Definite Integrals

Definition

(i) If $a \in Dom(f)$, then $\int_a^a f(x)\,dx = 0$

(ii) If f is integrable on $[a,b]$, then $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Theorem

If f and g are integrable on $[a,b]$ and k is a constant, then kf, $f+g$ and $f-g$ are integrable on $[a,b]$ and then

(i) $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

(ii) $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(iii) $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

In general,

$$\int_a^b [f_1(x) \pm f_2(x) \pm \cdots \pm f_n(x)]dx = \int_a^b f_1(x)\,dx \pm \int_a^b f_2(x)\,dx \pm \cdots \pm \int_a^b f_n(x)\,dx$$

Some properties of definite integrals can be motivated. For example, if f is continuous and nonnegative on the interval $[a, b]$, and if c is a point between a and b, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Some important properties for evaluating integrals

(1) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(2) $$\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & if\ \ f\ \ is\ even \\ 0 & if\ \ f\ \ is\ odd \end{cases}$$

(3) $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$

Solved Problems

1. Evaluate $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$

Solution: Let $I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$ (1)

Then $I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\{\cos(\pi - x)\}^2}dx = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}dx$ (2)

Adding equations (1) and (2), we get

$2I = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx = 2\pi\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}dx \Rightarrow I = \pi\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}dx$ (3)

Let $\cos x = u$. Then $-\sin x\, dx = du$. Also, if $x = 0$ then $u = 1$ and if $x = \pi/2$ then $u = 0$.

So that $I = -\pi\int_1^{0} \frac{1}{1+u^2}du = \pi[\tan^{-1} u]_0^1 = \pi(\pi/4 - 0) = \frac{\pi^2}{4}$.

2. Evaluate $\int_0^{\pi} \frac{x\tan x}{\sec x + \cos x}dx$;

Solution: Let $I = \int_0^{\pi} \frac{x\tan x}{\sec x + \cos x}dx = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = \pi^2/4$.

3. Evaluate $\int_0^{\pi} \frac{x}{1+\sin x}dx$;

Solution: Let $I = \int_0^{\pi} \frac{x}{1+\sin x}dx$ (1)

Then $I = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)}dx = \int_0^{\pi} \frac{\pi - x}{1+\sin x}dx$ (2)

Adding equations (1) and (2), we get

$2I = \pi\int_0^{\pi} \frac{1}{1+\sin x}dx = 2\pi\int_0^{\pi/2} \frac{1}{1+\sin(\pi/2 - x)}dx$

$\Rightarrow I = \pi\int_0^{\pi/2} \frac{1}{1+\cos x}dx = \pi\int_0^{\pi/2} \frac{1}{2\cos^2(x/2)}dx = \frac{\pi}{2}\int_0^{\pi/2} \sec^2(x/2)\,dx$

$= \frac{\pi}{2}\times 2\,[\tan\frac{x}{2}]_0^{\pi/2} = \pi\tan\frac{\pi}{4} = \pi$.

4. Evaluate $\int_0^{\pi} \frac{x \sin x}{1+\sin x} dx$

Solution: Let $I = \int_0^{\pi} \frac{x \sin x}{1+\sin x} dx$ (1)

Then $I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\sin(\pi - x)} dx = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\sin x} dx$ (2)

Adding equations (1) and (2), we get

$$2I = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x} dx = 2\pi \int_0^{\pi/2} \frac{\sin x}{1+\sin x} dx$$

$$\Rightarrow I = \pi \int_0^{\pi/2} \frac{\sin\{(\pi/2)-x\}}{1+\sin\{(\pi/2)-x\}} dx = \pi \int_0^{\pi/2} \frac{\cos x}{1+\cos x} dx = \pi \int_0^{\pi/2} \frac{(1+\cos x)-1}{1+\cos x} dx$$

$$= \pi \frac{\pi}{2} - \frac{\pi}{2} \times 2\,[\tan \frac{x}{2}]_0^{\pi/2} = \frac{\pi^2}{2} - \pi \tan \frac{\pi}{4} = \pi(\frac{\pi}{2} - 1).$$

