ON THE LARGEST PRIME FACTOR OF INTEGERS IN SHORT INTERVALS II

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ABSTRACT. The author sharpens a result of Baker and Harman (2009), showing that for sufficiently large x, the interval $[x, x + x^{\frac{1}{2}}]$ contains an integer with a prime factor larger than $x^{0.7437}$. Optimized bounds for multiple exponential sums and accurate numerical calculations are used for this improvement.

CONTENTS

1. INTRODUCTION

The Legendre's conjecture, which states that there is always a prime number between consecutive squares, is one of Landau's problems on prime numbers. Clearly this means that there is always a prime number in the interval $[x, x + x^{\frac{1}{2}}]$. However, we cannot prove it even on the Riemann Hypothesis. Assuming RH, one can only show that there is always a prime number in the interval $[x, x + x^{\frac{1}{2}} \log x]$. The best unconditional result is due to Li [15], where he showed the interval $[x, x + x^{0.52}]$ contains primes. Under a special condition concerning the existence of Siegel zeros, he [12] also showed that the interval $[x, x + x^{0.4923}]$ contains primes for long ranges of x.

Instead of relaxing the length of the short interval, one can attack this conjecture by relaxing our restriction of primes. A number with a large prime factor is a good approximation of prime numbers. Thus, we can try to find numbers with a large prime factor in the interval $[x, x + x^{\frac{1}{2}}]$ or $[x, x + x^{\frac{1}{2}+\varepsilon}]$. In [14] we considered the latter interval, and here we shall focus on the former one.

In this direction, Ramachandra [18] showed in 1969 that this interval contains a number with a prime factor larger than $x^{0.576}$. The exponent 0.576 has been improved to

$$0.625,\ 0.662,\ 0.675225,\ 0.692,\ 0.7,\ 0.71,\ 0.723,\ 0.728,\ 0.732,\ 0.738,\ 0.74 \text{ and } 0.7428$$

by Ramachandra [19], Graham [5], Zhu [22], Jia [7], Baker [1], Jia [8], Jia [9] (and Liu [16]), Jia [11], Baker and Harman [2], Liu and Wu [17], Harman [[6], Chapter 6] and Baker and Harman [3] respectively. In this paper, by optimizing the estimation of exponential sums and the sieve machinery, we obtain the following result.

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Theorem 1.1. *For sufficiently large x, the interval $[x, x+x^{\frac{1}{2}}]$ contains an integer with a prime factor larger than $x^{0.7437}$.*

Throughout this paper, we always suppose that ε is a sufficiently small positive constant. The letter p, with or without subscript, is reserved for prime numbers. Let $\eta = \exp(3/\varepsilon)$, $0.5 < \theta \leqslant 0.75$ and $\psi(\alpha) = \alpha - [\alpha] - \frac{1}{2}$. Let

$$N(d) = \sum_{\substack{x < n \leqslant x + x^{\frac{1}{2}} \\ d \mid n}} 1,$$

then by [[6], Section 6.2] we know that

$$\sum_{d<x} \Lambda(d) N(d) = \sum_{x < n \leqslant x + x^{\frac{1}{2}}} (\log n - \Lambda(n)) = x^{\frac{1}{2}} \log x + O\left(x^{\frac{1}{2}}\right), \tag{1}$$

$$\sum_{d \leqslant x^{0.6-\varepsilon}} \Lambda(d) N(d) = (0.6 - \varepsilon) x^{\frac{1}{2}} \log x + O\left(x^{\frac{1}{2}}\right), \tag{2}$$

$$\sum_{\substack{x^{0.6-\varepsilon} < d < x \\ d \text{ not prime}}} \Lambda(d) N(d) = O\left(x^{\frac{1}{2}}\right). \tag{3}$$

Now it suffices for the proof of Theorem 1.1 to show that

$$\sum_{x^{0.6-\varepsilon} < p \leqslant x^{0.7437}} (\log p) N(p) < 0.4 x^{\frac{1}{2}} \log x. \tag{4}$$

Let

$$\mathcal{A} = \{n : x^\theta < n \leqslant e x^\theta,\ N(n) = 1\}, \quad \mathcal{B} = \{n : x^\theta < n \leqslant e x^\theta\},$$
$$\mathcal{A}_d = \{a : ad \in \mathcal{A}\}, \quad P(z) = \prod_{p<z} p, \quad S(\mathcal{A}, z) = \sum_{\substack{a \in \mathcal{A} \\ (a, P(z)) = 1}} 1$$

and write $S(\theta) = S\left(\mathcal{A}, \left(e x^\theta\right)^{\frac{1}{2}}\right)$ that counts the number of primes in $\mathcal{A}$. Let $T(\theta) = S(\theta) x^{-\frac{1}{2}} \log x$. It is not hard to see that (4) follows from the bound

$$\int_{0.6-\varepsilon}^{0.7437} \theta T(\theta) d\theta < 0.4 \tag{5}$$

which we shall establish in the following sections.

2. Bounds for multiple exponential sums

In this section we shall give various estimations on the following two types of multiple exponential sums with 1-bounded coefficients:

$$\text{(Type-I)} \qquad \sum_{h \sim H} \sum_{m \sim M} \sum_{\substack{n \sim N \\ x^\theta < mn \leqslant e x^\theta}} a_m e\left(\frac{hx}{mn}\right)$$

and

$$\text{(Type-II)} \qquad \sum_{h \sim H} \sum_{m \sim M} \sum_{\substack{n \sim N \\ x^\theta < mn \leqslant e x^\theta}} a_m b_n e\left(\frac{hx}{mn}\right)$$

which will be very helpful for obtaining asymptotic formulas for some sieve functions. Now we define ϕ_j by the following table and put $J_j = [\phi_j, \phi_{j+1})$ as in [3].

ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9	ϕ_{10}	ϕ_{11}	ϕ_{12}	ϕ_{13}	ϕ_{14}
$\frac{3}{5}$	$\frac{73}{120}$	$\frac{11}{18}$	$\frac{161}{261}$	$\frac{13}{21}$	$\frac{5}{8}$	$\frac{47}{75}$	$\frac{35}{54}$	$\frac{2}{3}$	$\frac{90}{131}$	$\frac{226}{323}$	$\frac{547}{771}$	$\frac{23}{32}$	0.7437

We then write

$$\mathcal{J}(\theta) = [\theta - 0.5 + \varepsilon,\ \tau(\theta) - \varepsilon] \cup [\theta - \tau(\theta) + \varepsilon,\ 0.5 - \varepsilon],$$

where $\tau(\theta)$ is given by the next table.

Interval	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	J_9	J_{10}	J_{11}	J_{12}
$\tau(\theta)$	$2-3\theta$	$2-3\theta$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{9\theta-3}{17}$	$\frac{12\theta-5}{17}$	$\frac{55\theta-25}{17}$	$\frac{59\theta-28}{66}$	$\frac{245\theta-119}{261}$

For $\theta < 0.625-\varepsilon$, we also write

$$\mathcal{K}(\theta) = [2\theta-1+\varepsilon,\ 1.5-2\theta-\varepsilon] \cup [3\theta-1.5+\varepsilon,\ 1-\theta-\varepsilon].$$

Note that $\mathcal{J}(\theta)$ and $\mathcal{K}(\theta)$ are the "Type-II asymptotic regions" that will occur later. We shall first present our "Type-I" estimation, which gives a minor improvement over [[3], Lemma 1].

Lemma 2.1. *Suppose that* $0.6 \leqslant \theta < 0.75-\varepsilon$ *and* $\frac{1}{2} \leqslant H \leqslant [x^{\theta-\frac{1}{2}+4\eta}]$. *If either*

$$M \ll x^{\frac{2}{5}-\varepsilon}$$

or

$$x^{\min\left(6\theta-\frac{13}{4},\frac{110}{23}\theta-\frac{5}{2}\right)+\varepsilon} \ll M \ll x^{\frac{1}{2}-\varepsilon}$$

holds, then we have

$$\sum_{h\sim H}\sum_{m\sim M}\sum_{\substack{n\sim N\\ x^\theta<mn\leqslant ex^\theta}} a_m e\left(\frac{hx}{mn}\right) \ll x^{\theta-6\eta}.$$

Proof. The first case comes from [[17], Corollary 2]. For the second case, we apply [[21], Theorem 2] with $k=4$ or 5, $\alpha=\gamma=-1$, $\beta=1$ and replace (H,M,N,X) by $(N,H,M,Hx^{1-\theta})$. We then have the bounds

$$\sum_{h\sim H}\sum_{m\sim M}\sum_{\substack{n\sim N\\ x^\theta<mn\leqslant ex^\theta}} a_m e\left(\frac{hx}{mn}\right) \ll x^{\eta}\left(\left(\left(Hx^{1-\theta}\right)^{16}N^{52}H^{68}M^{60}\right)^{\frac{1}{80}} + \left(\left(Hx^{1-\theta}\right)N^2H^2M^4\right)^{\frac{1}{4}}\right.$$
$$\left. +NH+N(HM)^{\frac{1}{2}}+N^{\frac{1}{2}}HM+\left(Hx^{1-\theta}\right)^{-\frac{1}{2}}HMN\right)$$

and

$$\sum_{h\sim H}\sum_{m\sim M}\sum_{\substack{n\sim N\\ x^\theta<mn\leqslant ex^\theta}} a_m e\left(\frac{hx}{mn}\right) \ll x^{\eta}\left(\left(\left(Hx^{1-\theta}\right)^{32}N^{114}H^{147}M^{137}\right)^{\frac{1}{174}} + \left(\left(Hx^{1-\theta}\right)N^2H^2M^4\right)^{\frac{1}{4}}\right.$$
$$\left. +NH+N(HM)^{\frac{1}{2}}+N^{\frac{1}{2}}HM+\left(Hx^{1-\theta}\right)^{-\frac{1}{2}}HMN\right).$$

After verifying all conditions as in [3], the proof of Lemma 2.1 is completed. Note that Baker and Harman [3] only use $k=4$ to get the exponent $6\theta-\frac{13}{4}$, and an application of $k=5$ can be found in [[4], Section 2.4]. Using $k=3$ or 6 will not lead to any better estimations here. □

Next we shall present Baker and Harman's estimation for "Type-II" exponential sum, which mainly uses the work of Robert and Sargos [20].

Lemma 2.2. *([3], Lemma 2). Suppose that* $0.6 \leqslant \theta \leqslant \frac{23}{32}$ *and* $\frac{1}{2} \leqslant H \leqslant [x^{\theta-\frac{1}{2}+4\eta}]$. *If either*

$$x^{\theta-\frac{1}{2}+\varepsilon} \ll M \ll x^{\tau(\theta)-\varepsilon}$$

or

$$\theta < \frac{5}{8}-\varepsilon \quad \textit{and} \quad x^{2\theta-1+\varepsilon} \ll M \ll x^{\frac{3}{2}-2\theta-\varepsilon}$$

holds, then we have

$$\sum_{h\sim H}\sum_{m\sim M}\sum_{\substack{n\sim N\\ x^\theta<mn\leqslant ex^\theta}} a_m b_n e\left(\frac{hx}{mn}\right) \ll x^{\theta-6\eta}.$$

3. Sieve asymptotic formulas

In this section we shall give some asymptotic formulas for sieve functions of various types. We shall also use many results proved in [3] in our final decomposition and the readers can easily check them. We remark that for many applications of lemmas in this section, we omit the process of using a truncated Perron's formula to remove the dependencies between variables (conditions like $p_n < p_{n-1} < \cdots < p_1$). For the removing process one can see [[1], Lemma 11].

