

PRIMES IN ALMOST ALL SHORT INTERVALS

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ABSTRACT. The author sharpens a result of Jia (1996), showing that the interval $[n, n + n^{\frac{1}{21.5} + \varepsilon}]$ contains prime numbers for almost all n . Watt's mean value bound, a delicate sieve decomposition and more accurate estimates for integrals are used to good effect.

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1. INTRODUCTION

One of the famous topics in number theory is to find prime numbers in short intervals. In 1937, Cramér [7] conjectured that every interval $[n, n + f(n)(\log n)^2]$ contains prime numbers for some $f(n) \rightarrow 1$ as $n \rightarrow \infty$. The Riemann Hypothesis implies that for all large n , the interval $[n, n + n^\theta]$ contains $\sim n^\theta (\log n)^{-1}$ prime numbers for every $\frac{1}{2} + \varepsilon \leq \theta \leq 1$. The first unconditional result of this asymptotic formula was proved by Hoheisel [18] in 1930 with $\theta = 1 - \frac{1}{33000}$. After the works of Hoheisel [18], Heilbronn [17], Chudakov [6], Ingham [20] and Montgomery [35], Huxley [19] proved in 1972 that the above asymptotic formula holds when $\theta > \frac{7}{12}$ by his zero density estimate. In 2024, Guth and Maynard [9] improved this to $\theta > \frac{17}{30}$ by a new zero density estimate.

In 1979, Iwaniec and Jutila [21] first introduced a sieve method into this problem. They established a lower bound with correct order of magnitude (instead of an asymptotic formula) with $\theta = \frac{13}{23}$. After that breakthrough, many improvements were made and the value of θ was reduced successively to

$$\begin{aligned} \frac{5}{9} \approx 0.5556, \quad \frac{11}{20} = 0.5500, \quad \frac{17}{31} \approx 0.5484, \quad \frac{23}{42} \approx 0.5476, \\ \frac{1051}{1920} \approx 0.5474, \quad \frac{35}{64} \approx 0.5469, \quad \frac{6}{11} \approx 0.5455 \quad \text{and} \quad \frac{7}{13} \approx 0.5385 \end{aligned}$$

by Iwaniec and Jutila [21], Heath-Brown and Iwaniec [16], Pintz [37] [38], Iwaniec and Pintz [22], Mozzochi [36] and Lou and Yao [30] [32] [33] [34] respectively. In 1996, Baker and Harman [2] presented an alternative approach to this problem. They used the alternative sieve developed by Harman [11] [12] to reduce θ to 0.535. Finally, Baker, Harman and Pintz [4] further developed this sieve process and combined it with Watt's theorem and showed $\theta = 0.525$.

However, if we only consider the prime numbers in "almost all" intervals instead of "all" intervals, the intervals will be much shorter than $n^{0.525}$. In 1943, under the Riemann Hypothesis, Selberg [42] showed that Cramér's interval contains primes for almost all n if $f(n) \rightarrow \infty$ as $n \rightarrow \infty$. In the same paper, he also showed unconditionally that the interval $[n, n + n^{\frac{19}{77} + \varepsilon}]$ contains prime numbers for almost all n . In 1971, Montgomery [35] improved the exponent $\frac{19}{77}$ to $\frac{1}{5}$ with an asymptotic formula. The zero density estimate of Huxley [19] gives the exponent $\frac{1}{6}$ with an asymptotic formula, and the best asymptotic result now is also due to Guth and Maynard [9], where they proved the exponent $\frac{2}{15}$.

In 1982, Harman [10] used his alternative sieve method to showed that the interval $[n, n + n^{\frac{1}{10} + \varepsilon}]$ contains prime numbers for almost all n . His method can only provide a lower bound instead of an asymptotic formula. The exponent $\frac{1}{10}$ was reduced successively to

$$\begin{aligned} \frac{1}{12} = 0.0833, \quad \frac{14}{159} = 0.0881, \quad \frac{1}{13} = 0.0769, \quad \frac{17}{227} = 0.0749, \quad \frac{1}{13.5} = 0.0740, \\ \frac{1}{14} = 0.0714, \quad \frac{1}{15} = 0.0667, \quad \frac{1}{16} = 0.0625, \quad \frac{1}{18} = 0.0556 \quad \text{and} \quad \frac{1}{20} = 0.0500 \end{aligned}$$

2020 Mathematics Subject Classification. 11N05, 11N35, 11N36.

Key words and phrases. Prime, Sieve methods, Dirichlet polynomial.

by Harman [11] (and Heath-Brown [15]), Lou and Yao [31], Jia [24] [25], Lou and Yao [31], Li [28], Jia [23] (and Watt [44]), Li [29], Baker, Harman and Pintz [3], Wong [45] (and Jia [27], Harman [14], Chapter 9) and Jia [26] respectively. As the strongest result above, Jia [26] used many powerful tools and arguments in his proof, both analytic and combinatorial, and these are extremely complicated. In this paper, we further develop the sieve machinery used in [26] and obtain the following result.

Theorem 1.1. *The interval $[n, n + n^{\frac{1}{21.5} + \varepsilon}]$ contains prime numbers for almost all n . Specifically, suppose that B is a sufficiently large positive constant, ε is a sufficiently small positive constant and X is sufficiently large. Then for positive integers $n \in [X, 2X]$, except for $O(X(\log X)^{-B})$ values, the interval $[n, n + n^{\frac{1}{21.5} + \varepsilon}]$ contains $\gg n^{1/21.5 + \varepsilon} (\log n)^{-1}$ prime numbers.*

Throughout this paper, we always suppose that B is a sufficiently large positive constant, ε is a sufficiently small positive constant, X is sufficiently large and $x \in [X, 2X]$. The letter p , with or without subscript, is reserved for prime numbers. Let c_0, c_1 and c_2 denote positive constants which may have different values at different places, and we write $m \sim M$ to mean that $c_1 M < m \leq c_2 M$. Let $b = 1 + \frac{1}{\log X}$, $\delta = \varepsilon^{1/3}$ and $\eta = \frac{1}{2}X^{-\frac{20.5}{21.5} + \varepsilon}$. We use $M(s)$, $N(s)$, $H(s)$ and some other capital letters to denote 1-bounded Dirichlet polynomials

$$M(s) = \sum_{m \sim M} a_m m^{-s}, \quad N(s) = \sum_{n \sim N} b_n n^{-s}, \quad H(s) = \sum_{h \sim H} c_h h^{-s},$$

and we use $L(s)$ to denote a "zeta-factor"

$$L(s) = \sum_{l \sim L} l^{-s}.$$

2. AN OUTLINE OF THE PROOF

Let \mathcal{C} denote a finite set of positive integers, $p_j = X^{t_j}$ in the following sections and put

$$\begin{aligned} \mathcal{A} &= \{n : x < n \leq x + \eta x\}, \quad \mathcal{B} = \{n : x < n \leq 2x\}, \\ \mathcal{C}_d &= \{a : ad \in \mathcal{C}\}, \quad P(z) = \prod_{p < z} p, \quad S(\mathcal{C}, z) = \sum_{\substack{a \in \mathcal{C} \\ (a, P(z))=1}} 1. \end{aligned}$$

Then we have

$$\pi(x + \eta x) - \pi(x) = S(\mathcal{A}, (2X)^{\frac{1}{2}}). \quad (1)$$

Buchstab's identity is the equation

$$S(\mathcal{C}, z) = S(\mathcal{C}, w) - \sum_{w \leq p < z} S(\mathcal{C}_p, p),$$

where $2 \leq w < z$.

In order to prove Theorem 1.1, we only need to show that $S(\mathcal{A}, (2X)^{\frac{1}{2}}) > 0$. By Buchstab's identity, we have

$$\begin{aligned} S(\mathcal{A}, (2X)^{\frac{1}{2}}) &= S(\mathcal{A}, X^{\frac{79}{817}}) - \sum_{\substack{\frac{79}{817} \leq t_1 < \frac{1}{2} \\ \frac{79}{817} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1))}} S(\mathcal{A}_{p_1 p_2}, p_2) \\ &= S_1 - S_2 + S_3. \end{aligned} \quad (2)$$

Our aim is to show that the sparser set \mathcal{A} contains the expected proportion of primes compared to the larger set \mathcal{B} , which requires us to decompose $S(\mathcal{A}, (2X)^{\frac{1}{2}})$ using the above Buchstab's identity, prove asymptotic formulas (for almost $x \in [X, 2X]$ except for $O(X(\log X)^{-B})$ values) of the form

$$S(\mathcal{A}, z) = \eta(1 + o(1))S(\mathcal{B}, z) \quad (3)$$

for some parts of it, and drop the remaining positive parts.

In Sections 3 and 4 we provide some arithmetic information by proving mean value bounds for Dirichlet polynomials, and we shall use them to prove asymptotic formulas for terms of the form $S(\mathcal{A}_{p_1 \dots p_n}, X^\delta)$ and $S(\mathcal{A}_{p_1 \dots p_n}, p_n)$ in Section 5. In Section 6 we make further use of Buchstab's identity to decompose $S(\mathcal{A}, (2X)^{\frac{1}{2}})$ and prove Theorem 1.1.

3. ARITHMETIC INFORMATION I

In the following two sections we provide some arithmetic information which will help us prove the asymptotic formulas for sieve functions. In this section we only use the classical mean value estimate and Halász method. In next section we will provide a crucial result which will help us get asymptotic formulas for more sieve functions.

Lemma 3.1. *Suppose that $MH = X$, $b_h = \Lambda(h)$ and $X^\delta \ll H \ll X^{\frac{79}{817}}$. Then for $(\log X)^{B/\varepsilon} \leq T \leq X$, we have*

$$\left(\min\left(\eta, \frac{1}{T}\right) \right)^2 \int_T^{2T} |M(b+it)H(b+it)|^2 dt \ll \eta^2 (\log x)^{-10B}.$$

Proof. The proof is similar to that of [[26], Lemma 1]. Let $s = b + it$ and by the zero-free region of the ζ function, for $|t| \leq 2X$ we have

$$H(s) = \frac{(c_2 H)^{1-s} - (c_1 H)^{1-s}}{1-s} + O\left((\log x)^{-2B/\varepsilon}\right).$$

So, for $T_0 \leq |t| \leq 2X$ we have $H(s) \ll (\log x)^{-B/\varepsilon}$. According to the discussion in , there are $O((\log X)^2)$ sets $S(V, W)$, where $S(V, W)$ is the set of $t_k (k = 1, \dots, K)$ with the property $|t_r - t_s| \geq 1$ ($r \neq s$). Moreover,

$$V \leq M^{\frac{1}{2}} |M(b + it_k)| < 2V, \quad W \leq H^{\frac{1}{2}} |H(b + it_k)| < 2W,$$

where $X^{-1} \leq M^{-\frac{1}{2}} V, X^{-1} \leq H^{-\frac{1}{2}} W$ and $V \ll M^{\frac{1}{2}}, W \ll H^{\frac{1}{2}} (\log x)^{-B/\varepsilon}$. Then we have

$$\int_T^{2T} |M(b + it)H(b + it)|^2 dt \ll V^2 W^2 x^{-1} (\log x)^2 |S(V, W)| + O\left(x^{-2\varepsilon^2}\right),$$

where $S(V, W)$ is one of sets with the above properties.

Assume $X^{\frac{1}{k+1}} \leq H < X^{\frac{1}{k}}$, where k is a positive integer, $k \geq 10$ and $k\delta \ll 1$. Applying the mean value estimate to $M(s)$ and $H^k(s)$, we have

$$\begin{aligned} |S(V, W)| &\ll V^{-2} (M + T) (\log x)^d, \\ |S(V, W)| &\ll W^{-2k} (H^k + T) (\log x)^d, \end{aligned}$$

where $d = c/\delta^2$. Applying the Halász method to $M(s)$ and $H^k(s)$, we have

$$\begin{aligned} |S(V, W)| &\ll (V^{-2} M + V^{-6} M T) (\log x)^d, \\ |S(V, W)| &\ll (W^{-2k} H^k + W^{-6k} H^k T) (\log x)^d. \end{aligned}$$

Thus,

$$V^2 W^2 |S(V, W)| \ll V^2 W^2 F (\log x)^d,$$

where

$$F = \min \left\{ V^{-2} (M + T), V^{-2} M + V^{-6} M T, W^{-2k} (H^k + T), W^{-2k} H^k + W^{-6k} H^k T \right\}.$$

It will be proved that

$$\left(\min \left(\eta, \frac{1}{T} \right) \right)^2 V^2 W^2 F \ll \eta^2 x (\log x)^{-B/\varepsilon}.$$

We consider four cases.

