Section 3.2:

Problem 1. A. Find the degree 2 interpolating polynomial P2(x) through the points $(0, 0)$, $(\pi/2, 1)$, and $(\pi, 0)$. B. Calculate $P2(\pi/4)$, an approximation for $sin(\pi/4)$. C. Use Theorem 3.3 to give an error bound for the approximation in part B. D. Using a calculator or MATLAB, compare the actual error to your error bound.

Solution: A.

x	f(x)	f(xx)	f(xxx)
0	0	$2/\pi$	$-4/(\pi^2)$
$\pi/2$	1	$-2/\pi$	-
π	0	-	-

$P_2(x) = \frac{2}{\pi}(x) - \frac{4}{\pi^2}(x)(x-\pi)$

B. $P_2(\pi/4) = \frac{2}{\pi}(\pi/4) - \frac{4}{\pi^2}(\pi/4)(\pi/4 - \pi) = 3/4$
C. $|f(\pi/4) - P(\pi/4)| \le \frac{(\pi/4-0)(\pi/4-\pi/2)(\pi/4-\pi)}{3!} = \frac{\pi^3}{128}$
D. $|f(\pi/4) - P_2(\pi/4)| = |\frac{\sqrt{2}}{2} - \frac{3}{4}| \approx 0.043$ □

Problem 2. (a) Given the data points (1, 0), (2, ln 2), (4, ln 4), find the degree 2 interpolating polynomial. (b) Use the result of (a) to approximate ln 3. (c) Use Theorem 3.3 to give an error bound for the approximation in part (b). (d) Compare the actual error to your error bound.

Solution:

x	f(x)	f(xx)	f(xxx)
1	0	ln2	-ln2/6
2	ln2	ln2/2	-
4	ln4	-	-

$P_2(x) = ln2(-x^2/6 + 3/2x - 4/3)$

$P_2(3) = ln2(-(3)^2/6 + (3/2)3 - 4/3) \approx 1.15525...$
$|f(3) - P(3)| \le \frac{f^3(c)}{6}(3-1)(3-2)(3-4) \quad c = 1$ □

Problem 3. Assume that the polynomial $P_9(x)$ interpolates the function $f(x) = e^{-2x}$ at the 10 evenly spaced points $x = 0, 1/9, 2/9, 3/9, ..., 8/9, 1$. (a) Find an upper bound for the error $|f(1/2) - P9(1/2)|$. (b) How many decimal places can you guarantee to be correct if $P_9(1/2)$ is used to approximate e^{-1}?

Solution: $|f(x) - p2(x)| = \frac{(x-0)(x-1/9)(x-2/9)(x-3/9)(x-4/9)(x-5/9)(x-6/9)(x-7/9)(x-8/9)(x-1)}{18^10}$
$|f(1/2) - p2(1/2)| = \frac{(1/2-0)(1/2-1/9)(1/2-2/9)(1/2-3/9)(1/2-4/9)(1/2-5/9)(1/2-6/9)(1/2-7/9)(1/2-8/9)(1/2-1)}{18^10 * 10!}$
$\approx 6.892 * 10^{-14}$ is the upper bound.
If it's used to find e^{-1}; if would at least be 9 decimal places correct. □

Computer

Problem 3. The total world oil production in millions of barrels per day is shown in the table that follows. Determine and plot the degree 9 polynomial through the data. Use it to estimate 2010 oil production. Does the Runge phenomenon occur in this example? In your opinion, is the interpolating polynomial a good model of the data? Explain

