Chapter 1
The Foundations : Logic and Proofs
Kenneth H. Rosen 7th edition

Section 1.6 :Rules of Inference

Rules of Inference

- Infer means:
  - To deduce or conclude (something) from evidence and reasoning rather than from explicit statements.
- Proofs in mathematics are
  - Valid arguments
  - These arguments establish the truth of mathematical statements.
- By an ***argument***, we mean a sequence of statements that end with a conclusion.
- By ***valid***, we mean that the conclusion, or final statement of the ***argument***, must follow from the truth of the preceding statements, or ***premises*** of the argument.
- An argument is valid if the truth of all its premises implies that the conclusion is true
  - i.e. the conclusion is true if the premises are all true.

Rules of Inference(Contd.)

- From the definition of a valid argument form we see that the argument form with premises $p_1, p_2, \ldots, p_n$ and conclusion q is valid if and only if $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$ is a tautology.
- The key to showing that an argument in propositional logic is valid is to show that its argument form is valid.

Rules of Inference(Contd.)

Rule of Inference	***Name***	***Rule of Inference***	***Name***
p $p \to q$ $\therefore\ q$	*Modus ponens*	p $\therefore\ p \lor q$	*Addition*
$\neg q$ $p \to q$ $\therefore\ \neg p$	*Modus tollens*	$p \land q$ $\therefore\ p$	*Simplification*
$p \to q$ $q \to r$ $\therefore\ p \to r$	*Hypothetical syllogism*	p q $\therefore\ p \land q$	*Conjunction*
$p \lor q$ $\neg p$ $\therefore\ q$	*Disjunctive syllogism*	$p \lor q$ $\neg p \lor r$ $\therefore\ q \lor r$	*Resolution*

Rules of Inference(Contd.)

- **Example 1:**

  Consider the following argument:

  - "If you have a current password, then you can log onto the network."
  - "You have a current password."
  - Therefore,
  - "You can log onto the network."

Rules of Inference(Contd.)

- **Solution:**
  - Let,

    $p =$ "$You\ have\ a\ current\ password$."

    $q =$ "$You\ can\ log\ onto\ the\ network$."

    Then, the argument has the form

$$\begin{array}{ll} & p \rightarrow q \\ & p \\ \hline \therefore & q \end{array}$$

Here, we can see that the argument uses the form of Modus Ponens rule. The argument is thus a valid one. Now if, both $p \rightarrow q$ and p are $true,$ then q must also be true.

Rules of Inference(Contd.)

- What if, you have two premises, $p \rightarrow q$ and p and the conclusion as q where, not both of the premises are true?

Rules of Inference(Contd.)

- **<u>Example 2:</u>**

  Consider the following argument:

  - "If you have access to the network, then you can change your grades."
  - "You have access to the network."
  - Therefore,
  - "You can change your grades."

Rules of Inference(Contd.)

- **Solution:**
  - Let,

$p =$ "$You\ have\ access\ to\ the\ network.$"

$q =$ "$You\ can\ change\ your\ grades.$"

Then, the argument has the form

$$\begin{array}{ll} & p \rightarrow q \\ & p \\ \hline \therefore & q \end{array}$$

Here, we can see that the argument uses the form of Modus Ponens rule. The argument is thus a valid one. Now , both $p \rightarrow q$ and q are not $true,$ namely the first one is $false$. Thus, we cannot conclude that q is $true$.

Rules of Inference(Contd.)

- **Example 3:**
- Determine whether the argument given here is valid and determine whether its conclusion must be true because of the validity of the argument.
  - If $\sqrt{2} > \frac{3}{2}$, then $\left(\sqrt{2}\right)^2 > \left(\frac{3}{2}\right)^2$. We know that $\sqrt{2} > \frac{3}{2}$. Consequently $\left(\sqrt{2}\right)^2 = 2 > \left(\frac{3}{2}\right)^2 = \frac{9}{4}$.

