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Section 2.1:

Problem 6. Assume that your computer completes a 5000 equation back substitution in 0.005 seconds. Use the approximate operation counts n2 for back substitution and 2n3/3 for elimination to estimate how long it will take to do a complete Gaussian elimination of this size. Round your answer to the nearest second.

Solution: Operation Count $= n^2$

$$(5000)t^2 = .005 \rightarrow t = \frac{.005}{5000^2}$$

$$\frac{2(5000)^3 + 3(5000)^2}{3}t \quad simplify$$

$$\frac{2(5000)^3 + 3(5000)^2}{3} * \frac{.005}{5000^2} \quad simplify$$

total time = 16.67s

$\square$

Problem 7. Assume that a given computer requires 0.002 seconds to complete back substitution on a 4000 $\times$ 4000 upper triangular matrix equation. Estimate the time needed to solve a general system of 9000 equations in 9000 unknowns. Round your answer to the nearest second

Solution:

$$(4000)^3 t = 0.002$$

$$\frac{2}{3}\frac{(9000)^3 * (0.002)}{4000^2}$$

total time = 60.75s

$\square$

Section 2.2:

Problem 2a. Find the LU factorization of the given matrices. Check by matrix multiplication.

$$\begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix} \tag{1}$$

Solution:

$$\begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix}$$

$\rightarrow R2 = R2 - 2R1 \rightarrow R3 = R3 - R1 \quad \rightarrow$

$$\begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

□

Problem 4a. Solve the system by finding the LU factorization and then carrying out the two-step back substitution.

$$\begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \tag{2}$$

Solution: Solution =

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

□

Section 2.3:

Problem 1b. Find the norm $||A||_\infty$ of each of the following matrices:

$$\begin{bmatrix} 1 & 5 & 1 \\ -1 & 2 & -3 \\ 1 & -7 & 0 \end{bmatrix} \tag{3}$$

Solution:

$$||A||_\infty = 1 + 5 + 1 = 7 \tag{4}$$

□

Problem 2b.

$$\begin{bmatrix} 1 & 2.01 \\ 3 & 6 \end{bmatrix} \tag{5}$$

Solution:

$$\begin{bmatrix} 1 & 2.01 \\ 3 & 6 \end{bmatrix}$$

$$\frac{1}{ad - bc}$$

$$\begin{bmatrix} 6 & -2.01 \\ -3 & 1 \end{bmatrix}$$

$=$

$$\begin{bmatrix} -600 & 201 \\ 300 & -100 \end{bmatrix}$$

$norm(A) = 9 \; norm(A^{-1}) = 200$
$||A||_\infty * ||A^{-1}||_\infty = 1800$

□

Problem 3e. Find the forward and backward errors, and the error magnification factor (in the infinity norm) for the following approximate solutions xa of the system in Example 2.11 [-2,4.0001]

Solution:

$$\begin{cases} x_1 + x_2 = 2 \\ 1.0001x_1 + x_2 = 2.0001 \end{cases}$$

$FE = x - x_a$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ 4.0001 \end{bmatrix} = \begin{bmatrix} 3 \\ -3.0001 \end{bmatrix} = -3.0001$$

$BE = b - Ax_a$

$$\begin{bmatrix} 2 \\ 2.0001 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1.0001 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 4.0001 \end{bmatrix} = \begin{bmatrix} 0.0001 \\ 0.0002 \end{bmatrix} = 0.0002$$

$$EMF = \frac{\frac{||x - x_a||_\infty}{||x||_\infty}}{\frac{||r||_\infty}{||b||_\infty}}$$

$$EMF = \frac{\frac{3.0001}{1}}{\frac{0.0002}{2.0001}}$$

EMF = 30002.50005

$\square$

Section 2.5:

Problem 1b.

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \tag{6}$$

Solution:

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 2 \\ 0 & -1 & 2 & 0 \end{bmatrix}$$

$$u_{k+1} = \frac{v_k}{2}$$
$$v_{k+1} = \frac{v_k + w_k + 2}{2}$$
$$w_{k+1} = \frac{v_k}{2}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_0, \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_1, \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \end{bmatrix}_2$$

Gauss Seidel equations

$$u_{k+1} = \frac{v_k}{2}$$
$$v_{k+1} = \frac{v_{k+1} + w_k + 2}{2}$$
$$w_{k+1} = \frac{v_{k+1}}{2}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_0, \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix}_1, \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/2 \\ 3/4 \end{bmatrix}_2$$

□

Problem 2b. Rearrange the equations to form a strictly diagonally dominant system. Apply two steps of the Jacobi and Gauss–Seidel Methods from starting vector [0,...,0].

