A REMARK ON THE DISTRIBUTION OF $\sqrt{p}$ MODULO ONE INVOLVING PRIMES OF SPECIAL TYPE II

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Abstract. Let P_r denote an integer with at most r prime factors counted with multiplicity. In this paper we prove that for some $\lambda < \frac{1}{12}$, the inequality $\{\sqrt{p}\} < p^{-\lambda}$ has infinitely many solutions in primes p such that $p+2 = P_r$, where $r = 4, 5, 6, 7$. Specially, when $r = 4$ we obtain $\lambda = \frac{1}{15.1}$, which improves Cai's $\frac{1}{15.5}$.

Contents

1. Introduction

Let $[x]$ denote the largest integer not greater than x and write $\{x\} = x - [x]$. Beginning with Vinogradov [11], many mathematicians have studied the inequality $\{\sqrt{p}\} < p^{-\lambda}$ with prime solutions. Now the best result is due to Harman and Lewis [7]. In [7] they proved that there are infinitely many solutions in primes p to the inequality $\{\sqrt{p}\} < p^{-\lambda}$ with $\lambda = 0.262$, which improved the previous results of Vinogradov [11], Kaufman [9], Harman [5] and Balog [1].

On the other hand, one of the famous problems in prime number theory is the twin primes problem, which states that there are infinitely many primes p such that $p+2$ is also a prime. Let P_r denote an integer with at most r prime factors counted with multiplicity. Now the best result in this aspect is due to Chen [3], who showed that there are infinitely many primes p such that $p + 2 = P_2$.

In 2013, Cai [2] combined those two problems and considered a mixed version.

Definition 1.1. Let $M(\lambda, r)$ denotes the following statement: The inequality

$$\{\sqrt{p}\} < p^{-\lambda} \tag{1}$$

holds for infinitely many primes p such that $p + 2 = P_r$.

In his paper [2], he also showed that

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Theorem 1.2. *$M(\frac{1}{15.5}, 4)$ holds true.*

In 2017, Dunn [4] considered a similar problem and improved Cai's result concerning the number of prime divisors of $p+2$. Let $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq 0$, and let $\|x\|$ denote the distance from x to the nearest integer. He obtained that if $0 < \gamma < 1$ and $\theta < \frac{\gamma}{10}$, then there are infinitely many primes p such that

$$\|\alpha p^{\gamma} + \beta\| < p^{-\theta} \quad \text{and} \quad p + 2 = P_3.$$

In 2024, Li [10] generalized Cai's result to a wider range of λ. He got

Theorem 1.3. *$M(\lambda, \lfloor \frac{8}{1-4\lambda} \rfloor)$ holds true for all $0 < \lambda < \frac{1}{4}$.*

In [10], Li mentioned that Cai [2] actually prove a new mean value theorem (see [[2], Lemma 5]) for this problem and it may be useful on improving the results. In the present paper, we shall make use of this mean value theorem and improve previous results.

Theorem 1.4. *$M(\frac{1}{15.1}, 4)$, $M(\frac{1}{12.4}, 5)$, $M(\frac{1}{12.03}, 6)$ and $M(\frac{1}{12.01}, 7)$ hold true.*

We mention that $\lambda = \frac{1}{12}$ is near the limit of our method that we will explain later.

2. Preliminary lemmas

Let $\mathcal{A}$ denote a finite set of positive integers and $z \geqslant 2$. For square-free d, put

$$\mathcal{P} = \{p : (p, 2) = 1\}, \quad \mathcal{P}(r) = \{p : p \in \mathcal{P},\ (p, r) = 1\},$$

$$P(z) = \prod_{\substack{p \in \mathcal{P} \\ p < z}} p, \quad \mathcal{A}_d = \{a : a \in \mathcal{A},\ d \mid a\}, \quad S(\mathcal{A}; \mathcal{P}, z) = \sum_{\substack{a \in \mathcal{A} \\ (a, P(z)) = 1}} 1.$$