5. Evaluate $\int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$

Solution: Let $I = \int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$ (1)

Then $I = \int_0^{\pi/2} \frac{(\pi/2)-x}{\sin((\pi/2)-x) + \cos((\pi/2)-x)} dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{(\pi/2)-x}{\cos x + \sin x} dx \qquad (2)$$

Therefore, adding (1) and (2), we get

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sec^2(x/2)}{1-\tan^2(x/2) + 2\tan(x/2)} dx \qquad (3)$$

Let $\tan(x/2) = y \Rightarrow (1/2)\sec^2(x/2)\,dx = dy \Rightarrow \sec^2(x/2)\,dx = 2dy$

Here, $x = 0 \Rightarrow y = 0$ and $x = \pi/2 \Rightarrow y = 1$

Then $2I = \frac{\pi}{2} \int_0^1 \frac{2dy}{1-y^2+2y} = \pi \int_0^1 \frac{dy}{(\sqrt{2})^2 - (y-1)^2} = \pi \times \frac{1}{2\sqrt{2}} \left[\log \frac{\sqrt{2}+y-1}{\sqrt{2}-y+1} \right]_0^1$

$$= \pi \times \frac{1}{2\sqrt{2}} \left[\log \frac{\sqrt{2}}{\sqrt{2}} - \log \frac{\sqrt{2}-1}{\sqrt{2}+1} \right] = \pi \times \frac{1}{2\sqrt{2}} \log \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$= \pi \times \frac{1}{2\sqrt{2}} \log \frac{(\sqrt{2}+1)^2}{2-1} = \pi \times \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$$

Thus, $I = \int_0^{\pi/2} \frac{x}{\sin x + \cos x}\,dx = \frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$

5. Evaluate $\int_0^{\frac{\pi}{2}} \log(\cos x)\,dx$

Solution: Let $I = \int_0^{\frac{\pi}{2}} \log(\cos x)\,dx \qquad (1)$

Then $I = \int_0^{\frac{\pi}{2}} \log\{\cos(\frac{\pi}{2} - x)\}\,dx = \int_0^{\frac{\pi}{2}} \log \sin x\,dx \qquad (2)$

Adding equations (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \log(\cos x)\,dx + \int_0^{\frac{\pi}{2}} \log(\sin x)\,dx = \int_0^{\frac{\pi}{2}} \log(\sin x \cos x)\,dx = \int_0^{\frac{\pi}{2}} \log(\frac{\sin 2x}{2})\,dx$$

$$= \int_0^{\frac{\pi}{2}} \log(\sin 2x)\,dx - \int_0^{\frac{\pi}{2}} \log(2)\,dx = I_1 - \frac{\pi}{2}\log 2$$

For I_1, let $2x = z$. Then $2dx = dz \Rightarrow dx = \frac{1}{2}dz$

So, $I_1 = \int_0^{\frac{\pi}{2}} \log(\sin 2x)\,dx = \int_0^{\pi} \log(\sin z)\frac{1}{2}dz = \frac{1}{2} \times 2\int_0^{\frac{\pi}{2}} \log(\sin x)\,dx = I$

Hence, $2I = I - \frac{\pi}{2}\log 2 \;\; \Rightarrow I = -\frac{\pi}{2}\log 2 = \frac{\pi}{2}\log\frac{1}{2}$.

