Exerice 1

$M = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & \sqrt{2} & c \\ a & \sqrt{2} & d \\ \sqrt{3} & b & e \end{pmatrix}$

$u_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} \\ a \\ \sqrt{3} \end{pmatrix}, u_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ b \end{pmatrix}, u_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} c \\ d \\ e \end{pmatrix}$

$||u_1||^2 =< u_1|u_1 >= \frac{1}{6}(3 + a^2 + 3) = 1 + \frac{a^2}{6}$

$||u_2||^2 =< u_2|u_2 >= \frac{1}{6}(2 + 2 + b^2) = \frac{2}{3} + \frac{b^2}{6}$

$||u_3||^2 =< u_3|u_3 >= \frac{1}{6}(c^2 + d^2 + e^2)$

Or pour que $M \in O_3(\mathbb{R})$ il faut que $||u_1|| = ||u_2|| = ||u_3|| = 1$

$||u_1|| = 1 \Leftrightarrow ||u_1||^2 = 1 \Leftrightarrow 1 + \frac{a^2}{6} = 1 \Leftrightarrow a = 0$

$||u_2|| = 1 \Leftrightarrow ||u_2||^2 = 1 \Leftrightarrow \frac{2}{3} + \frac{b^2}{6} = 1$

$\Leftrightarrow b^2 = 2 \Leftrightarrow b \pm \sqrt{2}$

$M = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & \sqrt{2} & c \\ 0 & \sqrt{2} & d \\ \sqrt{3} & \pm\sqrt{2} & e \end{pmatrix}$

On a aussi que $< u_1|u_2 >= 0$

$< u_1|u_2 >= \sqrt{3} * \sqrt{2} + 0 * sqrt(2) \pm \sqrt{3} * \sqrt{2}$

$< u_1|u_2 >= \sqrt{6} \pm \sqrt{6}$

Donc $b = -\sqrt{2}$

$M = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & \sqrt{2} & c \\ 0 & \sqrt{2} & d \\ \sqrt{3} & -\sqrt{2} & e \end{pmatrix}$

On a finalement $\begin{cases} ||u_3|| = 1 \\ < u_1|u_3 >= 0 \\ < u_2|u_3 >= 0 \end{cases} \Leftrightarrow \begin{cases} ||u_3||^2 = 1 \\ < u_1|u_3 >= 0 \\ < u_2|u_3 >= 0 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{6}(c^2 + d^2 + e^2) = 1 \\ \sqrt{3} * c + \sqrt{3} * e = 0 \\ \sqrt{2} * c + \sqrt{2} * d - \sqrt{2} * e = 0 \end{cases}$

$$\Leftrightarrow \begin{cases} c^2 + d^2 + e^2 = 6 \\ \sqrt{3} * c = -\sqrt{3} * e \\ \sqrt{2} * c + \sqrt{2} * d - \sqrt{2} * e = 0 \end{cases} \Leftrightarrow \begin{cases} c^2 + d^2 + e^2 = 6 \\ c = -e \\ -\sqrt{2} * e - \sqrt{2} * e = -\sqrt{2} * d \end{cases} \Leftrightarrow \begin{cases} (-e)^2 + d^2 + e^2 = 6 \\ c = -e \\ 2 * e = d \end{cases}$$

$$\Leftrightarrow \begin{cases} (-e)^2 + (2e)^2 + e^2 = 6 \\ c = -e \\ d = 2e \end{cases} \Leftrightarrow \begin{cases} e^2 + 4e^2 + e^2 = 6 \\ c = -e \\ d = 2e \end{cases} \Leftrightarrow \begin{cases} 6e^2 = 6 \\ c = -e \\ d = 2e \end{cases} \Leftrightarrow \begin{cases} e^2 = 1 \\ c = -e \\ d = 2e \end{cases}$$

$$\Leftrightarrow \begin{cases} e = 1 \\ c = -1 \\ d = 2 \end{cases} \text{ou} \begin{cases} e = -1 \\ c = 1 \\ d = -2 \end{cases}$$

Donc $M = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & \sqrt{2} & -1 \\ 0 & \sqrt{2} & 2 \\ \sqrt{3} & -\sqrt{2} & 1 \end{pmatrix}$ ou $M = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & \sqrt{2} & 1 \\ 0 & \sqrt{2} & -2 \\ \sqrt{3} & -\sqrt{2} & -1 \end{pmatrix}$

$\bullet M \in SO_3(\mathbb{R}) \Leftrightarrow det(M) = 1$

<u>$Cas1$</u> :

$det(M) = (\frac{1}{\sqrt{(6)}})^3 * det \begin{pmatrix} \sqrt{3} & \sqrt{2} & -1 \\ 0 & \sqrt{2} & 2 \\ \sqrt{3} & -\sqrt{2} & 1 \end{pmatrix} = (\frac{1}{\sqrt{(6)}})^3 * (\sqrt{3} * det \begin{pmatrix} \sqrt{2} & 2 \\ -\sqrt{2} & 1 \end{pmatrix} + \sqrt{3} * det \begin{pmatrix} \sqrt{2} & -1 \\ \sqrt{2} & 2 \end{pmatrix})$

$= (\frac{1}{\sqrt{(6)}})^3 * (\sqrt{3}(\sqrt{2} * 1 + 2 * \sqrt{2}) + \sqrt{3} * (\sqrt{2} * 2 + \sqrt{2} * 1))$

$= (\frac{1}{\sqrt{(6)}})^3 * (3 * \sqrt{6} + 3 * \sqrt{6})$

$= (\frac{1}{\sqrt{(6)}})^3 * (6 * \sqrt{6}) = 1$

Donc dans ce cas $M \in SO_3(\mathbb{R})$

$\underline{Cas2}$:

$$det(M) = (\frac{1}{\sqrt{(6)}})^3 * det \begin{pmatrix} \sqrt{3} & \sqrt{2} & 1 \\ 0 & \sqrt{2} & -2 \\ \sqrt{3} & -\sqrt{2} & -1 \end{pmatrix} = (\frac{1}{\sqrt{(6)}})^3 * (\sqrt{3} * det \begin{pmatrix} \sqrt{2} & -2 \\ -\sqrt{2} & -1 \end{pmatrix} + \sqrt{3} * det \begin{pmatrix} \sqrt{2} & 1 \\ \sqrt{2} & -2 \end{pmatrix})$$

$= (\frac{1}{\sqrt{(6)}})^3 * (\sqrt{3}(\sqrt{2} * -1 - 2 * \sqrt{2}) + \sqrt{3} * (\sqrt{2} * -2 - \sqrt{2} * 1))$
$= (\frac{1}{\sqrt{(6)}})^3 * (-3 * \sqrt{6} - 3 * \sqrt{6})$
$= (\frac{1}{\sqrt{(6)}})^3 * (-6 * \sqrt{6}) = -1$
Donc dans ce cas $M \notin SO_3(\mathbb{R})$