What is a DIFFERENTIAL EQUATION??

2 First-Order Differential Equations

2.1 Solution Curves Without a Solution

2.1.1 Direction Fields

2.1.2 Autonomous First-Order DEs

The Geometric Meaning of Differential Equations // Slope Fields, Integral Curves & Isoclines

In Problems 5–12 use computer software to obtain a direction field for the given differential equation. By hand, sketch an approximate solution curve passing through each of the given points.

9. $\dfrac{dy}{dx} = 0.2x^2 + y$

(a) $y(0) = \frac{1}{2}$

(b) $y(2) = -1$

10. $\dfrac{dy}{dx} = xe^y$

(a) $y(0) = -2$

(b) $y(1) = 2.5$

Autonomous Equations, Equilibrium Solutions, and Stability

20. Consider the autonomous first-orde differential equation $dy/dx = y^2 - y^4$ and the initial condition $y(0) = y_0$. By hand, sketch the graph of a typical solution $y(x)$ when y_0 has the given values.

(a) $y_0 > 1$

(b) $0 \quad y_0 \quad 1$

(c) $-1 \quad y_0 \quad 0$

(d) $y_0 < -1$

The Logistic Growth Differential Equation

2.2 Separable Equations

Separation of Variables

EXAMPLE 3 **Losing a Solution**

Solve $\dfrac{dy}{dx} = y^2 - 4$.

EXAMPLE 4 **An Initial-Value Problem**

Solve $(e^{2y} - y) \cos x \dfrac{dy}{dx} = e^y \sin 2x, \quad y(0) = 0$.

EXAMPLE 1 **Bacterial Growth**

A culture initially has P_0 number of bacteria. At $t = 1$ h the number of bacteria is measured to be $\frac{3}{2}P_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at time t, determine the time necessary for the number of bacteria to triple.

1. The population of a community is known to increase at a rate proportional to the number of people present at time t. If an initial population P_0 has doubled in 5 years, how long will it take to triple? To quadruple?

2. Suppose it is known that the population of the community in Problem 1 is 10,000 after 3 years. What was the initial population P_0? What will be the population in 10 years? How fast is the population growing at $t = 10$?

Newton's Law of Cooling

Newton's Law of Cooling/Warming In equation (3) of Section 1.3 we saw that the mathematical formulation of Newton's empirical law of cooling/warming of an object is given by the linear first-order di ferential equation

$$\frac{dT}{dt} = k(T - T_m), \tag{2}$$

where k is a constant of proportionality, $T(t)$ is the temperature of the object for $t > 0$, and T_m is the ambient temperature—that is, the temperature of the medium around the object. In Example 4 we assume that T_m is constant.

EXAMPLE 4 Cooling of a Cake

When a cake is removed from an oven, its temperature is measured at 300° F. Three minutes later its temperature is 200° F. How long will it take for the cake to cool off to a room temperature of 70° F?

14. A thermometer is taken from an inside room to the outside, where the air temperature is 5° F. After 1 minute the thermometer reads 55° F, and after 5 minutes it reads 30° F. What is the initial temperature of the inside room?

15. A small metal bar, whose initial temperature was 20° C, is dropped into a large container of boiling water. How long will it take the bar to reach 90° C if it is known that its temperature increases 2° in 1 second? How long will it take the bar to reach 98° C?

2.3 Linear Equations

DEFINITION 2.3.1 Linear Equation

A first-order di ferential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \tag{1}$$

is said to be a linear equation in the variable y.

$$\frac{dy}{dx} + P(x)y = f(x).$$

$$\mu(x) = e^{\int P(x)dx}$$

is called an **integrating factor** for equation (2).

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y = e^{\int P(x)dx}f(x)$$

$$\frac{d}{dx}\left[e^{\int P(x)dx}y\right] = e^{\int P(x)dx}f(x).$$

[Linear Differential Equations & the Method of Integrating Factors]

[The Method of Integrating Factors for Linear 1st Order ODEs **full example**]

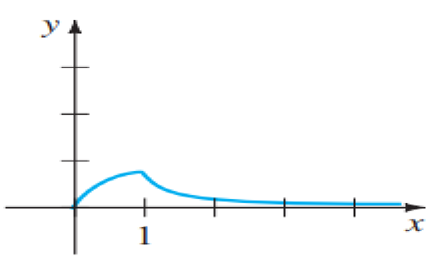
EXAMPLE 3 General Solution

Solve $x\dfrac{dy}{dx} - 4y = x^6e^x$.

