Note on Condensed matter

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Contents

1 Drude model and Sommerfeld model

A classical model, neglecting electron-electron interactions and most quantum effects.

- Free carriers (e.g. electrons) and "collision" rate $1/\tau$
- Local equilibrium after the "collision"
  - Drude: Maxwell distribution
  - Sommerfeld: Fermi-Dirac distribution

The classical equation of motion gives DC conductivity and Hall effect:

$$m\dot{\boldsymbol{v}} = 0 = q\boldsymbol{E} - \frac{m\boldsymbol{v}}{\tau} \qquad \Leftrightarrow \qquad \boldsymbol{j} = qn\boldsymbol{v} = \frac{nq^2\tau}{m}\boldsymbol{E} = \sigma\boldsymbol{E} \tag{1.1}$$

$$m\boldsymbol{v} = 0 = q(\boldsymbol{E} + \times\boldsymbol{B}) - \frac{m\boldsymbol{v}}{\tau} \qquad \Leftrightarrow \qquad \boldsymbol{E} = \frac{m}{nq^2\tau}\boldsymbol{j} - \frac{\boldsymbol{j}\times\boldsymbol{B}}{nq} \tag{1.2}$$

AC conductivity:

$$-\mathrm{i}\omega m\boldsymbol{v} = q\boldsymbol{E} - \frac{m\boldsymbol{v}}{\tau} \qquad \Leftrightarrow \qquad \boldsymbol{j} = \frac{nq^2\tau}{m}\frac{1}{1-\mathrm{i}\omega\tau}\boldsymbol{E} \tag{1.3}$$

gives complex conductivity, with in-phase dissipated part (real part) and out-of-phase inductive part (imaginary part)

1.1 Plasma frequency

AC field produce net surface charge and result in a frequency dependent permittivity $E_{\text{in}} = E_{\text{out}}/\epsilon(\omega)$, with

$$\epsilon(\omega) = 1 + \frac{\mathrm{i}\sigma}{\omega\epsilon_0} = 1 - \frac{nq^2\tau^2}{m\epsilon_0(1+\omega^2\tau^2)} + \mathrm{i}\frac{nq^2\tau}{m\omega\epsilon(1+\omega^2\tau^2)} \tag{1.4}$$

In the limit $\omega\tau \gg 1$,

$$\epsilon(\omega) \approx 1 - \frac{nq^2}{m\epsilon_0 + \omega^2} \equiv 1 - \frac{\omega_p^2}{\omega^2} \tag{1.5}$$

with the **plasma frequency** $\omega_p = \sqrt{nq^2/m\epsilon_0}$. In good metal plasma frequency is in the ultraviolet. Here the metal becomes more transparent instead of absorbing or reflecting E-M radiation.

1.2 Heat current carried by charge carriers

Average energy carried by the charge carrier gives the heat conductivity:

$$\boldsymbol{j}_Q = -n\langle v_x^2\rangle\tau\frac{\mathrm{d}\varepsilon}{\mathrm{d}T}\nabla T \tag{1.6}$$

with v_x meaning velocity along direction of ∇T

	Drude model	Sommerfeld model
$\frac{1}{2}\langle v_x^2\rangle$	$\frac{1}{2}k_BT$	$\frac{1}{3}E_F$
$\mathrm{d}\varepsilon/\mathrm{d}T$	$\frac{3}{2}k_B$	$\frac{\pi^2}{2}k_B^2T/E_F$
product	$\frac{3}{4}k_B^2T$	$\frac{\pi^2}{6}k_B^2T$

2 Structure of crystal

Gas and liquid fully respect the translational, reflectional and rotational symmetry of space in their *time-averaged structure*. Crystal breaks both rotational and translational symmetries of space from continuous to discrete, and has *Long-range order*.

2.1 Classification

- 1-D crystals
  - 1 Bravais lattices
  - 2 crystal symmetries: with(out) inversion symmetry
- 2-D crystals
  - 5 Bravais lattices: minimum symmetry, rectangular, rhombic, square, hexagonal.
- 3-D crystals:
  - 14 Bravais lattices: simple cubic, body centered cubic (BCC), face centered cubic (FCC)...

2.2 Observation (scattering)

Real-space pictures: electron microscope, STM or X-ray microscope, etc. But most widely-used is X-ray scattering (can also scatter neutron, electron, atom, ion, ...).

To scatter from a cell at $\boldsymbol{R}$, the scattering wave

$$\psi_s(\boldsymbol{r}) \sim \frac{f}{|\boldsymbol{r}-\boldsymbol{R}|}\sum_{\boldsymbol{R}} \exp\left[\mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{R} + \boldsymbol{k}_f\cdot(\boldsymbol{r}-\boldsymbol{R})\right] \tag{2.1}$$

$$\approx \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}_f\cdot\boldsymbol{r}}}{|\boldsymbol{r}|}\sum_{\boldsymbol{R}} f\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{R}} = \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}_f\cdot\boldsymbol{r}}}{|\boldsymbol{r}|}\int \mathrm{d}\boldsymbol{R}\,\rho\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{R}} \tag{2.2}$$

with $\boldsymbol{q} = \boldsymbol{k}_i - \boldsymbol{k}_f$ the momentum transfer. So the amplitude

$$I = |\psi_s|^2 \sim \frac{1}{r^2}\int \mathrm{d}\boldsymbol{R}\,\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{R}}\int \mathrm{d}\boldsymbol{R}_2\,\rho(\boldsymbol{R}_2)\rho(\boldsymbol{R}_2+\boldsymbol{R}) \tag{2.3}$$

is proportional to the Fourier transform of density-density correlation.

- In liquid, there's small peak showing typical distance between atoms
- In solid, δ functions in $\boldsymbol{q} = \boldsymbol{G}$ ("Bragg peaks") with diffuse scattering from defects.
  $\boldsymbol{G}$: reciprocal lattice
  - Finite crystals: $N = L\times L\times L$
    $I\sim N^2$, $|\Delta q|\sim 1/L$, "volume" of the Bragg peak $\sim N$
  - Diffuse scattering: $\rho(\boldsymbol{r}) = \rho_0(\boldsymbol{r}) + \delta\rho(\boldsymbol{r},t)$
    $I\sim\sqrt{N}$

3 non-interacting Electrons and bands

First approximation: *non-interacting electrons* in a perfect crystal; treats electron-electron and electron-ion interactions only on average ("mean-field")

- Discrete translational symmetry ⇒ Bloch form gives "crystal momentum" $\boldsymbol{q}$ in the first Brillouin zone (FBZ).
- Nearly free electron model
$$\psi_{n,\boldsymbol{q}}(\boldsymbol{r}) = \sum_{\boldsymbol{G}} \mathrm{e}^{\mathrm{i}(\boldsymbol{G}+\boldsymbol{q})\cdot\boldsymbol{r}}\hat{u}_{n,\boldsymbol{q}}(\boldsymbol{G}) \tag{3.1}$$
- Tight binding model
$$H = \sum_{\boldsymbol{R},n}\epsilon^{(0)}_{\boldsymbol{R},n}\hat{c}^\dagger_{\boldsymbol{R},n}\hat{c}_{\boldsymbol{R},n} + \sum_{(\boldsymbol{R},n)\neq(\boldsymbol{R}',n')} t_{\boldsymbol{R},\boldsymbol{R}',n,n'}\hat{c}^\dagger_{\boldsymbol{R}',n'}\hat{c}_{\boldsymbol{R},n} \tag{3.2}$$
- Band insulator: E_F in a band gap.
  - For large gap, electron-electron/phonon interactions does not change much qualitatively
- Semiconductors
  "Dope" a band insulator: to put some electrons in the conduction band(s) and/or some holes in the valence band(s):
  - "Intrinsic" Semiconductors: doped by thermal excitations, especially in narrow-gap semiconductors
  - Optically: Excite carriers by photons
  - Gating: Thin sample, put large enough voltage and draw in carriers electrostatically

