Chapter 2
Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Section 2.3 : Functions

Functions

- A function f from A to B is an assignment of exactly one element of B to each element of A.
- We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.
- Iff is a function from $A\ to\ B$, we write $f : A \rightarrow B$.

Functions (Contd.)

- If f is a function from A to B, then,
  - A is the domain of f
  - B is the codomain of f.
  - f maps A to B.
- If $f(a) = b$, then,
  - b is the image of a
  - a is a pre-image of b.
  - The range/image of f is the set of all images of elements of A.

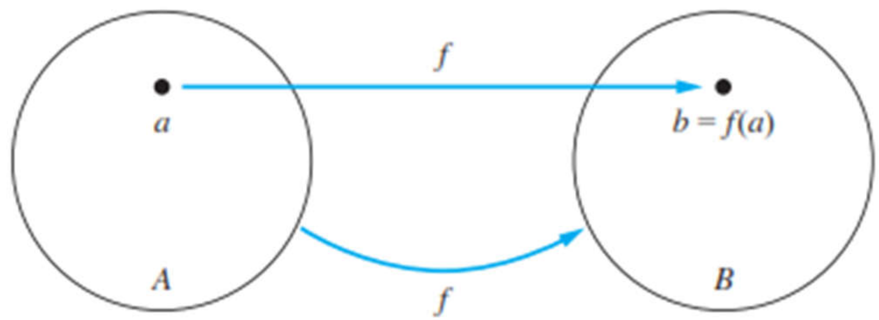


FIGURE 2 The Function f Maps A to B.

Functions (Contd.)

- Example 1:
  - Suppose that each student in a discrete mathematics class is assigned a letter grade from the set $\{A, B, C, D, F\}$. And suppose that the grades are A for $Adams$, C for $Chou$, B for $Goodfriend$, A for $Rodriguez$, and F for $Stevens$.
  - What are the $domain$, $codomain$, and $range$ of the function that assigns grades to students described in the first paragraph of the introduction of this section?

Functions (Contd.)

- Solution:
  - Let,

    Let G be the function that assigns a grade to a student in our discrete mathematics class.

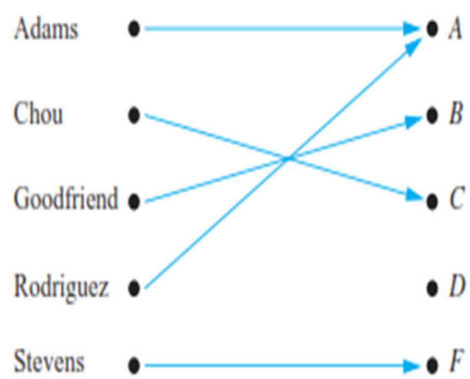
    

    FIGURE 1 Assignment of Grades in a Discrete Mathematics Class.

    Note that $G(Adams) = A$.

    The domain of G is the set $\{Adams, Chou, Goodfriend, Rodriguez, Stevens\}$

    The codomain is the set $\{A, B, C, D, F\}$.

    The range of G is the set $\{A, B, C, F\}$, because each grade except D is assigned to some student.

Functions(Contd.)

- Let $f1$ and $f2$ be functions from A to R.
- Then $f1 + f2$ and $f1f2$ are also functions from A to R defined for all $x \in A$ by
- $(f1 + f2)(x) = f1(x) + f2(x)$
- $(f1f2)(x) = f1(x)f2(x)$.

Functions(Contd.)

- Example 2:
  - Let $f1$ and $f2$ be functions from R to R such that $f1(x) = x^2$ and $f2(x) = x - x^2$. What are the functions $f1 + f2$ and $f1f2$?
- Solution :

From the definition of the sum and product of functions, it follows that

$$(f1 + f2)(x) = f1(x) + f2(x) = x^2 + (x - x^2) = x$$

and

$$(f1f2)(x) = x^2(x - x^2) = x^3 - x^4.$$

When f is a function from A to B, the image of a subset of A can also be defined.

One-to-One Functions/Injunctions

- A function f is said to be one-to-one, or an injunction, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be injective if it is one-to-one.
- Note that a function f is also one-to-one if and only if $f(a) \neq f(b)$whenever $a \neq b$.