(a) $F \leq 2V^{-2} M, F \leq 2W^{-2k} H^k$. Then

$$\begin{aligned} V^2 W^2 F &\ll V^2 W^2 \min \left\{ V^{-2} M, W^{-2k} H^k \right\} \\ &\leq V^2 W^2 (V^{-2} M)^{1-\frac{1}{2k}} (W^{-2k} H^k)^{\frac{1}{2k}} \\ &= V^{\frac{1}{k}} W M^{1-\frac{1}{2k}} H^{\frac{1}{2}} \\ &\ll x (\log x)^{-B/\varepsilon} \end{aligned}$$

and so

$$\left(\min \left(\eta, \frac{1}{T} \right) \right)^2 V^2 W^2 F \ll \eta^2 x (\log x)^{-B/\varepsilon}.$$

(b) $F > 2V^{-2} M, F > 2W^{-2k} H^k$. Then

$$\begin{aligned} V^2 W^2 F &\ll V^2 W^2 \min \left\{ V^{-2} T, V^{-6} M T, W^{-2k} T, W^{-6k} H^k T \right\} \\ &\leq V^2 W^2 (V^{-2})^{1-\frac{3}{2k}} (V^{-6} M)^{\frac{1}{2k}} (W^{-2k})^{\frac{1}{k}} T \\ &= M^{\frac{1}{2k}} T. \end{aligned}$$

Since $k \geq 10$, we have $H \geq X^{\frac{1}{k+1}} \geq X^{1-\frac{k}{11}}, M^{\frac{1}{2k}} \ll X^{\frac{1}{22}}$, and so

$$\left(\min \left(\eta, \frac{1}{T} \right) \right)^2 V^2 W^2 F \ll \frac{\eta}{T} x^{\frac{1}{22}} T \ll \eta^2 x^{1-\varepsilon^2}.$$

(c) $F \leq 2V^{-2} M, F > 2W^{-2k} H^k$. Then

$$\begin{aligned} V^2 W^2 F &\ll V^2 W^2 \min \left\{ V^{-2} M, W^{-2k} T, W^{-6k} H^k T \right\} \\ &\leq V^2 W^2 (V^{-2} M)^{1-\frac{1}{3k}} (W^{-6k} H^k T)^{\frac{1}{3k}} \\ &\ll M H^{\frac{1}{3}} T^{\frac{1}{3k}}, \end{aligned}$$

since $V \ll M^{\frac{1}{2}}$. As $H \geq X^{\frac{1}{k+1}} \geq X^{\frac{21}{44k} + \varepsilon}$ and $M \leq X^{1 - \frac{21}{44k}}$, we have

$$\begin{aligned} \left(\min \left(\eta, \frac{1}{T} \right) \right)^2 V^2 W^2 F &\ll \eta^{2 - \frac{1}{3k}} \left(\frac{1}{T} \right)^{\frac{1}{3k}} (MH)^{\frac{1}{3}} M^{\frac{2}{3}} T^{\frac{1}{3k}} \\ &\ll x^{\frac{1}{3} - \frac{20.5}{21.5}(2 - \frac{1}{3k}) + \frac{2}{3}(1 - \frac{21}{44k})} \\ &\ll \eta^2 x^{1 - \varepsilon^2}. \end{aligned}$$

(d) $F > 2V^{-2}M, F \leq 2W^{-2k}H^k$. Then

$$\begin{aligned} V^2 W^2 F &\ll V^2 W^2 \min \{ V^{-2}T, V^{-6}MT, W^{-2k}H^k \} \\ &\leq V^2 W^2 (V^{-2}T)^{1 - \frac{3}{2k}} (V^{-6}MT)^{\frac{1}{2k}} (W^{-2k}H^k)^{\frac{1}{k}} \\ &= M^{\frac{1}{2k}} HT^{1 - \frac{1}{k}}. \end{aligned}$$

If $k \geq 11$, then $H \leq X^{\frac{1}{k}} \leq X^{1 - \frac{21(k-1)}{11(2k-1)}}$, $M \gg X^{\frac{21(k-1)}{11(2k-1)}}$, and so

$$\begin{aligned} \left(\min \left(\eta, \frac{1}{T} \right) \right)^2 V^2 W^2 F &\ll \eta^{1 + \frac{1}{k}} \left(\frac{1}{T} \right)^{1 - \frac{1}{k}} (MH) M^{-\frac{2k-1}{2k}} T^{1 - \frac{1}{k}} \\ &\ll x^{1 - \frac{20.5}{21.5}(1 + \frac{1}{k}) - \frac{21(k-1)}{22k}} \\ &\ll \eta^2 x^{1 - \varepsilon^2}. \end{aligned}$$

If $k = 10$, then $X^{\frac{1}{11}} \leq H \ll X^{\frac{79}{817}}$, $M \gg X^{\frac{738}{817}}$, and so

$$\begin{aligned} \left(\min \left(\eta, \frac{1}{T} \right) \right)^2 V^2 W^2 F &\ll \eta^{\frac{11}{10}} \left(\frac{1}{T} \right)^{\frac{9}{10}} (MH) M^{-\frac{19}{20}} T^{\frac{9}{10}} \\ &\ll x^{1 - \frac{11}{10} \cdot \frac{20.5}{21.5} - \frac{19}{20} \cdot \frac{738}{817}} \\ &\ll \eta^2 x^{1 - \varepsilon^2}. \end{aligned}$$

Combining the above cases, Lemma 3.1 is proved. \square

Lemma 3.2. Suppose that $MHK = X$, and M and H satisfy one of the following 9 conditions:

- (1) $MH \ll X^{\frac{673}{1247}}, X^{\frac{82}{473}} \ll H, M^{29}H^{-1} \ll X^{\frac{427}{43}}, X^{\frac{246}{817}} \ll M, M^{-1}H^{29} \ll X^{\frac{263}{43}}, X^{\frac{246}{43}} \ll M^{12}H^{11};$
- (2) $MH \ll X^{\frac{223}{387}}, M^{29}H^{19} \ll X^{\frac{632}{43}}, X^{\frac{328}{387}} \ll M^2H, M^2H^{11} \ll X^{\frac{120}{43}}, X^{\frac{1230}{43}} \ll M^{58}H^{49};$
- (3) $MH \ll X^{\frac{268}{473}}, X^{\frac{41}{215}} \ll H, M^6H \ll X^{\frac{94}{43}}, X^{\frac{82}{301}} \ll M \ll X^{\frac{16}{43}}, MH^8 \ll X^{\frac{98}{43}}, X^{\frac{123}{43}} \ll M^6H^5;$
- (4) $MH \ll X^{\frac{571}{817}}, X^{\frac{82}{473}} \ll H, M^{12}H \ll X^{\frac{270}{43}}, X^{\frac{574}{1247}} \ll M, M^{-1}H^{19} \ll X^{\frac{161}{43}}, X^{\frac{328}{43}} \ll M^{12}H^{11};$
- (5) $M^2H \ll X^{\frac{446}{387}}, M^{58}H^9 \ll X^{\frac{1264}{43}}, X^{\frac{164}{387}} \ll M, MH^5 \ll X^{\frac{60}{43}}, X^{\frac{615}{43}} \ll M^{29}H^{10};$
- (6) $X^{\frac{27}{43}} \ll MH \ll X^{\frac{219}{301}}, X^{\frac{41}{215}} \ll H, M^6H \ll X^{\frac{135}{43}}, X^{\frac{205}{473}} \ll M, M^{-1}H^7 \ll X^{\frac{55}{43}}, X^{\frac{164}{43}} \ll M^6H^5;$
- (7) $MH \ll X^{\frac{268}{473}}, M^{35}H^{23} \ll X^{\frac{767}{43}}, X^{\frac{410}{473}} \ll M^2H, M^2H^{13} \ll X^{\frac{130}{43}}, X^{\frac{1476}{43}} \ll M^{70}H^{59};$
- (8) $MH \ll X^{\frac{313}{559}}, M^{41}H^{27} \ll X^{\frac{902}{43}}, X^{\frac{492}{559}} \ll M^2H, M^2H^{15} \ll X^{\frac{138}{43}}, X^{\frac{1722}{43}} \ll M^{82}H^{69};$
- (9) $M^2H \ll X^{\frac{536}{473}}, M^{70}H^{11} \ll X^{\frac{1534}{43}}, X^{\frac{205}{473}} \ll M, MH^6 \ll X^{\frac{65}{43}}, X^{\frac{738}{43}} \ll M^{35}H^{12}.$

Assume that for $(\log X)^{B/\varepsilon} \leq |t| \leq 2X, M(b+it) \ll (\log x)^{-B/\varepsilon}$ and $H(b+it) \ll (\log x)^{-B/\varepsilon}$. Then for $(\log X)^{B/\varepsilon} \leq T \leq X$, we have

$$\left(\min \left(\eta, \frac{1}{T} \right) \right)^2 \int_T^{2T} |M(b+it)H(b+it)K(b+it)|^2 dt \ll \eta^2 (\log x)^{-10B}.$$

Proof. The proof is similar to that of [[26], Lemmas 5,7,9,10,12]. Using the method of Lemma 3.1, we only need to show that for $T = 1/\eta = 2X^{\frac{20.5}{21.5} - \varepsilon}$,

$$I = \int_T^{2T} |M(b+it)H(b+it)K(b+it)|^2 dt \ll (\log x)^{-10B}.$$

Assuming the condition (1). By applying the mean value estimate and Halász method to $M^3(s), H^5(s)$ and $K^2(s)$, we get

$$I \ll U^2 V^2 W^2 x^{-1} F(\log x)^c$$

where

$$F = \min \{ V^{-6} (M^3 + T), V^{-6} M^3 + V^{-18} M^3 T, W^{-10} (H^5 + T), W^{-10} H^5 + W^{-30} H^5 T, U^{-4} (K^2 + T), U^{-4} K^2 + U^{-12} K^2 T \}.$$

We consider 8 cases:

(a) $F \leq 2V^{-6}M^3, F \leq 2W^{-10}H^5, F \leq 2U^{-4}K^2$. Then

$$\begin{aligned} U^2 V^2 W^2 F &\ll U^2 V^2 W^2 \min \{ V^{-4} W^{-2} M^2 H, W^{-10} H^5, U^{-4} K^2 \} \\ &\leq U^2 V^2 W^2 (V^{-4} W^{-2} M^2 H)^{\frac{3}{8}} (W^{-10} H^5)^{\frac{1}{8}} (U^{-4} K^2)^{\frac{1}{2}} \end{aligned}$$

$$= V^{\frac{1}{2}} M^{\frac{3}{4}} HK \\ \ll x(\log x)^{-11B}.$$

(b) $F \leq 2V^{-6}M^3, F \leq 2W^{-10}H^5, F > 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-6}M^3, W^{-10}H^5, U^{-4}T, U^{-12}K^2T\} \\ &\leq U^2V^2W^2 (V^{-6}M^3)^{\frac{1}{3}} (W^{-10}H^5)^{\frac{1}{5}} (U^{-4}T)^{\frac{9}{20}} (U^{-12}K^2T)^{\frac{1}{60}} \\ &= T^{\frac{7}{15}} MHK^{\frac{1}{30}} \\ &= T^{\frac{7}{15}} (MHK)^{\frac{1}{30}} \left(M^{\frac{29}{30}} H^{\frac{29}{30}} \right) \\ &\ll x^{1-\varepsilon^2}, \end{aligned}$$

where $MH \ll X^{\frac{673}{1247}}$ is required.

(c) $F \leq 2V^{-6}M^3, F > 2W^{-10}H^5, F \leq 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-6}M^3, W^{-10}T, W^{-30}H^5T, U^{-4}K^2\} \\ &\leq U^2V^2W^2 (V^{-6}M^3)^{\frac{1}{3}} (W^{-10}T)^{\frac{3}{20}} (W^{-30}H^5T)^{\frac{1}{60}} (U^{-4}K^2)^{\frac{1}{2}} \\ &= T^{\frac{1}{6}} MH^{\frac{1}{12}} K \\ &= T^{\frac{1}{6}} (MHK) H^{-\frac{11}{12}} \\ &\ll x^{1-\varepsilon^2}, \end{aligned}$$

where $X^{\frac{82}{473}} \ll H$ is required.

(d) $F \leq 2V^{-6}M^3, F > 2W^{-10}H^5, F > 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-6}M^3, W^{-10}T, W^{-30}H^5T, U^{-4}T, U^{-12}K^2T\} \\ &\leq U^2V^2W^2 (V^{-6}M^3)^{\frac{1}{3}} (W^{-10}T)^{\frac{1}{5}} (U^{-4}T)^{\frac{9}{20}} (U^{-12}K^2T)^{\frac{1}{60}} \\ &= T^{\frac{2}{3}} MK^{\frac{1}{30}} \\ &= T^{\frac{2}{3}} (MHK)^{\frac{1}{30}} \left(M^{\frac{29}{30}} H^{-\frac{1}{30}} \right) \\ &\ll x^{1-\varepsilon^2}, \end{aligned}$$

where $M^{29}H^{-1} \ll X^{\frac{427}{43}}$ is required.

(e) $F > 2V^{-6}M^3, F \leq 2W^{-10}H^5, F \leq 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-6}T, V^{-18}M^3T, W^{-10}H^5, U^{-4}K^2\} \\ &\leq U^2V^2W^2 (V^{-6}T)^{\frac{17}{60}} (V^{-18}M^3T)^{\frac{1}{60}} (W^{-10}H^5)^{\frac{1}{5}} (U^{-4}K^2)^{\frac{1}{2}} \\ &= T^{\frac{3}{10}} M^{\frac{1}{20}} HK \\ &= T^{\frac{3}{10}} (MHK) M^{-\frac{19}{20}} \\ &\ll x^{1-\varepsilon^2}, \end{aligned}$$

where $X^{\frac{246}{817}} \ll M$ is required.

(f) $F > 2V^{-6}M^3, F \leq 2W^{-10}H^5, F > 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-6}T, V^{-18}M^3T, W^{-10}H^5, U^{-4}T, U^{-12}K^2T\} \\ &\leq U^2V^2W^2 (V^{-6}T)^{\frac{1}{3}} (W^{-10}H^5)^{\frac{1}{5}} (U^{-4}T)^{\frac{9}{20}} (U^{-12}K^2T)^{\frac{1}{60}} \\ &= T^{\frac{4}{5}} HK^{\frac{1}{30}} \\ &= T^{\frac{4}{5}} (MHK)^{\frac{1}{30}} \left(M^{-\frac{1}{30}} H^{\frac{29}{30}} \right) \\ &\ll x^{1-\varepsilon^2}, \end{aligned}$$

where $M^{-1}H^{29} \ll X^{\frac{263}{43}}$ is required.

(g) $F > 2V^{-6}M^3, F > 2W^{-10}H^5, F \leq 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-6}T, V^{-18}M^3T, W^{-10}T, W^{-30}H^5T, U^{-4}K^2\} \\ &\leq U^2V^2W^2 (V^{-6}T)^{\frac{1}{3}} (W^{-10}T)^{\frac{3}{20}} (W^{-30}H^5T)^{\frac{1}{60}} (U^{-4}K^2)^{\frac{1}{2}} \\ &= T^{\frac{1}{2}} H^{\frac{1}{12}} K \end{aligned}$$

$$\begin{aligned}
&= T^{\frac{1}{2}}(MHK) \left(M^{-1} H^{-\frac{11}{12}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $X^{\frac{246}{43}} \ll M^{12} H^{11}$ is required.

(h) $F > 2V^{-6}M^3, F > 2W^{-10}H^5, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{ V^{-6}, V^{-18}M^3, W^{-10}, W^{-30}H^5, U^{-4}, U^{-12}K^2 \} T \\
&\leq U^2V^2W^2 (V^{-6})^{\frac{170}{60}} (V^{-18}M^3)^{\frac{1}{60}} (W^{-10})^{\frac{1}{5}} (U^{-4})^{\frac{1}{2}} T \\
&= TM^{\frac{1}{20}} \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $M \ll X^{\frac{5025}{13717}}$ and $TM^{\frac{1}{20}} \ll X^{\frac{20.5}{21.5} + \frac{1}{20} \cdot \frac{5025}{13717}} \ll X^{\frac{53321}{54868}}$ are used. The former one can be obtained by using $MH \ll X^{\frac{673}{1247}}$ and $X^{\frac{82}{473}} \ll H$.