Rules of Inference(Contd.)

- **Solution:**
  - Let,

$$p = \sqrt{2} > \frac{3}{2}$$

$$q = \left(\sqrt{2}\right)^2 > \left(\frac{3}{2}\right)^2$$

The argument can be represented as

$$\begin{array}{ll} & p \to q \\ & p \\ \hline \therefore & q \end{array}$$

The argument is valid as it is constructed using modus ponens. But, we cannot conclude that the conclusion is true. Because, the premises p is $false$. Also by observation, we can see that the conclusion, q is also $false$.

Rules of Inference(Contd.)

- **Example 4:**
- State which rule of inference is the basis of the following argument:
  - $\text{“}It\ is\ below\ freezing\ now. Therefore, it\ is\ either\ below\ freezing\ or\ raining\ now.\text{”}$
- **Solution:**
  - Let,

  $p =$ “$It\ is\ below\ freezing\ now$.”

  $q =$ “$It\ is\ raining\ now$.”

  The argument can be represented as,

$$\frac{p}{\therefore \quad p \vee q}$$

  This argument uses the addition rule.

Rules of Inference(Contd.)

- **Example 5:**
- State which rule of inference is used in the argument:
  - $If\ it\ rains\ today, then\ we\ will\ not\ have\ a\ barbecue\ today. If\ we\ do\ not\ have\ a\ barbecue\ today, then\ we\ will\ have\ a\ barbecue\ tomorrow. Therefore, if\ it\ rains\ today, then\ we\ will\ have\ a\ barbecue\ tomorrow.$
- **Solution:**
  - Let,

    $p =$ "$It\ is\ raining\ today.$"

    $q =$ "$We\ will\ not\ have\ a\ berbecue\ today.$"

    $r =$ "$We\ will\ have\ a\ berbecue\ tomorrow.$"

Rules of Inference(Contd.)

The argument can be represented as,

$$\begin{array}{rl} & p \to q \\ & q \to r \\ \hline \therefore & p \to r \end{array}$$

The argument thus uses the hypothetical syllogism rule.

Using Rules of Inference to Build Arguments

- **Example 1:**
- Show that the premises
  - "It is not sunny this afternoon and it is colder than yesterday,"
  - "We will go swimming only if it is sunny,"
  - "If we do not go swimming, then we will take a canoe trip,"
  - "If we take a canoe trip, then we will be home by sunset"

Lead to the conclusion

  - "We will be home by sunset."

Using Rules of Inference to Build Arguments(Contd.)

- **Solution:**
  - Let,

    $p =$ "*It is sunny this afternoon.*"

    $q =$ "*It is colder than yesterday.*"

    $r =$ "*We will go swimming.*"

    $s =$ "*We will take a canoe trip.*"

    $t =$ "*We will be home by sunset.*"

    Then, the premises become,

    1. $\neg p \wedge q$
    2. $r \rightarrow p$
    3. $\neg r \rightarrow s$
    4. $s \rightarrow t$

    The conclusion is simply t.

Using Rules of Inference to Build Arguments(Contd.)

We construct an argument to show that our premises lead to the desired conclusion as follows,

Steps	***Reasons***
1. $\neg p \wedge q$	$premise$
2. $\neg p$	$Simplification\ using\ (1)$
3. $r \rightarrow p$	$premise$
4. $\neg r$	$Modulus\ tollens\ using\ (2) and\ (3)$
5. $\neg r \rightarrow s$	$premise$
6. s	$Modulus\ ponens\ using\ (4) and\ (5)$
7. $s \rightarrow t$	$premise$
8. t	$Modulus\ ponens\ using\ (6) and\ (7)$

Thus, we can see that our premises lead to the desired conclusion.

Using Rules of Inference to Build Arguments(Contd.)