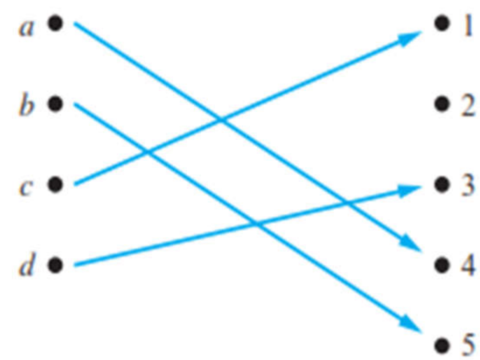


FIGURE 3 A One-to-One Function.

One-to-One Functions/Injunctions (Contd.)

- Example 3:
  - Determine whether the function f from {a, b, c, d} to $\{1, 2, 3, 4, 5\}$ $with\ f(a) = 4, f(b) = 5, f(c) = 1,$ and $f(d) = 3$ is one-to-one.

- Solution :

  The function f is one-to-one because f takes on different values at the four elements of its domain.

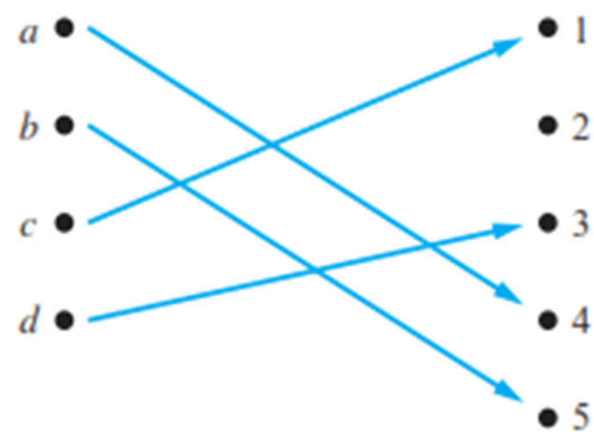


FIGURE 3 A One-to-One Function.

One-to-One Functions/Injunctions (Contd.)

- Example 4:
  - Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

- Solution :

  The function $f(x) = x^2$ is not one-to-one because, for instance,

  $f(1) = f(-1) = 1$,

  but $1 \neq -1$.

One-to-One Function/Injunctions (Contd.)

- **Example 5:**
  - Determine whether the function $f(x) = x + 1$ from the set of real numbers to itself is one-to-one.

- **Solution:**

  The function $f(x) = x + 1$ is a one-to-one function. To demonstrate this, note that $x + 1 \neq y + 1$ when $x \neq y$.

Onto Functions/Surjections

- A function f from A to B is called onto, or a surjection, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is also called surjective if it is onto.

Onto Functions/Surjections (Contd.)

- Example 6:
  - Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3, f(b) = 2, f(c) = 1$ and $f(d) = 3$. Is f an onto function?

- Solution:

  Because all three elements of the codomain are images of elements in the domain, we see that f is onto. Note that if the codomain were $\{1, 2, 3, 4\}$, then f would not be onto.

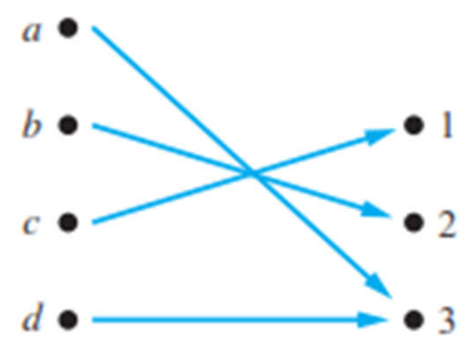


FIGURE 4 An Onto Function.

Onto Functions/Surjections (Contd.)

- Example 7:

  Determine whether the following functions are onto functions from the set of integers to the set of integer?

$$f(x) = x^2$$
$$g(x) = x + 1$$

- Solution :

  The function f is not onto because there is no integer x with $x^2 = -1$, for instance.