Assuming the condition (2). By applying the mean value estimate and Halász method to $M^2(s)H(s), H^5(s)$ and $K^2(s)$, we get

$$I \ll U^2V^2W^2x^{-1}F(\log x)^c$$

where

$$\begin{aligned}
F &= \min \{ V^{-4}W^{-2} (M^2H + T), V^{-4}W^{-2}M^2H + V^{-12}W^{-6}M^2HT, \\
&\quad W^{-10} (H^5 + T), W^{-10}H^5 + W^{-30}H^5T, U^{-4} (K^2 + T), U^{-4}K^2 + U^{-12}K^2T \}.
\end{aligned}$$

We consider 8 cases:

(a) $F \leq 2V^{-4}W^{-2}M^2H, F \leq 2W^{-10}H^5, F \leq 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{ V^{-4}W^{-2}M^2H, W^{-10}H^5, U^{-4}K^2 \} \\
&\leq U^2V^2W^2 (V^{-4}W^{-2}M^2H)^{\frac{3}{8}} (W^{-10}H^5)^{\frac{1}{8}} (U^{-4}K^2)^{\frac{1}{2}} \\
&= V^{\frac{1}{2}} M^{\frac{3}{4}} HK \\
&\ll x(\log x)^{-11B}.
\end{aligned}$$

(b) $F \leq 2V^{-4}W^{-2}M^2H, F \leq 2W^{-10}H^5, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{ V^{-4}W^{-2}M^2H, W^{-10}H^5, U^{-4}T, U^{-12}K^2T \} \\
&\leq U^2V^2W^2 (V^{-4}W^{-2}M^2H)^{\frac{1}{2}} (W^{-10}H^5)^{\frac{1}{10}} (U^{-4}T)^{\frac{7}{20}} (U^{-12}K^2T)^{\frac{1}{20}} \\
&= T^{\frac{2}{5}} MHK^{\frac{1}{10}} \\
&= T^{\frac{2}{5}} (MHK)^{\frac{1}{10}} \left(M^{\frac{9}{10}} H^{\frac{9}{10}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $MH \ll X^{\frac{223}{387}}$ is required.

(c) $F \leq 2V^{-4}W^{-2}M^2H, F > 2W^{-10}H^5, F \leq 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{ V^{-4}W^{-2}M^2H, W^{-10}T, W^{-30}H^5T, U^{-4}K^2 \} \\
&\leq U^2V^2W^2 (V^{-4}W^{-2}M^2H)^{\frac{1}{2}} (U^{-4}K^2)^{\frac{1}{2}} \\
&= WMH^{\frac{1}{2}} K \\
&\ll x(\log x)^{-11B}.
\end{aligned}$$

(d) $F \leq 2V^{-4}W^{-2}M^2H, F > 2W^{-10}H^5, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{ V^{-4}W^{-2}M^2H, W^{-10}T, W^{-30}H^5T, U^{-4}T, U^{-12}K^2T \} \\
&\leq U^2V^2W^2 (V^{-4}W^{-2}M^2H)^{\frac{1}{2}} (W^{-30}H^5T)^{\frac{1}{30}} (U^{-4}T)^{\frac{9}{20}} (U^{-12}K^2T)^{\frac{1}{10}} \\
&= T^{\frac{1}{2}} MH^{\frac{2}{3}} K^{\frac{1}{30}} \\
&= T^{\frac{1}{2}} (MHK)^{\frac{1}{30}} \left(M^{\frac{29}{30}} H^{\frac{19}{30}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $M^{29}H^{19} \ll X^{\frac{632}{43}}$ is required.

(e) $F > 2V^{-4}W^{-2}M^2H, F \leq 2W^{-10}H^5, F \leq 2U^{-4}K^2$. Then

$$U^2V^2W^2F \ll U^2V^2W^2 \min \{ V^{-4}W^{-2}T, V^{-12}W^{-6}M^2HT, W^{-10}H^5, U^{-4}K^2 \}$$

$$\begin{aligned}
&\leq U^2 V^2 W^2 (V^{-4} W^{-2} T)^{\frac{7}{20}} (V^{-12} W^{-6} M^2 H T)^{\frac{1}{20}} (W^{-10} H^5)^{\frac{1}{10}} (U^{-4} K^2)^{\frac{1}{2}} \\
&= T^{\frac{2}{5}} M^{\frac{1}{10}} H^{\frac{11}{20}} K \\
&= T^{\frac{2}{5}} (M H K) \left(M^{-\frac{9}{10}} H^{-\frac{9}{20}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $X^{\frac{328}{387}} \ll M^2 H$ is required.

(f) $F > 2V^{-4} W^{-2} M^2 H, F \leq 2W^{-10} H^5, F > 2U^{-4} K^2$. Then

$$\begin{aligned}
U^2 V^2 W^2 F &\ll U^2 V^2 W^2 \min \{V^{-4} W^{-2} T, V^{-12} W^{-6} M^2 H T, W^{-10} H^5, U^{-4} T, U^{-12} K^2 T\} \\
&\leq U^2 V^2 W^2 (V^{-4} W^{-2} T)^{\frac{7}{20}} (V^{-12} W^{-6} M^2 H T)^{\frac{1}{20}} (W^{-10} H^5)^{\frac{1}{10}} (U^{-4} T)^{\frac{1}{2}} \\
&= T^{\frac{9}{10}} \left(M^{\frac{1}{10}} H^{\frac{11}{20}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $M^2 H^{11} \ll X^{\frac{122}{43}}$ is required. This can be obtained by using $M^2 H^{11} \ll X^{\frac{120}{43}}$.

(g) $F > 2V^{-4} W^{-2} M^2 H, F > 2W^{-10} H^5, F \leq 2U^{-4} K^2$. Then

$$\begin{aligned}
U^2 V^2 W^2 F &\ll U^2 V^2 W^2 \min \{V^{-4} W^{-2} T, V^{-12} W^{-6} M^2 H T, W^{-10} T, W^{-30} H^5 T, U^{-4} K^2\} \\
&\leq U^2 V^2 W^2 (V^{-4} W^{-2} T)^{\frac{9}{20}} (V^{-12} W^{-6} M^2 H T)^{\frac{1}{60}} (W^{-30} H^5 T)^{\frac{1}{30}} (U^{-4} K^2)^{\frac{1}{2}} \\
&= T^{\frac{1}{2}} M^{\frac{1}{30}} H^{\frac{11}{60}} K \\
&= T^{\frac{1}{2}} (M H K) \left(M^{-\frac{29}{30}} H^{-\frac{49}{60}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $X^{\frac{1230}{43}} \ll M^{58} H^{49}$ is required.

(h) $F > 2V^{-4} W^{-2} M^2 H, F > 2W^{-10} H^5, F > 2U^{-4} K^2$. Then

$$\begin{aligned}
U^2 V^2 W^2 F &\ll U^2 V^2 W^2 \min \{V^{-4} W^{-2}, V^{-12} W^{-6} M^2 H, W^{-10}, W^{-30} H^5, U^{-4}, U^{-12} K^2\} T \\
&\leq U^2 V^2 W^2 (V^{-4} W^{-2})^{\frac{9}{20}} (V^{-12} W^{-6} M^2 H)^{\frac{1}{60}} (W^{-30} H^5)^{\frac{1}{30}} (U^{-4})^{\frac{1}{2}} T \\
&= T \left(M^{\frac{1}{30}} H^{\frac{11}{60}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $M^2 H^{11} \ll X^{\frac{120}{43}}$ is required.

Assuming the condition (3). By applying the mean value estimate and Halász method to $M^3(s), H^4(s)$ and $K^2(s)$, we get

$$I \ll U^2 V^2 W^2 x^{-1} F (\log x)^c$$

where

$$\begin{aligned}
F &= \min \{V^{-6} (M^3 + T), V^{-6} M^3 + V^{-18} M^3 T, W^{-8} (H^4 + T), \\
&\quad W^{-8} H^4 + W^{-24} H^4 T, U^{-4} (K^2 + T), U^{-4} K^2 + U^{-12} K^2 T\}.
\end{aligned}$$

We consider 8 cases:

(a) $F \leq 2V^{-6} M^3, F \leq 2W^{-8} H^4, F \leq 2U^{-4} K^2$. Then

$$\begin{aligned}
U^2 V^2 W^2 F &\ll U^2 V^2 W^2 \min \{V^{-6} M^3, W^{-8} H^4, U^{-4} K^2\} \\
&\leq U^2 V^2 W^2 (V^{-6} M^3)^{\frac{1}{4}} (W^{-8} H^4)^{\frac{1}{4}} (U^{-4} K^2)^{\frac{1}{2}} \\
&= V^{\frac{1}{2}} M^{\frac{3}{4}} H K \\
&\ll x (\log x)^{-11B}.
\end{aligned}$$

(b) $F \leq 2V^{-6} M^3, F \leq 2W^{-8} H^4, F > 2U^{-4} K^2$. Then

$$\begin{aligned}
U^2 V^2 W^2 F &\ll U^2 V^2 W^2 \min \{V^{-6} M^3, W^{-8} H^4, U^{-4} T, U^{-12} K^2 T\} \\
&\leq U^2 V^2 W^2 (V^{-6} M^3)^{\frac{1}{3}} (W^{-8} H^4)^{\frac{1}{4}} (U^{-4} T)^{\frac{3}{8}} (U^{-12} K^2 T)^{\frac{1}{24}} \\
&= T^{\frac{5}{12}} M H K^{\frac{1}{12}} \\
&= T^{\frac{5}{12}} (M H K)^{\frac{1}{12}} \left(M^{\frac{11}{12}} H^{\frac{11}{12}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $M H \ll X^{\frac{268}{473}}$ is required.

(c) $F \leq 2V^{-6}M^3, F > 2W^{-8}H^4, F \leq 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-6}M^3, W^{-8}T, W^{-24}H^4T, U^{-4}K^2\} \\
&\leq U^2V^2W^2 (V^{-6}M^3)^{\frac{1}{3}} (W^{-8}T)^{\frac{1}{8}} (W^{-24}H^4T)^{\frac{1}{24}} (U^{-4}K^2)^{\frac{1}{2}} \\
&= T^{\frac{1}{6}} M H^{\frac{1}{6}} K \\
&= T^{\frac{1}{6}} (MHK) H^{-\frac{5}{6}} \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $X^{\frac{41}{215}} \ll H$ is required.

(d) $F \leq 2V^{-6}M^3, F > 2W^{-8}H^4, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-6}M^3, W^{-8}T, W^{-24}H^4T, U^{-4}T, U^{-12}K^2T\} \\
&\leq U^2V^2W^2 (V^{-6}M^3)^{\frac{1}{3}} (W^{-8}T)^{\frac{1}{8}} (W^{-24}H^4T)^{\frac{1}{24}} (U^{-4}T)^{\frac{1}{2}} \\
&= T^{\frac{2}{3}} (MH^{\frac{1}{6}}) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $M^6H \ll X^{\frac{94}{43}}$ is required.

(e) $F > 2V^{-6}M^3, F \leq 2W^{-8}H^4, F \leq 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-6}T, V^{-18}M^3T, W^{-8}H^4, U^{-4}K^2\} \\
&\leq U^2V^2W^2 (V^{-6}T)^{\frac{5}{24}} (V^{-18}M^3T)^{\frac{1}{24}} (W^{-8}H^4)^{\frac{1}{4}} (U^{-4}K^2)^{\frac{1}{2}} \\
&= T^{\frac{1}{4}} M^{\frac{1}{8}} HK \\
&= T^{\frac{1}{4}} (MHK) M^{-\frac{7}{8}} \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $X^{\frac{82}{301}} \ll M$ is required.

(f) $F > 2V^{-6}M^3, F \leq 2W^{-8}H^4, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-6}T, V^{-18}M^3T, W^{-8}H^4, U^{-4}T, U^{-12}K^2T\} \\
&\leq U^2V^2W^2 (V^{-6}T)^{\frac{5}{24}} (V^{-18}M^3T)^{\frac{1}{24}} (W^{-8}H^4)^{\frac{1}{4}} (U^{-4}T)^{\frac{1}{2}} \\
&= T^{\frac{3}{4}} (M^{\frac{1}{8}}H) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $MH^8 \ll X^{\frac{98}{43}}$ is required.

(g) $F > 2V^{-6}M^3, F > 2W^{-8}H^4, F \leq 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-6}T, V^{-18}M^3T, W^{-8}T, W^{-24}H^4T, U^{-4}K^2\} \\
&\leq U^2V^2W^2 (V^{-6}T)^{\frac{1}{3}} (W^{-8}T)^{\frac{1}{8}} (W^{-24}H^4T)^{\frac{1}{24}} (U^{-4}K^2)^{\frac{1}{2}} \\
&= T^{\frac{1}{2}} H^{\frac{1}{6}} K \\
&= T^{\frac{1}{2}} (MHK) (M^{-1}H^{-\frac{5}{6}}) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $X^{\frac{123}{43}} \ll M^6H^5$ is required.

(h) $F > 2V^{-6}M^3, F > 2W^{-8}H^4, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-6}, V^{-18}M^3, W^{-8}, W^{-24}H^4, U^{-4}, U^{-12}K^2\} T \\
&\leq U^2V^2W^2 (V^{-6})^{\frac{5}{24}} (V^{-18}M^3)^{\frac{1}{24}} (W^{-8})^{\frac{1}{4}} (U^{-4})^{\frac{1}{2}} T \\
&= TM^{\frac{1}{8}} \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $M \ll X^{\frac{16}{43}}$ is required.

Assuming the condition (4). By applying the mean value estimate and Halász method to $M^2(s), H^5(s)$ and $K^3(s)$, we get

$$I \ll U^2V^2W^2x^{-1}F(\log x)^c$$

where

$$F = \min \{ V^{-4} (M^2 + T), V^{-4} M^2 + V^{-12} M^2 T, W^{-10} (H^5 + T), \\ W^{-10} H^5 + W^{-30} H^5 T, U^{-6} (K^3 + T), U^{-6} K^3 + U^{-18} K^3 T \}.$$

We consider 8 cases:

(a) $F \leq 2V^{-4} M^2, F \leq 2W^{-10} H^5, F \leq 2U^{-6} K^3$. Then

$$\begin{aligned} U^2 V^2 W^2 F &\ll U^2 V^2 W^2 \min \{ V^{-4} M^2, W^{-10} H^5, U^{-6} K^3 \} \\ &\leq U^2 V^2 W^2 (V^{-4} M^2)^{\frac{1}{2}} (W^{-10} H^5)^{\frac{1}{6}} (U^{-6} K^3)^{\frac{1}{3}} \\ &= W^{\frac{1}{3}} M H^{\frac{5}{6}} K \\ &\ll x(\log x)^{-11B}. \end{aligned}$$

(b) $F \leq 2V^{-4} M^2, F \leq 2W^{-10} H^5, F > 2U^{-6} K^3$. Then

$$\begin{aligned} U^2 V^2 W^2 F &\ll U^2 V^2 W^2 \min \{ V^{-4} M^2, W^{-10} H^5, U^{-6} T, U^{-18} K^3 T \} \\ &\leq U^2 V^2 W^2 (V^{-4} M^2)^{\frac{1}{2}} (W^{-10} H^5)^{\frac{1}{6}} (U^{-6} T)^{\frac{17}{60}} (U^{-18} K^3 T)^{\frac{1}{60}} \\ &= T^{\frac{3}{10}} M H K^{\frac{1}{20}} \\ &= T^{\frac{3}{10}} (M H K)^{\frac{1}{20}} \left(M^{\frac{19}{20}} H^{\frac{19}{20}} \right) \\ &\ll x^{1-\varepsilon^2}, \end{aligned}$$

where $M H \ll X^{\frac{571}{817}}$ is required.

