Subject : Math-4241 Date : 12/01/2024

Prof. KFU

1. Integral Calculus (IC)
2. Ordinary Differential Equation (ODE)
3. Series solution of ODE (SS)
4. Partial differential equation (PDE)

Textbook:

1) Calculus : Early Transcendentals by Howard Anton, Irl Bivens and Stephen Davis

11) Ordinary Differential Equation by D.G. Zill

S.L Roos

Integration by parts

Integration is an inverse process of differentiation.

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

Integrand

$$\int \cos x \, dx = \sin x + c$$

$$\int \sin x \, dx = -\cos x + c$$

Antiderivatives

$\frac{1}{3}x^2 + c$

$\frac{1}{3}x^2 + \ln c$

$\frac{1}{3}x^2 + \sin c$

C.E.BT 7.1-7.5

$u \equiv u(x)$

$v \equiv v(x)$

Integration by parts

$$\int uv\,dx = u\int v\,dx - \int\left(\frac{d}{dx}(u)\cdot\int v\,dx\right)dx$$

✓ Integration by substitution

✓ Standard Integrals

7.1, 7.2, 7.3 i) Reduction formula

7.5 ii) Partial fraction decomposition

7.4 iii) Trigonometric substitution

6.9 iv) Hyperbolic functions ; Hanging Cables

$$e^x = \underbrace{\frac{e^x+e^{-x}}{2}}_{\cosh x} + \underbrace{\frac{e^x-e^{-x}}{2}}_{\sinh x}$$

$$\cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x} = c_1e^x + c_2e^{-x}$$

Every hyperbolic function is a linear combination of exponential functions e^x and e^{-x}.

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh x = \frac{1}{\operatorname{cosech} x}$$

$$\coth x = \frac{1}{\tanh x}$$

$$\cosh x = \frac{1}{\operatorname{sech} x}$$

Subject : .... Date : ....

✓ Definite Integrals

6.1, 6.2, 6.3, 6.4, 6.5

✓ Improper integrals 7.8

Asymptotes, Limit

✓ Definite Integrals

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Antiderivative

Here a, b are finite and real constants —

$$\int_{-\infty}^{+\infty} f(x)\,dx, \quad \int_{\infty}^{b} f(x)\,dx, \quad \int_{a}^{+\infty} f(x)\,dx$$

$$\int_1^3 \frac{dx}{x-1}, \quad \int_1^3 \frac{dx}{x-3}, \quad \int_1^3 \frac{dx}{x-2}$$

Shajahan Products

Pattern recognition:

Type-1

2. $\int \frac{xe^x}{(1+x)^2}\,dx$

$= xe^x \int \frac{1}{(1+x)^2}dx - \int \left(\frac{d}{dx}(xe^x) \cdot \int \frac{1}{(1+x)^2}dx\right) dx$

$= -\frac{xe^x}{1+x} - \int (x+1)e^x\left(-\frac{1}{1+x}\right) dx$

$= -\frac{xe^x}{1+x} + e^x + c$

3. $\int (3x^2 - x + 2)e^{-x}\,dx$

Tabular integration by parts

Repeated diff.		Repeated integration
$3x^2 - x + 2$	$+$	e^{-x}
$6x - 1$	$-$	$-e^{-x}$
6	$+$	e^{-x}
0		$-e^{-x}$

$I = -(3x^2 - x + 2)e^{-x} - (6x-1)e^{-x} - 6e^{-x} + c$

$$\int e^{ax}\cos bx\,dx$$

$$= \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right) + c$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right) + c$$

Type-II

1. $$\int \frac{e^{m\tan^{-1}x}}{(1+x^2)^2}\,dx = \int \frac{e^{mz}}{1+\tan^2 z}\,dz = \int e^{mz}\cos^2 z\,dz$$

Let, $\tan^{-1}x = z$

$\Rightarrow \frac{1}{1+x^2}dx = dz$

$x = \tan z$

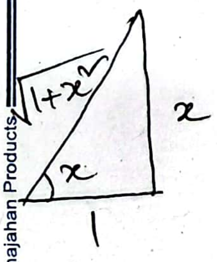
$$= \frac{1}{2}\int e^{mz}\cdot 2\cos^2 z\,dz$$

$$= \frac{1}{2}\int e^{mz}(1+\cos 2z)\,dz$$

$$= \frac{1}{2}\int e^{mz}dz + \frac{1}{2}\int e^{mz}\cos 2z\,dz$$

$$= \frac{1}{2m}e^{mz} + \frac{1}{2}\cdot\frac{e^{mz}}{m^2+2^2}\left(m\cos 2z + 2\sin 2z\right) + c$$

$$= \frac{1}{2m}e^{m\tan^{-1}x} + \frac{1}{2}\cdot\frac{e^{m\tan^{-1}x}}{m^2+4}\left[m\cos(2\tan^{-1}x) + 2\sin(2\tan^{-1}x)\right] + c$$