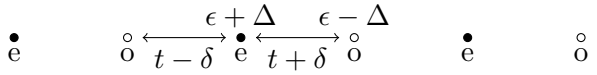
  - Chemically: Put in other element from different columns of periodic table
    * "n-type", negative charge, carriers in conduction band
    * "p-type", positive charge, holes in valence band

- Metals: In the non-interacting electron approximation, electron per unit cell is not even or bans has overlap near E_F
  Electron-electron interactions may open gaps at low T:
  - Superconductivity
  - Charge-density wave (CDW)
  - Spin-density wave (SDW)
  - Mott insulator

Topology of FBZ

- 1D, a circle
- 2D, a torus
- 3D, a "3-torus"

3.1 Single-particle Bloch states



For given momentum q, this gives a 2×2 Hamiltonian, written in terms of $H_q = \epsilon + \boldsymbol{h}_q \cdot \boldsymbol{\sigma}$

Topology of a band: the topological properties of the continuous mapping FBZ $\mapsto$ "Bloch sphere"

- 1D, circle $\mapsto$ circle, no topological special cases, but add symmetry

  : $\Delta = 0$, $\boldsymbol{h}_q$ on $x-y$ plane, there's topological distinct "winding number"
  * $|t+\delta| > |t-\delta|$: winding number $= 0$
  * $|t+\delta| < |t-\delta|$: winding number $= \pm 1$
  * Topological transition at $\delta = 0$: gap closes at $q = \pi/2a$ (edge of FBZ), where $\boldsymbol{h}_q = 0$

3.2 Edge states

Topological insulators often have edge states with energies in the band gap.

For example 1D Kitaev model, the Majorana fermions, arose in superconductors, are forced to be there by particle-hole symmetry.

3.3 Direct measurement of bands

ARPES: Angle-Resolved Photo-Emission Spectroscopy

Scatter electrons by photons from a clean, flat surface. With known $\boldsymbol{q}_{\text{in}}, \epsilon_{\text{in}}, \epsilon_{\text{out}}, \boldsymbol{q}_{\text{out}}$, and the scattering conservs $\boldsymbol{q}_{\parallel}$ to surface, we can learn $\epsilon_n(\boldsymbol{q})$ of occupied bands. Best for quasi-2D materials (like hight-T_c superconductors) and surface states (like topological insulators).

To get unoccupied bands: Inverse photo-emission (under development).

4 Density of states (DOS) of electrons in a crystal

- Number of quantum states = one per "volume" $h^d = (2\pi\hbar)^d$ in $\boldsymbol{r}$-$\boldsymbol{p}$ space = one per $(2\pi)^d$ in $\boldsymbol{r}$-$\boldsymbol{k}$ space

$$g(\epsilon) = \sum_{n\text{bands}} \frac{1}{(2\pi)^d} \int_{\text{FBZ}} \mathrm{d}\boldsymbol{k}\, \delta(\epsilon - \epsilon_n(\boldsymbol{k})) \tag{4.1}$$

4.1 van Hore singularities

- At the edge of a band, $\mathrm{d}\epsilon_n(\boldsymbol{k})/\mathrm{d}\boldsymbol{k} = \boldsymbol{0}$

$$|\epsilon(\boldsymbol{k}) - \epsilon_m| \sim k^2 \tag{4.2}$$
$$k \sim \sqrt{|\epsilon - \epsilon_m|} \tag{4.3}$$
$$\text{“volume” or No. of states} \sim |\epsilon - \epsilon_m|^{d/2} \tag{4.4}$$
$$\text{DOS} \sim |\epsilon - \epsilon_m|^{(d-2)/2} \tag{4.5}$$

  - Quasi-1D(2D): much stronger hopping along one(two) direction(s)

- At saddle points ?TBD?
  - 3D: $(g - g_s) \sim \sqrt{\epsilon - \epsilon_s}$
  - 2D: $g \sim |\log|\epsilon - \epsilon_s||$

$g(E_F)$ is important for the low-T properties of metals.

4.2 Pauli para-magnetism

Spin energy $E = \mu_B \boldsymbol{H} \cdot \boldsymbol{\sigma}$, which gives different Fermi surface for different spins, and result in polarization (to the 1st order):

$$M \approx g(E_F)\mu_B^2 H; \qquad \chi \equiv \frac{\mathrm{d}M}{\mathrm{d}H} = g(E_F)\mu_B^2 \tag{4.6}$$

Interactions between the spins can make big changes: magnetism and magnetic ordering, or gaps opening suppressing this

4.3 Low T electronic specific heat of metal

$$u(T) = \int \mathrm{d}\epsilon\, \epsilon g(\epsilon) f\left(\frac{\epsilon - \mu(T)}{k_B T}\right) \tag{4.7}$$
$$\approx u(T=0) + \frac{\pi^2}{6}(k_B T)^2 g(E_F) + \mathcal{O}\left(\frac{k_B T}{E_F}\right)^3 \tag{4.8}$$
$$c_{\text{el}} = \frac{\mathrm{d}u}{\mathrm{d}T} = \frac{\pi^2}{3} k_B^2 T g(E_F) \tag{4.9}$$

Other main contribution to specific heat of a crystal: phonons. $c_p \sim T^2$ in 3D. Other effects due to defects can also give a linear-in-T specific heat, even in insulators.

5 Semi-classical dynamics of electrons in bands

Note the group velocity and dynamics of momentum:

$$\boldsymbol{v}(\boldsymbol{q}_0) = \frac{1}{\hbar}\nabla_{\boldsymbol{q}}\epsilon_n|_{\boldsymbol{q}_0}; \qquad F = \hbar\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} \tag{5.1}$$

Semi-classical dynamics:

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{v} = \frac{1}{\hbar}\nabla_{\boldsymbol{q}}\epsilon_n(\boldsymbol{q}) \tag{5.2}$$
$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = -\frac{e}{\hbar}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{5.3}$$

For electrons at the top of a band $\epsilon_n = \hbar q^2/2m$ with $m < 0$, $-e/m = e/|m|$ is the same as positive mass and positive charge (holes).

5.1 Bloch oscillations

Electron in band + uniform $\boldsymbol{E}$ field. $\boldsymbol{q}$ steadily "increases" but just winds around FBZ, with oscillations of velocity and position.

- The phenomenon can be observed for atoms in optical lattice
- For electrons in crystals it's difficult: need very clean + low T + weak coupling to phonons

5.2 Landau levels in a band

The magnetic field does no work $\hbar\dot{\boldsymbol{q}} = e\boldsymbol{B}\times\boldsymbol{v}$, so the particle follows "contour" of constant ϵ_n (e.g. follows Fermi surface).