(c) $F \leq 2V^{-4} M^2, F > 2W^{-10} H^5, F \leq 2U^{-6} K^3$. Then

$$\begin{aligned} U^2 V^2 W^2 F &\ll U^2 V^2 W^2 \min \{ V^{-4} M^2, W^{-10} T, W^{-30} H^5 T, U^{-6} K^3 \} \\ &\leq U^2 V^2 W^2 (V^{-4} M^2)^{\frac{1}{2}} (W^{-10} T)^{\frac{3}{20}} (W^{-30} H^5 T)^{\frac{1}{60}} (U^{-6} K^3)^{\frac{1}{3}} \\ &= T^{\frac{1}{6}} M H^{\frac{1}{12}} K \\ &= T^{\frac{1}{6}} (M H K) H^{-\frac{11}{12}} \\ &\ll x^{1-\varepsilon^2}, \end{aligned}$$

where $X^{\frac{82}{473}} \ll H$ is required.

(d) $F \leq 2V^{-4} M^2, F > 2W^{-10} H^5, F > 2U^{-6} K^3$. Then

$$\begin{aligned} U^2 V^2 W^2 F &\ll U^2 V^2 W^2 \min \{ V^{-4} M^2, W^{-10} T, W^{-30} H^5 T, U^{-6} T, U^{-18} K^3 T \} \\ &\leq U^2 V^2 W^2 (V^{-4} M^2)^{\frac{1}{2}} (W^{-10} T)^{\frac{3}{20}} (W^{-30} H^5 T)^{\frac{1}{60}} (U^{-6} T)^{\frac{1}{3}} \\ &= T^{\frac{1}{2}} \left(M H^{\frac{1}{12}} \right) \\ &\ll x^{1-\varepsilon^2}, \end{aligned}$$

where $M^{12} H \ll X^{\frac{270}{43}}$ is required.

(e) $F > 2V^{-4} M^2, F \leq 2W^{-10} H^5, F \leq 2U^{-6} K^3$. Then

$$\begin{aligned} U^2 V^2 W^2 F &\ll U^2 V^2 W^2 \min \{ V^{-4} T, V^{-12} M^2 T, W^{-10} H^5, U^{-6} K^3 \} \\ &\leq U^2 V^2 W^2 (V^{-4} T)^{\frac{9}{20}} (V^{-12} M^2 T)^{\frac{1}{60}} (W^{-10} H^5)^{\frac{1}{6}} (U^{-6} K^3)^{\frac{1}{3}} \\ &= T^{\frac{7}{15}} M^{\frac{1}{30}} H K \\ &= T^{\frac{7}{15}} (M H K) M^{-\frac{29}{30}} \\ &\ll x^{1-\varepsilon^2}, \end{aligned}$$

where $X^{\frac{574}{1247}} \ll M$ is required.

(f) $F > 2V^{-4} M^2, F \leq 2W^{-10} H^5, F > 2U^{-6} K^3$. Then

$$\begin{aligned} U^2 V^2 W^2 F &\ll U^2 V^2 W^2 \min \{ V^{-4} T, V^{-12} M^2 T, W^{-10} H^5, U^{-6} T, U^{-18} K^3 T \} \\ &\leq U^2 V^2 W^2 (V^{-4} T)^{\frac{1}{2}} (W^{-10} H^5)^{\frac{1}{6}} (U^{-6} T)^{\frac{17}{60}} (U^{-18} K^3 T)^{\frac{1}{60}} \\ &= T^{\frac{4}{5}} H K^{\frac{1}{20}} \\ &= T^{\frac{4}{5}} (M H K)^{\frac{1}{20}} \left(M^{-\frac{1}{20}} H^{\frac{19}{20}} \right) \\ &\ll x^{1-\varepsilon^2}, \end{aligned}$$

where $M^{-1}H^{19} \ll X^{\frac{161}{43}}$ is required.

(g) $F > 2V^{-4}M^2, F > 2W^{-10}H^5, F \leq 2U^{-6}K^3$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-4}T, V^{-12}M^2T, W^{-10}T, W^{-30}H^5T, U^{-6}K^3\} \\ &\leq U^2V^2W^2 (V^{-4}T)^{\frac{1}{2}} (W^{-10}T)^{\frac{3}{20}} (W^{-30}H^5T)^{\frac{1}{60}} (U^{-6}K^3)^{\frac{1}{3}} \\ &= T^{\frac{2}{3}} H^{\frac{1}{12}} K \\ &= T^{\frac{2}{3}} (MHK) \left(M^{-1} H^{-\frac{11}{12}} \right) \\ &\ll x^{1-\varepsilon^2}, \end{aligned}$$

where $X^{\frac{328}{43}} \ll M^{12}H^{11}$ is required.

(h) $F > 2V^{-4}M^2, F > 2W^{-10}H^5, F > 2U^{-6}K^3$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-4}, V^{-12}M^2, W^{-10}, W^{-30}H^5, U^{-6}, U^{-18}K^3\} T \\ &\leq U^2V^2W^2 (V^{-4})^{\frac{1}{2}} (W^{-10})^{\frac{1}{5}} (U^{-6})^{\frac{17}{60}} (U^{-18}K^3)^{\frac{1}{60}} T \\ &= TK^{\frac{1}{20}} \\ &= T(MHK)^{\frac{1}{20}} \left(M^{-\frac{1}{20}} H^{-\frac{1}{20}} \right) \\ &\ll x^{1-\varepsilon^2}, \end{aligned}$$

where $X^{\frac{3}{43}} \ll MH$ is required. This can be obtained by using $X^{\frac{574}{1247}} \ll M$ and $X^{\frac{82}{473}} \ll H$.

Assuming the condition (5). By applying the mean value estimate and Halász method to $M^2(s), H^5(s)$ and $K^2(s)H(s)$, we get

$$I \ll U^2V^2W^2x^{-1}F(\log x)^c$$

where

$$\begin{aligned} F = \min \{ &V^{-4} (M^2 + T), V^{-4}M^2 + V^{-12}M^2T, W^{-10} (H^5 + T), \\ &W^{-10}H^5 + W^{-30}H^5T, U^{-4}W^{-2} (K^2H + T), U^{-4}W^{-2}K^2H + U^{-12}W^{-6}K^2HT \}. \end{aligned}$$

We consider 8 cases:

(a) $F \leq 2V^{-4}M^2, F \leq 2W^{-10}H^5, F \leq 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-4}M^2, W^{-10}H^5, U^{-4}W^{-2}K^2H\} \\ &\leq U^2V^2W^2 (V^{-4}M^2)^{\frac{1}{2}} (U^{-4}W^{-2}K^2H)^{\frac{1}{2}} \\ &= WMH^{\frac{1}{2}} K \\ &\ll x(\log x)^{-11B}. \end{aligned}$$

(b) $F \leq 2V^{-4}M^2, F \leq 2W^{-10}H^5, F > 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-4}M^2, W^{-10}H^5, U^{-4}W^{-2}T, U^{-12}W^{-6}K^2HT\} \\ &\leq U^2V^2W^2 (V^{-4}M^2)^{\frac{1}{2}} (W^{-10}H^5)^{\frac{1}{10}} (U^{-4}W^{-2}T)^{\frac{7}{20}} (U^{-12}W^{-6}K^2HT)^{\frac{1}{20}} \\ &= T^{\frac{2}{5}} MH^{\frac{11}{20}} K^{\frac{1}{10}} \\ &= T^{\frac{2}{5}} (MHK)^{\frac{1}{10}} \left(M^{\frac{9}{10}} H^{\frac{9}{20}} \right) \\ &\ll x^{1-\varepsilon^2}, \end{aligned}$$

where $M^2H \ll X^{\frac{446}{387}}$ is required.

(c) $F \leq 2V^{-4}M^2, F > 2W^{-10}H^5, F \leq 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-4}M^2, W^{-10}T, W^{-30}H^5T, U^{-4}W^{-2}K^2H\} \\ &\leq U^2V^2W^2 (V^{-4}M^2)^{\frac{1}{2}} (U^{-4}W^{-2}K^2H)^{\frac{1}{2}} \\ &= WMH^{\frac{1}{2}} K \\ &\ll x(\log x)^{-11B}. \end{aligned}$$

(d) $F \leq 2V^{-4}M^2, F > 2W^{-10}H^5, F > 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-4}M^2, W^{-10}T, W^{-30}H^5T, U^{-4}W^{-2}T, U^{-12}W^{-6}K^2HT\} \\ &\leq U^2V^2W^2 (V^{-4}M^2)^{\frac{1}{2}} (W^{-30}H^5T)^{\frac{1}{30}} (U^{-4}W^{-2}T)^{\frac{9}{20}} (U^{-12}W^{-6}K^2HT)^{\frac{1}{60}} \\ &= T^{\frac{1}{2}} MH^{\frac{11}{60}} K^{\frac{1}{30}} \end{aligned}$$

$$\begin{aligned}
&= T^{\frac{1}{2}} (MHK)^{\frac{1}{30}} \left(M^{\frac{29}{30}} H^{\frac{9}{60}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $M^{58}H^9 \ll X^{\frac{1264}{43}}$ is required.

(e) $F > 2V^{-4}M^2, F \leq 2W^{-10}H^5, F \leq 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-4}T, V^{-12}M^2T, W^{-10}H^5, U^{-4}W^{-2}K^2H\} \\
&\leq U^2V^2W^2 (V^{-4}T)^{\frac{7}{20}} (V^{-12}M^2T)^{\frac{1}{20}} (W^{-10}H^5)^{\frac{1}{10}} (U^{-4}W^{-2}K^2H)^{\frac{1}{2}} \\
&= T^{\frac{2}{5}} M^{\frac{1}{10}} HK \\
&= T^{\frac{2}{5}} (MHK) M^{-\frac{9}{10}} \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $X^{\frac{164}{387}} \ll M$ is required.

(f) $F > 2V^{-4}M^2, F \leq 2W^{-10}H^5, F > 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-4}T, V^{-12}M^2T, W^{-10}H^5, U^{-4}W^{-2}T, U^{-12}W^{-6}K^2HT\} \\
&\leq U^2V^2W^2 (V^{-4}T)^{\frac{7}{20}} (V^{-12}M^2T)^{\frac{1}{20}} (W^{-10}H^5)^{\frac{1}{10}} (U^{-4}W^{-2}T)^{\frac{1}{2}} \\
&= T^{\frac{9}{10}} \left(M^{\frac{1}{10}} H^{\frac{1}{2}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $MH^5 \ll X^{\frac{61}{43}}$ is required. This can be obtained by using $MH^5 \ll X^{\frac{60}{43}}$.

(g) $F > 2V^{-4}M^2, F > 2W^{-10}H^5, F \leq 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-4}T, V^{-12}M^2T, W^{-10}T, W^{-30}H^5T, U^{-4}W^{-2}K^2H\} \\
&\leq U^2V^2W^2 (V^{-4}T)^{\frac{9}{20}} (V^{-12}M^2T)^{\frac{1}{60}} (W^{-30}H^5T)^{\frac{1}{30}} (U^{-4}W^{-2}K^2H)^{\frac{1}{2}} \\
&= T^{\frac{1}{2}} M^{\frac{1}{30}} H^{\frac{2}{3}} K \\
&= T^{\frac{1}{2}} (MHK) \left(M^{-\frac{29}{30}} H^{-\frac{1}{3}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $X^{\frac{615}{43}} \ll M^{29}H^{10}$ is required.

(h) $F > 2V^{-4}M^2, F > 2W^{-10}H^5, F > 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-4}, V^{-12}M^2, W^{-10}, W^{-30}H^5, U^{-4}W^{-2}, U^{-12}W^{-6}K^2H\} T \\
&\leq U^2V^2W^2 (V^{-4})^{\frac{9}{20}} (V^{-12}M^2)^{\frac{1}{60}} (W^{-30}H^5)^{\frac{1}{30}} (U^{-4}W^{-2})^{\frac{1}{2}} T \\
&= T \left(M^{\frac{1}{30}} H^{\frac{1}{6}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $MH^5 \ll X^{\frac{60}{43}}$ is required.

Assuming the condition (6). By applying the mean value estimate and Halász method to $M^2(s), H^4(s)$ and $K^3(s)$, we get

$$I \ll U^2V^2W^2x^{-1}F(\log x)^c$$

where

$$\begin{aligned}
F &= \min \{V^{-4} (M^2 + T), V^{-4}M^2 + V^{-12}M^2T, W^{-8} (H^4 + T), \\
&\quad W^{-8}H^4 + W^{-24}H^4T, U^{-6} (K^3 + T), U^{-6}K^3 + U^{-18}K^3T\}.
\end{aligned}$$

We consider 8 cases:

(a) $F \leq 2V^{-4}M^2, F \leq 2W^{-8}H^4, F \leq 2U^{-6}K^3$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-4}M^2, W^{-8}H^4, U^{-6}K^3\} \\
&\leq U^2V^2W^2 (V^{-4}M^2)^{\frac{1}{2}} (W^{-8}H^4)^{\frac{1}{6}} (U^{-6}K^3)^{\frac{1}{3}} \\
&= W^{\frac{2}{3}} M H^{\frac{2}{3}} K \\
&\ll x(\log x)^{-11B}.
\end{aligned}$$

(b) $F \leq 2V^{-4}M^2, F \leq 2W^{-8}H^4, F > 2U^{-6}K^3$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-4}M^2, W^{-8}H^4, U^{-6}T, U^{-18}K^3T\} \\
&\leq U^2V^2W^2 (V^{-4}M^2)^{\frac{1}{2}} (W^{-8}H^4)^{\frac{1}{4}} (U^{-6}T)^{\frac{5}{24}} (U^{-18}K^3T)^{\frac{1}{24}}
\end{aligned}$$

$$\begin{aligned}
&= T^{\frac{1}{4}} M H K^{\frac{1}{8}} \\
&= T^{\frac{1}{4}} (M H K)^{\frac{1}{8}} \left(M^{\frac{7}{8}} H^{\frac{7}{8}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $MH \ll X^{\frac{219}{301}}$ is required.

(c) $F \leq 2V^{-4}M^2, F > 2W^{-8}H^4, F \leq 2U^{-6}K^3$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-4}M^2, W^{-8}T, W^{-24}H^4T, U^{-6}K^3\} \\
&\leq U^2V^2W^2 (V^{-4}M^2)^{\frac{1}{2}} (W^{-8}T)^{\frac{1}{8}} (W^{-24}H^4T)^{\frac{1}{24}} (U^{-6}K^3)^{\frac{1}{3}} \\
&= T^{\frac{1}{6}} M H^{\frac{1}{6}} K \\
&= T^{\frac{1}{6}} (M H K) H^{-\frac{5}{6}} \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $X^{\frac{41}{215}} \ll H$ is required.