Lemma 3.1. *([3], Lemma 5]). Suppose that $0.6 \leqslant \theta \leqslant \frac{23}{32}$. If we can group $\{1, \cdots, n\}$ into I and J with*

$$\prod_{i \in I} p_i \sim M \quad \textit{and} \quad \prod_{j \in J} p_j \sim N$$

such that either

$$x^{\theta-\frac{1}{2}+\varepsilon} \ll M \ll x^{\tau(\theta)-\varepsilon}$$

or

$$\theta < \frac{5}{8} - \varepsilon \quad \textit{and} \quad x^{2\theta-1+\varepsilon} \ll M \ll x^{\frac{3}{2}-2\theta-\varepsilon}$$

holds, then we have the asymptotic formula

$$\sum_{p_1, \cdots, p_n} S\left(\mathcal{A}_{p_1 \cdots p_n}, p_n\right) = (1+o(1)) \sum_{p_1, \cdots, p_n} S\left(\mathcal{B}_{p_1 \cdots p_n}, p_n\right).$$

Lemma 3.2. *([3], Lemma 6]). Suppose that $0.6 \leqslant \theta \leqslant \frac{23}{32}$. If we can group $\{1, \cdots, n\}$ into I and J with*

$$\prod_{i \in I} p_i \sim M \quad \textit{and} \quad \prod_{j \in J} p_j \sim N$$

such that

$$M \ll x^{\theta-\frac{1}{2}+\varepsilon}, \quad N \ll x^{\gamma-\theta+\frac{1}{2}-\varepsilon}$$

holds, where

$$\gamma = \gamma(\theta) = \begin{cases} \frac{1}{2} - 2\varepsilon & \textit{for } 0.6 \leqslant \theta \leqslant \frac{73}{120}, \\ \frac{2}{5} - 2\varepsilon & \textit{for } \frac{73}{120} \leqslant \theta \leqslant \frac{23}{32}, \end{cases}$$

then we have the asymptotic formula

$$\sum_{p_1, \cdots, p_n} S\left(\mathcal{A}_{p_1 \cdots p_n}, x^{\tau(\theta)-\theta+\frac{1}{2}-2\varepsilon}\right) = (1+o(1)) \sum_{p_1, \cdots, p_n} S\left(\mathcal{B}_{p_1 \cdots p_n}, x^{\tau(\theta)-\theta+\frac{1}{2}-2\varepsilon}\right).$$

Lemma 3.3. *Suppose that $0.6 \leqslant \theta \leqslant \frac{5}{8} - \varepsilon$. If we have either*

$$\theta < \frac{73}{120} - \varepsilon \quad \textit{and} \quad \prod_{1 \leqslant i \leqslant n} p_i \ll x^{\theta-\tau(\theta)+\varepsilon}$$

or

$$\theta < \frac{161}{261} \quad \textit{and} \quad x^{\min\left(6\theta-\frac{13}{4}, \frac{110}{23}\theta-\frac{5}{2}\right)+\varepsilon} \ll \prod_{1 \leqslant i \leqslant n} p_i \ll x^{\theta-\tau(\theta)+\varepsilon},$$

then we have the asymptotic formula

$$\sum_{p_1, \cdots, p_n} S\left(\mathcal{A}_{p_1 \cdots p_n}, x^{\tau(\theta)-\theta+\frac{1}{2}-2\varepsilon}\right) = (1+o(1)) \sum_{p_1, \cdots, p_n} S\left(\mathcal{B}_{p_1 \cdots p_n}, x^{\tau(\theta)-\theta+\frac{1}{2}-2\varepsilon}\right).$$

Lemma 3.4. *Suppose that $0.6 \leqslant \theta \leqslant \frac{5}{8} - \varepsilon$. If we have*

$$\prod_{1 \leqslant i \leqslant n} p_i \ll x^{3\theta-\frac{3}{2}+\varepsilon},$$

then we have the asymptotic formula

$$\sum_{p_1, \cdots, p_n} S\left(\mathcal{A}_{p_1 \cdots p_n}, x^{\frac{5}{2}-4\theta-2\varepsilon}\right) = (1+o(1)) \sum_{p_1, \cdots, p_n} S\left(\mathcal{B}_{p_1 \cdots p_n}, x^{\frac{5}{2}-4\theta-2\varepsilon}\right).$$

Note that we have $\tau(\theta)-\theta+\frac{1}{2} = \frac{5}{2}-4\theta$ for $\theta \leqslant \frac{11}{18}$, and we have $\tau(\theta)-\theta+\frac{1}{2} > \frac{5}{2}-4\theta$ for $\frac{11}{18} < \theta < \frac{5}{8}$.

Proof. These two lemmas can be proved by the essentially same process as in [[3], Lemma 7], using our Lemma 2.1 instead of [[3], Lemma 1]. Note that we have

$$\min\left(6\theta-\frac{13}{4}, \frac{110}{23}\theta-\frac{5}{2}\right)<\theta-\tau(\theta)$$

when $\theta<\frac{161}{261}$. $\square$

Lemma 3.5. *([3], Lemma 8]). Suppose that $\theta \leqslant 0.65-\varepsilon$ and*

$$p_1 \ll x^{2\tau(\theta)-2\varepsilon}.$$

Then we have the asymptotic formula

$$\sum_{p_1} S\left(\mathcal{A}_{p_1}, x^{\tau(\theta)-\theta+\frac{1}{2}-2\varepsilon}\right)=(1+o(1))\sum_{p_1} S\left(\mathcal{B}_{p_1}, x^{\tau(\theta)-\theta+\frac{1}{2}-2\varepsilon}\right).$$

Moreover, by a simple calculation we know that

$$\frac{\theta}{2}<2\tau(\theta)$$

holds for $0.6 \leqslant \theta \leqslant 0.65$.

Lemma 3.6. *([3], Lemma 9]). Suppose that $0.6 \leqslant \theta \leqslant \frac{5}{8}-\varepsilon$ and no combination of variables satisfy the conditions of Lemma 2.1. If we have either*

$$\theta<\frac{73}{120}-\varepsilon$$

or

$$p_1 \ll x^{\frac{1-\theta}{2}-\varepsilon},$$

then we have the asymptotic formula

$$\sum_{\substack{x^{\tau(\theta)-\theta+\frac{1}{2}-2\varepsilon}\leqslant p_2\\ p_2<p_1<\left(ex^{\theta}\right)^{\frac{1}{2}}\\ p_1p_2^2\ll x^{\theta}}} S\left(\mathcal{A}_{p_1p_2}, x^{\frac{5}{2}-4\theta-2\varepsilon}\right)=(1+o(1))\sum_{\substack{x^{\tau(\theta)-\theta+\frac{1}{2}-2\varepsilon}\leqslant p_2\\ p_2<p_1<\left(ex^{\theta}\right)^{\frac{1}{2}}\\ p_1p_2^2\ll x^{\theta}}} S\left(\mathcal{B}_{p_1p_2}, x^{\frac{5}{2}-4\theta-2\varepsilon}\right).$$

4. The final decomposition

In this section, we ignore the presence of ε for clarity. Let $\omega(u)$ denotes the Buchstab function determined by the following differential-difference equation

$$\begin{cases}\omega(u)=\frac{1}{u}, & 1 \leqslant u \leqslant 2,\\ (u\omega(u))'=\omega(u-1), & u \geqslant 2.\end{cases}$$

Moreover, we have the upper and lower bounds for $\omega(u)$:

$$\omega(u)\geqslant\omega_0(u)=\begin{cases}\frac{1}{u}, & 1\leqslant u<2,\\ \frac{1+\log(u-1)}{u}, & 2\leqslant u<3,\\ \frac{1+\log(u-1)}{u}+\frac{1}{u}\int_2^{u-1}\frac{\log(t-1)}{t}dt \geqslant 0.5607, & 3\leqslant u<4,\\ 0.5612, & u\geqslant 4,\end{cases}$$

$$\omega(u)\leqslant\omega_1(u)=\begin{cases}\frac{1}{u}, & 1\leqslant u<2,\\ \frac{1+\log(u-1)}{u}, & 2\leqslant u<3,\\ \frac{1+\log(u-1)}{u}+\frac{1}{u}\int_2^{u-1}\frac{\log(t-1)}{t}dt \leqslant 0.5644, & 3\leqslant u<4,\\ 0.5617, & u\geqslant 4.\end{cases}$$

We shall use $\omega_0(u)$ and $\omega_1(u)$ to give numerical bounds for some sieve functions discussed below. We shall also use the simple upper bound $\omega(u)\leqslant\max(\frac{1}{u}, 0.5672)$ (see Lemma 8(iii) of [10]) to estimate high-dimensional integrals. Let $p_j \sim x^{t_j}$ and $\phi_1 \leqslant \theta \leqslant \phi_{14}$. We shall split this range of θ to several subranges and use different methods to treat them and obtain good bounds for $T(\theta)$.

Write $\nu_1(\theta)=\tau(\theta)-\theta+\frac{1}{2}$ and $\nu_2(\theta)=\frac{5}{2}-4\theta$. We begin our final decomposition with, using Buchstab's identity,

$$S(\theta)=S\left(\mathcal{A},\left(ex^{\theta}\right)^{\frac{1}{2}}\right)=S\left(\mathcal{A},x^{\nu_1(\theta)}\right)-\sum_{\nu_1(\theta)\leqslant t_1<\frac{\theta}{2}} S\left(\mathcal{A}_{p_1},p_1\right). \tag{6}$$

We can give an asymptotic formula for the first sum on the right-hand side.

