LEGENDRE POLYNOMIALS

The differential equation of the form

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + m(m+1)y = 0 \tag{1}$$

where m is a positive integer, is called Legendre's differential equation.

Let us first identify the nature of the singular points of this equation.

Let in (1) $p(x) = \frac{-2x}{1-x^2}$ and $q(x) = \frac{m(m+1)}{1-x^2}$.

Since neither of these functions is analytic at $x = \pm 1$, so these points are the singular points of (1).

Next $(x-1)p(x) = \frac{2x}{1+x}$ and $(x-1)^2 q(x) = m(m+1)\frac{x-1}{x+1}$ are both analytic at $x = 1$. It follows that $x = 1$ is a regular singular point of (1).

Similarly, we see that $x = -1$ is also a regular singular point of (1).

Thus, a power series solution of (1) exists in the interval $-1 < x < 1$. Any solution of (1) is called a Legendre polynomial or Legendre function.

Let the series solution of (1) be

$$y(x) = \sum_{n=0}^{\infty} a_n x^{c-n}, a_0 \neq 0 \tag{2}$$

be the trial solution of (1).

Therefore, $\frac{dy}{dx} = \sum_{n=0}^{\infty}(c-n)a_n x^{c-n-1}$ and $\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(c-n)(c-n-1)a_n x^{c-n-2}$

Then substituting the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$(1-x^2)\sum_{n=0}^{\infty}(c-n)(c-n-1)a_n x^{c-n-2} - 2x\sum_{n=0}^{\infty}(c-n)a_n x^{c-n-1}$$

$$+m(m+1)\sum_{n=0}^{\infty} a_n x^{c-n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}(c-n)(c-n-1)a_n x^{c-n-2} - \sum_{n=0}^{\infty}\{(c-n)(c-n-1)+2(c-n)-m(m+1)\}a_n x^{c-n} = 0 \tag{3}$$

Now, the coefficient of x^{c-n} equals

$(c-n)^2 - (c-n) + 2(c-n) - m(m+1) = (c-n)^2 - m^2 + (c-n) - m$

$= (c-n-m)(c-n+m+1)$.

Hence (3) becomes

$$\sum_{n=0}^{\infty}(c-n)(c-n-1)a_n x^{c-n-2} - \sum_{n=0}^{\infty}(c-n-m)(c-n+m+1)a_n x^{c-n} = 0 \tag{4}$$

To get the indicial equation, equate the coefficient of x^c to zero and obtain

$$(c-m)(c+m+1)a_0 = 0, \quad a_0 \neq 0 \tag{5}$$

which implies $c = m$ or $c = -(m+1)$.

Next, equating the coefficient of x^{c-1} to zero to obtain

$$(c-1-m)(c+m)a_1 = 0 \tag{6}$$

Since $c = m$ or $c = -(m+1)$, neither $(c-1-m)$ nor $(c+m)$ is zero and then we get from (6), $a_1 = 0$.

Finally, equating the coefficient of x^{c-n} to zero to obtain

$$(c-n+2)(c-n+1)a_{n-2} - (c-n-m)(c-n+m+1)a_n = 0$$

$$\Rightarrow a_n = \frac{(c-n+2)(c-n+1)}{(c-n-m)(c-n+m+1)}a_{n-2}; \quad n \geq 2 \tag{7}$$

Putting $n = 3, 5, 7, \ldots$ in (7), we get

$$a_1 = a_3 = a_5 = a_7 = \cdots = 0 \tag{8}$$

To obtain $a_2, a_4, a_6, \ldots$, etc., consider the following two cases:

Case I: When $c = m$, equation (7) becomes

$$a_n = \frac{(m-n+2)(m-n+1)}{(-n)(2m-n+1)}a_{n-2} \tag{9}$$

Putting $n = 2, 4, 6, \ldots$ in (9), we get

$a_2 = -\frac{m(m-1)}{2(2m-1)}a_0$, $a_4 = -\frac{(m-2)(m-3)}{4(2m-3)}a_2 = \frac{m(m-1)(m-2)(m-3)}{2.4.(2m-1)(2m-3)}a_0$, and so on.

