Section 9.1:

Problem 1. Find the period of the linear congruential generator defined by (a) a = 2, b = 0, m = 5 (b) a = 4,b = 1,m = 9.

Solution:

$$x_i = 2x_{i-1} \quad mod(5)$$
$$u_i = \frac{x_i}{5}$$

X_0	U_0
1	1/5
2	2/5
4	4/5
3	3/5
1	1/5

The Period is 4 because it's repeated.

□

Solution:

$$x_i = 4x_{i-1} + 1 \quad mod(9)$$
$$u_i = \frac{x_i}{9}$$

X_0	U_0
1	1/9
5	5/9
3	3/9
4	4/9
8	8/9
6	6/9
7	7/9
2	2/9
0	0/9

The Period is 9 because it's repeated. □

Problem 2. Find the period of the LCG defined by a = 4,b = 0,m = 9. Does the period depend on the seed?

Solution:

$$x_i = 4x_{i-1} \quad mod(5)$$

$$u_i = \frac{x_i}{5}$$

X_0	U_0
1	1/5
4	4/5
1	1/5
4	4/5

The Period is 2 because it's repeated.

X_0	U_0
2	2/5
3	3/5
2	2/5
3	3/5

The Period is 2 because it's repeated; Therefore, It does not matter what the seed is, it will continue to repeat after 2 iterations. □

Problem 3. Approximate the area under the curve $y = x^2$ for $0 \le x \le 1$, using the LCG with (a) a = 2,b = 0,m = 5 (b) a = 4,b = 1,m = 9.

Solution:
You can gather an approximate answer through

$$\frac{1}{n}\Sigma_{k=1}^{n} F(u_k)$$

$$x_i = 2x_{i-1} \quad mod(5)$$

$$u_i = \frac{x_i}{5}$$

X_0	U_0
1	1/5
2	2/5
4	4/5
3	3/5
1	1/5

$\frac{1}{4} * [\left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2 + \left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{1}{5}\right)^2] = \frac{3}{10}$ $\square$

Solution:

$$x_i = 4x_{i-1} + 1 \quad mod(9)$$

$$u_i = \frac{x_i}{9}$$

X_0	U_0
1	1/9
5	5/9
3	3/9
4	4/9
8	8/9
6	6/9
7	7/9
2	2/9
0	0/9

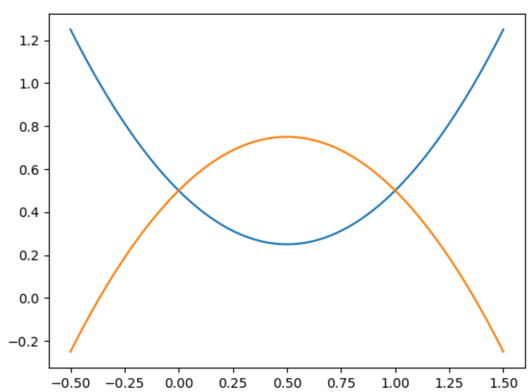
$\frac{1}{9} * [\left(\frac{1}{9}\right)^2 + \left(\frac{5}{9}\right)^2 + \left(\frac{3}{9}\right)^2 + \left(\frac{4}{9}\right)^2 + \left(\frac{8}{9}\right)^2 + \left(\frac{6}{9}\right)^2 + \left(\frac{7}{9}\right)^2 + \left(\frac{2}{9}\right)^2 + \left(\frac{0}{9}\right)^2] = \frac{68}{243} \approx .28$ $\square$

Computer

Problem 3. (a) Using calculus, find the area bounded by the two parabolas P1(x) = x2 - x + 1/2 and P2(x) = -x2 + x + 1/2. (b) Estimate the area as a Type 1 Monte Carlo simulation, by finding the average value of P2(x) - P1(x) on [0,1]. Find estimates for n = 10^i for $2 \leq i \leq 6$. (c) Same as (b), but estimate as a Type 2 Monte Carlo problem: Find the proportion of points in the square [0, 1] × [0, 1] that lie between the parabolas. Compare the efficiency of the two Monte Carlo approaches.

Solution:

$$\int_0^1 P_2(x) - P_1(x)\,dx \;= \int_0^1 -x^2 + x + 0.5 - x^2 + x - 0.5 = -\frac{2}{3}x^3 + 2x \quad = 1/3$$

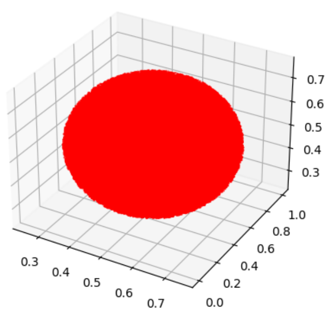


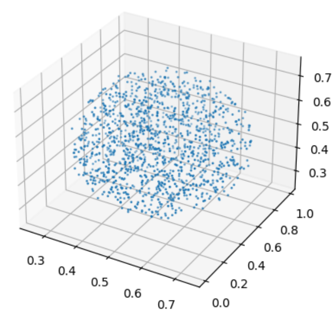
Number	**Monte Carto(type I):**	**Monte Carto(type II):**
2	0.31989997416025223	0.29292929292929293
3	0.33471903062092195	0.33733733733733734
4	0.33469846188664665	0.3473347334733473
5	0.3343791518519886	0.33707337073370736
6	0.33332828373508067	0.33334633334633335

□

Problem 6. Use n = 10^4 pseudo-random points to estimate the interior volume of the ellipsoid defined by $2+4x^2+4z^2+y^2 \leq 4x+4z+y$, contained in the unit cube $0 \leq x, y, z \leq 1$. Compare your estimate with the correct volume $\pi/24$, and report the error. Repeat with n = 10^6 points.

Solution:





Volume of unit sphere:
Expected:0.1308996938995747

Blue(N) = 10,000
MC(type II): 0.1327265453090618
Error: 0.0018268514094871013

Red(N) = 1,000,000
MC(type II): 0.13238026476052953
Error : 0.0014805708609548218

□

Section 9.3:

Computer

Problem 1 & 2. Design a Monte Carlo simulation to estimate the probability of a random walk reaching the top a of the given interval [-b,a]. Carry out n = 10000 random walks. Calculate the error by comparing with the correct answer.
Calculate the mean escape time for the random walks in Computer Problem 1.

(a) [-2,5] (b) [-5,3] (c) [-8,3]

Solution:

For 10,000 trials

(a) Exit time (one trial): 10

$$\begin{aligned} Sample\ Mean(Exit\ Time)\ &:10.04 \\ Variance\ &:90.20642064206284 \\ Standard\ Error\ &:9.497706072629477 \\ \\ Expected\,probability\ &:0.2857142857142857 \\ Actual\ probability\ &:0.2894 \end{aligned}$$

(b) Exit time (one trial): 9

$$\begin{aligned} Sample\ Mean(Exit\ Time)\ &:15.028 \\ Variance\ &:163.11472747274937 \\ Standard\ Error\ &:12.771637619066295 \\ \\ Expected\ probability\ &:0.625 \\ Actual\ probability\ &:0.6307 \end{aligned}$$

(c) Exit time (one trial): 5

$$\begin{aligned} Sample\ Mean(Exit\ Time)\ &:24.1239 \\ Variance\ &:563.6523140213797 \\ Standard\ Err\ &:23.741362935210347 \\ \\ Expected\ probability\ &:0.7272727272727273 \\ Actual.\ probability\ &:0.7273 \end{aligned}$$

□