EXAMPLE 6 An Initial-Value Problem

Solve $\dfrac{dy}{dx} + y = f(x)$, $y(0) = 0$ where $f(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & x > 1. \end{cases}$

$$y = \begin{cases} 1 - e^{-x}, & 0 \le x \le 1, \\ (e-1)e^{-x}, & x > 1 \end{cases}$$



33. $(x + 1)\dfrac{dy}{dx} + y = \ln x, \quad y(1) = 10$

34. $x(x + 1)\dfrac{dy}{dx} + xy = 1, \quad y(e) = 1$

35. $y' - (\sin x)y = 2 \sin x, \quad y(\ \ /2) = 1$

36. $y' + (\tan x)y = \cos^2 x, \quad y(0) = -1$

40. $(1 + x^2)\dfrac{dy}{dx} + 2xy = f(x), y(0) = 0$, where

$$f(x) = \begin{cases} x, & 0 \leq x \quad 1 \\ -x, & x \geq 1 \end{cases}$$

53. Heart Pacemaker A heart pacemaker consists of a switch, a battery of constant voltage E_0, a capacitor with constant capacitance C, and the heart as a resistor with constant resistance R. When the switch is closed, the capacitor charges; when the switch is open, the capacitor discharges, sending an electrical stimulus to the heart. During the time the heart is being stimulated, the voltage E across the heart satisfies the linear differential equation

$$\frac{dE}{dt} = -\frac{1}{RC}E.$$

Solve the DE, subject to $E(4) = E_0$.

2.4 Exact Equations

Differential of a Function of Two Variables If $z = f(x, y)$ is a function of two variables with continuous first partial derivatives in a region R of the xy-plane, then its differential is

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy. \tag{1}$$

In the special case when $f(x, y) = c$, where c is a constant, then (1) implies

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0. \tag{2}$$

DEFINITION 2.4.1 Exact Equation

A differential expression $M(x, y)\,dx + N(x, y)\,dy$ is an **exact differential** in a region R of the xy-plane if it corresponds to the differential of some function $f(x, y)$ defined in R. A first-order di ferential equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is said to be an **exact equation** if the expression on the left-hand side is an exact differential.

THEOREM 2.4.1 Criterion for an Exact Differential

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a \quad x \quad b$, $c \quad y \quad d$. Then a necessary and sufficient condition that $M(x, y)\,dx + N(x, y)\,dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{4}$$

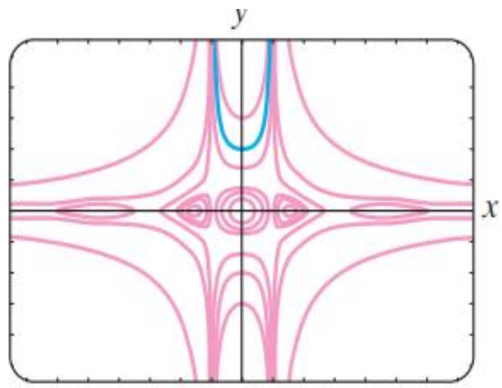
EXAMPLE 2 Solving an Exact DE

Solve $(e^{2y} - y\cos xy)\,dx + (2xe^{2y} - x\cos xy + 2y)\,dy = 0$.

$$xe^{2y} - \sin xy + y^2 + c = 0.$$

EXAMPLE 3 **An Initial-Value Problem**

Solve $\dfrac{dy}{dx} = \dfrac{xy^2 - \cos x \sin x}{y(1 - x^2)}, \quad y(0) = 2.$



$y^2(1 - x^2) - \cos^2 x = 3.$

Exact Differential Equations

We summarize the results for the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0. \qquad (12)$$

- If $(M_y - N_x)/N$ is a function of x alone, then an integrating factor for (12) is

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}. \qquad (13)$$

- If $(N_x - M_y)/M$ is a function of y alone, then an integrating factor for (12) is

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}. \qquad (14)$$

EXAMPLE 4 **A Nonexact DE Made Exact**

The nonlinear first-order di ferential equation

$$xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$$

is not exact. With the identifications $M = xy$, $N = 2x^2 + 3y^2 - 20$, we find the partial derivatives $M_y = x$ and $N_x = 4x$. The first quotient from (13) gets us nowhere, since

The integrating factor is then $e^{\int 3dy/y} = e^{3\ln y} = e^{\ln y^3} = y^3$. After we multiply the given DE by $\mu(y) = y^3$, the resulting equation is

$$xy^4\,dx + (2x^2y^3 + 3y^5 - 20y^3)\,dy = 0.$$

You should verify that the last equation is now exact as well as show, using the method of this section, that a family of solutions is $\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = c$.

In Problems 1–20 determine whether the given differential equation is exact. If it is exact, solve it.

18. $(2y \sin x \cos x - y + 2y^2 e^{xy^2})\, dx$

$$= (x - \sin^2 x - 4xye^{xy^2})\, dy$$

25. $(y^2 \cos x - 3x^2 y - 2x)\, dx$
$+ (2y \sin x - x^3 + \ln y)\, dy = 0, \quad y(0) = e$

In Problems 27 and 28 find the value of k so that the given differential equation is exact.