Re-writing (2),we have for $c = m$,

$$y = a_0x^m + a_1x^{m-1} + a_2x^{m-2} + a_3x^{m-3} + a_4x^{m-4} + \cdots \tag{10}$$

Therefore, with the above values of $a_0, a_1, a_2, a_3, a_4, a_5, a_6, \ldots$, (10) becomes with $a_0 = a$,

$$y = a[x^m - \frac{m(m-1)}{2(2m-1)}x^{m-2} + \frac{m(m-1)(m-2)(m-3)}{2.4.(2m-1)(2m-3)}x^{m-4} - \cdots] \tag{11}$$

where a is an arbitrary constant.

Case II: When $c = -(m+1)$, then equation (7) becomes

$$a_n = \frac{(m+n-1)(m+n)}{n(2m+n+1)}a_{n-2} \tag{12}$$

Putting $n = 2, 4, 6, \ldots$ in (12), we get

$a_2 = \frac{(m+1)(m+2)}{2(2m+3)} a_0$, $a_4 = \frac{(m+3)(m+4)}{4(2m+5)} a_2 = \frac{(m+1)(m+2)(m+3)(m+4)}{2.4.(2m+3)(2m+5)} a_0$, and so on.

Hence, for $c = -(m+1)$, equation (2) becomes,

$$y = a_0 x^{-m-1} + a_1 x^{-m-2} + a_2 x^{-m-3} + a_3 x^{-m-4} + a_4 x^{-m-5} + \cdots] \tag{13}$$

So that with the above values of $a_0, a_1, a_2, a_3, a_4, a_5, a_6, \ldots$, (13) gives with $a_0 = b$,

$$y = b[x^{-m-1} + \frac{(m+1)(m+2)}{2(2m+3)} x^{-m-3} + \frac{(m+1)(m+2)(m+3)(m+4)}{2.4.(2m+3)(2m+5)} x^{-m-5} + \cdots \tag{14}$$

where b is an arbitrary constant.

Thus, the independent solutions of (1) are given by (11) and (14).

Taking $a = \frac{1.3.5.\cdots.(2m-1)}{m!}$, the solution (11) is denoted by $P_m(x)$ and it is called *Legendre's function of the first kind* or *Legendre's polynomial of degree m*. So, $P_m(x)$ is a solution of (1).

Again, if we take $b = \frac{m!)}{1.3.5.\cdots.(2m+1)}$, the solution (14) is denoted by $Q_m(x)$ and it is called *Legendre's function of the second kind* or *Legendre's polynomial of degree m.*
Thus, $P_m(x)$ and $Q_m(x)$ are the linearly independent solutions of (1) and finally, the general solution of (1) is

$$y = c_1 P_m(x) + c_2 Q_m(x) \tag{15}$$

where c_1 and c_2 are arbitrary constants.

Legendre's polynomial of degree n is denoted and defined by

$$P_n(x) = \frac{1.3.5.\ldots.(2n-1)}{n!}[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)} x^{n-4} - \ldots] \tag{16}$$

We now re-write (16) in compact form. The general term of (16) is given by

$\frac{1.3.5.\ldots.(2n-1)}{n!}(-1)^r \frac{n(n-1)(n-2)\cdots(n-2r+1)}{2.4.\ldots.2r(2n-1)(2n-3)\cdots(2n-2r+1)} x^{n-2r}$.

Now,

$$1.3.5.\ldots.(2n-1) = \frac{1.2.3.4.5.6.\ldots.(2n-1)2n}{2.4.6.\ldots.2n} = \frac{(2n)!}{(2.1)(2.2)(2.3)\ldots(2.n)} = \frac{(2n)!}{2^n n!} \tag{17}$$

Also,

$$n(n-1)\cdots(n-2r+1) = \frac{n(n-1)\cdots(n-2r+1)(n-2r)(n-2r-1)\ldots 3.2.1}{(n-2r)(n-2r-1)\cdots 3.2.1} = \frac{n!}{(n-2r)!} \tag{18}$$

$$2.4.6.\ldots.2r = (2.1)(2.2)(2.3)\ldots(2.r) = 2^r r! \tag{19}$$

and $(2n-1)(2n-3)\ldots(2n-2r+1) = \frac{2n(2n-1)(2n-2)\cdots(2n-2r+2)(2n-2r+1)}{2n(2n-2)(2n-4)\cdots(2n-2r+2)} \times \frac{(2n-2r)!}{(2n-2r)!}$

