Section 4.1:

Problem 1a. Solve the normal equations to find the least squares solution and 2-norm error for the following inconsistent systems:

$$\begin{bmatrix}1 & 2\\ 0 & 1\\ 2 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}3\\ 1\\ 1\end{bmatrix}$$

Solution:

$$A^T = \begin{bmatrix}1 & 0 & 2\\ 2 & 1 & 1\end{bmatrix}$$

Normal Matrix

$$\begin{bmatrix}5 & 4\\ 4 & 6\end{bmatrix}\begin{bmatrix}X_1\\ X_2\end{bmatrix} = \begin{bmatrix}5\\ 8\end{bmatrix}$$

r = b - Ax = $\begin{bmatrix}\frac{2}{7}\\ -\frac{3}{7}\\ -\frac{1}{-7}\end{bmatrix}$

$$||e||_2 = \sqrt{(2/7)^2 + (-3/7)^2 + (-1/7)^2} = \frac{\sqrt{14}}{7}$$

□

Problem 2b. Find the least squares solutions and RMSE of the following systems

$$\begin{bmatrix}1 & 0 & 1\\ 1 & 0 & 2\\ 1 & 1 & 1\\ 2 & 1 & 1\end{bmatrix}\begin{bmatrix}X_1\\ X_2\\ X_3\end{bmatrix} = \begin{bmatrix}2\\ 3\\ 1\\ 2\end{bmatrix}$$

Solution:

$$A^T = \begin{bmatrix}1 & 1 & 1 & 2\\ 0 & 0 & 1 & 1\\ 1 & 2 & 1 & 1\end{bmatrix}$$

Normal Matrix =

$$\begin{bmatrix}7 & 3 & 6\\ 3 & 2 & 2\\ 5 & 2 & 6\end{bmatrix}\begin{bmatrix}X_1\\ X_2\\ X_3\end{bmatrix} = \begin{bmatrix}10\\ 3\\ 9\end{bmatrix} \rightarrow \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$e = b - Ax \rightarrow 0 \therefore ||e||_2 = 0$$

□

Problem 8c. Find the best line through each set of data points, and find the RMSE:
(c) (0,5),(1,3),(2,3),(3,1).

Solution:

$$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 3 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} 12 \\ 12 \end{bmatrix} \rightarrow X = \begin{bmatrix} \frac{24}{5} \\ -\frac{6}{5} \end{bmatrix}$$

$$r = \begin{bmatrix} 5 \\ 3 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \frac{24}{5} \\ -\frac{6}{5} \end{bmatrix} \rightarrow \begin{bmatrix} 0.2 \\ -0.6 \\ 0.6 \\ -0.2 \end{bmatrix}$$

RMSE $= \sqrt{\frac{(1/5)^2+(-3/5)^2+(3/5)^2+(1/5)^2}{4}} = \frac{\sqrt{5}}{5}$

□

Problem 9c. Find the best parabola through each data point set in Exercise 8, and compare the RMSE with the best-line fit.

Solution:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ 24 \end{bmatrix}$$

$$X = \begin{bmatrix} 4.8 \\ -1.2 \\ 0 \end{bmatrix}$$

$$r = \begin{bmatrix} 5 \\ 3 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} 4.8 \\ -1.2 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 24/5 \\ -6/5 \\ 0 \end{bmatrix}$$

RMSE $= \sqrt{\frac{(1/5)^2+(-3/5)^2+(3/5)^2+(1/5)^2}{4}} = \frac{\sqrt{5}}{5}$ It's the same as 8c.

□

Computer

Problem 1a. Form the normal equations, and compute the least squares solution and 2-norm error for the following inconsistent systems:

Solution:

```
(env) ~/Documents/_School/Depaul/MAT385/code > python3 LeastSquares.py
Solving Inconsistent System Ax=b
 A= [[ 3 -1  2]
 [ 4  1  0]
 [-3  2  1]
 [ 1  1  5]
 [-2  0  3]]
b= [10 10 -5 15  0]

Normal system:

Transpose(A)*A
 [[39 -4  2]
 [-4  7  5]
 [ 2  5 39]]

Transpose(A)*b
 [100   5  90]

Least-Squares Solution: [2.5246085  0.66163311 2.09340045]

2-norm error =  2.413492090641353

Root mean squared error =  1.0793464755664297
```

□

Problem 3. Consider the world population data of Computer Problem 3.1.1. Find the best least squares (a) line, (b) parabola through the data points, and the RMSE of the fit. In each case, estimate the 1980 population. Which fit gives the best estimate?

Solution:

```
(env) ~/Documents/_School/Depaul/MAT385/code > python3 LeastSquares.py


 Associated Matrix A (line fit):
 [[1.00e+00 1.96e+03]
 [1.00e+00 1.97e+03]
 [1.00e+00 1.99e+03]
 [1.00e+00 2.00e+03]]

Coefficients of line of best fit c1+c2*x:
 -147026357794.99933 76542140.14999966

 Root Mean Square Error =  36751088.1624098

Evaluate line at x = 1980.0 : y = 4527079702.0


 Associated Matrix A (parabola fit):
 [[1.0000e+00 1.9600e+03 3.8416e+06]
 [1.0000e+00 1.9700e+03 3.8809e+06]
 [1.0000e+00 1.9900e+03 3.9601e+06]
 [1.0000e+00 2.0000e+03 4.0000e+06]]

Coefficients of parabola of best fit c1+c2*x+c3*x^2:
 702727531186.6881 -781849860.2493656 216765.6566665244

 Root Mean Square Error =  17129714.187417276

Evaluate parabola at x = 1980.0 : y = 4472888288.386475
```

□