- **Example 2:**
- Show that the premises
  - "If you send me an e-mail message, then I will finish writing the program."
  - "If you do not send me an e-mail message, then I will go to sleep early."
  - "If I go to sleep early, then I will wake up feeling refreshed."
- Lead to the conclusion
  - "If I do not finish writing the program, then I will wake up feeling refreshed."

Using Rules of Inference to Build Arguments(Contd.)

- **<u>Solution:</u>**
  - Let,

$p =$ "$You\ send\ me\ an\ e - mail\ message$."

$q =$ "$I\ will\ finish\ writing\ the\ program$."

$r =$ "$I\ will\ go\ to\ sleep\ early$."

$s =$ "$I\ will\ wake\ up\ feeling\ refreshed$."

Then the premises become,

1. $p \rightarrow q$
2. $\neg p \rightarrow r$
3. $r \rightarrow s$

The conclusion is $\neg q \rightarrow s$

Using Rules of Inference to Build Arguments(Contd.)

We construct an argument to show that our premises lead to the desired conclusion as follows,

Steps	***Reasons***
1. $p \rightarrow q$	*premise*
2. $\neg q \rightarrow \neg p$	*Contrapositive rule*
3. $\neg p \rightarrow r$	*premise*
4. $\neg q \rightarrow r$	*Hypothetical syllogism using* (3)
5. $r \rightarrow s$	*premise*
6. $\neg q \rightarrow s$	*Hypothetical syllogism using* (3)

Thus, we can see that our premises lead to the desired conclusion.

Resolution

- Resolution is nothing but a rule of inference based on the tautology,

$$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$$

- Using this resolution, we can derive rule of inference.
  - Let,

  $r = False.$

  Then the resolution becomes,

  $$\big((p \vee q) \wedge \neg p\big) \rightarrow q$$

  This is the same as disjunctive syllogism.

Resolution(Contd.)

- **Example 1:**
- Use resolution to show that the hypotheses
  - "Jasmine is skiing or it is not snowing"
  - "It is snowing or Bart is playing hockey"
- Imply that
  - "Jasmine is skiing or Bart is playing hockey."

Resolution(Contd.)

- **Solution:**
  - Let,

  $p =$ "*It is snowing*"

  $r =$ "*Jasmine is skiing*"

  $q =$ "*Bart is playing hockey*."

  The hypotheses can be represented as follows

  1. $\neg p \vee r$
  2. $p \vee q$

  The resolution suggests that

  $$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$$

  Thus, the hypotheses implies,

  $$(q \vee r)$$

Resolution(Contd.)

- **Example 2:**
- Show that the premises $(p \wedge q) \vee r$ and $r \rightarrow s$ imply the conclusion $p \vee s$.

Resolution(Contd.)

- **Solution:**
  - We can rewrite the premises $(p \land q) \lor r$ as two clauses, $p \lor r$ and $q \lor r$. We can also replace $r \rightarrow s$ by the equivalent clause $\neg r \lor s$. Using the two clauses $p \lor r$ and $\neg r \lor s$, we can use resolution to conclude $p \lor s$.

Rules of Inference for Quantifiers

- Like the rules of inference for propositions, we now we see the rules of inference for quantified statements.

Rules of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore P(c)}$	*Universal instantiation*
$\dfrac{P(c)\ \text{for an arbitrary } c}{\therefore \forall x P(x)}$	*Universal generalization*
$\dfrac{\exists x P(x)}{\therefore P(c)\ \text{for some element } c}$	*Existential instantiation*
$\dfrac{P(c)\ \text{for some element } c}{\therefore \exists x P(x)}$	*Existential generalization*

Rules of Inference for Quantifiers(Contd.)

- Universal Instantiation
  - $\boldsymbol{P(c)}$ is true, where $\boldsymbol{c}$ is a particular member of the domain, given the premise $\forall \boldsymbol{xP(x)}$.