27. $(y^3 + kxy^4 - 2x)\, dx + (3xy^2 + 20x^2y^3)\, dy = 0$

28. $(6xy^3 + \cos y)\, dx + (2kx^2y^2 - x \sin y)\, dy = 0$

In Problems 37 and 38 solve the given initial-value problem by finding as in Example 4, an appropriate integrating factor.

37. $x\, dx + (x^2 y + 4y)\, dy = 0, \quad y(4) = 0$

38. $(x^2 + y^2 - 5)\, dx = (y + xy)\, dy, \quad y(0) = 1$

2.5 Solutions by Substitutions

Homogeneous Differential Equations

Homogeneous Equations If a function f possesses the property $f(tx, ty) = t^{\alpha} f(x, y)$ for some real number α, then f is said to be a **homogeneous function** of degree α. For example, $f(x, y) = x^3 + y^3$ is a homogeneous function of degree 3, since

$$f(tx, ty) = (tx)^3 + (ty)^3 = t^3(x^3 + y^3) = t^3 f(x, y),$$

whereas $f(x, y) = x^3 + y^3 + 1$ is not homogeneous. A first-order DE in differential form

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{1}$$

is said to be **homogeneous*** if both coefficient functions M and N are homogeneous functions of the *same* degree. In other words, (1) is homogeneous if

$$M(tx, ty) = t^{\alpha} M(x, y) \qquad \text{and} \qquad N(tx, ty) = t^{\alpha} N(x, y).$$

EXAMPLE 1 Solving a Homogeneous DE

Solve $(x^2 + y^2)\,dx + (x^2 - xy)\,dy = 0$.

In Problems 1–10 solve the given differential equation by using an appropriate substitution.

9. $-y\,dx + \left(x + \sqrt{xy}\right)dy = 0$

10. $x\dfrac{dy}{dx} = y + \sqrt{x^2 - y^2}, \quad x > 0$

The Bernoulli Equation

Bernoulli's Equation The differential equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n, \tag{4}$$

where n is any real number, is called **Bernoulli's equation.** Note that for $n = 0$ and $n = 1$, equation (4) is linear. For $n \neq 0$ and $n \neq 1$ the substitution $u = y^{1-n}$ reduces any equation of form (4) to a linear equation.

EXAMPLE 2 Solving a Bernoulli DE

Solve $x\dfrac{dy}{dx} + y = x^2y^2$.

In Problems 21 and 22 solve the given initial-value problem.

21. $x^2\dfrac{dy}{dx} - 2xy = 3y^4, \quad y(1) = \frac{1}{2}$

22. $y^{1/2}\dfrac{dy}{dx} + y^{3/2} = 1, \quad y(0) = 4$

4 Higher-Order Differential Equations

4.3 Homogeneous Linear Equations with Constant Coefficients

How to Solve Constant Coefficient Homogeneous Differential Equations

Constant Coefficient ODEs: Real & Distinct vs Real & Repeated vs Complex Pair

EXAMPLE 1 **A Second Solution by Reduction of Order**

Given that $y_1 = e^x$ is a solution of $y'' - y = 0$ on the interval $(-\infty, \quad)$, use reduction of order to find a second solution y_2.

EXAMPLE 1 **Second-Order DEs**

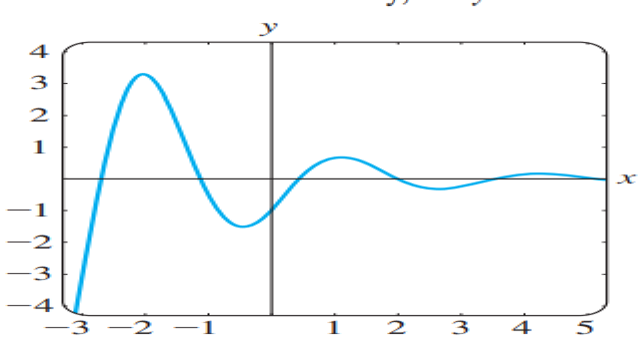
Solve the following differential equations.

(a) $2y'' - 5y' - 3y = 0$ **(b)** $y'' - 10y' + 25y = 0$ **(c)** $y'' + 4y' + 7y = 0$

EXAMPLE 2 **An Initial-Value Problem**

Solve $4y'' + 4y' + 17y = 0$, $y(0) = -1$, $y'(0) = 2$.

Hence the solution of the IVP is $y = e^{-x/2}(-\cos 2x + \frac{3}{4}\sin 2x)$. In Figure 4.3.1 we see that the solution is oscillatory, but $y \to 0$ as $x \to \quad$.



EXAMPLE 3 **Third-Order DE**

Solve $y''' + 3y'' - 4y = 0$.

EXAMPLE 4 **Fourth-Order DE**

Solve $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = 0$.

4.4 Undetermined Coefficients—Superposition Approach

Undetermined Coefficients: Solving non-homogeneous ODEs

EXAMPLE 1 **General Solution Using Undetermined Coefficient**

Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6$. (2)

EXAMPLE 2 **Particular Solution Using Undetermined Coefficient**

Find a particular solution of $y'' - y' + y = 2 \sin 3x$.