- "Open orbits" do not close, so are not quantized in Landau levels, and do not contribute to the Hall voltage.
- "closed orbits" are quantized: $\Delta\epsilon = \hbar\omega_c$. The period

$$T = \frac{2\pi}{\omega_c} = \oint \frac{|\mathrm{d}\boldsymbol{q}|}{|\dot{\boldsymbol{q}}|} = \frac{\hbar}{eB}\oint \frac{|\mathrm{d}\boldsymbol{q}|}{|\boldsymbol{v}_\perp|} = \frac{\hbar^2}{eB}\oint |\mathrm{d}\boldsymbol{q}| \left|\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}\epsilon}\right| = \frac{\hbar^2}{eB}\frac{\mathrm{d}A_q}{\mathrm{d}\epsilon} \tag{5.4}$$

  which gives quantization of "area" in $\boldsymbol{q}$ space and **flux quanta** from $\Delta\epsilon = \hbar\omega_c$

$$\Delta A_q = 2\pi\frac{eB}{\hbar} = \frac{B}{\Phi_0} \quad \text{with} \quad \Phi_0 = \frac{h}{e} \tag{5.5}$$

  - Motion $\parallel \boldsymbol{B}$ remains "free" and gives 1D-like van Hove singularity.
  - Number of states per Landau level: $D = \Phi/\Phi_0$
  - Number of occupied Landau levels: $\nu = \Phi_0 A_q/B$

5.2.1 Shubnikov-de Hass van Alphen oscillations

We get peaks DOS at E_F periodically in $1/B$. Can be seen in specific heat, resistivity and most qualities.

- Low T: $f(\epsilon)$ is sharp compared to Landau-level spacing
- Clean material: To complete orbits before scattering

If there are more than one pieces of Fermi surface, get sum of contributions each with its period in $1/B$: One way to measuring Fermi surface shape.

6 Lattice vibrations: phonons

Nuclei are heavy: at low T in most crystals, a classical, harmonic approximation is a good first approximation (except hydrogen, helium).

For crystals with N atoms per unit cell in d dimension, there are dN modes at each $\boldsymbol{q}$

- Acoustic mode: longitudinal sound with $\hat{\epsilon}$ closer to $\parallel \boldsymbol{q}$; transverse sound with $\hat{epsilon}$ closer to $\perp \boldsymbol{q}$. *Usually* $c_l > c_t$, crystal is "stiffer" to compression than to shear.
- Optic mode: with more than 1 atom per unit cell, has $q = 0$ vibrations. In ionic photon's $\boldsymbol{E}$ couples strongly to optic phonons, because of large electric dipole moment. See optic phonons as inelastic light scattering.

6.1 Heat capacity due to phonons

- Classical regime $k_BT \gg \hbar\omega_n(\boldsymbol{q})$, the heat capacity

$$C_{\text{phonon}} = 3k_BN_{\text{nuclei}} + \text{anharmonic terms} \tag{6.1}$$

  Interpreted as $3k_B$ per atom or k_B per oscillator mode.

- At low T in 3 dimension

$$C_{\text{phonon}} \sim k_BN\left(\frac{T}{\theta_D}\right)^3 \tag{6.2}$$

  Only acoustic phonons with $\hbar c|\boldsymbol{q}| \lesssim k_BT$ contribute.

Typically heat capacity of a metal $C \sim aT + bT^3$ with 1st term the electron kinetic energy and 2nd term phonons.

6.2 Structural instability of a crystal

- A sound speed $\to 0$: unit cell disorts (usually to lower symmetry at lower T)
- A phonon mode's $\omega(\boldsymbol{q}) \to 0$ at some $\boldsymbol{q} \neq 0$
  - At $\boldsymbol{q} = \boldsymbol{G}/n$: crystal distorts to have a larger unit cell
  - At irrational fraction of $\boldsymbol{G}$: "incommensurate crystal", no unit celll any more

These structural transitions can all be ciewed as a type of Bose condensate, but total distortion is limited by anharmonicities.

6.2.1 Peierls instability

A 1D equally spaced chain with one electron per ion is unstable. Distort phonon $k = 2k_F$:

1. Coupling between $\epsilon(k)$ and $\epsilon(k - 2k_F)$ gives energy shift

$$\Delta\epsilon \sim -\frac{A^2}{|\epsilon_0(k - 2k_F) - \epsilon(k)|} \sim -\frac{A^2}{\big||k| - |k_F|\big|\hbar v_F} \tag{6.3}$$

2. Energy shift over all states in FBZ (cut off $\Omega \sim A$

$$\Delta E \sim -A^2\left|\log\left(\frac{Aa}{\hbar v_F}\right)\right| \tag{6.4}$$

3. Elestic energy $\sim A^2$. The band energy shift dominants, therefore there's a minimum at $A > 0$.

This illustrates the much more general phenomenon that Fermi surfaces in metals can be unstable to anything that opens a gap (superconducting being another example).

Treating the phonons fully *quantum* means this instability may **not happen** when it is very weak.

7 Electron-electron interactions

7.1 Hubbard model

To capture short-range part of electron-electron interactions within one band. The "simplest" version: a one-band tight-binding model with on-site electron-electron repulsion:

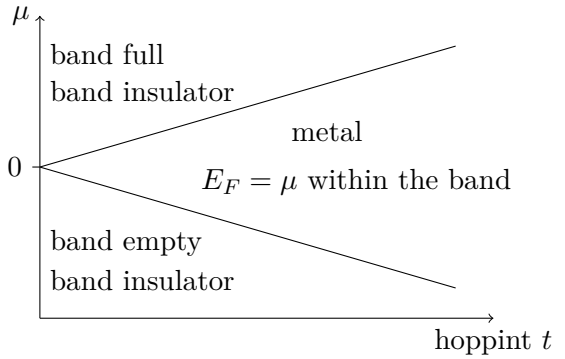
$$H = U\sum_i \hat{n}_{i\downarrow}\hat{n}_{i\uparrow} - t\sum_{\langle i,j\rangle,\sigma}\left(\hat{c}^\dagger_{i,\sigma}\hat{c}_{j,\sigma} + \hat{c}^\dagger_{j,\sigma}\hat{c}_{i,\sigma}\right) \tag{7.1}$$

Hubbard model is usually appropriate for compounds with one band crossing teh Fermi energy.

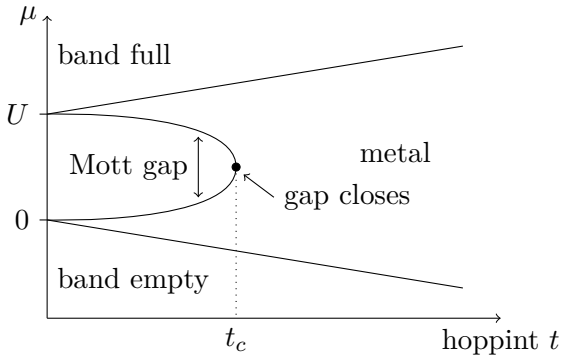
7.1.1 Mott insulator

For $U \gg t$ in Hubbard model, ground state can have one electron per site so band is only half-full. For $t = 0$ (atomic limit) ground state for N sites is 2^N-fold degenerate ($\uparrow$ and $\downarrow$ spins).

