CHAPTER 7

PRINCIPLES OF INTEGRAL EVALUATION

7.1 AN OVERVIEW OF INTEGRATION METHODS

Methods for Approaching Integration Problems

A Review of Familiar Integration Formulas

References

Table 5.2.1 Integration Formulas (page 325, Howard Anton, 10th edition)

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x) = 1 \qquad \int dx = \int 1\, dx = x + c$$

$$\frac{d}{dx}(x^n) = nx^{n-1} \qquad \int x^n dx = \frac{x^{n+1}}{n+1} + c,\ \ n \neq -1; \qquad \int \frac{1}{x^n} dx = \frac{-1}{(n-1)x^{n-1}} + c,\ n \neq 1$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}} \qquad \int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + c$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e \qquad \int \frac{1}{x} dx = \ln|x| + c$$

$$\frac{d}{dx}(e^x) = e^x \qquad \int e^x dx = e^x + c$$

$$\frac{d}{dx}(e^{mx}) = me^{mx} \qquad \int e^{mx} dx = \frac{e^{mx}}{m} + c$$

$$\frac{d}{dx}(a^x) = a^x \ln a \qquad \int a^x dx = \frac{a^x}{\ln a} + c$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \int \cos x\, dx = \sin x + c$$

$$\frac{d}{dx}(\sin mx) = m\cos mx \qquad \int \sin mx\, dx = -\frac{1}{m}\cos mx + c$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \int \sin x\, dx = -\cos x + c$$

$$\frac{d}{dx}(\cos mx) = -m\sin mx \qquad \int \cos mx\, dx = \frac{1}{m}\sin mx + c$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \int \sec^2 x\, dx = \tan x + c$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \qquad \int \operatorname{cosec}^2 x\, dx = -\cot x + c$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \int \sec x \tan x\, dx = \sec x + c$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x \qquad \int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c$$

$$\frac{d}{dx}(\ln \sec x) = \tan x \qquad \int \tan x \, dx = \ln|\sec x| + c$$

$$\frac{d}{dx}(\log \sin x) = \cot x \qquad \int \cot x \, dx = \ln|\sin x| + c$$

$$\frac{d}{dx}[\ln(\sec x + \tan x)] = \sec x \qquad \int \sec x \, dx = \ln|\sec x + \tan x| + c$$

$$\frac{d}{dx}[\ln \tan(\frac{\pi}{4} + \frac{x}{2})] = \sec x \qquad \int \sec x \, dx = \ln\left|\tan(\frac{\pi}{4} + \frac{x}{2})\right| + c$$

$$\frac{d}{dx}\ln(\operatorname{cosec} x + \cot x) = \operatorname{cosec} x \qquad \int \operatorname{cosec} x \, dx = \ln|\operatorname{cosec} x - \cot x| + c$$

$$\frac{d}{dx}\ln \tan\frac{x}{2} = \operatorname{cosec} x \qquad \int \operatorname{cosec} x \, dx = \ln\left|\tan\frac{x}{2}\right| + c$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c, \ |x| < 1$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \qquad \int \frac{1}{\sqrt{1-x^2}} dx = -\cos^{-1} x + c, \ |x| < 1$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \int \frac{1}{1+x^2} dx = \tan^{-1} x + c$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2} \qquad \int \frac{-1}{1+x^2} dx = \cot^{-1} x + c$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}} \qquad \int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1}|x| + c$$

$$\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}} \qquad \int \frac{-1}{x\sqrt{x^2-1}} dx = \operatorname{cosec}^{-1} x + c$$

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \int \cosh x \, dx = \sinh x + c$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \int \sinh x \, dx = \cosh x + c$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \int \operatorname{sech}^2 x \, dx = \tanh x + c$$

$$\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x \qquad \int \operatorname{cosech}^2 x \, dx = -\coth x + c$$

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}} \qquad \int \frac{1}{\sqrt{1+x^2}} dx = \sinh^{-1} x + c$$