Lemma 2.1. *([8], Pages 205–209]). Suppose that every $|\mathcal{A}_d|$ can be written as*

$$|\mathcal{A}_d| = \frac{\varpi(d)}{d} X_{\mathcal{A}} + \eta(X_{\mathcal{A}}, d),$$

where $\varpi(d)$ is a multiplicative function, $0 \leqslant \varpi(p) < p$, $X_{\mathcal{A}} > 1$ is independent of d. Assume further that

$$\sum_{z_1 \leqslant p < z_2} \frac{\varpi(p)}{p} = \log \frac{\log z_2}{\log z_1} + O\left(\frac{1}{\log z_1}\right), \quad z_2 > z_1 \geqslant 2.$$

Then

$$S(\mathcal{A}; \mathcal{P}, z) \geqslant X_{\mathcal{A}} W(z) \left\{ f\left(\frac{\log D}{\log z}\right) + O\left(\frac{1}{\log^{\frac{1}{3}} D}\right) \right\} - \sum_{\substack{d \leqslant D \\ d \mid P(z)}} |\eta(X_{\mathcal{A}}, d)|,$$

$$S(\mathcal{A}; \mathcal{P}, z) \leqslant X_{\mathcal{A}} W(z) \left\{ F\left(\frac{\log D}{\log z}\right) + O\left(\frac{1}{\log^{\frac{1}{3}} D}\right) \right\} + \sum_{\substack{d \leqslant D \\ d \mid P(z)}} |\eta(X_{\mathcal{A}}, d)|,$$

where D is a power of z,

$$W(z) = \prod_{p \mid P(z)} \left(1 - \frac{\varpi(p)}{p}\right),$$

and $f(s)$ and $F(s)$ are determined by the following differential-difference equation

$$\begin{cases} F(s) = \frac{2e^\gamma}{s}, \quad f(s) = 0, & 0 < s \leqslant 2, \\ (sF(s))' = f(s-1), \quad (sf(s))' = F(s-1), & s \geqslant 2. \end{cases}$$

Lemma 2.2. *([[2], Lemma 4]). For any given constant $A > 0$ and $0 < \lambda < \frac{1}{4}, 0 < \theta < \frac{1}{4} - \lambda$ we have*

$$\sum_{d \leqslant x^\theta} \max_{(l,d)=1} \left| \sum_{\substack{x < p \leqslant 2x \\ \{\sqrt{p}\} < p^{-\lambda} \\ p \equiv l (\bmod d)}} 1 - \frac{(2x)^{1-\lambda} - x^{1-\lambda}}{\varphi(d)(1-\lambda)\log x} \right| \ll \frac{x^{1-\lambda}}{\log^A x}.$$

Lemma 2.3. *([[2], Lemma 5]). Let*

$$\mathcal{N} = \left\{ p_1p_2p_3p_4m : x^{\frac{1}{14}} \leqslant p_1 < p_2 < p_3 < p_4,\ x < p_1p_2p_3p_4m \leqslant 2x,\ (m, P(p_4)) = 1 \right\}.$$

Then for any given constant $A > 0$ and $0 < \lambda < \frac{1}{8}, 0 < \theta < \frac{1}{4} - \lambda$ we have

$$\sum_{d \leqslant x^\theta} \max_{(l,d)=1} \left| \sum_{\substack{n \in \mathcal{N} \\ n \equiv l (\bmod d) \\ \{\sqrt{n-2}\} < (n-2)^{-\lambda}}} 1 - \frac{1}{\varphi(d)} \sum_{\substack{n \in \mathcal{N} \\ (n,d)=1}} n^{-\lambda} \right| \ll \frac{x^{1-\lambda}}{\log^A x}.$$

Moreover, the lower bound $x^{\frac{1}{14}}$ for prime variables can be replaced by $x^{\frac{1}{12}}$, and the proof is similar to that in [2].

