CHAPTER 10

Partial Differential Equations

In the previous chapters of this text our attention has been focused upon ordinary differential equations. In this chapter we will limit our consideration to a special kind of differential equation known as partial differential equation. The subject of partial differential equation is a vast one.

"A partial differential equation is one which contains more than one independent variables." This type of equation plays a very important role in sciences and engineering dealing with wave motions, such as heat, light, electricity, magnetism, radio, radar, television and weather etc.

In general, z is taken as dependent variable whereas x and y are treated as independent variables so that

$$z = f(x, y).$$

The partial derivatives of z with respect to x and y are denoted by p and q respectively, so that

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y} \tag{10.1}$$

we will also use

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial p}{\partial x} \quad ; \quad s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}, \; t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial q}{\partial y} \tag{10.2}$$

Order and Degree: The order of a PDE is the order of the highest derivative of the equation. The degree of the PDE is the power of the highest derivative of the equation.

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$\therefore$ Equation $\dfrac{\partial^2 z}{\partial x^2}+5\dfrac{\partial z}{\partial x}+6\dfrac{\partial z}{\partial y}=0$ is the 2nd order first degree PDE.

Formation: The PDE may be derived in the following two ways

i) By eliminating arbitrary constants.

ii) By eliminating arbitrary functions.

10.1 Formation of PDE by eliminating arbitrary constants

Let z be a function of two independent variables x and y connected by

$$f(x, y, z, a, b)=0 \tag{10.3}$$

where a and b are arbitrary constants. By differentiating (10.3) partially with respect to x and y we obtain

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial x}=\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}=0 \tag{10.4}$$

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial y}=\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}=0 \tag{10.5}$$

Now arbitrary constants may be eliminated applying (10.3), (10.4), (10.5) yielding a partial differential equation of order one given by

$$f(x, y, z, p, q)=0$$

Example 10.1 Eliminate the arbitrary constants a and b from

$$z=ax^3+by^3+ab.$$

Solution Differentiating

$$z=ax^3+by^3+ab$$

partially with respect to x and y we have

$$p=3ax^2$$

or

$$a=\frac{p}{3x^2}$$

and

$$q=3by^2$$

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or $$b = \frac{q}{3y^2}$$

putting the values of a and b in the given equation, we get

$$z = \frac{p}{3x^2}.x^3 + \frac{q}{3y^2}y^3 + \frac{pq}{9x^2y^2}$$

or $$9x^2y^2z = 3px^3y^2 + 3qx^2y^3 + pq.$$

Note: If the number of arbitrary constants is less than or equal to the number of independent variables then the differential equation formed by the elimination of arbitrary constants would be of the first order.

Example 10.2 Eliminate a, b from $2z = (ax+y)^2 + b$.

Solution Differentiating partially the given equation with respect to x and y, we get

$$2p = 2(ax+y)a \quad \text{and} \quad 2q = 2(ax+y)$$

or $$p = (ax+y)a \quad \text{or,} \quad q = ax+y.$$

Now $$px + qy = (ax+y)^2 = q^2.$$

Example 10.3 Eliminate the constants a, b, c from the equation of ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution Differentiating $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{10.6}$$

partially with respect to x and y, we get

$$\frac{x}{a^2} + \frac{z}{c^2}p = 0 \quad \text{or} \quad \frac{c^2}{a^2} = -\frac{pz}{x} \tag{10.7}$$

and $$\frac{y}{b^2} + \frac{z}{c^2}q = 0 \quad \text{or} \quad \frac{c^2}{b^2} = -\frac{qz}{y} \tag{10.8}$$

Differentiating (10.7) again partially with respect to x, we get

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$$\frac{1}{a^2}+\frac{1}{c^2}(p^2+zr)=0 \text{ or } \frac{c^2}{a^2}=-\left(p^2+zr\right) \tag{10.9}$$

Thus
$$\frac{c^2}{a^2}=-\left(p^2+zr\right)=-\frac{pz}{x}$$

or
$$pz=p^2x+xzr$$

which is a differential equation of order 2.

Similarly, differentiating (10.8) with respect to y we get

$$\frac{1}{b^2}+\frac{1}{c^2}(q^2+zt)=0.$$

$$\therefore \quad \frac{c^2}{b^2}=-(q^2+zt)=-\frac{qz}{y}$$

or
$$qz=yq^2+yzt.$$

Note-1: If the number of arbitrary constants are more than the number of independent variables then the differential equation will be of the minimum second order.

Note-2: The PDE formed by eliminating arbitrary constants is not always unique.

10.2 Elimination of arbitrary functions

Let u, v be two functions of x, y, z connected by the relation

$$\phi(u, v)=0 \tag{10.10}$$

Regarding z as the dependent variable and differentiating (10.10) partially with respect to x and y, we obtain

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0$$

and
$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0$$

Eliminating
$$\frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial v}$$

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from these, we get $\begin{vmatrix} \dfrac{\partial u}{\partial x}+p\dfrac{\partial u}{\partial z} & \dfrac{\partial v}{\partial x}+p\dfrac{\partial v}{\partial z} \\ \dfrac{\partial u}{\partial y}+q\dfrac{\partial u}{\partial z} & \dfrac{\partial v}{\partial y}+q\dfrac{\partial v}{\partial z} \end{vmatrix}=0$

or
$$p\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right)+q\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}\right)$$
$$+\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)=0 \qquad (10.11)$$

which can be rearranged in the form
$$pP+qQ=R.$$

where
$$\frac{P}{\begin{vmatrix} \dfrac{\partial v}{\partial y} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial z} & \dfrac{\partial u}{\partial z} \end{vmatrix}}=\frac{Q}{\begin{vmatrix} \dfrac{\partial v}{\partial z} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial u}{\partial x} \end{vmatrix}}=\frac{R}{\begin{vmatrix} \dfrac{\partial v}{\partial x} & \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} & \dfrac{\partial u}{\partial y} \end{vmatrix}} = K.$$

Equation (10.11) is a partial differential equation linear in p, q and free of the arbitrary function $\phi(u, v)$.

Example 10.4 Find the differential equation arising from
$$\phi=(x-y+z,\ x^2+2y^2-3z^2)=0.$$

Solution Let $u=x-y+z$, $v=x^2+2y^2-3z^2$

$$\frac{\partial u}{\partial x}=1 \qquad \frac{\partial v}{\partial x}=2x$$
$$\frac{\partial u}{\partial y}=-1 \qquad \frac{\partial v}{\partial y}=4y$$
$$\frac{\partial u}{\partial z}=1 \qquad \frac{\partial v}{\partial z}=-6z$$

Now $\qquad \phi(u, v)=0.$

Applying equation (10.11) we get
$$p(4y+6z)+q(-6z-2x)+(4y+2x)=0$$

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or $$p(2y+3z)-q(x+3z)+(x+2y)=0.$$

is the required differential equation.

Example 10.5 Eliminate arbitrary function ϕ from the equation

$$\phi(\tan x+\sin^{-1} y-\log z,\ e^x-\sec y+z^3)=0$$

Solution Let $u=\tan x+\sin^{-1} y-\log z$

$$v=e^x-\sec y+z^3$$

$$\frac{\partial u}{\partial x}=\sec^2 x,\ \frac{\partial u}{\partial y}=\frac{1}{\sqrt{1-y^2}},\ \frac{\partial u}{\partial z}=-\frac{1}{z}$$

$$\frac{\partial v}{\partial x}=e^x,\ \frac{\partial v}{\partial y}=-\sec y\tan y,\ \frac{\partial v}{\partial z}=3z^2$$

Now $$\phi\,(u,v)=0.$$

Applying equation (10.11) we get,

$$p\left(\frac{1}{z}\sec y\tan y-\frac{1}{\sqrt{1-y}}3z^2\right)+q\left(\sec^2 x.3z^2+\frac{1}{z}.e^x\right)$$

$$+\left(-\sec^2 x\sec y\tan y-\frac{1}{\sqrt{1-y^2}}e^x\right)=0.$$

Example 10.6 Eliminate the arbitrary functions $f(x)$ and $g(y)$ from

$$z=yf(x)+xg(y).$$

Solution Differentiating

$$z=yf(x)+xg(y) \tag{10.12}$$

with respect to x and y, we have

$$p=yf'(x)+g(y)$$

and $$q=f(x)+xg'(y) \tag{10.13}$$

Again differentiating p and q with respect to x we get,

$$r=yf''(x);\quad s=f'(x)+g'(y)$$

$$f'(x)=\frac{p-g(y)}{y};\quad g'(y)=\frac{q-f(x)}{x}$$

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$$s = f'(x) + g'(y) = \frac{p - g(y)}{y} + \frac{q - f(x)}{x}$$

or
$$xys = px + qy - [yf(x) + xg(y)]$$
$$= px + qy - z$$

is the required partial differential equation.

Example 10.7 Obtain the partial differential equation by eliminating the arbitrary functions f and g from

$$z = f(x\cos\alpha + y\sin\alpha - at) + g(x\cos\alpha + y\sin\alpha + at).$$

Solution We have

$$\frac{\partial z}{\partial x} = \cos\alpha f'(x\cos\alpha + y\sin\alpha - at)$$
$$+ \cos\alpha g'(x\cos\alpha + y\sin\alpha + at)$$

$$\frac{\partial^2 z}{\partial x^2} = \cos^2\alpha f''(x\cos\alpha + y\sin\alpha - at)$$
$$+ \cos^2\alpha g''(x\cos\alpha + y\sin\alpha + at)$$

similarly
$$\frac{\partial^2 z}{\partial y^2} = \sin^2\alpha f''(x\cos\alpha + y\sin\alpha - at)$$
$$+ \sin^2\alpha g''(x\cos\alpha + y\sin\alpha + at)$$

$$\frac{\partial^2 z}{\partial t^2} = a^2 f''(x\cos\alpha + y\sin\alpha - at)$$
$$+ a^2 g''(x\cos\alpha + y\sin\alpha + at)$$

or
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{a^2}\frac{\partial^2 z}{\partial t^2}$$

the required differential equation.

Example 10.8 Eliminate the functions f and g from

$$z = f(x) + e^y g(x).$$

Solution: We get, $q = e^y g(x)$

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$$t = e^y g(x)$$

$$\therefore \qquad t - q = 0$$

is the required differential equation.

Example 10.9 Eliminate the functions from $x = f(z) + g(y)$.

Solution Differentiating $x = f(z) + g(y)$ partially with respect to x and y, we get

$$1 = pf'(z) \tag{10.14}$$

$$0 = qf'(z) + g'(y) \tag{10.15}$$

Differentiating (10.14) with respect to x.

$$0 = p^2 f''(z) + rf'(z) \tag{10.16}$$

Differentiating (10.14) with respect to y.

$$0 = pqf''(z) + sf'(z) \tag{10.17}$$

From (10.14) $f'(z) = \dfrac{1}{p}$.

From (10.16) $\qquad f''(z) = -\dfrac{r}{p^2} f'(z) = -\dfrac{r}{p^3}$.

Equation (10.17) becomes $0 = -pq\dfrac{r}{p^3} + \dfrac{s}{p}$

or $$\frac{s}{p} - \frac{pqr}{p^3} = 0$$

or $$p^2 s - pqr = 0$$

or $$ps - qr = 0.$$

is the required partial differential equation.

Example 10.10 Eliminate constant and function from

$$z = ax^2 + g(y).$$

Solution

$$p = 2ax$$

$$q = g'(y)$$

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$$r = 2a$$

$$s = 0$$

$$p = 2a.x = rx$$

$$\therefore \quad p - xr = 0$$

and $s = 0$ are the required partial differential equation.

10.3 Classification of integrals

There are different forms of solutions of partial differential equation. For simplicity, we will consider by taking two independent variables. But this can be extended to any number of variables. Solutions of partial differential equations appear in almost four forms, namely

1. Complete solution; 2. Particular solution; 3. Singular solution and 4. General solution

Four types of solutions will be discussed with examples.

Complete solution: The differential equation formed by eliminating arbitrary constants a, b from

$$z = ax^3 + by^3 + ab \tag{10.18}$$

is

$$3px^3y^2 + 2qy^3x^2 + pq = 9x^2y^2z \tag{10.19}$$

Equation (10.18) contains two arbitrary constants equal to the number of independent variables of differential equation (10.19) and is called the complete solution of (10.19).

Definition: A solution which contains as many arbitrary constants as the number of independent variables of a differential equation is called the complete solution.

Particular integral: Assigning particular values for arbitrary constants, particular integrals of the differential equation are obtained.

$$z = x^3 + y^3 + 1$$

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is a particular integral of (10.19) which is obtained by letting $a = 1$ and $b = 1$ in the complete solution.

Singular solution: Differentiating (10.18) partially with respect to a and b we get,

$$0 = x^3 + b$$

or

$$b = -x^3$$

$$0 = y^3 + a$$

or

$$a = -y^3$$

and eliminating a and b we obtain

$$z = -x^3y^3 - x^3y^3 + x^3y^3$$

or

$$z + x^3y^3 = 0$$

is the singular solution.

Singular solution: Singular solution is a solution of differential equation which is independent of arbitrary constants and which is obtained by following a definite rule.

General solution: General solution does not contain any arbitrary constant but is different from singular solution. In equation (10.18), making b a function of a, say $b = a$, we obtain

$$z = a(x^3 + y^3 + a)$$

Differentiating with respect to a,

$$0 = x^3 + y^3 + 2a$$

or

$$a = -\frac{1}{2}(x^3 + y^3)$$

Eliminating a we have $z = -\frac{1}{2}(x^3 + y^3)\left[x^3 + y^3 - \frac{1}{2}(x^3 + y^3)\right]$

$$= -\frac{1}{2}(x^3 + y^3)\frac{x^3 + y^3}{2}$$

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or $$4z + (x^3 + y^3)^2 = 0.$$

which is a general solution and is different from singular solution.

Note: An arbitrary functional relation between a and b is of the form

$$b = \phi(a).$$

Here the elimination becomes rather difficult. So we can choose any convenient relation between the arbitrary constants. Hence there are infinite number of general solutions.

10.4 Linear partial differential equation of first order

Definition: A partial differential equation involving partial derivatives p and q only and not higher partial derivatives is called a first order partial differential equation.

If the degree of p and q is unity, then it is called a linear partial differential equation of order one. Thus

$$px^2 + qy^2 = z^2$$

is a linear equation of first order.

On the otherhand, if the degrees of p and q are higher than the first, then the equation is called non-linear.

$$p^2x^3 + pqxy + q^2y^2 = z^2$$

is non-linear first order partial differential equation.

Lagrange's Linear Equation: A linear differential equation of the form $pP + qQ = R$ where P, Q, R are functions of x, y, z respectively is known as Lagrange's linear partial differential equation. To obtain the solution of the above equation we have the following rule.

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Lagrange's method: Let u, v be functions of x, y, z connected by an arbitrary relation between them, say of the form

$$\phi(u, v) = 0. \tag{10.20}$$

Differentiating (10.20) partially with respect to x and y, we have

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

and
$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0.$$

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$, we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y} + q\frac{\partial v}{vz} \end{vmatrix} = 0.$$

or
$$p\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right) + q\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}\right) = \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}.$$

which can be rearranged in the form

$$pP + qQ = R$$

where
$$P = \frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}$$

or
$$\frac{P}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}} = 1$$

and
$$Q = \frac{\partial u}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}$$

or
$$\frac{Q}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}} = 1$$

and
$$R = \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}$$

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or
$$\frac{R}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}}=1$$

Obviously,
$$\frac{P}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}}=\frac{Q}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}}=\frac{R}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}} \quad (10.21)$$

Let us consider the equations, $u = a$ and $v = b$ where a and b are arbitrary constants. Differentiating these integrals, we get

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=0$$

$$\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz=0$$

Solving above equations for dx, dy, dz, we have

$$\frac{dx}{-\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right)}=\frac{dy}{-\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)}$$

$$=\frac{dz}{-\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)} \quad (10.22)$$

Comparing this with (10.21), we obtain the equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \quad (10.23)$$

which are known as Lagrange's equations or auxiliary equations.

10.5 Working Rule for Solving $Pp+Qq=R$ by Langrange's Method

Step-1: Put the given first order linear partial differential equation in the standard form

$$Pp+Qq=R. \quad (10.24)$$

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Step-2: Write down Lagrange's auxiliary equation for (10.24) namely,

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \tag{10.25}$$

Step-3: Find the two independent solutions of (10.25) in the form

$$u(x, y, z)=a$$

and

$$v(x, y, z)=b$$

Step-4: The general solution or integral of (10.24) is then written in one of the following three equivalent forms:

$$\phi(u, v)=0,\ u=\phi(a),\ v=\phi(b)$$

Cor. If the differential equation is of the form

$$P\frac{\partial z}{\partial x}+Q\frac{\partial z}{\partial y}+R\frac{\partial z}{\partial t}=s \tag{10.26}$$

having three independent, variables x, y, t, then the Lagrange's auxiliary equation are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dt}{R}=\frac{dz}{s} \tag{10.27}$$

The general solutions of (10.26) is

$$\phi(u, v, w)=0 \quad \text{where } u, v, w$$

are three independent solutions of (10.27). The system can be generalized to any number of independent variables.

Example 10.11 Solve $yzp+zxq=xy$.

Solution Auxiliary equations are, $\dfrac{dx}{yz}=\dfrac{dy}{zx}=\dfrac{dz}{xy}$

From first two,

$$\frac{dx}{y}=\frac{dy}{x}.$$

Integrating

$$x^2-y^2=c_1.$$

Similarly from last two, we have,

$$y^2-z^2=c_2.$$

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Hence the required general solution is

$$\phi(x^2 - y^2, y^2 - z^2) = 0$$

where ϕ is an arbitrary function.

Example 10.12 Solve $(y+z)p + (z+x)q = x+y$.

Solution: The subsidiary equations are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \tag{10.28}$$

By the theory of proportion, we find

$$\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{-(x-y)}$$

$$\therefore \quad \log(x+y+z) + 2\log(x-y) = \log c_1$$

$$\therefore \quad (x+y+z)(x-y)^2 = c_1.$$

Again we have $\quad \dfrac{dx-dy}{x-y} = \dfrac{dy-dz}{y-z}$

integrating $\quad \log(x-y) = \log(y-z) + \log c_2$

or $\quad \dfrac{x-y}{y-z} = c_2.$

$\therefore$ The general solution is

$$\phi\left[(x+y+z)(x-y)^2, \frac{x-y}{y-z}\right] = 0.$$

Example 10.13 Solve $pz - qz = z^2 + (x+y)^2$.

Solution The subsidiary equations are

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2}.$$

Taking the first and the second members we have

$$dx + dy = 0.$$

$$\therefore \quad x + y = c_1.$$

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Taking the first and the last members, we have

$$\frac{zdz}{z^2+(x+y)^2}=dx$$

or

$$\frac{2zdz}{z^2+c_1^2}=2dx.$$

Integrating, we have $\log(z^2+c_1^2)=2x+c_2$

$$\log(z^2+x^2+y^2+2xy)-2x=c_2.$$

Hence the complete integral is

$$\phi\left[x+y,\ \log\{(z^2+(x+y)^2\}-2x\right]=0.$$

Example 10.14 Solve $p\cos(x+y)+q\sin(x+y)=z$.