In grand canonical ensemble, ground state of $H - \mu N$ is



$U = 0$



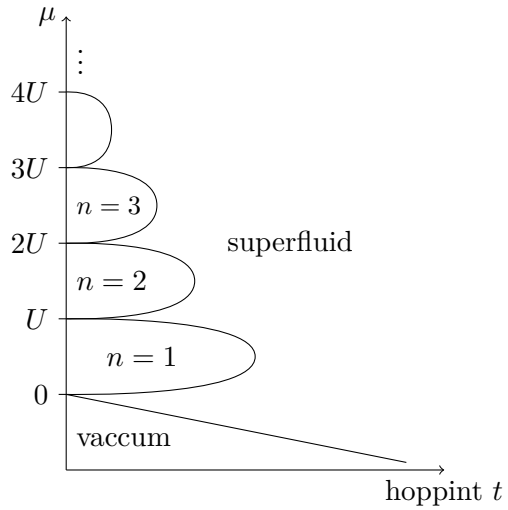
$U > 0$

- Mott insulator has E_F in "Mott gap", and the band is half-filled
- For $t < t_c$, there's upper(lower) Hubbard band
- Many materials are Mott insulators due to strong electron-electron interactions
- Mott insulator has a "charge gap" but not a "spin gap"

Mott insulators and the Hubbard model are also realized with cold neutral atoms in optical lattices. The neutral atoms don't have Coulomb interaction, only "contact" interaction. Atoms can be either fermions or bosons, and may have many spin (hyperfine) state.

Bose-Hubbard model allows any numver of bosons on site:

$$H = \frac{U}{2} \sum_i n_i(n_i - 1) - t \sum_{\langle i,j \rangle} \left(\hat{c}_i^\dagger \hat{c}_j + \hat{c}_j^\dagger \hat{c}_i \right) \tag{7.2}$$



7.1.2 Magnetism

Spin interactions in a Hubbard model (Eq. 7.1). Simplest case: 2 electrons on two sites, 6 two-particle states:

- total spin zero: $|2,0\rangle$, $|0,2\rangle$, $(|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle)/\sqrt{2}$
- total spin one: $|\uparrow,\uparrow\rangle$, $|\downarrow,\downarrow\rangle$ $(|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle)/\sqrt{2}$

The ground state:

- $U = 0$:
  - single-particle $(|\sigma, 0\rangle + |0, \sigma\rangle)/\sqrt{2}$ with $\epsilon = -t$
  - Two-particle $(|2, 0\rangle + |0, 2\rangle + |\uparrow, \downarrow\rangle - |\downarrow, \uparrow\rangle)/2$ with $\epsilon = -2t$

  Pauli exclusion produces anti-ferromagnetic spin corelations, which does not rely on interactions
- $t = 0$: 4-fold degenerate ground state of $E = 0$, the states $|\uparrow, \downarrow\rangle$, $|\downarrow, \uparrow\rangle$, $|\uparrow, \uparrow\rangle$, $|\downarrow, \downarrow\rangle$.
- General U, t, total spin $S = 1$ gives $E = 0$, $S = 0$ states:

$$E = \frac{U}{2} \pm \sqrt{\left(\frac{U}{2}\right)^2 + 4t^2} \tag{7.3}$$

Ferromagnetism requires:

- Degenerate and partially filled orbitals (p, d, f... orbitals)
- Coulomb interaction: anti-symmetry space wave-function

From three-site Hubbard model:

- $U = 0$ single-particle eigenstates:
  - $l = 0$, $(|A\rangle + |B\rangle + |C\rangle)/\sqrt{3}$, $\epsilon = 2t$
  - $l = 1$, $(|A\rangle + w\,|B\rangle + w^*\,|C\rangle)/\sqrt{3}$, $\epsilon = -t$ [1]
  - $l = -1$, $(|A\rangle + w^*\,|B\rangle + w\,|C\rangle)/\sqrt{3}$, $\epsilon = -t$
- Weak U, total S, S_z, l can be good quantum number for two electron. For $l = \pm 1$ single-particle states:
  - Total $S = 0$ states all have doubly-occupied sites, total $l = \pm 2$ or 0, 3 states, higher energy for $U > 0$;
  - Total $S = 1$ has no doubly-occupied sites, total $l = 0$, 3 states

  Total $S = 1$ ground state for weak U
- Strong U, starting from single occupied with one $\uparrow$, the other $\downarrow$, and add hopping. Ground state $E = -2t$ is still total $S = 1$

7.2 Spin models

A spin $\boldsymbol{S}_i$ on each atom:

$$H_s = \sum_{\langle i,j \rangle} \boldsymbol{S}_i \overleftrightarrow{J}_{ij} \boldsymbol{S}_j + \sum_i \boldsymbol{h}_i \cdot \boldsymbol{S}_i \tag{7.4}$$

[1] $w = \exp(2\mathrm{i}\pi/3)$

7.2.1 Heisenberg model

For spin-1/2, isotropic interactions:

$$H = \sum_{\langle i,j \rangle} J_{ij} \boldsymbol{S}_i \cdot \boldsymbol{S}_j \tag{7.5}$$

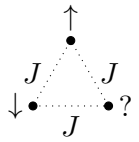
- If $J_{ij} < 0$, ground state is ferromagnetic. Ground state $\sum S = \sum S_z = N/2$
  - "Spin waves": Excited states with one spin flipped:

$$\boldsymbol{S}_i \cdot \boldsymbol{S}_j = S_{iz} S_{jz} + \frac{1}{2} \left(S_{i+} S_{j-} + S_{i-} S_{j+} \right) \tag{7.6}$$

    spin-flip is a "hard-core" boson. The wave $\omega(q) \sim q^2$ for small q

  Most ferromagnets J comes from Coulomb interaction + Pauli exclusion. Weaker but long-range ($\sim 1/t^3$) magnetic dipole interaction contributes at long length scale: **"shape anisotropy"**

- Anti-ferromagnets: spins anti-align in ground state.
  - The Simplest case: two sub-lattices $\uparrow, \downarrow$, doubles the unit cell and a new Bragg peak appear (*magnetic Bragg peak*.
  - Spin-waves: $\omega(q) \sim q$ (Anderson, 1952)
  - Order parameter: $\boldsymbol{S}_A - \boldsymbol{S}_B$
  - **Frustrated anti-ferromangets**: can't minimize energy of all interactions. The Simplest example: 3 spins on a trangle:



    is minimized by minimal total spin: $H = J/2(\sum \boldsymbol{S}_i)^2 - J/2(\sum \boldsymbol{S}_i^2)$.
    * Kagome lattice: For spin configuration, there's no long range order, no anti-ferromangets — **spin liquid**.

8 Superconductivity and superfluidity

8.1 Phenomena

- Zero Ohmic resistivity: super currents that flow without dissipation
- Meissner effect: magnetic fields are either expelled from the material, or confined to flux lines on vortices
- Josephson effect
- .....

8.2 Pairing and ground state

In some superconductors, such as high-T_c, the mechanism by which electrons attract is still not known; in many superconductors it is known that electron-phonon interactions produce the attraction.