  The function g is onto, because for every integer y there is an integer x such that $g(x) = y$. To see this, note that $g(x) = y$ if and only if $x + 1 = y$, which holds if and only if $x = y - 1$.

One-to-One Correspondence/Bijections

- The function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto.
- We also say that such a function is bijective

One-to-One Correspondence/Bijections(Contd.)

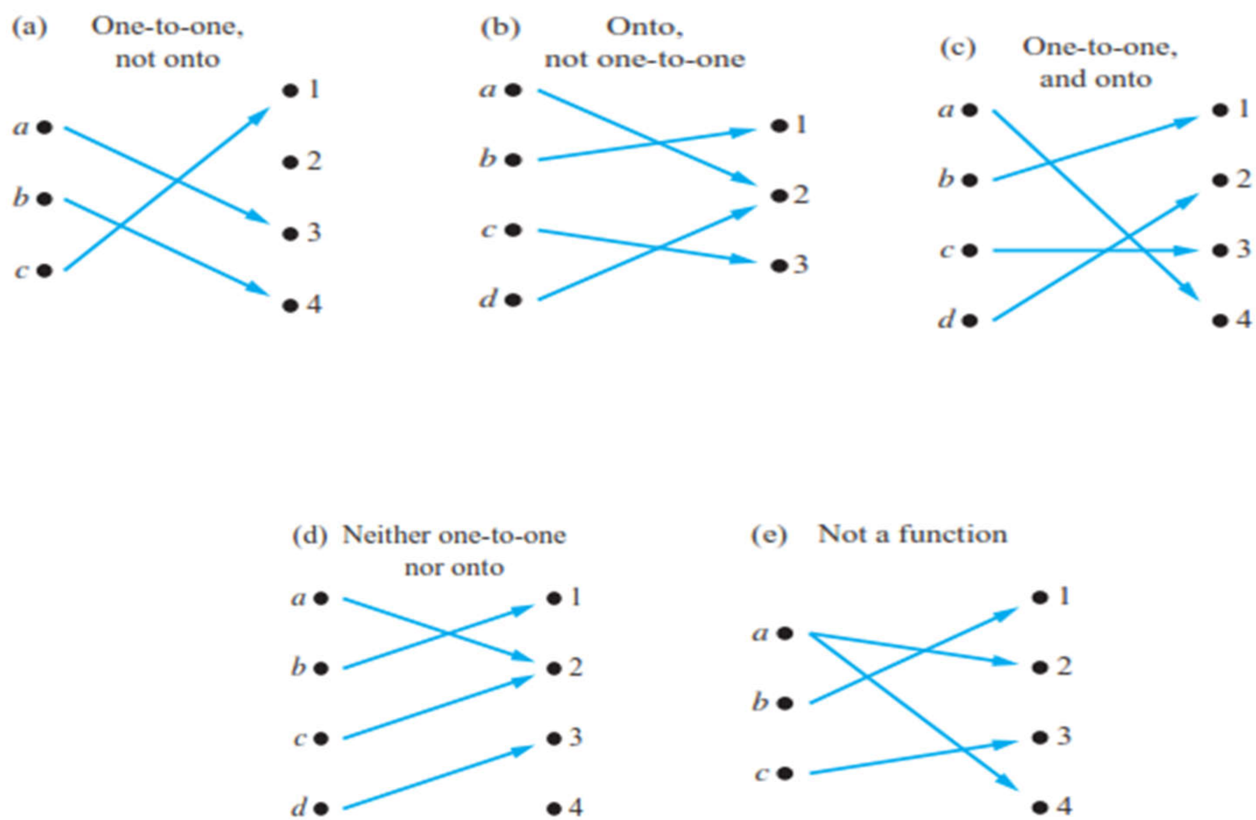
- Example 8:
  - Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4, f(b) = 2, f(c) = 1$, and $f(d) = 3$. Is f a bijection?

- Solution :

  The function f is one-to-one and onto. It is one-to-one because no two values in the domain are assigned the same function value.

  It is onto because all four elements of the codomain are images of elements in the domain. Hence, f is a bijection.