Solution Lagrange's auxiliary equations are

$$\frac{dx}{\cos(x+y)}=\frac{dy}{\sin(x+y)}=\frac{dz}{z}$$

$$\therefore\ \frac{dx+dy}{\cos(x+y)+\sin(x+y)}=\frac{dx-dy}{\cos(x+y)-\sin(x+y)}$$

or

$$\frac{[\cos(x+y)-\sin(x+y)](dx+dy)}{\cos(x+y)+\sin(x+y)}=dx-dy.$$

Integrating, $\log[\cos(x+y)+\sin(x-y)]=x-y+\log a$

or

$$[\cos(x+y)+\sin(x+y)]e^{y-x}=a$$

Again

$$\frac{dx+dy}{\cos(x+y)+\sin(x+y)}=\frac{dz}{z}$$

$$\therefore\ \frac{\frac{1}{\sqrt{2}}(dx+dy)}{\sin\frac{\pi}{4}\cos(x+y)+\cos\frac{\pi}{4}\sin(x+y)}=\frac{dz}{z}$$

or

$$\frac{\frac{1}{\sqrt{2}}(dx+dy)}{\sin\left(x+y+\frac{\pi}{4}\right)}=\frac{dz}{z}$$

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Integrating, $\frac{1}{\sqrt{2}}\cos\tan\left(\frac{x+y}{2}+\frac{\pi}{8}\right)=\log z+\log b$

$$\therefore \qquad \tan\left(\frac{x+y}{2}+\frac{\pi}{8}\right)z^{-\sqrt{2}}=b$$

Hence $\phi\left[\{\cos(x+y)+\sin(x+y)\}\,e^{y-x},\ z^{-\sqrt{2}}\tan\left(\frac{x+y}{2}+\frac{\pi}{8}\right)\right]=0$

is the required solution.

Example 10.15 Solve $p+3q=5z+\tan(y-3x)$.

Solution Auxiliary equations are

$$\frac{dx}{1}=\frac{dy}{3}=\frac{dz}{5z+\tan(y-3x)}.$$

Taking the first two members, we have

$$dy-3dx=0;\quad y-3x=a.$$

Again taking the first and the last members, we have

$$\frac{dx}{1}=\frac{dz}{5z+\tan a}.$$

$$\therefore \qquad x=\frac{1}{5}\log(5z+\tan a)-\log c_1$$

$$\therefore \qquad 5x=\log(5z+\tan a)-\log b$$

or $$[5z+\tan(y-3x)]e^{-5x}=b.$$

The general solution is

$$\phi[y-3x,\ e^{-5x}\{5z+\tan(y-3x)\}]=0.$$

Example 10.16 Solve $(z^2-2yz-y^2)p+(xy+zx)q=xy-zx$.

Solution The auxiliary equations are

$$\frac{dx}{z^2-2yz-y^2}=\frac{dy}{xy+zx}=\frac{dz}{xy-zx}.$$

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Taking x, y, z as multipliers, we have each relation

$$= \frac{xdx + ydy + zdz}{0}.$$

$$\therefore \qquad xdx + ydy + zdz = 0$$

Integrating $\qquad x^2 + y^2 + z^2 = a$

Again taking the last two members, we have

$$\frac{dy}{y+z} = \frac{dz}{y-z}.$$

or $\qquad ydy - (ydz + zdy) - zdz = 0$

Integrating $\qquad \frac{y^2}{2} - yz - \frac{z^2}{2} = c_1$

$$\therefore \qquad y^2 - 2yz - z^2 = b.$$

The general solution is

$$x^2 + y^2 + z^2 = \phi(y^2 - 2yz - z^2) \tag{10.29}$$

Example 10.17 Solve $(3x + y - z)p + (x + y - z)q = 2(z - y)$.

Solution The auxiliary equations are

$$\frac{dx}{3x+y-z} = \frac{dy}{x+y-z} = \frac{dz}{2(z-y)}.$$

Taking 1, –3, –1 as multipliers, we have

each fraction $= \dfrac{dx - 3dy - dz}{3x + y - z - 3x - 3y + 3z - 2z + 2y}$

or $\qquad dx - 3dy - dz = 0.$

Integrating $\qquad x - 3y - z = c_1.$

Again $\qquad \dfrac{dx - dy + dz}{2(x - y + z)} = \dfrac{dx + dy - dz}{4(x + y - z)}$

or $\qquad 2\log(x - y + z) = \log(x + y - z) + \log c_2; \quad \dfrac{(x - y + z)^2}{x + y - z} = c_2.$

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$\therefore$ The solution is, $\phi\left[x-3y-z, \dfrac{(x-y+z)^2}{x+y-z}\right]=0$.

Example 10.18 Solve $z(px-qy)=y^2-x^2$.

Solution $zxp-zyq=y^2-x^2$.

The subsidiary equations are

$$\frac{dx}{zx}=\frac{dy}{-zy}=\frac{dz}{y^2-x^2}.$$

Using x, y, z as multipliers, we have each fraction

$$=\frac{xdx+ydy+zdz}{0}.$$

$$\therefore \qquad xdx+ydy+zdz=0.$$

Integrating $\qquad x^2+y^2+z^2=c_1$.

From the first two numbers, we have

$$ydx+xdy=0.$$

Integrating $\qquad xy=c_2$.

$\therefore$ The general solution is

$$\phi(xy,\ x^2+y^2+z^2)=0.$$

Example 10.19 Find the integral surface of the partial differential equation

$$(x-y)p+(y-z-x)q=z$$

through the circle $\quad z=1,\ x^2+y^2=1$.

Solution Given $(x-y)p+(y-z-x)q=z$.

Lagrange's auxiliary equations are

$$\frac{dx}{x-y}=\frac{dy}{y-z-x}=\frac{dz}{z} \tag{10.30}$$

Each fraction $=\dfrac{dx+dy+dz}{0}$.

So that $\qquad x+y+z=c_1 \qquad (10.31)$

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Taking the last two fractions of (10.30) and using (10.31) we get

$$\frac{dy}{y-(c_1-y)}=\frac{dz}{z}$$

or
$$\frac{2dy}{2y-c_1}-\frac{2dz}{z}=0.$$

Integrating, $\log(2y-c_1)-2\log z=\log c_2$

or
$$\frac{2y-c_1}{z^2}=c_2 \tag{10.32}$$

or
$$\frac{2y-x-y-z}{z^2}=c_2$$

or
$$\frac{y-x-z}{z^2}=c_2 \tag{10.33}$$

The given curve is given by

$$z=1,\ x^2+y^2=1.$$

Putting $z = 1$ in (10.31) and (10.33), we get

$$x+y=c_1-1 \text{ and } y-x=c_2+1 \tag{10.34}$$

But
$$2(x^2+y^2)=(x+y)^2+(y-x)^2. \tag{10.35}$$

Using (10.34), (10.35) becomes

$$2=(c_1-1)^2+(c_2+1)^2$$

or
$$c_1^2+c_2^2-2c_1+2c_2=0$$

Putting the values of c_1 and c_2 from (10.32) and (10.33) we get the required surface as

$$(x+y+z)^2+\frac{(y-x-z)^2}{z^4}-2(x+y+z)+\frac{2(y-x-z)}{z^2}=0$$

or
$$z^4(x+y+z)^2+(y-z-x)^2-2z^4(x+y+z)$$
$$+2z^2(y-z-x)=0.$$

is the required partial differential equation.

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10.6 Lagrange's method for more than two independent variables

Let the linear equation with n independent variables be

$$P_1p_1 + P_2p_2 + P_3p_3 + P_np_n = R \tag{10.36}$$

where $$p_i = \frac{\partial z}{\partial x_i}, \qquad i = 1, 2, 3, \cdots\cdots, n$$

and $$P_1, P_2, P_3, \cdots\cdots, P_n$$

and R are functions of $x_1, x_2, x_3, \cdots\cdots, x_n$ and z.

The corresponding auxiliary equations are

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \frac{dx_3}{P_3} = \cdots\cdots = \frac{dx_n}{P_n} = \frac{dz}{R}$$

Let $$u_1 = a_1, u_2 = a_2, u_3 = a_3, u_n = a_n$$

be n independent integrals where, a_1, a_2, a_3, a_n are arbitrary constants. Then the general integral of (10.36) is

$$\phi(u_1, u_2, u_3, \cdots\cdots, u_n) = 0$$

where ϕ is an arbitrary function.

It is to be noted that the above procedure is generalization of Lagrange's method.

Examples 10.20 Solve $x_2x_3p_1 + x_3x_1p_2 + x_1x_2p_3 + x_1x_2x_3 = 0$.

Solution Re-writing the given equation in standard form

$$x_2x_3p_1 + x_3x_1p_2 + x_1x_2p_3 = -x_1x_2x_3$$

The auxiliary equations are

$$\frac{dx_1}{x_2x_3} = \frac{dx_2}{x_3x_1} = \frac{dx_3}{x_1x_2} = \frac{dz}{-x_1x_2x_3}$$

Taking the 1st and 4th members, we have

$$x_1dx_1 + dz = 0$$

so that $$x_1^2 + 2z = c_1$$

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Similarly taking 1st and 2nd, 1st and 3rd members, we have respectively

$$x_1^2 - x_2^2 = c_2$$

and $$x_1^2 - x_3^2 = c_3$$

Thence the required general solution is,

$$\phi\left(x_1^2 + 2z,\ x_1^2 - x_2^2,\ x_1^2 - x_3^2\right) = 0$$

Examples 10.21 Solve

$$(t+y+z)\frac{\partial t}{\partial x} + (t+z+x)\frac{\partial t}{\partial y} + (t+x+y)\frac{\partial t}{\partial z} = x+y+z.$$

Solution The auxiliary equations are

$$\frac{dx}{t+y+z} = \frac{dy}{t+z+x} = \frac{dz}{t+x+y} = \frac{dt}{x+y+z}$$

Each of these ratios

$$= \frac{dx+dy+dz+dt}{3(x+y+z+t)} = \frac{dx-dt}{-(x-t)} = \frac{dy-dt}{-(y-t)} = \frac{dz-dt}{-(z-t)}$$

Integrating we have

$$(x+y+z+t)^{\frac{1}{3}}(x-t) = a$$

$$(x+y+z+t)^{\frac{1}{3}}(y-t) = b$$

$$(x+y+z+t)^{\frac{1}{3}}(z-t) = c$$

Hence the general solution is,

$$\left[\phi(x+v+z+t)^{\frac{1}{3}}(x-t),\ (x+y+z+t)^{\frac{1}{3}}(y-t),\ (x+y+z+t)^{\frac{1}{3}}(z-t)\right] = 0$$

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10.7 Non-Linear partial differential equation of the first order

The equations which involve p and q other than in the first degree are called non-linear partial differential equations of the first order. For such equations, the complete solution consists of only two arbitrary constants (i.e. equal to the number of independent variables involved) and particular solution is obtained by assigning particular values to the constants.

Before proceeding to indicate a method of integration which is applicable to the most general equation of the first order, it will be advisable to consider a few standard form of equations involving two independent variables. An equation is not considered fully solved unless its general, singular as well as complete integrals are obtained.

10.8 Standard form I

Equations involving only p and q and not x, y, z. Under this standard form we consider the equation as

$$f(p, q) = 0 \tag{10.37}$$

A solution of (10.37) is given by

$$z = ax + by + c \tag{10.38}$$

where a and b are connected by the relation

$$f(a, b) = 0 \tag{10.39}$$

Since $$p = \frac{\partial z}{\partial x} = a$$

and $$q = \frac{\partial z}{\partial y} = b$$

which when substituted in (10.39), gives (10.37). From (10.39) we may find b in terms of a i.e. $b = \phi(a)$ say. The complete integral of (10.37) is

$$z = ax + \phi(a).y + c \tag{10.40}$$

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10.9 General integral

Taking $c = \psi(a)$ where ψ is an arbitrary function, we have

$$z = ax + \phi(a).y + \psi(a) \tag{10.41}$$

Differentiating (10.41) w.r.t. a, we have

$$0 = x + \phi'(a)y + \psi'(a) \tag{10.42}$$

Eliminating a between (10.41) and (10.42) we get the general integral.

Singular integral: Singular integral (if exists) is obtained by eliminating arbitrary constants a and c. Differentiating (10.40) w.r.t a and c we have

$$0 = x + \phi'(a)y$$

and

$$0 = 1.$$

2nd result is inadmissible which shows that such equations have no singular integral.

Note: Sometimes, change of variables reduces the equations to the form of standard 1.

Examples 10.22 Solve $q = 3p^2$.

Solution The equation is of the form $f(p,q) = 0$.

$\therefore$ The complete solution is given by

$$z = ax + by + c$$

where

$$b = 3a^2.$$

$\therefore$ The complete solution is

$$z = ax + 3a^2 y + c.$$

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Examples 10.23 Solve $p^2x^2 + q^2y^2 = z^2$.

Solution The equations may be written as

$$\left(\frac{x}{z}p\right)^2 + \left(\frac{y}{z}q\right)^2 = 1.$$

Putting $$\frac{dx}{x} = dX;\ \frac{dy}{y} = dY;\ \frac{dz}{z} = dZ$$

i.e. $$X = \log x,\quad Y = \log y,\quad Z = \log z$$

the equation becomes.

$$P^2 + Q^2 = 1$$

were $$P = \frac{\partial Z}{\partial X},\ Q = \frac{\partial Z}{\partial Y}$$

which is of the form of standard 1.

The complete solution is given by

$$Z = aX + bY + c_1$$

where $$a^2 + b^2 = 1$$

or $$b = \sqrt{1 - a^2}$$

$$\therefore \quad Z = aX + \sqrt{1 - a^2}\,Y = c_1$$

or $$\log z = a\log x + \sqrt{1 - a^2}\,\log y + c_1$$

If we put $a = \cos\alpha$, then $\log z = \cos\alpha \log x + \sin\alpha \log y + \log c$

where $c_1 = \log c$

$$\therefore \quad z = cx^{\cos\alpha}.y^{\sin\alpha}$$

is the required solution.

Examples 10.24 Solve $pq = x^m y^n z^{21}$.

Solution The equation may be re-written as

$$\frac{pz^{-l}}{x^m}\frac{qz^{-l}}{y^n} = 1.$$

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let $$Z=\frac{z^{1-l}}{1-l},\quad X=\frac{x^{m+1}}{m+1},\quad Y=\frac{y^{n+1}}{n+1}$$

Then $$\frac{\partial Z}{\partial X}=\frac{\partial Z}{\partial x}\frac{dx}{dX}=z^{-l}p\frac{1}{x^m}$$

$$\frac{\partial Z}{\partial Y}=\frac{\partial Z}{\partial y}\frac{dy}{dY}=z^{-l}q\frac{1}{y^n}$$

$\therefore$ The equation becomes, $$\frac{\partial Z}{\partial X}.\frac{\partial Z}{\partial Y}=1$$

or $$PQ=1.$$

$\therefore$ Solution is $$Z=aX+bY+c$$

where $$ab=1.$$

$\therefore$ $$Z=aX+\frac{1}{a}Y+C$$

or $$\frac{z^{1-l}}{1-l}=a\frac{x^{m+1}}{m+1}+\frac{y^{n+1}}{a(n+1)}+c.$$

10.10 Standard form II

This is analogous to Clairaut's form in ordinary differential equation. The form of the equation for two independent variables is

$$z=px+qy+f(p,q) \tag{10.43}$$

The complete integral is

$$z=ax+by+f(a,b) \tag{10.44}$$

which contains two arbitrary constants a and b. Notice that the complete integral of (10.43) is obtained replacing p and q by a and b respectively.

Singular integral: Differentiating (10.44) with respect to a and b, we get

$$0=x+\frac{\partial f}{\partial a};\quad 0=y+\frac{\partial f}{\partial b} \tag{10.45}$$

Eliminating a and b form (10.45) we get the singular integral

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General integral: Let $b = \phi(a)$ where ϕ denotes an arbitrary function. Then (10.44) becomes

$$z = ax + \phi(a)y + f[a, \phi(a)] \tag{10.46}$$

Differentiating (10.46) with respect to a, we have

$$O = x + y\phi'(a) + f'[a, \phi(a)] \tag{10.47}$$

The general integral is obtained by eliminating a from (10.46) and (10.47).

Examples 10.25 Solve $z = px + qy + pq$.

Solution The complete integral is

$$z = ax + by + ab \tag{10.48}$$

Singular integral: Differentiating (10.48) partially w.r.t. a and b

$$0 = x + b$$

and

$$0 = y + a \tag{10.49}$$

Eliminating a and b between (10.48) and (10.49), we get

$$z = -xy - xy + xy$$

i.e.

$$z = xy$$

which is the required singular solution, as it satisfies the given differential equation.

Examples 10.26 Solve $z = px + qy + 3p^{\frac{1}{3}}q^{\frac{1}{3}}$.

Solution A complete solution is

$$z = ax + by + 3a^{\frac{1}{3}}b^{\frac{1}{3}} \tag{10.50}$$

The derivatives with respect to a and b are

$$x + a^{-\frac{2}{3}}b^{\frac{1}{3}} = 0 \text{ and } y + a^{\frac{1}{3}}b^{-\frac{2}{3}} = 0 \tag{10.51}$$

From (10.51) $\quad ax + by = -a^{\frac{1}{3}}b^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} = -2a^{\frac{1}{3}}b^{\frac{1}{3}}$

$$xy = a^{-\frac{2}{3}}b^{\frac{1}{3}}.a^{\frac{1}{3}}b^{-\frac{2}{3}} = a^{-\frac{1}{3}}b^{-\frac{1}{3}}$$

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and substituting in the complete solution, we obtain the singular solution as

$$z = a^{\frac{1}{3}} b^{\frac{1}{3}} = \frac{1}{xy}$$

or

$$xyz = 1.$$

Examples 10.27 Solve $z = px + qy + p^2 + pq + q^2$.

Solution A complete solution is $z = ax + by + a^2 + ab + b^2$.

Differentiating the complete solution with respect to a and b, we have

$$0 = x + 2a + b, \quad 0 = y + a + 2b$$

Solution gives $\quad a = \dfrac{y - 2x}{3}, \; b = \dfrac{x - 2y}{3}$

and substituting in the complete solution, the singular solution is

$$3z = xy - x^2 - y^2.$$

10.11 Standard form III

The equation is of the form

$$F(z, \; p, \; q) = 0$$

where the independent variables do not occur explicitly, but the dependent variables are involved.

We assume a tentative solution as

$$Z = f(x + ay) = f(X)$$

where a is an arbitrary constant. Then

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{dx} = \frac{dz}{dX}.\frac{\partial X}{dx} = \frac{dz}{dX}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{dX}\frac{\partial X}{dy} = a\frac{dz}{dX}$$

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On substitution, the given equation becomes

$$F\left(z, \frac{dz}{dX}, a\frac{dz}{dX}\right) = 0$$

which is an ordinary differential equation of first order and may be easily solved.

Procedure: In an equation of the type

$$F(z, p, q) = 0$$

put (i) $\frac{dz}{dX}$ for p, $a\frac{dz}{dX}$ for q.

(ii) Integrate the resulting ordinary differential equation.

(iii) Put $x + ay$ for X.

This gives a complete solution. The general solution and singular solution may be found in the usual way.

Examples 10.28 Solve $pz = 1 + q^2$.

Solution This is of the form $f(z, p, q) = 0$.