$= \frac{2n(2n-1)(2n-2)\cdots(2n-2r+1)(2n-2r)(2n-2r-1)\cdots.3.2.1}{2n.2(n-1).2(n-2)\cdots 2(n-r+1).(2n-2r)!}$

$= \frac{(2n)!}{2^n.n(n-1)(n-2)\cdots(n-r+1)(2n-2r)!}$

$= \frac{(2n)!}{2^n(2n-2r)!} \times \frac{(n-r)(n-r-1)\cdots 3.2.1}{n(n-1)(n-2)\cdots(n-r+1)(n-r)(n-r-1)\cdots 3.2.1}$

Finally,

$$(2n-1)(2n-3)\ldots(2n-2r+1) = \frac{(2n)!}{2^n(2n-2r)!} \times \frac{(n-r)!}{n!} \tag{20}$$

Using (17), (18), (19)and (20), the most simplified form of the general term in (16) is given by

$$\frac{(2n)!}{2^n n!}(-1)^r \frac{n!}{(n-2r)!} \times \frac{1}{2^r r!} \times \frac{2^n(2n-2r)!n!}{(2n)!(n-r)!} x^{n-2r} = (-1)^r \frac{(2n-2r)!}{2^n r!(n-r)!(n-2r)!} x^{n-2r} \tag{21}$$

Since (16) is a polynomial of degree n, r must be chosen so that $n-2r \geq 0$, i.e., $r \leq \frac{n}{2}$.

Thus, if n is even, r goes from 0 to $\frac{n}{2}$, while if n is odd, r goes from 0 to $\frac{n-1}{2}$. We denote it by $[\frac{n}{2}]$.

Hence, the Legendre polynomial in x of degree n is given by

$$P_n(x) = \sum_{r=0}^{[\frac{n}{2}]} (-1)^r \frac{(2n-2r)!}{2^n r!(n-r)!(n-2r)!} x^{n-2r} \tag{22}$$

Generating function for Legendre polynomials

The function

$$(1-2xh+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x) \tag{23}$$

is known as the generating function of Legendre polynomials.

Problems

1. Prove that $P_n(-x) = (-1)^n P_n(x)$.

Proof: Generating function for Legendre polynomials is given by

$$(1-2xh+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x) \qquad (1)$$

Replacing x by $-x$ in (23), we get

$$1+2xh+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(-x) \qquad (2)$$

Next, replacing h by $-h$ in (1), we get

$$(1+2xh+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} (-h)^n P_n(x) \qquad (3)$$

From (2) and (3), we get

$$\sum_{n=0}^{\infty} h^n P_n(-x) = \sum_{n=0}^{\infty} (-h)^n P_n(x) \tag{4}$$

Equating the coefficient of h^n, we get $P_n(-x) = (-1)^n P_n(x)$.

2. Prove that $\frac{1-h^2}{(1-2xh+h^2)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)h^n P_n(x)$.

Proof: Generating function for Legendre polynomials is given by

$$(1-2xh+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x) \tag{1}$$

Differentiating both sides of (1) with respect to h, we get

$$-\tfrac{1}{2}(1-2xh+h^2)^{-\frac{3}{2}}(-2x+2h) = \sum_{n=0}^{\infty} nh^{n-1}P_n(x)$$

$$\Rightarrow (x-h)(1-2xh+h^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} nh^{n-1}P_n(x) \tag{2}$$

Next, multiply both sides of (2) by $2h$, we get

$$2h(x-h)(1-2xh+h^2)^{-\frac{3}{2}} = 2\sum_{n=0}^{\infty} nh^n P_n(x) \tag{3}$$

Adding (1) and (3), we get

$$\frac{1}{(1-2xh+h^2)^{\frac{1}{2}}} + \frac{2h(x-h)}{(1-2xh+h^2)^{\frac{3}{2}}} = \sum_{n=0}^{\infty} h^n P_n(x) + \sum_{n=0}^{\infty} 2nh^n P_n(x)$$

$$\Rightarrow \frac{1-2xh+h^2+2h(x-h)}{(1-2xh+h^2)^{\frac{3}{2}}} = \sum_{n=0}^{\infty}(2n+1)h^n P_n(x)$$

Hence, $\frac{1-h^2}{(1-2xh+h^2)^{\frac{3}{2}}} = \sum_{n=0}^{\infty}(2n+1)h^n P_n(x)$.