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}} \qquad \int \frac{1}{\sqrt{x^2-1}} dx = \cosh^{-1} x + c, \ x > 1$$

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2} \qquad \int \frac{1}{1-x^2} dx = \tanh^{-1} x + c, \ x < 1$$

$$\frac{d}{dx}\log f(x) = \frac{f'(x)}{f(x)} \qquad \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c \qquad \int f'(x)(f(x))^n dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

$$\frac{d}{dx}2\sqrt{f(x)}=\frac{f'(x)}{\sqrt{f(x)}} \qquad \int\frac{f'(x)}{\sqrt{f(x)}}dx=2\sqrt{f(x)}+c$$

$$\int\frac{1}{x^2-a^2}dx=\frac{1}{2a}\ln\frac{x-a}{x+a}+c \qquad \int\frac{1}{a^2-x^2}dx=\frac{1}{2a}\ln\frac{a+x}{a-x}+c$$

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$$\int\frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\frac{x}{a}+c$$

$$\frac{d}{dx}(\tan^{-1}\frac{x}{a})=\frac{a}{a^2+x^2} \qquad \int\frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}\frac{x}{a}+c$$

$$\int\frac{1}{x\sqrt{x^2-a^2}}dx=\frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right|+c$$

$$\int\frac{1}{\sqrt{x^2+a^2}}dx=\ln(x+\sqrt{x^2+a^2})+c=\sinh^{-1}\frac{x}{a}+c$$

$$\int\frac{1}{\sqrt{x^2-a^2}}dx=\ln(x+\sqrt{x^2-a^2})+c=\cosh^{-1}\frac{x}{a}+c$$

$$\int\sqrt{a^2-x^2}\,dx=\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c$$

$$\int\sqrt{x^2+a^2}\,dx=\frac{x\sqrt{x^2+a^2}}{2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a}+c$$

$$=(1/2)[x\sqrt{x^2+a^2}+a^2\ln\{x+\sqrt{x^2+a^2}\}]+c$$

$$\int\sqrt{x^2-a^2}\,dx=\frac{x\sqrt{x^2-a^2}}{2}+\frac{a^2}{2}\cosh^{-1}\frac{x}{a}+c$$

$$=(1/2)[x\sqrt{x^2-a^2}-a^2\ln\{x+\sqrt{x^2-a^2}\}]+c$$

$$\frac{d}{dx}[u(x)v(x)]=u(x)\frac{d}{dx}v(x)+v(x)\frac{d}{dx}u(x) \qquad \int uv\,dx=u\int v\,dx-\int\{\frac{du}{dx}\int v\,dx\}dx$$

$$\frac{d}{dx}[\frac{u(x)}{v(x)}]=\frac{v(x)\frac{d}{dx}u(x)-u(x)\frac{d}{dx}v(x)}{[v(x)]^2}$$

$$\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{\sqrt{a^2+b^2}}(a\cos bx+b\sin bx)+c$$

$$\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx-\tan^{-1}\frac{b}{a}\right)+c$$

$$\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{\sqrt{a^2+b^2}}(a\sin bx-b\cos bx)+c$$

