n} t^{\frac{1}{n}-1}\, dt$.
So that if $x=0$ then $t=0$ and if $x=1$ then $t=1$.

Therefore, $\int_0^1 x^m(1-x^n)^p\, dx = \int_0^1 t^{m/n}(1-t)^p \frac{1}{n} t^{\frac{1}{n}-1}\, dt = \frac{1}{n}\int_0^1 t^{\frac{m+1}{n}-1}(1-t)^p\, dt$

$$= \frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right) = \frac{1}{n}\frac{\Gamma(p+1)\Gamma\left(\frac{m+1}{n}\right)}{\Gamma\left(p+1+\frac{m+1}{n}\right)}.$$

7. Show that $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}.$

Solution: Let $I_1 = \int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}}$ and $I_2 = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$.

To compute I_1, let $x^2 = \sin\theta$. Then $dx = \frac{\cos\theta}{2\sqrt{\sin\theta}}\, d\theta$.

So that if $x=0$ then $\theta=0$ and if $x=1$ then $\theta=\pi/2$.

Therefore, $I_1=\int_0^1 \frac{x^2dx}{\sqrt{1-x^4}}=\frac{1}{2}\int_0^{\pi/2}\frac{\sin\theta}{\cos\theta}\frac{\cos\theta}{\sqrt{\sin\theta}}d\theta=\frac{1}{2}\int_0^{\pi/2}\sin^{1/2}\theta\cos^0\theta\, d\theta$

$$=\frac{1}{2}\frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}+0+2}{2}\right)}=\frac{1}{4}\frac{\Gamma(\frac{3}{4})\sqrt{\pi}}{\Gamma\left(\frac{5}{4}\right)}=\frac{1}{4}\frac{\Gamma(\frac{3}{4})\sqrt{\pi}}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}=\frac{\Gamma(\frac{3}{4})\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)}.$$

To compute I_2, let $x^2=\tan\theta$. Then $dx=\frac{\sec^2\theta}{2\sqrt{\tan\theta}}d\theta$.

So that if $x=0$ then $\theta=0$ and if $x=1$ then $\theta=\pi/4$.

Therefore, $I_2=\int_0^1\frac{dx}{\sqrt{1+x^4}}=\frac{1}{2}\int_0^{\pi/4}\frac{\sec^2\theta}{\sec\theta\sqrt{\tan\theta}}d\theta=\frac{1}{\sqrt{2}}\int_0^{\pi/4}\frac{1}{\sqrt{\sin 2\theta}}\,d\theta$

Let $2\theta=z$. Then $d\theta=\frac{1}{2}dz$. So that if $\theta=0$ then $z=0$ and if $\theta=\pi/4$ then $z=\pi/2$.

Therefore, $I_2=\frac{1}{2\sqrt{2}}\int_0^{\pi/2}\frac{1}{\sqrt{\sin z}}\,dz=\frac{1}{2\sqrt{2}}\int_0^{\pi/2}\sin^{-1/2}z\cos^0 z\,dz$

$$=\frac{1}{2\sqrt{2}}\frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+0+2}{2}\right)}=\frac{1}{4\sqrt{2}}\frac{\Gamma(\frac{1}{4})\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)}.$$

Thus, $I_1I_2=\int_0^1\frac{x^2dx}{\sqrt{1-x^4}}\times\int_0^1\frac{dx}{\sqrt{1+x^4}}=\frac{\Gamma(\frac{3}{4})\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)}\times\frac{1}{4\sqrt{2}}\frac{\Gamma(\frac{1}{4})\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)}=\frac{\pi}{4\sqrt{2}}.$

8. Evaluate $\int_0^\infty\frac{dx}{1+x^4}$.

Solution: Let $x^4=t$. Then $x=t^{1/4}$ and $dx=\frac{1}{4}t^{-3/4}dt$.

Therefore, $\int_0^\infty\frac{dx}{1+x^4}=\frac{1}{4}\int_0^\infty\frac{t^{-3/4}}{1+t}\,dt=\frac{\pi}{4\sin(\pi/4)}=\frac{\pi\sqrt{2}}{4}.$

Exercises

1. Evaluate each of the following

   (i) $\dfrac{\Gamma(7)}{2\Gamma(4)\Gamma(3)}$, (ii) $\dfrac{\Gamma(3)\Gamma(3/2)}{\Gamma(9/2)}$, (iii) $\Gamma(1/2)\Gamma(3/2)\Gamma(5/2)$.