2. $\int \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\, dx$

Let $\sin^{-1}x = z \Rightarrow dz = \frac{1}{\sqrt{1-x^2}}\, dx$

$\Rightarrow x = \sin z$

$$\int \frac{\sin^{-1}x}{\sqrt{1-x^2}\,.\,(1-x^2)}\, dx = \int \frac{z}{1-\sin^2 z}\, dz$$

$$= \int \frac{z}{\cos^2 z}\, dz$$

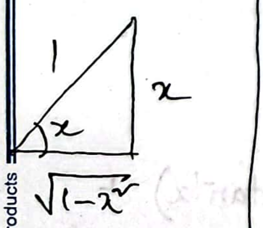
$$= \int z \sec^2 z \, dz$$

$$= z \int \sec^2 z \, dz - \int 1.\tan z \, dz$$

$$= z \tan z - \ln(\cos z) + c$$

$$= (\sin^{-1}x).\tan(\sin^{-1}x) + \ln(\cos(\sin^{-1}x)) + c$$

$$= (\sin^{-1}x).\left(\tan \tan^{-1} \frac{x}{\sqrt{1-x^2}}\right) + \ln\left(\cos \cos^{-1}\sqrt{1-x^2}\right) + c$$



$$= \frac{x}{\sqrt{1-x^2}} \sin^{-1}x + \frac{1}{2}\ln(1-x^2) + c$$

Subject : .... Date : ....

3. $\int \sin^{-1}\sqrt{\frac{x}{x+a}}\, dx$

Let $x = a\tan^2 z$

$\Rightarrow dx = a.\, 2\tan z.\sec^2 z\, dz$

$\therefore \sqrt{\frac{x}{x+a}} = \sqrt{\frac{a\tan^2 z}{a\tan^2 z + a}} = \frac{\tan z}{\sec z} = \sin z$

$\therefore \int \sin^{-1}\sqrt{\frac{x}{x+a}}\, dx = \int \sin^{-1}(\sin z)\,(2a\tan z.\sec^2 z)\, dz$

$= 2a\int z.\tan z.\sec^2 z\, dz$

$= 2a\left[z.\int \tan z.\sec^2 z\, dz - \int\left(1.\int \tan z.\sec^2 z\, dz\right)dz\right]$

$= 2a\left[z.\frac{1}{2}\tan^2 z - \frac{1}{2}\int \tan^2 z\, dz\right]$

$= az\tan^2 z - a\int(\sec^2 z - 1)\, dz$

$= az\tan^2 z - a(\tan z - z) + c$

$= a.\left[\tan^{-1}\sqrt{\frac{x}{a}}\right]\frac{x}{a} - a\left[\sqrt{\frac{x}{a}} - \tan^{-1}\sqrt{\frac{x}{a}}\right] + c$

$= x\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + a\tan^{-1}\sqrt{\frac{x}{a}} + c$

$= (x+a)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + c$

$\int \underbrace{\tan z}_{u}.\underbrace{\sec^2 z\, dz}_{du}$

$= \int u\, du$

$= \frac{u^2}{2} + c$

$= \frac{1}{2}\tan^2 z + c$

Shajahan Products

CamScanner

4. $\int \frac{2}{(2-x)^2} \cdot \sqrt[3]{\frac{2-x}{2+x}}\, dx$

Result:

$\frac{3}{4}\left(\frac{2-x}{2+x}\right)^{2/3} + c$

Let, $\frac{2-x}{2+x} = t^3$

$\Rightarrow -\frac{4}{(2+x)^2}\, dx = 3t^2 dt$

$\Rightarrow \frac{2}{(2-x)^2} dx = -\frac{3}{2}\left(\frac{2+x}{2-x}\right)^2 . t^2 dt$

$\therefore \frac{2}{(2-x)^2} dx = -\frac{3}{2} t^{-4} dt$

$dx = -\frac{3}{4} t^2 (2+x)^2 dt$

Type-III

1. $\int \frac{dx}{1+\sin x + \cos x}$

$= \int \frac{\frac{1}{\cos x} dx}{\frac{1}{\cos x} + \frac{\sin x}{\cos x} + \frac{\cos x}{\cos x}}$

$= \int \frac{\sec x\, dx}{\sec x + \tan x + 1}$

Let, $\sec x + \tan x + 1 = z$

$\Rightarrow dz = (\sec x \tan x . + \sec^2 x) dx$

$\Rightarrow dz = \sec x . dx (\sec x + \tan x)$

$\Rightarrow \sec x . dx = \frac{dz}{z-1}$

$\therefore \int \frac{\sec x\, dx}{\sec x + \tan x + 1} = \int \frac{dz}{z(z-1)} = \int \left(\frac{1}{z-1} - \frac{1}{z}\right) dz$

Subject : .... Date : ....

$= \ln|z-1| - \ln|z| + c$

$= \ln\left|\frac{z-1}{z}\right| + c$

$= \ln\left|\frac{\sec x + \tan x}{\sec x + \tan x + 1}\right| + c$

$\sin 2x = \frac{2\tan x}{1+\tan^2 x}$

$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$

$\tan x = z$

Ans: $\ln\left|1+\tan\frac{x}{2}\right| + c$

2. $\int \frac{1+\sin x}{\sin x(1+\cos x)}\,dx$

$= \int \frac{dx}{\sin x(1+\cos x)} + \int \frac{\cancel{\sin x}}{\cancel{\sin x}(1+\cos x)}\,dx$

$= I_1 + I_2$

$I_2 = \int \frac{dx}{1+\cos x} = \int \frac{dx}{2\cos^2\frac{x}{2}} = \frac{1}{2}\int \sec^2\frac{x}{2}\,dx$

$= \frac{1}{2}\cdot 2\cdot \tan\frac{x}{2} + c$

$= \tan\frac{x}{2} + c$

$I_1 = \int \frac{dx}{\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\left(1+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}$