Putting $$p = \frac{dz}{dX}, \; q = a\frac{dz}{dX}$$

where $$X = x + ay$$

the required equation becomes

$$z\frac{dz}{dX} = 1 + a^2\left(\frac{dz}{dX}\right)^2$$

or $$a^2\left(\frac{dz}{dX}\right)^2 - z\frac{dz}{dX} + 1 = 0$$

$$\therefore \quad \frac{dz}{dX} = \frac{z \pm \sqrt{z^2 - 4a^2}}{2a^2}$$

or $$\frac{dz}{z \pm \sqrt{z^2 - 4a^2}} = \frac{dX}{2a^2}$$

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or
$$\frac{\left(z \mp \sqrt{z^2-4a^2}\right)dz}{\left(z \pm \sqrt{z^2-4a^2}\right)\left(z \mp \sqrt{z^2-4a^2}\right)} = \frac{dX}{2a^2}$$

or
$$\frac{\left(z \mp \sqrt{z^2-4a^2}\right)dz}{4a^2} = \frac{dX}{2a^2}$$

or
$$\left(z \mp \sqrt{z^2-4a^2}\right)dz = 2dX$$

Integrating, $\frac{z^2}{2} \mp \left[\frac{z\sqrt{z^2-4a^2}}{2} - \frac{4a^2}{2}\log\left(z+\sqrt{z^2-4a^2}\right)\right] = 2X + c$

or
$$z^2 \mp \left[z\sqrt{z^2-4a^2} - 4a^2\log\left\{z+\sqrt{z^2-4a^2}\right\}\right]$$
$$= 4(x+ay) + 2c$$

is the complete solution.

Examples 10.29 Solve $p(1-q^2) = q(1-z)$.

Solution: Assume $z = F(x+ay) = F(u)$ (10.52)

Then
$$p = \frac{dz}{du},\ q = a\frac{dz}{du}$$

and the given equation becomes

$$\frac{dz}{du}\left[1 - a^2\left(\frac{dz}{du}\right)^2\right] = a\frac{dz}{du}(1-z)$$

or
$$\frac{dz}{du}\left[1 - a + az - a^2\left(\frac{dz}{du}\right)^2\right] = 0$$

Thus
$$\frac{dz}{du} = 0 \text{ giving } z = c.$$

or
$$1 - a + az - a^2\left(\frac{dz}{du}\right)^2 = 0$$

or
$$\frac{adz}{\sqrt{1-a+az}} = du$$

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Integrating, $2\sqrt{1-a+az} = u+b = x+ay+b$

or $$4(1-a+az) = (x+ay+b)^2$$

Each of $z = c$ and $4(1-a+az) = (x+ay+b)^2$

is a solution. The later is a complete solution. Using it, the equation for the singular solutions are

$$g = 4(1-a+az)-(x+ay+b)^2 = 0$$

$$\frac{\partial g}{\partial a} = 4(-1+z)-2y(x+ay+b) = 0$$

$$\frac{\partial g}{\partial b} = -2(x+ay+b) = 0$$

There is no singular solution.

Examples 10.30 Solve $p^2 = z^2(1-pq)$.

Solution Putting $p = \dfrac{dz}{dX}$, $q = a\dfrac{dz}{dX}$, the given equation becomes

$$\left(\frac{dz}{dX}\right)^2 = z^2\left[1-a\left(\frac{dz}{dX}\right)^2\right]$$

or $$\frac{dz}{dX} = \frac{z}{\sqrt{1+az^2}}$$

or $$\frac{\sqrt{1+az^2}}{z}dz = dX$$

or $$dX = \frac{1+az^2}{z\sqrt{1+az^2}}dz$$

$$dX = \frac{1}{z\sqrt{1+az^2}}dz + \frac{azdz}{\sqrt{1+az^2}}$$

Integration yields

$$X+c = \frac{1}{\sqrt{a}}\log\left[z\sqrt{a}+\sqrt{1+az^2}\right]+\sqrt{1+az^2}.$$

Putting $X = x+ay$ we get the complete solution.

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10.12 Standard IV

The equation is of the form

$$f_1(x, p) = f_2(y, q)$$

where z does not occur explicitly, while x is associated with p and y with q. Here $F = 0$ is

$$f_1(x, p) - f_2(y, q) = 0,$$

so the

$$\frac{\partial F}{\partial x} = \frac{\partial f_1}{\partial x}, \frac{\partial F}{\partial y} = -\frac{\partial f_2}{\partial y}, \frac{\partial F}{\partial z} = 0$$

$$\frac{\partial F}{\partial p} = \frac{\partial f_1}{\partial p}, \frac{\partial F}{\partial q} = -\frac{\partial f_2}{\partial q}.$$

The auxiliary equations are

$$\frac{dp}{\frac{\partial f_1}{\partial x}} = \frac{dx}{-\frac{\partial f_1}{\partial p}}$$

From these two, we have

$$\frac{\partial f_1}{\partial p} dp + \frac{\partial f_2}{\partial x} dx = 0$$

which on integration gives

$$f_1(x, p) = a$$

where a is an arbitrary constant.

Therefore from the given equation

$$f_2(y, q) = a.$$

Solving these for p and q we have

$$p = \theta_1(x, a), \; q = \theta_2(y, a)$$

Substituting in $\quad dz = pdx + qdy$

we have $\quad dz = \theta_1(x, a)dx + \theta_2(y, a)dy$

which on integration gives

$$z = \int \theta_1(x, a)dx + \int \theta_2(y, a)dy + b$$

where b is another arbitrary constant.

Thus this is the complete integral.

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Examples 10.31 Solve $\sqrt{p}-\sqrt{q}+3x=0$.

Solution $\sqrt{p}+3x=\sqrt{q}$

Set $\sqrt{p}+3x=a$ and $\sqrt{q}=a$

Then $p=(a-3x)^2$ and $q=a^2$

A complete solution is $z=\int pdx+\int qdy+b$

or $z=\int(a-3x)^2dx+a^2\int dy+b$

or $z=-\frac{1}{9}(a-3x)^3+a^2y+b$

There is no singular solution.

Examples 10.32 Solve $pe^{-x}=qe^{-y}$.

Solution Set $pe^{-x}=qe^{-y}=a$ (say)

or $p=ae^x,\ q=ae^y$

Putting these values in $dz=pdx+qdy$

we get $dz=a(e^xdx+e^ydy)$.

Integrating, we have $z=a(e^x+e^y)+b$

as the complete solution.

Examples 10.33 Solve $p-3x^2=q^2-y$.

Solution $p-3x^2=q^2-y=a$ (say)

$\therefore$ $p=3x^2+a;\ q=\pm\sqrt{a+y}$

A complete solution is

$$z=\int pdx+\int qdy+b=(3x^2+a)dx\pm\sqrt{a+y}dy+b$$

$$=x^3+ax\pm\frac{2}{3}(a+y)^{\frac{3}{2}}+b$$

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10.13 Transformations

There are many equations, which though not of the standard forms, could be converted into one of the four standard forms by suitable transformations. The following transformations are used in general.

For the combination px, the transformation is $X = \log x$

For the combination qy, the transformation is $Y = \log y$

For the combination $\frac{p}{z}, \frac{q}{z}$, the transformation is $Z = \log z$

Sometimes transformation into polar coordinates is found suitable.

Examples which can be reduced to the form $f(p, q) = 0$.

Examples 10.34 Solve $x^2p^2 + y^2q^2 = z^2$.

Solution Re-writing the given equation, we have

$$\left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1 \tag{10.53}$$

Now, let as put

$$\left.\begin{aligned} \frac{dz}{z} &= dZ \text{ i.e. } \log z = Z \\ \frac{dy}{y} &= dY \text{ i.e. } \log y = Y \\ \frac{dx}{x} &= dX \text{ i.e. } \log x = X \end{aligned}\right\} \tag{10.54}$$

Then the given equation becomes

$$\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1 \tag{10.55}$$

which is of the standard form 1.

Notice that in equation (10.55) we have variables X, Y and Z in place of x, y and z respectively of form 1. Hence the complete solution of (10.55) is

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$$X = aX + bY + c$$

where $$a^2 + b^2 = 1$$

i.e. $$Z = aX + \sqrt{1-a^2}\,Y + c$$

It we put $$a = \cos\alpha$$

so that $$\sqrt{1-a^2} = \sin\alpha$$

the complete solution can be written as

$$\log z = \cos\alpha \log x + \sin\alpha \log y + \log c' \text{ taking } c = \log c'$$

or $$z = c' x^{\cos\alpha} y^{\sin\alpha}$$

Examples 10.35 Solve $(x^2 + y^2)(p^2 + q^2) = 1$.

Solution Put $x = r\cos\theta,\ y = r\sin\theta$

$$\therefore \qquad x^2 + y^2 = r^2,\ \theta = \tan^{-1}\frac{y}{x}$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial z}{\partial r}\cos\theta + \frac{\partial z}{\partial \theta}\left(-\frac{\sin\theta}{r}\right)$$

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$= \frac{\partial z}{\partial r}\sin\theta + \frac{\partial z}{\partial \theta}\left(-\frac{\cos\theta}{r}\right)$$

Putting the values of p and q in

$$(x^2 + y^2)(p^2 + q^2) = 1$$

and simplifying we get

$$r^2\left[\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2\right] = 1$$

i.e. $$\left(r\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = 1 \qquad (10.56)$$

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Put $$\frac{dr}{r} = R$$

so that $\log r = R$ then (10.56) gives

$$\left(\frac{\partial z}{\partial R}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = 1$$

$$\therefore \qquad z = aR + b\theta + c \tag{10.57}$$

where $$a^2 + b^2 = 1$$

i.e. $$b = \sqrt{1 - a^2}.$$

Therefore (10.57) becomes

$$z = a \log r + \sqrt{1 - a^2}\theta + c$$

where $$r = \sqrt{x^2 + y^2}$$

and $$\theta = \tan^{-1}\left(\frac{y}{x}\right).$$

10.14 Examples which transforms to standard form II

Examples 10.36 Solve $4xyz = pq + 2px^2y + 2qxy^2$.

Solution Put $x = X^{\frac{1}{2}}$, $y = Y^{\frac{1}{2}}$ / then

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{dX}{dx} = 2X^{\frac{1}{2}}\frac{\partial z}{\partial X}, \quad q = 2Y^{\frac{1}{2}}\frac{\partial z}{\partial Y}$$

$\therefore$ the equation becomes

$$z = X\frac{\partial z}{\partial X} + Y\frac{\partial z}{\partial Y} + \frac{\partial z}{\partial X}.\frac{\partial z}{\partial Y}.$$

$\therefore$ complete solution is $z = aX + bY + ab$

or $$z = ax^2 + by^2 + ab. \tag{10.58}$$

Singular solution: Differentiating (10.58) partially w.r.t. a and b, we get

$$0 = x^2 + b, \quad 0 = y^2 + a.$$

Eliminating a, b the singular solution is $2 + x^2y^2 = 0$.

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Examples 10.37 Find complete and singular integrals of

$$z^2(p^2z^2+q^2)=1.$$

Solution Give $$z^2(p^2z^2+q^2)=1 \tag{10.59}$$

Take $$q=ap \tag{10.60}$$

Puting the value of q from (10.60) in (10.59), we get

$$p^2z^2(z^2+a^2)=1 \text{ or } p=\pm\frac{1}{\left\{z\sqrt{(z^2+a^2)}\right\}} \tag{10.61}$$

But $$dz=pdx+qdy=pdx+apdy=p(dx+ady)$$

or $$dz=\pm\frac{dz+ady}{z\sqrt{(z^2+a^2)}}, \text{ by (10.61)}$$

or $$dx+ady=\pm\frac{1}{2}(z^2+a^2)^{1/2},\ (2z)dz$$

Integrating it, $$x+ay+b=\pm\frac{1}{2}\frac{(z^2+a^2)^{3/2}}{\frac{3}{2}}$$

or $$3(x+ay+b)=\pm(z^2+a^2)^{3/2}$$

or $$9(x+ay+b)^2=(z^2+a^2)^3 \tag{10.62}$$

is the required complete integral.

Singular integral: Differentiating (10.62) partially w.r.t. a and b, we get

$$18(x+ay+b)=3(z^2+a^2)^2.2a \tag{10.63}$$

and $$18(x+ay+b)=0 \tag{10.64}$$

From (10.63) and (10.64), $a=0$. Also from (10.64), $x+ay+b=0$.

With these values (10.62) gives

$$9\times 0=(z^2+0)^3 \text{ or } z=0 \tag{10.65}$$

From (10.65), $$p=\frac{dt}{dx}=0,\ q=\frac{dz}{dy}=0 \tag{10.66}$$

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Values of z, p and q given by (10.65) and (10.66), reduce (10.59) to the form $0 = 1$. Hence $z = 0$ is not S. I.

10.15 Charpit's method

We will now give a general method for obtaining the complete solution of a nonlinear partial differential equation. This method is due to Charpit and is applicable to all nonlinear partial differential equations of first order.

Let the given equation be

$$F(x, y, z, p, q) = 0 \tag{10.67}$$

If we are able to find another relation connecting the variables and the derivatives, we may regard these two as a pair of simultaneous equations and solving them we can obtain p and q in terms of x, y, z. We know that $z = z(x, y)$.

$$\therefore \qquad dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$\therefore \qquad dz = pdx + qdy \tag{10.68}$$

Substituting the values of p and q in (10.68) we get an equation containing x, y, z. The resulting equation is either easily integrable or is integrated by the help of integrating factor. Let the other relation be

$$\phi(x, y, z, p, q) = 0 \tag{10.69}$$

Differentiating (10.67) and (10.69) partially with respect to x and y, we have

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0 \tag{10.70}$$

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z} p + \frac{\partial \phi}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial \phi}{\partial q}\frac{\partial q}{\partial x} = 0 \tag{10.71}$$

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$$\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}q+\frac{\partial F}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial y}=0 \tag{10.72}$$

$$\frac{\partial \phi}{\partial y}+\frac{\partial \phi}{\partial z}q+\frac{\partial \phi}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial \phi}{\partial q}\frac{\partial q}{\partial y}=0 \tag{10.73}$$

Multiplying (10.70) by $\frac{\partial \phi}{\partial p}$, (10.71) by $\frac{\partial F}{\partial p}$ and subtracting to eliminate $\frac{\partial p}{\partial x}$, we get

$$\left(\frac{\partial F}{\partial x}\frac{\partial \phi}{\partial p}-\frac{\partial F}{\partial p}\frac{\partial \phi}{\partial x}\right)+p\left(\frac{\partial F}{\partial z}\frac{\partial \phi}{\partial p}-\frac{\partial F}{\partial p}\frac{\partial \phi}{\partial z}\right)$$

$$+\frac{\partial q}{\partial x}\left(\frac{\partial F}{\partial q}\frac{\partial \phi}{\partial p}-\frac{\partial F}{\partial p}\frac{\partial \phi}{\partial q}\right)=0 \tag{10.74}$$

Multiplying (10.72) by $\frac{\partial \phi}{\partial q}$, (10.73) by $\frac{\partial F}{\partial q}$ and subtracting to eliminate $\frac{\partial q}{\partial y}$, we get

$$\left(\frac{\partial F}{\partial y}\frac{\partial \phi}{\partial q}-\frac{\partial F}{\partial q}\frac{\partial \phi}{\partial y}\right)+q\left(\frac{\partial F}{\partial z}\frac{\partial \phi}{\partial q}-\frac{\partial F}{\partial q}\frac{\partial \phi}{\partial z}\right)$$

$$+\frac{\partial p}{\partial y}\left(\frac{\partial F}{\partial p}\frac{\partial \phi}{\partial q}-\frac{\partial F}{\partial q}\frac{\partial \phi}{\partial p}\right)=0 \tag{10.75}$$

Since $$\frac{\partial q}{\partial x}=\frac{\partial^2 z}{\partial x \partial y}=\frac{\partial^2 z}{\partial u \partial x}=\frac{\partial p}{\partial y}$$

adding (10.74) and (10.75) and rearranging, we have

$$\left(\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}\right)\frac{\partial \phi}{\partial p}+\left(\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}\right)\frac{\partial \phi}{\partial q}+\left(-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}\right)\frac{\partial \phi}{\partial z}$$

$$+\left(-\frac{\partial F}{\partial p}\right)\frac{\partial \phi}{\partial x}+\left(-\frac{\partial F}{\partial q}\right)\frac{\partial \phi}{\partial y}=0 \tag{10.76}$$

which is a linear equation of the first order with

$$x, y, z, p, q$$

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as independent variables and ϕ as dependent variable. The corresponding auxiliary equations (by Lagrange's method) are

$$\frac{dp}{\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}}=\frac{dq}{\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}}=\frac{dz}{-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}}$$

$$=\frac{dx}{-\frac{\partial F}{\partial p}}=\frac{dy}{-\frac{\partial F}{\partial q}}=\frac{d\phi}{\partial} \qquad (10.77)$$

It is evident that these equations can hold only if

$$d\phi=0$$

i.e.

$$\phi=A$$

an arbitrary constant.

The simplest relation involving at least one of p and q may be taken as

$$\phi=0.$$

Now from $\phi=0$

and $F=0$

the values of p and q should be found in terms of x and y and substituting in (10.68) and integrating, the solution is obtained.

Note: The equation (10.75) can be easily obtained from equation (10.74) by interchanging x, y and p, q.

10.16 Working rule while using Charpit's method

Step 1: Transfer all terms of the given equation to L.H.S. and denote the entire expression by F.

Step 2: Write down the Charpit's auxiliary equation (10.77).

Step 3: Using the value of F in step 1 write down the values of

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$$

etc. occurring in step 2 and put these in Charpit's equations.

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Step 4: After simplifying the step 3, select two proper fractions so that the resulting integral may come out to be the simplest relation involving at least one of p and q.

Step 5: The simplest relation of step 4 is solved along with the given equation to determine p and q. Put these values of p and q in equation $dz = pdx + qdy$ which on integration gives the complete solution of the given equation since the solution will contain as many arbitrary constants i.e. two which is equal to the number of independent variables.

The Singular and General integral may be obtained in the usual manner.

Examples 10.38 Find complete integral of

$$2zx - px^2 - 2qxy + pq = 0.$$

Solution Here $F = 0$ is $2zx - px^2 - 2qxy + pq = 0$

so that
$$\frac{\partial F}{\partial p} = -x^2 + q, \quad \frac{\partial F}{\partial q} = -2xy + p$$

$$\frac{\partial F}{\partial z} = 2x, \quad \frac{\partial F}{\partial x} = 2z - 2px - 2qy$$

and
$$\frac{\partial F}{\partial y} = -2qx.$$

Taking the two simplest ratios of Charpit's equations, we have

$$\frac{dp}{2z - 2px - 2qy + 2px} = \frac{dq}{-2qx + 2qx}$$

or
$$\frac{dp}{2z - 2qy} = \frac{dq}{0}.$$

Integrating, we get $q = a$ an arbitrary constant.

Substituting in the given equation, we get

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$$p = \frac{2xz - 2axy}{x^2 - a} = \frac{2x(z - ay)}{x^2 - a}$$

$$\therefore \quad dz = pdx + qdy = \frac{2x(z - ay)dx}{x^2 - a} + ady$$

or
$$\frac{dz - ady}{z - ay} = \frac{2xdx}{x^2 - a}$$

Integrating, $\log(z - ay) = \log(x^2 - a) + \log b$

or
$$z - ay = b(x^2 - a)$$

i. e.
$$z = ay + b(x^2 - a),$$

where b is another arbitrary constant. This is the complete integral.

Examples 10.39 Solve $px + qy = pq$.

Solution: Here $F = px + qy - pq$ (10.78)

$$\frac{\partial F}{\partial x} = p; \ \frac{\partial F}{\partial y} = q; \ \frac{\partial F}{\partial z} = o; \ \frac{\partial F}{\partial p} = x - q; \ \frac{\partial F}{\partial q} = y - p.$$

The Charpit's auxiliary equations are

$$\frac{dp}{p} = \frac{dq}{q} = \frac{dz}{-p(x-q) - q(y-p)}$$

$$= \frac{dx}{-(x-q)} = \frac{dy}{-(y-p)} = \frac{d\phi}{0}$$

Taking the first two members, we have

$$\frac{dp}{p} = \frac{dq}{q}.$$

Integrating, $\log p = \log q + \log a$

or
$$p = aq$$

where a is an arbitrary constant. Putting aq for p in the given equation, we get

$$aqx + qy = aq \qquad \therefore \ q = \frac{y + ax}{a}$$

$$\therefore \qquad p = aq = y + ax$$

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Putting the values of p and q in $dz = pdx + qdy$, we get

$$dz = (y + ax)dx + \frac{(y + ax)}{a}dy$$

or $$adz = (y + ax)(dy + adx)$$

Integrating, $$az = \frac{1}{2}(y + ax)^2 + b$$

which is the complete integral, as b is another arbitrary constant.