Lemma 2.4. *Let*

$$z = x^{\frac{1}{u}}, \quad 0 \leqslant y \leqslant x, \quad Q(z) = \prod_{p < z} p.$$

Then for $u > 1$, we have

$$\sum_{\substack{x < n \leqslant x+y \\ (n, Q(z))=1}} 1 = (1 + o(1))\omega(u)\frac{y}{\log z},$$

where $\omega(u)$ is the Buchstab function determined by the following differential-difference equation

$$\begin{cases} \omega(u) = \frac{1}{u}, & 1 \leqslant u \leqslant 2, \\ (u\omega(u))' = \omega(u-1), & u \geqslant 2. \end{cases}$$

Proof. Lemma 2.4 can be proved by Prime Number Theorem with Vinogradov's error term and the inductive arguments in [[6], Chapter A.2]. ∎

3. Proof of Theorem 1.4

In this section, we define the function ϖ as $\varpi(p) = 0$ for $p = 2$ and $\varpi(p) = \frac{p}{p-1}$ for other primes. Note that every odd, square-free d can be written as $d = q_1 q_2 \cdots q_n$ with prime factors $q_i > 2$, we have

$$\frac{\varpi(d)}{d} = \frac{\frac{q_1 q_2 \cdots q_n}{(q_1-1)(q_2-1)\cdots(q_n-1)}}{q_1 q_2 \cdots q_n} = \frac{1}{(q_1-1)(q_2-1)\cdots(q_n-1)} = \frac{1}{\varphi(d)}. \tag{2}$$

Put

$$D = x^{\frac{1}{4}-\lambda-\varepsilon}, \quad \mathcal{A} = \left\{p+2 : x < p \leqslant 2x,\ \{\sqrt{p}\} < p^{-\lambda}\right\},$$

$$\mathcal{M} = \left\{p_1 p_2 \cdots p_r m_1 : x^{\frac{1}{12}} \leqslant p_1 < p_2 < \cdots < p_r,\ x < p_1 p_2 \cdots p_r m_1 \leqslant 2x,\ (m_1, P(p_r)) = 1\right\},$$

$$\mathcal{B}^1 = \left\{n-2 : n \in \mathcal{N},\ \{\sqrt{n-2}\} < (n-2)^{-\lambda}\right\},$$

$$\mathcal{B}^2 = \left\{n-2 : n \in \mathcal{M},\ \{\sqrt{n-2}\} < (n-2)^{-\lambda}\right\}.$$

Let γ denote Euler's constant, $4 \leqslant r \leqslant 7$ and S_r denote the number of prime solutions to the inequality (1) such that $p + 2 = P_r$, then we have

$$\begin{aligned} S_4 \geqslant{} & S\left(\mathcal{A}; \mathcal{P}, x^{\frac{1}{14}}\right) - \sum_{x^{\frac{1}{14}} \leqslant p_1 < p_2 < p_3 < p_4 < \left(\frac{2x}{p_1 p_2 p_3}\right)^{\frac{1}{2}}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}; \mathcal{P}(p_1 p_2 p_3), p_4\right) + O\left(x^{\frac{13}{14}}\right) \\ ={} & S_{4,1} - S_{4,2} + O\left(x^{\frac{13}{14}}\right), \end{aligned} \tag{3}$$

and

$$\begin{aligned} S_r \geqslant{} & S\left(\mathcal{A}; \mathcal{P}, x^{\frac{1}{12}}\right) - \sum_{x^{\frac{1}{12}} \leqslant p_1 < \cdots < p_r < \left(\frac{2x}{p_1 \cdots p_{r-1}}\right)^{\frac{1}{2}}} S\left(\mathcal{A}_{p_1 \cdots p_r}; \mathcal{P}(p_1 \cdots p_{r-1}), p_r\right) + O\left(x^{\frac{11}{12}}\right) \\ ={} & S_{r,1} - S_{r,2} + O\left(x^{\frac{11}{12}}\right) \end{aligned} \tag{4}$$

for $5 \leqslant r \leqslant 7$.

In order to get a lower bound for S_r, we need to get a lower bound for $S_{r,1}$ and an upper bound for $S_{r,2}$. Now we ignore the presence of ε for clarity.

