Math 385
Section 000
Semester winter23
Carlos Tapia Assignment HW

Section 1.2:

Problem 2a. Find all fixed points of the following g(x).

$$(a)x + 6/3x - 2$$

Solution: put equation in form f(x) = x;

$$\frac{x+6}{3x-2} = x$$

solve for x

$$3x^2 - 3x - 6 = 0$$

$$(x-2)(x+1) = 0$$

Therefore, fixed points are at $x = 2, -1$.

□

Problem 7. Use Theorem 1.6 to determine whether Fixed-Point Iteration of g(x) is locally convergent to the given fixed point r = 0.

$$(a)g(x) = (2x-1)^(1/3), r = 1$$

$$(b)g(x) = (x^3+1)/2, r = 1$$

$$(c)g(x) = sin(x) + x, r = 0$$

Solution:
(a)

$$g(x) = (2x-1)^{1/3}, r = 1$$

$$g\prime(x) = (2/3)(2x-1)^{-2/3}; g\prime(r) = 2/3 < 1$$

Therefore the (a) will converge
(b)

$$g(x) = (x^3+1)/2, r = 1$$

$$g\prime(x) = (3/2)x^2; g\prime(r) = 3/2 > 1$$

Therefore the (b) will diverge
(c)

$$g(x) = sin(x) + x, r = 0$$

$$g\prime(x) = sin(x) + x, g\prime(r) = 2 > 1$$

Therefore the (c) will diverge □

Problem 10. Find each fixed point and decide whether Fixed-Point Iteration is locally convergent to it.

$$(a)g(x) = x^2 - (3/2)x + 32$$

$$(b)g(x) = x^2 + (1/2)x - (1/2)$$

Solution: (a) set g(x) = x, solve for x;

$$g(x) = x^2 - (3/2)x + 32 = x$$

$$f(x) = x^2 - (5/2)x + 3/2 = 0$$

$x = 1, 3/2$

$$g\prime(x) = 2x - (3/2); g\prime(1) = (1/2) < 1$$

$$g\prime(x) = 2x - (3/2); g\prime(3/2) = (3/2) > 1$$

At fixed point $x = 1$ it will converge; At fixed point $x = 3/2$ it will diverge;
(b)

$$g(x) = x^2 + (1/2)x - (1/2) = x$$

$$x^2 - (1/2)x - (1/2) = 0$$

$x = 1, -1/2$

$$g\prime(x) = 2x + (1/2); g\prime(1) = 5/2 > 1$$

$$g\prime(x) = 2x + (1/2); g\prime(1) = 1/2 < 1$$

At fixed point $x = 1$ it will diverge; At fixed point $x = -1/2$ it will converge;

□

Problem 12. Consider the Fixed-Point Iteration (a) Do you expect Fixed-Point Iteration to calculate the root -0.2, say, to 10 or to correct decimal places, faster or slower than the Bisection Method? (b) Find the other fixed point. Will FPI converge to it?

$$x \to g(x) = x^2 - 0.24$$

Solution:
(a) FPI will calculate the root -0.2 faster than bisection because $g\prime(.02) = .4$ while bisection takes 0.5 per step.
(b) $g(x) = x$

$$x^2 - 0.24 \to x^2 - x - 0.24 = 0 \to x = -0.2, 1.2$$

fixed points are $x = -0.2, 1.2$ therefore $g\prime(-0.2) = .4 < 1$ implying that it converges; $g\prime(1.2) = 2.4 > 1$ implying that it diverges.

□

Problem 15. Which of the following three Fixed-Point Iterations converge to 5? Rank the ones that converge from fastest to slowest. (A) x → (4/5)*x + (1/x) (B) x → (x/2) + (5/2x) (C) x → (x+5)/(x+1)

Solution:

$$x \to (5/5) * x + (1/x)$$

$$g\prime(x) = (4/5) - 1/(x^2)$$

$$g\prime(5^{1/2}) = (3/5) < 1$$

$$x \to (x/2) + (5/2x)$$
$$g\prime(x) = (1/2) - 5/(2x^2)$$
$$g\prime(5^{1/2}) = (0) < 1$$
$$x \to (x+5)/(x+1)$$
$$g\prime(x) = (1/2) - 5/(2x^2)$$
$$g\prime(5^{1/2}) = ((5^{1/2} - 3)/2) < 1$$

Therefore $B < C < A$. B is the fastest.

