Lecture 04 and Lecture 05

Chapter 1: Introduction

Math 4441: Probability and Statistics

Reference: Goodman & Yates – Introduction to Probability and Stochastic Processes, 3^{rd} Edition

Conditional Probability: Consider the following three examples:

Example 1.15: Roll a six-sided fair die. Before you observe the outcome, someone tells that the outcome is an even number. After knowing that the outcome is an even number, what is the probability the outcome is the square of an integer?

Example 1.16: A box contains 8 red balls and 5 blue balls. Balls are mixed together. Pick a ball randomly, observe the color, and return the ball. Pick the ball again, observe the color and return the ball.

$B \triangleq$ Event that the first ball is red

$A \triangleq$ Event that the second ball is red

Example 1.17: A box contains 8 red balls and 5 blue balls. Balls are mixed together. Pick a ball randomly, observe the color, and keep it. Pick a ball again, observe the color and keep it as well.

$B \triangleq$ Event that the first ball is red

$A \triangleq$ Event that the second ball is red

Conditional Probability:

- Partial information is given, which changes the sample space.
- The probabilities are calculated in the modified sample space usually with less number of elements.
- Conditional probability is dented by $P[A|B]$
- Probability of occurring event A, given that event B has already occurred
- Probability of A, given B

Proportion of the outcomes in B that are represented by joint outcome $A \cap B$

Proportion of the $P[B]$ represented by $P[AB]$

Conditional Probability

The conditional probability of occurring event A, given that another event B has already occurred is defined by

$$P[A|B] =$$

Law of Multiplication (Product Rule)

Defines the joint probability of occurring two events simultaneously

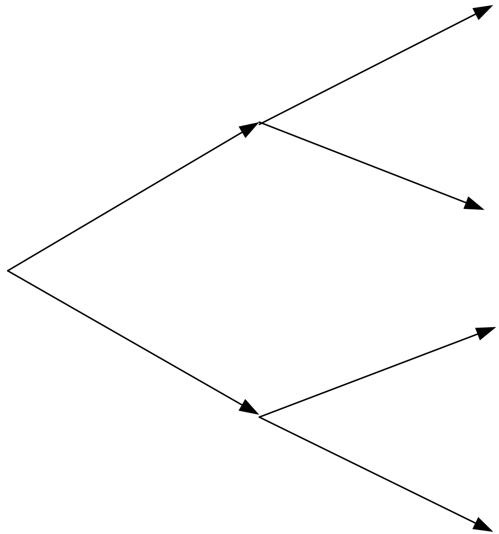
Example 1.18: A box contains 8 red balls and 5 blue balls. Balls are mixed together. Pick a ball randomly, observe the color, and keep it. Pick a ball again, observe the color and keep it as well.

$R_i \triangleq$ Event that the i-th ball is red

$B_i \triangleq$ Event that the i-th ball is blue

$P[R_1 R_2] =$

$P[R_1 B_2] =$



Law of Multiplication

If A and B are two events, then the joint probability of occurring these two events

$P[AB] =$

The Chain Rule

The joint probability of occurring more than two events can be found by extending the product rule. If A, B, C is a set n events, then

$$P[ABC] =$$

The Chain Rule

The joint probability of occurring more than two events can be found by extending the product rule. If $A_1, A_2, \dots, A_n$ is a set n events, then

$$P[A_1 A_2 \dots A_n] = P[A_1]P[A_2|A_1]P[A_3|A_1 A_2] \dots P[A_n|A_1 A_2 \dots A_{n-1}$$

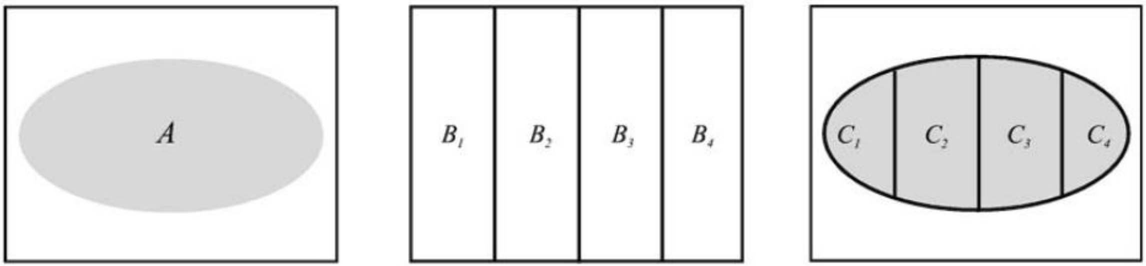
$$= \prod_{i=1}^{n} P[A_i| \cap_{j=1}^{i-1} A_j]$$

Example 1.19: A box contains 8 red balls and 5 blue balls. Balls are mixed together. Pick a ball randomly, observe the color, and keep it. Now assume you pick 4 balls successively without replacement. Find the probability that the sequence of colors of the picked balls is Red, Blue, Red, and Blue.