4.1. **Case 1.** $\phi_1\leqslant\theta\leqslant\phi_2$**.** In this case we have $\nu_1(\theta)=\nu_2(\theta)$ and

$$\frac{\theta}{2}-\frac{1}{4}<\nu_1(\theta)\leqslant\frac{\tau(\theta)}{2}\leqslant\theta-\frac{1}{2}<\tau(\theta)\leqslant 2\theta-1<\frac{3}{2}-2\theta\leqslant\frac{\theta}{2}.$$

We start by splitting the summation range of the second sum on the right-hand side of (6). Now (6) becomes that

$$\begin{aligned}
S(\theta)&=S\left(\mathcal{A},x^{\nu_1(\theta)}\right)-\sum_{\nu_1(\theta)\leqslant t_1<\frac{\theta}{2}} S\left(\mathcal{A}_{p_1},p_1\right)\\
&=S\left(\mathcal{A},x^{\nu_1(\theta)}\right)-\sum_{\nu_1(\theta)\leqslant t_1<\frac{\tau(\theta)}{2}} S\left(\mathcal{A}_{p_1},p_1\right)-\sum_{\frac{\tau(\theta)}{2}\leqslant t_1<\theta-\frac{1}{2}} S\left(\mathcal{A}_{p_1},p_1\right)-\sum_{\theta-\frac{1}{2}\leqslant t_1<\tau(\theta)} S\left(\mathcal{A}_{p_1},p_1\right)\\
&\quad-\sum_{\tau(\theta)\leqslant t_1<2\theta-1} S\left(\mathcal{A}_{p_1},p_1\right)-\sum_{2\theta-1\leqslant t_1<\frac{3}{2}-2\theta} S\left(\mathcal{A}_{p_1},p_1\right)-\sum_{\frac{3}{2}-2\theta\leqslant t_1<\frac{\theta}{2}} S\left(\mathcal{A}_{p_1},p_1\right)\\
&=S_{11}-S_{121}-S_{122}-S_{123}-S_{124}-S_{125}-S_{126}.
\end{aligned} \tag{7}$$

By Lemma 3.1, we can give asymptotic formulas for S_{123} and S_{125}. We can also give an asymptotic formula for S_{121} by the discussion in [[6], Section 6.7.1]. For the remaining sums, write

$$S_{12}=S_{122}+S_{124}+S_{126}=\sum_{\substack{\frac{\tau(\theta)}{2}\leqslant t_1<\frac{\theta}{2}\\ t_1\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1},p_1\right). \tag{8}$$

Let $\boldsymbol{T}_n$ denote a positive partial sum of $t_1+t_2+\cdots+t_n$ (corresponding to a partial product of $p_1p_2\cdots p_n$ except 1) and $\boldsymbol{T}_n'$ denote a positive partial sum of $t_1+t_2+\cdots+t_{n-1}+2t_n$. Clearly we must have $\boldsymbol{T}_1=t_1$. We define the Type-I asymptotic regions corresponding to Lemmas 3.2-3.4 as

$$\mathcal{L}_1(\theta)=\left[0,\ \theta-\frac{1}{2}\right],\qquad \mathcal{L}_2(\theta)=\left[0,\ \gamma-\theta+\frac{1}{2}\right],$$

$$\mathcal{M}_1(\theta)=\left[0,\ \theta-\tau(\theta)\right],\qquad \mathcal{M}_2(\theta)=\left[\min\left(6\theta-\frac{13}{4},\frac{110}{23}\theta-\frac{5}{2}\right),\ \theta-\tau(\theta)\right],\qquad \mathcal{N}(\theta)=\left[0,\ 3\theta-\frac{3}{2}\right]$$

and $\mathcal{J}(\theta)$ and $\mathcal{K}(\theta)$ defined in Section 2 are Type-II asymptotic regions. Note that in **Case 1** all five regions are valid.

Using Buchstab's identity on S_{12}, we have

$$\begin{aligned}
S_{12}&=\sum_{\substack{\frac{\tau(\theta)}{2}\leqslant t_1<\frac{\theta}{2}\\ t_1\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1},p_1\right)\\
&=\sum_{\substack{\frac{\tau(\theta)}{2}\leqslant t_1<\frac{\theta}{2}\\ t_1\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1},x^{\nu_1(\theta)}\right)-\sum_{\substack{\frac{\tau(\theta)}{2}\leqslant t_1<\frac{\theta}{2}\\ t_1\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)\\ \nu_1(\theta)\leqslant t_2<\min\left(t_1,\frac{1}{2}(\theta-t_1)\right)}} S\left(\mathcal{A}_{p_1p_2},p_2\right)\\
&=\sum_{\substack{\frac{\tau(\theta)}{2}\leqslant t_1<\frac{\theta}{2}\\ t_1\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1},x^{\nu_1(\theta)}\right)-\sum_{\substack{\frac{\tau(\theta)}{2}\leqslant t_1<\frac{\theta}{2}\\ t_1\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)\\ \nu_1(\theta)\leqslant t_2<\min\left(t_1,\frac{1}{2}(\theta-t_1)\right)\\ \boldsymbol{T}_2\in\mathcal{J}(\theta)\cup\mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1p_2},p_2\right)
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
&= \sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1}, x^{\nu_1(\theta)}\right) - \sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \in \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
&- \sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2}, x^{\nu_1(\theta)}\right) + \sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \in \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right) \\
&+ \sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right) \\
&= S_{131} - S_{132} - S_{133} + S_{134} + S_{135}. \qquad (9)
\end{aligned}
$$

Lemma 3.5 yields an asymptotic formula for S_{131}. By Lemma 3.6 (case 1), we can give an asymptotic formula for S_{133}. We can also give asymptotic formulas for S_{132} and S_{134} by Lemma 3.1. For S_{135}, we can perform a straightforward decomposition by using Buchstab's identity twice if any of the following 2 conditions holds:

(A1) $t_1 + t_2 + 2t_3 \in \mathcal{M}_1(\theta)$;

(A2) $\boldsymbol{T}_3' \in \mathcal{L}_1(\theta)$ and $t_1 + t_2 + 2t_3 - \boldsymbol{T}_3' \in \mathcal{L}_2(\theta)$.

For those parts of S_{135} that satisfy either (A1) or (A2), we can get

$$
\begin{aligned}
&\sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \text{either (A1) or (A2) holds}}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right) \\
= &\sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \text{either (A1) or (A2) holds}}} S\left(\mathcal{A}_{p_1 p_2 p_3}, x^{\nu_1(\theta)}\right) - \sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \text{either (A1) or (A2) holds} \\ \nu_1(\theta) \leqslant t_4 < \min\left(t_3, \frac{1}{2}(\theta - t_1 - t_2 - t_3)\right) \\ \boldsymbol{T}_4 \in \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \text{either (A1) or (A2) holds} \\ \nu_1(\theta) \leqslant t_4 < \min\left(t_3, \frac{1}{2}(\theta - t_1 - t_2 - t_3)\right) \\ \boldsymbol{T}_4 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, x^{\nu_1(\theta)}\right) + \sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \text{either (A1) or (A2) holds} \\ \nu_1(\theta) \leqslant t_4 < \min\left(t_3, \frac{1}{2}(\theta - t_1 - t_2 - t_3)\right) \\ \boldsymbol{T}_4 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_5 < \min\left(t_4, \frac{1}{2}(\theta - t_1 - t_2 - t_3 - t_4)\right) \\ \boldsymbol{T}_5 \in \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5\right) \\
&+ \sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \text{either (A1) or (A2) holds} \\ \nu_1(\theta) \leqslant t_4 < \min\left(t_3, \frac{1}{2}(\theta - t_1 - t_2 - t_3)\right) \\ \boldsymbol{T}_4 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_5 < \min\left(t_4, \frac{1}{2}(\theta - t_1 - t_2 - t_3 - t_4)\right) \\ \boldsymbol{T}_5 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5\right) \\
&= S_{1351} - S_{1352} - S_{1353} + S_{1354} + S_{1355}. \qquad (10)
\end{aligned}
$$

We can give asymptotic formulas for S_{1352} and S_{1354} by Lemma 3.1. Note that $t_1+t_2+t_3+t_4 < t_1+t_2+2t_3$. If (A1) holds, then we can use Lemma 3.3 (case 1) to give asymptotic formulas for S_{1353} and S_{1351}. If (A2) holds, then we can use Lemma 3.2 to give asymptotic formulas for S_{1353} and S_{1351}. We discard the whole of S_{1355} since the corresponding loss is small enough, and it is not worth to consider the seven-dimensional sum after two further Buchstab iteration. We also discard the remaining parts of S_{135} (in which neither (A1) nor (A2) holds). Here we don't use the reversed Buchstab's identity since the possible savings are quite small. By summing up the loss from these two sums, we get the following upper bound for $T(\theta)$:

$$
\begin{aligned}
T(\theta) \leqslant \frac{1}{\theta} &+ \left(\int_{(t_1,t_2,t_3) \in L_{11}} \frac{\omega\left(\frac{\theta - t_1 - t_2 - t_3}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 dt_2 dt_1 \right) \\
&+ \left(\int_{(t_1,t_2,t_3,t_4,t_5) \in L_{12}} \frac{\omega\left(\frac{\theta - t_1 - t_2 - t_3 - t_4 - t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1 \right),
\end{aligned}
$$

where

$$
\begin{aligned}
L_{11}(t_1, t_2, t_3) := \Bigg\{ & \frac{\tau(\theta)}{2} \leqslant t_1 < \frac{\theta}{2},\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
& \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right),\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
& \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right),\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
& \text{neither (A1) nor (A2) holds} \Bigg\}, \\
L_{12}(t_1, t_2, t_3, t_4, t_5) := \Bigg\{ & \frac{\tau(\theta)}{2} \leqslant t_1 < \frac{\theta}{2},\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
& \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right),\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
& \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right),\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),
\end{aligned}
$$

$$\text{either (A1) or (A2) holds,}$$
$$\nu_1(\theta) \leqslant t_4 < \min\left(t_3, \frac{1}{2}(\theta - t_1 - t_2 - t_3)\right), \ \boldsymbol{T}_4 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),$$
$$\left. \nu_1(\theta) \leqslant t_5 < \min\left(t_4, \frac{1}{2}(\theta - t_1 - t_2 - t_3 - t_4)\right), \ \boldsymbol{T}_5 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \right\}.$$

Numerical calculation gives that

$$\int_{\phi_1}^{\phi_2} \theta T(\theta) d\theta < 0.008349. \tag{11}$$

4.2. **Case 2.** $\phi_2 \leqslant \theta \leqslant \phi_3$**.** In this case we have $\nu_1(\theta) = \nu_2(\theta)$ and

$$\frac{\theta}{2} - \frac{1}{4} \leqslant \nu_1(\theta) < \frac{\tau(\theta)}{2} < \theta - \frac{1}{2} < \tau(\theta) < \frac{1-\theta}{2} \leqslant \theta - \left(6\theta - \frac{13}{4}\right) < 2\theta - 1 < \frac{3}{2} - 2\theta < \frac{9}{10} - \theta < \frac{\theta}{2}.$$

By a similar process as in **Case 1**, (6) becomes that

$$\begin{aligned}
S(\theta) &= S\left(\mathcal{A}, x^{\nu_1(\theta)}\right) - \sum_{\nu_1(\theta) \leqslant t_1 < \frac{\theta}{2}} S\left(\mathcal{A}_{p_1}, p_1\right) \\
&= S\left(\mathcal{A}, x^{\nu_1(\theta)}\right) - \sum_{\nu_1(\theta) \leqslant t_1 < \frac{\tau(\theta)}{2}} S\left(\mathcal{A}_{p_1}, p_1\right) - \sum_{\frac{\tau(\theta)}{2} \leqslant t_1 < \theta - \frac{1}{2}} S\left(\mathcal{A}_{p_1}, p_1\right) - \sum_{\theta - \frac{1}{2} \leqslant t_1 < \tau(\theta)} S\left(\mathcal{A}_{p_1}, p_1\right) \\
&\quad - \sum_{\tau(\theta) \leqslant t_1 < \frac{1-\theta}{2}} S\left(\mathcal{A}_{p_1}, p_1\right) - \sum_{\frac{1-\theta}{2} \leqslant t_1 < \theta - \left(6\theta - \frac{13}{4}\right)} S\left(\mathcal{A}_{p_1}, p_1\right) - \sum_{\theta - \left(6\theta - \frac{13}{4}\right) \leqslant t_1 < 2\theta - 1} S\left(\mathcal{A}_{p_1}, p_1\right) \\
&\quad - \sum_{2\theta - 1 \leqslant t_1 < \frac{3}{2} - 2\theta} S\left(\mathcal{A}_{p_1}, p_1\right) - \sum_{\frac{3}{2} - 2\theta \leqslant t_1 < \frac{9}{10} - \theta} S\left(\mathcal{A}_{p_1}, p_1\right) - \sum_{\frac{9}{10} - \theta \leqslant t_1 < \frac{\theta}{2}} S\left(\mathcal{A}_{p_1}, p_1\right) \\
&= S_{21} - S_{221} - S_{222} - S_{223} - S_{224} - S_{225} - S_{226} - S_{227} - S_{228} - S_{229}.
\end{aligned} \tag{12}$$

