Exerice

$$\sum_{n\geq 0} \frac{n^2-10n+1}{n!}x^n$$

$$a_n = \frac{n^2-10n+1}{n!}$$

Étudions $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2-10(n+1)+1}{(n+1)!} * \frac{n!}{n^2-10n+1}$

$$= \frac{n^2+2n+1-10n-10+1}{n!(n+1)} * \frac{n!}{n^2-10n+1}$$

$$= \frac{n^2-8n-8}{(n+1)(n^2-10n+1)}$$

$$= \frac{n^2-10n+1}{(n+1)(n^2-10n+1)} + \frac{2n-9}{(n+1)(n^2-10n+1)}$$

$$= \frac{1}{n+1} + \frac{2(n+1)}{(n+1)(n^2-10n+1)} - \frac{11}{(n+1)(n^2-10n+1)}$$

$= \frac{1}{n+1} + \frac{2}{(n^2-10n+1)} - \frac{11}{(n+1)(n^2-10n+1)}$ or

$$= \underbrace{\frac{1}{n+1}}_{\underset{n\to\infty}{\longrightarrow} 0} + \underbrace{\frac{2}{(n^2-10n+1)}}_{\underset{n\to\infty}{\longrightarrow} 0} - \underbrace{\frac{11}{(n+1)(n^2-10n+1)}}_{\underset{n\to\infty}{\longrightarrow} 0}$$

Donc $\frac{a_{n+1}}{a_n} \underset{n\to\infty}{\longrightarrow} 0$

Or d'après le critère d'Alembert on a R $= \frac{1}{0}$ qui par convention donne R $= \infty$

On cherche une fonction tel que $f^{(n)}(0) = n^2-10n+1$

Soit S(x) $= \sum_{n\geq 0} \frac{n^2-10n+1}{n!}x^n$

$n^2-10n+1 = n(n-1)+n-10n+1 = n(n-1)-9n+1$

$$\frac{n^2-10n+1}{n!} = \frac{n(n-1)}{n!} - 9\frac{n}{n!} + \frac{1}{n!}$$

Pour $n \geq 2$

$$= \frac{1}{(n-2)!} - 9\frac{1}{(n-1)!} + \frac{1}{n!}$$

$a_0 = 1$

$a_1 = -8$

$$\sum_{n\geq 0} \frac{n^2-10n+1}{n!}x^n = 1-8x+\sum_{n\geq 2}(\frac{1}{(n-2)!} - 9\frac{1}{(n-1)!} + \frac{1}{n!})x^n$$

$$= 1-8x+\sum_{n\geq 2} x^2\frac{x^{n-2}}{(n-2)!} - 9\sum_{n\geq 2} x\frac{x^{n-1}}{(n-1)!} + \sum_{n\geq 2}\frac{x^n}{n!}$$

$$= 1-8x+x^2\sum_{j\geq 0}\frac{x^j}{j!} - 9x\sum_{n\geq 1} x\frac{x^k}{k!} + \sum_{n\geq 2}\frac{x^n}{n!}$$

$$
\begin{aligned}
&= 1 - 8x + x^2 \sum_{j \geq 0} \frac{x^j}{j!} - 9x(\sum_{n \geq 0} x \frac{x^k}{k!} - 1) + \sum_{n \geq 0} \frac{x^n}{n!} - 1 - x \\
&= 1 - 8x + x^2 e^x - 9x(e^x - 1) + e^x - 1 - x \\
&= (x^2 - 9x + 1)e^x
\end{aligned}
$$