EXAMPLE 3 **Forming y_p by Superposition**

Solve $y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$.

EXAMPLE 4 **A Glitch in the Method**

Find a particular solution of $y'' - 5y' + 4y = 8e^x$.

EXAMPLE 6 **Forming y_p by Superposition—Case I**

Determine the form of a particular solution of

$$y'' - 9y' + 14y = 3x^2 - 5 \sin 2x + 7xe^{6x}.$$

EXAMPLE 7 **Particular Solution—Case II**

Find a particular solution of $y'' - 2y' + y = e^x$.

EXAMPLE 8 **An Initial-Value Problem**

Solve $y'' + y = 4x + 10 \sin x$, $y(\pi) = 0$, $y'(\pi) = 2$.

EXAMPLE 9 **Using the Multiplication Rule**

Solve $y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$.

EXAMPLE 10 **Third-Order DE—Case I**

Solve $y''' + y'' = e^x \cos x$.

4.5 Undetermined Coefficients—Annihilator Approach

How to use the Annihilator Method to Solve a Differential Equation

EXAMPLE 1 **Annihilator Operators**

Find a differential operator that annihilates the given function.

(a) $1 - 5x^2 + 8x^3$ **(b)** e^{-3x} **(c)** $4e^{2x} - 10xe^{2x}$

EXAMPLE 2 **Annihilator Operator**

Find a differential operator that annihilates $5e^{-x} \cos 2x - 9e^{-x} \sin 2x$.

EXAMPLE 3 **General Solution Using Undetermined Coefficient**

Solve $y'' + 3y' + 2y = 4x^2$. (9)

EXAMPLE 4 **General Solution Using Undetermined Coefficient**

Solve $y'' - 3y' = 8e^{3x} + 4 \sin x$.

EXAMPLE 5 **General Solution Using Undetermined Coefficient**

Solve $y'' + y = x \cos x - \cos x$.

4.6 Variation of Parameters

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y = u_1(x)y_1(x) + u_2(x)y_2(x)$$

$$u_1' = \frac{W_1}{W} = -\frac{y_2 f(x)}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W} = \frac{y_1 f(x)}{W}, \tag{9}$$

where
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}. \tag{10}$$

Variation of Parameters || How to solve non-homogeneous ODEs

EXAMPLE 1 General Solution Using Variation of Parameters

Solve $y'' - 4y' + 4y = (x + 1)e^{2x}$.

EXAMPLE 2 General Solution Using Variation of Parameters

Solve $4y'' + 36y = \csc 3x$.

EXAMPLE 3 General Solution Using Variation of Parameters

Solve $y'' - y = \dfrac{1}{x}$.

4.7 Cauchy-Euler Equation

Cauchy-Euler Equation A linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1 x\frac{dy}{dx} + a_0 y = g(x),$$

Cauchy Euler Differential Equation (equidimensional equation)

EXAMPLE 1 Distinct Roots

Solve $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 4y = 0$.

EXAMPLE 2 Repeated Roots

Solve $4x^2 \frac{d^2y}{dx^2} + 8x \frac{dy}{dx} + y = 0$.

EXAMPLE 3 An Initial-Value Problem

Solve $4x^2y'' + 17y = 0$, $y(1) = -1$, $y'(1) = -\frac{1}{2}$.

EXAMPLE 5 Variation of Parameters

Solve $x^2y'' - 3xy' + 3y = 2x^4e^x$.

23. $x^2y'' + xy' - y = \ln x$ **24.** $x^2y'' + xy' - y = \dfrac{1}{x+1}$

30. $x^2y'' - 5xy' + 8y = 8x^6$, $y\left(\frac{1}{2}\right) = 0$, $y'\left(\frac{1}{2}\right) = 0$

Solving…Cauchy's Linear Equations –by converting to constant coefficients

EXAMPLE 6 Changing to Constant Coefficient

Solve $x^2y'' - xy' + y = \ln x$.

In Problems 31–36 use the substitution $x = e^t$ to transform the given Cauchy-Euler equation to a differential equation with constant coefficients. Solve the original equation

34. $x^2y'' - 4xy' + 6y = \ln x^2$

35. $x^2y'' - 3xy' + 13y = 4 + 3x$

36. $x^3y''' - 3x^2y'' + 6xy' - 6y = 3 + \ln x^3$

4.10 NONLINEAR DIFFERENTIAL EQUATIONS

EXAMPLE 1 Dependent Variable *y* Is Missing

Solve $y'' = 2x(y')^2$.