Singular integral: Differentiating the complete integral partially w.r.t. a and b, we have

$$z = x(y + ax) \text{ and } 0 = 1 \text{ (absurd)}$$

Hence there is no singular integral.

General integral : Writing $b = \phi(a)$ we have

$$az = \frac{1}{2}(y + ax)^2 + \phi(a) \tag{10.79}$$

Differentiating (10.79) partially w.r.t. a we have

$$z = x(y + ax) + \phi'(a) \tag{10.80}$$

General integral is obtained by eliminating a from (10.79) and (10.80).

Examples 10.40 Find the complete integral of $pxy + pq + qy = yz$.

Solution Here $F \equiv pxy + pq + qy - yz = 0$

Charpits auxiliary equations are

$$\frac{dp}{py + p(-y)} = \frac{dq}{px + q + qy}$$

$\therefore$ $$dp = 0$$

or $$p = a.$$

Then from the given equation

$$q = \frac{y(z - ax)}{a + y}.$$

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Substituting these values of p and q in

$$dz = pdx + qdy$$

we have
$$dz = adx + \frac{y(z-ax)dy}{a+y}$$

$$\therefore \quad \frac{dz - adx}{z - ax} - \frac{ydy}{a+y} = \left(1 - \frac{a}{a+y)}\right)dy$$

Integrating, $\log(z-ax) = y - a\log(a+y) + \log b$

or
$$(z-ax)(a+y)^a = be^y.$$

Examples 10.41 Apply Charpit's method to find complete integral of

$$z = px + qy + p^2 + q^2.$$

Solution Charpit's auxiliary equations are

$$\frac{dp}{-p+p} = \frac{dq}{-q+q} = \cdots\cdots$$

Here
$$dp = 0, dq = 0$$

give
$$p = a, q = b$$

where a and b are arbitrary constants. Putting these values in the given equations we get $z = ax + by + a^2 + b^2$ which is the complete integral.

Examples 10.42 Find the complete integral of the equation

$$2(z + xp + yq) = yp^2.$$

Solution $F \equiv 2(z + xp + yq) - yp^2 = 0$

$\therefore$ The Charpit's auxiliary equations are

$$\frac{dp}{2p+2p} = \frac{dq}{2q - p^2 + 2q} = \frac{dz}{-p(2x-2yp) - 2qy}$$

$$= \frac{dx}{-2x+2yp} = \frac{dy}{-2y}.$$

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Taking the first and fifth members, we have

$$\frac{dp}{p}+2\frac{dy}{y}=0.$$

Integrating $$py^2=a.$$

$$\therefore \quad p=\frac{a}{y^2}$$

Putting $$p=\frac{a}{y^2}$$

in the given equation, we get

$$z+\frac{ax}{y^2}+yq=\frac{a^2}{2y^3}$$

$$\therefore \quad q=-\frac{z}{y}-\frac{ax}{y^3}+\frac{a^2}{2y^4}$$

From $$dz=pdx+qdy$$

we have $$dz=\frac{a}{y^2}dx-\frac{z}{y}dy-\frac{ax}{y^2}dy+\frac{a^2}{2y^4}dy$$

$$\therefore \quad ydz+zdy=a\left(\frac{ydx-xdy}{y^2}\right)+\frac{a^2}{2y^3}dy$$

Integrating, $$yz-\frac{ax}{y}+\frac{a^2}{4y^2}=b$$

$$\therefore \quad z=\frac{ax}{y^2}-\frac{a^2}{4y^3}+\frac{b}{y}$$

is the compete integral.

Examples 10.43 Solve $p(q^2+1)+(b-z)q=0.$

Solution Here

$$\frac{dp}{pq}=\frac{dq}{q^2}=\frac{dz}{3pq^2+p+(b-z)q}=\frac{dx}{q^2+1}=\frac{dy}{-z+b+zpq}$$

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From the first two fractions, after integration, we get $q = ap$ where a is an arbitrary constant. This and the given equation determine the values of p and q as

$$p = \frac{\sqrt{a(z-b)-1}}{a}, \quad q = \sqrt{a(z-b)-1}$$

Substituting these values in $dz = pdx + ady$, we obtain

$$dz = \sqrt{a(z-b)-1}\left(\frac{dx}{a} + dy\right).$$

Separating the variables and integrating, we get the solution as

$$2\sqrt{a(z-b)-1} = x + ay + b, \; b$$

is another arbitrary constant.

Test your knowledge

Form the partial differential equations by eliminating arbitrary constants from:

1. $z = (x^2 + a^2)(x^2 + b^2)$.
2. $z = xy + y\sqrt{x^2 - a^2} + b$.
3. $ax^2 + by^2 + z^2 = 1$.
4. Find the differential equation of the family of spheres of radius 5 with centres on the plane $x = y$.

Form the differential equations by eliminating the arbitrary functions from

5. $x = f(z) + qy$.
6. $z = y^2 + 2\phi\left(\frac{1}{x} + \log y\right)$.
7. $lx + my + nz = f(x^2 + y^2 + z^2)$.
8. $xyz = \phi(x + y + z)$.

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9. Find the differential equation of all surfaces of revolution having the z-axis as axis of rotation.

[**Hints:** Eliminate ϕ from $z=\phi\left(\sqrt{x^2+y^2}\right)=\psi(x^2+y^2)$.]

10. $z=e^{ax+by}f(ax-by)$.

Solve the following differential equations:

11. $(mz-ny)p+(nx-lz)q=ly-mx$.

12. $(x^2-y^2-z^2)p+2xyq=2xz$.

13. $x^2p+y^2q=(x+y)z$.

14. $x(y^2-z^2)p+y(z^2-x^2)q=z(x^2-y^2)$.

15. $p\tan x+q\tan y=\tan z$.

16. $yp-xq+x^2-y^2=0$.

17. $x(y^2-z^2)p+y(z^2-x^2)q=z(x^2-y^2)$.

18. $(x+2z)p+(4zx-y)q=2x^2+y$.

19. $x(y^n-z^n)p+y(z^n-x^n)q=z(x^n-y^n)$.

20. $x^2(y-z)p+y^2(z-x)q=z^2(x-y)$.

21. Find the integral surface of the linear differential equation $x(y^2+z)p-y(x^2+z)q=(x^2-y^2)z$, which contains the line $x+y=0,\ z=1$.

22. $P+q=\sin x+\sin y$.

23. $p^2x+q^2y=z$.

24. $px+qy=z(1+pq)^{\frac{1}{2}}$.

25. $p=(qy+z)^2$.

26. $(p^2+q^2)y=qz$.

27. $z=qp$.

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28. $2x(z^2q^2+1)=pz.$

29. $yzp^2=q.$

30. $p^2=4ze^{\frac{qy}{z}}.$

31. $z=px+qy+\sqrt{1+p^2+q^2}.$

32. $z^2=pqxy.$

33. $p(1+q^2)=q(z-a).$

34. $(p^2+q^2)x=pz.$

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CHAPTER 11

Linear Partial Differential Equations

11.1 Homogeneous partial differential equations

An equation of the form

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \cdots\cdots + k_n \frac{\partial^n z}{\partial y^n} = F(x, y) \qquad (11.1)$$

where $k_1, k_1, \cdots\cdots, k_n$ are constants, is called a linear homogeneous partial differential equation of order n with constant coefficients. Equation (11.1) can be written in symbolic form as

$$(D^n + k_1 D^{n-1} D' + \cdots\cdots + k_n D'^n) z = F(x, y) \qquad (11.2)$$

or $$(D_x^n + k_1 D_x^{n-1} D_y + \cdots\cdots + k_n D_y^n) z = F(x, y)$$

or $$f(D_x, D_y) z = F(x, y)$$

where $$D_x^n = \frac{\partial^n}{\partial x^n}, \; D_y^n = \frac{\partial^n}{\partial y^n}.$$

Like ordinary linear differential equations with constant coefficients, the complete solution of equation (11.1) consists of the complementary function (C. F.) and particular integral (P. I). The complementary function is the complete solution of $f(D_x, D_y) z = 0$ which contains n arbitrary constants. The particular integral is the particular solution of equation (11.2).

11.2 Methods of finding complementary functions

Let $F(D_x, D_y) z = (D_x - m_1 D_y)(D_x - m_2 D_y) \cdots\cdots (D_x - m_n D_y) z$

where $$m_1, m_2, \cdots\cdots, m_n$$

are real and distinct.

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Consider $$(D_x - m_r D_y)z = 0.$$

or $$\frac{\partial z}{\partial x} - m_r \frac{\partial z}{\partial y} = 0 \text{ or, } p - m_r q = 0$$

which is a Lagrange's linear equation. The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_r} = \frac{dz}{0}$$

First two relations give $dy + m_r dx = 0$

$$\therefore \qquad y + m_r x = c_1$$

and $$\frac{dz}{0} = \frac{dx}{1} \text{ yielding } z = c_2.$$

Hence $$z = \phi_r(y + m_r x)$$

is a solution of $$(D_x - m_r D_y)z = 0$$

where ϕ_r is an arbitrary function.

Hence if $$m_1, m_2, \cdots\cdots, m_n$$

are all distinct the complete solution is

$$z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \cdots\cdots + \phi_n(y + m_n x)$$

Cor: Let $$z = \phi(y + mx)$$

be a solution of $$F(D_x, D_y)z = 0$$

then $$D_x^r z = m^r \phi^{(r)}(y + mx)$$

and $$D_y^r z = \phi^{(r)}(y + mx).$$

Thus $$F(D_x, D_y)z = (A_0 D^n + A_1 D^{n-1} D_y + A_n D_y^n)z$$

$$= (A_0 m^n + A_1 m^{n-1} + A_n)\phi(y + mx)$$

The equation $$A_0 m^n + A_1 m^{n-1} + \cdots\cdots + A_n = 0$$

is called the auxiliary equation. The auxiliary equation is simply obtained by replacing $D_x z$ by m and $D_y z$ by 1 in

$$F(D_x, D_y)z = 0.$$

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Example 11.1 Solve $2\frac{\partial^2 z}{\partial x^2}+5\frac{\partial^2 z}{\partial x\partial y}+2\frac{\partial^2 z}{\partial y^2}=0$

Solution The given equation can be written as

$$\left(2D_x^2+5D_xD_y+2D_y^2\right)z=0.$$

The auxiliary equation $2m^2+5m+2=0$

has the roots $$m=-2,\ -\frac{1}{2}.$$

Hence the complete solution is

$$z=\phi_1(y-2x)+\phi_2\left(y-\frac{1}{2}x)\right).$$

Example 11.2 Solve $\frac{\partial^4 z}{\partial x^4}-\frac{\partial^4 z}{\partial y^4}=0.$

Solution the equation is $\left(D_x^4-D_y^4\right)z=0.$

The auxiliary equation is $m^4-1=0$ giving $m=\pm 1,\pm i.$

Thus the complete solution is

$$z=\phi_1(y+x)+\phi_2(y-x)+\phi_3(y+ix)+\phi_4(y-ix).$$

11.3 For repeated roots of auxiliary equation

If the roots of the auxiliary equation are equal say m then

$$(D_x-mD_y)(D_x-mD_y)z=0 \tag{11.3}$$

Putting $(D_x-mD_y)z=u$

in (11.3), we have $(D_x-mD_y)u=0$

which gives $u=\phi_1(y+mx)$.

Thus equation (11.3) becomes

$$(D_x-mD_y)z=\phi_1(y+mx)$$

or $$p-mq=\phi_1(y+mx)$$

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This is again Lagrange's linear equation and the subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\phi_1(y+mx)}$$

which gives $\quad y + mx = a$

and $\quad dz = \phi_1(a)dx$

i.e. $\quad z = x\phi_1(a) + b.$

Therefore the complete solution of (11.3) is

$$z = x\phi_1(y+mx) + \phi_2(y+mx)$$

$$\therefore \quad \text{If } (D_x - mD_y)z = 0 = (D_x - mD_y)^k = 0$$

then $\quad z = \phi_1(y+mx) + x\phi_2(y+mx) + x^2\phi_3(y+mx)$

$$+ \cdots\cdots x^{k-1}\phi_k(y+mx)$$

Example 11.3 Solve $r + 8s + 16t = 0$.

Solution The equation can be write as

$$\left(D_x^2 + 8D_xD_y + 16D_y^2\right)z = 0.$$

The auxiliary equation $m^2 + 8m + 16 = 0$

has two equal roots $\quad m = -4, -4$

and the complete solution is $z = \phi_1(y-4x) + x\phi_2(y-4x)$.

Particular integral

To find the particular integral, consider the second order partial differential equation

$$\left(D_x^2 + k_1D_xD_y + k_2D_y^2\right)z = F(x, y)$$

or $\quad f\left(D_x, D_y\right)z = F(x, y).$

$$\text{Particular integral} = \frac{1}{f\left(D_x, D_y\right)}F(x, y).$$

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We have the following 4 cases.

Case(a): When $F(x, y) = e^{ax+by}$.

$$D_x e^{ax+by} = ae^{ax+by}, \quad D_y^a e^{ax+by} = be^{ax+by}$$

$$D_x^2 e^{ax+by} = a^2 e^{ax+by}, \quad D_y^2 e^{ax+by} = b^2 e^{ax+by}$$

and $$D_x D_y e^{ax+by} = abe^{ax+by}$$

Thus $$\left(D_x^2 + k_1 D_x D_y + k_2 D_y^2\right)e^{ax+by} = \left(a^2 + k_1 ab + k_2 b^2\right)e^{ax+by}$$

or $$f(D_x, D_y)e^{ax+by} = f(a, b)e^{ax+by}$$

Operating both sides by $\dfrac{1}{f(D_x, D_y)}$

we get $$\text{P. I} = \frac{1}{f(D_x, D_y)} F(x, y) = \frac{1}{f(a, b)} e^{ax+by},$$

provided $$f(a, b) \neq 0.$$

Working Rule to evaluate $\dfrac{1}{f(D_x, D_y)} e^{ax+by}$ when $f(a, b) = 0$

(i) Differentiate $f(D_x, D_y)$

with respect to D_x partially and multiply the expression by x, so that

$$\frac{1}{f(D_x, D_y)} e^{ax+by} = x \frac{1}{f'(D_x, D_y)} e^{ax+by}.$$

(ii) If $f'(a, b)$ is also zero, differentiate $f'(D_x, D_y)$

with respect to D_x partially and multiply by x again so that

$$\frac{1}{f(D_x, D_y)} e^{ax+by} = x^2 \frac{1}{f''(D_x, D_y)} e^{ax+by}.$$

Proceed differentiating in this way every time and multiplying by x as long as the derivative of $f(D_x, D_y)$ vanishes when $D_x = a$ and $D_y = b$.

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Case(b): When $f(x, y)$ be of the form $\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(ax+by)$

Here $$P.I = \frac{1}{f\left(D_x^2, D_xD_y, D_y^2\right)} \cdot \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(ax+by)$$

$$= \frac{1}{f\left(-a^2, -ab, -b^2\right)} \cdot \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(ax+by)$$

provided $$f\left(-a^2, -ab, -b^2\right) \neq 0.$$

If $$f\left(-a^2, -ab, -b^2\right) = 0$$

then P. I. is obtained in similar way as in the case of

$$\frac{1}{f(D_x, D_y)} e^{ax+by}.$$

Case(c): When $f(x, y) = x^a y^b$, a, b are constants.

Here $$P.I = \frac{1}{f\left(D_x, D_y\right)} x^a y^b = \left[f\left(D_x, D_y\right)\right]^{-1} x^a y^b$$

and to evaluate it expand $\left[f\left(D_x, D_y\right)\right]^{-1}$

in ascending power of D_x or D_y using the binomial theorem and then operate on $x^a y^b$ term by term.

Case (d): When $f(x, y)$ is any function of x and y.

Here $$P.I = \frac{1}{f\left(D_x, D_y\right)} f(x, y)$$

and to evaluate it, resolve $\dfrac{1}{f\left(D_x, D_y\right)}$

into partial fractions considering $f\left(D_x, D_y\right)$

as a function of D_x only and operate each partial fraction on $f(x, y)$ noting that

$$\frac{1}{D_x - mD_y} f(x, y) = \int f(x, c - mx)dx$$

where c is replaced by $y + mx$ after integration.

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Example 11.4 Solve $\left(D_x^2 - D_x D_y - 2D_y^2\right)z = 2x + 3y.$

Solution The auxiliary equation is $m^2 - m - 2 = 0$ giving $m = 2, -1.$

Hence the complementary function is

$$z_c = \phi_1(y + 2x) + \phi_2(y - x)$$

We now proceed to find particular integral.

First method $z_p = \dfrac{1}{\left(D_x^2 - D_x D_y - 2D_y^2\right)}(2x + 3y)$

$$= \frac{1}{D_x^2} \cdot \frac{1}{1 - \dfrac{D_y}{D_x} - \dfrac{2D_y^2}{D_x^2}}(2x + 3y)$$

$$= \frac{1}{D_x^2}\left[\left\{1 + \frac{D_y}{D_x} + \cdots\cdots\right\}(2x + 3y)\right]$$

$$= \frac{1}{D_x^2}[2x + 3y + 3x] = \frac{1}{D_x^2}[5x + 3y]$$

$$= \frac{5}{6}x^3 + \frac{3x^2 y}{2}$$

Now $z = z_c + z_p$

Second method $z_p = \dfrac{1}{\left(D_x^2 - D_x D_y - 2D_y^2\right)}(2x + 3y)$

$$= \frac{1}{\left(D_x - 2D_y\right)}\left[\frac{1}{D_x + D_y}(2x + 3y)\right]$$

Put $u = \dfrac{1}{D_x + D_y}(2x + 3y)$

i.e. $(D_x + D_y)u = 2x + 3y$

Auxiliary equation is $\dfrac{dx}{1} = \dfrac{dy}{1} = \dfrac{du}{2x + 3y}$

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giving $$y - x = a$$

or $$y = x + a$$

Then $$\frac{dx}{1} = \frac{du}{5x + 3a}.$$

On integrating we get $u = \dfrac{5x^2}{2} + 3ax = \dfrac{5x^2}{2} + 3x(y - x)$

$$= 3xy - \frac{x^2}{2}.$$

Now $$z_p = \frac{1}{(D_x - 2D_y)}\left(3xy - \frac{x^2}{2}\right)$$

i.e. $$(D_x - 2D_y)z_p = 3xy - \frac{x^2}{2}$$

Auxiliary equation is $\dfrac{dx}{1} = \dfrac{dy}{-2} = \dfrac{dz_p}{3xy - \dfrac{x^2}{2}}.$

From the first two ratios we get $y + 2x = b$

i.e. $$y = b - 2x.$$

Then $$\frac{dx}{1} = \frac{dz_p}{3bx - \left(\dfrac{13}{2}x^2\right)}$$

$$\therefore \quad z_p = \frac{3bx^2}{2} - \frac{13x^3}{6}$$

$$= \frac{3}{2}x^2(y + 2x) - \frac{13x^3}{6}$$

$$= \frac{5x^3}{6} + \frac{3}{2}x^2 y.$$

$$\therefore \quad z = z_c + z_p$$

Example 11.5 Solve $\left(D_x^2 + 4D_xD_y + 4D_y^2\right)z = e^{2x+y}$.

