ON THE PIATETSKI–SHAPIRO PRIME NUMBER THEOREM

RUNBO LI

ABSTRACT. The author sharpens a result of Rivat and Wu (2000), showing that for sufficiently large n, there are infinitely many primes of the form $[n^c]$ for $1 < c < \frac{211}{178}$.

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1. INTRODUCTION

The Euler's conjecture, which states that there are infinitely many primes of the form $n^2 + 1$, is one of Landau's problems on prime numbers. There are several ways to attack this conjecture. One way is to relax the number of prime factors of $f(n)$, and the best result in this way is due to Iwaniec [4]. Building on the previous work of Richert [11], he showed that for any irreducible polynomial $f(n) = an^2 + bn + c$ with $a > 0$ and $c \equiv 1(\bmod 2)$, there are infinitely many x such that $f(x)$ has at most 2 prime factors.

Another possible way is to consider the degree of the polynomial. In 1953, Piatetski–Shapiro [10] has proposed to investigate the prime numbers of the form $[n^c]$, where $c > 1$ and $[n^c]$ denotes the integer part of n^c. Clearly $[n^c]$ can be regarded as "polynomials of degree c". Define

$$\pi_c(x) := \left|\{n \leqslant x : [n^c] \text{ is a prime number}\}\right|,$$

then he has shown that $\pi_c(x) \sim x(c \log x)^{-1}$ holds for any $1 < c < \frac{12}{11} \approx 1.0909$ as $x \to \infty$. This range has been improved by many authors, and the best record now is due to Rivat and Sargos [13], where they proved the above asymptotic formula holds for any $1 < c < \frac{2817}{2426} \approx 1.1612$.

In 1992, Rivat [12] first introduced a sieve method into this problem. He established a lower bound with correct order (instead of an asymptotic formula) with $1 < c < \frac{7}{6} \approx 1.1616$. After this, many improvements were made and the range of c was enlarged successively to

$$1 < c < \frac{20}{17} \approx 1.1765,\ 1 < c < \frac{13}{11} \approx 1.1818,\ 1 < c < \frac{45}{38} \approx 1.1842 \text{ and } 1 < c < \frac{243}{205} \approx 1.18536$$

by Jia [6] (and Baker, Harman and Rivat [1]), Jia [5], Kumchev [7] and Rivat and Wu [14] respectively. In this paper, we obtain the following result.

Theorem 1.1. *For sufficiently large x and $1 < c < \frac{211}{178} \approx 1.18539$, we have $\pi_c(x) \gg x(c \log x)^{-1}$.*

Throughout this paper, we always suppose that x is a sufficiently large integer, γ and θ_0–θ_6 are positive numbers which will be fixed later. Let $\frac{37}{44} < \gamma < \frac{28}{33}$ and $c = \frac{1}{\gamma}$. The letter p, with or without subscript, is reserved for prime numbers. We define the sets $\mathcal{A}$ and $\mathcal{B}$ as

$$\mathcal{A} = \{m : m = [n^c],\ x \leqslant n^c < 2x\}, \quad \mathcal{B} = \{n : x \leqslant n < 2x\},$$

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and we put

$$\mathcal{A}_d=\{a: ad\in\mathcal{A}\},\quad \mathcal{B}_d=\{b: bd\in\mathcal{A}\},\quad P(z)=\prod_{p<z}p,\quad S(\mathcal{A},z)=\sum_{\substack{a\in\mathcal{A}\\(a,P(z))=1}}1,\quad S(\mathcal{B},z)=\sum_{\substack{b\in\mathcal{B}\\(b,P(z))=1}}1.$$

Then we only need to show that $S\left(\mathcal{A},(2x)^{\frac{1}{2}}\right)>0$. Our aim is to show that the sparser set $\mathcal{A}$ contains the expected proportion of primes compared to the bigger set $\mathcal{B}$, which requires us to decompose $S\left(\mathcal{A},(2x)^{\frac{1}{2}}\right)$ and prove asymptotic formulas of the form

$$S(\mathcal{A},z)=(1+o(1))x^{\gamma-1}\left(2^{\gamma}-1\right)S(\mathcal{B},z) \tag{1}$$

for some parts of it, and drop the other positive parts. The asymptotic formulas will be given in the next section.