Let, $\tan\frac{x}{2} = z$

$\Rightarrow dz = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$= \int \frac{(1+\tan^2\frac{x}{2})\,\sec^2\frac{x}{2}\,dx}{(2\tan\frac{x}{2})(2)}$

$= \int \frac{1+\cancel{z^2}}{2z\cdot\cancel{2}}\,\cancel{2}dz$

$= \frac{1}{2}\int \frac{1+z^2}{z}\,dz$

~~$\frac{1}{2}\int\frac{dz}{z} = \frac{1}{2}\ln|z| + c$~~

$= \frac{1}{2}\ln|z| + \frac{z^2}{4} + c$

$= \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + \frac{\tan^2\frac{x}{2}}{4} + c$

Subject : .... Date : ....

3. $\int \frac{dx}{\sin x(2+\cos x-2\sin x)}$

$= \int \frac{(1+\tan^2 \frac{x}{2})(\sec^2 \frac{x}{2})\,dx}{(\tan \frac{x}{2})(3+\tan^2 \frac{x}{2}-4\tan \frac{x}{2})}$

Let, $z=\tan \frac{x}{2}$

$\Rightarrow dz = \frac{1}{2}\sec^2 \frac{x}{2}\,dx$

$= \int \frac{1+z^2}{z(3+z^2-4z)}\,dz$

$\frac{1+z^2}{z(z-1)(z-3)} = \frac{2/3}{z} + \frac{-1}{z-1} + \frac{10/6}{z-3}$

Subject : Math-4241 Date : 19/01/2024

Type - III $f(\sin x, \cos x)$

1. $\int \frac{\cos x}{5-3\cos x}\,dx$

$m(5-3\cos x)+n=\cos x$

2. $\int \frac{dx}{\cos 3x-\cos x}$

$= \int \frac{dx}{2\sin 2x \sin x}$

$= \int \frac{1}{2} \frac{dx}{2\sin x \cos x \sin x}$

$= \frac{1}{4}\int \frac{\sin^2 x+\cos^2 x}{\sin^2 x \cos x}\,dx$

$= \frac{1}{4}\int \sec x\,dx + \frac{1}{4}\int \operatorname{cosec} x \cdot \cot x\,dx$

$= \frac{1}{4}\ln|\sec x+\tan x| - \frac{1}{4}\operatorname{cosec} x + c$

3. $\int \frac{x^2}{(x\sin x+\cos x)^2}\,dx$

$$\boxed{\int e^x\{f(x)+f'(x)\}\,dx = e^x f(x) + c}$$

1. $\int e^x \frac{2-\sin 2x}{1-\cos 2x}\,dx$

2. $\int e^x \frac{x^2+1}{(x+1)^2}\,dx$

Homework

1. $\int (x+1)\sqrt{x^2-2x+5}\,dx$

2. $\int (x-3)^3\sqrt{6x-x^2}\,dx$

3. $\int \frac{x^2+1}{x^4+x^2+1}\,dx$

4. $\int \frac{x^3+x^2+x+1}{\sqrt{x^2+2x+3}}\,dx$

5. $\int \frac{dx}{(x+1)^3\sqrt{x^2+3x+2}}$

6. $\int \frac{2x-1}{\sqrt{4x^2+4x+2}}\,dx$

7. $\int \frac{dx}{(3+4x^2)\sqrt{4-3x^2}}$

8. $\int \frac{dx}{(x+1)\sqrt{1+2x-x^2}}$

Subject : .... Date : ....

Formulas :

1. $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right| + c$

2. $\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left|x+\sqrt{a^2+x^2}\right| + c = \sinh^{-1}\frac{x}{a} + c$

3. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + c = \cosh^{-1}\frac{x}{a} + c$

Relation :

$\sinh^{-1}x = \ln\left|x+\sqrt{x^2+1}\right|$

$\cosh^{-1}x = \ln\left|x+\sqrt{x^2-1}\right|$

Ch. 6.9 [C-E.T]

(1) $\int (x+1)\sqrt{x^2-2x+5}\,dx$

$x+1 = m.\frac{d}{dx}(x^2-2x+5) + n$

$= m(2x-2)+n$

$= 2mx - 2m + n$

$\therefore m = \frac{1}{2}$

$n = 2$

$\therefore \int (x+1)\sqrt{x^2-2x+5}\,dx = \int\left\{\frac{1}{2}(2x-2)+2\right\}\sqrt{x^2-2x+5}\,dx$

$= \frac{1}{2}\int (2x-2)\sqrt{x^2-2x+5}\,dx + 2\int\sqrt{x^2-2x+5}\,dx$

$= I_1 + I_2$

Subject : .... Date : ....