For an event space $B = \{B_1, B_2, \dots, B_n\}$ and any event A in the sample space. Let $C_i = A \cap B_i$, for $i = 1,2, \dots, n$.

For $i \neq j$, the events C_i and C_j are mutually exclusive and

$$A = C_1 \cup C_2 \cup \dots \cup C_n$$



Law of Total Probability (Sum Rule of Probability):

Let B be an event with$P[B] \geq 0$ and $P[B^c] \geq 0$, then for an event A

$$P[A] = P[A|B]P[B] + P[A|B^c]P[B^c]$$

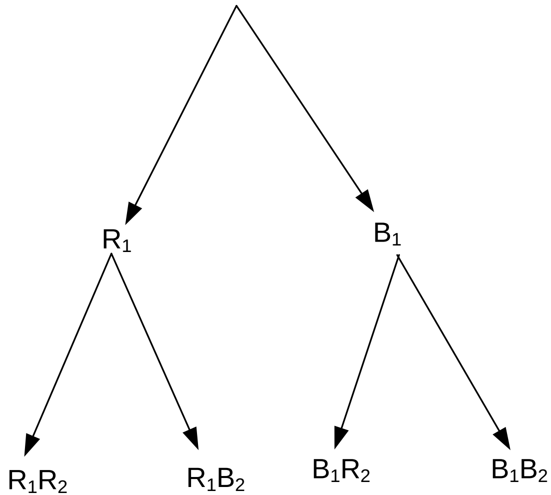
Law of Total Probability (Sum Rule of Probability):

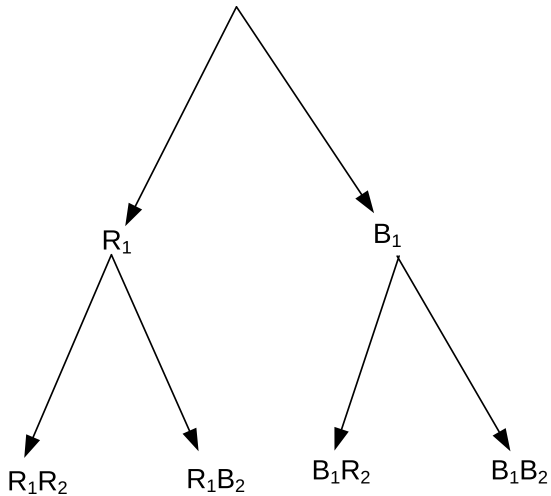
Let $B = \{B_1, B_2, \dots, B_n$ be an event space, then for an event A

$$P[A] = P[A|B_1]P[B_1] + P[A|B_2]P[B_2] + \ \dots + P[A|B_n]P[B_n]$$

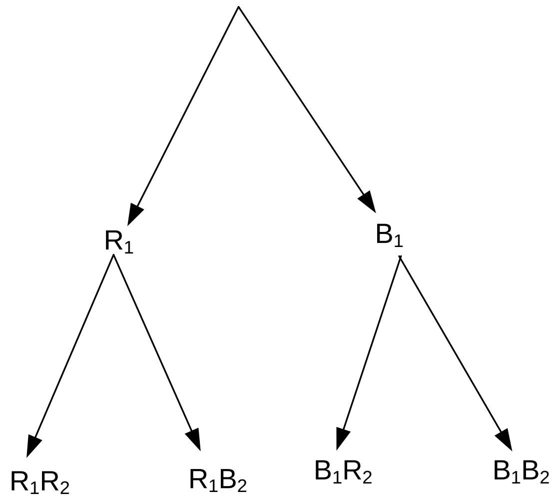
$$= \sum_{i=1}^{n} P[A|B_i]\, P[B_i]$$

Example 1.20: A box contains 8 red balls and 5 blue balls. Balls are mixed together. Pick two balls randomly without replacement and observe their color. Find the probability that the second ball is red.




R_1
B_1
R_1R_2
R_1B_2
B_1R_2
B_1B_2

Bayes Theorem



Example 1.20: A box contains 8 red balls and 5 blue balls. Balls are mixed together. Pick two balls randomly without replacement and observe their color. If the color of the second ball is red, find the probability that the first was also red.