In the absence of a Fermi sea, two quantum particles with a *weak* attraction do always form a bound state in 1D and 2D, but not in 3D or higher; but in the presence of a filled Fermi seam 2 fermions do bind.

$$H = \sum_{\boldsymbol{q}} \epsilon_{\boldsymbol{q}} (\hat{n}_{\boldsymbol{q},\uparrow} + \hat{n}_{\boldsymbol{q},\downarrow}) \tag{8.1}$$

The interaction term:

$$U = \sum_{\boldsymbol{R}} \sum_{\boldsymbol{r}} U(\boldsymbol{r}) \hat{c}^\dagger_{\boldsymbol{R},\uparrow} \hat{c}_{\boldsymbol{R},\uparrow} \hat{c}^\dagger_{\boldsymbol{R}+\boldsymbol{r},\downarrow} \hat{c}_{\boldsymbol{R}+\boldsymbol{r},\downarrow} \tag{8.2}$$

$$= \frac{1}{2N} \sum_{\boldsymbol{q}_1+\boldsymbol{q}_2=\boldsymbol{q}_3+\boldsymbol{q}_4} \tilde{U}(\boldsymbol{q}_2-\boldsymbol{q}_3) \hat{c}^\dagger_{\boldsymbol{q}_1,\uparrow} \hat{c}^\dagger_{\boldsymbol{q}_2,\downarrow} \hat{c}_{\boldsymbol{q}_3,\downarrow} \hat{c}_{\boldsymbol{q}_4,\uparrow} \tag{8.3}$$

The Cooper problem with $\tilde{U}(\boldsymbol{q}) = U < 0$ is constant, which corresponds to $U(\boldsymbol{r}) = U\delta(\boldsymbol{r})$

8.2.1 Single particle pairing

For coupling only above Fermi sea, the pairing of the form:

$$|\psi\rangle = \sum_{\boldsymbol{q}} a_{\boldsymbol{q}} c^\dagger_{\boldsymbol{q},\uparrow} c^\dagger_{-\boldsymbol{q},\downarrow} |\text{Fermi sea}\rangle \tag{8.4}$$

has binding energy $E \approx 2E_F - 2(\Omega - E_F)\exp[2/Ug(E_F)]$. Note that the result is non-purtubative in U.

In general, weak attraction and filled Fermi sea give binding of Cooper pairs. If $U(\boldsymbol{q})$ is not always negative, the lowest energy Cooper pair may be in some other "pairing channel" such as triplet, p-wave; or singlet, d-wave (high-T_c superconductors pair this way), so Cooper pairs are not always s-wave, total spin 0.

8.2.2 Full many-body BCS pairing

$$H - \mu N = \sum_{\boldsymbol{q}\sigma} (\epsilon_{\boldsymbol{q}} - \mu) \hat{c}^\dagger_{\boldsymbol{q},\sigma} \hat{c}_{\boldsymbol{q},\sigma} + \frac{1}{2N} \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{q}} \tilde{U}(\boldsymbol{q}) \hat{c}^\dagger_{\boldsymbol{k}_1,\uparrow} \hat{c}^\dagger_{\boldsymbol{k}_2+\boldsymbol{q},\downarrow} \hat{c}_{\boldsymbol{k}_2,\downarrow} \hat{c}_{\boldsymbol{k}_1+\boldsymbol{q},\uparrow} \tag{8.5}$$

$$= \sum_{\boldsymbol{q}\sigma} (\epsilon_{\boldsymbol{q}} - \mu) \hat{c}^\dagger_{\boldsymbol{q},\sigma} \hat{c}_{\boldsymbol{q},\sigma} + \frac{1}{2N} \sum_{\boldsymbol{k},\boldsymbol{q}} \tilde{U}(\boldsymbol{k}-\boldsymbol{q}) \hat{c}^\dagger_{\boldsymbol{k},\uparrow} \hat{c}^\dagger_{-\boldsymbol{k},\downarrow} \hat{c}_{-q,\downarrow} \hat{c}_{\boldsymbol{q},\uparrow} + \text{"non-paired" terms} \tag{8.6}$$

The BCS (variational) wavefunction is of the form:

$$|\Psi_{\text{BCS}}\rangle = \prod_{\boldsymbol{q}} \left(\sin\theta_{\boldsymbol{q}} + \cos\theta_{\boldsymbol{q}} \mathrm{e}^{\mathrm{i}\phi_{\boldsymbol{q}}} \hat{c}^\dagger_{\boldsymbol{q},\uparrow} \hat{c}^\dagger_{-\boldsymbol{q},\downarrow} \right) |0\rangle \tag{8.7}$$

In this wavefunction $\langle$"non-paired terms"$\rangle = 0$

$$\langle H - \mu N \rangle = \sum_{\boldsymbol{q}} \cos^2\theta_{\boldsymbol{q}} (\epsilon_{\boldsymbol{q}} - \mu) + \frac{1}{2N} \sum_{\boldsymbol{k},\boldsymbol{q}} \tilde{U}(\boldsymbol{k}-\boldsymbol{q}) \mathrm{e}^{\mathrm{i}(\phi_{\boldsymbol{q}} - \phi_{\boldsymbol{k}})} \cos\theta_{\boldsymbol{k}} \sin\theta_{\boldsymbol{k}} \cos\theta_{\boldsymbol{q}} \sin\theta_{\boldsymbol{q}} \tag{8.8}$$

- If $U(\boldsymbol{q}) < 0$, for all $|\boldsymbol{q}| \le 2k_F$ minimum energy has $\phi_{\boldsymbol{q}} = \phi_{\boldsymbol{k}} = $ const.: s-wave paring (most common)
- If $U((\boldsymbol{q}) > 0$, for some $|\boldsymbol{q}| < 2k_F$ minimum energy might have p-wave, d-wave, etc.

For $\tilde{U}(\boldsymbol{q}) = U < 0$, minimization of $\langle H - \mu N \rangle$ gives:

$$\frac{\sin 2\theta_{\boldsymbol{q}}}{\cos 2\theta_{\boldsymbol{q}}} = \frac{\Delta}{\mu - \epsilon} \qquad \Delta = -\frac{U}{2N} \sum_{\boldsymbol{q}} \sin 2\theta_{\boldsymbol{q}} \tag{8.9}$$

- Far below $\epsilon_q = \epsilon_F$, $\theta_q \to 0$, electron present
- Far above $\epsilon_q = \epsilon_F$, $\theta_q \to 0$, electron absent
- At $\epsilon_q = \epsilon_F$, $\theta_q = \pi/4$, equal amplitude

- The occupation number $\langle \hat{n}_q \rangle = \cos^2 \theta_q$

$$\langle \hat{n}_q \rangle = \frac{1}{2}\left(1 + \frac{\mu - \epsilon_q}{\sqrt{\Delta^2 + (\mu - \epsilon_q)^2}}\right) \tag{8.10}$$

  has $\sim \Delta$ width near "Fermi surface" and thus no sharp jump; $\sim \left[\Delta/(\epsilon - \epsilon_F)\right]^2$ as $\epsilon \gg \epsilon_F$, longer tails than Fermi-Dirac function

The "gap" Δ should satisfy the gap equation:

$$1 = \frac{|U|}{2N}\sum_{\boldsymbol{q}} \frac{1}{\sqrt{\Delta^2 + (\mu - \epsilon_{\boldsymbol{q}}^2)}} = |U| \int \mathrm{d}\epsilon \frac{g(\epsilon)}{\sqrt{\Delta^2 - (\epsilon - \epsilon_F)^2}} \tag{8.11}$$

- Weak gap/weak coupling:

$$\Delta \sim \Omega \mathrm{e}^{-1/g(\epsilon_F)|U|} \tag{8.12}$$

  with $\Omega \sim \hbar\omega_{\text{phonon}}$ being the cut off

- Δ and pairing mean field:

$$\Delta(\boldsymbol{q}) \equiv \frac{1}{N}\sum_{\boldsymbol{k}} \tilde{U}(\boldsymbol{k} - \boldsymbol{q}) \langle \hat{c}_{-\boldsymbol{k},\downarrow} \hat{c}_{\boldsymbol{k},\uparrow} \rangle \tag{8.13}$$

  is consistent with the definition before.