(d) $F \leq 2V^{-4}M^2, F > 2W^{-8}H^4, F > 2U^{-6}K^3$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-4}M^2, W^{-8}T, W^{-24}H^4T, U^{-6}T, U^{-18}K^3T\} \\
&\leq U^2V^2W^2 (V^{-4}M^2)^{\frac{1}{2}} (W^{-8}T)^{\frac{1}{8}} (W^{-24}H^4T)^{\frac{1}{24}} (U^{-6}T)^{\frac{1}{3}} \\
&= T^{\frac{1}{2}} \left(M H^{\frac{1}{6}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $M^6H \ll X^{\frac{135}{43}}$ is required.

(e) $F > 2V^{-4}M^2, F \leq 2W^{-8}H^4, F \leq 2U^{-6}K^3$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-4}T, V^{-12}M^2T, W^{-8}H^4, U^{-6}K^3\} \\
&\leq U^2V^2W^2 (V^{-4}T)^{\frac{3}{8}} (V^{-12}M^2T)^{\frac{1}{24}} (W^{-8}H^4)^{\frac{1}{4}} (U^{-6}K^3)^{\frac{1}{3}} \\
&= T^{\frac{5}{12}} M^{\frac{1}{12}} H K \\
&= T^{\frac{5}{12}} (M H K) M^{-\frac{11}{12}} \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $X^{\frac{205}{473}} \ll M$ is required.

(f) $F > 2V^{-4}M^2, F \leq 2W^{-8}H^4, F > 2U^{-6}K^3$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-4}T, V^{-12}M^2T, W^{-8}H^4, U^{-6}T, U^{-18}K^3T\} \\
&\leq U^2V^2W^2 (V^{-4}T)^{\frac{1}{2}} (W^{-8}H^4)^{\frac{1}{4}} (U^{-6}T)^{\frac{5}{24}} (U^{-18}K^3T)^{\frac{1}{24}} \\
&= T^{\frac{3}{4}} H K^{\frac{1}{8}} \\
&= T^{\frac{3}{4}} (M H K)^{\frac{1}{8}} \left(M^{-\frac{1}{8}} H^{\frac{7}{8}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $M^{-1}H^7 \ll X^{\frac{55}{43}}$ is required.

(g) $F > 2V^{-4}M^2, F > 2W^{-8}H^4, F \leq 2U^{-6}K^3$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-4}T, V^{-12}M^2T, W^{-8}T, W^{-24}H^4T, U^{-6}K^3\} \\
&\leq U^2V^2W^2 (V^{-4}T)^{\frac{1}{2}} (W^{-8}T)^{\frac{1}{8}} (W^{-24}H^4T)^{\frac{1}{24}} (U^{-6}K^3)^{\frac{1}{3}} \\
&= T^{\frac{2}{3}} H^{\frac{1}{6}} K \\
&= T^{\frac{2}{3}} (M H K) \left(M^{-1} H^{-\frac{5}{6}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $X^{\frac{164}{43}} \ll M^6H^5$ is required.

(h) $F > 2V^{-4}M^2, F > 2W^{-8}H^4, F > 2U^{-6}K^3$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-4}, V^{-12}M^2, W^{-8}, W^{-24}H^4, U^{-6}, U^{-18}K^3\} T \\
&\leq U^2V^2W^2 (V^{-4})^{\frac{1}{2}} (W^{-8})^{\frac{1}{4}} (U^{-6})^{\frac{5}{24}} (U^{-18}K^3)^{\frac{1}{24}} T \\
&= T K^{\frac{1}{8}}
\end{aligned}$$

$$\begin{aligned}
&= T(MHK)^{\frac{1}{8}} \left(M^{-\frac{1}{8}} H^{-\frac{1}{8}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $X^{\frac{27}{43}} \ll MH$ is required.

Assuming the condition (7). By applying the mean value estimate and Halász method to $M^2(s)H(s)$, $H^6(s)$ and $K^2(s)$, we get

$$I \ll U^2 V^2 W^2 x^{-1} F(\log x)^c$$

where

$$\begin{aligned}
F = \min \{ &V^{-4}W^{-2} (M^2H + T), V^{-4}W^{-2}M^2H + V^{-12}W^{-6}M^2HT, \\
&W^{-12} (H^6 + T), W^{-12}H^6 + W^{-36}H^6T, U^{-4} (K^2 + T), U^{-4}K^2 + U^{-12}K^2T \}.
\end{aligned}$$

We consider 8 cases:

(a) $F \leq 2V^{-4}W^{-2}M^2H$, $F \leq 2W^{-12}H^6$, $F \leq 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{ V^{-4}W^{-2}M^2H, W^{-12}H^6, U^{-4}K^2 \} \\
&\leq U^2V^2W^2 (V^{-4}W^{-2}M^2H)^{\frac{1}{2}} (U^{-4}K^2)^{\frac{1}{2}} \\
&= WMH^{\frac{1}{2}}K \\
&\ll x(\log x)^{-11B}.
\end{aligned}$$

(b) $F \leq 2V^{-4}W^{-2}M^2H$, $F \leq 2W^{-12}H^6$, $F > 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{ V^{-4}W^{-2}M^2H, W^{-12}H^6, U^{-4}T, U^{-12}K^2T \} \\
&\leq U^2V^2W^2 (V^{-4}W^{-2}M^2H)^{\frac{1}{2}} (W^{-12}H^6)^{\frac{1}{12}} (U^{-4}T)^{\frac{3}{8}} (U^{-12}K^2T)^{\frac{1}{24}} \\
&= T^{\frac{5}{12}} M H K^{\frac{1}{12}} \\
&= T^{\frac{5}{12}} (MHK)^{\frac{1}{12}} \left(M^{\frac{11}{12}} H^{\frac{11}{12}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $MH \ll X^{\frac{268}{473}}$ is required.

(c) $F \leq 2V^{-4}W^{-2}M^2H$, $F > 2W^{-12}H^6$, $F \leq 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{ V^{-4}W^{-2}M^2H, W^{-12}T, W^{-36}H^6T, U^{-4}K^2 \} \\
&\leq U^2V^2W^2 (V^{-4}W^{-2}M^2H)^{\frac{1}{2}} (U^{-4}K^2)^{\frac{1}{2}} \\
&= WMH^{\frac{1}{2}}K \\
&\ll x(\log x)^{-11B}.
\end{aligned}$$

(d) $F \leq 2V^{-4}W^{-2}M^2H$, $F > 2W^{-12}H^6$, $F > 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{ V^{-4}W^{-2}M^2H, W^{-12}T, W^{-36}H^6T, U^{-4}T, U^{-12}K^2T \} \\
&\leq U^2V^2W^2 (V^{-4}W^{-2}M^2H)^{\frac{1}{2}} (W^{-36}H^6T)^{\frac{1}{36}} (U^{-4}T)^{\frac{11}{24}} (U^{-12}K^2T)^{\frac{1}{72}} \\
&= T^{\frac{1}{2}} M H^{\frac{2}{3}} K^{\frac{1}{36}} \\
&= T^{\frac{1}{2}} (MHK)^{\frac{1}{36}} \left(M^{\frac{35}{36}} H^{\frac{23}{36}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $M^{35}H^{23} \ll X^{\frac{767}{43}}$ is required.

(e) $F > 2V^{-4}W^{-2}M^2H$, $F \leq 2W^{-12}H^6$, $F \leq 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{ V^{-4}W^{-2}T, V^{-12}W^{-6}M^2HT, W^{-12}H^6, U^{-4}K^2 \} \\
&\leq U^2V^2W^2 (V^{-4}W^{-2}T)^{\frac{3}{8}} (V^{-12}W^{-6}M^2HT)^{\frac{1}{24}} (W^{-12}H^6)^{\frac{1}{12}} (U^{-4}K^2)^{\frac{1}{2}} \\
&= T^{\frac{5}{12}} M^{\frac{1}{12}} H^{\frac{13}{24}} K \\
&= T^{\frac{5}{12}} (MHK) \left(M^{-\frac{11}{12}} H^{-\frac{11}{24}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $X^{\frac{410}{473}} \ll M^2H$ is required.

(f) $F > 2V^{-4}W^{-2}M^2H$, $F \leq 2W^{-12}H^6$, $F > 2U^{-4}K^2$. Then

$$U^2V^2W^2F \ll U^2V^2W^2 \min \{ V^{-4}W^{-2}T, V^{-12}W^{-6}M^2HT, W^{-12}H^6, U^{-4}T, U^{-12}K^2T \}$$

$$\begin{aligned}
&\leq U^2 V^2 W^2 (V^{-4} W^{-2} T)^{\frac{3}{8}} (V^{-12} W^{-6} M^2 H T)^{\frac{1}{24}} (W^{-12} H^6)^{\frac{1}{12}} (U^{-4} T)^{\frac{1}{2}} \\
&= T^{\frac{11}{12}} \left(M^{\frac{1}{12}} H^{\frac{13}{24}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $M^2 H^{13} \ll X^{\frac{130}{43}}$ is required.

(g) $F > 2V^{-4} W^{-2} M^2 H, F > 2W^{-12} H^6, F \leq 2U^{-4} K^2$. Then

$$\begin{aligned}
U^2 V^2 W^2 F &\ll U^2 V^2 W^2 \min \{ V^{-4} W^{-2} T, V^{-12} W^{-6} M^2 H T, W^{-12} T, W^{-36} H^6 T, U^{-4} K^2 \} \\
&\leq U^2 V^2 W^2 (V^{-4} W^{-2} T)^{\frac{11}{24}} (V^{-12} W^{-6} M^2 H T)^{\frac{1}{72}} (W^{-36} H^6 T)^{\frac{1}{36}} (U^{-4} K^2)^{\frac{1}{2}} \\
&= T^{\frac{1}{2}} M^{\frac{1}{36}} H^{\frac{13}{72}} K \\
&= T^{\frac{1}{2}} (M H K) \left(M^{-\frac{35}{36}} H^{-\frac{59}{72}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $X^{\frac{1476}{43}} \ll M^{70} H^{59}$ is required.

(h) $F > 2V^{-4} W^{-2} M^2 H, F > 2W^{-12} H^6, F > 2U^{-4} K^2$. Then

$$\begin{aligned}
U^2 V^2 W^2 F &\ll U^2 V^2 W^2 \min \{ V^{-4} W^{-2}, V^{-12} W^{-6} M^2 H, W^{-12}, W^{-36} H^6, U^{-4}, U^{-12} K^2 \} T \\
&\leq U^2 V^2 W^2 (V^{-4} W^{-2})^{\frac{1}{2}} (W^{-12})^{\frac{1}{12}} (U^{-4})^{\frac{3}{8}} (U^{-12} K^2)^{\frac{1}{24}} T \\
&= T K^{\frac{1}{12}} \\
&= T (M H K)^{\frac{1}{12}} \left(M^{-\frac{1}{12}} H^{-\frac{1}{12}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $X^{\frac{19}{43}} \ll M H$ is required. This can be obtained by using $X^{\frac{738}{1505}} \ll M H^{\frac{59}{70}} \ll M H$.

Assuming the condition (8). By applying the mean value estimate and Halász method to $M^2(s)H(s), H^7(s)$ and $K^2(s)$, we get

$$I \ll U^2 V^2 W^2 x^{-1} F (\log x)^c$$

where

$$\begin{aligned}
F &= \min \{ V^{-4} W^{-2} (M^2 H + T), V^{-4} W^{-2} M^2 H + V^{-12} W^{-6} M^2 H T, \\
&\quad W^{-14} (H^7 + T), W^{-14} H^7 + W^{-42} H^7 T, U^{-4} (K^2 + T) U^{-4} K^2 + U^{-12} K^2 T \}.
\end{aligned}$$

We consider 8 cases:

(a) $F \leq 2V^{-4} W^{-2} M^2 H, F \leq 2W^{-14} H^7, F \leq 2U^{-4} K^2$. Then

$$\begin{aligned}
U^2 V^2 W^2 F &\ll U^2 V^2 W^2 \min \{ V^{-4} W^{-2} M^2 H, W^{-14} H^7, U^{-4} K^2 \} \\
&\leq U^2 V^2 W^2 (V^{-4} W^{-2} M^2 H)^{\frac{1}{2}} (U^{-4} K^2)^{\frac{1}{2}} \\
&= W M H^{\frac{1}{2}} K \\
&\ll x (\log x)^{-11B}.
\end{aligned}$$

(b) $F \leq 2V^{-4} W^{-2} M^2 H, F \leq 2W^{-14} H^7, F > 2U^{-4} K^2$. Then

$$\begin{aligned}
U^2 V^2 W^2 F &\ll U^2 V^2 W^2 \min \{ V^{-4} W^{-2} M^2 H, W^{-14} H^7, U^{-4} T, U^{-12} K^2 T \} \\
&\leq U^2 V^2 W^2 (V^{-4} W^{-2} M^2 H)^{\frac{1}{2}} (W^{-14} H^7)^{\frac{1}{14}} (U^{-4} T)^{\frac{11}{28}} (U^{-12} K^2 T)^{\frac{1}{28}} \\
&= T^{\frac{3}{7}} M H K^{\frac{1}{14}} \\
&= T^{\frac{3}{7}} (M H K)^{\frac{1}{14}} \left(M^{\frac{13}{14}} H^{\frac{13}{14}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $M H \ll X^{\frac{313}{559}}$ is required.

(c) $F \leq 2V^{-4} W^{-2} M^2 H, F > 2W^{-14} H^7, F \leq 2U^{-4} K^2$. Then

$$\begin{aligned}
U^2 V^2 W^2 F &\ll U^2 V^2 W^2 \min \{ V^{-4} W^{-2} M^2 H, W^{-14} T, W^{-42} H^7 T, U^{-4} K^2 \} \\
&\leq U^2 V^2 W^2 (V^{-4} W^{-2} M^2 H)^{\frac{1}{2}} (U^{-4} K^2)^{\frac{1}{2}} \\
&= W M H^{\frac{1}{2}} K \\
&\ll x (\log x)^{-11B}.
\end{aligned}$$

(d) $F \leq 2V^{-4}W^{-2}M^2H, F > 2W^{-14}H^7, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-4}W^{-2}M^2H, W^{-14}T, W^{-42}H^7T, U^{-4}T, U^{-12}K^2T\} \\
&\leq U^2V^2W^2 (V^{-4}W^{-2}M^2H)^{\frac{1}{2}} (W^{-42}H^7T)^{\frac{1}{42}} (U^{-4}T)^{\frac{13}{28}} (U^{-12}K^2T)^{\frac{1}{84}} \\
&= T^{\frac{1}{2}} M H^{\frac{2}{3}} K^{\frac{1}{42}} \\
&= T^{\frac{1}{2}} (MHK)^{\frac{1}{42}} \left(M^{\frac{41}{42}} H^{\frac{9}{14}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $M^{41}H^{27} \ll X^{\frac{902}{43}}$ is required.

