Lectures 01, 02 and 03

Chapter 1: Introduction

Math 4441: Probability and Statistics

Reference: Goodman & Yates – Introduction to Probability and Stochastic Processes, 3rd Edition

Course Outcomes (Cos)

CO1	Understand the knowledge of fundamental probability concepts, including probability of events, conditional probability, sum and product rules, Bayes rule, random variables, correlation and covariance
CO2	Apply knowledge of probability to model solutions for engineering problems and mathematically explain uncertain phenomenon
CO3	Use an appropriate probability model to describe and analyze observed data and draw conclusions concerning interesting parameters

Probability vs Statistics

Probability	Statistics
Given a problem statement	Given a set of data
Develop a Probability Model	Analyze the data
Identify the uncertainty	Estimate the probability Model
Predict the future behavior	Estimate Parameter values and draw conclusions

Problem 1 (Disease or Not)

1. The probability that a randomly selected person has a disease is 0.8 percent (8 out of 1,000).
2. The probability that a person with the disease will have a positive test result is 90 percent.
3. The probability that a person with the disease will have a positive test result is 7 percent.

Problem 2 (Monty Hall Dilemma)

The dilemma is taken from an old American game show called Let’s Make a Deal. The original host of the show, Monty Hall, would select a member of the audience and show that person three large closed doors labeled 1, 2, and 3. Behind one of the doors was a new car. Behind the remaining two doors were joke prizes, such as a live goat.

The contestant was asked to pick a door. Then, Hall would ask that one of the doors the contestant didn’t pick be opened, naturally one that didn’t have a car behind it. After the audience stopped laughing at whatever joke prize was behind that door, Hall would ask the contestant if they wanted to keep the originally selected door, or if they would rather change their selection to the remaining door. The dilemma is simply that: do they keep their original guess, or do they switch to the remaining door?

Problem 3 (Birthdays Problem)

We want to determine the probability that in a class size of r two or more birth dates match. We shall assume that the year consists of 365 days

Probability

A probability is a number between zero and one and represents the likelihood of occurring something in an uncertain phenomenon.

A probability quantifies the chances by a number.

If it is guaranteed that some phenomenon will occur then the probability of occurring it is 1 and if it is unlike to occur some the probability of occurring it is closed to zero.

Otherwise, the probability is related to the percentage of occurring the phenomenon

<u>Probabilistic Models or Probability Models</u>

A probabilistic model is a mathematical description of an uncertain situation.

Quantifies the uncertainty by measurable values, i.e., numbers

Certainty vs Uncertainty (<u>Number</u> of possible distinct results)

An uncertain situation is characterized by more than one possible result and one of which appears as outcomes with chances.

Random Experiment

Produces exactly one result out of several possible results.

Experiments consist of two ingredients:

Procedure: Real world phenomenon that produces uncertain outcomes.

Observation: Expected results obtained by running an experiment

Each of the distinct observations of an experiment is an **outcome**.

A set of outcomes is an **event**

If any outcome associated with an event occurs, we say the event has occurred

Example 1.1. Sending a data packet from one computer (sender) to another computer (receiver). Success occurs if the packet is Delivered (D) and failure occurs if the packet is not delivered (F)

Procedure: Send the data packet from the sender to the receiver

Observation: Observe the success or failure of the delivery.

Probability Model: Likelihood of **D** is three times than that of **F**. Each of the deliveries is independent of the others → One does not affect others

Similarity of Experiments

Two experiments are similar iif both procedures and observations are similar.

Different if either or both the procedure or the observation is different.

Experiment 1.2

Procedure: Send 3 packets from a sender to a receiver.
Observation: Sequence of successes and failures of the deliveries.

Experiment 1.3

Procedure: Send 3 packets from a sender to a receiver.
Observation: Number of successes.