SOLUTION If we let $u = y'$, then $du/dx = y''$. After substituting, the second-order equation reduces to a first-order equation with separable variables; the independent variable is x and the dependent variable is u:

$$\frac{du}{dx} = 2xu^2 \quad \text{or} \quad \frac{du}{u^2} = 2x\,dx$$

$$\int u^{-2}\,du = \int 2x\,dx$$

$$-u^{-1} = x^2 + c_1^2.$$

The constant of integration is written as c_1^2 for convenience. The reason should be obvious in the next few steps. Because $u^{-1} = 1/y'$, it follows that

$$\frac{dy}{dx} = -\frac{1}{x^2 + c_1^2},$$

and so $$y = -\int \frac{dx}{x^2 + c_1^2} \quad \text{or} \quad y = -\frac{1}{c_1}\tan^{-1}\frac{x}{c_1} + c_2.$$

EXAMPLE 2 Independent Variable *x* Is Missing

Solve $yy'' = (y')^2$.

SOLUTION With the aid of $u = y'$, the Chain Rule shown above, and separation of variables, the given differential equation becomes

$$y\left(u\frac{du}{dy}\right) = u^2 \quad \text{or} \quad \frac{du}{u} = \frac{dy}{y}.$$

Integrating the last equation then yields $\ln|u| = \ln|y| + c_1$, which, in turn, gives $u = c_2 y$, where the constant $\pm e^{c_1}$ has been relabeled as c_2. We now resubstitute $u = dy/dx$, separate variables once again, integrate, and relabel constants a second time:

$$\int \frac{dy}{y} = c_2\int dx \quad \text{or} \quad \ln|y| = c_2x + c_3 \quad \text{or} \quad y = c_4e^{c_2x}.$$

EXERCISES 4.10

In Problems 3–8 solve the given differential equation by using the substitution $u = y'$.

3. $y'' + (y')^2 + 1 = 0$
4. $y'' = 1 + (y')^2$
5. $x^2y'' + (y')^2 = 0$
6. $(y + 1)y'' = (y')^2$
7. $y'' + 2y(y')^3 = 0$
8. $y^2y'' = y'$

In Problems 9 and 10 solve the given initial-value problem.

9. $2y'y'' = 1,\ y(0) = 2,\ y'(0) = 1$
10. $y'' + x(y')^2 = 0,\ y(1) = 4,\ y'(1) = 2$

In Problems 13 and 14 show that the substitution $u = y'$ leads to a Bernoulli equation. Solve this equation (see Section 2.5).

13. $xy'' = y' + (y')^3$
14. $xy'' = y' + x(y')^2$

METHOD OF FACTORISATION OF DIFFERENTIAL OPERATOR (ODE)

Method of operational factor how to solve second order differential equation (part-4)

5 Modeling with Higher-Order Differential Equations

Mass Spring System

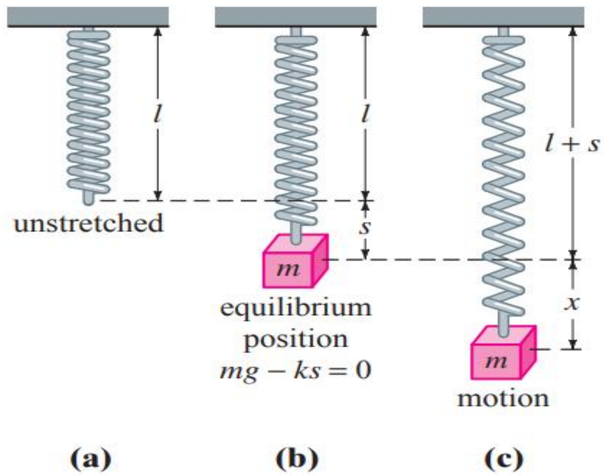


FIGURE 5.1.1 Spring/mass system

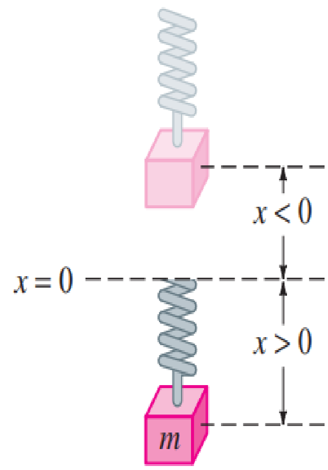


FIGURE 5.1.2 Direction below the equilibrium position is positive.

$$m\frac{d^2x}{dt^2} = -k(s + x) + mg = -\,kx + \underbrace{mg - ks}_{\text{zero}} = -kx.$$

5.1.1 Spring/Mass Systems: Free Undamped Motion

Modeling of Undamped Mass Spring system

EXAMPLE 1 Free Undamped Motion

A mass weighing 2 pounds stretches a spring 6 inches. At $t = 0$ the mass is released from a point 8 inches below the equilibrium position with an upward velocity of $\frac{4}{3}$ ft/s. Determine the equation of motion.