(e) $F > 2V^{-4}W^{-2}M^2H, F \leq 2W^{-14}H^7, F \leq 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-4}W^{-2}T, V^{-12}W^{-6}M^2HT, W^{-14}H^7, U^{-4}K^2\} \\
&\leq U^2V^2W^2 (V^{-4}W^{-2}T)^{\frac{11}{28}} (V^{-12}W^{-6}M^2HT)^{\frac{1}{28}} (W^{-14}H^7)^{\frac{1}{14}} (U^{-4}K^2)^{\frac{1}{2}} \\
&= T^{\frac{3}{7}} M^{\frac{1}{14}} H^{\frac{15}{28}} K \\
&= T^{\frac{3}{7}} (MHK) \left(M^{-\frac{13}{14}} H^{-\frac{13}{28}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $X^{\frac{492}{559}} \ll M^2H$ is required.

(f) $F > 2V^{-4}W^{-2}M^2H, F \leq 2W^{-14}H^7, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-4}W^{-2}T, V^{-12}W^{-6}M^2HT, W^{-14}H^7, U^{-4}T, U^{-12}K^2T\} \\
&\leq U^2V^2W^2 (V^{-4}W^{-2}T)^{\frac{11}{28}} (V^{-12}W^{-6}M^2HT)^{\frac{1}{28}} (W^{-14}H^7)^{\frac{1}{14}} (U^{-4}T)^{\frac{1}{2}} \\
&= T^{\frac{13}{14}} \left(M^{\frac{1}{14}} H^{\frac{15}{28}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $M^2H^{15} \ll X^{\frac{138}{43}}$ is required.

(g) $F > 2V^{-4}W^{-2}M^2H, F > 2W^{-14}H^7, F \leq 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-4}W^{-2}T, V^{-12}W^{-6}M^2HT, W^{-14}T, W^{-42}H^7T, U^{-4}K^2\} \\
&\leq U^2V^2W^2 (V^{-4}W^{-2}T)^{\frac{13}{88}} (V^{-12}W^{-6}M^2HT)^{\frac{1}{84}} (W^{-42}H^7T)^{\frac{1}{42}} (U^{-4}K^2)^{\frac{1}{2}} \\
&= T^{\frac{1}{2}} M^{\frac{1}{42}} H^{\frac{5}{28}} K \\
&= T^{\frac{1}{2}} (MHK) \left(M^{-\frac{41}{42}} H^{-\frac{23}{28}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $X^{\frac{1722}{43}} \ll M^{82}H^{69}$ is required.

(h) $F > 2V^{-4}W^{-2}M^2H, F > 2W^{-14}H^7, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min \{V^{-4}W^{-2}, V^{-12}W^{-6}M^2H, W^{-14}, W^{-42}H^7, U^{-4}, U^{-12}K^2\} T \\
&\leq U^2V^2W^2 (V^{-4}W^{-2})^{\frac{1}{2}} (W^{-14})^{\frac{1}{14}} (U^{-4})^{\frac{11}{28}} (U^{-12}K^2)^{\frac{1}{28}} T \\
&= TK^{\frac{1}{14}} \\
&= T(MHK)^{\frac{1}{14}} \left(M^{-\frac{1}{14}} H^{-\frac{1}{14}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $X^{\frac{15}{43}} \ll MH$ is required. This can be obtained by using $X^{\frac{246}{559}} \ll MH^{\frac{1}{2}} \ll MH$.

Assuming the condition (9). By applying the mean value estimate and Halász method to $M^2(s), H^6(s)$ and $K^2(s)H(s)$, we get

$$I \ll U^2V^2W^2x^{-1}F(\log x)^c$$

where

$$\begin{aligned}
F = \min \{ &V^{-4} (M^2 + T), V^{-4}M^2 + V^{-12}M^2T, W^{-12} (H^6 + T), W^{-12}H^6 + W^{-36}H^6T, \\
&U^{-4}W^{-2} (K^2H + T), U^{-4}W^{-2}K^2H + U^{-12}W^{-6}K^2HT \}.
\end{aligned}$$

We consider 8 cases:

(a) $F \leq 2V^{-4}M^2, F \leq 2W^{-12}H^6, F \leq 2U^{-4}W^{-2}K^2H$. Then

$$U^2V^2W^2F \ll U^2V^2W^2 \min \{V^{-4}M^2, W^{-12}H^6, U^{-4}W^{-2}K^2H\}$$

$$\begin{aligned}
&\leq U^2 V^2 W^2 (V^{-4} M^2)^{\frac{1}{2}} (U^{-4} W^{-2} K^2 H)^{\frac{1}{2}} \\
&= W M H^{\frac{1}{2}} K \\
&\ll x(\log x)^{-11B}.
\end{aligned}$$

(b) $F \leq 2V^{-4} M^2, F \leq 2W^{-12} H^6, F > 2U^{-4} W^{-2} K^2 H$. Then

$$\begin{aligned}
U^2 V^2 W^2 F &\ll U^2 V^2 W^2 \min \{V^{-4} M^2, W^{-12} H^6, U^{-4} W^{-2} T, U^{-12} W^{-6} K^2 H T\} \\
&\leq U^2 V^2 W^2 (V^{-4} M^2)^{\frac{1}{2}} (W^{-12} H^6)^{\frac{1}{12}} (U^{-4} W^{-2} T)^{\frac{3}{8}} (U^{-12} W^{-6} K^2 H T)^{\frac{1}{24}} \\
&= T^{\frac{5}{12}} M H^{\frac{13}{24}} K^{\frac{1}{12}} \\
&= T^{\frac{5}{12}} (M H K)^{\frac{1}{12}} \left(M^{\frac{11}{12}} H^{\frac{11}{24}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $M^2 H \ll X^{\frac{536}{473}}$ is required.

(c) $F \leq 2V^{-4} M^2, F > 2W^{-12} H^6, F \leq 2U^{-4} W^{-2} K^2 H$. Then

$$\begin{aligned}
U^2 V^2 W^2 F &\ll U^2 V^2 W^2 \min \{V^{-4} M^2, W^{-12} T, W^{-36} H^6 T, U^{-4} W^{-2} K^2 H\} \\
&\leq U^2 V^2 W^2 (V^{-4} M^2)^{\frac{1}{2}} (U^{-4} W^{-2} K^2 H)^{\frac{1}{2}} \\
&= W M H^{\frac{1}{2}} K \\
&\ll x(\log x)^{-11B}.
\end{aligned}$$

(d) $F \leq 2V^{-4} M^2, F > 2W^{-12} H^6, F > 2U^{-4} W^{-2} K^2 H$. Then

$$\begin{aligned}
U^2 V^2 W^2 F &\ll U^2 V^2 W^2 \min \{V^{-4} M^2, W^{-12} T, W^{-36} H^6 T, U^{-4} W^{-2} T, U^{-12} W^{-6} K^2 H T\} \\
&\leq U^2 V^2 W^2 (V^{-4} M^2)^{\frac{1}{2}} (W^{-36} H^6 T)^{\frac{1}{36}} (U^{-4} W^{-2} T)^{\frac{11}{24}} (U^{-12} W^{-6} K^2 H T)^{\frac{1}{72}} \\
&= T^{\frac{1}{2}} M H^{\frac{13}{72}} K^{\frac{1}{36}} \\
&= T^{\frac{1}{2}} (M H K)^{\frac{1}{36}} \left(M^{\frac{35}{36}} H^{\frac{11}{72}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $M^{70} H^{11} \ll X^{\frac{1534}{43}}$ is required.

(e) $F > 2V^{-4} M^2, F \leq 2W^{-12} H^6, F \leq 2U^{-4} W^{-2} K^2 H$. Then

$$\begin{aligned}
U^2 V^2 W^2 F &\ll U^2 V^2 W^2 \min \{V^{-4} T, V^{-12} M^2 T, W^{-12} H^6, U^{-4} W^{-2} K^2 H\} \\
&\leq U^2 V^2 W^2 (V^{-4} T)^{\frac{3}{8}} (V^{-12} M^2 T)^{\frac{1}{24}} (W^{-12} H^6)^{\frac{1}{12}} (U^{-4} W^{-2} K^2 H)^{\frac{1}{2}} \\
&= T^{\frac{5}{12}} M^{\frac{1}{12}} H K \\
&= T^{\frac{5}{12}} (M H K) M^{-\frac{11}{12}} \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $X^{\frac{205}{473}} \ll M$ is required.

(f) $F > 2V^{-4} M^2, F \leq 2W^{-12} H^6, F > 2U^{-4} W^{-2} K^2 H$. Then

$$\begin{aligned}
U^2 V^2 W^2 F &\ll U^2 V^2 W^2 \min \{V^{-4} T, V^{-12} M^2 T, W^{-12} H^6, U^{-4} W^{-2} T, U^{-12} W^{-6} K^2 H T\} \\
&\leq U^2 V^2 W^2 (V^{-4} T)^{\frac{3}{8}} (V^{-12} M^2 T)^{\frac{1}{24}} (W^{-12} H^6)^{\frac{1}{12}} (U^{-4} W^{-2} T)^{\frac{1}{2}} \\
&= T^{\frac{11}{12}} \left(M^{\frac{1}{12}} H^{\frac{1}{2}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $M H^6 \ll X^{\frac{65}{43}}$ is required.

(g) $F > 2V^{-4} M^2, F > 2W^{-12} H^6, F \leq 2U^{-4} W^{-2} K^2 H$. Then

$$\begin{aligned}
U^2 V^2 W^2 F &\ll U^2 V^2 W^2 \min \{V^{-4} T, V^{-12} M^2 T, W^{-12} T, W^{-36} H^6 T, U^{-4} W^{-2} K^2 H\} \\
&\leq U^2 V^2 W^2 (V^{-4} T)^{\frac{11}{24}} (V^{-12} M^2 T)^{\frac{1}{72}} (W^{-36} H^6 T)^{\frac{1}{36}} (U^{-4} W^{-2} K^2 H)^{\frac{1}{2}} \\
&= T^{\frac{1}{2}} M^{\frac{1}{36}} H^{\frac{2}{3}} K \\
&= T^{\frac{1}{2}} (M H K) \left(M^{-\frac{35}{36}} H^{-\frac{1}{3}} \right) \\
&\ll x^{1-\varepsilon^2},
\end{aligned}$$

where $X_{43}^{\frac{738}{43}} \ll M^{35}H^{12}$ is required.

(h) $F > 2V^{-4}M^2, F > 2W^{-12}H^6, F > 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-4}, V^{-12}M^2, W^{-12}, W^{-36}H^6, U^{-4}W^{-2}, U^{-12}W^{-6}K^2H\}T \\ &\leq U^2V^2W^2 (V^{-4})^{\frac{3}{8}} (V^{-12}M^2)^{\frac{1}{24}} (W^{-12})^{\frac{1}{12}} (U^{-4}W^{-2})^{\frac{1}{2}} T \\ &= TM^{\frac{1}{12}} \\ &\ll x^{1-\varepsilon^2}, \end{aligned}$$

where $M \ll X_{43}^{\frac{24}{43}}$ is required. This can be obtained by using $M \ll MH_{70}^{\frac{11}{70}} \ll X_{1505}^{\frac{767}{1505}}$.

Finally, by combining all the cases above, Lemma 3.2 is proved. \square

Lemma 3.3. Suppose that $NQRK = X$, and N, Q and R satisfy one of the following 10 conditions:

- (1) $X_{56994}^{\frac{27367}{56994}} \ll NQ \ll X_{559}^{\frac{231}{559}}, X_{817}^{\frac{79}{817}} \ll R \ll Q \ll (NQ)^{-\frac{35}{23}} X_{859}^{\frac{767}{859}};$
- (2) $X_{559}^{\frac{231}{559}} \ll NQ \ll X_{7826}^{\frac{3275}{7826}}, X_{817}^{\frac{79}{817}} \ll R \ll Q \ll (NQ)^{-1} X_{559}^{\frac{313}{559}};$
- (3) $X_{7826}^{\frac{3275}{7826}} \ll NQ \ll X_{33497}^{\frac{15005}{33497}}, X_{817}^{\frac{79}{817}} \ll R \ll Q \ll (NQ)^{-\frac{41}{27}} X_{1161}^{\frac{902}{1161}};$
- (4) $X_{28595}^{\frac{13074}{28595}} \ll NQ \ll X_{43}^{\frac{29}{43}}, X_{817}^{\frac{79}{817}} \ll R \ll Q \ll (NQ)^{-\frac{1}{5}} X_{43}^{\frac{12}{43}};$
- (5) $X_{43}^{\frac{29}{43}} \ll NQ \ll X_{86}^{\frac{41}{86}}, X_{817}^{\frac{79}{817}} \ll R \ll Q \ll (NQ)^{\frac{1}{7}} X_{301}^{\frac{55}{301}};$
- (6) $X_{86}^{\frac{41}{86}} \ll NQ \ll X_{473}^{\frac{227}{473}}, X_{817}^{\frac{79}{817}} \ll R \ll Q \ll (NQ)^{-1} X_{301}^{\frac{219}{301}};$
- (7) $X_{473}^{\frac{227}{473}} \ll NQ \ll X_{4333}^{\frac{233}{4333}}, X_{817}^{\frac{79}{817}} \ll R \ll Q \ll (NQ)^{-\frac{58}{9}} X_{387}^{\frac{1264}{387}};$
- (8) $X_{4333}^{\frac{233}{4333}} \ll NQ \ll X_{5203}^{\frac{2501}{5203}}, X_{817}^{\frac{79}{817}} \ll R \ll Q \ll (NQ)^{-\frac{1}{6}} X_{258}^{\frac{65}{258}};$
- (9) $X_{5203}^{\frac{2501}{5203}} \ll NQ \ll X_{1032}^{\frac{499}{1032}}, X_{817}^{\frac{79}{817}} \ll R \ll Q \ll (NQ)^{-2} X_{473}^{\frac{536}{473}};$
- (10) $X_{1032}^{\frac{499}{1032}} \ll NQ \ll X_{57150}^{\frac{28277}{57150}}, X_{817}^{\frac{79}{817}} \ll R \ll Q \ll (NQ)^{-\frac{70}{11}} X_{473}^{\frac{1534}{473}}.$