Solution:

```
(env) ~/Documents/_School/Depaul/MAT385/code > python3 newtdd.py
Newton interpolating polynomial:
  67.052 +
0.956(x-1994.000) +
0.420(x-1994.000)(x-1995.000) +
-0.069(x-1994.000)(x-1995.000)(x-1996.000) +
-0.036(x-1994.000)(x-1995.000)(x-1996.000)(x-1997.000) +
0.002(x-1994.000)(x-1995.000)(x-1996.000)(x-1997.000)(x-1998.000) +
0.012(x-1994.000)(x-1995.000)(x-1996.000)(x-1997.000)(x-1998.000)(x-1999.000) +
-0.008(x-1994.000)(x-1995.000)(x-1996.000)(x-1997.000)(x-1998.000)(x-1999.000)(x-2000.000) +
0.003(x-1994.000)(x-1995.000)(x-1996.000)(x-1997.000)(x-1998.000)(x-1999.000)(x-2000.000)(x-2001.000) +
-0.001(x-1994.000)(x-1995.000)(x-1996.000)(x-1997.000)(x-1998.000)(x-1999.000)(x-2000.000)(x-2001.000)(x-2002.000)

 y value at x=2010 is -1,951,646.134
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□

Section 3.4:

Problem 2a. (a) Check the spline conditions for Cubic Spline.

$$\begin{aligned} S1(x) &= 1 + 2x + 3x^2 + 4x^3[0, 1] \\ S2(x) &= 10 + 20(x-1) + 15(x-1)^2 + 4(x-1)^3[1, 2] \end{aligned} \tag{1}$$

(b) Regardless of your answer to (a), decide whether any of the following extra conditions are satisfied for this example: natural, parabolically terminated, not-a-knot.

Solution:
$S1\prime = 2 + 6x + 12x^2$
$S1\prime = 6 + 24x$

$S2\prime = 20 + 30(x-1) + 12(x-1)^2$
$S2\prime\prime = 30 + 24(x-1)$

check $S_1(1) = S_2(1)$ is true
$S_1\prime(1) = S_2\prime(1)$ is true
$S_1\prime\prime(1) = S_2\prime\prime(1)$ is true
therefore it is a cubic spline

□

Problem 3a. Find c in the following cubic splines. Which of the three end conditions—natural, parabolically terminated, or not-a-knot if any, are satisfied?

Solution:
$s1\prime(x) = -11/4 + 9/4(x^2) =$
$S2\prime(x) = -1/2(x-1) + 2C(x-1) - 9/4(x-1)^2$

$s1\prime\prime(x) = (9/2)x =$
$S2\prime\prime(x) = 2C - 9/4(x-1)^2$

Solving for c = 9/4;

$S1\prime\prime(0) = 0 \quad and \quad s2\prime\prime(2) = 0$ Therefore it is Natural. □

Problem 3c.

Solution:
$S_1(x) = -2 - (3/2)x + (7/2)x^2 - x^3$
$S_1\prime(x) = -3/2 + 7x - 3x^2$
$S_1\prime\prime(x) = 7 - 6x$

$S_2(x) = -1 + c(x-1) + 1/2(x-1)^2 - (x-1)^3$
$S_2\prime(x) = c + (x-1) - 3(x-1)^2$

$S_2''(x) = (x-1) - 6(x-1)$

$S_3(x) = 1 + 1/2(x-2) - 5/2(x-2)^2 - (x-2)^3$
$S_3'(x) = 1/2 - 5(x-2) - 3(x-2)^2$
$S_3''(x) = 5 - 2(x-2)$

Setting $S_1'(x) = \S_2'(x)$ and solving for C; We find C = 5/2;