3.1. The evaluation of $S_{r,1}$.

We take

$$X_{\mathcal{A}} = \frac{(2x)^{1-\lambda} - x^{1-\lambda}}{(1-\lambda)\log x}. \tag{5}$$

Now, by (2) and the definition of $\eta(X_{\mathcal{A}}, d)$ in Lemma 2.1, we have

$$\begin{aligned} \eta(X_{\mathcal{A}}, d) ={} & |\mathcal{A}_d| - \frac{\varpi(d)}{d} X_{\mathcal{A}} \\ ={} & \sum_{\substack{a \in \mathcal{A} \\ d \mid a}} 1 - \frac{1}{\varphi(d)} X_{\mathcal{A}} \\ ={} & \sum_{\substack{x < p \leqslant 2x \\ \{\sqrt{p}\} < p^{-\lambda} \\ p \equiv -2 (\bmod d)}} 1 - \frac{(2x)^{1-\lambda} - x^{1-\lambda}}{\varphi(d)(1-\lambda)\log x}. \end{aligned} \tag{6}$$

By Lemma 2.2 and (6), we can easily show that

$$\sum_{\substack{d\leqslant D\\ d|P(x^{\frac{1}{14}})}} |\eta(X_{\mathcal{A}},d)| \ll \sum_{d\leqslant D} \mu^2(d)\,|\eta(X_{\mathcal{A}},d)| \ll x^{1-\lambda}(\log x)^{-5}. \tag{7}$$

We know that

$$\begin{aligned} W(z) &= \prod_{p|P(z)}\left(1-\frac{\varpi(p)}{p}\right)\\ &= \left(1+O\left(\frac{1}{\log z}\right)\right)\frac{e^{-\gamma}}{\log z}\cdot\prod_{p}\left(1-\frac{\varpi(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-1}\\ &= \left(1+O\left(\frac{1}{\log z}\right)\right)\frac{e^{-\gamma}}{\log z}\cdot 2\prod_{p>2}\left(\frac{p-2}{p-1}\right)\left(\frac{p}{p-1}\right)\\ &= (1+o(1))2C_2\frac{e^{-\gamma}}{\log z}, \end{aligned} \tag{8}$$

where

$$C_2 := \prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right). \tag{9}$$

Hence

$$W\left(x^{\frac{1}{14}}\right) = (1+o(1))2C_2\frac{e^{-\gamma}}{\frac{1}{14}\log x}. \tag{10}$$

Then by Lemma 2.1 and (7)–(10), we have

$$\begin{aligned} S_{4,1} &\geqslant X_{\mathcal{A}}W\left(x^{\frac{1}{14}}\right)\left\{f\left(\frac{\log D}{\log x^{\frac{1}{14}}}\right)+O\left(\frac{1}{\log^{\frac{1}{3}} D}\right)\right\} - \sum_{\substack{d\leqslant D\\ d|P\left(x^{\frac{1}{14}}\right)}} |\eta(X_{\mathcal{A}},d)|\\ &\geqslant (1+o(1))X_{\mathcal{A}}\cdot\frac{2C_2e^{-\gamma}}{\frac{1}{14}\log x}\cdot f\left(\frac{\frac{1}{4}-\lambda}{\frac{1}{14}}\right)\\ &= (1+o(1))\frac{2C_2X_{\mathcal{A}}}{\log D}\cdot\frac{e^{-\gamma}}{\left(\frac{1}{14}/\left(\frac{1}{4}-\lambda\right)\right)}f\left(\frac{\frac{1}{4}-\lambda}{\frac{1}{14}}\right). \end{aligned} \tag{11}$$