□

Problem 19. Explore the idea of Example 1.6 for cube roots. If x is a guess that is smaller than $A^{1/3}$, then A/x^2 will be larger than $A^{1/3}$, so that the average of the two will be a better approximation than x. Suggest a Fixed-Point Iteration on the basis of this fact, and use Theorem 1.6 to decide whether it will converge to the cube root of A.

Solution: $x < A^{1/3} \Rightarrow \frac{A}{x^2} > A^{1/3}$; Implies that we will approach $A^{1/3}$ as we improve the x_0; which can be written as the average of $x_0 + A/x^2$. A formula that works should be...

$$x_{i+1} = \frac{x_i + \frac{A}{x_i^2}}{2}$$

$$g(x) = \frac{x + \frac{A}{x^2}}{2}$$
$$g\prime(x) = \frac{1 - \frac{2A}{x^3}}{2}$$
$$g\prime(A^{1/3}) = \frac{1 - \frac{2A}{(A^{1/3})^3}}{2}$$
$$g\prime(A^{1/3}) = (1/2) * (1-2) = 1/2 < 1$$

Since the $g(A^{1/3})$ is less than 1; It will converge.

□

Computer

Problem 1a. Apply Fixed-Point Iteration to find the solution of each equation to eight correct decimal places.

$$(a)x^3 = 2x + 2$$

Solution: See Attached Code; FPI Approx: 1.76929235; iterations: 14

□

Problem 1c. Apply Fixed-Point Iteration to find the solution of each equation to eight correct decimal places.

$$(c)ex + sinx = 4.$$

Solution: See Attached Code; FPI Approx: 1.12998050 iterations: 11

□

Problem 4c. Calculate the cube roots of the following numbers to eight correct decimal places, by using Fixed-Point Iteration with $g(x) = (2x + A/x^2)/3$, where A is (a) 2 (b) 3 (c) 5. State your initial guess and the number of steps needed.

Solution: See Attached Code; FPI Approx: 1.70997595 iterations: 9

□

Section 1.4:

Problem 1a. Apply two steps of Newton's Method with initial guess $x^0 = 0$.

$$(a)x^3 + x - 2 = 0$$

Solution: (a)

$$x^3 + x - 2 = 0$$
$$f(x) = x^3 + x - 2$$
$$f\prime(x) = 3x^2 + 1$$
$$x_{i+1} = x_n - \frac{x_n^2 + x_n - 2}{3x_n^2 + 1}$$

Two steps of newtons method
$x_1 = -2; x_2 = \frac{18}{13}$

□

Problem 3a. Use Theorem 1.11 or 1.12 to estimate the error ei+1 in terms of the previous error ei as Newton's Method converges to the given roots. Is the convergence linear or quadratic?

$$(a)x^5 - 2x^4 + 2x^2 - x = 0; r = -1, r = 0, r = 1$$

Solution:

$$f(x) = x^5 - 2x^4 + 2x^2 - x = 0; r = -1, r = 0, r = 1$$
$$f\prime(x) = 5x^4 - 8x^3 + 4x - 1$$
$$f\prime\prime(x) = 20x^3 - 24x^2 + 4$$
$$f\prime\prime\prime(x) = 60x^2 - 48x^1$$

To check if Convergent or divergent, check if $\frac{f\prime\prime(r)}{2f\prime(r)}$ is less than infinity. At $r = -1 \to -2$ which implies its quadratic. At $r = 0 \to 5/2$ which implies its quadratic. At $r = 1 \to 0/0$ which implies its linear therefore keep checking. $f\prime\prime\prime(r = 1) = 12$ therefore step is 2/3.