Bayes Theorem

Let $B_1; B_2; \dots, B_n$ be an event space of the sample space S of an experiment. If for $i = 1, 2, \dots, n$, $P[B_i] > 0$ then for any event A of S with $P[A] > 0$,

$$P[B_k|A] = \frac{P[A|B_k]P[B_k]}{P[A|B_1]P[B_1] + P[A|B_2]P[B_2] + \cdots + P[A|B_n]P[B_n]}$$

In statistical applications of Bayes' theorem, $B_1, B_2, \dots, B_n$ are called the **hypothesis**, $P(B_i)$ is called the **prior** probability of B_i, and the conditional probability $P[B_i|A]$ is called the **posterior** probability of B_i after the occurrence of A.

Example 1.21 You enter a chess tournament where your probability of winning a game is 0.3 against half the players (call them type 1), 0.4 against a quarter of the players (call them type 2), and 0.5 against the remaining quarter of the players (call them type 3). Suppose you play a game against a randomly chosen opponent, and you win. What is the probability that you had an opponent of type 1?

Independence

Example 1.27: Toss a fair coin twice observe the sequence of heads and tails

$S = \{TT, TH, HT, HH\}$

Let

$$A \triangleq \{\text{Head on second toss}\} = \{TH, HH\}$$

$$P[A] = \frac{1}{2}$$

$$B \triangleq \{\text{Tail on first toss}\} = \{TT, TH\}$$

$$P[B] = \frac{1}{2}$$

$$P[A \cap B] = \frac{1}{4}$$

$$P[A|B] = \frac{P[AB]}{P[B]} = \frac{1}{4} \times \frac{2}{1} = \frac{1}{2}$$

$$P[A|B] = P[A]$$

Independence

Knowing that B has occurred, does not change the probability of occurring event A

$P[A|B] = P[A]$, $P[B] > 0$

$A \perp B \triangleq A$ is independed of B

$A \perp B = B \perp A$

2^{nd} Definition: Independence is defined from the Law of multiplication

$$P[AB] = P[A|B]P[B] = P[A]P[B]$$

$$P[A \cap B] = P[A]P[B]$$

Two events are independent, if their joint probability is the product of their individual probabilities.

Example 1.22 Consider an experiment involving two successive rolls of a 4-sided die in which all 16 possible outcomes are equally likely and have probability 1/16.

1. Are the events $A_i = \{\text{1st roll results in } i\}$, $B_j = \{\text{2nd roll results in } j\}$, independent?

2. Are the events $A = \{1st \text{ roll is a } 1\}$, $B = \{\text{sum of the two rolls is a } 5\}$, independent?

3. Are the events $A = \{\text{maximum of the two rolls is } 2\}$,
$B = \{\text{minimum of the two rolls is } 2\}$, independent?

If A is a set of events, $A = \{A_1, A_2, \dots, A_n\}$ are independent, if

If for every subset of $A_i \subset A$

For every subset $A_i \subset A$ with k element, $k = 2, 3, \dots, n$

$$P\left[A_{i_1}, A_{i_2}, \dots, A_{i_k}\right] = P\left[A_{i_1}\right] P[A_{i_2}] \ \dots \ P[A_{i_k}]$$

For example, If the set of events consists of A, B and C then, A, B and C are independent, when

For example, If the set of events consists of A, B and C then, A, B and C are independent, when

$P[AB] = P[A]P[B]$

$P[AC] = P[A]P[C]$

$P[BC] = P[B]P[C]$, and

$P[ABC] = P[A]P[B]P[C]$

It is possible that the first 3 conditions are satisfied but the last condition is not satisfied and vice versa

Example 1.23: A fair coin is tossed twice and sequence of heads and tails are observed

$S = \{TT, TH, HT, HH\}$

$A = \{head\ on\ first\ toss\} = \{HT, HH\}$

$B = \{head\ on\ 2nd\ toss\} = \{TH, HH\}$

$C = \{1st\ and\ 2nd\ toss\ produce\ same\ outcome\} = \{TT, HH\}$

Conditional Probabilities Behave Just Like Probabilities

1. Conditional probabilities follow the axioms of probabilities
   - Probability of every event will have conditional probability conditioned on another event
   - It satisfies all 3 axioms of probability

2. Every conditional probability can have its conditional probabilities
   - Conditional probability any event A, given another event B, can be conditioned on another event C
   - $P[A|B]$
   - $P[A|BC] =$

