Chapter 2
Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Section 2.4 : Sequences and Summations

Sequences

- A sequence is a discrete structure used to represent an ordered list.
- It is defined as a function from a subset of the set of integers (usually either the set $\{0, 1, 2, \ldots\}$ or the set $\{1, 2, 3, \ldots\}$) to a set S.
- We use the notation a_n to denote the image of the integer n.
- We call a_n a *term* of the sequence.

Sequences (Contd.)

- **Example 1:**

  Consider the sequence $\{a_n\}$, where,

$$a_n = \frac{1}{n}$$

- **Solution:**

  The list of the terms of this sequence, beginning with a_1 [a_0 is not possible as $\frac{1}{0} = \infty$ and the domain of the function is $\{1, 2, 3, \ldots n\}$] are, $a_1, a_2, a_3, a_4, \ldots$

  Thus, the sequence will be $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

Progressions

- Geometric Progression:
  - A sequence of the form $a, ar, ar^2, \dots, ar^n, \dots$
  - The **initial term** a and the **common ratio** r are real numbers.
  - It is a discrete analogue of the exponential function $f(x) = ar^x$.

- Arithmetic Progression:
  - A sequence of the form $a, a + d, a + 2d, \dots, a + nd, \dots$
  - The **initial term** a and the **common difference** d are real numbers.
  - It is a discrete analogue of the linear function $f(x) = dx + a$.

Progressions (Contd.)

- **Example 2:**

Consider the following sequences

a. $\{b^n\}$ with $b^n = (-1)^n$
b. $\{c^n\}$ with $c^n = 2 \cdot 5^n$
c. $\{d^n\}$ with $d^n = 6 \cdot \left(\frac{1}{3}\right)^n$

Find out the following,

1. What type of progressions are they?
2. What are the initial terms and common factors?
3. Find the list of terms in the sequences and their values.

Progressions (Contd.)

- Solution:
  1. The progressions in a, b, c are all geometric progressions.
  2. The initial terms and the common ratios are listed below,

	Initial Terms	Common Ratio
a.	1	-1
b.	2	5
c.	6	$\frac{1}{3}$

  3. The list of terms and their values are listed below,

	Terms	Values
a.	$b_0, b_1, b_2, b_3, b_4, \ldots$	$1, -1, 1, -1, 1, \ldots$
b.	$c_0, c_1, c_2, c_3, c_4, \ldots$	$2, 10, 50, 250, 1250, \ldots$
c.	$d_0, d_1, d_2, d_3, d_4, \ldots$	$6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots$

Progressions (Contd.)

- **Example 3:**

Consider the following sequences

a. $\{s_n\}$ with $s_n = -1 + 4n$
b. $\{t_n\}$ with $t_n = 7 - 3n$

Find out the following,

1. What type of progressions are they?
2. What are the initial terms and common factors?
3. Find the list of terms in the sequences and their values.

Progressions (Contd.)

- Solution:
  1. The progressions in a, b are all arithmetic progressions.
  2. The initial terms and the common differences are listed below,

	Initial Terms	Common Difference
a.	-1	4
b.	7	-3

  3. The list of terms and their values are listed below,

	Terms	Values
a.	$s_0, s_1, s_2, s_3, \ldots$	$-1,3,7,11, \ldots$
b.	$t_0, t_1, t_2, t_3, \ldots$	$7,4,1,-2, \ldots$

Recurrence Relations

- A $Recurrence\ Relation$ for the sequence $\{a_n\}$ is
  - An equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.
- A sequence is called a $solution$ of a recurrence relation if its terms satisfy the recurrence relation.
- A recurrence relation is said to $recursively\ define$ a sequence.

Recurrence Relations (Contd.)

- **Example 4:**

  Consider the sequence $\{a_n\}$ which satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots$ and suppose that $a_0 = 2$. What are a_1, a_2, a_3?

- **Solution :**

  From the recurrence relation it is clear that,

  - $a_1 = a_0 + 3 = 2 + 3 = 5$
  - $a_2 = a_1 + 3 = 5 + 3 = 8$
  - $a_3 = a_2 + 3 = 8 + 3 = 11$

Recurrence Relations (Contd.)

- **Example 5:**

  Consider the sequence $\{a_n\}$ which satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$ and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2, a_3, a_4, a_5?