Assume further that for $(\log X)^{B/\varepsilon} \leq |t| \leq 2X, N(b+it)Q(b+it) \ll (\log x)^{-B/\varepsilon}$ and $R(b+it) \ll (\log x)^{-B/\varepsilon}$. Then for $(\log X)^{B/\varepsilon} \leq T \leq X$, we have

$$\left(\min\left(\eta, \frac{1}{T}\right)\right)^2 \int_T^{2T} |N(b+it)Q(b+it)R(b+it)K(b+it)|^2 dt \ll \eta^2 (\log x)^{-10B}.$$

Proof. The proof is similar to that of [[26], Lemma 13]. Let $M(s) = N(s)Q(s)$ and $H(s) = R(s)$. Then by Lemma 3.2, Lemma 3.3 is proved. \square

Lemma 3.4. Suppose that $NQRK = X$, and N, Q and R satisfy $X_{817}^{\frac{79}{817}} \ll R \ll Q$ and one of the following 9 conditions:

- (1) $(NR)Q \ll X_{1247}^{\frac{673}{1247}}, X_{473}^{\frac{82}{473}} \ll Q, (NR)^{29}Q^{-1} \ll X_{43}^{\frac{427}{43}}, X_{817}^{\frac{246}{817}} \ll (NR), (NR)^{-1}Q^{29} \ll X_{43}^{\frac{263}{43}}, X_{43}^{\frac{246}{43}} \ll (NR)^{12}Q^{11};$
- (2) $(NR)Q \ll X_{387}^{\frac{223}{387}}, (NR)^{29}Q^{19} \ll X_{43}^{\frac{632}{43}}, X_{387}^{\frac{328}{387}} \ll (NR)^2Q, (NR)^2Q^{11} \ll X_{43}^{\frac{120}{43}}, X_{43}^{\frac{1230}{43}} \ll (NR)^{58}Q^{49};$
- (3) $(NR)Q \ll X_{473}^{\frac{268}{473}}, X_{215}^{\frac{41}{215}} \ll Q, (NR)^6Q \ll X_{43}^{\frac{94}{43}}, X_{301}^{\frac{82}{301}} \ll (NR) \ll X_{43}^{\frac{16}{43}}, (NR)Q^8 \ll X_{43}^{\frac{98}{43}}, X_{43}^{\frac{123}{43}} \ll (NR)^6Q^5;$
- (4) $(NR)Q \ll X_{817}^{\frac{571}{817}}, X_{473}^{\frac{82}{473}} \ll Q, (NR)^{12}Q \ll X_{43}^{\frac{270}{43}}, X_{1247}^{\frac{574}{1247}} \ll (NR), (NR)^{-1}Q^{19} \ll X_{43}^{\frac{161}{43}}, X_{43}^{\frac{328}{43}} \ll (NR)^{12}Q^{11};$
- (5) $(NR)^2Q \ll X_{387}^{\frac{446}{387}}, (NR)^{58}Q^9 \ll X_{43}^{\frac{1264}{43}}, X_{387}^{\frac{164}{387}} \ll (NR), (NR)Q^5 \ll X_{43}^{\frac{60}{43}}, X_{43}^{\frac{615}{43}} \ll (NR)^{29}Q^{10};$
- (6) $X_{43}^{\frac{27}{43}} \ll (NR)Q \ll X_{301}^{\frac{219}{301}}, X_{215}^{\frac{41}{215}} \ll Q, (NR)^6Q \ll X_{43}^{\frac{135}{43}}, X_{473}^{\frac{205}{473}} \ll (NR), (NR)^{-1}Q^7 \ll X_{43}^{\frac{55}{43}}, X_{43}^{\frac{164}{43}} \ll (NR)^6Q^5;$
- (7) $(NR)Q \ll X_{473}^{\frac{268}{473}}, (NR)^{35}Q^{23} \ll X_{43}^{\frac{767}{43}}, X_{473}^{\frac{410}{473}} \ll (NR)^2Q, (NR)^2Q^{13} \ll X_{43}^{\frac{130}{43}}, X_{43}^{\frac{1476}{43}} \ll (NR)^{70}Q^{59};$
- (8) $(NR)Q \ll X_{559}^{\frac{313}{559}}, (NR)^{41}Q^{27} \ll X_{43}^{\frac{902}{43}}, X_{559}^{\frac{492}{559}} \ll (NR)^2Q, (NR)^2Q^{15} \ll X_{43}^{\frac{138}{43}}, X_{43}^{\frac{1722}{43}} \ll (NR)^{82}Q^{69};$
- (9) $(NR)^2Q \ll X_{473}^{\frac{536}{473}}, (NR)^{70}Q^{11} \ll X_{43}^{\frac{1534}{43}}, X_{473}^{\frac{205}{473}} \ll (NR), (NR)Q^6 \ll X_{43}^{\frac{65}{43}}, X_{43}^{\frac{738}{43}} \ll (NR)^{35}Q^{12}.$

Assume further that for $(\log X)^{B/\varepsilon} \leq |t| \leq 2X, N(b+it)R(b+it) \ll (\log x)^{-B/\varepsilon}$ and $Q(b+it) \ll (\log x)^{-B/\varepsilon}$. Then for $(\log X)^{B/\varepsilon} \leq T \leq X$, we have

$$\left(\min\left(\eta, \frac{1}{T}\right)\right)^2 \int_T^{2T} |N(b+it)Q(b+it)R(b+it)K(b+it)|^2 dt \ll \eta^2 (\log x)^{-10B}.$$

Proof. Let $M(s) = N(s)R(s)$ and $H(s) = Q(s)$ and by Lemma 3.2, Lemma 3.4 is proved. \square

Lemma 3.5. Suppose that $NQRK = X$, and N, Q and R satisfy $X_{817}^{\frac{79}{817}} \ll R \ll Q$ and one of the following 9 conditions:

- (1) $(QR)N \ll X_{1247}^{\frac{673}{1247}}, X_{473}^{\frac{82}{473}} \ll N, (QR)^{29}N^{-1} \ll X_{43}^{\frac{427}{43}}, X_{817}^{\frac{246}{817}} \ll (QR), (QR)^{-1}N^{29} \ll X_{43}^{\frac{263}{43}}, X_{43}^{\frac{246}{43}} \ll (QR)^{12}N^{11};$
- (2) $(QR)N \ll X_{387}^{\frac{223}{387}}, (QR)^{29}N^{19} \ll X_{43}^{\frac{632}{43}}, X_{387}^{\frac{328}{387}} \ll (QR)^2N, (QR)^2N^{11} \ll X_{43}^{\frac{120}{43}}, X_{43}^{\frac{1230}{43}} \ll (QR)^{58}N^{49};$
- (3) $(QR)N \ll X_{473}^{\frac{268}{473}}, X_{215}^{\frac{41}{215}} \ll N, (QR)^6N \ll X_{43}^{\frac{94}{43}}, X_{301}^{\frac{82}{301}} \ll (QR) \ll X_{43}^{\frac{16}{43}}, (QR)N^8 \ll X_{43}^{\frac{98}{43}}, X_{43}^{\frac{123}{43}} \ll (QR)^6N^5;$
- (4) $(QR)N \ll X_{817}^{\frac{571}{817}}, X_{473}^{\frac{82}{473}} \ll N, (QR)^{12}N \ll X_{43}^{\frac{270}{43}}, X_{1247}^{\frac{574}{1247}} \ll (QR), (QR)^{-1}N^{19} \ll X_{43}^{\frac{161}{43}}, X_{43}^{\frac{328}{43}} \ll (QR)^{12}N^{11};$
- (5) $(QR)^2N \ll X_{387}^{\frac{446}{387}}, (QR)^{58}N^9 \ll X_{43}^{\frac{1264}{43}}, X_{387}^{\frac{164}{387}} \ll (QR), (QR)N^5 \ll X_{43}^{\frac{60}{43}}, X_{43}^{\frac{615}{43}} \ll (QR)^{29}N^{10};$
- (6) $X_{43}^{\frac{27}{43}} \ll (QR)N \ll X_{301}^{\frac{219}{301}}, X_{215}^{\frac{41}{215}} \ll N, (QR)^6N \ll X_{43}^{\frac{135}{43}}, X_{473}^{\frac{205}{473}} \ll (QR), (QR)^{-1}N^7 \ll X_{43}^{\frac{55}{43}}, X_{43}^{\frac{164}{43}} \ll (QR)^6N^5;$
- (7) $(QR)N \ll X_{473}^{\frac{268}{473}}, (QR)^{35}N^{23} \ll X_{43}^{\frac{767}{43}}, X_{473}^{\frac{410}{473}} \ll (QR)^2N, (QR)^2N^{13} \ll X_{43}^{\frac{130}{43}}, X_{43}^{\frac{1476}{43}} \ll (QR)^{70}N^{59};$
- (8) $(QR)N \ll X_{559}^{\frac{313}{559}}, (QR)^{41}N^{27} \ll X_{43}^{\frac{902}{43}}, X_{559}^{\frac{492}{559}} \ll (QR)^2N, (QR)^2N^{15} \ll X_{43}^{\frac{138}{43}}, X_{43}^{\frac{1722}{43}} \ll (QR)^{82}N^{69};$
- (9) $(QR)^2N \ll X_{473}^{\frac{536}{473}}, (QR)^{70}N^{11} \ll X_{43}^{\frac{1534}{43}}, X_{473}^{\frac{205}{473}} \ll (QR), (QR)N^6 \ll X_{43}^{\frac{65}{43}}, X_{43}^{\frac{738}{43}} \ll (QR)^{35}N^{12}.$

Assume further that for $(\log X)^{B/\varepsilon} \leq |t| \leq 2X$, $Q(b+it)R(b+it) \ll (\log x)^{-B/\varepsilon}$ and $N(b+it) \ll (\log x)^{-B/\varepsilon}$. Then for $(\log X)^{B/\varepsilon} \leq T \leq X$, we have

$$\left(\min \left(\eta, \frac{1}{T} \right) \right)^2 \int_T^{2T} |N(b+it)Q(b+it)R(b+it)K(b+it)|^2 dt \ll \eta^2 (\log x)^{-10B}.$$

Proof. Let $M(s) = Q(s)R(s)$ and $H(s) = N(s)$ and by Lemma 3.2, Lemma 3.5 is proved. \square

Lemma 3.6. Suppose that $NQRK = X$, and N , Q and R satisfy $X^{\frac{79}{817}} \ll R \ll Q$ and one of the following 9 conditions:

- (1) $N(QR) \ll X^{\frac{673}{1247}}$, $X^{\frac{82}{473}} \ll (QR)$, $N^{29}(QR)^{-1} \ll X^{\frac{427}{43}}$, $X^{\frac{246}{817}} \ll N$, $N^{-1}(QR)^{29} \ll X^{\frac{263}{43}}$, $X^{\frac{246}{43}} \ll N^{12}(QR)^{11}$;
- (2) $N(QR) \ll X^{\frac{223}{387}}$, $N^{29}(QR)^{19} \ll X^{\frac{632}{43}}$, $X^{\frac{328}{387}} \ll N^2(QR)$, $N^2(QR)^{11} \ll X^{\frac{120}{43}}$, $X^{\frac{1230}{43}} \ll N^{58}(QR)^{49}$;
- (3) $N(QR) \ll X^{\frac{268}{473}}$, $X^{\frac{41}{215}} \ll (QR)$, $N^6(QR) \ll X^{\frac{94}{43}}$, $X^{\frac{82}{301}} \ll N \ll X^{\frac{16}{43}}$, $N(QR)^8 \ll X^{\frac{98}{43}}$, $X^{\frac{123}{43}} \ll N^6(QR)^5$;
- (4) $N(QR) \ll X^{\frac{571}{817}}$, $X^{\frac{82}{473}} \ll (QR)$, $N^{12}(QR) \ll X^{\frac{270}{43}}$, $X^{\frac{574}{1247}} \ll N$, $N^{-1}(QR)^{19} \ll X^{\frac{161}{43}}$, $X^{\frac{328}{43}} \ll N^{12}(QR)^{11}$;
- (5) $N^2(QR) \ll X^{\frac{446}{387}}$, $N^{58}(QR)^9 \ll X^{\frac{1264}{43}}$, $X^{\frac{164}{387}} \ll N$, $N(QR)^5 \ll X^{\frac{60}{43}}$, $X^{\frac{615}{43}} \ll N^{29}(QR)^{10}$;
- (6) $X^{\frac{27}{43}} \ll N(QR) \ll X^{\frac{219}{301}}$, $X^{\frac{41}{215}} \ll (QR)$, $N^6(QR) \ll X^{\frac{135}{43}}$, $X^{\frac{205}{473}} \ll N$, $N^{-1}(QR)^7 \ll X^{\frac{55}{43}}$, $X^{\frac{164}{43}} \ll N^6(QR)^5$;
- (7) $N(QR) \ll X^{\frac{268}{473}}$, $N^{35}(QR)^{23} \ll X^{\frac{767}{43}}$, $X^{\frac{410}{473}} \ll N^2(QR)$, $N^2(QR)^{13} \ll X^{\frac{130}{43}}$, $X^{\frac{1476}{43}} \ll N^{70}(QR)^{59}$;
- (8) $N(QR) \ll X^{\frac{313}{559}}$, $N^{41}(QR)^{27} \ll X^{\frac{902}{43}}$, $X^{\frac{492}{559}} \ll N^2(QR)$, $N^2(QR)^{15} \ll X^{\frac{138}{43}}$, $X^{\frac{1722}{43}} \ll N^{82}(QR)^{69}$;
- (9) $N^2(QR) \ll X^{\frac{536}{473}}$, $N^{70}(QR)^{11} \ll X^{\frac{1534}{43}}$, $X^{\frac{205}{473}} \ll N$, $N(QR)^6 \ll X^{\frac{65}{43}}$, $X^{\frac{738}{43}} \ll N^{35}(QR)^{12}$.

Assume further that for $(\log X)^{B/\varepsilon} \leq |t| \leq 2X$, $Q(b+it)R(b+it) \ll (\log x)^{-B/\varepsilon}$ and $N(b+it) \ll (\log x)^{-B/\varepsilon}$. Then for $(\log X)^{B/\varepsilon} \leq T \leq X$, we have

$$\left(\min \left(\eta, \frac{1}{T} \right) \right)^2 \int_T^{2T} |N(b+it)Q(b+it)R(b+it)K(b+it)|^2 dt \ll \eta^2 (\log x)^{-10B}.$$

Proof. Let $M(s) = N(s)$ and $H(s) = Q(s)R(s)$ and by Lemma 3.2, Lemma 3.6 is proved. \square

4. ARITHMETIC INFORMATION II

In this section we are looking for more Type-I information. In [26], Jia used a mean value bound of Deshouillers and Iwaniec [8], which makes an approximation to the sixth-power moment of the Riemann zeta-function. Now we shall use another mean value bound of Watt [43], which is stronger than that of Deshouillers and Iwaniec when the length of interval is $n^{\frac{1}{21.5} + \varepsilon}$.

