INTEGRAL CALCULUS

Indefinite integrals

Integration is the inverse process of differentiation. This chapter is devoted to indefinite integrals which is the second major problem of calculus.

Antiderivatives

Definition: A function F is called an *antiderivative* of a function f on a given interval if $F'(x) = f(x)$ for all values of x in that interval. The functions $(1/3)x^3$, $(1/3)x^3 + \sqrt{5}$, $(1/3)x^3 + C$, etc. are antiderivatives of $f(x) = x^2$ on the interval $(-\infty, +\infty)$.

The process of computing antiderivatives is called *antidifferentiation* or *integration*. If $\frac{d}{dx}F(x) = f(x)$ then the functions of the form $F(x) + c$ are the antiderivatives of $f(x)$. It is denoted by $\int f(x)dx = F(x) + c$, where the symbol $\int$ is called the *integral sign* and $f(x)$ is the *integrand*. The constant c is called the *integrating constant*. Since the right side of this relation is not a definite function the term '*indefinite*' is used.

Properties

Let $F(x)$ and $G(x)$ be two antiderivatives of $f(x)$ and $g(x)$, respectively and let c be a constant, real or complex. Then the following properties hold:

(i) $\int cf(x)dx = cF(x) + C$

(ii) $\int [f(x) + g(x)]\, dx = F(x) + G(x) + C$

(iii) $\int [f(x) - g(x)]\, dx = F(x) - G(x) + C$

Problems

For necessary formulas see page 324, Table 5.2.1 of the textbook Calculus by Howard Anton, Irl Bivens and Stephen Davis (10th Edition)

1. Evaluate $\int \frac{t^2 - 2t^4}{t^4} dt$

Solution: We can write $\int \frac{t^2 - 2t^4}{t^4} dt = \int \left(\frac{t^2}{t^4} - \frac{2t^4}{t^4} \right) dt = \int (t^{-2} - 2)\, dt = -\frac{1}{t} - 2t + C$

2. Evaluate (i) $\int \frac{1 - 2t^3}{t^3} dt$, (ii) $\int \left(\frac{1}{2t} - \sqrt{2}e^t \right) dt$

3. Evaluate $\int \frac{x^2}{x^2 + 1} dx$

Solution: We can write $\int \frac{(x^2 + 1) - 1}{x^2 + 1} dx = \int 1\, dx - \int \frac{1}{x^2 + 1} dx = x - \tan^{-1} x + C$

4. Evaluate $\int [\text{cosec}^2 t - \sec t \tan t]\, dt$

Solution: We can write $\int [\text{cosec}^2 t - \sec t \tan t]\, dt = -\cot t - \sec t + C$

5. Evaluate $\int \text{cosec}\, x(\sin x + \cot x)\, dx$

Solution: We can write $\int \text{cosec}\, x(\sin x + \cot x)\, dx = \int (1 + \cos ecx \cot x)\, dx = x - \cos ecx + C$

6. Evaluate $\int \frac{\sec x + \cos x}{2\cos x} dx$

Solution: We can write $\int \frac{\sec x + \cos x}{2\cos x} dx = \frac{1}{2}\int (\sec^2 x + 1)\, dx = \frac{1}{2}(\tan x + x) + \text{C}$

7. Evaluate $\int \left[\frac{4}{x\sqrt{x^2-1}} + \frac{1+x+x^3}{1+x^2} \right] dx$

Solution: We can write

$$\int \left[\frac{4}{x\sqrt{x^2-1}} + \frac{1+x+x^3}{1+x^2} \right] dx = 4\sec^{-1} x + \int \frac{1}{1+x^2} dx + \int \frac{x(1+x^2)}{1+x^2} dx$$

$$= 4\sec^{-1} x + \tan^{-1} x + \frac{1}{2}x^2 + C$$

Integration by Parts and Standard Integrals

1. Evaluate $\int \ln x\, dx$

Solution: We can write $\int \ln x\, dx = \ln x \int 1 dx - \int \frac{1}{x} x\, dx = x \ln x - x + c$

2. Evaluate $\int x^2 \tan^{-1} x\, dx$

Solution: We can write

$$\int x^2 \tan^{-1} x\, dx = \tan^{-1} x \int x^2\, dx - \int \frac{1}{1+x^2} \frac{x^3}{3} dx$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3}\int \frac{x^3}{1+x^2} dx = \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int \frac{x(x^2+1) - x}{1+x^2} dx + c$$

$$= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{3} \times \frac{x^2}{2} + \frac{1}{3}\int \frac{x}{1+x^2} dx + c$$

$$= \frac{1}{3}x^3 \tan^{-1} x - \frac{x^2}{6} + \frac{1}{6}\ln(1+x^2) + c$$

3. $\int \frac{e^{m\tan^{-1} x}}{(1+x^2)^2} dx$

Solution. Let $\tan^{-1} x = z \Rightarrow \frac{1}{1+x^2} dx = dz$ and $x = \tan z$