Recurrence relations

Form I: $nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$.

Proof: Generating function for Legendre polynomials is given by

$$(1-2xh+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x) \tag{1}$$

Differentiating both sides of (1) with respect to h, we get

$$-\tfrac{1}{2}(1-2xh+h^2)^{-\frac{3}{2}}(-2x+2h) = \sum_{n=0}^{\infty} nh^{n-1}P_n(x)$$

$$\Rightarrow (x-h)(1-2xh+h^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} nh^{n-1}P_n(x) \tag{2}$$

Multiplying both sides of (2) by $1-2xh+h^2$, we get

$$(x-h)(1-2xh+h^2)^{-\frac{1}{2}} = (1-2xh+h^2)\sum_{n=0}^{\infty} nh^{n-1}P_n(x)$$

$$\Rightarrow (x-h)\sum_{n=0}^{\infty} h^n P_n(x) = (1-2xh+h^2)\sum_{n=0}^{\infty} nh^{n-1}P_n(x)$$

Equating the coefficient of h^{n-1}, we get

$xP_{n-1}(x) - P_{n-2}(x) = nP_n(x) - 2(n-1)xP_{n-1}(x) + (n-2)P_{n-2}(x)$

Thus, $nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$.

Form II: $xP_n'(x) - P_{n-1}'(x) = nP_n(x)$.

Proof: Generating function for Legendre polynomials is given by

$$(1-2xh+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x) \qquad (1)$$

Differentiating both sides of (1) with respect to h, we get

$$-\tfrac{1}{2}(1-2xh+h^2)^{-\frac{3}{2}}(-2x+2h) = \sum_{n=0}^{\infty} nh^{n-1}P_n(x)$$

$$\Rightarrow (x-h)(1-2xh+h^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} nh^{n-1}P_n(x) \qquad (2)$$

Differentiating (1) with respect to x, we get

$$-\tfrac{1}{2}(1-2xh+h^2)^{-\frac{3}{2}}(-2h) = \sum_{n=0}^{\infty} h^n P_n'(x)$$

$$\Rightarrow h(1-2xh+h^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} h^n P_n'(x) \qquad (3)$$

Now divide (2) by (3), we get

$$\frac{x-h}{h} = \frac{\sum_{n=0}^{\infty} nh^{n-1}P_n(x)}{\sum_{n=0}^{\infty} h^n P_n'(x)}$$

$$\Rightarrow (x-h)\sum_{n=0}^{\infty} h^n P_n'(x) = \sum_{n=0}^{\infty} nh^n P_n(x)$$

Equating the coefficient of h^n, we get

$$xP_n'(x) - P_{n-1}'(x) = nP_n(x).$$

Form III: $P_n'(x) - xP_{n-1}'(x) = nP_{n-1}(x)$.

Proof: We have $nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$ (Recurrence Formula I)
Differentiating with respect to x, we get

$$nP_n'(x) = (2n-1)P_{n-1}(x) + (2n-1)xP_{n-1}'(x) - (n-1)P_{n-2}'(x)$$

$$\Rightarrow n[P_n'(x) - xP_{n-1}'(x)] - (n-1)[xP_{n-1}'(x) - P_{n-2}'(x)] = (2n-1)P_{n-1}(x)$$

$$\Rightarrow n[P_n'(x) - xP_{n-1}'(x)] = [(n-1)^2 + (2n-1)]P_{n-1}(x) = n^2 P_{n-1}(x)$$

(by Recurrence Formula II)

Thus, $P_n'(x) - xP_{n-1}'(x) = nP_{n-1}(x)$.

Form IV: $P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x)$.