→ 1.2 and 1.3 have the same procedure but different observations

Experiment 1.4

Procedure: Keep sending packets from a sender to a receiver until 1 packet is delivered
Observation: Number of attempts

Experiment 1.5

Procedure: Keep sending packets from a sender to a receiver until 3 packets are delivered
Observation: Number of attempts

→ 1.4 and 1.5 have the same observation but different procedures

Outcomes, Events and Probability

Outcome: Every possible result of an experiment
An outcome is an observation

Event: A set of one or more outcomes

Probability: Percentage of time an outcome of event occurs

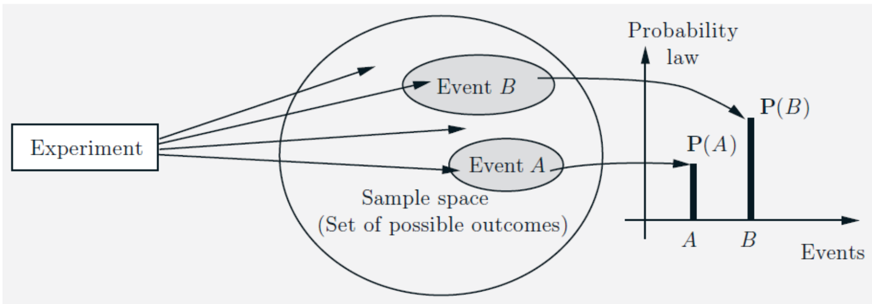
$P[A]$ → probability of occurring the event A

$P[\omega]$ → Probability of occurring the outcome ω

Elements of Probability Model

→Sample Space (S): set of all possible outcomes

→Probability of outcomes/events: Assignment of probabilities to each outcome of event



Sample Space Set of all possible outcomes is said to be the sample space

→ if the outcomes individually/collectively satisfy 3 properties

1. Finest Grain: each outcome is distinguishable, non-overlapping, smallest but most detailed
2. Mutually Exclusive: if one appears as an outcome then no other can appear, $\omega_i \cap \omega_j = \phi$
3. Collectively Exhaustive: Every possible outcome must be in the Sample space

Example 1.6 Tossing a coin and rolling a die

Sample Space

1. Discrete Set
2. Continuous Set

Experiment 1.2

Procedure: Send 3 packets from a sender to a receiver.
Observation: Sequence of successes and failures of the deliveries.

Possible results: $\{FFF, FFD, \ldots, DDD\}$

$$S = \{FFF, FFD, FDF, FDD, DFF, DFD, DDF, DDD\}$$

Experiment 1.3

Procedure: Send 3 packets from a sender to a receiver.
Observation: Number of successes.

Possible results $\{0, 1, 2, 3\}$

What is meant by 1 success? Can it uniquely identify an outcome?

$$S = \{FFF, FFD, FDF, FDD, DFF, DFD, DDF, DDD\}$$

Experiment 1.4

Procedure: Keep sending packets from a sender to a receiver until 1 packet is delivered

Observation: Number of attempts

Possible results: $\{1, 2, \ldots\}$

$$S = \{D, FD, FFD, FFFD, \ldots\}$$

Experiment 1.5

Procedure: Keep sending packets from a sender to a receiver until 3 packets are delivered
Observation: Number of attempts

Possible results: $\{3, 4, 5, \ldots\}$

$$S = \{DDD, FDDD, DFDD, DDFD, FFDDD, \ldots\}$$

Event Space

Collectively exhaustive and mutually exclusive set of events

$E = \{E_1, E_2, \ldots, E_n\}$ is an event space, if

1. Mutually exclusive: Two events are mutually exclusive, they cannot occur together
   $E_i \cap E_j = \phi, i \neq j$
2. Collectively exhaustive: set of events are collectively exhaustive, if they include all possible outcomes
   $\cup_{i=1}^{n} E_i = S$

Experiment 1.3

Procedure: Send 3 packets from a sender to a receiver.

Observation: Number of successes.

$S = \{FFF, FFD, FDF, FDD, DFF, DFD, DDF, DDD\}$

$E_i = \{i\ successes\}, \qquad i = 0, 1, 2, 3$

$E_0 = \{0\ sucess\} = \{FFF\}$

$E_1 = \{1\ sucess\} = \{FFD, FDF, DFF\}$

$E_2 = \{2\ sucesses\} = \{FDD, DFD, DDF\}$

$E_3 = \{3\ sucesses\} = \{DDD\}$

$E = \{E_0, E_1, E_2, E_3\}$ is an event space or note?