Solution:

$$\begin{cases} u - 8v - 2w = 1 \\ u + v + 5w = 4 \\ 3u - v + w = -2 \end{cases} \quad \textit{Rearrange equations} \quad \begin{cases} 3u - v + w = -2 \\ u - 8v - 2w = 1 \\ u + v + 5w = 4 \end{cases} \tag{7}$$

Jacobi method equations

$$u_{k+1} = \frac{(-2 + v_k - w_k)}{3}$$
$$v_{k+1} = \frac{-(1 - v_k + 2w_k)}{8}$$

$w_{k+1} = \dfrac{(4 - u_k - v_k)}{5}$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_0, \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -39/400 \\ -109/320 \\ 23/24 \end{bmatrix}_2$$

Gauss method equations
$u_{k+1} = \dfrac{(-2 + v_k - w_k)}{3}$
$v_{k+1} = \dfrac{-(1 - v_{k+1} + 2w_k)}{8}$
$w_{k+1} = \dfrac{(4 - u_{k+1} - v_{k+1})}{5}$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_0, \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -49/180 \\ -29/72 \\ 187/200 \end{bmatrix}_2$$

$\square$

Problem 3b. Apply two steps of SOR to the systems in Exercise 1. Use starting vector [0, . . . , 0] and w = 1.5

Solution:
$u_{k+1} = (1 - w)u_k + w\frac{v_k}{2}$
$v_{k+1} = (1 - w)v_k + w\frac{u_{k+1}+w_k+2}{2}$
$w_{k+1} = (1 - w)w_k + w\frac{w_{k+1}}{2}$
With the given $w = 1.5$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_0, \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 9/8 \\ 39/16 \\ 81/64 \end{bmatrix}_2$$

$\square$

Computer

Problem 1. Use the Jacobi Method to solve the sparse system within six correct decimal places (forward error in the infinity norm) for n = 100 and n = 100000. The correct solution is [1,...,1]. Report the number of steps needed and the backward error.

Solution:

```
~/Documents/_School/Depaul/MAT385/code > python3 jacobi.py
Matrix A:
 [[ 3. -1.  0.  0.  0.  0.  0.  0.  0.  0.]
 [-1.  3. -1.  0.  0.  0.  0.  0.  0.  0.]
 [ 0. -1.  3. -1.  0.  0.  0.  0.  0.  0.]
 [ 0.  0. -1.  3. -1.  0.  0.  0.  0.  0.]
 [ 0.  0.  0. -1.  3. -1.  0.  0.  0.  0.]
 [ 0.  0.  0.  0. -1.  3. -1.  0.  0.  0.]
 [ 0.  0.  0.  0.  0. -1.  3. -1.  0.  0.]
 [ 0.  0.  0.  0.  0.  0. -1.  3. -1.  0.]
 [ 0.  0.  0.  0.  0.  0.  0. -1.  3. -1.]
 [ 0.  0.  0.  0.  0.  0.  0.  0. -1.  3.]]
Vector b:
 [2. 1. 1. 1. 1. 1. 1. 1. 1. 2.]
Number of iterations needed: 34
Solution is:
 [[0.99999991]
 [0.99999983]
 [0.99999976]
 [0.99999971]
 [0.99999968]
 [0.99999968]
 [0.99999971]
 [0.99999976]
 [0.99999983]
 [0.99999991]]
Backward error: 3.419073248966953e-07
```

```
~/Documents/_School/Depaul/MAT385/code > python3 jacobi.py
Matrix A:
 [[ 3. -1.  0. ...  0.  0.  0.]
 [-1.  3. -1. ...  0.  0.  0.]
 [ 0. -1.  3. ...  0.  0.  0.]
 ...
 [ 0.  0.  0. ...  3. -1.  0.]
 [ 0.  0.  0. ... -1.  3. -1.]
 [ 0.  0.  0. ...  0. -1.  3.]]
Vector b:
 [2. 1. 1. ... 1. 1. 2.]
Number of iterations needed: 36
Solution is:
 [[0.99999994]
 [0.99999988]
 [0.99999982]
 ...
 [0.99999982]
 [0.99999988]
 [0.99999994]]
Backward error: 4.5785176450152676e-07
```

□