Proof: We have $nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$ (Recurrence Formula I)
Replacing n by $n+1$, we get

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

Differentiating with respect to x, we get

$$\begin{aligned}(n+1)P_{n+1}'(x) &= (2n+1)P_n(x) + (2n+1)xP_n'(x) - nP_{n-1}'(x)\\ &= (2n+1)P_n(x) + (2n+1)[nP_n(x) + P_{n-1}'(x)] - nP_{n-1}'(x)\\ &\qquad \text{(by Recurrence Formula II)}\\ &= (2n+1)(n+1)P_n(x) + (n+1)P_{n-1}'(x).\end{aligned}$$

Thus, $P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x)$.

Form V: $(x^2-1)P_n'(x) = n[xP_n(x) - P_{n-1}(x)]$.

Proof: We have $P_n'(x) - xP_{n-1}'(x) = nP_{n-1}(x)$ (Recurrence Formula III)

$$xP_n'(x) - P_{n-1}'(x) = nP_n(x) \text{ (Recurrence Formula II)}$$

Multiplying Formula II by x and then subtracting Formula III, we get

$$(x^2-1)P_n'(x) = n[xP_n(x) - P_{n-1}(x)]$$

Form VI: $(x^2-1)P_n'(x) = (n+1)[P_{n+1}(x) - xP_n(x)]$.

Proof: We have $nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$ (Recurrence Formula I)

Replacing n by $n+1$, we get

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

$$\Rightarrow (n+1)[P_{n+1}(x) - xP_n(x)] = n[xP_n(x) - P_{n-1}(x)] = (x^2-1)P_n'(x),$$

(by Recurrence Formula V)

Thus, $(x^2-1)P_n'(x) = (n+1)[P_{n+1}(x) - xP_n(x)]$.

Orhtogonal properties of Legendre polynomials

(A) Prove that $\int_{-1}^{1} P_m(x)P_n(x)dx = 0$ if $m \neq n$.

(B) Prove that $\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}$ if $m = n$.

Proof: Since $P_m(x)$ and $P_n(x)$ satisfy Legendre's differential equation, we have

$$(1-x^2)P_m''(x) - 2xP_m'(x) + m(m+1)P_m(x) = 0 \qquad (1)$$

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0 \qquad (2)$$

$(1)\times P_n(x)-$ $(2)\times P_m(x) \Rightarrow$

$$(1-x^2)[P_n(x)P''_m(x)-P_m(x)P''_n(x)]-2x[P_n(x)P'_m(x)-P_m(x)P'_n(x)]$$
$$+[m(m+1)-n(n+1)]P_m(x)P_n(x)=0$$

$$\Rightarrow (1-x^2)\tfrac{d}{dx}[P_n(x)P'_m(x)-P_m(x)P'_n(x)]-2x[P_n(x)P'_m(x)-P_m(x)P'_n(x)]$$
$$+[m^2+m-n^2-n]P_m(x)P_n(x)=0$$

$$\Rightarrow \tfrac{d}{dx}(1-x^2)[P_n(x)P'_m(x)-P_m(x)P'_n(x)]=(n-m)(n+m+1)P_m(x)P_n(x)$$

Integrating both sides with respect to x from -1 to 1, we get

$$(n-m)(n+m+1)\int_{-1}^{1}P_m(x)P_n(x)dx=[(1-x^2)\{P_n(x)P'_m(x)-P_m(x)P'_n(x)\}]_{-1}^{1}$$

Thus, $\int_{-1}^{1}P_m(x)P_n(x)dx=0$ if $m\neq n$.