So $\int \frac{e^{m\tan^{-1} x}}{(1+x^2)^2} dx = \int \frac{e^{mz}}{1+\tan^2 z} dz = \int e^{mz} \cos^2 z\, dz = \frac{1}{2}\int e^{mz}(1 + \cos 2z)\, dz$

$$= \frac{1}{2}\int e^{mz}\, dz + \frac{1}{2}\int e^{mz} \cos 2z\, dz$$

$$= \frac{1}{2m} e^{mz} + \frac{1}{2} \times \frac{e^{mz}}{m^2+4}\left[m\cos 2z + 2\sin 2z\right] + c$$

$$= \frac{1}{2m} e^{m\tan^{-1}x} + \frac{e^{m\tan^{-1}x}}{2(m^2+4)}\left[m\cos(2\tan^{-1}x) + 2\sin(2\tan^{-1}x)\right] + c$$

4. Evaluate $\int \sqrt{1+x^2}\, x^5\, dx$

Solution: We can write $\int \sqrt{1+x^2}\, x^5\, dx = \int \sqrt{1+x^2}\, x^4 . x\, dx$

Let $1+x^2 = u.$ Then $2x\, dx = du$ giving $x\, dx = (1/2)du$

Therefore, $\int \sqrt{1+x^2}\, x^5\, dx = \frac{1}{2}\int \sqrt{u}\,(u-1)^2\, du = \frac{1}{2}\int \sqrt{u}\,(\mathrm{u}^2 - 2u + 1)\, du$

$= \frac{1}{2}\int (\mathrm{u}^{5/2} - 2u^{3/2} + u^{1/2})\, du$

5. Evaluate $\int \frac{dx}{(1-x)\sqrt{1+x}} dx$

Solution: Substitute $1 + x = u^2 \Rightarrow dx = 2udu$

So that we can write $\int \frac{dx}{(1-x)\sqrt{1+x}} dx = \int \frac{2u}{(1-u^2+1)u} du = 2\int \frac{1}{2-u^2} du$

$= 2\int \frac{1}{(\sqrt{2})^2 - (u)^2} du = 2(1/2\sqrt{2}) \log \frac{\sqrt{2}+u}{\sqrt{2}-u} + c = (1/\sqrt{2}) \log \frac{\sqrt{2}+\sqrt{1+x}}{\sqrt{2}-\sqrt{1+x}} + c$

6. Evaluate $\int \frac{x^2+1}{x^4+x^2+1} dx$

Solution: We can write $\int \frac{x^2+1}{x^4+x^2+1} dx = \int \frac{x^2\left(1+\frac{1}{x^2}\right)}{x^2\left(x^2+1+\frac{1}{x^2}\right)} dx = \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+3} dx$

Let $x - \frac{1}{x} = z \Rightarrow \left(1+\frac{1}{x^2}\right) dx = dz$

Therefore, we get

$$\int \frac{x^2+1}{x^4+x^2+1} dx = \int \frac{dz}{z^2+3} = \frac{1}{\sqrt{3}} \tan^{-1} \frac{z}{\sqrt{3}} + c = \frac{1}{\sqrt{3}} \tan^{-1} \frac{x-\frac{1}{x}}{\sqrt{3}} + c$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \frac{x^2-1}{\sqrt{3}\, x} + c$$

7. Evaluate $\int \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx$

Solution: Let $\sin^{-1} x = z \Rightarrow \frac{1}{\sqrt{1-x^2}} dx = dz$ and $x = \sin z$.

We can write $\int \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx = \int \frac{\sin^{-1} x}{(1-x^2)^{1/2}(1-x^2)} dx = \int \frac{z}{1-\sin^2 z} dz$

$= \int z \sec^2 z\, dz = z \tan z - \int \tan z\, dz + c = z \tan z + \ln(\cos z) + c$

$= \sin^{-1} x \tan(\sin^{-1} x) + \ln \cos(\sin^{-1} x) + c$

$= \frac{x}{\sqrt{1-x^2}} \sin^{-1} x + \ln\sqrt{1-x^2} + c$

8. $\int \frac{1}{1+\sin x+\cos x} dx$

Solution: Let $I = \int \frac{1}{1+\sin x+\cos x} dx.$

Dividing both numerator and denominator by $\cos x,$ we get $I = \int \frac{\sec x}{\sec x+\tan x+1} dx.$

Let $\sec x + \tan x + 1 = z.$ Then $(\sec x \tan x + \sec^2 x)\, dx = dz$ and so we get

$\sec x(\sec x + \tan x)\, dx = dz \Rightarrow \sec x\, dx = \frac{dz}{z^2 - 1}$

Therefore, we get

$I = \int \frac{dz}{z(z-1)} = \int \left(\frac{1}{z-1} - \frac{1}{z} \right) dz = \ln|z-1| - \ln|z| + C$