Example 1.6: Monitor 3 consecutive phone calls going through a telephone switching office. Classify each one as a voice call (v) if someone is speaking, or a data call (d) if the call is carrying a modem or fax signal. Your observation is a sequence of three letters (each letter is either v or d). For example, two voice calls followed by one data call corresponds to vvd. Write the elements of the following sets:

1. $A_1 =$ {first call is a voice call}

2. $B_1 =$ {first call is a data call}

3. A_2 = {second call is a voice call}

4. B_2 = {second call is a data call}

5. A_3 = {all calls are the same}

6. B_3 = {voice and data alternate}

7. A_4 = {one or more voice calls}

8. B_4 = {two or more data calls}

For each pair of events A_1 and B_1, A_2 and B_2, and so on, identify whether the pair of events is either mutually exclusive or collectively exhaustive or both.

<u>Probabilities of Events/Outcomes</u>:

➔ How probabilities are assigned to the outcomes and events

1. Probability Axioms
    a. Rules related to the assignment of probabilities, and every assignment must satisfy
    b. Ensures a number between 0 and 1 to each outcome and event
2. Assignment of Probabilities
    a. Classical Approach
    b. Relative Frequency Approach
    c. Subjective Approach

Probability Axioms

Axiom 1 (Non-negativity): If A is an event, then the probability of A satisfies

$$P[A] \geq 0$$

Axiom 2 (Additivity): If $A_1$and A_2 are two disjoint events, then

$$P[A_1 \cup A_2] = P[A_1] + P[A_2]$$

Axiom 3 (Normalization): The probability of all elements in the sample space must be sum to 1

$$P[S] = P[\cup_{i=1}^{n} \omega_i] = \sum_{i=1}^{n} P[\omega_i] = 1$$

Simple Extension of the Axioms

1. If $A_1, A_2, \dots, A_n$ is a disjoint set, then
$$P[\cup_{i=1}^{n} A_i] = P[A_1] + P[A_2] + \cdots + P[A_n]$$

2. If $A \cap B \neq \phi$, A and B are not mutually exclusive, then
$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

3. $P[A^c] = 1 - P[A]$

4. If $A \subset B$, then $P[A] \leq P[B]$

Assignment of Probabilities

1. Classical Approach

Assume that all outcomes in an experiment are equally likely

- If the coin is fair, heads and tails are equally likely to appear
- If the die is fair, number of dots $1, 2, \dots, 6$ are equally likely to appear

If there are n elements in the sample space,

$S = \{\omega_1, \omega_2, \dots, \omega_n\}$ and

$P[\omega_i] \geq 0$ [axiom 1]

Since ω_is are elements of S, $\omega_i \cap \omega_j = \phi$

$P[\omega_1 \cup \omega_2 \cup \dots \cup \omega_n] = P[\omega_1] + P[\omega_2] + \cdots + P[\omega_n] = 1$ [axiom 2, 3]

Since the outcomes are equally likely

$$P[\omega_1] = P[\omega_2] = \cdots = P[\omega_n]$$

Finally,

$$P[\omega_i] = \frac{1}{n}, i = 1, 2, \dots, n$$

<u>For the experiment of rolling a die</u>

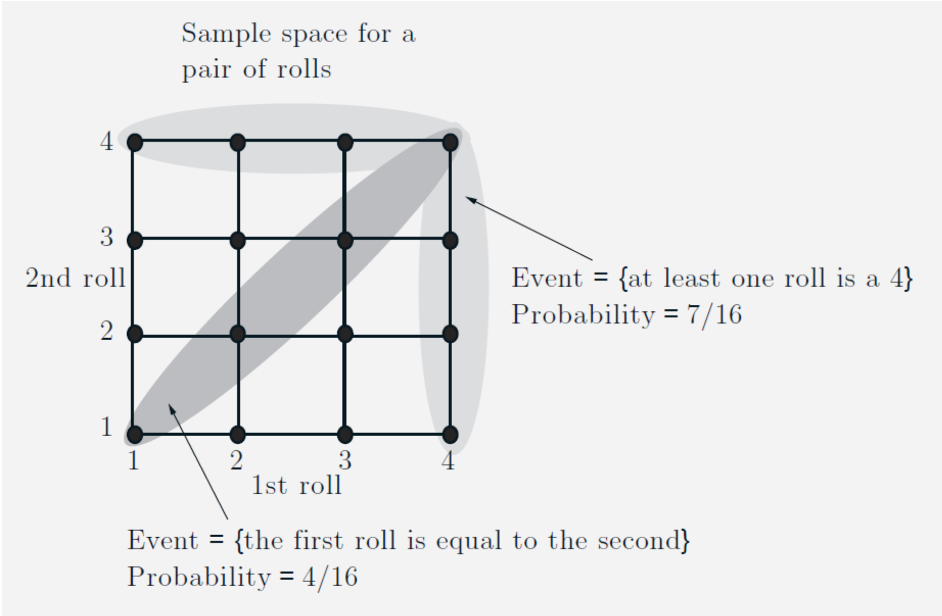
Probabilities of Events: The probability of occurring event A is,

1. The probability is defined by the proportion of S represented by A
2. If the outcomes are equally likely, the ratio of the number of outcomes in A to the number of outcomes in S
3. If the outcomes are not equally likely?