By Lemma 3.1, we can give asymptotic formulas for S_{223} and S_{227}. We can also give an asymptotic formula for S_{221} by the discussion in [[6], Section 6.7.1]. For the remaining sums, write

$$S_{22} = S_{222} + S_{224} = \sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1}, p_1\right). \tag{13}$$

We shall consider S_{22}, S_{225}, S_{226}, S_{228} and S_{229} respectively. For S_{22}, we perform a straightforward decomposition just like the decomposition of S_{12} in **Case 1**. By Buchstab's identity, we have

$$\begin{aligned}
S_{22} &= \sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1}, p_1\right) \\
&= \sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1}, x^{\nu_1(\theta)}\right) - \sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \in \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
&\quad - \sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2}, x^{\nu_1(\theta)}\right) + \sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \in \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right)
\end{aligned}$$

$$
+ \sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right)
$$

$$
= S_{231} - S_{232} - S_{233} + S_{234} + S_{235}. \tag{14}
$$

We have asymptotic formulas for S_{231} by Lemma 3.5 and for S_{232} and S_{234} by Lemma 3.1. By Lemma 3.6 (case 2), we know that S_{233} also has an asymptotic formula. For S_{235}, we can perform a straightforward decomposition by using Buchstab's identity twice if any of the following 3 conditions holds:

(B1) $\boldsymbol{T}_3' \in \mathcal{L}_1(\theta)$ and $t_1 + t_2 + 2t_3 - \boldsymbol{T}_3' \in \mathcal{L}_2(\theta)$;

(B2) $t_1 + t_2 + t_3 + \nu_1(\theta) \in \mathcal{M}_2(\theta)$ and $t_1 + t_2 + 2t_3 \in \mathcal{M}_2(\theta)$;

(B3) $t_1 + t_2 + 2t_3 \in \mathcal{N}(\theta)$.

Note that in **Case 2** $\mathcal{M}_1(\theta)$ is invalid. For those parts of S_{235} that satisfy (B1) or (B2) or (B3), we have

$$
\sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \text{(B1) or (B2) or (B3) holds}}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right)
$$

$$
= \sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \text{(B1) or (B2) or (B3) holds}}} S\left(\mathcal{A}_{p_1 p_2 p_3}, x^{\nu_1(\theta)}\right) - \sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \text{(B1) or (B2) or (B3) holds} \\ \nu_1(\theta) \leqslant t_4 < \min\left(t_3, \frac{1}{2}(\theta - t_1 - t_2 - t_3)\right) \\ \boldsymbol{T}_4 \in \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right)
$$

$$
- \sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \text{(B1) or (B2) or (B3) holds} \\ \nu_1(\theta) \leqslant t_4 < \min\left(t_3, \frac{1}{2}(\theta - t_1 - t_2 - t_3)\right) \\ \boldsymbol{T}_4 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, x^{\nu_1(\theta)}\right) + \sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \text{(B1) or (B2) or (B3) holds} \\ \nu_1(\theta) \leqslant t_4 < \min\left(t_3, \frac{1}{2}(\theta - t_1 - t_2 - t_3)\right) \\ \boldsymbol{T}_4 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_5 < \min\left(t_4, \frac{1}{2}(\theta - t_1 - t_2 - t_3 - t_4)\right) \\ \boldsymbol{T}_5 \in \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5\right)
$$

$$
+ \sum_{\substack{\frac{\tau(\theta)}{2} \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \text{(B1) or (B2) or (B3) holds} \\ \nu_1(\theta) \leqslant t_4 < \min\left(t_3, \frac{1}{2}(\theta - t_1 - t_2 - t_3)\right) \\ \boldsymbol{T}_4 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_5 < \min\left(t_4, \frac{1}{2}(\theta - t_1 - t_2 - t_3 - t_4)\right) \\ \boldsymbol{T}_5 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5\right)
$$

$$
= S_{2351} - S_{2352} - S_{2353} + S_{2354} + S_{2355}. \tag{15}
$$

We can give asymptotic formulas for S_{1352} and S_{1354} by Lemma 3.1. Note that we have

$$t_1+t_2+t_3+\nu_1(\theta)<t_1+t_2+t_3+t_4<t_1+t_2+2t_3.$$

If (B1) holds, then we can use Lemma 3.2 to give asymptotic formulas for S_{2353} and S_{2351}. If (B2) holds, then we can use Lemma 3.3 (case 2) to give asymptotic formulas for S_{2353} and S_{2351}. If (B3) holds, then we can use Lemma 3.4 to give asymptotic formulas for S_{2353} and S_{2351}. We discard the whole of S_{2355} and the remaining parts of S_{235}, leading to two integrals of loss.

For S_{228} we can do a similar decomposition as (14). Using Buchstab's identity twice, we have

$$\begin{aligned}
S_{228}=&\sum_{\substack{\frac{3}{2}-2\theta\leqslant t_1<\frac{9}{10}-\theta\\ t_1\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1},p_1\right)\\
=&\sum_{\substack{\frac{3}{2}-2\theta\leqslant t_1<\frac{9}{10}-\theta\\ t_1\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1},x^{\nu_1(\theta)}\right)-\sum_{\substack{\frac{3}{2}-2\theta\leqslant t_1<\frac{9}{10}-\theta\\ t_1\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)\\ \nu_1(\theta)\leqslant t_2<\min\left(t_1,\frac{1}{2}(\theta-t_1)\right)\\ \boldsymbol{T}_2\in\mathcal{J}(\theta)\cup\mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1p_2},p_2\right)\\
&-\sum_{\substack{\frac{3}{2}-2\theta\leqslant t_1<\frac{9}{10}-\theta\\ t_1\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)\\ \nu_1(\theta)\leqslant t_2<\min\left(t_1,\frac{1}{2}(\theta-t_1)\right)\\ \boldsymbol{T}_2\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1p_2},x^{\nu_1(\theta)}\right)+\sum_{\substack{\frac{3}{2}-2\theta\leqslant t_1<\frac{9}{10}-\theta\\ t_1\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)\\ \nu_1(\theta)\leqslant t_2<\min\left(t_1,\frac{1}{2}(\theta-t_1)\right)\\ \boldsymbol{T}_2\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)\\ \nu_1(\theta)\leqslant t_3<\min\left(t_2,\frac{1}{2}(\theta-t_1-t_2)\right)\\ \boldsymbol{T}_3\in\mathcal{J}(\theta)\cup\mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right)\\
&+\sum_{\substack{\frac{3}{2}-2\theta\leqslant t_1<\frac{9}{10}-\theta\\ t_1\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)\\ \nu_1(\theta)\leqslant t_2<\min\left(t_1,\frac{1}{2}(\theta-t_1)\right)\\ \boldsymbol{T}_2\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)\\ \nu_1(\theta)\leqslant t_3<\min\left(t_2,\frac{1}{2}(\theta-t_1-t_2)\right)\\ \boldsymbol{T}_3\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right)\\
=&\,S_{2281}-S_{2282}-S_{2283}+S_{2284}+S_{2285}.
\end{aligned}\tag{16}$$

We have asymptotic formulas for S_{2282} and S_{2284} by Lemma 3.1. For S_{2283}, we know that if $t_2>\tau(\theta)$, then we have

$$\theta>t_1+2t_2>\left(\frac{3}{2}-2\theta\right)+2\tau(\theta)=\left(\frac{3}{2}-2\theta\right)+2(2-3\theta)=\frac{11}{2}-8\theta\geqslant\theta$$

when $\theta\leqslant\frac{11}{18}$, which is a contradiction. Hence we must have $t_2\leqslant\tau(\theta)$. Because $t_2\notin\mathcal{J}(\theta)$, we have $t_2<\theta-\frac{1}{2}$ in S_{2283}. Now by Lemma 3.2, we can give asymptotic formulas for S_{2283} and S_{2281}. For S_{2285}, we can perform a further decomposition if some parts of it satisfy (B1) or (B2) or (B3). Working as in S_{235}, we can get a loss of two integrals.

For S_{225} we can use a variable role-reversal mentioned in [3]. We refer the readers to [15], [14] and [13] for more applications of role-reversals. By [[3], Lemma 11(i)], the loss of S_{225} is just the three-dimentional sum

$$\sum_{\substack{\frac{1-\theta}{2}\leqslant t_1<\theta-\left(6\theta-\frac{13}{4}\right)\\ t_1\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)\\ \nu_1(\theta)\leqslant t_2<\min\left(t_1,\frac{1}{2}(\theta-t_1)\right)\\ \boldsymbol{T}_2\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)\\ \nu_1(\theta)\leqslant t_3<\frac{1}{2}t_1\\ \boldsymbol{U}_3\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)}} S\left(\mathcal{A}_{\beta p_2p_3},p_3\right),\tag{17}$$

where $\beta\sim x^{\theta-t_1-t_2}=x^{t_0}$, $(\beta,P(p_2))=1$ and $\boldsymbol{U}_3$ denotes a positive partial sum of $t_0+t_2+t_3$.