$$I_1 = \frac{1}{2} \cdot \frac{2}{3}\left(x^2-2x+5\right)^{3/2}$$

$$= \frac{1}{3}\left(x^2-2x+5\right)^{3/2} + c_1$$

$$I_2 = 2 \quad \frac{x-1}{2}\sqrt{x^2-2x+5} + \frac{2^2}{2}\cdot \ln\left|(x-1)+\sqrt{x^2-2x+5}\right| + c_2$$

formula:

$$\int [f(x)]^n \cdot f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

6. $$\int \frac{2x-1}{\sqrt{4x^2+4x+2}}\,dx$$

$$= \int \frac{\frac{1}{4}(8x+4)-2}{\sqrt{4x^2+4x+2}}\,dx$$

$$= \frac{1}{4}\int \frac{8x+4}{\sqrt{4x^2+4x+2}}\,dx - 2\int \frac{dx}{2\sqrt{(x+\frac{1}{2})^2+\frac{1}{4}}}$$

$$= \frac{1}{2}\sqrt{4x^2+4x+2} - \ln\left|(x+\tfrac{1}{2}) + \sqrt{4x^2+4x+2}\right| + c$$

$$2x-1 = m\cdot\frac{d}{dx}(4x^2+4x+2)+n$$

$$= m(8x+4)+n$$

$$\therefore m = \frac{1}{4}$$

$$n = -2$$

$$\sqrt{4x^2+4x+2}$$

$$= 2\sqrt{x^2+x+\frac{1}{2}}$$

$$= 2\sqrt{(x+\tfrac{1}{2})^2+(\tfrac{1}{2})^2}$$

formula:

$$\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + c$$

Quiz 01

Date: Jan 26, 2024

Syllabus: Indefinite integrals

7.2, 7.4, 7.5 already finished

Subject : .... Date : ....

7.3 Reduction ~~problems~~ formula

$$\int \sin^m x \cos^n x \, dx$$

Derivation will be available in the handouts only.

$m, n \to$ odd or even Table 7.3.1

*. $\int \sin^3 x \cos^3 x \, dx$

$= \int \sin^3 x . \cos^2 x . \cos x dx$

$= \int \sin^3 x (1 - \sin^2 x) \cos x \, dx$

Let, $u = \sin x$

$\Rightarrow du = \cos x \, dx$

$= \int u^3 (1 - u^2) du$

$= \int u^3 du - \int u^5 du$

$= \frac{u^4}{4} - \frac{u^{5+1}}{5+1} + c$

$= \frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x + c$

*. $\int \sin^4 x \cos^4 x \, dx$

$\sin^4 x = (\sin^2 x)^2 = \frac{1}{4}(2 \sin^2 x)^2$

$= \frac{1}{4}(1 - \cos 2x)^2$

$\cos^4 x = (\cos^2 x)^2 = \frac{1}{4}(2 \cos^2 x)^2$

$= \frac{1}{4}(1 + \cos 2x)^2$

Formula:

$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x . \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$

$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x . \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$

$\therefore \sin^4 x.\cos^4 x = \frac{1}{16}(1-\cos^2 2x)^2$

$= \frac{1}{16}(\sin^2 2x)^2$

$= \frac{1}{16}\sin^4 2x$

$\therefore \int \sin^4 x.\cos^4 x\, dx = \frac{1}{16}\int \sin^4 2x\, dx$

$= \frac{1}{16}\cdot\frac{1}{2}\int \sin^4 u\, du$

$= \frac{1}{32}\left[-\frac{1}{4}\sin^3 u.\cos u + \frac{3}{4}\int \sin^2 u\, du\right]$

$= \frac{1}{32}\left[-\frac{1}{4}\sin^3 u.\cos u + \frac{3}{4}\left(\frac{1}{2}u - \frac{1}{4}\sin 2u\right)\right] + c$

Let,
$u = 2x$
$\Rightarrow du = 2.dx$

*. $\int \sin^3 x \cos^2 x\, dx$ (m odd)

$= \int \sin^2 x.\cos^2 x.\sin x\, dx$

$= \int (1-\cos^2 x).\cos^2 x.\sin x\, dx$

$= -\int (1-u^2)\, u^2\, du$

$= -\frac{1}{3}u^3 + \frac{1}{5}u^5 + c$

$\Rightarrow = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + c$

Let, $u = \cos x$
$\Rightarrow du = -\sin x\, dx$
$\Rightarrow -du = \sin x.dx$

Subject : .... Date : ....

$\int \tan^m x \cdot \sec^n x \, dx$ Table 7.3.2

1. $\int \tan^4 x \cdot \sec x \, dx$

$= \int (\tan^2 x)^2 \cdot \sec x \, dx$

$= \int (\sec^2 x - 1)^2 \sec x \, dx$

$= \int (\sec^4 x - 2\sec^2 x + 1) \sec x \cdot dx$

$= \int \sec^5 x \, dx - 2\int \sec^3 x \, dx + \int \sec x \, dx$

$= \dfrac{\sec^3 x \cdot \tan x}{4} + \dfrac{3}{4}\int \sec^3 x \, dx$

$- 2\int \sec^3 x \, dx + \int \sec x \, dx$

Reduction formula:

$$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$$

$$\int \sec^n x \, dx = \frac{\sec^{n-2} x \cdot \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x \, dx$$

$= \frac{1}{4}\sec^3 x \tan x + \left(\frac{3}{4} - 2\right)\int \sec^3 x \, dx + \int \sec x \, dx$

$= \frac{1}{4}\sec^3 x \tan x - \frac{5}{4}\left[\frac{1}{2}\sec x \cdot \tan x + \frac{1}{2}\int \sec x \, dx\right] + \int \sec x \, dx$

$= \frac{1}{4}\sec^3 x \tan x - \frac{5}{8}\sec x \cdot \tan x - \left(\frac{5}{8} - 1\right)\int \sec x \, dx$