Comparison of One-to-One, Onto, Bijections



Inverse Functions

- Let f be a one-to-one correspondence (both one-to-one and onto) from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$.
- The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

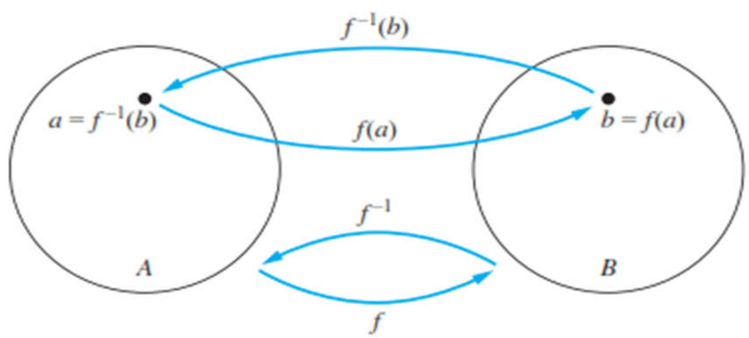


FIGURE 6 The Function f^{-1} Is the Inverse of Function f.

Inverse Functions(Contd.)

- Example 9:
  - Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2, f(b) = 3$ and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

- Solution:

  The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f ,so $f^{-1}(1) = c, f^{-1}(2) = a,$ and $f^{-1}(3) = b.$

Inverse Functions(Contd.)

- Example 10:
  - Let $f : Z \rightarrow Z$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?
- Solution:

The function f has an inverse because it is a one-to-one correspondence, as follows from $Examples$ 5 and 7.

To reverse the correspondence, suppose that

$$f(x) = y = x + 1$$
$$Or, x = f^{-1}(y)$$

$$Again, y = x + 1$$
$$Or, x = y - 1$$

$$So, f^{-1}(y) = y - 1$$

$$\therefore f^{-1}(x) = x - 1$$

Inverse Functions(Contd.)

- Example 11:
  - Let f be the function from R to R with $f(x) = x^2$. Is f invertible?

- Solution:

  Because $f(-2) = f(2) = 4$, f is not one-to-one. If an inverse function were defined, it would have to assign two elements to 4. Hence, f is not invertible. (Note we can also show that f is not invertible because it is not onto.)

Composition of Functions

- Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, denoted for all $a \in A$ by fog, is defined by $(fog)(a) = f(g(a))$.

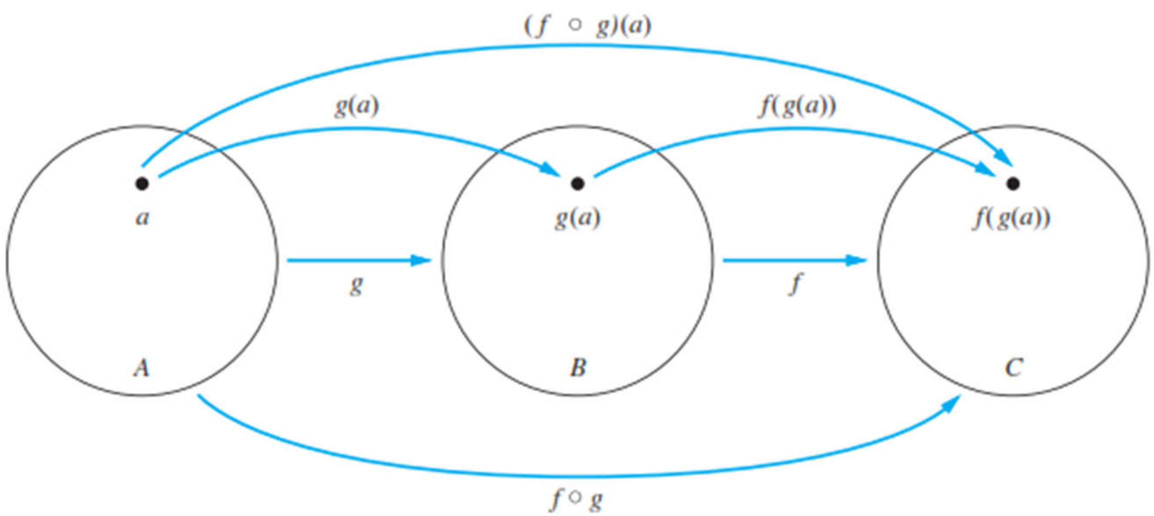


FIGURE 7 The Composition of the Functions f and g.