6. Evaluate $\int_3^{29} \frac{(x-2)^{2/3}}{(x-2)^{2/3}+3}\,dx$.

Solution: Let $(x-2)^{2/3} = y^2$. Then $x - 2 = y^3 \Rightarrow dx = 3y^2 dy$

When $x = 3$ then $y = 1$ and when $x = 29$ then $y = 3$

Therefore, we can write

$$\int_3^{29} \frac{(x-2)^{2/3}}{(x-2)^{2/3}+3}\,dx = \int_3^{29}\left(1 - \frac{3}{(x-2)^{2/3}+3}\right)dx = \int_1^3\left(1 - \frac{3}{y^2+3}\right)3y^2 dy$$

$$= 3\int_1^3\left(y^2 - \frac{3y^2}{y^2+3}\right)dy = 3\int_1^3 y^2\,dy - 9\int_1^3\left(1 - \frac{3}{y^2+3}\right)dy$$

$$= 3\left[\frac{y^3}{3}\right]_1^3 - 9\left[y - \frac{3}{\sqrt{3}}\tan^{-1}\frac{y}{\sqrt{3}}\right]_1^3 = 26 - 9\left(2 - \sqrt{3}\tan^{-1}\sqrt{3} + \sqrt{3}\tan^{-1}\frac{1}{\sqrt{3}}\right)$$

$$= 26 - 9\left(2 - \sqrt{3}\frac{\pi}{3} + \sqrt{3}\frac{\pi}{6}\right) = 8 + \frac{3\sqrt{3}}{2}\pi .$$

7. Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{(a^2\cos^2 x + b^2\sin^2 x)^2}$

Solution: We can write

$$\int_0^{\frac{\pi}{2}} \frac{dx}{(a^2\cos^2 x + b^2\sin^2 x)^2} = \int_0^{\frac{\pi}{2}} \frac{\sec^4 x dx}{(a^2 + b^2\tan^2 x)^2}$$

Let $b\tan x = a\tan y \Rightarrow \sec^2 x\,dx = (a/b)\sec^2 y\,dy$

Here, $x = 0 \Rightarrow y = 0$ and $x = \pi/2 \Rightarrow y = \pi/2$

Therefore, we get $\int_0^{\frac{\pi}{2}} \frac{dx}{(a^2\cos^2 x + b^2\sin^2 x)^2} = \int_0^{\frac{\pi}{2}} \frac{(\sec^2 x)(\sec^2 x)\,dx}{(a^2 + b^2\tan^2 x)^2}$

$$= \int_0^{\frac{\pi}{2}} \frac{(1 + \frac{a^2}{b^2}\tan^2 y)(a/b)\sec^2 y\,dy}{(a^2 + a^2\tan^2 y)^2} = \int_0^{\frac{\pi}{2}} \frac{(b^2 + a^2\tan^2 y)}{a^3 b^3 \sec^2 y}\,dy = \frac{1}{a^3b^3}\int_0^{\frac{\pi}{2}} (b^2\cos^2 y + a^2\sin^2 y)\,dy$$

Let $I = \frac{1}{a^3b^3}\int_0^{\frac{\pi}{2}} (b^2\cos^2 y + a^2\sin^2 y)\,dy$

$$= \frac{1}{a^3b^3}\int_0^{\frac{\pi}{2}} \{b^2\cos^2(\frac{\pi}{2} - y) + a^2\sin^2(\frac{\pi}{2} - y)\}\,dy = \frac{1}{a^3b^3}\int_0^{\frac{\pi}{2}} (b^2\sin^2 y + a^2\cos^2 y)\,dy$$

So that $2I = \frac{1}{a^3b^3}\int_0^{\frac{\pi}{2}} (a^2 + b^2)\,dy = \frac{a^2 + b^2}{a^3b^3}\frac{\pi}{2}$

Thus, we get $\int_0^{\frac{\pi}{2}} \frac{dx}{(a^2\cos^2 x + b^2\sin^2 x)^2} = \frac{a^2 + b^2}{a^3b^3}\frac{\pi}{4}$