$$\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)+c$$

$$\int e^x\{f(x)+f'(x)\}dx = e^x f(x)+c$$

Trigonometric Formulas

$\sin(x+y)=\sin x\cos y+\cos x\sin y$

$\sin(x-y)=\sin x\cos y+\cos x\sin y$

$\cos(x+y)=\cos\cos y-\sin x\sin y$

$\cos(x-y)=\cos\cos y+\sin x\sin y$

$\tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\tan y}$

$\sin^2 x+\cos^2 x=1$

$\sin 2x=2\sin x\cos x$

$\cos 2x=\cos^2 x-\sin^2 x$

$\cos 2x=2\cos^2 x-1$

$\cos 2x=1-2\sin^2 x$

$\sin^2 x=\dfrac{1}{2}(1-\cos 2x)$

$\tan^2 x=\dfrac{1-\cos 2x}{1+\cos 2x}$

$\cos 3x=4\cos^3 x-3\cos x$

$2\sin x\cos y=\sin(x+y)+\sin(x-y)$

$2\cos x\sin y=\sin(x+y)-\sin(x-y)$

$2\cos x\cos y=\cos(x+y)+\cos(x-y)$

$2\sin x\sin y=\cos(x-y)-\cos(x+y)$

$\tan(x-y)=\dfrac{\tan x-\tan y}{1+\tan x\tan y}$

$\sec^2 x=1+\tan^2 x$

$\mathrm{cosec}^2 x=1+\cot^2 x$

$\sin 2x=\dfrac{2\tan x}{1+\tan^2 x}$

$\cos 2x=\dfrac{1-\tan^2 x}{1+\tan^2 x}$

$\tan 2x=\dfrac{2\tan x}{1-\tan^2 x}$

$\cos^2 x=\dfrac{1}{2}(1+\cos 2x)$

$\sin 3x=3\sin x-4\sin^3 x$

$\tan 3x=\dfrac{3\tan x-\tan^3 x}{1-3\tan^2 x}$

$\sin 4x=2\sin 2x\cos 2x=4\sin x\cos x(2\cos^2 x-1)=8\cos^3 x\sin x-4\sin x\cos x$

$\sin 4x=4\sin x\cos x(1-2\sin^2 x)=4\sin x\cos x-8\sin^3 x\cos x$

Law of Cosines

For a triangle with sides a,b,c and opposite angles $A,B,C,$ respectively, we have

$$c^2=a^2+b^2-2ab\cos C$$
$$b^2=a^2+c^2-2ac\cos B$$
$$a^2=b^2+c^2-2bc\cos A$$

Heron's Formula

The area A of a triangle with sides a,b,c is

$$A=\sqrt{s(s-a)(s-b)(s-c)}$$

where $s=\dfrac{1}{2}(a+b+c)$.

Hyperbolic functions

These functions are the linear combinations of exponential functions e^x and e^{-x}. These are essential in problems like hanging telegraph cable. These types of functions are called *hyperbolic trigonometric* functions.

We have $\cosh x=\dfrac{e^x+e^{-x}}{2}$ and $\sinh x=\dfrac{e^x-e^{-x}}{2}$.

So that $\operatorname{sec}hx = \dfrac{2}{e^x + e^{-x}}$, $\operatorname{cos}echx = \dfrac{2}{e^x - e^{-x}}$, $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ and $\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

Basic properties

From the above definitions, we have

$$\cosh^2 x - \sinh^2 x = 1 \qquad (1)$$

It is known that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad (2)$$

Therefore, $\cos(ix) = \dfrac{e^{-x} + e^x}{2} = \cosh x$ and $\sin(ix) = \dfrac{e^x - e^{-x}}{2} = i \sinh x$ (3)

Similarly, we can show that

$$\cosh(ix) = \cos x \quad \text{and} \quad \sinh(ix) = i \sin x \qquad (4)$$

From the definition, it can easily be proved that

$$1 - \tanh^2 x = \operatorname{sec}h^2 x, \quad \coth^2 x = 1 - \operatorname{cos}ech^2 x,$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y,$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y,$$

$$\sinh(2x) = 2 \sinh x \cosh x, \qquad \cosh(2x) = \cosh^2 x + \sinh^2 x.$$

Relation between inverse hyperbolic and logarithmic functions

$$\sinh^{-1} x = \ln\left|x + \sqrt{x^2 + 1}\right|, \qquad \cosh^{-1} x = \ln\left|x + \sqrt{x^2 - 1}\right|,$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|, \quad \operatorname{sec}h^{-1} x = \ln\left|\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right|,$$

$$\operatorname{cos}ech^{-1} x = \ln\left|\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right|$$