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Solution The auxiliary equation is $m^2 + 4m + 4 = 0$

giving $$m = -2, -2.$$

Hence $$C.F = \phi_1(y - 2x) + x\phi_2(y - 2x)$$

Also $$P.I = \frac{1}{D_x^2 + 4D_xD_y + 4D_y^2} e^{2x+y}.$$

Here e^{2x+y} is a function of the form $\phi(ax + by)$

and $$D_x^2 + 4D_xD_y + 4D_y^2$$

is a homogeneous function of D_x, D_y of degree 2.

Integrating e^{2x+y} twice with respect to $2x + y$, we get $\dfrac{e^{2x+y}}{4}$.

Also putting 2 for D_x and 1 for D_y

$$\text{P.I} = \frac{1}{4 + 4.2.1 + 4.1^2} e^{2x+y} = \frac{e^{2x+y}}{16}.$$

Therefore the complete solution is

$$z = \phi_1(y - 2x) + x\phi_2(y - 2x) + \frac{1}{16} e^{2x+y}.$$

Example 11.6 Solve $\left(D_x^3 - 2D_x^2D_y - D_xD_y^2 + 2D_y^3\right)z = e^{x+y}$.

Solution The auxiliary equation is

$$m^3 - 2m^2 - m + 2 = 0$$

giving $$m = 1, -1, 2$$

$$\therefore \quad C.F. = \phi_1(y + x) + \phi_2(y - x) + \phi_3(y + 2x)$$

$$P.I = \frac{1}{D_x^3 - 2D_x^2D_y - D_xD_y^2 + 2D_y^3} e^{x+y}.$$

Here denominator vanishes where

$$D_x = 1, D_y = 1$$

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$$\therefore \text{ P. I} = x\frac{1}{3D_x^2 - 4D_xD_y - D_y^2}e^{x+y}$$

$$= x.\frac{1}{3.1 - 4.1 - 1}e^{x+y} = -\frac{1}{2}xe^{x+y}$$

The complete solution is

$$z = \phi_1(y+x) + \phi_2(y-x) + \phi_3(y+2x) - \frac{x}{2}e^{x+y}$$

Example 11.7 Solve $(D_x - D_y)^2 z = x + \phi(x+y)$.

Solution The auxiliary equation is

$$(m-1)^2 = 0, \text{ giving } m = 1, 1$$

$$\therefore \quad \text{C. F.} = \phi_1(y+x) + x\phi_2(y+x)$$

$$\text{P. I.} = \frac{1}{(D_x - D_y)^2}x[(x+\phi)(x+y)]$$

$$= \frac{1}{(D_x - D_y)^2}x + \frac{1}{(D_x - D_y)^2}\phi(x+y)$$

$$= \frac{1}{D_x^2(1 - D_y)^2}x + \frac{x}{2(D_x - D_y)}\phi(x+y)$$

$$= \frac{x^3}{6} + \frac{x^2}{2}\phi(x+y)$$

$\therefore$ The complete solution is

$$z = \phi_2(y+x) + x\phi_2(y+x) + \frac{x^3}{6} + \frac{x^2}{2}\phi(x+y)$$

Example 11.8 Solve $(D_x^2 - 5D_xD_y + 4D_y^2)z = \sin(2x+y)$.

Solution The roots of auxiliary equation are

$$m = 1, 4$$

$$\therefore \quad \text{C. F.} = \phi_1(y+x) + \phi_2(y+4x)$$

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$$\text{P. I.} = \frac{1}{D_x^2 - 5D_xD_y + 4D_y^2)}\sin(2x+y)$$

$$= \frac{1}{-4+10-4}\sin(2x+y)$$

$$= \frac{1}{2}\sin(2x+y).$$

The solution is $z = \phi_1(y+x) + \phi_2(y+4x) + \frac{1}{2}\sin(2x+y).$

Example 11.9 Solve $(D_x^3 - 4D_x^2D_y + 4D_xD_y^2)z = 4\cos(2x+y).$

Solution The auxiliary equation is $m^3 - 4m^2 + 4m = 0$

giving $m = 0.\ 2,\ 2$

$$\therefore\ \text{C. F.} = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x)$$

$$\text{P. I.} = \frac{1}{(D_x^3 - 4D_x^2D_y + 4D_xD_y^2)}.4\cos(2x+y)$$

The denominator becomes zero when $D_x = 2,\ D_y = 1.$

Differentiating denominator w.r. to D_x and multiplying numerator by x, we get

$$\text{P. I.} = \frac{x}{3D_x^2 - 8D_xD_y + 4D_y^2)}4\cos(2x+y)$$

The denominator again vanishes when $D_x = 2,\ D_y = 1.$

Therefore, again differentiating the denominator with respect to D_x and multiplying numerator by x we get

$$\text{P. I.} = x^2\frac{1}{6D_x - 8D_y}4\cos(2x+y)$$

The denominator is of order 1 and does not vanish when

$$D_x = 2,\ D_y = 1.$$

$$\therefore\quad \text{P. I.} = \frac{4x^2}{6.2 - 8.1}\int\cos t\,dt$$

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when
$$t = 2x + y$$
$$= x^2 \sin t = x^2 \sin(2x + y)$$

$$\therefore z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x) + x^2 \sin(2x + y).$$

Example-11.10 Solve $\left(D_x - D_x D_y - 6D_y^2\right)z = e^{2y} \sin 3x.$

Solution C. F. $= \phi_1(y + 2x) + \phi_2(y - 3x)$

$$\text{P. I.} = \frac{1}{D_x^2 - D_x D_y - 6D_y^2} e^{2y} \sin 3x$$

$$= e^{2y} \frac{1}{D_x^2 - D_x(D_y + 2) - 6(D_y + 2)^2} \sin 3x$$

$$= e^{2y} \frac{1}{D_x^2 - D_x D_y - 2D_x - 6D_y^2 - 24D_y - 24} \sin 3x$$

$$= e^{2y} \frac{1}{-9 - 2D_x - 24D_y - 24} \sin 3x$$

$$= -e^{2y} \frac{1}{2D_x + 24D_y + 33} \sin 3x$$

$$= -e^{2y} \frac{2D_x + 24D_y - 33}{(2D_x + 24D_y)^2 - 33^2} \sin 3x$$

$$= -e^{2y} \frac{6\cos 3x - 33\sin 3x}{4(-9) - 33^2}$$

$$= e^{2y} \frac{6\cos 3x - 33\sin 3x}{1125}$$

$$= e^{2y} \frac{2\cos 3x - 11\sin 3x}{375}$$

General solution is

$$z = \phi_1(y + 2x) + \phi_2(y - 3x) + e^{2y} \frac{2\cos 3x - 11\sin 3x}{375}.$$

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Example-11.11 Solve $(D_x^3 - 4D_x^2 D_y + 5D_x D_y^2 - 2D_y^3)z = (y+x)^{\frac{1}{2}}$.

Solution The auxiliary equation is $m^3 - 4m^2 + 5m - 2 = 0$

giving $$m = 2, 1, 1$$

$$\text{C. F.} = \phi_1(y+2x) + \phi_2(y+x) + x\phi_3(y+x)$$

$$\text{P. I.} = \frac{1}{(D_x^3 - 4D_x^2 D_y + 5D_x D_y^2 - 2D_y^3)}(y+x)^{\frac{1}{2}}$$

$$= \frac{1}{(D_x - D_y)^2 (D_x - 2D_y)}\sqrt{y+x}$$

$$= \frac{1}{(D_x - D_y)^2}\int [c - 2x + x]^{\frac{1}{2}} dx \text{ as corresponding to}$$

$$(D_x - 2D_y)z = 0,\ y + 2x = c$$

$$= \frac{1}{(D_x - D_y)^2}\sqrt{c - x}\ dx$$

$$= -\frac{2}{3}\frac{1}{(D_x - D_y)^2}(y+x)^{\frac{3}{2}} \text{ as } c = y + 2x$$

$$= -\frac{2}{3}.\frac{1}{D_x - D_y}\int (c_1 - x + x)^{\frac{3}{2}}\ dx \text{ as corresponding to}$$

$$(D_x - D_y) = 0,\ y + x = c_1$$

$$= -\frac{2}{3}.\frac{1}{D_x - D_y} c_1^{\frac{3}{2}} x$$

$$= -\frac{2}{3}.\frac{1}{D_x - D_y}(y+x)^{\frac{3}{2}} x$$

$$= -\frac{2}{3}\int (c_2 - x + x)^{\frac{3}{2}} x\ dx$$

$$= -\frac{2}{3}\int c_2^{\frac{3}{2}} x\ dx$$

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$$= -\frac{2}{3}.c_2^{\frac{3}{2}}\frac{x^2}{2} = -\frac{1}{3}(x+y)^{\frac{3}{2}}x^2$$

$$\therefore z = \phi_1(y+2x) + \phi_2(y+x) + x\phi_3(y+x) - \frac{x^2}{3}(y+x)^{\frac{3}{2}}x.$$

Example-11.12 Solve $(D_x^2 - 4D_y^2)z = \dfrac{4x}{y^2}$

Solution The auxiliary equation is $m^2 - 4 = 0$ giving $m = 2, -2$

$\therefore$ C. F $= \phi_1(y+2x) + \phi_2(y-3x)$

$$\text{P. I.} = \frac{1}{(D_x^2 - 4D_y^2)}.\frac{4x}{y^2}$$

$$= \frac{1}{(D_x + 2D_y)(D_x - 2D_y)}\frac{x}{y^2}$$

$$= \frac{1}{(D_x + 2D_y)}\int\frac{4x}{(c-2x)^2}dx \text{ as corresponding to}$$

$$(D_x - 2D_y)z = 0,\ y + 2x = c$$

$$= \frac{1}{(D_x + 2D_y)}\int\left[-\frac{2}{(c-2x)} + \frac{2c}{(c-2x)^2}\right]dx$$

$$= \frac{1}{(D_x + 2D_y)}\left[\log(c-2x) + \frac{c}{c-2x}\right]$$

$$= \frac{1}{(D_x + 2D_y)}\left[\log y + \frac{y+2x}{y}\right]$$

$$= \int\left[\log(c'+2x) + \frac{c'+2x}{c'+2x}\right]dx \text{ as corresponding to}$$

$$(D_x + 2D_y)z = 0,\ y - 2x = c'$$

$$= \int\left[\log(c'+2x) + 2 - \frac{c'}{c'+2x}\right]dx$$

$$= x\log(c'+2x) - \int\frac{2x}{c'+2x}dx + 2x - \frac{1}{2}\log(c'+2x)$$

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$$= x\log(c'+2x) - \int \frac{c'+2x-c'}{c'+2x}dx + 2x - \frac{1}{2}\log(c'+2x)$$

$$= x\log(c'+2x) - x + \frac{c'}{2}\log(c'+2x) + 2x - \frac{1}{2}\log(c'+2x)$$

$$= x\log y - x + \frac{y-2x}{2}\log y + 2x - \frac{1}{2}\log y$$

$$= \left(x + \frac{y-2x}{2} - \frac{1}{2}\right)\log y + x$$

General solution is

$$z = \phi_1(y+2x) + \phi_2(y-3x) + \left(x + \frac{y-2x}{2} - \frac{1}{2}\right)\log y + x$$

Non-homogeneous linear partial differential equation of higher order

An equation of the form $f(D_x, D_y)z = F(x, y)$

where $f(D_x, D_y)z = \left[A_0 D_x^n + A_1 D_x^{n-1} D_y + A_n D_x^n + B_0 D_y^{n-1}\right]$

$$+ B_1 D_x^{n-2} D_y + \cdots\cdots + B_{n-1} D_y^{n-1} + L_0 D_x + L_1 D_y + M\Big] z$$

is called a non-homogeneous linear differential equation of order n and its complete solution consists of a complementary function and a particular integral. The problems are classified into two types.

(i) If $f(D_x, D_y)z$

can be expressed as a product of the linear factors of the form

$$(a_1 D_x + b_1 D_y + c_1)(a_2 D_x + b_2 D_y + c_2)\cdots\cdots(a_n D_x + b_n D_y + c_n)z$$

then the equation is said to be reducible.

(ii) If $f(D_x, D_y)z$

can not be factorised in linear factors of D_x and D_y then the equation is called irreducible equation. Linear factor such as $(D_x^2 + D_y)z = e^{x+y}$ is an example of irreducibl equation.

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11.4 Reducible equations

Complementary function. Let the equation be

$$f(D_x, D_y)z = (a_1 D_x + b_1 D_y + c_1)(a_2 D_x + b_2 D_y + c_2)$$

or $$(a_r D_x + b_r D_y + c_r) = F(x, y)$$

where $a_r, b_r, c_n (n = 1, n)$ are constants.

Let us consider the equation

$$(a_1 D_x + b_1 D_y + c_1)z = 0$$

or $$a_1 p + b_1 q + c_1 z = 0$$

The auxiliary equations are $\dfrac{dx}{a_1} = \dfrac{dy}{b_1} = \dfrac{dz}{-c_1 z}$

From the first two relations we get, $a_1 y - b_1 x = A$

Case-1: If $a_1 \neq 0$ then from $\dfrac{dx}{a_1} = \dfrac{dz}{-c_1 z}$

we get $$\frac{dz}{z} = -\frac{c_1}{a} dx$$

or $$\log z = -\frac{c_1 x}{a} + \log B$$

or $$z = Be^{-\frac{c_1 x}{a_1}}$$

or $$ze^{-\frac{c_1 x}{a_1}} = B$$

$\therefore$ The general solution is

$$z = e^{-\frac{c_1 x}{a_1}} \phi(a_1 y - b_1 x) \tag{11.14}$$

Case-2: If $b_1 \neq 0$ then from $\dfrac{dy}{b_1} = \dfrac{dz}{-c_1 z}$

we get $$ze^{-\frac{c_1 y}{b_1}} = c$$

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$\therefore$ The general solution may be expressed as

$$z = e^{-\frac{c_1 y}{b_1}} \phi(a_1 y - b_1 x) \tag{11.15}$$

Thus if no two factors are linearly dependent (i.e. no factor is a multiple of another) the general solution of

$$f(D_x, D_y)z = 0$$

is the sum of n arbitrary functions of the types mentioned above. In the case of repeated factors, for example

$$(D_x - mD_y - a)^3 z = 0$$

we have $z = e^{ax}\phi_1(y + mx) + xe^{ax}\phi_2(y + mx) + x^2 e^{ax}\phi_3(y + mx)$

Case-3: If $c_1 = 0$, the equation becomes homogeneous linear equation and its solution is obtained if we put $c_1 = 0$ in equation (11.14) or (11.15) which reduces to $z = \phi(a_1 y - b_1 x)$.

11.5 Irreducible equation

If complementary function.

When $F(D_x, D_y)z$ can not be factorized into linear factors we apply for complementary function the trial method given below:

Let us consider the linear equation with constant coefficients

$$f(D_x, D_y)z = 0 \tag{11.16}$$

Since

$$D_x^r D_y^s \left(Ce^{ax+by}\right) = ca^r b^r e^{ax+by} \tag{11.17}$$

were a, b, c are constants. The trial substitution

$$z = ce^{ax+by}$$

in (11.16) gives

$$cf(a, b)e^{ax+by}$$

is a solution of (11.16) provided

$$f(a, b) = 0 \tag{11.18}$$

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with c arbitrary. Now for any chosen value of a (or b) one or more values of (11.18). Thus, there exist infinitely many pairs of number (a_r, b_r) satisfying the relation (11.16) and so

$$z = \sum_{r=1}^{\infty} c_r e^{a_r x + b_r y}$$

where $$f(a_r, b_r) = 0$$

is a solution of the given equation, i.e. the complementary function of the equation

$$f(D_x, D_y)z = F(x, y).$$

Example 11.13 Solve

$$(3D_x + 2D_y - 6)(D_x + 2D_y)(D_x + 3)(D_y - 2)z = 0.$$

Solution The solution is

$$z = e^{2x}\phi_1(3y - 2x) + \phi_2(y - 2x) + e^{-3x}\phi_3(y) + e^{2y}\phi_4(x).$$

Example 11.14 Solve

$$(D_x^2 + 2D_xD_y + D_y^2 - 2D_x - 2D_y)z = \cos(x + 2y).$$

Solution Here $f(D_x, D_y)z = (D_x + D_y)(D_x + D_y - 2)z.$

Therefore $$\text{C. F} = \phi_1(y - x) + e^{2x}\phi_2(y - x)$$

$$\text{P. I} = \frac{1}{D_x^2 + 2D_xD_y + D_y^2 - 2D_x - 2D_y}\cos(x + 2y)$$

$$= \frac{1}{-1 + 2(-2) + (-4) - 2D_x - 2D_y}\cos(x + 2y)$$

$$= -\frac{1}{2(D_x + D_y) + 9}\cos(x + 2y)$$

$$= -\frac{2(D_x + D_y) - 9}{4(D_x^2 + 2D_xD_y + D_y^2) - 81}\cos(x + 2y)$$

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$$= -\frac{2(D_x + D_y) - 9}{4[-1 + 2(-2) - 4] - 81}\cos(x + 2y)$$

$$= \frac{1}{117}[-2\sin(x + 2y) - 4\sin(x + 2y)\} - 9\cos(x + 2y)]$$

$$= -\frac{1}{117}[6\sin(x + 2y) + 9\cos(x + 2y)]$$

$$= -\frac{3}{117}[2\sin(x + 2y) + 3\cos(x + 2y)]$$

Hence the general solution is

$$z = \phi_1(y - x) + e^{2x}\phi_2(y - x) - \frac{3}{117}[2\sin(x + 2y) + 3\cos(x + 2y)].$$

Example 11.15 Solve

$$(D_x^2 - D_xD_y - 2D_y^2 + 2D_x + 2D_y)z = e^{3x+4y} + xy.$$

Solution $(D_x^2 - D_xD_y - 2D_y^2 + 2D_x + 2D_y)z = e^{3x+4y} + xy$

or $$(D_x + D_y)(D_x - 2D_y + 2)z = e^{3x+4y} + xy.$$

The C. F is obtained from

$$(D_x + D_y)\,(D_x - 2D_y + 2)z = 0$$

and is found to be $z_1 = \phi_1(y - x) + e^{-2x}\phi_2(2x + y)$

Then P. I for e^{3x+4y}

$$= \frac{1}{D_x^2 - D_xD_y - 2D_y^2 + 2D_x + 2D_y}e^{3x+4y}$$

$$= \frac{1}{3^2 - 3.4 - 2.4^2 + 2.3 + 2.4}e^{3x+4y} = -\frac{1}{21}e^{3x+4y}$$

and for $xy = \dfrac{1}{(D_x + D_y)(D_x - 2D_y + 2)}xy$

$$= \frac{1}{2}\frac{1}{(D_x + D_y)}\left[1 - \left(D_y - \frac{D_x}{2}\right)\right]^{-1}xy$$

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$$= \frac{1}{2}\frac{1}{(D_x + D_y)}\left[1 + D_y - \frac{1}{2}D_x + D_y^2 - D_x D_y + \frac{1}{4}D_x^2 + \cdots\cdots\right]xy$$

$$= \frac{1}{2}\frac{1}{(D_x + D_y)}\left[xy + x - \frac{1}{2}y - 1\right]$$

$$= \frac{1}{2}\frac{1}{D_x}\left(1 + \frac{D_y}{D_x}\right)^{-1}\left(xy + x - \frac{1}{2}y - 1\right)$$

$$= \frac{1}{2}\frac{1}{D_x}\left(1 - \frac{D_y}{D_x}\right)\left(xy + x - \frac{1}{2}y - 1\right)$$

$$= \frac{1}{2}\frac{1}{D_x}\left(xy + x - \frac{1}{2}y - 1 - \frac{x^2}{2} + \frac{x}{2}\right)$$

$$= \frac{1}{2}\frac{1}{D_x}\left(xy - \frac{1}{2}y - \frac{x^2}{2} + \frac{3x}{2} - 1\right)$$

$$= \frac{1}{2}\left(\frac{x^2 y}{2} - \frac{xy}{2} - \frac{x^3}{6} + \frac{3x^2}{4} - x\right)$$

$$= \frac{x}{24}\left(6xy - 6y - 2x^2 + 9x - 12\right)$$

Hence the general solution is

$$z = \phi_1(y) + e^{-2x}\phi_2(2x + y) - \frac{1}{21}e^{3x+4y} + \frac{x}{24}(6xy - 6y - 2x^2 + 9x - 12)$$

Example 11.16 Solve $\left(D_x^2 + D_x D_y - 6D_y^2\right)z = x^2\cos(x + y)$.