Example 1.7 A pair of four-sided fair dice are rolled and the number of dots on the side facing upward in each dice is counted. Each outcome of this experiment is a pair of numbers in the range $1, 2, \ldots, 4$. There are 16 possible outcomes of this experiment. If we assume, if the outcomes are equally likely to occur, then each outcome is assigned a probability of $\frac{1}{16}$.

Now, suppose we want to find the probability of the event

A = {sum of two dice = 5}



Sample space for a
pair of rolls
4
3
2nd roll
2
1
1
2
3
4
1st roll
Event = {at least one roll is a 4}
Probability = 7/16
Event = {the first roll is equal to the second}
Probability = 4/16

How to calculate the probability of the events?

1. Identify the outcomes in each event
2. Apply axiom 2 to calculate the probability of the event.

Probabilities of some events:

1. P[the sum of the rolls is even] =

2. P[the sum of the rolls is odd] =

3. P[the first roll is equal to the second] =

4. P[the first roll is larger than the second] =

5. P[at least one roll is equal to 4] =

2. <u>Relative Frequency Approach</u>
a. If the outcomes in the sample space are not equally likely
b. We repeat the experiment a large number of times (say, n and $n \to \infty$)
c. Count the number of times outcome ω occurs, n_ω

The probability of occurring outcome ω is

$$P[\omega] = \lim_{n \to \infty} \frac{n_\omega}{n}$$

Similarly, the probability of occurring any event A can be calculated by counting the number of times event A occurs n_A if the experiment is repeated n times,

$$P[A] = \lim_{n \to \infty} \frac{n_A}{n}$$

Example 1.8 Consider two six-sided fair dice are rolled. We define event $E_k = \{\text{sum of the two dice} = k\}$. We can simulate the experiment by a computer program, and the results are shown below for the event $A = E_5$:

n	1000	2000	3000	4000	5000	6000	7000	8000	9000	10000
n_A	96	200	314	408	521	630	751	859	970	1095
n_A/n	0.096	0.10	0.105	0.102	0.104	0.105	0.107	0.107	0.108	0.110

$$
\begin{aligned}
P[A] &= P[\{1,4\} \cup \{2,3\} \cup \{3,2\} \cup \{4,1\}] \\
&= P[\{1,4\}] + P[\{2,3\}] + P[\{3,2\}] + P[\{4,1\}] \\
&= \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{4}{36} = \frac{1}{9} = 0.111
\end{aligned}
$$

3. Subjective Approach

a. Probabilities are assigned by engaging subject experts

b. Manually assign the probabilities

c. Vary from expert to expert.

Joint Probability

The probability of the intersection of two events, $P[A \cap B]$ or $P[A, B]$ or $P[AB]$

It is the simultaneous occurrence of two events

If events A and B are mutually exclusive, they two cannot occur together and the probability $P[AB] = 0$, because $A \cap B = \phi$

Calculation of the joint Probability:

1. Classical Approach: Find $A \cap B = \{\text{elements common in both events}\}$
   $P[A \cap B] = P[\bigcup_{w_i \in A \cap B} \omega_i] = \sum_{\omega_i \in A \cap B} P[\omega_{i]}]$

2. Use the relative frequency approach. If $n_{A,B}$is the number of times events A and B occur simultaneously in n repetition, then
   $P[A, B] = \lim_{n \to \infty} \frac{n_{A,B}}{n}$

Example 1.9: A standard deck of playing cards has 52 cards that can be divided in several manners. There are four suits (spades, hearts, diamonds, and clubs) each of which has 13 cards (Ace, $2, 3, \ldots, 10$, jack, queen, and king). There are two red suits (hearts and diamonds) and two black suits (spades and clubs). Also, the jacks, queens, and kings are referred to as face cards, while the others are number cards.

