Bessel's Function

In standard form, Bessel's differential equation is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \tag{1}$$

where $n \geq 0$ is a real number.

Another useful form of Bessel's differential equation is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (m^2x^2 - n^2)y = 0 \tag{2}$$

When developing the properties of Bessel's function, the form (1) will be used. Bessel's differential equation has a regular singular point at $x = 0$.

Let $y(x) = \sum_{r=0}^{\infty} a_r x^{r+c}$ be the trial solution of (1). Then substituting the value of y, y' and y'' in (1), we get

$\sum_{r=0}^{\infty}(r+c)(r+c-1)a_r x^{r+c} + \sum_{r=0}^{\infty}(r+c)a_r x^{r+c} +$

$\sum_{r=0}^{\infty} a_r x^{r+c+2} - n^2 \sum_{r=0}^{\infty} a_r x^{r+c} = 0$

Shifting the summation index in the third summation and collecting terms under a single summation gives

$(c^2 - n^2)a_0 x^c + [(c+1)^2 - n^2]a_1 x^{c+1} + \sum_{r=2}^{\infty}[(r+c+n)(r+c-n)a_r + a_{r-2}]x^{r+c} = 0$

Equating the coefficients of powers of x^c, x^{c+1} and x^{r+c} to zero shows the following:
Coefficient of x^c: $(c^2 - n^2)a_0 = 0$ with $a_0 \neq 0$ gives $c = \pm n$.
Coefficient of x^{c+1}: $[(c+1)^2 - n^2]a_1 = 0$.
Coefficient of x^{r+c}:

$$[(r+c)^2 - n^2]a_r + a_{r-2} = 0 \tag{3}$$

Since with $c = \pm n, (c+1)^2 - n^2 \neq 0$, we must have $a_1 = 0$. It follows from (3) that $a_1 = a_3 = a_5 = \cdots = 0$, i.e., $a_r = 0$ for all odd integer r. As only even indices r are involved, in the recurrence relation we set $r = 2m$ with $m = 0, 1, 2, \ldots$.

Substituting $c = n$ in (3), we get

$$a_{2m} = -\frac{1}{4m(m+n)}a_{2m-2}, \quad m = 1, 2, 3, \ldots \tag{4}$$

Since a_0 is arbitrary, set $a_0 = \frac{1}{2^n\Gamma(n+1)}$, where $\Gamma(n+1)$ is the gammma function. We define

$\Gamma(n) = \int_0^{\infty} e^{-t}t^{n-1}dt$, where $n > 0$

We have $\Gamma(n+1) = n\Gamma(n) = n!$ for $n = 0, 1, 2, \ldots$. Therefore,

$a_2 = -\frac{a_0}{2^2(1+n)} = -\frac{1}{2^{2+n}1!\Gamma(2+n)}$, $a_4 = -\frac{a_0}{2^2 2(2+n)} = \frac{1}{2^{4+n}2!\Gamma(3+n)}$, and in general, $a_{2m} = \frac{(-1)^m a_0}{2^{2m+n} m!\Gamma(m+1+n)}$ for $m = 1, 2, \ldots$.

By inserting these coefficients in $y(x) = \sum\limits_{r=0}^{\infty} a_r x^{r+c}$, we obtain a particular solution of (1) which is denoted by $J_n(x)$ and is defined as

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+n)!} (\frac{x}{2})^{2m+n}, \quad n = 0, 1, 2, \ldots, \tag{5}$$

which is known as the Bessel function of order n.
Bessel function of the first kind of order zero:

$J_0(x) = \sum\limits_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} (\frac{x}{2})^{2m} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \cdots$

Bessel function of the first kind of order one:

$J_1(x) = \sum\limits_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!} (\frac{x}{2})^{2m+1} = \frac{x}{2} - \frac{x^3}{2^3 1!2!} + \frac{x^5}{2^5 2!3!} - \frac{x^7}{2^7 3!4!} + \cdots$

Problem: Prove that (i) $J_{\frac{1}{2}}(x) = \sqrt{(\frac{2}{\pi x})} sinx$; (ii) $J_{-\frac{1}{2}}(x) = \sqrt{(\frac{2}{\pi x})} cosx$.
Proof: We have

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} [1 - \frac{x^2}{2 \times 2(n+1)} + \frac{x^4}{2 \times 4 \times 2^2 (n+1)(n+2)} - \cdots \tag{6}$$

(i) Substituting $n = \frac{1}{2}$ in (6), we get

$J_{\frac{1}{2}}(x) = \frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}} \Gamma(\frac{1}{2}+1)} [1 - \frac{x^2}{2 \times 2(\frac{1}{2}+1)} + \frac{x^4}{2 \times 4 \times 2^2 (\frac{1}{2}+1)(\frac{1}{2}+2)} - \cdots]$

$= \frac{\sqrt{(x)}}{\sqrt{(2)} \Gamma(\frac{3}{2})} [1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots] = \sqrt{(\frac{x}{2})} \frac{1}{\frac{1}{2}\Gamma(\frac{1}{2})} \frac{1}{x} [x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots]$

$= \sqrt{(\frac{2}{\pi x})} sinx$ with $\Gamma(\frac{1}{2}) = \sqrt{(\pi)}$.