For S_{229} we also need to use a role-reversal. By [[3], Lemma 10(ii)], the loss of S_{225} is just the three-dimentional sum

$$\sum_{\substack{\frac{9}{10}-\theta \leqslant t_1 < \frac{\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \frac{1}{2} t_1 \\ \boldsymbol{U}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{\beta p_2 p_3}, p_3\right). \tag{18}$$

Finally, we discard the whole of S_{226} and summing up the total loss. The value of $T(\theta)$ in **Case 2** can be bounded by

$$\begin{aligned}
\frac{1}{\theta} &+ \left(\int_{(t_1,t_2,t_3) \in L_{21}} \frac{\omega\left(\frac{\theta - t_1 - t_2 - t_3}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 dt_2 dt_1\right) \\
&+ \left(\int_{(t_1,t_2,t_3,t_4,t_5) \in L_{22}} \frac{\omega\left(\frac{\theta - t_1 - t_2 - t_3 - t_4 - t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1\right) \\
&+ \left(\int_{(t_1,t_2,t_3) \in L_{23}} \frac{\omega\left(\frac{\theta - t_1 - t_2 - t_3}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 dt_2 dt_1\right) \\
&+ \left(\int_{(t_1,t_2,t_3,t_4,t_5) \in L_{24}} \frac{\omega\left(\frac{\theta - t_1 - t_2 - t_3 - t_4 - t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1\right) \\
&+ \left(\int_{(t_1,t_2,t_3) \in L_{25}} \frac{\omega\left(\frac{t_1 - t_3}{t_3}\right) \omega\left(\frac{\theta - t_1 - t_2}{t_2}\right)}{t_2^2 t_3^2} dt_3 dt_2 dt_1\right) \\
&+ \left(\int_{(t_1,t_2,t_3) \in L_{26}} \frac{\omega\left(\frac{t_1 - t_3}{t_3}\right) \omega\left(\frac{\theta - t_1 - t_2}{t_2}\right)}{t_2^2 t_3^2} dt_3 dt_2 dt_1\right) \\
&+ \left(\int_{\theta - \left(6\theta - \frac{13}{4}\right)}^{2\theta - 1} \frac{\omega\left(\frac{\theta - t_1}{t_1}\right)}{t_1^2} dt_1\right),
\end{aligned}$$

where

$$\begin{aligned}
L_{21}(t_1, t_2, t_3) := \Bigg\{ & \frac{\tau(\theta)}{2} \leqslant t_1 < \frac{1-\theta}{2},\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
& \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right),\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
& \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right),\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
& \text{None of (B1), (B2) and (B3) holds}\Bigg\}, \\
L_{22}(t_1, t_2, t_3, t_4, t_5) := \Bigg\{ & \frac{\tau(\theta)}{2} \leqslant t_1 < \frac{1-\theta}{2},\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
& \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right),\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
& \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right),\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
& \text{One of (B1), (B2) or (B3) holds},
\end{aligned}$$

$$
\begin{aligned}
&\nu_1(\theta) \leqslant t_4 < \min\left(t_3, \frac{1}{2}(\theta - t_1 - t_2 - t_3)\right), \ \boldsymbol{T}_4 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
&\nu_1(\theta) \leqslant t_5 < \min\left(t_4, \frac{1}{2}(\theta - t_1 - t_2 - t_3 - t_4)\right), \ \boldsymbol{T}_5 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
L_{23}(t_1, t_2, t_3) := \Bigg\{ &\frac{3}{2} - 2\theta \leqslant t_1 < \frac{9}{10} - \theta, \ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
&\nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right), \ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
&\nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right), \ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
&\text{None of (B1), (B2) and (B3) holds}\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
L_{24}(t_1, t_2, t_3, t_4, t_5) := \Bigg\{ &\frac{3}{2} - 2\theta \leqslant t_1 < \frac{9}{10} - \theta, \ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
&\nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right), \ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
&\nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right), \ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
&\text{One of (B1), (B2) or (B3) holds}, \\
&\nu_1(\theta) \leqslant t_4 < \min\left(t_3, \frac{1}{2}(\theta - t_1 - t_2 - t_3)\right), \ \boldsymbol{T}_4 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
&\nu_1(\theta) \leqslant t_5 < \min\left(t_4, \frac{1}{2}(\theta - t_1 - t_2 - t_3 - t_4)\right), \ \boldsymbol{T}_5 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
L_{25}(t_1, t_2, t_3) := \Bigg\{ &\frac{1-\theta}{2} \leqslant t_1 < \theta - \left(6\theta - \frac{13}{4}\right), \ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
&\nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right), \ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
&\nu_1(\theta) \leqslant t_3 < \frac{1}{2} t_1, \ \boldsymbol{U}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
L_{26}(t_1, t_2, t_3) := \Bigg\{ &\frac{9}{10} - \theta \leqslant t_1 < \frac{\theta}{2}, \ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
&\nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right), \ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
&\nu_1(\theta) \leqslant t_3 < \frac{1}{2} t_1, \ \boldsymbol{U}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)\Bigg\}.
\end{aligned}
$$

Numerical calculation gives that

$$
\int_{\phi_2}^{\phi_3} \theta T(\theta) d\theta < 0.003204. \tag{19}
$$

4.3. **Case 3.** $\phi_3 \leqslant \theta \leqslant \phi_4$**.** In this case we have $\nu_1(\theta) \geqslant \nu_2(\theta) > 0$ and

$$
\nu_2(\theta) \leqslant \nu_1(\theta) < \theta - \frac{1}{2} < \tau(\theta) < \frac{1-\theta}{2} < 2\theta - 1 < \frac{3}{2} - 2\theta < \frac{9}{10} - \theta < \frac{\theta}{2}.
$$

By a similar process as in **Case 1**, (6) becomes that

$$
\begin{aligned}
S(\theta) &= S\left(\mathcal{A}, x^{\nu_1(\theta)}\right) - \sum_{\nu_1(\theta) \leqslant t_1 < \frac{\theta}{2}} S\left(\mathcal{A}_{p_1}, p_1\right) \\
&= S\left(\mathcal{A}, x^{\nu_1(\theta)}\right) - \sum_{\nu_1(\theta) \leqslant t_1 < \theta - \frac{1}{2}} S\left(\mathcal{A}_{p_1}, p_1\right) - \sum_{\theta - \frac{1}{2} \leqslant t_1 < \tau(\theta)} S\left(\mathcal{A}_{p_1}, p_1\right) - \sum_{\tau(\theta) \leqslant t_1 < \frac{1-\theta}{2}} S\left(\mathcal{A}_{p_1}, p_1\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{\frac{1-\theta}{2} \leqslant t_1 < 2\theta-1} S\left(\mathcal{A}_{p_1}, p_1\right) - \sum_{2\theta-1 \leqslant t_1 < \frac{3}{2}-2\theta} S\left(\mathcal{A}_{p_1}, p_1\right) - \sum_{\frac{3}{2}-2\theta \leqslant t_1 < \frac{\theta}{2}} S\left(\mathcal{A}_{p_1}, p_1\right) \\
&= S_{31} - S_{321} - S_{322} - S_{323} - S_{324} - S_{325} - S_{326}. \qquad (20)
\end{aligned}
$$

By Lemma 3.1, we can give asymptotic formulas for S_{322} and S_{325}. For the remaining sums, write

$$
S_{32} = S_{321} + S_{323} = \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1}, p_1\right) \qquad (21)
$$

and use Buchstab's identity twice to reach

$$
\begin{aligned}
S_{32} =& \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1}, p_1\right) \\
=& \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1}, x^{\nu_1(\theta)}\right) - \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \in \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
&- \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \theta - \frac{1}{2} \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) - \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \theta - \frac{1}{2} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
=& \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1}, x^{\nu_1(\theta)}\right) - \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \in \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
&- \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \theta - \frac{1}{2} \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2}, x^{\nu_1(\theta)}\right) + \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \theta - \frac{1}{2} \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \in \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right) \\
&+ \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \theta - \frac{1}{2} \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right) \\
&- \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \theta - \frac{1}{2} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2}, x^{\nu_2(\theta)}\right) + \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \theta - \frac{1}{2} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_2(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \in \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right) \\
&+ \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \theta - \frac{1}{2} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_2(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right)
\end{aligned}
$$

$$= S_{331} - S_{332} - S_{333} + S_{334} + S_{335} - S_{336} + S_{337} + S_{338}. \tag{22}$$

Lemma 3.5 gives an asymptotic formula for S_{331}. We have asymptotic formulas for S_{332}, S_{334} and S_{337} by Lemma 3.1. By Lemma 3.2, we know that S_{333} also has an asymptotic formula. By Lemma 3.6 (case 2), we know that S_{336} also has an asymptotic formula. For S_{335}, we can perform a straightforward decomposition by using Buchstab's identity twice if any of the following 3 conditions holds:

(C1) $\boldsymbol{T}_3' \in \mathcal{L}_1(\theta)$ and $t_1 + t_2 + 2t_3 - \boldsymbol{T}_3' \in \mathcal{L}_2(\theta)$;

(C2) $t_1 + t_2 + t_3 + \nu_1(\theta) \in \mathcal{M}_2(\theta)$ and $t_1 + t_2 + 2t_3 \in \mathcal{M}_2(\theta)$;

(C3) $t_1 + t_2 + 2t_3 \in \mathcal{N}(\theta)$.

Note that in **Case 3** $\mathcal{M}_1(\theta)$ is invalid. For those parts of S_{335} that satisfy either (C1) or (C2), we can perform a similar decomposition as (15), with lower bound $\nu_1(\theta)$ for p_4 and p_5. For other parts of S_{335} that satisfy neither (C1) nor (C2) but satisfy (C3), we can perform a similar decomposition as (15) but with $\nu_2(\theta)$ as the lower bound for p_4 and p_5 instead of $\nu_1(\theta)$. We discard the remaining parts of S_{335} and the corresponding two five-dimensional sums after the above two decompositions. For S_{338} we can repeat the above procedure, but the lower bound for p_4 and p_5 must be $\nu_2(\theta)$ regardless of which condition the sum satisfies. To sum up, for S_{32} we get five integrals of loss, two of those are three-dimensional and the remaining three are five-dimensional. We choose to discard the whole of S_{324} and S_{326}. Putting them together, the value of $T(\theta)$ in **Case 3** is no more than

$$\begin{aligned}
\frac{1}{\theta} &+ \left(\int_{(t_1,t_2,t_3) \in L_{31}} \frac{\omega\left(\frac{\theta - t_1 - t_2 - t_3}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 dt_2 dt_1 \right) \\
&+ \left(\int_{(t_1,t_2,t_3,t_4,t_5) \in L_{32}} \frac{\omega\left(\frac{\theta - t_1 - t_2 - t_3 - t_4 - t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
&+ \left(\int_{(t_1,t_2,t_3,t_4,t_5) \in L_{33}} \frac{\omega\left(\frac{\theta - t_1 - t_2 - t_3 - t_4 - t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
&+ \left(\int_{(t_1,t_2,t_3) \in L_{34}} \frac{\omega\left(\frac{\theta - t_1 - t_2 - t_3}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 dt_2 dt_1 \right) \\
&+ \left(\int_{(t_1,t_2,t_3,t_4,t_5) \in L_{35}} \frac{\omega\left(\frac{\theta - t_1 - t_2 - t_3 - t_4 - t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
&+ \left(\int_{\frac{1-\theta}{2}}^{2\theta - 1} \frac{\omega\left(\frac{\theta - t_1}{t_1}\right)}{t_1^2} dt_1 \right) + \left(\int_{\frac{3}{2} - 2\theta}^{\frac{\theta}{2}} \frac{\omega\left(\frac{\theta - t_1}{t_1}\right)}{t_1^2} dt_1 \right),
\end{aligned}$$

where

$$\begin{aligned}
L_{31}(t_1, t_2, t_3) := \Big\{ & \nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2},\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
& \nu_1(\theta) \leqslant t_2 < \theta - \frac{1}{2},\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
& \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right),\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
& \text{None of (C1), (C2) and (C3) holds} \Big\}, \\
L_{32}(t_1, t_2, t_3, t_4, t_5) := \Big\{ & \nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2},\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
& \nu_1(\theta) \leqslant t_2 < \theta - \frac{1}{2},\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),
\end{aligned}$$

$$\nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right), \ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),$$
Either (C1) or (C2) holds,
$$\nu_1(\theta) \leqslant t_4 < \min\left(t_3, \frac{1}{2}(\theta - t_1 - t_2 - t_3)\right), \ \boldsymbol{T}_4 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),$$
$$\left.\nu_1(\theta) \leqslant t_5 < \min\left(t_4, \frac{1}{2}(\theta - t_1 - t_2 - t_3 - t_4)\right), \ \boldsymbol{T}_5 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)\right\},$$

$$L_{33}(t_1, t_2, t_3, t_4, t_5) := \left\{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2}, \ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),\right.$$
$$\nu_1(\theta) \leqslant t_2 < \theta - \frac{1}{2}, \ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),$$
$$\nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right), \ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),$$
Neither (C1) nor (C2) holds, but (C3) holds,
$$\nu_2(\theta) \leqslant t_4 < \min\left(t_3, \frac{1}{2}(\theta - t_1 - t_2 - t_3)\right), \ \boldsymbol{T}_4 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),$$
$$\left.\nu_2(\theta) \leqslant t_5 < \min\left(t_4, \frac{1}{2}(\theta - t_1 - t_2 - t_3 - t_4)\right), \ \boldsymbol{T}_5 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)\right\},$$

$$L_{34}(t_1, t_2, t_3) := \left\{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2}, \ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),\right.$$
$$\theta - \frac{1}{2} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right), \ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),$$
$$\nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right), \ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),$$
None of (C1), (C2) and (C3) holds$\}$,

$$L_{35}(t_1, t_2, t_3, t_4, t_5) := \left\{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2}, \ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),\right.$$
$$\theta - \frac{1}{2} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right), \ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),$$
$$\nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right), \ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),$$
One of (C1), (C2) or (C3) holds,
$$\nu_1(\theta) \leqslant t_4 < \min\left(t_3, \frac{1}{2}(\theta - t_1 - t_2 - t_3)\right), \ \boldsymbol{T}_4 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),$$
$$\left.\nu_1(\theta) \leqslant t_5 < \min\left(t_4, \frac{1}{2}(\theta - t_1 - t_2 - t_3 - t_4)\right), \ \boldsymbol{T}_5 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)\right\}.$$