Composition of Functions(Contd.)

- Example 12:
  - Let f and g be the functions from the set of integers to the set of integers defined by

$$f(x) = 2x + 3$$
$$g(x) = 3x + 2.$$

What is the composition of f and g? What is the composition of g and f ?

Composition of Functions(Contd.)

- **Solution :**

Both the compositions fog and gof are defined. Moreover,

$$
\begin{aligned}
(fog)(x) &= f\big(g(x)\big) \\
&= f(3x + 2) \\
&= 2(3x + 2) + 3 \\
&= 6x + 7
\end{aligned}
$$

and

$$
\begin{aligned}
(gof)(x) &= g\big(f(x)\big) \\
&= g(2x + 3) \\
&= 3(2x + 3) + 2 \\
&= 6x + 11
\end{aligned}
$$

Floor & Ceiling Functions

- Floor function:
  - The floor function assigns to the real number x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$.
- Ceiling function:
  - The ceiling function assigns to the real number x the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$.

Floor & Ceiling Functions(Contd.)

TABLE 1 Useful Properties of the Floor and Ceiling Functions. (n is an integer, x is a real number)
(1a) $\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$ (1b) $\lceil x \rceil = n$ if and only if $n - 1 < x \leq n$ (1c) $\lfloor x \rfloor = n$ if and only if $x - 1 < n \leq x$ (1d) $\lceil x \rceil = n$ if and only if $x \leq n < x + 1$
(2) $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$
(3a) $\lfloor -x \rfloor = -\lceil x \rceil$ (3b) $\lceil -x \rceil = -\lfloor x \rfloor$
(4a) $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ (4b) $\lceil x + n \rceil = \lceil x \rceil + n$

Floor & Ceiling Functions(Contd.)

- Example 13:
  - Prove that if x is a real number, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor$.

- Solution :

  Let, $x = n + e$ where, n is an integer and $0 \leq e < 1$. There are two cases to consider, depending on whether $e < \frac{1}{2}$ or $e > \frac{1}{2}$ or $e = \frac{1}{2}$.

Floor & Ceiling Functions(Contd.)

We first consider the case when $0 \leq e < \frac{1}{2}$

In this case, $2x = 2n + 2e$ and $\lfloor 2x \rfloor = 2n$ because $0 \leq 2e < 1$.

Similarly, $x + \frac{1}{2} = n + (\frac{1}{2} + e)$,

So, $\left\lfloor x + \frac{1}{2} \right\rfloor = n$, because $0 < \frac{1}{2} + e < 1$.

Consequently, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor = n + n = 2n$.

Next, we consider the case when $\frac{1}{2} \leq e < 1$.

In this case, $2x = 2n + 2e = (2n + 1) + (2e - 1)$.

Because $0 \leq 2e - 1 < 1$, it follows that $\lfloor 2x \rfloor = 2n + 1$.

As, $\left\lfloor x + \frac{1}{2} \right\rfloor = \left\lfloor n + \left(\frac{1}{2} + e\right) \right\rfloor = \left\lfloor n + 1 + \left(e - \frac{1}{2}\right) \right\rfloor$ and $0 \leq e - \frac{1}{2} < 1$

Floor & Ceiling Functions(Contd.)

It follows that, $\left\lfloor x + \frac{1}{2} \right\rfloor = n + 1$

Consequently,

$$\lfloor 2x \rfloor = 2n + 1 \; and \; \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor = n + (n + 1) = 2n + 1.$$

This concludes the proof.

Partial Functions

- A partial function f from a set A to a set B is an assignment to each element a in a subset of A, called the domain of definition of f , of a unique element b in B.
- The sets A and B are called the domain and codomain of f , respectively.
- We say that f is undefined for elements in A that are not in the domain of definition of f .
- When the domain of definition of f equals A, we say that f is a total function.

THE END