$$\frac{d^2x}{dt^2} + 64x = 0.$$

The initial displacement and initial velocity are $x(0) = \frac{2}{3}$, $x'(0) = -\frac{4}{3}$,

$$x(t) = \frac{2}{3}\cos 8t - \frac{1}{6}\sin 8t.$$

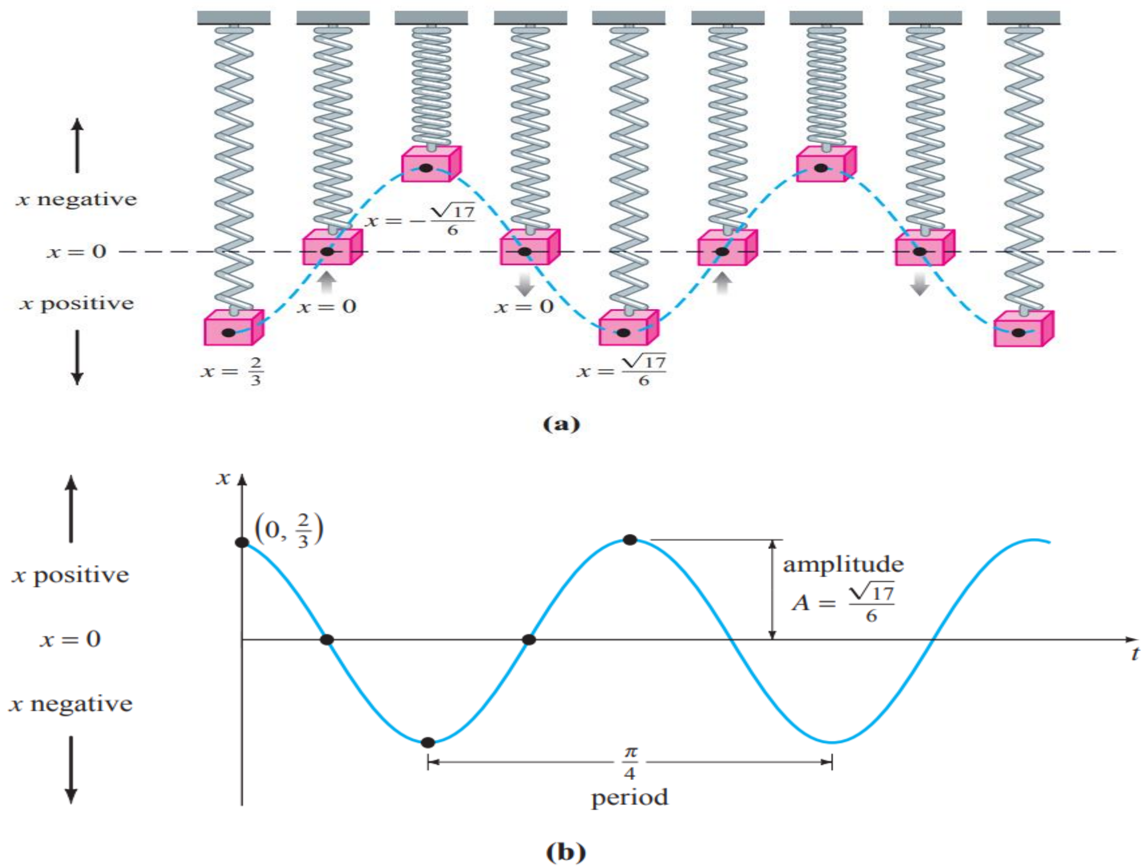


FIGURE 5.1.4 Simple harmonic motion

11. A mass weighing 64 pounds stretches a spring 0.32 foot. The mass is initially released from a point 8 inches above the equilibrium position with a downward velocity of 5 ft/s.

(a) Find the equation of motion.

(b) What are the amplitude and period of motion?

(c) How many complete cycles will the mass have completed at the end of 3π seconds?

(d) At what time does the mass pass through the equilibrium position heading downward for the second time?

(e) At what times does the mass attain its extreme displacements on either side of the equilibrium position?

(f) What is the position of the mass at $t = 3$ s?

(g) What is the instantaneous velocity at $t = 3$ s?

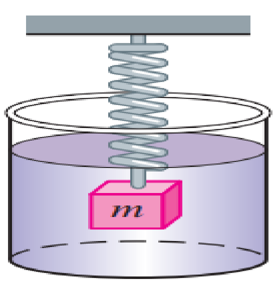
(h) What is the acceleration at $t = 3$ s?

(i) What is the instantaneous velocity at the times when the mass passes through the equilibrium position?

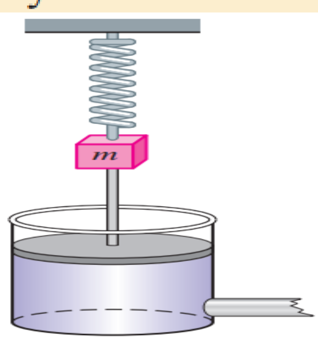
(j) At what times is the mass 5 inches below the equilibrium position?

(k) At what times is the mass 5 inches below the equilibrium position heading in the upward direction?