(ii) Substituting $n = -\frac{1}{2}$ in (6), we get

$J_{-\frac{1}{2}}(x) = \frac{x^{-\frac{1}{2}}}{2^{-\frac{1}{2}} \Gamma(-\frac{1}{2}+1)} [1 - \frac{x^2}{2 \times 2(-\frac{1}{2}+1)} + \frac{x^4}{2 \times 4 \times 2^2 (-\frac{1}{2}+1)(-\frac{1}{2}+2)} - \cdots]$

$= \frac{\sqrt{(2)}}{\sqrt{(x)} \Gamma(\frac{1}{2})} [1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots] = \sqrt{(\frac{2}{\pi x})} cosx$.

Recurrence relations

Form I: $xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$.
Proof: We have

$J_n(x) = \sum\limits_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+n+1)} (\frac{x}{2})^{2m+n}$

Differentiating with respect to x, we get

$J_n'(x) = \sum\limits_{m=0}^{\infty} \frac{(-1)^m (2m+n)}{m!\Gamma(m+n+1)} (\frac{x}{2})^{2m+n-1} \cdot \frac{1}{2}$

$xJ_n'(x) = n \sum\limits_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+n+1)} (\frac{x}{2})^{2m+n}$

$+x \sum\limits_{m=1}^{\infty} \frac{(-1)^m (2m)}{2(m!)\Gamma(m+n+1)} (\frac{x}{2})^{2m+n-1}$

$$= nJ_n(x) + x\sum_{m=1}^{\infty} \frac{(-1)^m}{(m-1)!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2m+n-1}$$

Putting $m-1=p$, we get

$$xJ_n'(x) = nJ_n(x) + x\sum_{p=0}^{\infty} \frac{(-1)^{p+1}}{p!\Gamma(p+n+2)}\left(\frac{x}{2}\right)^{2p+n+1}$$

$$= nJ_n(x) - x\sum_{p=0}^{\infty} \frac{(-1)^{p}}{p!\Gamma(p+(n+1)+1)}\left(\frac{x}{2}\right)^{2p+(n+1)}$$

$$= nJ_n(x) - xJ_{n+1}(x).$$

Form II: $xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x)$.
Proof: We have

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2m+n}$$

Differentiating with respect to x, we get

$$J_n'(x) = \sum_{m=0}^{\infty} \frac{(-1)^m(2m+n)}{m!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2m+n-1}\cdot\frac{1}{2}$$

$$xJ_n'(x) = \sum_{m=0}^{\infty} \frac{(-1)^m(2m+n)}{m!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2m+n}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m\{2(m+n)-n\}}{m!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2m+n}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m(2m+2n)}{m!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2m+n} - n\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2m+n}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m\cdot 2}{m!\Gamma(m+n)}\left(\frac{x}{2}\right)^{2m+n} - nJ_n(x)$$

$$= x\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma\{m+(n-1)+1\}}\left(\frac{x}{2}\right)^{2m+n-1} - nJ_n(x)$$

$$= xJ_{n-1}(x) - nJ_n(x).$$

Form III: $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$.
Proof: We have

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$$

and

$$xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x)$$

Adding the above two equations, we get

$$2xJ_n'(x) = -xJ_{n+1}(x) + xJ_{n-1}(x)$$

Thus the result follows.

Form IV: $2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$.
Proof: We have

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x) \qquad (1)$$

and

$$xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x) \qquad (2)$$

Subtracting (2) from (1), we get

$$0 = 2nJ_n(x) - xJ_{n+1}(x) - xJ_{n-1}(x).$$

Thus the result follows.

Form V: $\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$.
Proof: We have

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x) \qquad (1)$$

Multiplying (1) by x^{-n-1}, we get

$$x^{-n}J_n'(x) = nx^{-n-1}J_n(x) - x^{-n}J_{n+1}(x)$$

$$x^{-n}J_n'(x) - nx^{-n-1}J_n(x) = -x^{-n}J_{n+1}(x)$$

Thus $\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$.

Form VI: $\frac{d}{dx}[x^{n}J_n(x)] = x^{n}J_{n-1}(x)$.
Proof: We have

$$xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x) \qquad (2)$$

Multiplying (2) by x^{n-1}, we get

$$x^{n}J_n'(x) + nx^{n-1}J_n(x) = x^{n}J_{n-1}(x)$$

Thus $\frac{d}{dx}[x^{n}J_n(x)] = x^{n}J_{n-1}(x)$.

Orthogonality property of Bessel's function

(A) Show that

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = 0 \qquad (7)$$

where α and β are the roots of $J_n(x) = 0$.