Second Part

We have the generating function of Legendre polynomials

$$(1-2xh+h^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}h^nP_n(x) \qquad (3)$$

and

$$(1-2xh+h^2)^{-\frac{1}{2}}=\sum_{m=0}^{\infty}h^mP_m(x) \qquad (4)$$

Multiply (3) and (4) to get

$$(1-2xh+h^2)^{-1}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}h^{m+n}P_m(x)P_n(x)$$

Integrating both sides with respect to x from -1 to 1, we get

$$\int_{-1}^{1}(1-2xh+h^2)^{-1}dx=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}h^{m+n}\{\int_{-1}^{1}P_m(x)P_n(x)dx\}$$

Putting $m=n$, we get

$$\sum_{n=0}^{\infty}\{\int_{-1}^{1}[P_n^2(x)]dx\}h^{2n}=\int_{-1}^{1}\tfrac{dx}{1-2xh+h^2}$$
$$=[\tfrac{log(1-2xh+h^2)}{-2h}]_{-1}^{1}=-\tfrac{1}{2h}[log(1-h)^2-log(1+h)^2]$$
$$=-\tfrac{1}{2h}[2log(1-h)-2log(1+h)]=\tfrac{1}{h}[log(1+h)-log(1-h)]$$
$$=\tfrac{1}{h}[(h-\tfrac{h^2}{2}+\tfrac{h^3}{3}-\cdots)-(-h-\tfrac{h^2}{2}-\tfrac{h^3}{3}-\cdots)]$$
$$=\tfrac{2}{h}(h+\tfrac{h^3}{3}+\tfrac{h^5}{5}+\cdots)=\tfrac{2}{h}\sum_{n=0}^{\infty}\tfrac{h^{2n+1}}{2n+1}$$

Therefore,

$$\sum_{n=0}^{\infty}\{\int_{-1}^{1}[P_n^2(x)]dx\}h^{2n}=\sum_{n=0}^{\infty}\tfrac{2}{2n+1}h^{2n}$$

Equating the coefficient of h^{2n} from both sides, we get

$$\int_{-1}^{1}[P_n^2(x)]dx = \frac{2}{2n+1}.$$

Thus, $\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}$ if $m = n$.

Problem: Show that $\int_{-1}^{1} x^2 P_{n+1}(x)P_{n-1}(x)dx = \frac{2n(n+1)}{(2n+3)(4n^2-1)}$.

Proof: We have $(2n-1)xP_{n-1}(x) = nP_n(x) + (n-1)P_{n-2}(x)$ (1)
(Recurrence Formula I)

Replacing n by $n+2$, we get

$$(2n+3)xP_{n+1}(x) = (n+2)P_{n+2}(x) + (n+1)P_n(x) \quad (2)$$

Multiplying (1) and (2), we get
$(2n-1)(2n+3)x^2 P_{n+1}(x)P_{n-1}(x)$

$= n(n+2)P_n(x)P_{n+2}(x) + n(n+1)P_n^2(x) + (n-1)(n+2)P_{n-2}(x)P_{n+2}(x) + (n^2-1)P_{n-2}(x)P_n(x)$

Integrating both sides with respect to x from -1 to 1 and using Orthogonal Properties (A) and (B), we get

$(2n-1)(2n+3)\int_{-1}^{1} x^2 P_{n+1}(x)P_{n-1}(x)dx = 0 + n(n+1) \times \frac{2}{2n+1} + 0 + 0$

Hence, $\int_{-1}^{1} x^2 P_{n+1}(x)P_{n-1}(x)dx = \frac{2n(n+1)}{(2n+3)(4n^2-1)}$.

Exercises: Prove the following results:

1. $\int_{-1}^{1} xP_n(x)P_{n-1}(x)dx = \frac{2n}{4n^2-1}$.

2. $\int_{-1}^{1} x^2 P_n^2(x)dx = \frac{1}{8(2n-1)} + \frac{3}{4(2n+1)} + \frac{1}{8(2n+3)}$.

3. $\int_{-1}^{1} xP_n(x)P_n'(x)dx = \frac{2n}{2n+1}$.

Rodrigue's Formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

Proof: By definition of Legendre polynomial, we have

$$P_n(x) = \sum_{r=0}^{[\frac{n}{2}]}(-1)^r \frac{(2n-2r)!}{2^n r!(n-r)!(n-2r)!}x^{n-2r} \quad (1)$$

Now, $(x^2-1)^n$ can be expressed by binomial expansion as

$$(x^2-1)^n = \sum_{r=0}^{n} {}^nC_r (x^2)^{n-r}(-1)^r = \sum_{r=0}^{n} {}^nC_r (-1)^r x^{2n-2r}$$