5.1.2 Spring/Mass Systems: Free Damped Motion



(a)



(b)

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt},$$

Modeling and idea on Damped Mass Spring system

25. A force of 2 pounds stretches a spring 1 foot. A mass weighing 3.2 pounds is attached to the spring, and the system is then immersed in a medium that offers a damping force that is numerically equal to 0.4 times the instantaneous velocity.

(a) Find the equation of motion if the mass is initially released from rest from a point 1 foot above the equilibrium position.

(b) Express the equation of motion in the form given in (23).

(c) Find the first time at which the mass passes through the equilibrium position heading upward.

26. After a mass weighing 10 pounds is attached to a 5-foot spring, the spring measures 7 feet. This mass is removed and replaced with another mass that weighs 8 pounds. The entire system is placed in a medium that offers a damping force that is numerically equal to the instantaneous velocity.

(a) Find the equation of motion if the mass is initially released from a point $\frac{1}{2}$ foot below the equilibrium position with a downward velocity of 1 ft/s.

(b) Express the equation of motion in the form given in (23).

(c) Find the times at which the mass passes through the equilibrium position heading downward.

(d) Graph the equation of motion.

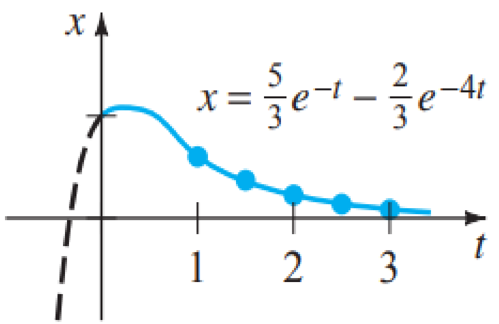
EXAMPLE 3 Overdamped Motion

It is readily verified that the solution of the initial-value proble

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x = 0, \quad x(0) = 1, \quad x'(0) = 1$$

is

$$x(t) = \frac{5}{3}e^{-t} - \frac{2}{3}e^{-4t}.$$

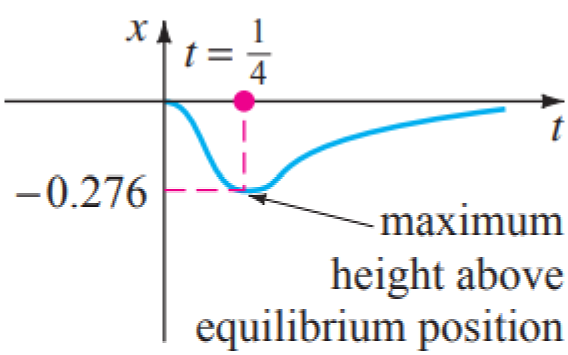


t	$x(t)$
1	0.601
1.5	0.370
2	0.225
2.5	0.137
3	0.083

EXAMPLE 4 Critically Damped Motion

A mass weighing 8 pounds stretches a spring 2 feet. Assuming that a damping force numerically equal to 2 times the instantaneous velocity acts on the system, determine the equation of motion if the mass is initially released from the equilibrium position with an upward velocity of 3 ft/s.

$$\frac{1}{4}\frac{d^2x}{dt^2} = -4x - 2\frac{dx}{dt} \quad x(0) = 0 \text{ and } x'(0) = -3,$$



EXAMPLE 5 Underdamped Motion

A mass weighing 16 pounds is attached to a 5-foot-long spring. At equilibrium the spring measures 8.2 feet. If the mass is initially released from rest at a point 2 feet above the equilibrium position, find the displacements $x(t)$ if it is further known that the surrounding medium offers a resistance numerically equal to the instantaneous velocity.

$$\frac{1}{2}\frac{d^2x}{dt^2} = -5x - \frac{dx}{dt} \qquad x(0) = -2 \text{ and } x'(0) = 0$$

$$x(t) = e^{-t}\left(-2\cos 3t - \frac{2}{3}\sin 3t\right).$$

5.1.3 Spring/Mass Systems: Driven Motion

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} + f(t).$$

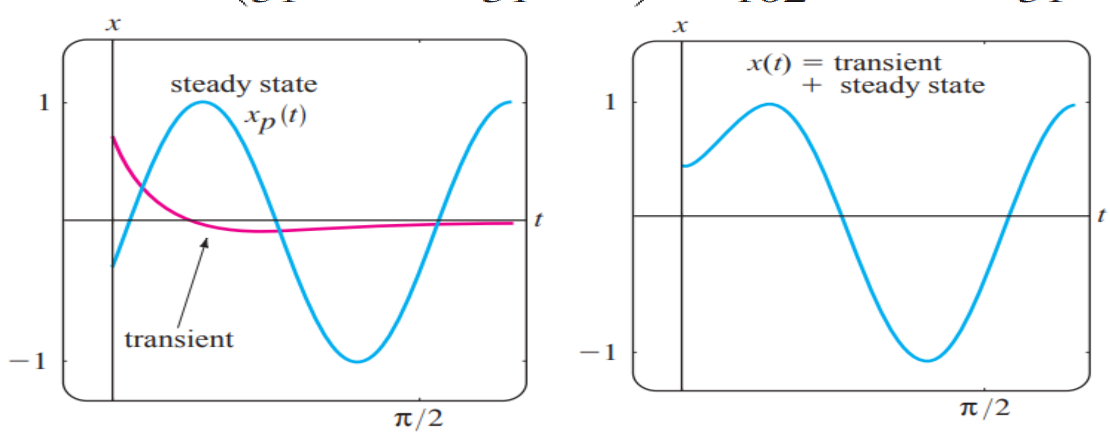
Spring / Mass Systems: Driven Motion Example