(B) Show that $\int_0^1 x[J_n(\alpha x)]^2 dx = \frac{1}{2}[J_{n+1}(\alpha)]^2$.

Proof: (A) We have

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\alpha^2x^2 - n^2)y = 0 \qquad (8)$$

and

$$x^2\frac{d^2z}{dx^2} + x\frac{dz}{dx} + (\beta^2x^2 - n^2)z = 0 \qquad (9)$$

Solutions of (8) and (9) are $J_n(\alpha x),\ J_n(\beta x)$, respectively.

Multiplying (8) by $\frac{z}{x}$, (9) by $\frac{-y}{x}$ and then adding the equations, we get

$$x(z\tfrac{d^2y}{dx^2} - y\tfrac{d^2z}{dx^2}) + (z\tfrac{dy}{dx} - y\tfrac{dz}{dx}) + (\alpha^2 - \beta^2)xyz = 0$$

$$\tfrac{d}{dx}[x(z\tfrac{dy}{dx} - y\tfrac{dz}{dx})] + (\alpha^2 - \beta^2)xyz = 0 \qquad (i)$$

Integrating (i) with respect to x and taking limit from 0 to 1, we get

$$x(z\tfrac{dy}{dx} - y\tfrac{dz}{dx})|_0^1 + (\alpha^2 - \beta^2)\textstyle\int_0^1 xyzdx = 0$$

$$\Rightarrow (\beta^2 - \alpha^2)\textstyle\int_0^1 xyzdx = [z\tfrac{dy}{dx} - y\tfrac{dz}{dx}]_{x=1} \qquad (ii)$$

We have $y = J_n(\alpha x)$ and $z = J_n(\beta x)$. So that $\frac{dy}{dx} = \alpha J_n'(\alpha x)$ and $\frac{dz}{dx} = \beta J_n'(\beta x)$
Therefore, (ii) becomes
$(\beta^2 - \alpha^2)\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = [\alpha J_n'(\alpha x)J_n(\beta x) - \beta J_n'(\beta x)J_n(\alpha x)]_{x=1}$

$$= \alpha J_n'(\alpha)J_n(\beta) - \beta J_n'(\beta)J_n(\alpha) \qquad (iii)$$

Since α and β are the roots of $J_n(x) = 0$, so $J_n(\alpha) = J_n(\beta) = 0$. Hence (iii) becomes

$(\beta^2 - \alpha^2)\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = 0$

Thus the result follows.

(B) We have
$(\beta^2 - \alpha^2)\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \alpha J_n'(\alpha)J_n(\beta) - \beta J_n'(\beta)J_n(\alpha) \qquad (iii)$
Putting $\alpha = \beta$. We have $J_n(\alpha) = 0$. Let β be a neighboring value of α, i.e., $\beta \to \alpha$. Then
$lim_{\beta\to\alpha}\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = lim_{\beta\to\alpha}\frac{0+\alpha J_n'(\alpha)J_n(\beta)}{\beta^2-\alpha^2}$
which is an indeterminate form of type $\frac{0}{0}$.
By L'Hospital's rule, we have
$\int_0^1 xJ_n^2(\alpha x)dx = lim_{\beta\to\alpha}\frac{\alpha J_n'(\alpha)J_n'(\beta)}{2\beta} = \frac{1}{2}[J_n'(\alpha)]^2$.
Again from recurrence relation, we know
$J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$
Therefore, $J_n'(\alpha) = \frac{n}{\alpha}J_n(\alpha) - J_{n+1}(\alpha)$
But $J_n(\alpha) = 0$. So that $J_n'(\alpha) = -J_{n+1}(\alpha)$. Thus
$\int_0^1 x[J_n(\alpha x)]^2dx = \frac{1}{2}[J_{n+1}(\alpha)]^2$. □

We know $J_{\frac{1}{2}}(x) = \sqrt{(\frac{2}{\pi x})}sinx$; and $J_{-\frac{1}{2}}(x) = \sqrt{(\frac{2}{\pi x})}cosx$.
Also from the recurrence formula IV for $J_n(x)$, we have $2nJ_n(x) = x[J_{n-1}(x)+J_{n+1}(x)]$.

Prove the following identities:
(i) $J_{\frac{3}{2}}(x) = \sqrt{(\frac{2}{\pi x})}[\frac{sinx}{x} - cosx]$.
(ii) $J_{-\frac{3}{2}}(x) = -\sqrt{(\frac{2}{\pi x})}[\frac{cosx}{x} + sinx]$.
(iii) $J_{\frac{5}{2}}(x) = \sqrt{(\frac{2}{\pi x})}[\frac{3}{x}(\frac{sinx}{x} - cosx) - sinx]$.
(iv) $J_{-\frac{5}{2}}(x) = \sqrt{(\frac{2}{\pi x})}[\frac{3}{x}(\frac{cosx}{x} + sinx) - cosx]$.