Similarly, for $5\leqslant r\leqslant 7$ we have

$$\begin{aligned} S_{r,1} &\geqslant X_{\mathcal{A}}W\left(x^{\frac{1}{12}}\right)\left\{f\left(\frac{\log D}{\log x^{\frac{1}{12}}}\right)+O\left(\frac{1}{\log^{\frac{1}{3}} D}\right)\right\} - \sum_{\substack{d\leqslant D\\ d|P\left(x^{\frac{1}{12}}\right)}} |\eta(X_{\mathcal{A}},d)|\\ &\geqslant (1+o(1))X_{\mathcal{A}}\cdot\frac{2C_2e^{-\gamma}}{\frac{1}{12}\log x}\cdot f\left(\frac{\frac{1}{4}-\lambda}{\frac{1}{12}}\right)\\ &= (1+o(1))\frac{2C_2X_{\mathcal{A}}}{\log D}\cdot\frac{e^{-\gamma}}{\left(\frac{1}{12}/\left(\frac{1}{4}-\lambda\right)\right)}f\left(\frac{\frac{1}{4}-\lambda}{\frac{1}{12}}\right). \end{aligned} \tag{12}$$

3.2. **The evaluation of $S_{r,2}$.** We first consider the case $r = 4$. By Chen's switching principle [3], we have

$$
\begin{aligned}
S_{4,2} &= \sum_{x^{\frac{1}{14}} \leqslant p_1 < p_2 < p_3 < p_4 < \left(\frac{2x}{p_1 p_2 p_3}\right)^{\frac{1}{2}}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}; \mathcal{P}(p_1 p_2 p_3), p_4\right) \\
&= S\left(\mathcal{B}^1; \mathcal{P}, (2x)^{\frac{1}{2}}\right). \qquad (13)
\end{aligned}
$$

The equation (13) comes from a simple observation: $S_{r,2}$ counts the number of primes p such that $p + 2 = n$ with $n \in \mathcal{N}$. Hence we have $p = n - 2$, and we can count "$n - 2$ that is prime" instead of "primes of the form $n - 2$". Now $S\left(\mathcal{B}^1; \mathcal{P}, (2x)^{\frac{1}{2}}\right)$ counts $n - 2$ with all prime factors larger than $(2x)^{\frac{1}{2}}$. If $n - 2$ has two or more prime factors, then their product will larger than $2x$, leading to a contradiction. Thus, the counted $n - 2$ must be prime, and the two sums are equal.

Since we have

$$
S\left(\mathcal{B}^1; \mathcal{P}, z\right) \leqslant S\left(\mathcal{B}^1; \mathcal{P}, w\right)
$$

for $w \leqslant z$, we have

$$
S_{4,2} = S\left(\mathcal{B}^1; \mathcal{P}, (2x)^{\frac{1}{2}}\right) \leqslant S\left(\mathcal{B}^1; \mathcal{P}, D^{\frac{1}{2}}\right). \qquad (14)
$$

We take

$$
X_{\mathcal{B}^1} = \sum_{n \in \mathcal{N}} n^{-\lambda}. \qquad (15)
$$

Now, by (2) and the definition of $\eta(X_{\mathcal{A}}, d)$ in Lemma 2.1, we have

$$
\begin{aligned}
\eta(X_{\mathcal{B}^1}, d) =& \ \left|\mathcal{B}^1_d\right| - \frac{\varpi(d)}{d} X_{\mathcal{B}^1} \\
=& \sum_{\substack{b \in \mathcal{B}^1 \\ d \mid b}} 1 - \frac{1}{\varphi(d)} X_{\mathcal{B}^1} \\
=& \sum_{\substack{n \in \mathcal{N} \\ n \equiv 2 (\bmod d) \\ \{\sqrt{n-2}\} < (n-2)^{-\lambda}}} 1 - \frac{1}{\varphi(d)} \sum_{n \in \mathcal{N}} n^{-\lambda} \\
=& \sum_{\substack{n \in \mathcal{N} \\ n \equiv 2 (\bmod d) \\ \{\sqrt{n-2}\} < (n-2)^{-\lambda}}} 1 - \frac{1}{\varphi(d)} \sum_{\substack{n \in \mathcal{N} \\ (n,d)=1}} n^{-\lambda} \\
&+ \sum_{\substack{n \in \mathcal{N} \\ n \equiv 2 (\bmod d) \\ \{\sqrt{n-2}\} < (n-2)^{-\lambda}}} 1 - \frac{1}{\varphi(d)} \sum_{\substack{n \in \mathcal{N} \\ (n,d)>1}} n^{-\lambda} \\
=& \ \eta_1(X_{\mathcal{B}^1}, d) + \eta_2(X_{\mathcal{B}^1}, d). \qquad (16)
\end{aligned}
$$