$S_1''(1) = S_3''(1)$ Therefore it is not-a-knot

□

Problem 7b. Solve equations (3.26) to find the natural cubic spline through the three points (b) (-1,1), (1,1), (2,4).

Solution: Splines in form: $y_2 = \quad S_1(x_2) = y_1 + b_1(x_2 - x_1) + c_1(x_2 - x_1)^2 + d_1(x_2 - x_1)^3$
$\delta_1 = y1 - y2 = 1 - 1 = 0$
$\Delta_1 = x1 - x2 = 1 + 1 = 2$
$\delta_2 = y3 - y2 = 4 - 1 = 3$
$\Delta_2 = x3 - x2 = 2 - 1 = 1$

$$A_{3\times3} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 6 & 1 & 9 \\ 0 & 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$d_i = \frac{c_{i+1} - c_i}{3\delta_i};$$
$$b_i = \frac{\Delta_i}{\delta_i} - \frac{\delta_i}{3}(2C_i + C_{i+1})$$

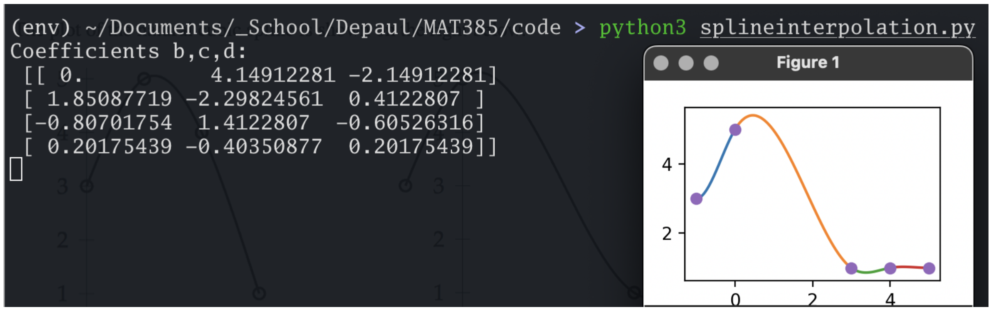
$d_1 = 1/4; \quad d_2 = -1/2$
$b_1 = -1; \quad b_2 = 2$

Answer =
$S_1(x) = 1 - (x+1) + 1/4(x+1)^3$
$S_2(x) = 1 + 2(x-1) + 3/2(x-1)^2 - 1/2(x-1)^3$ □

Computer

Problem 1b. Find the equations and plot the natural cubic spline that interpolates the data points (a) (0,3), (1,5),(2,4),(3,1) (b) (-1,3),(0,5),(3,1),(4,1),(5,1).

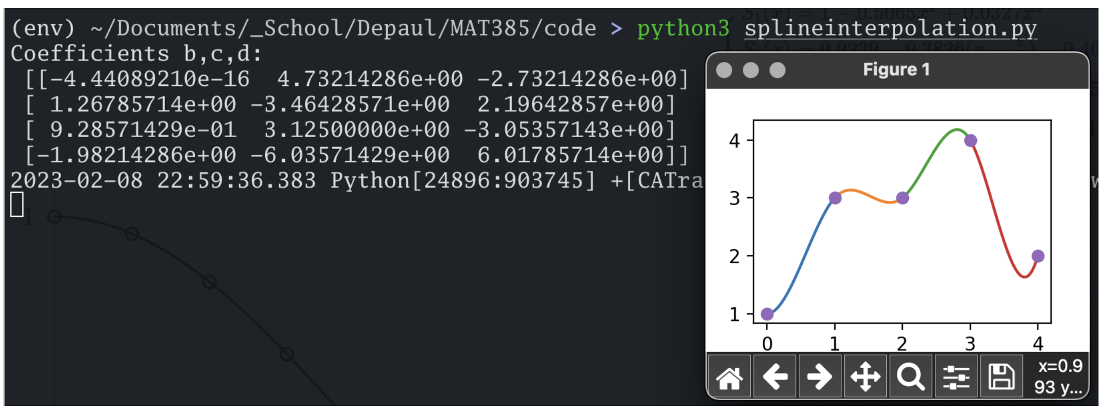
Solution:



□

Problem 5. Find and plot the cubic spline S satisfying S(0) = 1, S(1) = 3, S(2) = 3, S(3) = 4, S(4) = 2 and with S(0) = 0 and S(4) = 1

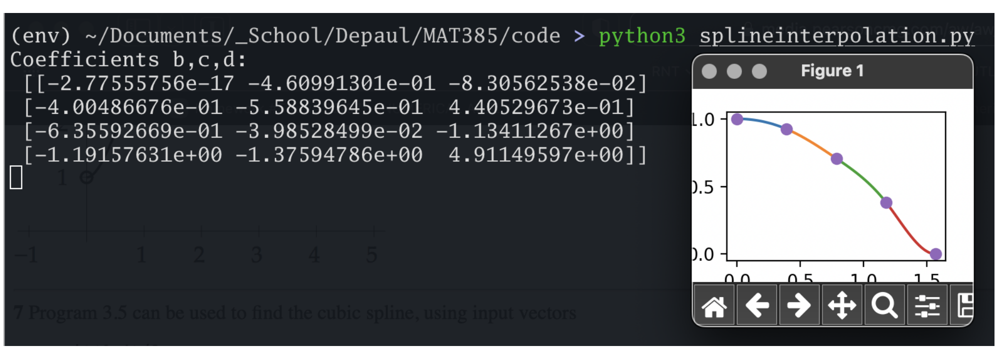
Solution:



□

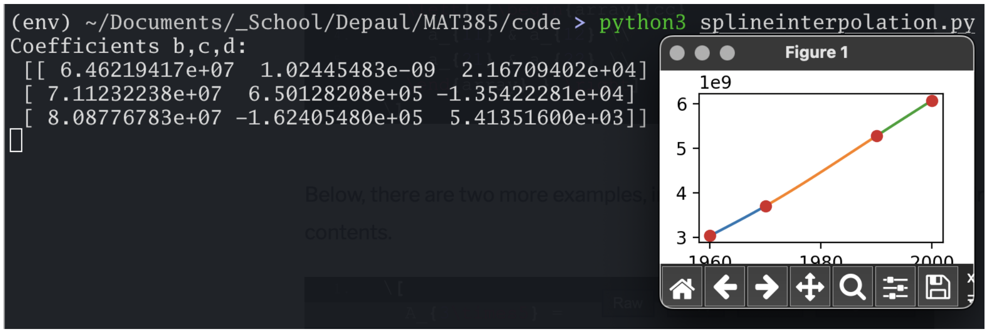
Problem 7. Find the clamped cubic spline that interpolates f(x)=cos x at five evenly spaced points in $[0, \pi/2]$, including the endpoints. What is the best choice for $S\prime(0)$ and $S\prime(\pi/2)$ to minimize interpolation error? Plot the spline and cos x on [0, 2].

Solution:



□

Problem 11. (a) Consider the natural cubic spline through the world population data points in Computer Problem 3.1.1. Evaluate the year 1980 and compare with the correct population. (b) Using a linear spline, estimate the slopes at 1960 and 2000, and use these slopes to find the clamped cubic spline through the data. Plot the spline and estimate the 1980 population. Which estimates better, natural or clamped?



Solution:

$S1(x) = 3039585530 + 6.46219417 * 10^7(x - 1960) + 2.1671 * 10^4(x - 1960)^3$

$S2(x) = 3707475887 + 7.1123224 * 10^7(x - 1970) + 6.50128(x - 1970)^2 - 1.3542 * 10^4(x - 1970)^3$

$S3(x) = 5281653820 + 8.0877678 * 10^7(x - 1990) - 1.62405(x - 1990)^2 + 5.414 * 10^3(x - 1990)^3$

$S2(1980) \approx 4470178717;$ $\square$