Exerice 8

$$\frac{x}{(x-1)(x-2)} = \frac{a}{x-1} + \frac{b}{x-2}$$

$$\frac{a(x-2)+b(x-1)}{(x-1)(x-2)} = \frac{ax-2a+bx-b}{(x-1)(x-2)} = \frac{x(a+b)-2a-b}{(x-1)(x-2)}$$

$$\begin{cases} a+b=1 \\ -2a-b=0 \end{cases} \Leftrightarrow \begin{cases} a=1-b \\ -2(1-b)-b=0 \end{cases} \Leftrightarrow \begin{cases} a=-1 \\ b=2 \end{cases}$$

Donc $\dfrac{x}{(x-1)(x-2)} = \dfrac{-1}{x-1} + \dfrac{2}{x-2}$

$$\frac{-1}{x-1} = \frac{1}{1-x} = \sum_{n\geq 0} x^n$$

$$\frac{2}{x-2} = -\frac{1}{1-\frac{x}{2}} = -\sum_{n\geq 0}\left(\frac{x}{2}\right)^n$$

$$\frac{x}{(x-1)(x-2)} = \sum_{n\geq 0} x^n - \sum_{n\geq 0}\left(\frac{x}{2}\right)^n = \sum_{n\geq 0} x^n - \left(\frac{x}{2}\right)^n$$

On a que le rayon de convergence de $\displaystyle\sum_{n\geq 0}\left(\frac{x}{2}\right)^n$ est 2

On a que le rayon de convergence de $\displaystyle\sum_{n\geq 0} x^n$ est 1

Donc le rayon de convergence est $\min(1,2)=1$

$$\frac{1}{1+x+x^2}$$

$$\begin{aligned} 1+x+x^2 &= \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} \\ &= \left(x+\frac{1}{2}\right) - \left(\frac{i\sqrt{3}}{2}\right)^2 \\ &= \left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right) \\ &= \left(x+\frac{1-i\sqrt{3}}{2}\right)\left(x+\frac{1+i\sqrt{3}}{2}\right) \end{aligned}$$

$$\frac{1}{1+x+x^2} = \frac{1}{\left(x+\frac{1-i\sqrt{3}}{2}\right)\left(x+\frac{1+i\sqrt{3}}{2}\right)}$$

$$\frac{1}{1+x+x^2} = \frac{a}{\left(x+\frac{1-i\sqrt{3}}{2}\right)} + \frac{b}{\left(x+\frac{1+i\sqrt{3}}{2}\right)}$$

$$\frac{1}{1+x+x^2} = \frac{\left(x+\frac{1+i\sqrt{3}}{2}\right)a + \left(x+\frac{1-i\sqrt{3}}{2}\right)b}{\left(x+\frac{1-i\sqrt{3}}{2}\right)\left(x+\frac{1+i\sqrt{3}}{2}\right)}$$

$$\frac{1}{1+x+x^2} = \frac{\left(x+\frac{1+i\sqrt{3}}{2}\right)a + \left(x+\frac{1-i\sqrt{3}}{2}\right)b}{1+x+x^2}$$

$$\frac{1}{1+x+x^2} = \frac{ax + a\frac{1+i\sqrt{3}}{2} + bx + b\frac{1-i\sqrt{3}}{2}}{1+x+x^2}$$

$$\begin{cases} a+b=0 \\ a\dfrac{1+i\sqrt{3}}{2} + b\dfrac{1-i\sqrt{3}}{2} = 1 \end{cases} \Leftrightarrow \begin{cases} a=-b \\ -b\dfrac{1+i\sqrt{3}}{2} + b\dfrac{1-i\sqrt{3}}{2} = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} a=-b \\ \dfrac{-b-ib\sqrt{3}}{2} + \dfrac{b-ib\sqrt{3}}{2} = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} a=-b \\ \dfrac{-2ib\sqrt{3}}{2} = 1 \end{cases} \Leftrightarrow \begin{cases} a=-b \\ ib\sqrt{3} = -1 \end{cases}$$

$$\Leftrightarrow \begin{cases} a=-b \\ b = \dfrac{i}{\sqrt{3}} \end{cases} \Leftrightarrow \begin{cases} a = \dfrac{-i\sqrt{3}}{3} \\ b = \dfrac{i\sqrt{3}}{3} \end{cases}$$

$$\frac{1}{1+x+x^2} = -\frac{\frac{i\sqrt{3}}{3}}{\left(x+\frac{1-i\sqrt{3}}{2}\right)} + \frac{\frac{i\sqrt{3}}{3}}{\left(x+\frac{1+i\sqrt{3}}{2}\right)}$$