Applying Lemma 2.3 directly, we can show that

$$\sum_{\substack{d\leqslant D\\ d\mid P\left(D^{\frac{1}{2}}\right)}} |\eta_1(X_{\mathcal{B}^1},d)| \ll \sum_{d\leqslant D}\mu^2(d)\,|\eta_1(X_{\mathcal{B}^1},d)| \ll x^{1-\lambda}(\log x)^{-5}. \tag{17}$$

The sum of $\eta_2(X_{\mathcal{B}^1},d)$ can be bounded trivially:

$$\sum_{\substack{d\leqslant D\\ d\mid P\left(D^{\frac{1}{2}}\right)}} |\eta_2(X_{\mathcal{B}^1},d)| \ll x^{1-\frac{1}{14}}\log x. \tag{18}$$

When $\lambda=\frac{1}{15.1}$, we have

$$\begin{aligned}\sum_{\substack{d\leqslant D\\ d\mid P\left(D^{\frac{1}{2}}\right)}} |\eta(X_{\mathcal{B}^1},d)| &\ll x^{1-\lambda}(\log x)^{-5}+x^{1-\frac{1}{14}}\log x\\ &\ll x^{1-\lambda}(\log x)^{-5}.\end{aligned} \tag{19}$$

Then by Lemma 2.1, (8) and (19), we have

$$\begin{aligned}S_{4,2} &\leqslant X_{\mathcal{B}^1}W\left(D^{\frac{1}{2}}\right)\left\{F\left(\frac{\log D}{\log D^{\frac{1}{2}}}\right)+O\left(\frac{1}{\log^{\frac{1}{3}}D}\right)\right\}+\sum_{\substack{d\leqslant D\\ d\mid P\left(D^{\frac{1}{2}}\right)}}|\eta(X_{\mathcal{B}^1},d)|\\ &\leqslant (1+o(1))X_{\mathcal{B}^1}\cdot\frac{2C_2e^{-\gamma}}{\frac{1}{2}\log D}\cdot F(2)\\ &=(1+o(1))\frac{4C_2X_{\mathcal{B}^1}}{\log D}.\end{aligned} \tag{20}$$

By Lemma 2.4, Prime Number Theorem and integration by parts we have

$$X_{\mathcal{B}^1}=(1+o(1))X_{\mathcal{A}}T_4, \tag{21}$$

where

$$T_4=\int_{\frac{1}{14}}^{\frac{1}{5}}\int_{t_1}^{\frac{1-t_1}{4}}\int_{t_2}^{\frac{1-t_1-t_2}{3}}\int_{t_3}^{\frac{1-t_1-t_2-t_3}{2}}\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1t_2t_3t_4^2}dt_4dt_3dt_2dt_1, \tag{22}$$

where $\omega(u)$ is defined in Lemma 2.4.

Combining (3), (11), (20) and (21), we have

$$S_4\geqslant(1+o(1))\frac{2C_2X_{\mathcal{A}}}{\log D}\left(\frac{e^{-\gamma}}{\left(\frac{1}{14}/\left(\frac{1}{4}-\lambda\right)\right)}f\left(\frac{\frac{1}{4}-\lambda}{\frac{1}{14}}\right)-2T_4\right). \tag{23}$$

Hence we only need

$$\frac{e^{-\gamma}}{\left(\frac{1}{14}/\left(\frac{1}{4}-\lambda\right)\right)}f\left(\frac{\frac{1}{4}-\lambda}{\frac{1}{14}}\right)-2T_4>0. \tag{24}$$

Numerical calculation shows that (24) holds for $\lambda=\frac{1}{15.1}$, hence $M(\frac{1}{15.1},4)$ holds true.