Error formula $= e_{i+1} \approx |\frac{f\prime\prime(x)}{2f\prime(x)}|e_i$ At $r = -1$; $\frac{5}{2}e^2$ At $r = 0$; $0e^2$ At $r = 1$; $\frac{2}{3}e$

□

Problem 4b. Estimateei+1 as in Exercise3.

$$(b)x^3 - x^2 - 5x - 3 = 0; r = -1, r = 3$$

Solution:

$$x^3 - x^2 - 5x - 3 = 0; r = -1, r = 3$$
$$f\prime(x) = 3x^2 - 2x - 5$$
$$f\prime\prime(x) = 6x - 2$$

To check if Convergent or divergent, check if $\frac{f\prime\prime(r)}{2f\prime(r)}$ is less than infinity. At $r = 3 \to 3$ which implies its quadratic. At $r = -1 \to 0/0$ which implies its linear therefore keep checking. $f\prime\prime\prime(r = 1) = 12$ therefore step is 2/3.

Error formula = $e_{i+1} \approx |\frac{f\prime\prime(x)}{2f\prime(x)}|e_i$ At $r = 3$; $1e^2$ At $r = -1$; $\frac{2}{3}e$ □

Problem 7. Let $f(x) = x^4 - 7x^3 + 18x^2 - 20x + 8$. Does Newton's Method converge quadratically to the root r = 2? Find lim ei+1/ei , where ei denotes the error at step i.

Solution:

$$f(x) = x^4 - 7x^3 + 18x^2 - 20x + 8$$
$$f\prime(x) = 4x^3 - 14x^2 + 36x - 20$$
$$f\prime\prime(x) = 12x^2 - 28x + 36$$
$$f\prime\prime\prime(x) = 24x - 28$$

At $f\prime(2) = f\prime\prime(2) = 0$ which implies its linearly therefore keep checking. $f\prime\prime\prime(2) = 20$ therefore step is 2/3.

Error is $\frac{2}{3}e$

□

Problem 10. Find the Fixed-Point Iteration produced by applying Newton's Method to $f(x) = x^3 - A$. See Exercise 1.2.10.

Solution:

$$f(x) = x^3 - A$$
$$f\prime(x) = 3x^2$$
$$x_{i+1} = x_i - \frac{x_i^3 - A}{3x_i^2}$$

simplify

$$x_{i+1} = \frac{2x_i + (A/x_i^2)}{3}$$

□

Computer

Problem 1a. Each equation has one root. Use Newton's Method to approximate the root to eight correctdecimalplaces.

$$(a)x^3 = 2x + 2$$

$$(b)ex + x = 7$$

$$(c)ex + sinx = 4$$

Solution: See Attached Code; FPI Approx: 1.76929235 iterations: 8

□

Problem 1c. Each equation has one root. Use Newton's Method to approximate the root to eight correctdecimalplaces.

$$(a)x^3 = 2x + 2$$

$$(b)ex + x = 7$$

$$(c)ex + sinx = 4$$

Solution: See Attached Code; FPI Approx: 1.12998050 iterations: 4

□

Problem 6. A 10-cm-high cone contains 60 cm3 of ice cream, including a hemispherical scoop on top. Find the radius of the scoop to four correct decimal places.

Solution: See Attached Code; FPI Approx: 2.02010732 iterations: 5

□

Problem 7. Consider the function $f(x) = esin3x + x^6 - 2x^4 - x^3 - 1$ on the interval [-2,2]. Plot the function on the interval, and find all three roots to six correct decimal places. Determine which roots converge quadratically, and find the multiplicity of the roots that converge linearly.

Solution: See Attached Code;
FPI Approx: 1.53013351 iterations: 7
FPI Approx: -1.19762372 iterations: 8
FPI Approx: 0.00008577 iterations: 31

$$f1(x)@2.0 = 0.00000000; fprime1(x)@2.0 = 14.97273117$$

$$f1(x)@ - 2.0 = 0.00000000; fprime1(x)@ - 2.0 = -4.92057686$$

$$f1(x)@0.3 = 0.00000000; fprime1(x)@0.3 = -0.00000000; f''1(x)@0.3 = -0.00000333;$$

Hence, f is quadratically convergent to 1.53013351 and -1.19762372; Linearly convergent to 0.00 with a step of 0.75 □