   Ans. (i) 30, (ii) $\dfrac{16}{105}$, (iii) $\dfrac{3}{8}\pi^{3/2}$.

2. Evaluate each of the following integrals:

   (i) $\int_0^\infty x^4 e^{-x}\,dx$, (ii) $\int_0^\infty x^6 e^{-3x}\,dx$, (iii) $\int_0^\infty x^2 e^{-2x^2}\,dx$, (iv) $\int_0^\infty e^{-x^3}\,dx$, (v) $\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\,dx$.

   Ans. (i) 24, (ii) $\dfrac{80}{243}$, (iii) $\dfrac{\sqrt{2\pi}}{16}$, (iv) $\dfrac{1}{3}\Gamma(1/3)$, (v) $\dfrac{3\sqrt{\pi}}{2}$.

3. Evaluate each of the following integrals:

   (i) $\int_0^1 x^4(1-x)^3\,dx$, (ii) $\int_0^2 \dfrac{x^2}{\sqrt{2-x}}\,dx$, (iii) $\int_0^a y^4\sqrt{a^2-y^2}\,dy$.

   Ans. (i) $\dfrac{1}{280}$, (ii) $\dfrac{64\sqrt{2}}{15}$, (iii) $\dfrac{\pi a^6}{16}$.

4. Prove that $\beta(m,n)=\int_0^\infty \dfrac{x^{m-1}}{(1+x)^{m+n}}\,dx$ and hence show that

   $$\beta(m,1-m)=\int_0^1 \frac{x^{m-1}+x^{-m}}{1+x}\,dx.$$

5. Prove that $\beta(m,n)=2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$.

   Hence show that $\int_0^{\frac{\pi}{2}} \sin^m\theta\,\cos^n\theta\,d\theta=\dfrac{1}{2}\beta\left(\dfrac{m+1}{2},\dfrac{n+1}{2}\right)$; $m>-1,\ n>-1$.

6. Evaluate (i) $\int_0^2 x\sqrt[3]{8-x^3}\,dx$, (ii) $\int_0^2 (4-x^2)^{3/2}\,dx$, (iii) $\int_0^4 u^{3/2}(4-u)^{5/2}\,du$,

   (iv) $\int_0^3 \dfrac{dt}{\sqrt{3t-t^2}}$.

   Ans. (i) $\dfrac{16\pi}{9\sqrt{3}}$, (ii) 3π, (iii) 12π, (iv) π.

7. Show that $\int_0^{\pi/2} \dfrac{d\phi}{\sqrt{1-(1/2)\sin^2\phi}}=\dfrac{\{\Gamma(1/4)\}^2}{4\sqrt{\pi}}$

8. Evaluate $\int_0^\infty x^m e^{-ax^n}\,dx$ where m,n,a are positive constants.

Ans. $\dfrac{1}{na^{(m+1)/n}}\Gamma\left(\dfrac{m+1}{n}\right).$

9. It is known that $\Gamma(m)\Gamma(1-m) = \beta(m,n) = \int_0^\infty \frac{x^{m-1}}{1+x}dx = \frac{\pi}{\sin m\pi}, \ 0 < m < 1.$

Hence show that (a) $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta = \frac{\pi}{\sqrt{2}},$ (b) $\int_0^1 \frac{dx}{(1-x^6)^{\frac{1}{6}}} = \frac{\pi}{3}.$

10. Evaluate each of the following integrals:

(i) $\int_0^{\frac{\pi}{2}} \sin^6 x\, dx,$ (ii) $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^5 x\, dx,$ (iii) $\int_0^{\pi} \cos^4 x\, dx.$

Ans. (i) $\frac{5\pi}{32},$ (ii) $\frac{8}{315},$ (iii) $\frac{3\pi}{8}.$

11. Prove that $\int_0^\infty \cos(bx^{\frac{1}{n}})\, dx = \frac{1}{b^n}\Gamma(n+1)\cos\frac{n\pi}{2}.$

12. Prove that $\int_0^a \frac{dy}{\sqrt{a^4 - y^4}} = \frac{\{\Gamma(1/4)\}^2}{4a\sqrt{2\pi}}.$

13. Prove that $\int_0^\infty \frac{u}{1+u^6} du = \frac{\pi}{3\sqrt{3}}.$

14. Prove that $\int_0^\infty \frac{x^2}{1+x^4} dx = \frac{\pi}{2\sqrt{2}}.$