- **Solution :**

  From the recurrence relation it is clear that,

  - $a_2 = a_1 - a_0 = 5 - 3 = 2$
  - $a_3 = a_2 - a_1 = 2 - 5 = -3$
  - $a_4 = a_3 - a_2 = -3 - 2 = -5$
  - $a_5 = a_4 - a_3 = -5 - (-3) = -2$

Recurrence Relations (Contd.)

- One of the most commonly used sequences defined by a recurrence relation is the $Fibonacci\ Sequence$.
- It is defined by
  - The initial conditions, $f_0 = 0$ and $f_1 = 1$.
  - The recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, 4, \ldots$
- The $Fibonacci\ Sequence$ is as follows,
  - $f_2 = f_1 + f_0 = 1 + 0 = 1,$
  - $f_3 = f_2 + f_1 = 1 + 1 = 2,$
  - $f_4 = f_3 + f_2 = 2 + 1 = 3,$
  - $f_5 = f_4 + f_3 = 3 + 2 = 5,$
  - $f_6 = f_5 + f_4 = 5 + 3 = 8. \ldots\ldots\ldots\ldots$

Recurrence Relations (Contd.)

- We say that we have solved the recurrence relation together with the initial conditions when we find an explicit formula, called a **closed formula**, for the terms of the sequence.

Recurrence Relations (Contd.)

- **Example 6:**

  Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every nonnegative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots.$

- **Solution:**

  Suppose that $a_n = 3n$ for every nonnegative integer n.
  Then, for $n \geq 2$, we see that,

$$\begin{aligned} 2a_{n-1} - a_{n-2} &= 2\big(3(n-1)\big) - 3(n-2) \\ &= 3n \\ &= a_n. \end{aligned}$$

  Therefore, $\{a_n\}$, where $a_n = 3n$, is a solution of the recurrence relation.

Recurrence Relations (Contd.)

- **Example 7:**

  Determine whether the sequence $\{a_n\}$, where $a_n = 2^n$ for every nonnegative integer n, is a solution of the recurrence relation

  $a_n = 2 \cdot a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$.

- **Solution:**

  Suppose that $a_n = 2^n$ for every nonnegative integer n.

  Note that, $a_0 = 1, a_1 = 2, a_2 = 4$.

  Then, for $n \geq 2$, we see that,

$$
\begin{aligned}
a_n &= 2 \cdot a_{n-1} - a_{n-2} \\
a_2 &= 2 \cdot a_1 - a_0 \\
&= 2 \cdot 2 - 1 \\
&= 3
\end{aligned}
$$

  But, $a_2 = 4$ from the equation $a_n = 2^n$.

  Therefore, $\{a_n\}$, where $a_n = 2^n$, is not a solution of the recurrence relation.

Recurrence Relations (Contd.)

- **Example 8:**

  Solve the recurrence relation $a_n = a_{n-1} + 3$ with the initial condition $a_1 = 2$.

- **Solution:**

  Starting with the initial condition $a_1 = 2$ and applying the recursive relation successively upward until we reach a_n, we will try to find a closed formula for the sequence $\{a_n\}$.

  $a_2 = a_1 + 3 = 2 + 3$
  $a_3 = a_2 + 3 = (2 + 3) + 3 = 2 + 3 \cdot 2$
  $a_4 = a_3 + 3 = (2 + 3 \cdot 2) + 3 = 2 + 3 \cdot 3$
  .
  .
  .
  $a_n = a_{n-1} + 3 = \big(2 + 3 \cdot (n-2)\big) + 3 = 2 + 3 \cdot (n-1)$

  Evaluating a_n at the initial condition i.e. a_1, we get, $a_1 = 2 + 3 \cdot (1 - 1) = 2$.

  Thus, the closed formula for the sequence $\{a_n\}$ is $a_n = 2 + 3 \cdot (n - 1)$

Summations

- Required for adding up the terms of a sequence.
- The following formula is used for calculating the sum of terms of a geometric progression. Where, a and r are real numbers and $r \neq 0$.

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r - 1} & , if\ r \neq 1 \\ (n+1)a & , if\ r = 1 \end{cases}$$

Summations (Contd.)

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1}-a}{r-1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

THE END