Similarly, for the case $5 \leqslant r \leqslant 7$ we have

$$
\begin{aligned}
S_{r,2} &= \sum_{x^{\frac{1}{12}} \leqslant p_1 < \cdots < p_r < \left(\frac{2x}{p_1 \cdots p_{r-1}}\right)^{\frac{1}{2}}} S\left(\mathcal{A}_{p_1 \cdots p_r}; \mathcal{P}(p_1 \cdots p_{r-1}), p_r\right) \\
&= S\left(\mathcal{B}^2; \mathcal{P}, (2x)^{\frac{1}{2}}\right) \leqslant S\left(\mathcal{B}^2; \mathcal{P}, D^{\frac{1}{2}}\right). \qquad (25)
\end{aligned}
$$

We take

$$
X_{\mathcal{B}^2} = \sum_{n \in \mathcal{M}} n^{-\lambda}. \qquad (26)
$$

Now, by (2) and the definition of $\eta(X_{\mathcal{A}}, d)$ in Lemma 2.1, we have

$$
\begin{aligned}
\eta(X_{\mathcal{B}^2}, d) &= \left|\mathcal{B}^2_d\right| - \frac{\varpi(d)}{d} X_{\mathcal{B}^2} \\
&= \sum_{\substack{b \in \mathcal{B}^2 \\ d \mid b}} 1 - \frac{1}{\varphi(d)} X_{\mathcal{B}^2} \\
&= \sum_{\substack{n \in \mathcal{M} \\ n \equiv 2 (\bmod d) \\ \{\sqrt{n-2}\} < (n-2)^{-\lambda}}} 1 - \frac{1}{\varphi(d)} \sum_{n \in \mathcal{M}} n^{-\lambda} \\
&= \sum_{\substack{n \in \mathcal{M} \\ n \equiv 2 (\bmod d) \\ \{\sqrt{n-2}\} < (n-2)^{-\lambda}}} 1 - \frac{1}{\varphi(d)} \sum_{\substack{n \in \mathcal{M} \\ (n,d)=1}} n^{-\lambda} \\
&\quad + \sum_{\substack{n \in \mathcal{M} \\ n \equiv 2 (\bmod d) \\ \{\sqrt{n-2}\} < (n-2)^{-\lambda}}} 1 - \frac{1}{\varphi(d)} \sum_{\substack{n \in \mathcal{M} \\ (n,d)>1}} n^{-\lambda} \\
&= \eta_1(X_{\mathcal{B}^2}, d) + \eta_2(X_{\mathcal{B}^2}, d). \qquad (27)
\end{aligned}
$$

Taking $m = p_5 \cdots p_r m_1$ in Lemma 2.3, we know that conditions

$$
p_4 < p_5 < \cdots < p_r \quad \text{and} \quad (m, P(p_4))
$$

are fulfilled. By Lemma 2.3 (with $x^{\frac{1}{14}}$ replaced by $x^{\frac{1}{12}}$), we can show that

$$
\sum_{\substack{d \leqslant D \\ d \mid P\left(D^{\frac{1}{2}}\right)}} \left|\eta_1(X_{\mathcal{B}^2}, d)\right| \ll \sum_{d \leqslant D} \mu^2(d) \left|\eta_1(X_{\mathcal{B}^2}, d)\right| \ll x^{1-\lambda} (\log x)^{-5}. \qquad (28)
$$

The sum of $\eta_2(X_{\mathcal{B}^2}, d)$ can be bounded trivially:

$$
\sum_{\substack{d \leqslant D \\ d \mid P\left(D^{\frac{1}{2}}\right)}} \left|\eta_2(X_{\mathcal{B}^2}, d)\right| \ll x^{1-\frac{1}{12}} \log x. \qquad (29)
$$