Therefore, $\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n = \frac{1}{2^n n!}\sum_{r=0}^{n} {}^nC_r(-1)^r \frac{d^n}{dx^n}x^{2n-2r}$ (2)

But $\frac{d^n}{dx^n}(x^m) = 0$ if $m < n$ and $\frac{d^n}{dx^n}(x^m) = \frac{m!}{(m-n)!}x^{m-n}$ if $m \geq n$ (3)

So that $\frac{d^n}{dx^n}(x^{2n-2r}) = 0$ if $2n-2r < n$, i.e., if $r > \frac{n}{2}$ (4)

Using (4) in (2), we see that we must replace $\sum\limits_{r=0}^{n}$ by $\sum\limits_{r=0}^{\frac{n}{2}}$ if n is even and by $\sum\limits_{r=0}^{\frac{n-1}{2}}$ if n is odd, i.e., we must replace $\sum\limits_{r=0}^{n}$ by $\sum\limits_{r=0}^{[\frac{n}{2}]}$.

Hence, (2) becomes,

$$\begin{aligned}
\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n &= \frac{1}{2^n n!}\sum_{r=0}^{[\frac{n}{2}]} {}^nC_r(-1)^r \frac{d^n}{dx^n}(x^{2n-2r}) \\
&= \frac{1}{2^n n!}\sum_{r=0}^{[\frac{n}{2}]} {}^nC_r(-1)^r \frac{(2n-2r)!}{(2n-2r-n)!}x^{2n-2r-n}, \text{ by (3)} \\
&= \sum_{r=0}^{[\frac{n}{2}]} \frac{1}{2^n n!}\frac{n!}{r!(n-r)!}(-1)^r\frac{(2n-2r)!}{(n-2r)!}x^{n-2r} \\
&= P_n(x), \text{ by (1)}.
\end{aligned}$$

Hence, $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$.

Problem

Use Rodrigue's formula to find $P_0(x), P_1(x), P_2(x)$ and $P_3(x)$.

Solution: We have the Rodrigue's formula $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$ $\qquad (\star)$

Putting $n=0$ in $(\star)$, we get $P_0(x) = \frac{1}{2^0 0!}(x^2-1)^0 = 1$

Putting $n=1$ in $(\star)$, we get $P_1(x) = \frac{1}{2^1 1!}\frac{d}{dx}(x^2-1)^1 = \frac{1}{2}(2x) = x$

Putting $n=2$ in $(\star)$, we get $P_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d}{dx}[2(x^2-1)(2x)]$

$$= \frac{1}{2}\frac{d}{dx}(x^3-x) = \frac{1}{2}(3x^2-1)$$

Putting $n=3$ in $(\star)$, we get $P_3(x) = \frac{1}{2^3 3!}\frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{48}\frac{d^2}{dx^2}[3(x^2-1)^2(2x)]$

$$\begin{aligned}
&= \frac{1}{8}\frac{d}{dx}[\frac{d}{dx}(x^5-2x^3+x)] = \frac{1}{8}\frac{d}{dx}(5x^4-6x^2+1) \\
&= \frac{1}{8}(20x^3-12x) = \frac{1}{2}(5x^3-3x).
\end{aligned}$$

Problem

Express $f(x) = 4x^3+6x^2+7x+2$ in terms of Legendre polynomials.

Solution: Let $f(x) = 4x^3+6x^2+7x+2 = aP_3(x)+bP_2(x)+cP_1(x)+dP_0(x)$

$$\begin{aligned}
&= \frac{a}{2}(5x^3-3x)+\frac{b}{2}(3x^2-1)+cx+d \\
&= \frac{5a}{2}x^3+\frac{3b}{2}x^2+(c-\frac{3a}{2})x-\frac{b}{2}+d
\end{aligned}$$

Equating the coefficient of the corresponding powers of x, we get

$$4 = \frac{5a}{2}, \;\; 6 = \frac{3b}{2}, \;\; 7 = c-\frac{3a}{2}, \;\; 2 = -\frac{b}{2}+d$$

$$\Rightarrow a = \frac{8}{5}, \;\; b = 4, \;\; c = \frac{47}{5}, \;\; d = 4.$$