Conditional Product Rule

Suppose that $A_1, A_2, \ldots, A_n, B$ are events such that $\Pr(B) > 0$ and $\Pr(A_1 \cap A_2$
$A_{n-1}|B) > 0$. Then

$$\Pr(A_1 \cap A_2 \cap \cdots \cap A_n|B) = \Pr(A_1|B)\Pr(A_2|A_1 \cap B) \cdots \times \Pr(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1} \cap B).$$

Conditional Sum Rule: The law of total probability has an analog conditional on another event *C*, namely,

$$\Pr(A|C) = \sum_{j=1}^{k} \Pr(B_j|C) \Pr(A|B_j \cap C).$$

Conditional Independence. We say that events $A_1, \ldots, A_k$ are *conditionally independent given* B if, for every subcollection $A_{i_1}, \ldots, A_{i_j}$ of j of these events ($j = 2, 3, \ldots, k$),

$$\Pr\left(A_{i_1} \cap \cdots \cap A_{i_j} \middle| B \right) = \Pr(A_{i_1}|B) \cdots \Pr(A_{i_j}|B).$$

If A_1, A_2, and B are events where $P[A_1B] > 0]$, then A_1 and A_2 are conditionally independent given B, iif

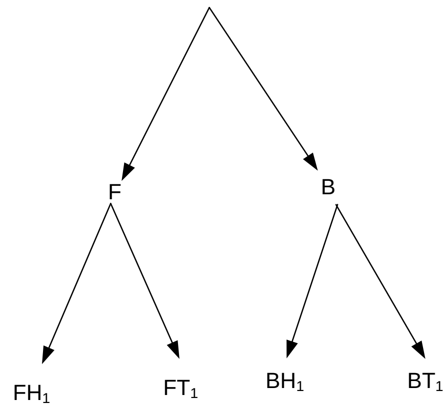
$$P[A_2|A_1B] = P[A_2|B]$$

Conditional Bayes' Theorem

$$\Pr(B_i|A \cap C) = \frac{\Pr(B_i|C)\Pr(A|B_i \cap C)}{\sum_{j=1}^{k} \Pr(B_j|C)\Pr(A|B_j \cap C)}$$

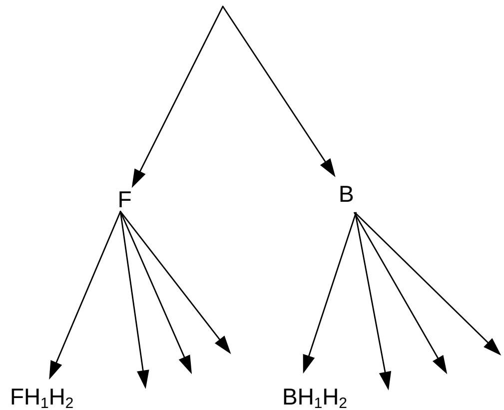
Example 1.24: Suppose a box contains one fair and one double headed coin. Suppose a coin is selected randomly and a head is obtained. Find the probability that the selected coin is the fair coin.

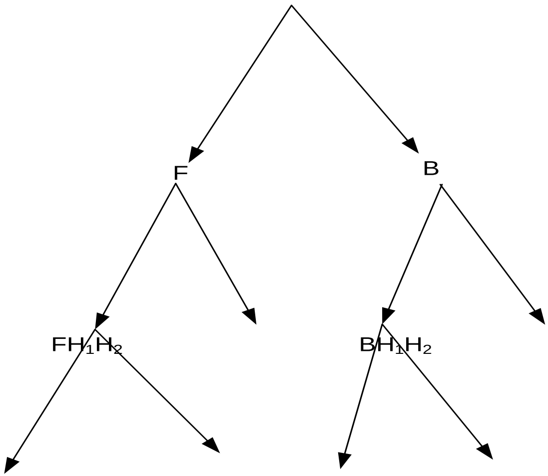
$P[F|H_1] =$



Example 1.24 (Continued): Suppose the same coin is tossed again and that the two tosses are conditionally independent given both F and B. Suppose further that another head is obtained. Find the probability that the selected coin is fair.

$P[F|H_1H_2] =$





$$\Pr(B_1|H_1 \cap H_2) = \frac{\Pr(B_1|H_1)\Pr(H_2|B_1 \cap H_1)}{\Pr(B_1|H_1)\Pr(H_2|B_1 \cap H_1) + \Pr(B_2|H_1)\Pr(H_2|B_2 \cap H_1)}$$

$$= \frac{(1/3)(1/2)}{(1/3)(1/2) + (2/3)(1)} = \frac{1}{5}.$$