2. Sieve asymptotic formulas

In this section we provide some asymptotic formulas for sieve functions. Let $\omega(u)$ denote the Buchstab function determined by the following differential–difference equation

$$\begin{cases}\omega(u)=\frac{1}{u}, & 1\leqslant u\leqslant 2,\\ (u\omega(u))'=\omega(u-1), & u\geqslant 2.\end{cases}$$

Following [14] directly, we set $\gamma=\frac{178}{211}$, $\theta_0=6\gamma-5$, $\theta_1=1-\gamma$, $\theta_2=\frac{61\gamma-49}{11}$, $\theta_3=3-3\gamma$, $\theta_4=3\gamma-2$, $\theta_5=\frac{60-61\gamma}{11}$, $\theta_6=\gamma$ and let $p_j=x^{t_j}$. We define the asymptotic region I as

$$\begin{aligned}I(m,n):=\ &\{\theta_1\leqslant m<\theta_2 \text{ or } \theta_3\leqslant m<\theta_4 \text{ or } \theta_5\leqslant m<\theta_6 \text{ or}\\ &\theta_1\leqslant m+n<\theta_2 \text{ or } \theta_3\leqslant m+n<\theta_4 \text{ or } \theta_5\leqslant m+n<\theta_6\}.\end{aligned}$$

Lemma 2.1. *We can give an asymptotic formula for*

$$\sum_{t_1\cdots t_n}S\left(\mathcal{A}_{p_1\cdots p_n},x^{\theta_0}\right)$$

if we have $t_1+\ldots+t_n<\theta_4$.

Lemma 2.2. *We can give an asymptotic formula for*

$$\sum_{t_1\cdots t_n}S\left(\mathcal{A}_{p_1\cdots p_n},p_n\right)$$

if we can group $(t_1,\ldots,t_n)$ into $(m,n)\in I$.

3. The final decomposition

Before decomposing, we define non–overlapping regions U_1–U_3 as

$$\begin{aligned}U_1(m,n)&:=\ \{(m,n)\notin I,\ m+2n<\theta_4\}\\ U_2(m,n)&:=\ \left\{(m,n)\notin I,\ m+2n\geqslant\theta_4,\ \frac{1-m-n}{n}<2\right\},\\ U_3(m,n)&:=\ \left\{(m,n)\notin I,\ m+2n\geqslant\theta_4,\ \frac{1-m-n}{n}\geqslant 2\right\}.\end{aligned}$$

We shall apply different techniques to the different regions above. By Buchstab's identity, we have

$$\begin{aligned}S\left(\mathcal{A},(2x)^{\frac{1}{2}}\right)&=S\left(\mathcal{A},x^{\theta_0}\right)-\sum_{\theta_0\leqslant t_1<\frac{1}{2}}S\left(\mathcal{A}_{p_1},x^{\theta_0}\right)+\sum_{\substack{\theta_0\leqslant t_1<\frac{1}{2}\\ \theta_0\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)}}S\left(\mathcal{A}_{p_1p_2},p_2\right)\\ &=S\left(\mathcal{A},x^{\theta_0}\right)-\sum_{\theta_0\leqslant t_1<\frac{1}{2}}S\left(\mathcal{A}_{p_1},x^{\theta_0}\right)+\sum_{\substack{\theta_0\leqslant t_1<\frac{1}{2}\\ \theta_0\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in I}}S\left(\mathcal{A}_{p_1p_2},p_2\right)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\substack{\theta_0\leqslant t_1<\frac{1}{2}\\ \theta_0\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in U_1}} S\left(\mathcal{A}_{p_1p_2},p_2\right)+\sum_{\substack{\theta_0\leqslant t_1<\frac{1}{2}\\ \theta_0\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in U_2}} S\left(\mathcal{A}_{p_1p_2},p_2\right)\\
&+\sum_{\substack{\theta_0\leqslant t_1<\frac{1}{2}\\ \theta_0\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in U_3}} S\left(\mathcal{A}_{p_1p_2},p_2\right)\\
&=S_1-S_2+S_I+S_{U1}+S_{U2}+S_{U3}.
\end{aligned} \tag{2}
$$

By Lemma 2.1 and Lemma 2.2, we can give asymptotic formulas for S_1, S_2 and S_I. For S_{U1}, we can use Buchstab's identity twice more to get