Solution The equation is $(D_x - 2D_y)(D_x + 3D_y)z = x^2\cos(x + y)$

$$\text{C. F.} = \phi_1(y + 2x) + \phi_2(y - 3x)$$

To evaluate the P. I. we take $\cos(x + y)$ as real part of $e^{i(x+y)}$

$$\therefore \quad \text{P. I.} = \text{Real part of } \frac{1}{D_x^2 + D_x D_y - 6D_y^2}x^2 e^{i(x+y)}$$

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$$= \text{Real part of } e^{i(x+y)} \frac{1}{(D_x+i)^2 + i(D_x+i) - 6i^2} x^2$$

putting $D_x + i$ for D_x and i for D_y.

$$= \text{Real part of } e^{i(x+y)} \frac{1}{D_x^2 + 3iD_x + 4} x^2$$

$$= \text{ Real part of } \frac{e^{i(x+y)}}{4}\left[1 + \frac{D_x^2 + 3iD_x}{4}\right]x^2$$

$$= \text{Real part of } \frac{e^{i(x+y)}}{4}\left[1 - \frac{3iD_x}{4} - \frac{D_x^2}{4} - \frac{9D_x^2}{16}\right]x^2$$

$$= \text{Real part of } \frac{e^{i(x+y)}}{4}\left(x^2 - \frac{3ix}{2} - \frac{13}{8}\right)$$

$$= \text{Real part of } \frac{\cos(x+y) + i\sin(x+y)}{4}\left(x^2 - \frac{3ix}{2} - \frac{13}{8}\right)$$

$$= \frac{1}{4}\left\{\left(x^2 - \frac{13}{8}\right)\cos(x+y) + \frac{3x}{2}\sin(x+y)\right\}$$

the general solution is

$$= z = \phi_1(y+2x) + \phi_2(y-3x)$$

$$+ \frac{1}{4}\left\{\left(x^2 - \frac{13}{8}\right)\cos(x+y) + \frac{3x}{2}\sin(x+y)\right\}$$

Remark In fact D_y should be replaced by $D_y + i$. But in the numerator D_y will be treated as dummy derivative since the differentiation will be carried out on x variable. So D_y is replaced by i alone.

Example 11.17 Solve $(3D_x^2 - 2D_y^2 + D_x - 1)z = 3e^{x+y}\sin(x+y)$.

Solution The equation is irreducible.

Here $\qquad f(a, b) = 3a^2 - 2b^2 + a - 1 = 0$

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so that for any $a = a_i,\ b_i = \dfrac{3a_i^2 + a_i - 1}{2}$.

Thus $$C.F = \sum_{i=1}^{n} C_i e^{aix+biy}$$

where $$3a_r^2 - 2b_r^2 + a_r - 1 = 0$$

$$\text{P. I} = \frac{1}{3D_x^2 - 2D_y^2 + D_x - 1} 3e^{x+y}\sin(x+y)$$

$$= 3e^{x+y}\frac{1}{3(D_x+1)^2 - 2(D_y+1)^2 + (D_x+1) - 1}\sin(x+y)$$

$$= 3e^{x+y}\frac{1}{3D_x^2 - 2D_y^2 + 7D_x - 4D_y + 1}\sin(x+y)$$

$$= 3e^{x+y}\frac{1}{-3+2+7D_x - 4D_y + 1}\sin(x+y)$$

$$= 3e^{x+y}\frac{7D_x + 4D_y}{49D_x^2 - 16D_y^2}\sin(x+y)$$

$$= 3e^{x+y}\frac{11\cos(x+y)}{-49+16}$$

$$= 3e^{x+y}\frac{\cos(x+y)}{-3}$$

$$= -e^{x+y}\cos(x+y)$$

$\therefore$ The general solution

$$z = \sum_{r=1}^{\infty} c_r e^{a_r x + b_r y} - e^{x+y}\cos(x+y)$$

where $$3a_r^2 - 2b_r^2 + a_r - 1 = 0.$$

Example 11.18 Solve $\left(3D_x^2 - 2D_y^2 + D_x + 4\right)z = 7\tan(x+2y)$

Solution the equation is irreducible

Here $$f(a, b) = 3a^2 - 2b^2 + a + 4 = 0$$

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sot that for any $a = ai, bi = \dfrac{3a_1^2 + a_1 + 4}{2}$

thus $\quad P.I = \dfrac{1}{3D_x^2 - 2D_y^2 + D_x + 4}.7\tan(x+2y)$

Where $D_x = 1$, and $D_y = 2$, $3D_x^2 - 2D_y^2 + D_x + 4 = 0$.

It is a case of failure.

Thus $\quad P.I = 7x.\dfrac{1}{6D_x + 1}\tan(x+2y)$

[multiplying numerator by x and differentiating the denominator w.r.t D_x]

$\therefore \quad$ P. I. $= \dfrac{7x}{7}\tan(x+2y)$

$\therefore$ Complete solution is $z = \sum_{i=1}^{n} c_i e^{a_i x + b_i y} + x\tan(x+2y)$

where $\quad 3a_r^2 - 2a_r^2 + a_r + 4 = 0$

Example 11.19 Solve

$$(D_x^2 - D_x D_y - 2D_y^2 + 6D_x - 9D_y + 5)z = (x^2 - y)e^{2x+y}.$$

Solution For C. F. the equation may be written as

$$(D_x^2 - D_x D_y - 2D_y^2 + 6D_x - 9D_y + 5)z = 0$$

or $\quad (D_x - 2D_y + 1)(D_x + D_y + 5)z = 0$

$$\therefore z_c = e^{-x}\phi_2(y+2x) + e^{-5x}\phi_1(y-x)$$

$$\text{P. I.} = \frac{1}{(D_x - 2D_y + 1)(D_x + D_y + 5)}.(x^2 - y)e^{2x+y}$$

$$= e^{2x+y}\frac{1}{\{(D_x+2) - 2(D_y+1) + 1\}\{D_x + 2 + D_y + 1 + 5\}}(x^2 - y)$$

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$$= e^{2x+y}\frac{1}{(D_x-2D_y+1)(D_x+D_y+8)}(x^2-y)$$

$$= \frac{1}{8}e^{2x+y}\frac{1}{(D_x-2D_y+1)}\left[1+\frac{D_x+D_y}{8}\right]^{-1}(x^2-y)$$

$$= \frac{1}{8}e^{2x+y}\frac{1}{(D_x-2D_y+1)}\left[1-\frac{D_x}{8}-\frac{D_y}{8}+\frac{D_x^2}{64}+\ldots\right](x^2-y)$$

$$= \frac{1}{8}e^{2x+y}\frac{1}{(D_x-2D_y+1)}\left[x^2-y-\frac{x}{4}+\frac{1}{8}+\frac{1}{32}\right]$$

$$= \frac{1}{8}e^{2x+y}(1+2D_y-D_x+D_x^2-\cdots\cdots)\left(x^2-\frac{x}{4}-y+\frac{5}{32}\right)$$

$$= \frac{1}{8}e^{2x+y}\left[x^2-\frac{x}{4}-y+\frac{5}{32}-2-2x+\frac{1}{4}+2\right]$$

$$= \frac{1}{8}e^{2x+y}\left(x^2-\frac{9x}{4}-y+\frac{5}{32}\right)$$

$$= \frac{1}{256}\left(32x^2-72x-32y+5\right)e^{2x+y}$$

General solution is

$$z = e^{-5x}\phi_1(y-x)+e^{-x}+\frac{1}{256}(32x^2-72x-32y+5)e^{2x+y}.$$

11.6 Homogeneous linear equation with variable coefficients

A partial differential equation of the form $F(xD_x,\ yD_y)=f(x,\ y)$ having variable coefficients can sometimes be reduced to equations with constant coefficients by suitable transformations. Let us change the independent variables x, y to u and v by the substitutions

$$x = e^u,\ y = e^v$$

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or $$u = \log x,\ v = \log y$$

$$D_x z = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = \frac{1}{x}\frac{\partial z}{\partial y}$$

or $$x\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \text{ or } xD_x z = D_u z$$

Again $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{1}{x}\frac{\partial z}{\partial u}\right)\frac{\partial u}{dx}$$

$$= \frac{\partial}{\partial u}\left(\frac{1}{x}\frac{\partial z}{\partial u}\right)\frac{1}{x}$$

$$= \frac{1}{x}\left[\frac{1}{x}\frac{\partial^2 z}{\partial u^2} - \frac{\partial z}{\partial u}.\frac{1}{x^2}\frac{\partial x}{\partial u}\right]\frac{1}{x} = \frac{1}{x^2}\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial u}\right)$$

$$x^2\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2} - \frac{\partial z}{\partial u} = D_u(D_u - 1)z$$

$$\therefore \qquad x^2\frac{\partial^2}{\partial x^2} = D_u(D_u - 1).$$

Similarly $$x^3\frac{\partial^3}{\partial x^3} = D_u(D_u - 1)(D_u - 2)$$

Proceeding in a similar way we obtain

$$y\frac{\partial}{\partial y} = D_v$$

$$y^2\frac{\partial^2}{\partial y^2} = D_v(D_v - 1)$$

$$y^3\frac{\partial^3}{\partial y^3} = D_v(D_v - 1)(D_v - 2) \text{ etc.}$$

Example 11.20 Solve $x^2\dfrac{\partial^2 z}{\partial x^2} - y^2\dfrac{\partial^2 z}{\partial y^2} = x^2 y.$

Solution The substitution $x = e^u$, $y = e^v$ transforms the given equation into

$$[D_u(D_u - 1) - D_v(D_v - 1)]z = e^{2u+v}$$

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A. E. is $$(D_u^2 - D_v^2 - D_u + D_v)z = 0$$

or $$(D_u - D_v)(D_u - D_v - 1)z = 0$$

$$C.F = \phi_1(v+u) + e^u \phi_2(v-u)$$

$$= \phi_1(\log y + \log x) + x\phi_2(\log y - \log x)$$

$$= f_1(xy) + xf_2\left(\frac{y}{x}\right)$$

$$\text{P. I.} = \frac{1}{(D_u - D_v)(D_u + D_v - 1)} . e^{2u+v}$$

$$= \frac{1}{2}e^{2u+v} = \frac{1}{2}x^2 y$$

General solution is $z = C.F + P.F. = f_1(xy) + xf_2\left(\frac{y}{x}\right) + \frac{1}{2}x^2 y.$

Example 11.21 Solve

$$\left(x^2\frac{\partial^2 z}{\partial x^2} - 4y^2\frac{\partial^2 z}{\partial y^2} - 4y\frac{\partial z}{\partial y} - 1\right)z = x^2 y^3 \log y.$$

Solution Substituting $x = e^u$, $y = e^v$ and denoting

$$\frac{\partial}{\partial u} \text{ and } \frac{\partial}{\partial v}$$

by D_u and D_v respectively the given equation reduces to

$$[D_u(D_u - 1) - 4D_v(D_v - 1) - 4D_v - 1]z = ve^{2u+3v}$$

$$\text{C. F. is } z = \sum_{i=1}^{\infty} c_i e^{a_i u + b_i v}$$

$$\text{P. I.} = \frac{1}{D_u^2 - 4D_v^2 - D_u - 1} ve^{2u+3v}$$

$$= e^{2u+3v}\frac{1}{(D_u+2)^2 - 4(D_v+3)^2 - (D_u+2) - 1} v$$

$$= e^{2u+3v}\frac{1}{D_u^2 - 4D_v^2 + 3D_u - 24D_v - 35} v$$

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$$= e^{2u+3v}\left[-\frac{v}{35}+\frac{24}{35^2}\right] = -\frac{1}{1225}e^{2u+3v}(35v-24)$$

$$\therefore \quad z = \sum_{i=1}^{\infty} c_i e^{a_i u + b_i v} - \frac{1}{1225}x^3 y^3 (35\log y - 24)$$

where $a_i^{\,2} - 4b_i^{\,2} - a_1' - 1 = 0$.

Example 11.22 Solve $yt - q = xy$.

Solution The given equation can be written as

$$y\frac{\partial^2 z}{\partial y^2} - \frac{\partial z}{\partial y} = xy$$

or
$$y^2\frac{\partial^2 z}{\partial y^2} - y\frac{\partial z}{\partial y} = xy^2$$

Substituting $\quad x = e^u, \; y = e^v$

we get $\quad [D_v(D_v - 1) - D_v]z = e^{u+2v}$

or $\quad D_v(D_v - 2)z = e^{u+2v}$

$$\text{C.F.} = f_1(\log x) + y^2 f_2(\log x)$$

$$= \phi_1(x) + y^2\phi_2(x)$$

$$\text{P. I.} = \frac{1}{D_v^2 - 2D_v}e^{u+2v}$$

$$= \frac{v}{2D_v - 2}e^{u+2v}$$

$$= \frac{v}{2}e^{u+2v} = \frac{1}{2}xy^2\cos y.$$

General solution is $z = \phi_1(x) + y^2\phi_2(x) + \frac{1}{2}xy^2\log y.$

Example 11.23 Solve

$$\left(x^2\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 y}{\partial x\partial y} - 3y^2\frac{\partial^2 z}{\partial y^2} + \frac{\partial y}{\partial x} - 3\frac{\partial z}{\partial y}\right) = x^2 y\sin(\log x^2)$$

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Solution Put $x = e^u$; $y = e^v$

$$\therefore [D_u(D_u - 1) - 2D_uD_v - 3D_v(D_v - 1) + D_u - 3D_v]z = e^{2u+v}\sin 2u$$

or $$(D_u^2 - 2D_uD_v - 3D_v^2)z = e^{2u+v}\sin 2u$$

or $$(D_u - 3D_v)(D_u + D_v)z = 2ue^{2u+v}$$

$$\text{C. F} = f_1(v + 3u) + f_2(v - u)$$

$$= \phi_1(x^3y) + \phi_2\left(\frac{y}{x}\right)$$

$$\text{P.I.} = \frac{1}{D_u^2 - 2D_uD_v - 3D_v^2}e^{2u+v}\sin 2u$$

$$= e^{2u+v}\frac{1}{(D_u + 2)^2 - 2(D_u + 2)(D_v + 1) - 3(D_v + 1)^2}\sin 2u$$

$$= e^{2u+v}\frac{1}{D_u^2 + 4D_u + 4 - 2D_uD_v - 2D_u - 4D_v - 4 - 3D_v^2 - 6D_v - 3}\sin 2u$$

$$= e^{2u+v}\frac{1}{D_u^2 - 3D_v^2 - 2D_uD_v + 2D_u - 10D_v - 3}\sin 2u$$

$$= e^{2u+v}\frac{1}{-4 + 2D_u - 3}\sin 2u$$

$$= e^{2u+v}\frac{1}{2D_x - 7}\sin 2u$$

$$= e^{2u+v}\frac{2D_u + 7}{4D_u^2 - 49}\sin 2u$$

$$= e^{2u+v}\frac{2D_u + 7}{-16 - 49}\sin 2u$$

$$= -\frac{e^{2u+v}}{65}[4\cos 2u + 7\sin 2u]$$

$$= -\frac{x^2y}{65}[4\cos(\cos x^2) + 4\sin(\cos x^2)]$$

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The general solution is

$$z = \phi_1(x^3 y) + \phi_2\left(\frac{y}{x}\right) - \frac{1}{65}x^2 y[4\cos(\cos x^2) + 7\sin(\cos x^2)].$$

Example 11.24 Solve $x^2\dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x \partial y} + y^2\dfrac{\partial^2 z}{\partial y^2} = (x^2+y^2)^{\frac{n}{2}}$

Solution $x = e^u$ and $y = e^v$ reduces the given equation to

$$[D_u(D_u - 1) + 2D_u D_v + D_v(D_v - 1)]z = \left(e^{2u} + e^{2v}\right)^{\frac{n}{2}}$$

or $$(D_u + D_v)(D_u + D_v - 1)z = \left(e^{2u} + e^{2v}\right)^{\frac{n}{2}}$$

$\therefore$ C. F. is $$f_1(v-u) + e^u f_2(v-u)$$

$$= f_1\left(\log\frac{y}{x}\right) + x f_2\left(\log\frac{y}{x}\right)$$

$$= \phi_1\left(\frac{y}{x}\right) + x\phi_2\left(\frac{y}{x}\right)$$

$$\text{P. I} = \frac{1}{(D_u + D_v)(D_u + D_v - 1)}\left(e^{2u} + e^{2v}\right)^{\frac{n}{2}}$$

$$= \frac{1}{(D_u + D_v)(D_u + D_v - 1)} e^{nu}\left[1 + e^{2(v-u)}\right]^{\frac{n}{2}}$$

$$= \frac{1}{(D_u + D_v)(D_u + D_v - 1)} e^{nu}\left[1 + \frac{n}{2}e^{2(v-u)} + \frac{\frac{n}{2}\left(\frac{n}{2}-1\right)}{1.2}e^{4(v-n)} + \cdots\cdots\right]$$

[using $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \cdots\cdots$]

$$= e^{nu}\frac{1}{n(n-1)}\left[1 + \frac{n}{2}e^{2(v-u)} + \cdots\cdots\right]$$

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$$= \frac{1}{n(n-1)}\left[e^{nu} + \frac{n}{2}e^{(n-2)u+2v} + \frac{\frac{n}{2}\left(\frac{n}{2}-1\right)}{1.2}e^{(n-4)u+4v}\right]$$

$$= \frac{1}{n(n-1)}e^{nu}\left[1+e^{2(v-u)}\right]^{\frac{n}{2}}$$

$$= \frac{\left(e^{2u}+e^{2v}\right)^{\frac{n}{2}}}{n(n-1)} = \frac{\left(x^2+y^2\right)^{\frac{n}{2}}}{n(n-1)}$$

G. S. is $z = \phi\left(\frac{y}{x}\right) + x\phi_2\left(\frac{y}{x}\right) + \frac{\left(x^2+y^2\right)^{\frac{n}{2}}}{n(n-1)}$

Example 11.25 Solve

$$x^2\left(\frac{\partial^2 z}{\partial x^2} - xy\frac{\partial^2 z}{\partial x \partial y} - 2y\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} - 2y\frac{\partial x}{\partial y}\right)z = \log\left(\frac{y}{x}\right) - \frac{1}{2}.$$