$= \frac{1}{4}\sec^3 x \tan x - \frac{5}{8}\sec x \cdot \tan x + \frac{3}{8}\ln\left|\sec x + \tan x\right| + C$

Subject : .... Date : ....

n even

$$\int \tan^2 x \cdot \sec^4 x \, dx$$

$$= \int \tan^2 x \cdot \sec^2 x \cdot \sec^2 x \, dx$$

$$= \int \tan^2 x \cdot (1+\tan^2 x) \sec^2 x \, dx$$

$$= \int u^2 (1+u^2) \, du$$

$$= \frac{1}{3}u^3 + \frac{1}{5}u^5 + c$$

$$= \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + c$$

Let, $u = \tan x$

$\Rightarrow du = \sec^2 x \, dx$

m odd

$$\int \tan^3 x \cdot \sec^3 x \, dx$$

$$= \int \tan^2 x \cdot \sec^2 x \cdot \sec x \cdot \tan x \cdot dx$$

$$= \int (\sec^2 x - 1) \sec^2 x \cdot \sec x \cdot \tan x \, dx$$

$$= \int (u^2 - 1) u^2 \, du$$

Let, $u = \sec x$

$\Rightarrow du = \sec x \cdot \tan x \, dx$

7.5 Check the solved problems

I) Lineare factor rule

II) Quadratic factor rule

You will learn quadratic factor rule for the solution of ordinary diff. eq^{n}s.

Definite Integrals

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

Prob. 5 $\displaystyle\int_0^{\pi/2} \log(\cos x)\,dx = \int_0^{\pi/2} \log(\sin x)\,dx = \frac{\pi}{2}\cdot\log\left(\frac{1}{2}\right)$

Prob. 6 Evaluate $\displaystyle\int_3^{29} \frac{(x-2)^{2/3}}{(x-2)^{2/3}+3}\,dx = \int_3^{29}\left[1-\frac{3}{(x-2)^{2/3}+3}\right]dx$

$$\boxed{\int \frac{x}{x+k}\,dx \qquad \frac{x}{x+k} = \frac{(x+k)-k}{x+k} = 1-\frac{k}{x+k}}$$

Let, $(x-2)^{2/3} = u^2$

$\Rightarrow x-2 = u^3$

$\Rightarrow dx = 3u^2\,du$

x	3	29
u	1	3

$$= \int_1^3 \left(1-\frac{3}{u^2+3}\right) 3u^2\,du$$

7. Evaluate

$$\int_0^{\pi/2} \frac{dx}{(a^2\cos^2 x + b^2\sin^2 x)^2}$$

$$= \int_0^{\pi/2} \frac{\sec^4 x}{(a^2+b^2\tan^2 x)^2}\,dx$$

$$= \int_0^{\pi/2} \frac{(1+\tan^2 x)(\sec^2 x)}{(a^2+b^2\tan^2 x)^2}\,dx$$

Let,

$b\tan x = a\tan y$

$\Rightarrow \tan x = \frac{a}{b}\tan y$

$\Rightarrow \sec^2 x\,dx = \frac{a}{b}\sec^2 y\,dy$

x	0	$\pi/2$
y	0	$\pi/2$

Subject : .... Date : ....

$$= \int_0^{\pi/2} \frac{1+\frac{a^2}{b^2}\tan^2 y}{(a^2+a^2\tan^2 y)^2} \cdot \frac{a}{b}\sec^2 y\, dy$$

$$= \frac{1}{a^3b^3}\int_0^{\pi/2} \frac{b^2+a^2\tan^2 y}{\sec^2 y}\, dy$$

$$= \frac{1}{a^3b^3}\int_0^{\pi/2} (b^2\cos^2 y + a^2\sin^2 y)\, dy$$

$$I = \frac{1}{a^3b^3}\int_0^{\pi/2} (b^2\cos^2 y + a^2\sin^2 y)\, dy \text{ - - - - - - - - } (1)$$

Also, $I = \frac{1}{a^3b^3}\int_0^{\pi/2} \left[b^2\cos^2\left(\frac{\pi}{2}-y\right) + a^2\sin^2\left(\frac{\pi}{2}-y\right)\right] dy$

$$= \frac{1}{a^3b^3}\int_0^{\pi/2} (b^2\sin^2 y + a^2\cos^2 y)\, dy \text{ - - - - - - - - } (2)$$

(1) + (2) $\Rightarrow$

$$2I = \frac{1}{a^3b^3}\int_0^{\pi/2} (a^2+b^2)\, dy = \frac{a^2+b^2}{a^3b^3}\int_0^{\pi/2} dy$$

$$\Rightarrow \boxed{I = \frac{\pi}{4} \cdot \frac{a^2+b^2}{a^3b^3}}$$

(Ans.)

Subject : .... Date : ....

8. $\displaystyle\int_0^\infty \frac{x\,dx}{(1+x)(1+x^2)}$

$x = \tan\theta$
$dx = \sec^2\theta \cdot d\theta$

9. $\displaystyle\int_0^\infty \frac{\log\left(x+\frac{1}{x}\right)}{1+x^2}\,dx$

x	0	∞
θ	0	$\pi/2$

$$= \int_0^{\pi/2} \frac{\log(\tan\theta+\cot\theta)}{1+\tan^2\theta}\sec^2\theta \cdot d\theta$$

$$= \int_0^{\pi/2} \log\left(\frac{1}{\sin\theta\cos\theta}\right) d\theta = \int_0^{\pi/2} \log\left(\frac{2}{\sin 2\theta}\right) d\theta$$

$$= \int_0^{\pi/2} \log 2\, d\theta - \int_0^{\pi/2} \log(\sin 2\theta)\, d\theta$$

Beta function

If $m>0$, $n>0$, then

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Advanced Calculus
Schaum's Outline series

Properties

1. $\beta(m,n) = \beta(n,m)$
2. $\beta(m,n) = 2\displaystyle\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta$
3. $\beta(m,n) = \displaystyle\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

Subject : .... Date : ....