Numerical calculation gives that
$$\int_{\phi_3}^{\phi_4} \theta T(\theta) d\theta < 0.00852. \tag{23}$$

4.4. **Case 4.** $\phi_4 \leqslant \theta \leqslant \phi_5$**.** In this case we have $\nu_1(\theta) > \nu_2(\theta) > 0$ and
$$\nu_2(\theta) < \nu_1(\theta) < \theta - \frac{1}{2} < \tau(\theta) < \frac{1-\theta}{2} < 2\theta - 1 < \frac{3}{2} - 2\theta < \frac{9}{10} - \theta < \frac{\theta}{2}.$$
By the same process as in **Case 3**, (6) becomes that
$$S(\theta) = S\left(\mathcal{A}, x^{\nu_1(\theta)}\right) - \sum_{\nu_1(\theta) \leqslant t_1 < \frac{\theta}{2}} S\left(\mathcal{A}_{p_1}, p_1\right)$$

$$
\begin{aligned}
&= S\left(\mathcal{A}, x^{\nu_1(\theta)}\right) - \sum_{\nu_1(\theta) \leqslant t_1 < \theta - \frac{1}{2}} S\left(\mathcal{A}_{p_1}, p_1\right) - \sum_{\theta - \frac{1}{2} \leqslant t_1 < \tau(\theta)} S\left(\mathcal{A}_{p_1}, p_1\right) - \sum_{\tau(\theta) \leqslant t_1 < \frac{1-\theta}{2}} S\left(\mathcal{A}_{p_1}, p_1\right) \\
&\quad - \sum_{\frac{1-\theta}{2} \leqslant t_1 < 2\theta - 1} S\left(\mathcal{A}_{p_1}, p_1\right) - \sum_{2\theta - 1 \leqslant t_1 < \frac{3}{2} - 2\theta} S\left(\mathcal{A}_{p_1}, p_1\right) - \sum_{\frac{3}{2} - 2\theta \leqslant t_1 < \frac{\theta}{2}} S\left(\mathcal{A}_{p_1}, p_1\right) \\
&= S_{41} - S_{421} - S_{422} - S_{423} - S_{424} - S_{425} - S_{426}. \qquad (24)
\end{aligned}
$$

By Lemma 3.1, we can give asymptotic formulas for S_{422} and S_{425}. For the remaining sums, write

$$
S_{42} = S_{421} + S_{423} = \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1}, p_1\right) \qquad (25)
$$

and use Buchstab's identity twice to reach

$$
\begin{aligned}
S_{42} &= \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1}, p_1\right) \\
&= \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1}, x^{\nu_1(\theta)}\right) - \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \in \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
&\quad - \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \theta - \frac{1}{2} \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) - \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \theta - \frac{1}{2} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
&= \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1}, x^{\nu_1(\theta)}\right) - \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \in \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
&\quad - \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \theta - \frac{1}{2} \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2}, x^{\nu_1(\theta)}\right) + \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \theta - \frac{1}{2} \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \in \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right) \\
&\quad + \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_2 < \theta - \frac{1}{2} \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right) \\
&\quad - \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \theta - \frac{1}{2} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2}, x^{\nu_2(\theta)}\right) + \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2} \\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \theta - \frac{1}{2} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_2(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \in \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right)
\end{aligned}
$$

$$+\sum_{\substack{\nu_1(\theta)\leqslant t_1<\frac{1-\theta}{2}\\ t_1\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)\\ \theta-\frac{1}{2}\leqslant t_2<\min\left(t_1,\frac{1}{2}(\theta-t_1)\right)\\ \boldsymbol{T}_2\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)\\ \nu_2(\theta)\leqslant t_3<\min\left(t_2,\frac{1}{2}(\theta-t_1-t_2)\right)\\ \boldsymbol{T}_3\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right)$$

$$= S_{431}-S_{432}-S_{433}+S_{434}+S_{435}-S_{436}+S_{437}+S_{438}. \tag{26}$$

Lemma 3.5 gives an asymptotic formula for S_{431}. We have asymptotic formulas for S_{432}, S_{434} and S_{437} by Lemma 3.1. By Lemma 3.2, we know that S_{433} also has an asymptotic formula. By Lemma 3.6 (case 2), we know that S_{436} also has an asymptotic formula. For S_{435}, we can perform a straightforward decomposition by using Buchstab's identity twice if any of the following 2 conditions holds:

(D1) $\boldsymbol{T}_3'\in\mathcal{L}_1(\theta)$ and $t_1+t_2+2t_3-\boldsymbol{T}_3'\in\mathcal{L}_2(\theta)$;

(D2) $t_1+t_2+2t_3\in\mathcal{N}(\theta)$.

Note that in **Case 4** both $\mathcal{M}_1(\theta)$ and $\mathcal{M}_2(\theta)$ are invalid. By a similar discussion as in **Case 3**, we can get five corresponding loss integrals for S_{42}. We remark that we can use the reversed Buchstab's identity to gain a small saving on parts of S_{42} if neither (D1) nor (D2) is satisfied. This can be seen as the following: if we have $t_3<\frac{1}{2}(\theta-t_1-t_2-t_3)$, then

$$\begin{aligned}\sum_{t_1,t_2,t_3} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right) =& \sum_{t_1,t_2,t_3} S\left(\mathcal{A}_{p_1p_2p_3},\left(\frac{ex^\theta}{p_1p_2p_3}\right)^{\frac{1}{2}}\right)\\ &+\sum_{\substack{t_1,t_2,t_3\\ t_3<t_4<\frac{1}{2}(\theta-t_1-t_2-t_3)\\ \boldsymbol{T}_4\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1p_2p_3p_4},p_4\right)\\ &+\sum_{\substack{t_1,t_2,t_3\\ t_3<t_4<\frac{1}{2}(\theta-t_1-t_2-t_3)\\ \boldsymbol{T}_4\in\mathcal{J}(\theta)\cup\mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1p_2p_3p_4},p_4\right).\end{aligned} \tag{27}$$

We cannot give an asymptotic formula for the sum on the left-hand side of (27), but we can give an asymptotic formula for the last sum on the right-hand side, hence we can subtract it from the loss. We refer the readers to [15], [14] and [13] for more applications of reversed Buchstab's identity. We discard the whole S_{424} and S_{426} just as in **Case 3**. The value of $T(\theta)$ in **Case 4** can thus be bounded by

$$\begin{aligned}\frac{1}{\theta}+&\left(\int_{(t_1,t_2,t_3)\in L_{41}}\frac{\omega\left(\frac{\theta-t_1-t_2-t_3}{t_3}\right)}{t_1t_2t_3^2}dt_3dt_2dt_1\right)\\ -&\left(\int_{(t_1,t_2,t_3,t_4)\in L_{42}}\frac{\omega\left(\frac{\theta-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1t_2t_3t_4^2}dt_4dt_3dt_2dt_1\right)\\ +&\left(\int_{(t_1,t_2,t_3,t_4,t_5)\in L_{43}}\frac{\omega\left(\frac{\theta-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1t_2t_3t_4t_5^2}dt_5dt_4dt_3dt_2dt_1\right)\\ +&\left(\int_{(t_1,t_2,t_3,t_4,t_5)\in L_{44}}\frac{\omega\left(\frac{\theta-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1t_2t_3t_4t_5^2}dt_5dt_4dt_3dt_2dt_1\right)\\ +&\left(\int_{(t_1,t_2,t_3)\in L_{45}}\frac{\omega\left(\frac{\theta-t_1-t_2-t_3}{t_3}\right)}{t_1t_2t_3^2}dt_3dt_2dt_1\right)\end{aligned}$$

$$
+\left(\int_{(t_1,t_2,t_3,t_4,t_5)\in L_{46}} \frac{\omega\left(\frac{\theta-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1t_2t_3t_4t_5^2} dt_5dt_4dt_3dt_2dt_1\right)
$$

$$
+\left(\int_{\frac{1-\theta}{2}}^{2\theta-1} \frac{\omega\left(\frac{\theta-t_1}{t_1}\right)}{t_1^2} dt_1\right)+\left(\int_{\frac{3}{2}-2\theta}^{\frac{\theta}{2}} \frac{\omega\left(\frac{\theta-t_1}{t_1}\right)}{t_1^2} dt_1\right),
$$