Solution $x = e^u$ and $y = e^v$ reduces the given equation to

$$[D_u(D_u-1) - D_uD_v - 2D_v(D_v-1) + D_u - 2D_v]z = v - u - \frac{1}{2}$$

$$\left(D_u^2 - D_uD_v - 2D_v^2\right)z = v - u - \frac{1}{2}$$

$$(D_u - 2D_v)(D_u + D_v)z = v - u - \frac{1}{2}$$

$$z_c = f_1(v+2u) + f_2(v-u)$$

$$= \phi_1(x^2y) + \phi_2\left(\frac{y}{x}\right)$$

$$\text{P. I} = \frac{1}{(D_u^2 - D_uD_v - 2D_v^2)}\left(v - u - \frac{1}{2}\right)$$

$$= \frac{1}{D_u^2 - D_uD_v - 2D_v^2}(v-u) - \frac{1}{2}\frac{1}{(D_u^2 - D_uD_v - 2D_v^2)}$$

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$$= \frac{1}{D_u^2}\left[1-\left(\frac{D_v}{D_u}+\frac{2D_v^2}{D_u^2}\right)\right]^{-1}(v-u)-\frac{1}{2}\cdot\frac{u^2}{2}$$

$$= \frac{1}{D_u^2}\left[1+\frac{D_v}{D_u}+\frac{2D_v^2}{D_u^2}\right](v-u)-\frac{u^2}{4}$$

$$= \frac{1}{D_u^2}[v-u+u]-\frac{u^2}{4}$$

$$= \frac{vu^2}{2}-\frac{u^2}{4}$$

$$= \frac{1}{2}(\log x)^2 \log y-\frac{1}{4}(\log x)^2$$

Therefore the complete solution is

$$z = \phi_1(x^2 y)+\phi_2\left(\frac{y}{x}\right)+\frac{1}{2}(\log x)^2 \log y-\frac{1}{4}(\log x)^2.$$

Example 11.26 Solve $\frac{1}{x^2}\frac{\partial^2 z}{\partial x^2}-\frac{1}{x^3}\frac{\partial z}{\partial x}=\frac{1}{y^2}\frac{\partial^2 z}{\partial y^2}-\frac{1}{y^3}\frac{\partial z}{\partial y}$

Solution Put $\frac{1}{2}x^2 = u$ and $\frac{1}{2}y^2 = v$

$$\therefore\ x\partial x = \partial u \quad ; \quad y\partial y = \partial v$$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}=\frac{1}{x}\frac{\partial z}{\partial x}$$

$$\frac{\partial^2 z}{\partial u^2}=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)=\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial u}$$

$$=\left[-\frac{1}{x^2}\frac{\partial z}{\partial x}+\frac{1}{x}\frac{\partial^2 z}{\partial x^2}\right]\frac{1}{x}$$

$$=-\frac{1}{x^3}\frac{\partial z}{\partial x}+\frac{1}{x^2}\frac{\partial^2 z}{\partial x^2}$$

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Similarly $$\frac{1}{y^2}\frac{\partial^2 z}{\partial y^2}-\frac{1}{y^3}\frac{\partial z}{\partial y}=\frac{\partial^2 z}{\partial y^2}$$

or $$\frac{\partial^2 z}{\partial u^2}-\frac{\partial^2 z}{\partial v^2}=0$$

or $$(D_u-D_v)(D_u+D_v)z=0$$

Therefore the solution is $z=f_1(v+u)+f_2(v-u)$

$$=f_1\left(\frac{y^2+x^2}{2}\right)+f_2\left(\frac{y^2-x^2}{2}\right)$$

$$=\phi_1\left(y^2+x^2\right)+\phi_2\left(y^2-x^2\right)$$

Example 11.27 Find a surface satisfying

$$2x^2r-5xys+2y^2t+2(px+qy)=0$$

and touching the hyperbolic paraboloid $z=x^2-y^2$ along its section by the plane $y=1$.

Solution The given equation can be written as

$$2x^2\frac{\partial^2 z}{\partial x^2}-5xy\frac{\partial^2 z}{\partial x\partial y}+2y^2\frac{\partial^2 z}{\partial y^2}+2\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)=0$$

$$x=e^u \text{ and } y=e^v$$

reduces the equation to

$$\left[2D_u(D_u-1)-5D_uD_v+2D_v(D_v-1)+2(D_u+D_v)z\right]=0$$

or $$(2D_u-D_v)(D_u-2D_v)z=0$$

Hence the solution is

$$z=f_1(2v+u)+f_2(v+2u)$$

$$=f_1(2\log y+\log x)+f_2(\log y+2\log x)$$

$$=f_1(\log xy^2)+f_2(\log x^2y)$$

$$=\phi_1(xy^2)+\phi_2(x^2y)$$

The other surface is $z=x^2-y^2$

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Since the two surfaces touch each other along the section by $y = 1$, the values of p and q for two surfaces must be equal at $y = 1$.

Now, $$p = \frac{\partial z}{\partial x} = y^2\phi_1'(xy^2) + 2xy\phi_2'(x^2y) = 2x \tag{11.19}$$

and $$q = \frac{\partial z}{\partial y} = 2xy\phi_1'(xy^2) + x^2\phi_2'(x^2y) = 2y \tag{11.20}$$

For $y = 1$, equation (11.19) and equation (11.20) reduces to

$$\phi_1'(x) + 2x\phi_2'(x^2) = 2x \tag{11.21}$$

and $$2x\phi_1'(x) + x^2\phi_2'(x^2) = -2 \tag{11.22}$$

Solving (11.21) and (11.22) we get

$$\phi_1'(x) = -\frac{2x}{3} - \frac{4}{3x} \tag{11.23}$$

and $$\phi_2'(x^2) = \frac{2}{3x^2} + \frac{4}{3} \tag{11.24}$$

Integrating (11.23) we get $\phi_1(x) = -\frac{x^2}{3} - \frac{4}{3}\log x$ (11.25)

Integrating (11.24) w.r.t. x^2 we obtain

$$\phi_2(x^2) = \frac{2}{3}\log x^2 + \frac{4}{3}x^2$$

$$= \frac{4}{3}\log x + \frac{4}{3}x^2 \tag{11.26}$$

Replacing x by xy^2 in (11.25) we get

$$\phi_1(xy^2) = -\frac{1}{3}x^2y^4 - \frac{4}{3}\log xy^2$$

$$= -\frac{1}{3}x^2y^4 - \frac{8}{3}\log y - \frac{4}{3}\log x$$

Replacing x^2 by x^2y, in (11.26) we get

$$\phi_2(x^2y) = \frac{4}{3}\log\left(x\sqrt{y}\right) + \frac{4}{3}x^2y$$

$$= \frac{4}{3}\log x + \frac{2}{3}\log y + \frac{4}{3}x^2y \tag{11.27}$$

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Putting the values of ϕ_1 and ϕ_2 in

$$z = \phi_1(xy^2) + \phi_2(x^2y), \text{ we get}$$

$$z = -\frac{1}{3}x^2y^4 - 2\log y + \frac{4}{3}x^2y + c_1$$

or $$3z = 4x^2y - x^2y^4 - 6\log y + c \tag{11.28}$$

where c is a constant.

But when $y = 1$

then $3z = 4x^2 - x^2 + c = 3x^2 + c$

But when $y = 1$, the other surface $z = x^2 - y^2$ becomes $z = x^2 - 1$.

Example 11.28 Solve $(D_x - D_y^2)z = \sin(x - 3y)$.

Solution For C. F, we have

$$(D_x - D_y^2)z = 0 \tag{11.29}$$

Let the solution be $z = Ae^{hx+ky}$

$\therefore$ $D_x z = hAe^{hx+ky}$

Similarly $D_y^2 z = k^2 Ae^{hx+ky}$

Now $(D_x - D_y^2)z = 0$, becomes $(h - k^2)Ae^{hx+ky} = 0$

Above will hold good if $h - k^2 = 0$

or $h = k^2$

Putting k^2 for h, the solution is given by $z = Ae^{k^2x+ky}$.

Since all values of k satisfy the given equation, a more general solution of the given equation is

$$z = \sum Ae^{k^2x+ky}$$

$$\text{P. I} = \frac{1}{D_x - D_y^2}\sin(x - 3y)$$

$$= \frac{1}{D_x + 9}\sin(x - 3y)$$

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$$= \frac{D_x - 9}{D_x^2 - 81}\sin(x-3y)$$

$$= \frac{D_x - 9}{-1-81}\sin(x-3y)$$

$$= -\frac{1}{82}[\cos(x-3y) - 9\sin(x-3y)]$$

$$= \frac{1}{82}[9\sin(x-3y) - \cos(x-3y)]$$

Hence the general solution is

$$z = \sum Ae^{k^2x+ky} + \frac{1}{82}[9\sin(x-3y) - \cos(x-3y)]$$

Example 11.29 Find the solution of the heat conduction problem,

$$u_{xx} = 4u_t,\ 0 < x < 2,\ t > 0;\quad u(0, t) = 0,\ u(2, t) = 0,\ t > 0$$

$$u(x, 0) = 2\sin\frac{\pi x}{2} - \sin \pi x + 4\sin 2\pi x,\ 0 \le x \le 2.$$

Solution We will solve the problem by the method of separation of variables. The given partial differential equation is

$$u_{xx} = 4u_t \quad \text{i. e.}\ \frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial t} \tag{11.30}$$

Let $u = XT$ where X depends only on x and T depends only on t.

Then $\dfrac{\partial u}{\partial x} = X'T,\ \dfrac{\partial^2 u}{\partial x^2} = X''T$ and $\dfrac{\partial u}{\partial t} = XT'$

Putting these values in (11.30), we get,

$$X''T = 4XT' \quad \text{or,}\ \frac{X''}{X} = \frac{4T'}{T} \tag{11.31}$$

Each side of (11.31) must be a constant, which we call $-\lambda^2$.

$$\frac{X''}{X} = \frac{4T'}{T} = -\lambda^2 \text{ (say)}$$

Then $X'' + \lambda^2 X = 0$ and $T' + \dfrac{\lambda^2}{4}T = 0$ whose solutions are,

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$$X = A_1 \cos \lambda x + B_1 \sin \lambda x \text{ and } T = C_1 e^{-\frac{\lambda^2}{4}t} \text{ respectively.}$$

Thus a solution of the partial differential equation is,

$$u(x, t) = XT = (A_1 \cos \lambda x + B_1 \sin \lambda x) C_1 e^{-\frac{\lambda^2 t}{4}}$$

$$= e^{-\frac{\lambda^2}{4}t} (A \cos \lambda x + B \sin \lambda x) \tag{11.32}$$

where $A_1 C_1 = A$ and $B_1 C_1 = B$.

Since $u(0, t) = 0$ $\quad \therefore\ e^{-\frac{\lambda^2}{4}t}(A + 0) = 0$

or, $Ae^{-\frac{\lambda^2}{4}t} = 0$ $\quad \therefore A = 0$, since $e^{-\frac{\lambda^2}{4}t} \neq 0$

Thus from (11.32), we have $u(x, t) = Be^{-\frac{\lambda^2}{4}t} \sin \lambda x$ (11.33)

since $u(2, t) = 0$, $\quad \therefore\ Be^{-\frac{\lambda^2}{4}t} \sin 2\lambda = 0$

So we must have $\sin 2\lambda = 0 \quad (B \neq 0)$

Or, $2\lambda = n\pi$ where $n = 0, \pm 1, \pm 2, \ldots\ldots$ $\quad \therefore\ \lambda = \dfrac{n\pi}{2}$.

Thus a solution is $u(x, t) = Be^{-\frac{n^2\pi^2 t}{16}} \sin \dfrac{n\pi x}{2}$ (11.34)

Now by the principle of superposition, we have,

$$u(x, t) = B_1 e^{-\frac{n_1^2\pi^2 t}{16}} \sin \frac{n_1 \pi x}{2} + B_2 e^{-\frac{n_2^2\pi^2 t}{16}} \sin \frac{n_2 \pi x}{2}$$

$$+ B_3 e^{-\frac{n_3^2\pi^2 t}{16}} \sin \frac{n_3 \pi x}{2} \tag{11.35}$$

Using the last boundary condition, we have

$$u(x, 0) = B_1 \sin \frac{n_1 \pi x}{2} + B_2 \sin \frac{n_2 \pi x}{2} + B_3 \sin \frac{n_3 \pi x}{2}$$

$$= 2 \sin \frac{\pi x}{2} - \sin \pi x + 4 \sin 2\pi x$$

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which is possible if and only if $B_1 = 2,\ n_1 = 1;$

$$B_2 = 1,\ n_2 = 2;\quad B_3 = 4,\ n_3 = 4.$$

Putting these values is (11.35), we get

$$u(x, t) = 2e^{-\frac{\pi^2 t}{16}} \sin\frac{\pi x}{2} - e^{-\frac{\pi^2 t}{4}} \sin \pi x + 4e^{-\pi^2 t} \sin 2\pi x$$

which is the required solution of the given heat conduction problem.

Test your knowledge

Solve

1. $\left(D_x^2 + 6D_xD_y + 9D_y^2\right)z = 0.$
2. $\left(D_x^3 - 2D_x^2D_y + 9D_y^2\right)z = 0.$
3. $\left(D_x^3 - 2D_x^2D_y\right)z = 3x^2y.$
4. $\left(D_x^2 + 2D_xD_y + D_y^2\right)z = x^2 + xy + y^2.$
5. $\left(D_x^2 + 5D_xD_y + 6D_y^2\right)z = e^{x-y}.$
6. $\left(2D_x^2 - D_xD_y - 3D_y^2\right)z = 5e^{x-y}.$
7. $\left(D_x - 2D_y\right)^2\left(D_x + 3D_y\right)z = e^{2x+y}.$
8. $\left(D_x^3 - 7D_xD_y^2 - 6D_y^3\right)z = \sin(x + 2y).$
9. $\left(D_x^2 - 3D_xD_y + 2D_y^2\right)z = \cos(x + 2y).$
10. $\left(D_x^3 - 4D_x^2D_y + 4D_xD_y^2\right)z = \sin(2x + y).$
11. $\left(D_x^3 + D_x^2D_y - D_xD_y^2\right) = e^x \cos 2y.$
12. $\left(D_x^2 - 5D_xD_y + 6D_y^2\right)z = e^{x-y}xy.$
13. $\left(4D_x^2 - 4D_xD_y + D_y^2\right)z = \log(x + 2y).$

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14. $\left(D_x^2 + 2D_xD_y - 8D_y^2\right)z = \sqrt{2x+3y}.$

15. $(4D_x - 2D_y + 3)(3D_x + 5D_y)(D_x - 4)(D_y + 7)z = 0.$

16. $(D_x + 1)(D_x + D_y - 1)z = \sin(x + 2y).$

17. $\left(D_x^3 + 3D_x^2D_y - 2D_x^2\right)z = (x^2 + 2y)e^{2x+y}.$

18. $\left(D_x^2 - D_xD_y - 2D_y^2 + 2D_x + 2D_y\right)z = xy + \sin(2x + y).$

19. $(2D_x - 4D_y + 3)(D_x - 2D_y - 2)z = xy^2\cos(2x + y).$

20. $(D_x + D_y - 1)(D_x + D_y - 3)(D_x + D_y)z = e^{x+y+2}\cos(2x - y).$

21. $\left(D_x^3 - D_xD_y^2 - D_y^2 + D_xD_y\right)z = \dfrac{x+2}{x^3}.$

22. $\left(D_x^2 + 2D_xD_y^2 - 2D_y + 3\right)z = e^{x+y}\cos(x + 2y).$

23. $\left(D_x^2 - D_xD_y + D_y - 1\right)z = \cos(x + 2y) + e^y.$

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CHAPTER 12

Second Order Partial Differential Equations with Variable Coefficients

12.1 Introduction

Second order partial differential equation contains at least one of the second order partial derivatives r, s and t. The first order partial derivatives may also appear in such equations. In general, the equations are non-linear, their forms being

$$f(x, y, z, p, q, r, s, t) = 0 \tag{12.1}$$

The above partial differential equation (12.1) can be integrated in some special cases. A wide class of such equations are solved by Monge's method. We will discuss the two Monge's method separately.

12.2 Monge's method of integrating

$$Rr + Ss + Tt = V$$

where r, s, t have their usual meaning and R, S, T, V are functions of x, y, z, p and q.

The given equation is $\quad Rr + Ss + Tt = V \qquad (12.2)$

We know that $\quad dp = \dfrac{\partial p}{\partial x}dx + \dfrac{\partial p}{\partial y}dy = rdx + sdy$

and $\quad dq = \dfrac{\partial q}{\partial x}dx + \dfrac{\partial q}{\partial y}dy = sdx + tdy$

which gives $\quad r = \dfrac{dp - sdy}{dx}$

and $\quad t = \dfrac{dq - sdx}{dy}.$

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Substituting these values of r and t in (12.2), we get

$$R\left(\frac{dp-sdy}{dx}\right)+Ss+T\left(\frac{dq-sdx}{dy}\right)=V$$

or, $(Rdpdy+Tdqdx-Vdxdy)-s(Rdy^2-Sdxdy+Tdx^2)=0$ (12.3)

Clearly any relation between

$$x, y, z, p, q$$

which satisfies (12.3) must also satisfy the following two simultaneous equations.

$$Rdpdy+Tdqdx-Vdxdy=0 \quad (12.4)$$

$$Rdy^2-Sdxdy+Tdx^2=0 \quad (12.5)$$

Equations (12.4) and (12.5) are called Monge's subsidiary equations.

Equation (12.5) being a quadratic, in general it can be resolved into two equations, say,

$$dy-m_1dx=0 \text{ and } dy-m_2dx=0 \quad (12.6)$$

Now two cases arise.

Case I: When m_1 and m_2 are distinct. In this case we combine

$$dy-m_1dx=0$$

with (12.4) and if necessary with

$$dz=pdx+qdy.$$

It will give us two integrals $u_1=a$

and $v_1=b.$

These give $u_1=f_1(v_1)$ (12.7)

where f_1 is an arbitrary function and is called an intermediate integral of (12.2). Similarly, taking another factor

$$dy-m_2dx=0$$

and proceeding as above we may have another intermediate integral of (12.2), say

$$u_2=f_2(v_2) \quad (12.8)$$

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Now we solve (12.7) and (12.8) to obtain the values of p and q in terms of x and y and then substitute these values in

$$dz = pdx + qdy$$

which on integration gives the desired comple integral of the given equation

$$Rr + Ss + Tt = V$$

Case II: When $m_1 = m_2$ i.e. when (12.5) is a perfect square, we get only one intermediate integral containing p, q and x, y, z of the form

$$Pp + Qq = R \tag{12.9}$$

The solution of (12.9) is obtained with the help of Lagrange's method.

Note: Sometimes in the first case also it is convenient to find the complete integral with the help of one intermediate integral only, as in the case when (12.5) is a perfect square.

We shall illustrate the above by taking suitable examples

Example 12.1 Solve $r = a^2 t$.

Solution We have $$r = \frac{dp - sdy}{dx}$$

and $$t = \frac{dq - sdx}{dy}$$

Putting these values of r and t in $r = a^2 t$, we get

$$\frac{dp - sdy}{dx} - a^2 \frac{dq - sdx}{dy} = 0$$

or $$(dpdy - a^2 dxdq) - s\{(dy)^2 - a^2 (dx)^2\} = 0$$

Hence, Monge's subsidiary equations are

$$(dy)^2 - a^2 (dx)^2 = 0 \tag{12.10}$$

and $$dpdy - a^2 dxdq = 0 \tag{12.11}$$

(12.10) can be written as

$$(dy - adx)(dy + adx) = 0$$

either $$dy = adx$$

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or $$dy = -adx \tag{12.12}$$

Taking $$dy = adx$$

i.e. $$y = ax + c$$

(12.11) gives $$dpadx - a^2dxdq = 0$$

or $$dp - adq = 0$$

or $$p - aq = \text{constant} = \phi(y - ax) \tag{12.13}$$

Similarly (12.10) and $dy = -adx$

give another intermediate integral as

$$p + aq = \psi(y + ax) \tag{12.14}$$

Solving (12.13) and (12.14), we get

$$p = \frac{1}{2}\{\phi(y - ax) + \psi(y + ax)\}$$

$$q = \frac{1}{2a}\{\psi(y + ax) - \phi(y - ax)\}$$

Hence we get, $dz = pdx + qdy$

$$= \frac{1}{2}\{\phi(y - ax) + \psi(y + ax)\}dx + \frac{1}{2a}\{\psi(y + ax) - \phi(y - ax)\}dy$$

$$= \frac{1}{2a}\psi(y + ax)(dy + adx) - \frac{1}{2a}\phi(y - ax)(dy - adx)$$

Integrating it, the complete solution is

$$z = \psi(y + ax) + \phi(y - ax).$$

Example 12.2 Solve $q^2r - 2pqs + p^2t = pq^2$.