EXAMPLE 6 Interpretation of an Initial-Value Problem

Interpret and solve the initial-value problem

$$\frac{1}{5}\frac{d^2x}{dt^2} + 1.2\frac{dx}{dt} + 2x = 5\cos 4t, \quad x(0) = \frac{1}{2}, \quad x'(0) = 0.$$

$$x(t) = e^{-3t}\left(\frac{38}{51}\cos t - \frac{86}{51}\sin t\right) - \frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t.$$



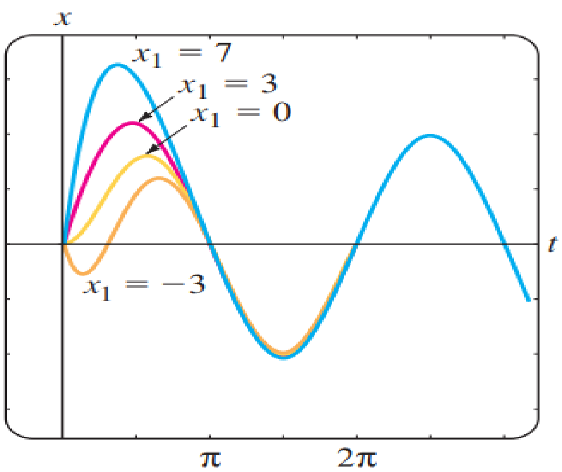
EXAMPLE 7 Transient/Steady-State Solutions

The solution of the initial-value problem

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 4\cos t + 2\sin t, \quad x(0) = 0, \quad x'(0) = x_1,$$

where x_1 is constant, is given by

$$x(t) = (x_1 - 2)\underbrace{e^{-t}\sin t}_{\text{transient}} + \underbrace{2\sin t}_{\text{steady-state}}.$$



31. A mass of 1 slug, when attached to a spring, stretches it 2 feet and then comes to rest in the equilibrium position. Starting at $t = 0$, an external force equal to $f(t) = 8\sin 4t$ is applied to the system. Find the equation of motion if the surrounding medium offers a damping force that is numerically equal to 8 times the instantaneous velocity.

32. In Problem 31 determine the equation of motion if the external force is $f(t) = e^{-t}\sin 4t$. Analyze the displacements for $t \longrightarrow \infty$.

33. When a mass of 2 kilograms is attached to a spring whose constant is 32 N/m, it comes to rest in the equilibrium position. Starting at $t = 0$, a force equal to $f(t) = 68e^{-2t}\cos 4t$ is applied to the system. Find the equation of motion in the absence of damping.

5.1.4 SERIES CIRCUIT ANALOGUE

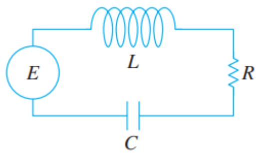


FIGURE 5.1.15 *LRC*-series circuit

***LRC*-Series Circuits** As was mentioned in the introduction to this chapter, many different physical systems can be described by a linear second-order differential equation similar to the differential equation of forced motion with damping:

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = f(t). \tag{32}$$

If $i(t)$ denotes current in the ***LRC*-series electrical circuit** shown in Figure 5.1.15, then the voltage drops across the inductor, resistor, and capacitor are as shown in Figure 1.3.4. By Kirchhoff's second law the sum of these voltages equals the voltage $E(t)$ impressed on the circuit; that is,

$$L\frac{di}{dt} + Ri + \frac{1}{C}q = E(t). \tag{33}$$

But the charge $q(t)$ on the capacitor is related to the current $i(t)$ by $i = dq/dt$, so (33) becomes the linear second-order differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t). \tag{34}$$

RLC Circuit Differential Equation

EXAMPLE 9 Underdamped Series Circuit

Find the charge $q(t)$ on the capacitor in an *LRC*-series circuit when $L = 0.25$ henry (h), $R = 10$ ohms (Ω), $C = 0.001$ farad (f), $E(t) = 0$, $q(0) = q_0$ coulombs (C), and $i(0) = 0$.

EXAMPLE 10 Steady-State Current

Find the steady-state solution $q_p(t)$ and the **steady-state current** in an *LRC*-series circuit when the impressed voltage is $E(t) = E_0 \sin \gamma t$.