$$
\begin{aligned}
S_{U1}=\sum_{\substack{\theta_0\leqslant t_1<\frac{1}{2}\\ \theta_0\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in U_1}} S\left(\mathcal{A}_{p_1p_2},p_2\right)=&\sum_{\substack{\theta_0\leqslant t_1<\frac{1}{2}\\ \theta_0\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in U_1}} S\left(\mathcal{A}_{p_1p_2},x^{\theta_0}\right)\\
&-\sum_{\substack{\theta_0\leqslant t_1<\frac{1}{2}\\ \theta_0\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in U_1\\ \theta_0\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ (t_1,t_2,t_3)\text{ can be partitioned into }(m,n)\in I}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right)\\
&-\sum_{\substack{\theta_0\leqslant t_1<\frac{1}{2}\\ \theta_0\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in U_1\\ \theta_0\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ (t_1,t_2,t_3)\text{ cannot be partitioned into }(m,n)\in I}} S\left(\mathcal{A}_{p_1p_2p_3},x^{\theta_0}\right)\\
&+\sum_{\substack{\theta_0\leqslant t_1<\frac{1}{2}\\ \theta_0\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in U_1\\ \theta_0\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ (t_1,t_2,t_3)\text{ cannot be partitioned into }(m,n)\in I\\ \theta_0\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)\\ (t_1,t_2,t_3,t_4)\text{ can be partitioned into }(m,n)\in I}} S\left(\mathcal{A}_{p_1p_2p_3p_4},p_4\right)\\
&+\sum_{\substack{\theta_0\leqslant t_1<\frac{1}{2}\\ \theta_0\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in U_1\\ \theta_0\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ (t_1,t_2,t_3)\text{ cannot be partitioned into }(m,n)\in I\\ \theta_0\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)\\ (t_1,t_2,t_3,t_4)\text{ cannot be partitioned into }(m,n)\in I}} S\left(\mathcal{A}_{p_1p_2p_3p_4},p_4\right)\\
&=S_{U11}-S_{U12}-S_{U13}+S_{U14}+S_{U15}.
\end{aligned} \tag{3}
$$

We can give asymptotic formulas for S_{U11}–S_{U14}. For S_{U15} we can perform Buchstab's identity more times to make savings, but we choose to discard all of it for the sake of simplicity. Combining the above cases, we get a loss from S_{U1} of

$$
\begin{aligned}
&\int_{\theta_0}^{\frac{1}{2}}\int_{\theta_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\theta_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\theta_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)} \mathbb{1}_{(t_1,t_2,t_3,t_4)\in U_{15}}\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1t_2t_3t_4^2}dt_4dt_3dt_2dt_1\\
&<0.001624,
\end{aligned} \tag{4}
$$

where

$$U_{15}(t_1,t_2,t_3,t_4) := \Big\{(t_1,t_2)\in U_1,\ \theta_0 \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \\ (t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in I, \\ \theta_0 \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\ (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in I\Big\}.$$

For S_{U2}, we cannot decompose further but have to discard the whole region giving the loss

$$\int_{\theta_0}^{\frac{1}{2}} \int_{\theta_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \mathbb{1}_{(t_1,t_2)\in U_2} \frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1 t_2^2} dt_2 dt_1 < 0.412666. \tag{5}$$

For S_{U3} we cannot use Buchstab's identity in a straightforward manner, but we can use Buchstab's identity in reverse to make almost–primes visible. The details of using Buchstab's identity in reverse are similar to those in [8] and [9]. By using Buchstab's identity in reverse twice, we have

$$\begin{aligned}
S_{U3} =& \sum_{\substack{\theta_0 \leqslant t_1 < \frac{1}{2} \\ \theta_0 \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2)\in U_3}} S\left(\mathcal{A}_{p_1p_2}, p_2\right) \\
=& \sum_{\substack{\theta_0 \leqslant t_1 < \frac{1}{2} \\ \theta_0 \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2)\in U_3}} S\left(\mathcal{A}_{p_1p_2}, \left(\frac{2x}{p_1p_2}\right)^{\frac{1}{2}}\right) \\
&+ \sum_{\substack{\theta_0 \leqslant t_1 < \frac{1}{2} \\ \theta_0 \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2)\in U_3 \\ t_2 < t_3 < \frac{1}{2}(1-t_1-t_2)}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right) \\
=& \sum_{\substack{\theta_0 \leqslant t_1 < \frac{1}{2} \\ \theta_0 \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2)\in U_3}} S\left(\mathcal{A}_{p_1p_2}, \left(\frac{2x}{p_1p_2}\right)^{\frac{1}{2}}\right) \\
&+ \sum_{\substack{\theta_0 \leqslant t_1 < \frac{1}{2} \\ \theta_0 \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2)\in U_3 \\ t_2 < t_3 < \frac{1}{2}(1-t_1-t_2) \\ (t_1,t_2,t_3) \text{ can be partitioned into } (m,n)\in I}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right) \\
&+ \sum_{\substack{\theta_0 \leqslant t_1 < \frac{1}{2} \\ \theta_0 \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2)\in U_3 \\ t_2 < t_3 < \frac{1}{2}(1-t_1-t_2) \\ (t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in I}} S\left(\mathcal{A}_{p_1p_2p_3}, \left(\frac{2x}{p_1p_2p_3}\right)^{\frac{1}{2}}\right) \\
&+ \sum_{\substack{\theta_0 \leqslant t_1 < \frac{1}{2} \\ \theta_0 \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2)\in U_3 \\ t_2 < t_3 < \frac{1}{2}(1-t_1-t_2) \\ (t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in I \\ t_3 < t_4 < \frac{1}{2}(1-t_1-t_2-t_3) \\ (t_1,t_2,t_3,t_4) \text{ can be partitioned into } (m,n)\in I}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\substack{\theta_0\leqslant t_1<\frac{1}{2}\\ \theta_0\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in U_3\\ t_2<t_3<\frac{1}{2}(1-t_1-t_2)\\ (t_1,t_2,t_3)\text{ cannot be partitioned into }(m,n)\in I\\ t_3<t_4<\frac{1}{2}(1-t_1-t_2-t_3)\\ (t_1,t_2,t_3,t_4)\text{ cannot be partitioned into }(m,n)\in I}} S\left(\mathcal{A}_{p_1p_2p_3p_4},p_4\right)\\
&= S_{U31}+S_{U32}+S_{U33}+S_{U34}+S_{U35}. \qquad (6)
\end{aligned}
$$