where

$$
\begin{aligned}
L_{41}(t_1,t_2,t_3):=\Big\{&\nu_1(\theta)\leqslant t_1<\frac{1-\theta}{2},\ t_1\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta),\\
&\nu_1(\theta)\leqslant t_2<\theta-\frac{1}{2},\ \boldsymbol{T}_2\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta),\\
&\nu_1(\theta)\leqslant t_3<\min\left(t_2,\frac{1}{2}(\theta-t_1-t_2)\right),\ \boldsymbol{T}_3\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta),\\
&\text{Neither (D1) nor (D2) holds}\Big\},\\
L_{42}(t_1,t_2,t_3,t_4):=\Big\{&\nu_1(\theta)\leqslant t_1<\frac{1-\theta}{2},\ t_1\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta),\\
&\nu_1(\theta)\leqslant t_2<\theta-\frac{1}{2},\ \boldsymbol{T}_2\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta),\\
&\nu_1(\theta)\leqslant t_3<\min\left(t_2,\frac{1}{2}(\theta-t_1-t_2)\right),\ \boldsymbol{T}_3\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta),\\
&\text{Neither (D1) nor (D2) holds,}\\
&t_3<t_4<\frac{1}{2}(\theta-t_1-t_2-t_3),\ \boldsymbol{T}_4\in\mathcal{J}(\theta)\cup\mathcal{K}(\theta)\Big\},\\
L_{43}(t_1,t_2,t_3,t_4,t_5):=\Big\{&\nu_1(\theta)\leqslant t_1<\frac{1-\theta}{2},\ t_1\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta),\\
&\nu_1(\theta)\leqslant t_2<\theta-\frac{1}{2},\ \boldsymbol{T}_2\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta),\\
&\nu_1(\theta)\leqslant t_3<\min\left(t_2,\frac{1}{2}(\theta-t_1-t_2)\right),\ \boldsymbol{T}_3\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta),\\
&\text{(D1) holds,}\\
&\nu_1(\theta)\leqslant t_4<\min\left(t_3,\frac{1}{2}(\theta-t_1-t_2-t_3)\right),\ \boldsymbol{T}_4\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta),\\
&\nu_1(\theta)\leqslant t_5<\min\left(t_4,\frac{1}{2}(\theta-t_1-t_2-t_3-t_4)\right),\ \boldsymbol{T}_5\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)\Big\},\\
L_{44}(t_1,t_2,t_3,t_4,t_5):=\Big\{&\nu_1(\theta)\leqslant t_1<\frac{1-\theta}{2},\ t_1\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta),\\
&\nu_1(\theta)\leqslant t_2<\theta-\frac{1}{2},\ \boldsymbol{T}_2\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta),\\
&\nu_1(\theta)\leqslant t_3<\min\left(t_2,\frac{1}{2}(\theta-t_1-t_2)\right),\ \boldsymbol{T}_3\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta),\\
&\text{(D1) does not hold, but (D2) holds,}\\
&\nu_2(\theta)\leqslant t_4<\min\left(t_3,\frac{1}{2}(\theta-t_1-t_2-t_3)\right),\ \boldsymbol{T}_4\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta),\\
&\nu_2(\theta)\leqslant t_5<\min\left(t_4,\frac{1}{2}(\theta-t_1-t_2-t_3-t_4)\right),\ \boldsymbol{T}_5\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
L_{45}(t_1,t_2,t_3) := \Bigg\{ & \nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2},\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
& \theta - \frac{1}{2} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right),\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
& \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right),\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
& \text{Neither (D1) nor (D2) holds}\Bigg\}, \\
L_{46}(t_1,t_2,t_3,t_4,t_5) := \Bigg\{ & \nu_1(\theta) \leqslant t_1 < \frac{1-\theta}{2},\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
& \theta - \frac{1}{2} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right),\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
& \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right),\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
& \text{Either (D1) or (D2) holds}, \\
& \nu_1(\theta) \leqslant t_4 < \min\left(t_3, \frac{1}{2}(\theta - t_1 - t_2 - t_3)\right),\ \boldsymbol{T}_4 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta), \\
& \nu_1(\theta) \leqslant t_5 < \min\left(t_4, \frac{1}{2}(\theta - t_1 - t_2 - t_3 - t_4)\right),\ \boldsymbol{T}_5 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)\Bigg\}.
\end{aligned}
$$

Numerical calculation gives that

$$
\int_{\phi_4}^{\phi_5} \theta T(\theta) d\theta < 0.003636. \tag{28}
$$

4.5. **Case 5.** $\phi_5 \leqslant \theta \leqslant \phi_6$**.** In this case we have $\nu_1(\theta) > \nu_2(\theta) \geqslant 0$ and

$$
\nu_2(\theta) < \nu_1(\theta) < \theta - \frac{1}{2} < \tau(\theta) < 2\theta - 1 \leqslant \frac{3}{2} - 2\theta < \frac{9}{10} - \theta < \frac{\theta}{2}.
$$

By a similar process as in **Case 3**, (6) becomes that

$$
\begin{aligned}
S(\theta) &= S\left(\mathcal{A}, x^{\nu_1(\theta)}\right) - \sum_{\nu_1(\theta) \leqslant t_1 < \frac{\theta}{2}} S\left(\mathcal{A}_{p_1}, p_1\right) \\
&= S\left(\mathcal{A}, x^{\nu_1(\theta)}\right) - \sum_{\nu_1(\theta) \leqslant t_1 < \theta - \frac{1}{2}} S\left(\mathcal{A}_{p_1}, p_1\right) - \sum_{\theta - \frac{1}{2} \leqslant t_1 < \tau(\theta)} S\left(\mathcal{A}_{p_1}, p_1\right) \\
&\quad - \sum_{\tau(\theta) \leqslant t_1 < 2\theta - 1} S\left(\mathcal{A}_{p_1}, p_1\right) - \sum_{2\theta - 1 \leqslant t_1 < \frac{3}{2} - 2\theta} S\left(\mathcal{A}_{p_1}, p_1\right) - \sum_{\frac{3}{2} - 2\theta \leqslant t_1 < \frac{\theta}{2}} S\left(\mathcal{A}_{p_1}, p_1\right) \\
&= S_{51} - S_{521} - S_{522} - S_{523} - S_{524} - S_{525}.
\end{aligned} \tag{29}
$$

By Lemma 3.1, we can give asymptotic formulas for S_{522} and S_{524}. For S_{521}, using Buchstab's identity twice we have

$$
\begin{aligned}
S_{521} &= \sum_{\nu_1(\theta) \leqslant t_1 < \theta - \frac{1}{2}} S\left(\mathcal{A}_{p_1}, p_1\right) \\
&= \sum_{\nu_1(\theta) \leqslant t_1 < \theta - \frac{1}{2}} S\left(\mathcal{A}_{p_1}, x^{\nu_1(\theta)}\right) - \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \theta - \frac{1}{2} \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \in \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
&\quad - \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \theta - \frac{1}{2} \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2}, x^{\nu_1(\theta)}\right) + \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \theta - \frac{1}{2} \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \in \mathcal{J}(\theta) \cup \mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right)
\end{aligned}
$$

$$+\sum_{\substack{\nu_1(\theta)\leqslant t_1<\theta-\frac{1}{2}\\ \nu_1(\theta)\leqslant t_2<\min\left(t_1,\frac{1}{2}(\theta-t_1)\right)\\ \boldsymbol{T}_2\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)\\ \nu_1(\theta)\leqslant t_3<\min\left(t_2,\frac{1}{2}(\theta-t_1-t_2)\right)\\ \boldsymbol{T}_3\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta)}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right)$$

$$= S_{531}-S_{532}-S_{533}+S_{534}+S_{535}. \tag{30}$$

We have asymptotic formulas for S_{531} by Lemma 3.5 and for S_{532} and S_{534} by Lemma 3.1. We remark that we have

$$t_2<t_1<\theta-\frac{1}{2}<\frac{9}{10}-\theta.$$

By Lemma 3.2, we know that S_{533} also has an asymptotic formula. For S_{535}, we can perform a straightforward decomposition by using Buchstab's identity twice if any of the following 2 conditions holds:

(E1) $\boldsymbol{T}_3'\in\mathcal{L}_1(\theta)$ and $t_1+t_2+2t_3-\boldsymbol{T}_3'\in\mathcal{L}_2(\theta)$;

(E2) $t_1+t_2+2t_3\in\mathcal{N}(\theta)$.

Note that in **Case 5** both $\mathcal{M}_1(\theta)$ and $\mathcal{M}_2(\theta)$ are invalid. By a similar discussion as in **Case 3**, we can get three corresponding loss integrals for S_{521}. We discard S_{523} and S_{525} and use the reversed Buchstab's identity to handle part of S_{521}. By the discussion in [[3], Section 5(v)], we know that there is an extra saving integral

$$\int_{\frac{1}{6}}^{\frac{1-\theta}{2}}\int_{\max\left(3\theta-\frac{3}{2}-t_1,t_1\right)}^{1-\theta-t_1}\frac{1}{t_1^2t_2^2}dt_2dt_1. \tag{31}$$

Combining the above loss integrals, the value of $T(\theta)$ in **Case 5** is less than

$$\begin{aligned}
\frac{1}{\theta}&+\left(\int_{(t_1,t_2,t_3)\in L_{51}}\frac{\omega\left(\frac{\theta-t_1-t_2-t_3}{t_3}\right)}{t_1t_2t_3^2}dt_3dt_2dt_1\right)\\
&-\left(\int_{(t_1,t_2,t_3,t_4)\in L_{52}}\frac{\omega\left(\frac{\theta-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1t_2t_3t_4^2}dt_4dt_3dt_2dt_1\right)\\
&+\left(\int_{(t_1,t_2,t_3,t_4,t_5)\in L_{53}}\frac{\omega\left(\frac{\theta-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1t_2t_3t_4t_5^2}dt_5dt_4dt_3dt_2dt_1\right)\\
&+\left(\int_{(t_1,t_2,t_3,t_4,t_5)\in L_{54}}\frac{\omega\left(\frac{\theta-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1t_2t_3t_4t_5^2}dt_5dt_4dt_3dt_2dt_1\right)\\
&+\left(\int_{\tau(\theta)}^{2\theta-1}\frac{\omega\left(\frac{\theta-t_1}{t_1}\right)}{t_1^2}dt_1\right)+\left(\int_{\frac{3}{2}-2\theta}^{\frac{\theta}{2}}\frac{\omega\left(\frac{\theta-t_1}{t_1}\right)}{t_1^2}dt_1\right)\\
&-\left(\int_{\frac{1}{6}}^{\frac{1-\theta}{2}}\int_{\max\left(3\theta-\frac{3}{2}-t_1,t_1\right)}^{1-\theta-t_1}\frac{1}{t_1^2t_2^2}dt_2dt_1\right),
\end{aligned}$$

where

$$\begin{aligned}
L_{51}(t_1,t_2,t_3):=\Big\{&\nu_1(\theta)\leqslant t_1<\theta-\frac{1}{2},\ t_1\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta),\\
&\nu_1(\theta)\leqslant t_2<\min\left(t_1,\frac{1}{2}(\theta-t_1)\right),\ \boldsymbol{T}_2\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta),\\
&\nu_1(\theta)\leqslant t_3<\min\left(t_2,\frac{1}{2}(\theta-t_1-t_2)\right),\ \boldsymbol{T}_3\notin\mathcal{J}(\theta)\cup\mathcal{K}(\theta),\\
&\text{Neither (E1) nor (E2) holds}\Big\},
\end{aligned}$$

$$
\begin{aligned}
L_{52}(t_1,t_2,t_3,t_4) := \Bigg\{ & \nu_1(\theta) \leqslant t_1 < \theta - \frac{1}{2},\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),\\
& \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right),\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),\\
& \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right),\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),\\
& \text{Neither (E1) nor (E2) holds,}\\
& t_3 < t_4 < \frac{1}{2}(\theta - t_1 - t_2 - t_3),\ \boldsymbol{T}_4 \in \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \Bigg\},\\
L_{53}(t_1,t_2,t_3,t_4,t_5) := \Bigg\{ & \nu_1(\theta) \leqslant t_1 < \theta - \frac{1}{2},\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),\\
& \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right),\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),\\
& \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right),\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),\\
& \text{(E1) holds,}\\
& \nu_1(\theta) \leqslant t_4 < \min\left(t_3, \frac{1}{2}(\theta - t_1 - t_2 - t_3)\right),\ \boldsymbol{T}_4 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),\\
& \nu_1(\theta) \leqslant t_5 < \min\left(t_4, \frac{1}{2}(\theta - t_1 - t_2 - t_3 - t_4)\right),\ \boldsymbol{T}_5 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \Bigg\},\\
L_{54}(t_1,t_2,t_3,t_4,t_5) := \Bigg\{ & \nu_1(\theta) \leqslant t_1 < \theta - \frac{1}{2},\ t_1 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),\\
& \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right)\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),\\
& \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right),\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),\\
& \text{(E1) does not hold, but (E2) holds,}\\
& \nu_2(\theta) \leqslant t_4 < \min\left(t_3, \frac{1}{2}(\theta - t_1 - t_2 - t_3)\right),\ \boldsymbol{T}_4 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta),\\
& \nu_2(\theta) \leqslant t_5 < \min\left(t_4, \frac{1}{2}(\theta - t_1 - t_2 - t_3 - t_4)\right),\ \boldsymbol{T}_5 \notin \mathcal{J}(\theta) \cup \mathcal{K}(\theta) \Bigg\}.
\end{aligned}
$$