When $\lambda < \frac{1}{12}$, we have

$$\sum_{\substack{d \leqslant D \\ d \mid P\left(D^{\frac{1}{2}}\right)}} |\eta(X_{\mathcal{B}^1}, d)| \ll x^{1-\lambda}(\log x)^{-5} + x^{1-\frac{1}{12}} \log x$$

$$\ll x^{1-\lambda}(\log x)^{-5}. \tag{30}$$

Then by Lemma 2.1, (8) and (30), for $5 \leqslant r \leqslant 7$ we have

$$\begin{aligned} S_{r,2} &\leqslant X_{\mathcal{B}^2} W\left(D^{\frac{1}{2}}\right)\left\{F\left(\frac{\log D}{\log D^{\frac{1}{2}}}\right) + O\left(\frac{1}{\log^{\frac{1}{3}} D}\right)\right\} + \sum_{\substack{d \leqslant D \\ d \mid P\left(D^{\frac{1}{2}}\right)}} |\eta(X_{\mathcal{B}^2}, d)| \\ &\leqslant (1+o(1)) X_{\mathcal{B}^2} \cdot \frac{2C_2 e^{-\gamma}}{\frac{1}{2}\log D} \cdot F(2) \\ &= (1+o(1)) \frac{4C_2 X_{\mathcal{B}^2}}{\log D}. \end{aligned} \tag{31}$$

Similar to the case $r = 4$, by Lemma 2.4, Prime Number Theorem and integration by parts we have

$$X_{\mathcal{B}^2} = (1+o(1)) X_{\mathcal{A}} T_r, \tag{32}$$

where

$$T_r = \int_{\frac{1}{12}}^{\frac{1}{r+1}} \int_{t_1}^{\frac{1-t_1}{r}} \cdots \int_{t_{r-1}}^{\frac{1-t_1-\cdots-t_{r-1}}{2}} \frac{\omega\left(\frac{1-t_1-\cdots-t_r}{t_r}\right)}{t_1 t_2 \cdots t_{r-1} t_r^2} dt_r \cdots dt_1. \tag{33}$$

Combining (4), (12), (31) and (32), for $5 \leqslant r \leqslant 7$ we have

$$S_r \geqslant (1+o(1)) \frac{2C_2 X_{\mathcal{A}}}{\log D}\left(\frac{e^{-\gamma}}{\left(\frac{1}{12}/\left(\frac{1}{4}-\lambda\right)\right)} f\left(\frac{\frac{1}{4}-\lambda}{\frac{1}{12}}\right) - 2T_r\right). \tag{34}$$

Hence we only need

$$\frac{e^{-\gamma}}{\left(\frac{1}{12}/\left(\frac{1}{4}-\lambda\right)\right)} f\left(\frac{\frac{1}{4}-\lambda}{\frac{1}{12}}\right) - 2T_r > 0. \tag{35}$$

When $r = 5, 6, 7$, numerical calculation shows that

$$\frac{e^{-\gamma}}{\left(\frac{1}{12}/\left(\frac{1}{4}-\frac{1}{12.4}\right)\right)} f\left(\frac{\frac{1}{4}-\frac{1}{12.4}}{\frac{1}{12}}\right) - 2T_5 > 0,$$

$$\frac{e^{-\gamma}}{\left(\frac{1}{12}/\left(\frac{1}{4}-\frac{1}{12.03}\right)\right)} f\left(\frac{\frac{1}{4}-\frac{1}{12.03}}{\frac{1}{12}}\right) - 2T_6 > 0$$

and

$$\frac{e^{-\gamma}}{\left(\frac{1}{12}/\left(\frac{1}{4}-\frac{1}{12.01}\right)\right)} f\left(\frac{\frac{1}{4}-\frac{1}{12.01}}{\frac{1}{12}}\right) - 2T_7 > 0.$$

Now Theorem 1.4 is proved. We remark that for positive λ, we have

$$f\left(\frac{\frac{1}{4}-\lambda}{\lambda}\right) > 0 \quad \text{or} \quad \frac{\frac{1}{4}-\lambda}{\lambda} > 2 \tag{36}$$

only when $\lambda < \frac{1}{12}$, so $\lambda = \frac{1}{12.01}$ is rather near the limit point.

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