4. $\int_0^{\pi/2} \sin^m\theta \cdot \cos^n\theta \, d\theta = \frac{1}{2} \cdot \beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$ $m > -1$, $n > -1$

4. Evaluate $\int_0^2 x^4 (8-x^3)^{-1/3} \, dx$

Let, $x^3 = 8y$

$\Rightarrow x = 2y^{1/3}$

$\Rightarrow dx = \frac{2}{3} y^{-2/3} \, dy$

x	0	2
y	0	1

$= \int_0^1 2^4 \cdot y^{4/3} (8-8y)^{-1/3} \cdot \frac{2}{3} y^{-2/3} \, dy$

$= \frac{16}{3} \int_0^1 y^{(2/3+1)-1} (1-y)^{(-1/3+1)-1} \, dy$

$= \frac{16}{3} \beta\left(\frac{5}{3}, \frac{2}{3}\right)$

Gamma Function

If $n > 0$, then $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, dx$

Properties:

1. $\Gamma(n+1) = n \cdot \Gamma(n) = n!$

2. $\Gamma(n) = 2 \int_0^\infty x^{2n-1} e^{-x^2} \, dx$

3. $\Gamma(1/2) = \sqrt{\pi}$

4. $\beta(m,n) = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)}$

Bessel Functions (Series solution)

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{m+1}{2}\right) \cdot \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)}$$

$$\Gamma(1) = 0! = 1$$

7. Given $\displaystyle\int_0^\infty \frac{x^{m-1}}{1+x} dx = \frac{\pi}{\sin(m\pi)}$

Laplace Transform

$$\Gamma(m) \cdot \Gamma(1-m) = \frac{\pi}{\sin(m\pi)}, \quad 0 < m < 1$$

$$\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)}$$

$$\therefore \beta(m, 1-m) = \frac{\Gamma(m) \cdot \Gamma(1-m)}{1} = \frac{\pi}{\sin(m\pi)} = \int_0^\infty \frac{x^{m-1}}{1+x} dx$$

3. Evaluate

$$\int_0^\infty \sqrt{y} \cdot e^{-y^3} dy$$

let, $y^3 = x$

$\Rightarrow 3y^2 \cdot dy = dx$

$y = x^{1/3}$

$dy = \frac{1}{3} x^{-2/3} dx$

$$= \frac{1}{3} \int_0^\infty x^{1/6} e^{-x} x^{-2/3} dx$$

$$= \frac{1}{3} \int_0^\infty e^{-x} x^{-1/2} dx$$

$$= \frac{1}{3} \int_0^\infty x^{(-1/2+1)-1} e^{-x} dx$$

$$= \frac{1}{3} \cdot \Gamma(1/2) = \frac{1}{3} \cdot \sqrt{\pi}$$

Gamma and Beta Functions

We have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \; ; \; m>0,\; n>0$$

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,dx \; ; \; n>0$$

$$\int_0^{\pi/2} \sin^m x \cdot \cos^n x\,dx = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right) \; ; \quad m>-1,\; n>-1$$

$$= \frac{\Gamma\left(\frac{m+1}{2}\right)\cdot\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(n+1) = n\cdot\Gamma(n)$$

$$\int_0^\infty \frac{x^{m-1}}{1+x}\,dx = \frac{\pi}{\sin(m\pi)} \; ; \quad 0<m<1$$

Example 1 :

$$\int_0^\infty \frac{dx}{1+x^4}$$

Let, $x^4 = t$

$\Rightarrow x = t^{1/4}$

$\Rightarrow dx = \frac{1}{4}t^{-3/4}\,dt$

x	0	∞
t	0	∞

$$= \int_0^\infty \frac{\frac{1}{4}t^{-3/4}\,dt}{1+t}$$

$$= \frac{1}{4}\int_0^\infty \frac{t^{1/4-1}}{1+t}\,dt$$

$$= \frac{1}{4}\cdot\frac{\pi}{\sin\left(\frac{1}{4}\pi\right)} = \frac{1}{4}\cdot\frac{\pi}{1/\sqrt{2}} = \frac{\sqrt{2}}{4}\pi$$

$m = \frac{1}{4} \in (0,1)$

Subject : .... Date : ....