1. $A = \{\text{red card selected}\}$,
2. $B = \{\text{number card selected}\}$, and
3. $C = \{\text{heart selected}\}$.

Example 1.10: Roll a six-sided fair die and observe the number of dots on the top side

$$S = \{1, 2, 3, 4, 5, 6\}$$

Define two events:

$$A \triangleq \{\text{Outcome is square of an integer}\} = \{1, 4\}$$

$$B \triangleq \{\text{Outcome is an even number}\} = \{2, 4, 6\}$$

$$P[A] =$$

$$P[B] =$$

$$P[AB] =$$

Simple Probability Models: Example 1.11 (Experiment 1.2)

Procedure: Send 3 packets from a sender to a receiver.
Observation: Sequence of successes and failures of the deliveries.

$$S = \{FFF, FFD, FDF, FDD, DFF, DFD, DDF, DDD\}$$

Probability that a packet is delivered is $\frac{1}{2}$

Probability that a delivery is failed is $\frac{1}{2}$

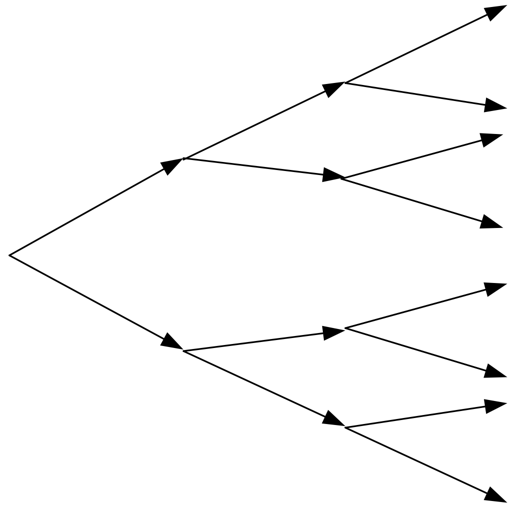
Probability of occurring a sequence of events

$P[FFF] =$

Probability that a packet is delivered is p

Probability that a delivery is failed is $1 - p$

$P[FFF] =$

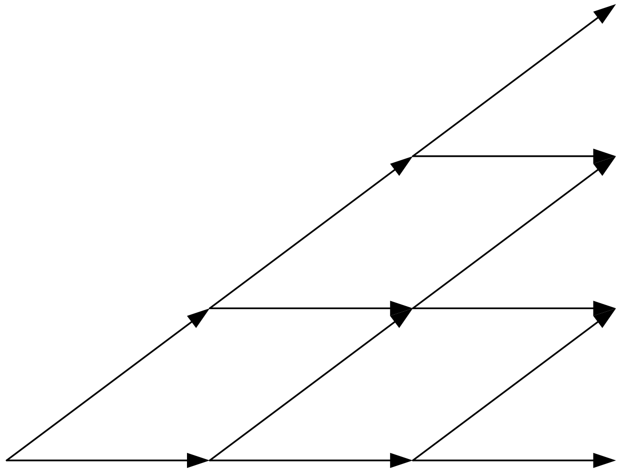


Example 1.12 (**Experiment 1.3)**

Procedure: Send 3 packets from a sender to a receiver.

Observation: Number of successes.

$$S = \{FFF, FFD, FDF, FDD, DFF, DFD, DDF, DDD\}$$

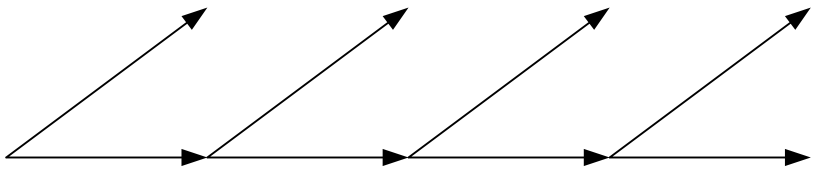


Example 1.13 (**Experiment 1.4)**

Procedure: Keep sending packets from a sender to a receiver until 1 packet is delivered

Observation: Number of attempts

$$S = \{D, FD, FFD, FFFD, \dots\}$$



Example 1.14 (**Experiment 1.5)**

Procedure: Keep sending packets from a sender to a receiver until 3 packets are delivered
Observation: Number of attempts

$S = \{DDD, FDDD, DFDD, DDFD, FFDDD, \ldots\}$