$= \ln\left|\frac{z-1}{z}\right| + C = \ln\left|\frac{\sec x + \tan x}{\sec x + \tan x + 1}\right| + C.$

Alternative method

We can write $1+\sin x+\cos x = 1 + \frac{2\tan(x/2)}{1+\tan^2(x/2)} + \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} = 2 + 2\tan(x/2)$

$I = \int \frac{1}{1+\sin x+\cos x} dx = \int \frac{1+\tan^2(x/2)}{2+2\tan(x/2)} dx = \frac{1}{2}\int \frac{\sec^2(x/2)}{1+\tan(x/2)} dx$

Let $1+\tan\frac{x}{2} = z \Rightarrow \frac{1}{2}\sec^2\frac{x}{2} dx = dz$

Therefore, $I = \int \frac{1}{z} dz = \ln|z| + C = \ln|1+\tan(x/2)| + C$

9. $\int \frac{1+\sin x}{\sin x(1+\cos x)} dx$

Solution: Given $\int \frac{1+\sin x}{\sin x(1+\cos x)} dx = \int \left(\frac{1}{\sin x(1+\cos x)} + \frac{\sin x}{\sin x(1+\cos x)} \right) dx$

$= \int \left(\frac{1}{\sin x(1+\cos x)} + \frac{1}{1+\cos x} \right) dx = I_1 + I_2,$ say

Now $I_1 = \int \frac{1}{\sin x(1+\cos x)} dx$

We can write $\sin x(1+\cos x) = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \left(1 + \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} \right)$

$$=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\left(\frac{1+\tan^2\frac{x}{2}+1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\left(\frac{2}{1+\tan^2\frac{x}{2}}\right)=\frac{4\tan\frac{x}{2}}{(1+\tan^2\frac{x}{2})(1+\tan^2\frac{x}{2})}$$

Let $\tan\frac{x}{2}=u \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx=du$

So $\quad I_1=\int\frac{1}{\sin x(1+\cos x)}\,dx=\int\frac{(1+\tan^2\frac{x}{2})\sec^2\frac{x}{2}}{4\tan\frac{x}{2}}dx$

$$=2\int\frac{(1+u^2)}{4u}du=\frac{1}{2}\int\frac{1}{u}du+\frac{1}{2}\int u\,du=\frac{1}{2}\ln|u|+\frac{u^2}{4}=\frac{1}{2}\ln\left|\tan\frac{x}{2}\right|+\frac{1}{4}\tan^2\frac{x}{2}$$

$I_2=\int\frac{1}{1+\cos x}dx=\frac{1}{2}\int\sec^2\frac{x}{2}dx=\tan\frac{x}{2}$

Hence, $I=I_1+I_2$.

10. Evaluate $\int\frac{1}{\cos 3x-\cos x}dx$

Solution: Let $I=\int\frac{x^2}{(x\sin x+\cos x)^2}dx=\int x\sec x\frac{x\cos x}{(x\sin x+\cos x)^2}dx$

$$=x\sec x\int\frac{x\cos x}{(x\sin x+\cos x)^2}dx-\int\left[\frac{d}{dx}(x\sec x)\int\frac{x\cos x}{(x\sin x+\cos x)^2}dx\right]dx$$

$=(x\sec x)I_1-\int(\sec x+x\sec x\tan x)I_1\,dx$ where $I_1=\int\frac{x\cos x}{(x\sin x+\cos x)^2}dx$

Let $x\sin x+\cos x=z \Rightarrow (x\cos x+\sin x-\sin x)dx=dz\Rightarrow x\cos x\,dx=dz$

Also, $\sec x+x\sec x\tan x=\frac{1}{\cos x}+x\frac{1}{\cos x}\frac{\sin x}{\cos x}=\frac{\cos x+x\sin x}{\cos^2 x}$

Therefore, we get $I_1=\int\frac{1}{z^2}dz=-\frac{1}{z}=-\frac{1}{x\sin x+\cos x}$

Hence, $I=-\frac{x\sec x}{x\sin x+\cos x}+\int\sec^2 x\,dx=-\frac{x\sec x}{x\sin x+\cos x}+\tan x+C$

Alternative method

We can write $\int\frac{1}{\cos 3x-\cos x}dx=\int\frac{1}{2\sin 2x\sin x}dx=\frac{1}{4}\int\frac{1}{\sin^2 x\cos x}dx$

$$=\frac{1}{4}\int\frac{\sin^2 x+cos^2x}{\sin^2 x\cos x}dx=\frac{1}{4}\int(\sec x\,dx+\mathrm{cs}\,cx\cot x)\,dx$$

$$=\frac{1}{4}\ln|\sec x+\tan x|-\frac{1}{4}\mathrm{cs}\,cx+C=\frac{1}{4}\ln\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|-\frac{1}{4}\mathrm{cs}\,cx+C$$