8.3 Excitations

In the mean field approximation, for states at $\boldsymbol{q},\uparrow$, $-\boldsymbol{q},\downarrow$:

$$(H - \mu N)_{\boldsymbol{q}} = (\epsilon_{\boldsymbol{q}} - \mu)(\hat{n}_{\boldsymbol{q},\uparrow} + \hat{n}_{-\boldsymbol{q},\downarrow}) - \Delta(\boldsymbol{q}) \hat{c}^\dagger_{\boldsymbol{q},\uparrow} \hat{c}^\dagger_{-\boldsymbol{q},\downarrow} - \Delta^*(\boldsymbol{q}) \hat{c}_{-\boldsymbol{q},\downarrow} \hat{c}_{\boldsymbol{q},\uparrow} \tag{8.14}$$

whose eigenenergy is $E = \epsilon_q - \mu$ (2-fold), $E = (\epsilon_q - \mu) \pm \sqrt{|\Delta(q)|^2 + (\epsilon_q - \mu)^2}$, with ground state we get before

$$(\sin\theta_q + \mathrm{e}^{\mathrm{i}\phi_q} \cos\theta_q \hat{c}^\dagger_{q,\uparrow} \hat{c}^\dagger_{-q,\downarrow}) |0\rangle \tag{8.15}$$

And the self-consistant gap equation

$$\Delta(\boldsymbol{q}) = \mathrm{e}^{\mathrm{i}\phi_{\boldsymbol{q}}} \sin 2\theta_{\boldsymbol{q}} \sqrt{|\Delta(\boldsymbol{q})|^2 + (\epsilon_{\boldsymbol{q}} - \mu)^2} \tag{8.16}$$

Here $|0\rangle$ means BCS ground state at other $\boldsymbol{q}$. The excitation can be described by quasiparticles (**Bogoliubov rotation**)

$$\hat{b}^\dagger_{\boldsymbol{q},\uparrow} = \sin\theta_{\boldsymbol{q}} \hat{c}^\dagger_{\boldsymbol{q},\uparrow} - \mathrm{e}^{-\mathrm{i}\phi_{\boldsymbol{q}}} \cos\theta_{\boldsymbol{q}} \hat{c}_{-\boldsymbol{q},\downarrow} \tag{8.17}$$

$$\hat{b}^\dagger_{-\boldsymbol{q},\downarrow} = \sin\theta_{\boldsymbol{q}} \hat{c}^\dagger_{-\boldsymbol{q},\downarrow} + \mathrm{e}^{-\mathrm{i}\phi_{\boldsymbol{q}}} \cos\theta_{\boldsymbol{q}} \hat{c}_{\boldsymbol{q},\uparrow} \tag{8.18}$$

so that

$$(H - \mu N)_{\boldsymbol{q}} = \sqrt{|\Delta(\boldsymbol{q})|^2 + (\epsilon_{\boldsymbol{q}} - \mu)^2} \left(\hat{b}^\dagger_{\boldsymbol{q},\uparrow} \hat{b}_{\boldsymbol{q},\uparrow} + \hat{b}^\dagger_{-\boldsymbol{q},\downarrow} \hat{b}_{-\boldsymbol{q},\downarrow}\right) + E_0 \tag{8.19}$$

- The quasi-particle energy/excitation gap $E_q = \sqrt{|\Delta(\boldsymbol{q})|^2 + (\epsilon_{\boldsymbol{q}} - \mu)^2} > |\Delta(\boldsymbol{k}_F)|$
- For $\Delta(\boldsymbol{q}) = \Delta = \text{const.}$, at $E = |\Delta|$, DOS of quasi-particles has 1D-like van Hove singularity. There're no states between $E = E_F \pm |\Delta|$ gap vs condensate?
  - This is seen in STM: no tunneling current when $|eV| < |\Delta|$
  - Not the case for p-wave or d-wave: there's node for $\Delta(\boldsymbol{q})$

8.3.1 Finite temperature

For excited states, $\langle c_{-\boldsymbol{q},\downarrow} c_{\boldsymbol{f},\uparrow}\rangle = \Delta/2E_{\boldsymbol{q}}$ (ground state, from Eq. (8.16), with $E_{\boldsymbol{q}}$ the quasi-particle energy) 0 (1 quasi-particle) or $-\Delta/2E_{\boldsymbol{q}}$. So the Boltzmann average of $\langle c_{-\boldsymbol{q},\downarrow} c_{\boldsymbol{f},\uparrow}\rangle$ gives:

$$\Delta = \frac{|U|}{N}\sum_{\boldsymbol{q}} \frac{\mathrm{e}^{E_k/T} - \mathrm{e}^{-E_k/T}}{\mathrm{e}^{E_k/T} + 2 + \mathrm{e}^{E_k/T}} \frac{\Delta}{2E_k} \sim \Delta |U| g(\epsilon_F) \ln\left(\frac{\Omega}{\max(T,\Delta)}\right) \tag{8.20}$$

- For T too high (roughly $T > \Delta(T=0)$), $\Delta = 0$ is the only solution: phase transition T_c. Near T_c, $\Delta \sim \sqrt{T_c - T}$

8.3.2 Total spin of pairs

For pair of $\boldsymbol{q}$ and $-\boldsymbol{q}$,

- If l is even ($\Delta(\boldsymbol{q}) = \Delta(-\boldsymbol{q})$, s-wave and d-wave), $S = 0$
- If l is odd ($\Delta(\boldsymbol{q}) = -\Delta(-\boldsymbol{q})$, p-wave), $S = 1$
  This occurs for superfluid ^{3}He and for electrons in some materials

8.4 Superfluid wavefunction

Phenomenally define the wavefunction $\Psi(\boldsymbol{r},t) \sim \Delta(\boldsymbol{r},t)$ with $\Psi(\boldsymbol{r},t) = |\Psi(\boldsymbol{r},t)|\mathrm{e}^{\mathrm{i}\phi(\boldsymbol{r},t)}$ and number density of bosons $n = |\Psi|^2$

- Number current:

$$\boldsymbol{j}_n = \frac{\hbar}{m^*}\Im\left[\Psi^*\left(\nabla - \mathrm{i}\frac{q}{\hbar}\boldsymbol{A}\right)\Psi\right] = |\Psi|^2\frac{\hbar}{m^*}\left(\nabla\phi - \frac{q}{\hbar}\boldsymbol{A}\right) = n\boldsymbol{v}_s \tag{8.21}$$

- London approximation: $|\Psi| = $ const. for $T \ll T_c$ which leads to **London equations**
- Ginzburg-Landau equations allow variation in $|\Psi|$

8.5 London equations and Meissner effect

From Maxwell equation

$$\nabla \times \boldsymbol{A} = \boldsymbol{B} \qquad \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} \tag{8.22}$$

and equation of current Eq. (8.21), $\boldsymbol{j} = q\boldsymbol{j}_n$, give London equation

$$\nabla \times \boldsymbol{j} = -\frac{nq^2}{m^*}\boldsymbol{B} \quad \text{or} \quad \boldsymbol{j} = -\frac{nq^2}{m^*}\boldsymbol{A} \tag{8.23}$$