8. Evaluate $\int_0^{\infty} \frac{x\,dx}{(1+x)(1+x^2)}$

Solution: Let $x = \tan y \Rightarrow dx = \sec^2 y\,dy$

Here, $x = 0 \Rightarrow y = 0$ and $x = \infty \Rightarrow y = \pi/2$

Therefore, $\int_0^{\infty} \frac{x\,dx}{(1+x)(1+x^2)} = \int_0^{\pi/2} \frac{\tan y \sec^2 y}{(1+\tan y)\sec^2 y}dy$

$$= \int_0^{\pi/2} \frac{\tan y}{1+\tan y}dy = \int_0^{\pi/2} \frac{\sin y}{\sin y + \cos y}dy = I, \text{ say}$$

Then $I = \int_0^{\pi/2} \frac{\sin((\pi/2) - y)}{\sin((\pi/2) - y) + \cos((\pi/2) - y)}dy = \int_0^{\pi/2} \frac{\cos y}{\cos y + \sin y}dy$

Thus, $2I = \int_0^{\pi/2} dy = \pi/2 \Rightarrow I = \int_0^{\infty} \frac{x\,dx}{(1+x)(1+x^2)} = \int_0^{\pi/2} \frac{\sin y}{\sin y + \cos y} dy = \pi/4$

9. Evaluate $\int_0^{\infty} \frac{\log(x+\frac{1}{x})}{1+x^2} dx$

Solution: Let $x = \tan y \Rightarrow dx = \sec^2 y\,dy$

Here, $x = 0 \Rightarrow y = 0$ and $x = \infty \Rightarrow y = \pi/2$

Therefore, $I = \int_0^{\infty} \frac{\log(x+\frac{1}{x})}{1+x^2} dx = \int_0^{\pi/2} \frac{\log(\tan y + \cot y)\sec^2 y}{1+\tan^2 y} dy$

$= \int_0^{\pi/2} \log\left[\frac{1+\tan^2 y}{\tan y}\right] dy = \int_0^{\pi/2} \log\left[\frac{\sec^2 y}{\tan y}\right] dy = \int_0^{\pi/2} \log\left[\frac{1}{\sin y \cos y}\right] dy$

$= \int_0^{\pi/2} \log\left[\frac{2}{2\sin y \cos y}\right] dy = \int_0^{\pi/2} \log 2\,dy - \int_0^{\pi/2} \log(\sin 2y)\,dy = \frac{\pi}{2}\log 2 - I_1,$

where $I_1 = \int_0^{\pi/2} \log(\sin 2y)\,dy$

Let $z = 2y \Rightarrow dz = 2\,dy \Rightarrow dy = (1/2)dz$

Here, $y = 0 \Rightarrow z = 0$ and $y = \pi/2 \Rightarrow z = \pi$

Therefore, $I_1 = \int_0^{\pi/2} \log(\sin 2y)\,dy = \frac{1}{2}\int_0^{\pi} \log(\sin z)\,dz$

$= \int_0^{\pi/2} \log(\sin z)\,dz = \frac{\pi}{2}\log\frac{1}{2}$

Thus, $I = \int_0^{\infty} \frac{\log(x+\frac{1}{x})}{1+x^2} dx = \frac{\pi}{2}\log 2 - \frac{\pi}{2}\log\frac{1}{2} = 2\times\frac{\pi}{2}\log 2 = \pi\log 2$

Exercises

Evaluate the following definite integrals:

1. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx$

2. $\int_0^{\frac{\pi}{2}} \frac{1}{1 + \tan x} dx$

3. $\int_0^{\frac{\pi}{2}} \log(\sin x)\, dx$

4. $\int_0^{\pi} \log(1 + \cos x)\, dx$

5. $\int_0^{\frac{\pi}{2}} \frac{\sin^{2m-1}\theta \cos^{2n-1}\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} d\theta$

6. $\int_0^{\infty} \frac{dx}{\sqrt{x}\,(x+1)}$

7. $\int_0^{1} \frac{\log(1+x)}{1+x^2} dx$

8. $\int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \sin x \cos x} dx$

9. $\int_0^{1} x^2 \sqrt{4 - x^2}\, dx$

10. $\int_0^{1} \frac{\log x}{\sqrt{1 - x^2}} dx$