- Universal Generalization
  - $\forall \boldsymbol{xP(x)}$ is true, given the premise that $\boldsymbol{P(c)}$is true for all elements c in the domain.
  - We show that $\forall \boldsymbol{xP(x)}$ is true by taking an arbitrary element $\boldsymbol{c}$ from the domain and showing that $\boldsymbol{P(c)}$ is true.
  - The element $\boldsymbol{c}$ that we select must be an arbitrary, and not a specific, element of the domain.

Rules of Inference for Quantifiers(Contd.)

- Existential Instantiation
  - Allows us to conclude that there is an element c in the domain for which $P(c)$ is true if we know that $\exists x P(x)$ is true.
  - We cannot select an arbitrary value of c here, but rather it must be a c for which $P(c)$ is true.

- Existential Generalization
  - Allows us to conclude that $\exists x P(x)$ is true when a particular element c with $P(c)$true is known.
  - That is, if we know one element c in the domain for which $P(c)$ is true, then we know that $\exists x P(x)$ is true.

Rules of Inference for Quantifiers(Contd.)

- **<u>Example 1:</u>**
- Show that the premises
  - "Everyone in this discrete mathematics class has taken a course in computer science"
  - "Marla is a student in this class"
- Imply the conclusion
  - "Marla has taken a course in computer science."

Rules of Inference for Quantifiers(Contd.)

- **Solution:**
  - Let,

$D(x) =$ " $x\ is\ in\ this\ Discrete\ Mathematics\ class$."

$C(x) =$ " $x\ has\ taken\ a\ course\ in\ Computer\ Science$."

Then, the premises can be represented as,

1. $\forall x(D(x) \rightarrow C(x))$
2. $D(Marla)$

The conclusion is simply, $C(Marla)$.

Rules of Inference for Quantifiers(Contd.)

We construct an argument to show that our premises lead to the desired conclusion as follows,

Steps	***Reasons***
1. $\forall x(D(x) \rightarrow C(x))$	*premise*
2. $D(Marla) \rightarrow C(Marla)$	$Universal\ Instantiation\ using\ (1)$
3. $D(Marla)$	*premise*
4. $C(Marla)$	$Modus\ ponens\ using\ (2) and\ (3)$

Thus, we can see that our premises lead to the desired conclusion.

Rules of Inference for Quantifiers(Contd.)

- **<u>Example 2:</u>**
- Show that the premises
  - “A student in this class has not read the book”
  - “Everyone in this class passed the first exam”
- Imply the conclusion
  - “Someone who passed the first exam has not read the book.”

Rules of Inference for Quantifiers(Contd.)

- **Solution:**
  - Let,

    $C(x) =$ "x is in $this$ $class$."

    $B(x) =$ "x has $read$ the $book$."

    $P(x) =$ "x $passed$ the $first$ $exam$."

    Then the premises can be represented as,

    1. $\exists x(C(x) \land \neg B(x))$
    2. $\forall x(C(x) \rightarrow P(x))$

    The conclusion is simply, $\exists x(P(x) \land \neg B(x))$.

Rules of Inference for Quantifiers(Contd.)

We construct an argument to show that our premises lead to the desired conclusion as follows,

Steps	***Reasons***
1. $\exists x(C(x) \wedge \neg B(x))$	*premise*
2. $C(a) \wedge \neg B(a)$	*Existential instantiation using* (1)
3. $C(a)$	*Simplification from* (2)
4. $\neg B(a)$	*Simplification from* (2)
5. $\forall x\big(C(x) \rightarrow P(x)\big)$	*premise*
6. $C(a) \rightarrow P(a)$	*Universal instantiation from* (5)
7. $P(a)$	*Modus ponens from* (3)*and* (6)
8. $P(a) \wedge \neg B(a)$	*Conjuntion from* (4) *and* (7)
9. $\exists x(P(x) \wedge \neg B(x))$	*Existential generalization from* (8)

Thus, we can see that our premises lead to the desired conclusion.

THE END