Solution We have $r = \dfrac{dp - sdy}{dx}$

$$t = \frac{dq - sdx}{dy}$$

Putting this value of r and t in the given equation we get

$$q^2\frac{dp - sdy}{dx} = 2pqs + p^2\frac{dq - sdx}{dy} = pq^2$$

or $$(dp - sdy)q^2dy - 2pqsdxdy + (dq - sdx)p^2dx - pq^2 = 0$$

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or $-s(q^2dy^2 + 2pqdxdy + p^2dx^2) + q^2dydp + p^2dxdq - pq^2dxdy = 0$

Therefore Monge's equations are

$$(qdy + pdx)^2 = 0$$

and $$q^2dydp + p^2dxdq - pq^2dxdy = 0$$

From the first equation $qdy + pdx = 0$

Then $$dz = pdx + qdy = 0$$

or $$z = a$$

Substituting $$qdy = -pdx$$

in the second we get, $-pdxqdp + p^2dxdq + pqdxpdx = 0$

or $$-qdp + pdq + pqdx = 0$$

or $$\frac{dp}{-p} + \frac{dq}{q} + dx = 0.$$

Integrating we get $$\frac{e^x q}{p} = b$$

Thus $$e^x q - pf(z) = 0$$

is an intermediate integral.

So Lagrange's subsidiary equations are

$$\frac{dx}{f(z)} = \frac{dy}{-e^x}, \quad dz = 0$$

From the second equation, $z = c$. Then the first becomes

$$\frac{dx}{f(c)} = \frac{dy}{-e^x}$$

with solution $$\frac{e^x}{f(c)} + y = d$$

The required solution is

$$y = -\frac{e^x}{f(z)} + \phi_2(z) = e^x\phi_1(z) + \phi_2(z)$$

where $$\phi_1(z) = -\frac{1}{f(z)}$$

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Example 12.3 Solve $2x^2r - 5xys + 2y^2t + 2(px + qy) = 0$

Solution We have $$r = \frac{dp - sdy}{dx}$$

and $$t = \frac{dq - sdx}{dy}$$

Putting for r and t in the given equation, we get

$$2x^2 \frac{dp - sdy}{dx} - 5xys + 2y^2 \frac{dq - sdx}{dy} + 2(px + qy) = 0$$

Hence the Monge's subsidiary equations are

$$2x^2 dpdy + 2y^2 dqdx + 2(px + qy)dxdy = 0 \tag{12.15}$$

and $$2x^2 dy^2 + 5xydxdy + 2y^2 dx^2 = 0 \tag{12.16}$$

Now (12.16) can be resolved into factors

$$xdy + 2ydx = 0 \tag{12.17}$$

$$2xdy + ydx = 0 \tag{12.18}$$

On integrating (12.17), we get

$$\log x^2 y = c$$

or $$x^2 y = A.$$

Now combining (12.17) with (12.15), we get

$$2xdp(xdy) + ydq(2ydx) + 2pdx(xdy) + qdy(2ydx) = 0$$

or $$2xdp(xdy) + ydq(-xdy) + 2pdx(xdy) + qdy(-xdy) = 0$$

by (12.17)

or $$2xdp - ydq + 2pdx - qdy = 0$$

or $$2(xdp + pdx) - (ydq + qdy) = 0$$

Integrating, we get $2px - qy =$ constant.

or $$2px - qy = f_1(x^2 y)$$

or $$x^2 y = A \tag{12.19}$$

which is of Lagrange's form. Lagrange's auxiliary equations are

$$\frac{dx}{2x} = \frac{dy}{-y} = \frac{dz}{f(x^2 y)}$$

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From the first two relations, we have

$$\frac{dx}{x}+\frac{2dy}{y}=0$$

or
$$xy^2=c \text{ or } x=\frac{c}{y^2} \tag{12.20}$$

From last two relations, we have

$$\frac{dy}{-y}=\frac{dz}{f\left(\frac{c^2}{y^4}y\right)}=\frac{dz}{f\left(\frac{c^2}{y^3}\right)}$$

or
$$dz=-\frac{1}{y}f\left(\frac{c^2}{y^3}\right)dy$$

or
$$dz=\left(\frac{y^4}{3y}\right)f\left(\frac{c^2}{y^3}\right)d\left(\frac{1}{y^3}\right)=0$$

since
$$\left[d\left(\frac{1}{y^3}\right)=-\frac{3dy}{y^4}\right]$$

or
$$dz=\left(\frac{1}{3}\right)\left(\frac{1}{c^4y^2}\right)f\left(\frac{c^2}{y^3}\right)d\left(\frac{c^2}{y^3}\right)=0$$

Integrating
$$z=F_1\left(\frac{c^2}{y^3}\right)+\text{ constant}$$

$$=F_1(x^2y)+F_2(xy^2)$$

$$\text{as } c=xy^2$$

is the complete solution.

Example 12.4 Solve $y^2r-2ys+t=p+6y$.

Solution
$$dp=rdx+sdy$$

and
$$dq=sdx+tdy$$

so that
$$r=\frac{dp-sdx}{dx}$$

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and $$t = \frac{dq - sdx}{dy}$$

Putting these values of r and t in the given equation, the Monge's subsidiary equations are

$$y^2 dpdy + dqdx - (p + 6y)dxdy = 0 \tag{12.21}$$

and $$y^2 dy^2 + 2ydydx + dx^2 = 0. \tag{12.22}$$

(12.22) gives $$(ydy + dx)^2 = 0$$

or $$ydy + dx = 0. \tag{12.23}$$

Combining (12.23) and (12.21), we get

$$ydp - dq + (p + 6y)dy = 0$$

or $$(ydp + pdy) - dq + 6ydy = 0.$$

Integrating $$py - q + 3y^2 = A.$$

Also integrating (12.23), we get

$$y^2 + 2x = B.$$

Thus the intermediate integral is

$$py - q + 3y^2 = f(y^2 + 2x). \tag{12.24}$$

Other intermediate integral cannot be found; hence we proceed to solve with the help of Lagrage's method.

(12.24) can be written as

$$py - q = f(y^2 + 2x) - 3y^2.$$

Hence Lagrange's subsidiary equations are

$$\frac{dx}{y} = \frac{dy}{-1} = \frac{dz}{f(y^2 + 2x) - 3y^2}$$

From second and third relations, we have

$$dz + [f(y^2 + 2x) - 3y^2]dy = 0$$

or $$dz + [f(B) - 3y^2]dy = 0,$$

as from two relations, $\frac{dx}{y} = \frac{dy}{-1}$, $y^2 + 2x = B$

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or $$z + yf(B) - y^2 = C = \phi(B).$$

Hence the complete solution is

$$z = y^3 - yf(y^2 + 2x) + \phi(y^2 + 2x).$$

Example 12.5 Solve the equation $x^2r + 2xys + y^2t = 0$.

Solution We have $dp = rdx + rdy$, $dq = sdx + tdy$

which give $$r = \frac{dp - sdy}{dx}$$

and $$t = \frac{dp - sdx}{dy}$$

Putting these values of r and t in the given equation, we have

$$x^2\left(\frac{dp - sdy}{dx}\right) + 2xys + y^2\left(\frac{dq - sdx}{dy}\right) = 0$$

or $$x^2dpdy + y^2dqdx - s(x^2dy^2 - 2xydxdy + y^2dx^2) = 0.$$

Thus Monge's subsidiary equations are

$$x^2dpdy + y^2dqdx = 0 \tag{12.25}$$

and $$x^2dy^2 - 2xydxdy + y^2dx^2 = 0 \tag{12.26}$$

(12.26) gives

$$(xdy - ydx)^2 = 0$$

i. e. $$xdy - ydx = 0 \text{ or } xdy = ydx \tag{12.27}$$

Applying (12.27) in (12.25) we get

$$xydxdp + y^2dqdx = 0$$

or $$xdp + ydq = 0$$

Adding $pdx + qdy$ on both sides we get

$$xdp + pdx + ydq + qdy = pdx + qdy$$

or $$xdp + pdx + ydq + qdy = dz.$$

Integrating $$px + qy = z + B.$$

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Also integrating (12.27), we get

$$\frac{y}{x} + A.$$

Thus the intermediate integral is

$$px + qy = z + f(A).$$

Hence by Lagrange's method, the subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z + f(A)}.$$

The first two relations give $\frac{y}{x} = A$

and last two relations give, $\log cy = \log[z + f(A)]$

or $$z + f(A) = Cy.$$

Hence the complete solution is

$$z = yf_1\left(\frac{y}{x}\right) + f_2\left(\frac{y}{x}\right).$$

Example 12.6 Solve $(1+q)^2 r - 2(1+p+q+pq)s + (1+p)^2 t = 0.$

Solution We have $dp = rdx + sdy$ and $dp = sdx + tdy$ so that $r = \frac{dp - sdy}{dx}$ and $t = \frac{dq - sdx}{dy}$.

Putting these values of r and t in the given equation, we get

$$(1+q)^2 \frac{dp - sdy}{dx} - 2(1+p+q+pq)s + (1+p)^2 \frac{dq - sdx}{dy} = 0$$

The Monge's subsidiary equations are

$$(1+q)^2 dpdy + (1+p)^2 dq\ dx = 0 \qquad (12.28)$$

and $$[(1+a)dy + (1+p)dx]^2 = 0 \qquad (12.29)$$

equation (12.29) gives only one relation

$$(1+q)dy + (1+p)dx = 0 \qquad (12.30)$$

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We shall get only one intermediate integral. To get it we combine (12.30) with (12.28) and we have

$$(1+q)dp-(1+p)dq=0 \text{ or, } \frac{dp}{1+p}-\frac{dq}{1+q}=0$$

Integration gives, $$\frac{1+p}{1+q}=A.$$

We may write (12.30) as

$$dx+dy+(p\,dx+q\,dy)=0$$

i.e. $$dx+dy+dz=0$$

This gives $$x+y+z=C$$

Therefore the intermediate integral is

$$(1+p)=(1+q)f(x+y+z)$$

or $$p-q\,f(x+y+z)=[f(x+y+z)-1]$$

This is of the form $P_p+Q_q=R$, and the Lagrange's subsidiary equations for this are

$$\frac{dx}{1}=\frac{dy}{-f(x+y+z)}=\frac{dz}{f(x+y+z)-1}=\frac{dx+dy+dz}{0}$$

One integral of it is $x+y+z=B$

Again first two relations give

$$\frac{dx}{1}+\frac{dy}{-f(B)} \text{ or } y+xf(B)=\text{cont.}=\phi(B), \text{ say}$$

Thus the complete integral is

$$y+xf(x+y+z)=\phi(x+y+z).$$

Example 12.7 Solve $pq=x(ps-qr)$.

Solution Given $xqr-xps+0.t=-pq$ (12.31)

As usual Monge's subsidiary equations are

$$xqdpdy+pqdxdy=0 \quad (12.32)$$

and $$xqdy^2+xpdxdy=0 \quad (12.33)$$

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which is of the Lagrange's form. Hence the Lagrange's subsidiary equations are

$$\frac{dx}{1}=\frac{dy}{-1}=\frac{dz}{(x-y)f(xy)}=\frac{f(xy)[ydx-xdy]+dz}{0}.$$

From the first two relations, we get

$$x+y=C.$$

And from the last relation, we get

$$dz+f(xy)d(xy)=0$$

or $$z=F_1(xy)+\text{const.}$$

Hence the complete solution is

$$z=F_1(xy)+F_2(x+y).$$

Test Your Memory

Solve the following differential Equations by Monge's method:

1. $r-t\cos^2 x+p\tan x=0.$
2. $z(qs-pt)=pq^2.$
3. $(r-s)y+(s-t)x+q-p=0.$
4. $y^2r+2xys+x^2t+px+qy=0.$
5. $(q+1)s=(p+1)t.$
6. $z(qs-pt)=pq^2.$
7. $t-r\sec^4 y=2q\tan y.$
8. Obtain the integral of $q^2r-2pqs+p^2f=0$ in the form $y+xf(x)=f(z)$. Show also that this solution represents a surface generated by straight lines that are parallel to a fixed plane.

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Chapter-10

1. $pq = 4xy.$

2. $px + qy = pq.$

3. $z(px + qy) = z^2 - 1.$

4. $(x - y)^2(p^2 + q^2 + 1) = 25(p - q)^2.$

5. $ps - qr = 0.$

6. $px^2 + qy = 2y^2.$

7. $(ny - mz)p + (lz - nx)q = mx - ly.$

8. $x(y - z)p + y(z - x)q = z(x - y).$

9. $yp - xq = 0.$

10. $bp - aq = 2abz.$

11. $\phi(lx + my + nz,\ x^2 + y^2 + z^2) = 0.$

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12. $\phi\left(\frac{y}{z}, x^2+y^2+z^2\right)=0.$ 13. $\phi\left(\frac{1}{y}-\frac{1}{x}, \frac{xy}{z}\right)=0.$

14. $\phi(x^2+y^2+z^2,\ xyz)=0.$ 15. $\phi\left(\frac{\sin x}{\sin y}, \frac{\sin x}{\sin z}\right)=0.$

16. $\phi(x^2+y^2,\ xy-z)=0.$ 17. $\phi(xyz,\ x^2+y^2+z^2)=0.$

18. $\phi(xy-z^2,\ x^2-y-z)=0.$ 19. $\phi(xyz,\ x^n+y^n+z^n)=0.$

20. $\phi(xyz,\ xy+yz+zx)=0.$ 21. $2xyz+x^2+y^2-2z+2=0.$

22. $z=a(x-y)-(\cos x+\cos y).$

23. $\sqrt{(1+a)z}=\sqrt{ax}+\sqrt{y}+b.$

24. $\log z+\frac{1}{2}u^2+\frac{1}{2}u\sqrt{u^2-1}+\frac{1}{2}\log\left[u+\sqrt{(u^2-1)}\right]=c$ where

$$u=\frac{t}{z}=\frac{ax+y}{z\sqrt{a}}.$$

25. $yz=ax+2\sqrt{ay}+c.$ 26. $z^2-a^2y^2=(ax+b)^2.$

27. $2\sqrt{c_1}z^{\frac{1}{2}}+c_2=c_1x+y.$ 28. $z^2=2(a^2+1)x^2+2ay+2b.$

29. $z^2=2ax+a^2y^2+b.$ 30. $z=(x+a)^2e^{by}.$

31. $z=ax+by+\sqrt{1+a^2+b^2}.$ 32. $z=ax^by^{\frac{1}{b}}.$

33. $2\sqrt{[c(z-a)-1]}=x+cy+d.$ 34. $z=bx^ny^{\frac{1}{a}}.$

Chapter-11

1. $z=\phi_1(y-3x)+x\phi_2(y-3x).$

2. $z=\phi_1(2y-3x)+x\phi_2(2y-3x).$

3. $z=\phi_1(y)+x\phi_2(y)+\phi_3(x+2y)+\frac{x^5y}{20}+\frac{x^6}{60}.$

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4. $z=\phi_1(y-x)+x\phi_2(y-x)+\frac{1}{4}\left(x^4-2x^3y+2x^2y^2\right)$

5. $z=\phi_1(y-2x)+\phi_2(y-3x)+\frac{1}{2}e^{x-y}$.

6. $z=\phi_1(2y+3x)+\phi_2(y-x)+xe^x-y$.

7. $z=\phi_1(y+2x)+x\phi_2(y+2x)+\phi_3(y-3x)+\frac{1}{10}x^2e^{2x+y}$.

8. $z=\phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)-\frac{1}{75}\cos(x+2y)$.

9. $z=\phi_1(y+2x)+\phi_2(y+x)-\frac{1}{3}\cos(x+2y)$.

10. $z=\phi_1(y)+\phi_2(y+2x)+x\phi_3(y+2x)-\frac{1}{4}x^2\cos(2x+y)$.

11. $z=\phi_1(y-x)+x\phi_2(y-x)+\phi_3(y+x)+\frac{1}{25}e^x(\cos 2y+2\sin 2y)$.

12. $z=\phi_1(y+2x)+\phi_2(y+3x)+\frac{1}{12}\left(xy+\frac{17}{12}x-\frac{7}{12}y+\frac{89}{72}\right)$.

13. $z=\phi_1(2y+x)+x\phi_2(2y+x)+\frac{1}{8}x^2\log(x+2y)$.

14. $z=\phi_1(y+2x)+\phi_2(y-4x)-\frac{1}{210}(2x+3y)^{\frac{5}{2}}$.

15. $e^{-\frac{3x}{4}}\phi_1(4y+2x)+\phi_2(3y-5x)+e^{4x}\phi_3(y)+e^{-7y}\phi_4(x)$.

16. $z=e^{-x}\phi_1(y)+e^x\phi_2(y-x)-\frac{1}{10}[\cos(x+2y)+2\sin(x+2y)]$.

17. $z=\phi_1(y)+x^2\phi_2\left(\frac{y}{x^2}\right)-\frac{1}{9}\frac{x^3}{y^2}$.

18. $z=\phi_1(y-x)+e^{-2x}\phi_2(y+2x)+\frac{x^2y}{4}-\frac{xy}{4}$

$$+\frac{3x^2}{8}-\frac{x}{2}-\frac{x^3}{12}-\frac{1}{6}\cos(2x+y)$$

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19. $z = e^{-\frac{3x}{2}}\phi_1(y+2x) + e^{2x}\phi_2(y+2x)$

$$-\frac{1}{6}\cos(2x+y)\left(xy^2 + \frac{2xy}{3} - \frac{y^2}{6} + \frac{26x}{9} - \frac{26y}{9} - \frac{25}{9}\right)$$

20. $z = e^x\phi_1(y-x) + e^{3x}\phi_2(y-x) + \phi_3(y-x)$

$$-\frac{1}{10}e^{x+y+2}[\sin(2x-y) + 2\cos(2x-y)]$$

21. $z = \phi_1(y) + \phi_2(y+x) + e^x\phi_3(y-x) + \log x.$

22. $z = \sum c_i e^{a_i x + b_i y} - e^{x+y}\cos(x+y),\ 3a_i^2 - 2b_i^2 + a_i - 1 = 0.$

23. $z = e^x\phi_1(y) + e^{-x}\phi_2(x+y) + \frac{1}{2}\sin(x+2y) - xe^y.$

Chapter - 12

1. $z = f_1(y+\sin x) + f_2(y-\sin x)$

2. $y = f_1(z) + zf_2(x)$

3. $z = f_1(x+y) + f_2(x^2 - y^2)$

4. $z = f_1(x^2+y^2)\log(y+x) + f_2(x^2-y^2)$

5. $z = f_1(x+y+z) + f_2(x)$

6. $y = z[f_1(z) + f_2(z)]$

7. $z = F(x - \tan y) + G(x + \tan y)$

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