The second form requires Coulomb gauge. This promises exponential decay of $\boldsymbol{B}$ into the bulk. The penetration length:

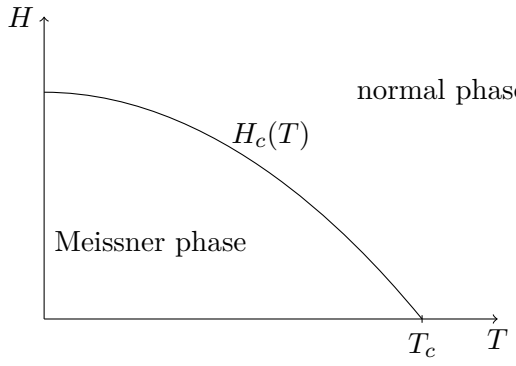
$$\lambda = \sqrt{\frac{m^*}{\mu_0 n q^2}} \tag{8.24}$$

8.5.1 Critical field H_c

The free energy of normal state $f_n(T)$ and superconductor state $f_s(T) + H^2/2\mu$ give the critical field $H_c = \sqrt{2\mu_0\big[f_n(T) - f_s(T)\big]}$: 1st order transition (vanishes linearly at $T \to T_c$)

- Type-I superconductors: the above happens
- Type-II superconductors have intermediate phase with vortices and flux lines

Shape and size effects: for example long slab with normal field.



Phase diagram of type-I superconductor

- For most superconductors, energy of currents and flux expulsion $\gg$ Zeeman energy, so coupling to electron spins can be ignored. Exceptions like thin film or wires in parallel fields.

8.5.2 Flux quantization (Aharonor-Bohm effect)

For theck loop of superconductor ($d \gg \lambda$), around path inside superconductor $\boldsymbol{j} = 0 = \nabla\phi - q\boldsymbol{A}/\hbar$. Wavefunction is single valued, so $\oint \nabla\phi \cdot \mathrm{d}\boldsymbol{l} = 2\pi k$ with $k \in \mathbb{Z}$

$$\frac{q}{\hbar}\oint \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{l} = \frac{q}{\hbar}\Phi = 2\pi k \qquad \Phi = k\Phi_0 = k\frac{h}{q} \tag{8.25}$$

where Φ is magnetic flux through the loop, Φ_0 is the quantum of magnetic flux, and for superconductor $\Phi_0 = h/2e$

8.5.3 Supercurrent in $\boldsymbol{E}$-field

Schrödinger equation of ϕ

$$\hbar\frac{\mathrm{d}\phi(\boldsymbol{r})}{\mathrm{d}t} = -2eV(\boldsymbol{r}) \tag{8.26}$$

Combining with Eq. (8.21) and $\boldsymbol{A} = 0$, $\boldsymbol{E} = -\nabla V$, we get $m\dot{\boldsymbol{v}}_s = 2e\boldsymbol{E}$ is free acceleration of the supercurrent. This happens until v_s gets high enough to produce quasi-particles or vortices: "critical current".

8.5.4 Josephson junction

In a superconductor 1 (with phase ϕ_1) – insulator – superconductor 2 (with phase ϕ_2) junction, the tunneling current

$$\boldsymbol{j} \sim \Im\left[(\mathrm{e}^{-\lambda z - \mathrm{i}\phi_2} + \mathrm{e}^{-\lambda z - \mathrm{i}\phi_1})\nabla(\mathrm{e}^{-\lambda z + \mathrm{i}\phi_2} + \mathrm{e}^{-\lambda z + \mathrm{i}\phi_1})\right] \sim 2\sin(\phi_1 - \phi_2)\hat{z} \tag{8.27}$$

is an unusual, nonlinear circuit element. So the total current (plus small Ohmic conductance)

$$I = I_0\sin(\phi_1 - \phi_2) + GV \approx I_0\sin(\phi_1 - \phi_2) \tag{8.28}$$

For fixed voltage, $\phi_1 - \phi_2 = -2eVt/\hbar$, so $I = I_0\sin(\omega_J t)$ with Josephson frequency $\omega_J = 2eV/\hbar$

8.6 Ginzburg-Landau theory

To go near T_c or near vortices, $|\Psi(\boldsymbol{r})|$ varies. Free energy density is defined:

$$f\big(\Psi(\boldsymbol{r}), \boldsymbol{B}(\boldsymbol{r})\big) = f_n + \alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4 + \frac{\hbar^2}{2m^*}\left|\left(\nabla - \mathrm{i}\frac{q}{\hbar}\boldsymbol{A}\right)\Psi\right|^2 + \frac{1}{2\mu_0}\left|\mu_0\boldsymbol{H} - \nabla\times\boldsymbol{A}\right|^2 \tag{8.29}$$

with f_n the normal state free energy density. $\alpha > 0$ for $T > T_c$. Assume $\beta > 0$.

The following for $T < T_c$:

- At $\boldsymbol{H} = 0$, minimized to constant $|\Psi| = \sqrt{|\alpha|/\beta}$
- In Meissner state, $f_s = f_n - \alpha^2/2\beta + \mu_0 H^2/2$. Transition at $f_s = f_n$ for type-I superconductors:

$$H_c(T) = \frac{|\alpha|}{\sqrt{\mu_0 \beta}} \tag{8.30}$$

8.6.1 Type-II superconductors

The perturbation form $f = f_n + \langle\psi|H|\psi\rangle + \mathcal{O}(|\psi|^4)$ suggests the Hamiltonian

$$H = \alpha + \frac{\hbar^2}{2m^*}\left(\nabla - \frac{q}{\hbar}\boldsymbol{A}\right)^2 = |\alpha|\left(-1 + \xi^2\left(\nabla - \frac{q}{\hbar}\boldsymbol{A}\right)^2\right) \tag{8.31}$$

with the **"coherence length"** ξ defined as:

$$\xi^2 = \frac{\hbar^2}{2m^*|\alpha|} \tag{8.32}$$

The kinetic term $\hbar^2/2m^*(\nabla - q\boldsymbol{A}/\hbar)^2$ suggests Landau level with frequency $\omega_c = qB/m^*$ and ground state energy $\frac{1}{2}\hbar\omega_c = \hbar q\mu_0 H/2m^*$ in which state

$$f_s = f_n + |\Psi|^2\left(-|\alpha| + \frac{\hbar q}{2m^*}\mu_0 H\right) + \mathcal{O}(|\Psi|^2) \tag{8.33}$$

This describes the free energy of BEC occupying the ground Landau level, with critical field H_{c2} defined as:

$$-|\alpha| + \frac{\hbar q}{2m^*}\mu_0 H = 0 \qquad \mu_0 H_{c2} = \frac{2m^*|\alpha|}{\hbar q} = \frac{\Phi_0}{2\pi\xi^2} \tag{8.34}$$

The critical field can be interpreted as flux quantum in an area of coherence length.