Example 02

$$\int_0^\infty \sqrt{y}\, e^{-y^3}\, dy$$

Let, $y^3 = x$

$\Rightarrow y = x^{1/3}$

$\Rightarrow dy = \frac{1}{3} x^{-2/3}\, dx$

x	0	∞
y	0	∞

$$= \frac{1}{3}\int_0^\infty x^{1/6} e^{-x} x^{-2/3}\, dx$$

$$= \frac{1}{3}\int_0^\infty x^{-1/2} e^{-x}\, dx$$

$$= \frac{1}{3}\int_0^\infty x^{1/2-1} e^{-x}\, dx$$

$$= \frac{1}{3}\cdot \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{3}$$

Example 03

Evaluate $\underbrace{\int_0^1 \frac{x^2\, dx}{\sqrt{1-x^4}}}_{(I_1)} \times \underbrace{\int_0^1 \frac{dx}{\sqrt{1+x^4}}}_{(I_2)}$

For I_1, let $x^2 = \sin\theta \Rightarrow x = \sin^{1/2}\theta$

$\Rightarrow dx = \frac{1}{2}\sin^{-1/2}\theta \,.\, \cos\theta\, d\theta$

x	0	1
θ	0	$\pi/2$

$$I_1 = \frac{1}{2}\int_0^{\pi/2} \frac{\sin\theta \,.\, \sin^{-1/2}\theta \cos\theta}{\sqrt{\cos^2\theta}}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta \,.\, \cos^0\theta\, d\theta$$

$$= \frac{1}{2}\cdot \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\cdot \Gamma\left(\frac{0+1}{2}\right)}{2\cdot\Gamma\left(\frac{\frac{1}{2}+0+2}{2}\right)} = \frac{1}{4}\cdot\frac{\Gamma(3/4)\cdot\Gamma(1/2)}{\Gamma(5/4)}$$

$$= \frac{1}{4}\cdot\frac{\Gamma(3/4)\,.\,\sqrt{\pi}}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} = \frac{\Gamma(3/4)\,.\,\sqrt{\pi}}{\Gamma(1/4)}$$

Shajahan Products

CamScanner

Subject : .... Date : ....

For I_2, Let, $x^2 = \tan\theta \Rightarrow x = \tan^{1/2}\theta$

$\Rightarrow dx = \frac{1}{2}\tan^{-1/2}\theta \cdot \sec^2\theta \, d\theta$

x	0	1
θ	0	$\pi/4$

$$\therefore I_2 = \frac{1}{2}\int_0^{\pi/4} \frac{\tan^{-1/2}\theta \sec^2\theta}{\sqrt{\sec^2\theta}} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{\sec\theta}{\sqrt{\tan\theta}} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{d\theta}{\sqrt{\sin\theta\cos\theta}}$$

$$= \frac{1}{\sqrt{2}} \cdot \int_0^{\pi/4} \frac{d\theta}{\sqrt{2\sin\theta\cos\theta}} = \frac{1}{\sqrt{2}} \cdot \int_0^{\pi/4} \frac{d\theta}{\sqrt{\sin 2\theta}}$$

Let, $2\theta = t$

$\Rightarrow \theta = \frac{1}{2}t$

$\Rightarrow d\theta = \frac{1}{2}dt$

θ	0	$\pi/4$
t	0	$\pi/2$

$$= \frac{1}{\sqrt{2}} \cdot \int_0^{\pi/2} \frac{\frac{1}{2}dt}{\sqrt{\sin t}}$$

$$= \frac{1}{2\sqrt{2}} \cdot \int_0^{\pi/2} \sin^{-1/2}t \cdot \cos^0 t \, dt$$

$$= \frac{1}{2\sqrt{2}} \cdot \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right) \cdot \Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+0+2}{2}\right)}$$

$$= \frac{1}{4\sqrt{2}} \cdot \frac{\Gamma(1/4) \cdot \sqrt{\pi}}{\Gamma(3/4)}$$

Thus, $$I_1 \times I_2 = \frac{\Gamma(3/4) \cdot \sqrt{\pi}}{\Gamma(1/4)} \cdot \frac{1}{4\sqrt{2}} \cdot \frac{\Gamma(1/4) \cdot \sqrt{\pi}}{\Gamma(3/4)}$$

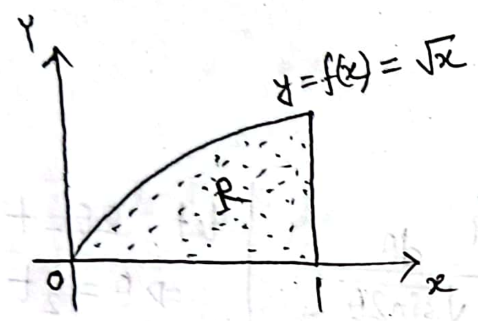
$$= \frac{\pi}{4\sqrt{2}}$$

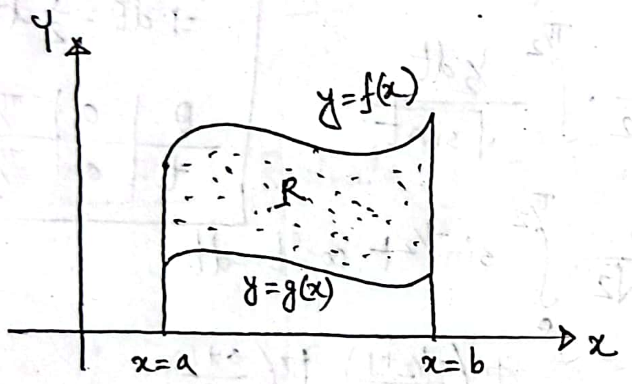
Subject : .... Date : ....