We can give asymptotic formulas for S_{U32} and S_{U34}, hence we can subtract them from the loss. In this way we obtain a loss from S_{U3} of

$$
\begin{aligned}
&\left(\int_{\theta_0}^{\frac{1}{2}}\int_{\theta_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)} \mathbb{1}_{(t_1,t_2)\in U_3}\frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1t_2^2}dt_2dt_1\right)\\
&-\left(\int_{\theta_0}^{\frac{1}{2}}\int_{\theta_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{t_2}^{\frac{1-t_1-t_2}{2}} \mathbb{1}_{(t_1,t_2,t_3)\in U_{32}}\frac{\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_1t_2t_3^2}dt_3dt_2dt_1\right)\\
&-\left(\int_{\theta_0}^{\frac{1}{2}}\int_{\theta_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{t_2}^{\frac{1-t_1-t_2}{2}}\int_{t_3}^{\frac{1-t_1-t_2-t_3}{2}} \mathbb{1}_{(t_1,t_2,t_3,t_4)\in U_{34}}\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1t_2t_3t_4^2}dt_4dt_3dt_2dt_1\right)\\
&<(0.98983-0.390798-0.020403)=0.578629, \qquad (7)
\end{aligned}
$$

where

$$
\begin{aligned}
U_{32}(t_1,t_2,t_3):=&\Big\{(t_1,t_2)\in U_3,\ t_2<t_3<\frac{1}{2}(1-t_1-t_2),\\
&(t_1,t_2,t_3)\text{ can be partitioned into }(m,n)\in I\Big\},\\
U_{34}(t_1,t_2,t_3,t_4):=&\Big\{(t_1,t_2)\in U_3,\ t_2<t_3<\frac{1}{2}(1-t_1-t_2),\\
&(t_1,t_2,t_3)\text{ cannot be partitioned into }(m,n)\in I,\\
&t_3<t_4<\frac{1}{2}(1-t_1-t_2-t_3),\\
&(t_1,t_2,t_3,t_4)\text{ can be partitioned into }(m,n)\in I\Big\}.
\end{aligned}
$$

Finally, by (2)–(7), the total loss is less than

$$0.001624+0.412666+0.578629<0.993$$

and the proof of Theorem 1.1 is completed.

4. Application: Piatetski–Shapiro–Vinogradov Theorem

In 1992, Balog and Friedlander [2] considered a hybrid of the Three Primes Theorem and the Piatetski–Shapiro prime number theorem. They proved that every sufficiently large odd integer can be written as the sum of three primes of the form $[n^{c_0}]$ for any fixed $1<c_0<\frac{21}{20}$, and every sufficiently large odd integer can be written as the sum of two normal primes and another prime of the form $[n^{c_1}]$ for any fixed $1<c_1<\frac{9}{8}$. Their result has been improved by many authors. Now the best range of c_1 is due to Cai [3], where he proved the above statement of c_1 holds for any fixed $1<c_1<\frac{243}{205}$. Using the same method but with our Theorem 1.1 instead of Rivat and Wu's result, we can easily deduce the following.

Theorem 4.1. *Every sufficiently large odd integer can be written as the sum of two normal primes and another prime of the form* $[n^{c_1}]$ *for any fixed* $1<c_1<\frac{211}{178}$.

We shall consider the range of c_0 in another paper.

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International Curriculum Center, The High School Affiliated to Renmin University of China, Beijing, China
Email address: `runbo.li.carey@gmail.com`