Note that the other length scale, the penetration length in Meissner state Eq. (8.24) can be written

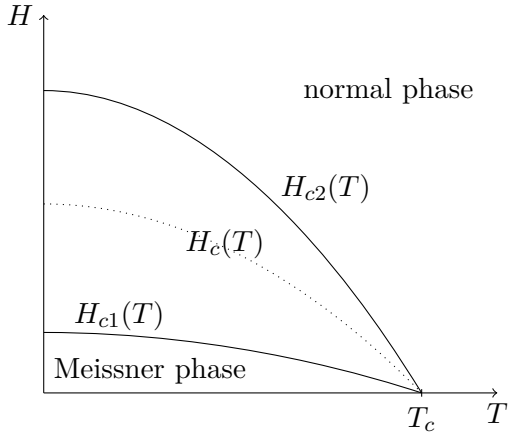
$$\lambda^2 = \frac{m^*\beta}{\mu_0|\alpha|q^2} \tag{8.35}$$

So the Meissner state critical field Eq. (8.30) can be written:

$$\mu_0 H_c = \frac{\Phi_0}{2\sqrt{2}\pi\xi\lambda} \tag{8.36}$$

Define $\kappa \equiv \lambda/\xi$

- Type-II superconductors $H_c < H_{c2}$, $\kappa > 1/\sqrt{2}$



Phase diagram of type-II superconductor

- Between H_{c1} and H_{c2} there's **"Abrikosov vortex lattice"**, partial flux expulsion of the field.
  - one vortex per flux quantum
  - $\Phi_0 \approx 20\,\mathrm{G}\,\mu\mathrm{m}^2$ which means 100 Å at 20 T ($\gg$ atomic spacing)
- H_{c1} is the transition of Meissner phase + 1 vortex line
  - $|\Psi|$ below $\sqrt{|\alpha|/\beta}$ within $r \lesssim \xi$
  - $\boldsymbol{j}$ increase within $r \lesssim \xi$ and decrease exponentially for $r \gg \lambda$

$$\mu_0 H_{c1} \approx \frac{\Phi_0}{4\pi\lambda^2} \ln \kappa \tag{8.37}$$

- Vortex motion $\Leftrightarrow$ voltage: if vortices can move, there's nonzero resistivity / Superconductivity only if vortices are immobile
- High-H_{c2} material: "pin" the vortices using local impurities/defects that are less superconducting.
- Thermal fluctuations of vortices in Abrikosov vortex lattice: melting transition from vortex solid(glass) to vortex liquid (resistive)
  - In vortex liquid $\langle\Psi\rangle = 0$ but $\langle|\Psi|^2\rangle > 0$

9 2D electron systems and Quantum Hall effect

- Some effective mass (tensor, but assumed to be isotropic in the following), can be $m^* \ll m_e$
- Low density, spacing $\gg$ unit cells

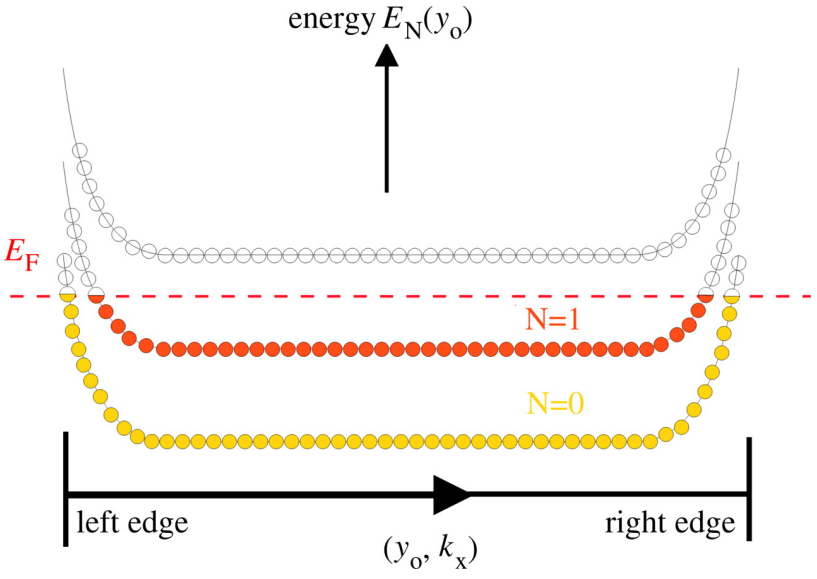
9.1 Integer Quantum Hall effect

Add magnetic field: Landau levels and Zeeman splitting:

$$\epsilon = \epsilon_0 + \hbar\omega_c \left\{ n + \frac{1}{2} \right\} + g\mu_B \boldsymbol{B} \cdot \boldsymbol{\sigma} \tag{9.1}$$

Note that the degeneracy of Landau level is one per flux quantum $j \leq \Phi/\Phi_0$, So the DOS is a sum of delta functions at $\hbar\omega_c(n + 1/2) \pm g\mu_B|B|$, and the filling of states $\nu =$ No. of electrons/No. of flux quanta.

- $\nu \in \mathbb{N}$: Integer quantum Hall effect. Adding the potential at edges, the band diagram:



- No gap with the edges, but states near E_F are all at edges
- Edges of 2D integer quantum Hall system are 1D "chiral metals"
- Gap in bulk (topological insulator), states at edge move only in one direction: $v_y = \mathrm{d}\epsilon/\hbar \mathrm{d}k_y$, $v_y > 0 (< 0)$ at right(left) edge
- scattering from imperfections at edge does *not* affect current, since all states have same direction of motion
- Net current in one Landau level:

$$I_y = \frac{e}{2\pi} \int \mathrm{d}k_y \frac{1}{\hbar} \frac{\partial \epsilon}{\partial k_y} = \frac{e}{h} \Delta\epsilon = \frac{e^2}{h} V_x \tag{9.2}$$

  which is robust to sample dimensions, disorder, weak interactions, but need to be at $T \ll \hbar\omega_c$. This now define the Ohm.

9.2 Wigner crystal of electrons

Add electron-electron interactions, but no $\boldsymbol{B}$. Coulomb potential and kinetic energy compete:

- High density: kinetic energy "wins", it's a Fermi liquid
- Low density: **Wigner crystal**. Seen on electrons on helium surface.
- $\boldsymbol{B} \neq =$: kinetic energy drops out, but Landau level $\hbar\omega_c$, and for low enough filling, **field-induced Wigner crystal**. Seen in 2D semiconductors devices at low filling and high field.

9.3 Fractional quantum Hall effect

At high filling, such as $\nu = 1/5, 1/3, 2/5, 2/3...$, ground state is **fractional quantum Hall effect**.

Under Coulomb gauge, the lowest Landau eigenfunctions are:

$$\Psi_m(z) = N_m z^m \mathrm{e}^{-|z|^2/4} = N_m r^m \mathrm{e}^{\mathrm{i}m\phi} \mathrm{e}^{-|r|^2/4} \tag{9.3}$$

Filled lowest Landau level wave function for n electrons with anti-symmetry is:

$$\Psi^{(1)}(z_1, z_2, \cdots z_n) = N \left[\prod_{i \neq j}^{n} (z_i - z_j) \right] \exp\left(- \sum_i \frac{|z_i|^2}{4} \right) \tag{9.4}$$

which is called **"quantum Hall droplet"**.

Laughlin(1983) gave wavefunction for filling $\nu = 1/m$ with $m = 1, 2, 3, ...$

$$\Psi^{(m)}(\{z_i\}) = N \left[\prod_{i \neq j}^{n} (z_i - z_j)^m \right] \exp\left(- \sum_i \frac{|z_i|^2}{4} \right) \tag{9.5}$$

is exact eigenfunction in limit of short-range repulsion, and is very good approximation for Coulomb.

- For large m, low filling $\nu = 1/m$: Wigner crystal
- For $m = 1, 3, 5, 7$, Laughlin liquid of fractional quantum Hall effect: fractional charged Laughlin quasi-particles.