Chapter 06 (C:E.T)

Applications of Definite Integrals in Geometry, Science and Engineering

6.1 Area between two curves





$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*) . \Delta x_k = \int_a^b f(x) . dx$$

Area between two curves

Type-I region

$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \left[f(x_k^*) - g(x_k^*) \right] \Delta x_k$$

$$= \int_a^b \left[f(x) - g(x) \right] dx$$

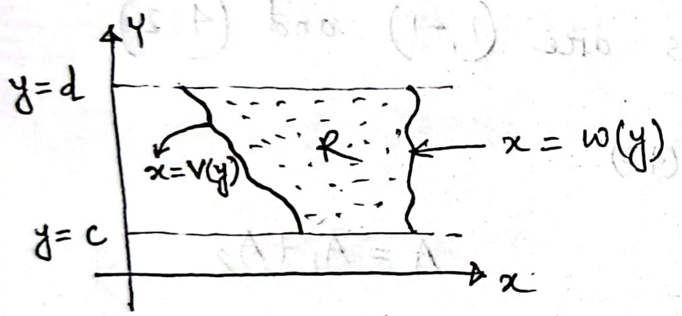
Subject : .... Date : ....

Find the area of the region bounded above by $y=f(x)$, below by $y=g(x)$, to the left by $x=a$, and to the right by $x=b$.

$f(x), g(x)$ should be i) continuous

ii) Non-negative (graph will lie above x-axis)

Type - II region :



$$A = \int_c^d [w(y) - v(y)]\, dy$$

Example 4 Find the area of the region enclosed by $y^2 = x$ and $y = x-2$

Example 5 [Same problem], different method

Subject : .... Date : ....

Soln:

To get the region R, we solve,

$$y^2 = y+2$$

$$\Rightarrow y^2 - y - 2 = 0$$

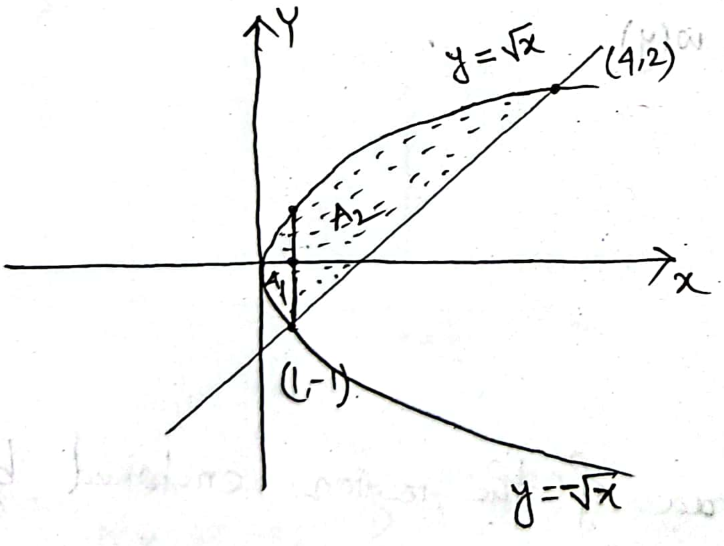
$$\Rightarrow (y-2)(y+1) = 0$$

$$\Rightarrow y = -1, 2$$

$y = -1$, $x = 1$

$y = 2$, $x = 4$

$\therefore$ the point of intersections are $(1,-1)$ and $(4,2)$.



$$A = A_1 + A_2$$

$$A_1 = \int_0^1 \left[\sqrt{x} - (-\sqrt{x})\right] dx$$

$$= 2\int_0^1 \sqrt{x}\, dx = 2 \times \frac{2}{3}\left[x^{3/2}\right]_0^1$$

$$= \frac{4}{3}(1-0) = \frac{4}{3}$$

$$A_2 = \int_1^4 \left[\sqrt{x} - (x-2)\right] dx$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_1^4$$

$$= \left(\frac{2}{3} \times 8 - 8 + 8\right) - \left(\frac{2}{3} - \frac{1}{2} + 2\right)$$

$$= \frac{19}{6}$$

$\therefore$ desired area is,

$$A = A_1 + A_2 = \frac{4}{3} + \frac{19}{6} = \frac{9}{2}$$

Subject : ....   Date : ....

Example : 5 (Type II region)

Sol^n:

$$A = \int_{-1}^{2} [(y+2) - y^2]\, dy$$

$$= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2}$$

$$= 9/2$$

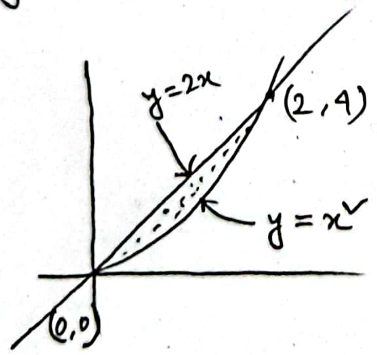
$x = y+2 \; (w(y))$

$x = y^2 \; (v(y))$

Exercises Find the area of the region enclosed by

1. $y^2 = 4x$, $y = 2x-4$

2. $y = 2x$ and $y = x^2$



$2x = x^2$

$x = 0, 2$

$$\text{Area} = \int_{0}^{2} (2x - x^2)\, dx$$

3. $y = x^2$, $y = \sqrt{x}$, $x = \frac{1}{4}$, $x = 1$

4. $x^2 = y$, $x = y - 2$

5. $y = x$, $y = 4x$, $y = -x+2$

6. $y = \sqrt{x}$, $y = -\frac{1}{4}x$

7. $x = 2 - y^2$, $x = -y$

LARSON
EDWARDS