Numerical calculation gives that

$$
\int_{\phi_5}^{\phi_6} \theta T(\theta) d\theta < 0.011398. \tag{32}
$$

4.6. **Case 6.** $\phi_6 \leqslant \theta \leqslant \phi_9$**.** In this case we have $\nu_1(\theta) > 0$, $\nu_2(\theta) \leqslant 0$ and

$$
\nu_1(\theta) < \theta - \frac{1}{2} < \tau(\theta) < \frac{9}{10} - \theta < \frac{\theta}{2}.
$$

Note that all of $\mathcal{K}(\theta)$, $\mathcal{M}_1(\theta)$, $\mathcal{M}_2(\theta)$ and $\mathcal{N}(\theta)$ are invalid in this case. By a similar process as in **Case 5**, (6) becomes that

$$
\begin{aligned}
S(\theta) &= S\left(\mathcal{A}, x^{\nu_1(\theta)}\right) - \sum_{\nu_1(\theta) \leqslant t_1 < \frac{\theta}{2}} S\left(\mathcal{A}_{p_1}, p_1\right)\\
&= S\left(\mathcal{A}, x^{\nu_1(\theta)}\right) - \sum_{\nu_1(\theta) \leqslant t_1 < \theta - \frac{1}{2}} S\left(\mathcal{A}_{p_1}, p_1\right) - \sum_{\theta - \frac{1}{2} \leqslant t_1 < \tau(\theta)} S\left(\mathcal{A}_{p_1}, p_1\right) - \sum_{\tau(\theta) \leqslant t_1 < \frac{\theta}{2}} S\left(\mathcal{A}_{p_1}, p_1\right)\\
&= S_{61} - S_{621} - S_{622} - S_{623}.
\end{aligned} \tag{33}
$$

We have an asymptotic formula for S_{622} by Lemma 2.1. We discard the whole of S_{623} and make further decompositions on S_{621}. Using Buchstab's identity, we have

$$
\begin{aligned}
S_{621} =& \sum_{\nu_1(\theta) \leqslant t_1 < \theta - \frac{1}{2}} S\left(\mathcal{A}_{p_1}, p_1\right) \\
=& \sum_{\nu_1(\theta) \leqslant t_1 < \theta - \frac{1}{2}} S\left(\mathcal{A}_{p_1}, x^{\nu_1(\theta)}\right) - \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \theta - \frac{1}{2} \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \in \mathcal{J}(\theta)}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
&- \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \theta - \frac{1}{2} \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta)}} S\left(\mathcal{A}_{p_1 p_2}, x^{\nu_1(\theta)}\right) + \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \theta - \frac{1}{2} \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \in \mathcal{J}(\theta)}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right) \\
&+ \sum_{\substack{\nu_1(\theta) \leqslant t_1 < \theta - \frac{1}{2} \\ \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right) \\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta) \\ \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right) \\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta)}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right) \\
=& S_{631} - S_{632} - S_{633} + S_{634} + S_{635}. \qquad (34)
\end{aligned}
$$

Since we have

$$t_2 < t_1 < \theta - \frac{1}{2} < \frac{9}{10} - \theta,$$

we have asymptotic formulas for S_{631} and S_{633} by Lemma 3.2. We also have asymptotic formulas for S_{632} and S_{634} by Lemma 3.1. For S_{635}, we can perform a straightforward decomposition by using Buchstab's identity twice if the following condition holds:

(F1) $\boldsymbol{T}_3' \in \mathcal{L}_1(\theta)$ and $t_1 + t_2 + 2t_3 - \boldsymbol{T}_3' \in \mathcal{L}_2(\theta)$.

By a similar decomposing process as (15), we can get a five-dimensional sum. For this sum, we can even use Buchstab's identity twice more if the following condition holds:

(F2) $\boldsymbol{T}_5' \in \mathcal{L}_1(\theta)$ and $t_1 + t_2 + t_3 + t_4 + 2t_5 - \boldsymbol{T}_5' \in \mathcal{L}_2(\theta)$.

Of course we will obtain a seven-dimensional sum after this decomposition. After using a reversed Buchstab's identity on the remaining part of S_{635} and combining all the losses, we can give an upper bound for $T(\theta)$ in **Case 6**.

$$
\begin{aligned}
T(\theta) \leqslant \frac{1}{\theta} &+ \left(\int_{(t_1,t_2,t_3) \in L_{61}} \frac{\omega\left(\frac{\theta - t_1 - t_2 - t_3}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 dt_2 dt_1\right) \\
&- \left(\int_{(t_1,t_2,t_3,t_4) \in L_{62}} \frac{\omega\left(\frac{\theta - t_1 - t_2 - t_3 - t_4}{t_4}\right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1\right) \\
&+ \left(\int_{(t_1,t_2,t_3,t_4,t_5) \in L_{63}} \frac{\omega\left(\frac{\theta - t_1 - t_2 - t_3 - t_4 - t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1\right) \\
&+ \left(\int_{(t_1,t_2,t_3,t_4,t_5,t_6,t_7) \in L_{64}} \frac{\omega\left(\frac{\theta - t_1 - t_2 - t_3 - t_4 - t_5 - t_6 - t_7}{t_7}\right)}{t_1 t_2 t_3 t_4 t_5 t_6 t_7^2} dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1\right) \\
&+ \left(\int_{\tau(\theta)}^{\frac{\theta}{2}} \frac{\omega\left(\frac{\theta - t_1}{t_1}\right)}{t_1^2} dt_1\right),
\end{aligned}
$$

where

$$
\begin{aligned}
L_{61}(t_1,t_2,t_3) := \Big\{ & \nu_1(\theta) \leqslant t_1 < \theta - \frac{1}{2},\ t_1 \notin \mathcal{J}(\theta), \\
& \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right),\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta), \\
& \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right),\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta), \\
& \text{(F1) does not hold}\Big\}, \\
L_{62}(t_1,t_2,t_3,t_4) := \Big\{ & \nu_1(\theta) \leqslant t_1 < \theta - \frac{1}{2},\ t_1 \notin \mathcal{J}(\theta), \\
& \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right),\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta), \\
& \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right),\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta), \\
& \text{(F1) does not hold}, \\
& t_3 < t_4 < \frac{1}{2}(\theta - t_1 - t_2 - t_3),\ \boldsymbol{T}_4 \in \mathcal{J}(\theta)\Big\}, \\
L_{63}(t_1,t_2,t_3,t_4,t_5) := \Big\{ & \nu_1(\theta) \leqslant t_1 < \theta - \frac{1}{2},\ t_1 \notin \mathcal{J}(\theta), \\
& \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right),\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta), \\
& \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right),\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta), \\
& \text{(F1) holds}, \\
& \nu_1(\theta) \leqslant t_4 < \min\left(t_3, \frac{1}{2}(\theta - t_1 - t_2 - t_3)\right),\ \boldsymbol{T}_4 \notin \mathcal{J}(\theta), \\
& \nu_1(\theta) \leqslant t_5 < \min\left(t_4, \frac{1}{2}(\theta - t_1 - t_2 - t_3 - t_4)\right),\ \boldsymbol{T}_5 \notin \mathcal{J}(\theta), \\
& \text{(F2) does not hold}\Big\}, \\
L_{64}(t_1,t_2,t_3,t_4,t_5,t_6,t_7) := \Big\{ & \nu_1(\theta) \leqslant t_1 < \theta - \frac{1}{2},\ t_1 \notin \mathcal{J}(\theta), \\
& \nu_1(\theta) \leqslant t_2 < \min\left(t_1, \frac{1}{2}(\theta - t_1)\right),\ \boldsymbol{T}_2 \notin \mathcal{J}(\theta), \\
& \nu_1(\theta) \leqslant t_3 < \min\left(t_2, \frac{1}{2}(\theta - t_1 - t_2)\right),\ \boldsymbol{T}_3 \notin \mathcal{J}(\theta), \\
& \text{(F1) holds}, \\
& \nu_1(\theta) \leqslant t_4 < \min\left(t_3, \frac{1}{2}(\theta - t_1 - t_2 - t_3)\right),\ \boldsymbol{T}_4 \notin \mathcal{J}(\theta), \\
& \nu_1(\theta) \leqslant t_5 < \min\left(t_4, \frac{1}{2}(\theta - t_1 - t_2 - t_3 - t_4)\right),\ \boldsymbol{T}_5 \notin \mathcal{J}(\theta), \\
& \text{(F2) holds}, \\
& \nu_1(\theta) \leqslant t_6 < \min\left(t_5, \frac{1}{2}(\theta - t_1 - t_2 - t_3 - t_4 - t_5)\right),\ \boldsymbol{T}_6 \notin \mathcal{J}(\theta), \\
& \nu_1(\theta) \leqslant t_7 < \min\left(t_6, \frac{1}{2}(\theta - t_1 - t_2 - t_3 - t_4 - t_5 - t_6)\right),\ \boldsymbol{T}_7 \notin \mathcal{J}(\theta)\Big\}.
\end{aligned}
$$

Numerical calculation gives that

$$\int_{\phi_6}^{\phi_9} \theta T(\theta) d\theta < 0.097444. \tag{35}$$

4.7. **Case 7.** $\phi_9 \leqslant \theta \leqslant \phi_{14}$. Just as in [3], we have

$$\int_{\phi_9}^{\phi_{13}} \theta T(\theta) d\theta < 0.17597 \tag{36}$$

and

$$\int_{\phi_{13}}^{\phi_{14}} \theta T(\theta) d\theta < \frac{5}{2}\left(0.7437^2 - \left(\frac{25}{32}\right)^2\right). \tag{37}$$

5. Proof of Theorem 1.1

By (11), (19), (23), (28), (32), (35), (36) and (37), we have

$$\begin{aligned}\int_{0.6-\varepsilon}^{0.7437} \theta T(\theta) d\theta < {} & 0.008349 + 0.003204 + 0.00852 + 0.003636 + 0.011398 \\ & + 0.097444 + 0.17597 + \frac{5}{2}\left(0.7437^2 - \left(\frac{25}{32}\right)^2\right) \\ < {} & 0.3998 < 0.4.\end{aligned}$$

Now (5) holds and the proof of Theorem 1.1 is completed.

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