Lemma 4.1. Let $T \geq 1$, then

$$\int_T^{2T} \left| L \left(\frac{1}{2} + it \right) \right|^4 \left| N \left(\frac{1}{2} + it \right) \right|^2 dt \ll \left(T + N^2 T^{\frac{1}{2}} + NL^2 T^{-2} \right) T^{\varepsilon^2}.$$

Proof. For $c_1 L \leq T^{\frac{1}{2}}$ and $N \leq T$, by the main theorem proved in [43] we have

$$\int_T^{2T} \left| L \left(\frac{1}{2} + it \right) \right|^4 \left| N \left(\frac{1}{2} + it \right) \right|^2 dt \ll \left(T + N^2 T^{\frac{1}{2}} \right) T^{\varepsilon^2}.$$

For $c_1 L \leq T^{\frac{1}{2}}$ and $N > T$, by the classical mean value estimate (see [26]) we have

$$\begin{aligned} \int_T^{2T} \left| L \left(\frac{1}{2} + it \right) \right|^4 \left| N \left(\frac{1}{2} + it \right) \right|^2 dt &\ll (L^2 N + T) (LN)^{\varepsilon^2} \\ &\ll \left(T + N^2 T^{\frac{1}{2}} \right) T^{\varepsilon^2}. \end{aligned}$$

When $T^{\frac{1}{2}} < c_1 L \leq 2T$, by a reflection principle based on an approximate functional equation, we can get the same bound as above. By this process we may replace L by $L_0 \sim T/L$, so that $L_0 \ll T^{\frac{1}{2}}$. For this, one can see [[8], Section 2], [[4], Lemma 2] or [[1], Lemma 5.2] for a detailed proof with $N \leq T$. For $N > T$, by the classical mean value estimate we have

$$\begin{aligned} \int_T^{2T} \left| L \left(\frac{1}{2} + it \right) \right|^4 \left| N \left(\frac{1}{2} + it \right) \right|^2 dt &\ll (L_0^2 N + T) (L_0 N)^{\varepsilon^2} + \text{Error} \\ &\ll \left(T + N^2 T^{\frac{1}{2}} \right) T^{\varepsilon^2}. \end{aligned}$$

When $2T < c_1 L$, by the method in [26] we have

$$\int_T^{2T} \left| L \left(\frac{1}{2} + it \right) \right|^4 \left| N \left(\frac{1}{2} + it \right) \right|^2 dt \ll NL^2 T^{-2}.$$

(Actually we have a better bound $L^2 T^{-4}(N+T)$ in this case, but this won't bring any improvement to the next lemma.)

Finally, by combining all the cases above, Lemma 4.1 is proved. \square

Lemma 4.2. Suppose that $MHL = X$, and M and H satisfy one of the following 2 conditions:

- (1) $M^2H \ll X^{2-\frac{20.5}{21.5}}$, $M^4H^6 \ll X^{4-\frac{20.5}{21.5}}$, $H^4 \ll X^{4-\frac{61.5}{21.5}}$;
- (2) $M \ll X^{\frac{45}{86}}$, $H \ll X^{\frac{41}{344}}$.

Then for $\sqrt{L} \leq T \leq X$, we have

$$\left(\min\left(\eta, \frac{1}{T}\right)\right)^2 \int_T^{2T} |M(b+it)H(b+it)L(b+it)|^2 dt \ll \eta^2 x^{-2\varepsilon^2}.$$

Proof. We can prove this by using Lemma 4.1 and the methods in [[26], Lemmas 3,4]. For condition (1) we apply Lemma 4.1 to L and N , while for condition (2) we apply Lemma 4.1 to L and N^2 . One can see [26] for an explanation. \square

Lemma 4.3. Suppose that $NQRL = X$, and N, Q and R satisfy $X^{\frac{79}{817}} \ll R \ll Q$ and one of the following 9 conditions:

- (1) $(NQ)^2R \ll X^{2-\frac{20.5}{21.5}}$, $(NQ)^4R^6 \ll X^{4-\frac{20.5}{21.5}}$, $R^4 \ll X^{4-\frac{61.5}{21.5}}$;
- (2) $(NQ) \ll X^{\frac{45}{86}}$, $R \ll X^{\frac{41}{344}}$;
- (3) $X^{\frac{55}{129}} \ll (NQ) \ll X^{\frac{20}{43}}$, $R \ll (NQ)^{-\frac{1}{5}} X^{\frac{12}{43}}$;
- (4) $X^{\frac{20}{43}} \ll (NQ) \ll X^{\frac{41}{86}}$, $R \ll (NQ)^{\frac{1}{7}} X^{\frac{55}{301}}$;
- (5) $X^{\frac{41}{86}} \ll (NQ) \ll X^{\frac{227}{473}}$, $R \ll (NQ)^{-1} X^{\frac{219}{301}}$;
- (6) $X^{\frac{227}{473}} \ll (NQ) \ll X^{\frac{2333}{4859}}$, $R \ll (NQ)^{-\frac{58}{9}} X^{\frac{1264}{387}}$;
- (7) $X^{\frac{2333}{4859}} \ll (NQ) \ll X^{\frac{2501}{5203}}$, $R \ll (NQ)^{-\frac{1}{6}} X^{\frac{65}{258}}$;
- (8) $X^{\frac{2501}{5203}} \ll (NQ) \ll X^{\frac{499}{1032}}$, $R \ll (NQ)^{-2} X^{\frac{536}{473}}$;
- (9) $X^{\frac{499}{1032}} \ll (NQ) \ll X^{\frac{28277}{57190}}$, $R \ll (NQ)^{-\frac{70}{11}} X^{\frac{1534}{473}}$.

Assume further that for $\sqrt{L} \leq |t| \leq 2X$, $N(b+it)Q(b+it) \ll (\log x)^{-B/\varepsilon}$ and $R(b+it) \ll (\log x)^{-B/\varepsilon}$. Then for $\sqrt{L} \leq T \leq X$, we have

$$\left(\min\left(\eta, \frac{1}{T}\right)\right)^2 \int_T^{2T} |N(b+it)Q(b+it)R(b+it)L(b+it)|^2 dt \ll \eta^2 (\log x)^{-10B}.$$

Proof. The proof is similar to that of [[26], Lemmas 14,15]. Let $M(s) = N(s)Q(s)$ and $H(s) = R(s)$. Then by Lemma 4.2 and Lemma 3.2, Lemma 4.3 is proved. \square

Lemma 4.4. Suppose that $NQRL = X$, and N, Q and R satisfy $X^{\frac{79}{817}} \ll R \ll Q$ and one of the following 9 conditions:

- (1) $(QR)^2N \ll X^{2-\frac{20.5}{21.5}}$, $(QR)^4N^6 \ll X^{4-\frac{20.5}{21.5}}$, $N^4 \ll X^{4-\frac{61.5}{21.5}}$;
- (2) $(QR) \ll X^{\frac{45}{86}}$, $N \ll X^{\frac{41}{344}}$;
- (3) $X^{\frac{55}{129}} \ll (QR) \ll X^{\frac{20}{43}}$, $N \ll (QR)^{-\frac{1}{5}} X^{\frac{12}{43}}$;
- (4) $X^{\frac{20}{43}} \ll (QR) \ll X^{\frac{41}{86}}$, $N \ll (QR)^{\frac{1}{7}} X^{\frac{55}{301}}$;
- (5) $X^{\frac{41}{86}} \ll (QR) \ll X^{\frac{227}{473}}$, $N \ll (QR)^{-1} X^{\frac{219}{301}}$;
- (6) $X^{\frac{227}{473}} \ll (QR) \ll X^{\frac{2333}{4859}}$, $N \ll (QR)^{-\frac{58}{9}} X^{\frac{1264}{387}}$;
- (7) $X^{\frac{2333}{4859}} \ll (QR) \ll X^{\frac{2501}{5203}}$, $N \ll (QR)^{-\frac{1}{6}} X^{\frac{65}{258}}$;
- (8) $X^{\frac{2501}{5203}} \ll (QR) \ll X^{\frac{499}{1032}}$, $N \ll (QR)^{-2} X^{\frac{536}{473}}$;
- (9) $X^{\frac{499}{1032}} \ll (QR) \ll X^{\frac{28277}{57190}}$, $N \ll (QR)^{-\frac{70}{11}} X^{\frac{1534}{473}}$.

Assume further that for $\sqrt{L} \leq |t| \leq 2X$, $Q(b+it)R(b+it) \ll (\log x)^{-B/\varepsilon}$ and $N(b+it) \ll (\log x)^{-B/\varepsilon}$. Then for $\sqrt{L} \leq T \leq X$, we have

$$\left(\min\left(\eta, \frac{1}{T}\right)\right)^2 \int_T^{2T} |N(b+it)Q(b+it)R(b+it)L(b+it)|^2 dt \ll \eta^2 (\log x)^{-10B}.$$

Proof. Let $M(s) = Q(s)R(s)$ and $H(s) = N(s)$. Then by Lemma 4.2 and Lemma 3.2, Lemma 4.4 is proved. \square

5. ASYMPTOTIC FORMULAS

Lemma 5.1. Suppose that $M \ll X^{\frac{45}{86}}$ and $H \ll X^{\frac{41}{344}}$. Then for real numbers $x \in [X, 2X]$, except for $O(X(\log X)^{-B})$ values, we have

$$\sum_{\substack{m \sim M \\ h \sim H}} a_m b_h \left(\sum_{x < mhl \leq x + \eta x} 1 - \frac{\eta x}{mh} \right) \ll \eta x (\log x)^{-B}.$$

Proof. The proof is similar to that of [[26], Lemma 16] where the condition (2) of Lemma 4.2 is used. \square

Lemma 5.2. Suppose that $NQRL = X$, and N, Q, R satisfy one of the 18 conditions in Lemma 4.3 and Lemma 4.4. Assume further that for $\sqrt{L} \leq |t| \leq 2X$, $N(b+it)Q(b+it)R(b+it) \ll (\log x)^{-B/\varepsilon}$. Then for real numbers $x \in [X, 2X]$, except for

$O(X(\log X)^{-B})$ values, we have

$$\sum_{\substack{n \sim N \\ q \sim Q \\ r \sim R}} a_n b_q c_r \left(\sum_{x < nqr \leq x + \eta x} 1 - \frac{\eta x}{nqr} \right) \ll \eta x (\log x)^{-B}.$$

Proof. The proof is similar to that of [[26], Lemma 17]. \square

Lemma 5.3. Suppose that $X^{\frac{738}{817}} \ll M \ll X^{1-\delta}$, $a_m \geq 0$, and $a_m = 0$ if m has a prime factor $< X^\delta$. Then for real numbers $x \in [X, 2X]$, except for $O(X(\log X)^{-B})$ values, we have

$$\sum_{\substack{x < mp \leq x + \eta x \\ m \sim M}} a_m = \eta \left(1 + O\left(\frac{1}{\log x}\right) \right) \sum_{\substack{x < mp \leq 2x \\ m \sim M}} a_m + O\left(\eta x (\log x)^{-B}\right).$$

Proof. The proof is similar to that of [[26], Lemma 18] where Lemma 3.1 is used. \square

Lemma 5.4. Suppose that $MHK = X$, $c_k = \Lambda(k)$, $X^{\frac{79}{817}} \ll MH \ll X^{\frac{738}{817}}$, and $a_m, b_h \geq 0$. Suppose that $a_m = 0$ if m has a prime factor $< X^\delta$, and $b_h = 0$ if h has a prime factor $< X^\delta$. If we have

$$\left(\min\left(\eta, \frac{1}{T}\right) \right)^2 \int_T^{2T} |M(b+it)H(b+it)K(b+it)|^2 dt \ll \eta^2 (\log x)^{-10B}$$

for $(\log X)^{B/\varepsilon} \leq T \leq X$, then for real numbers $x \in [X, 2X]$, except for $O(X(\log X)^{-B})$ values, we have

$$\sum_{\substack{x < mhp \leq x + \eta x \\ m \sim M \\ h \sim H}} a_m b_h = \eta \left(1 + O\left(\frac{1}{\log x}\right) \right) \sum_{\substack{x < mhp \leq 2x \\ m \sim M \\ h \sim H}} a_m b_h + O\left(\eta x (\log x)^{-B}\right).$$

Proof. The proof is similar to that of [[25], Lemma 4]. \square

Lemma 5.5. Suppose that $MNHK = X$, $d_k = \Lambda(k)$, and $a_m, b_n, c_h \geq 0$. Suppose that $a_m = 0$ if m has a prime factor $< X^\delta$, $b_n = 0$ if n has a prime factor $< X^\delta$, and $c_h = 0$ if h has a prime factor $< X^\delta$. If we have

$$\left(\min\left(\eta, \frac{1}{T}\right) \right)^2 \int_T^{2T} |M(b+it)N(b+it)H(b+it)K(b+it)|^2 dt \ll \eta^2 (\log x)^{-10B}$$

for $(\log X)^{B/\varepsilon} \leq T \leq X$, then for real numbers $x \in [X, 2X]$, except for $O(X(\log X)^{-B})$ values, we have

$$\sum_{\substack{x < mnhp \leq x + \eta x \\ m \sim M \\ n \sim N \\ h \sim H}} a_m b_n c_h = \eta \left(1 + O\left(\frac{1}{\log x}\right) \right) \sum_{\substack{x < mnhp \leq 2x \\ m \sim M \\ n \sim N \\ h \sim H}} a_m b_n c_h + O\left(\eta x (\log x)^{-B}\right).$$

Proof. The proof is similar to that of [[25], Lemma 4]. \square

Lemma 5.6. Suppose that $P_0 P_1 P_2 \cdots P_n = X$, $P_n < \cdots < P_2 < P_1 < X^{\frac{1}{2}}$ and $p_i \sim P_i$ for $1 \leq i$. Suppose that $P_1, P_2, \dots, P_n, X^{10^{-1000}}$ can be partitioned into M, H satisfy Lemma 5.1 or N, Q, R satisfy Lemma 5.2. Then for real numbers $x \in [X, 2X]$, except for $O(X(\log X)^{-B})$ values, we have

$$\sum_{(t_1, \dots, t_n)} S\left(\mathcal{A}_{p_1 p_2 \cdots p_n}, X^{\frac{79}{817}}\right) = \eta \sum_{(t_1, \dots, t_n)} S\left(\mathcal{B}_{p_1 p_2 \cdots p_n}, X^{\frac{79}{817}}\right) + O\left(\frac{\varepsilon \eta x}{\log x}\right).$$

Proof. By Buchstab's identity, we have

$$\sum_{(t_1, \dots, t_n)} S\left(\mathcal{A}_{p_1 p_2 \cdots p_n}, X^{\frac{79}{817}}\right) = \sum_{(t_1, \dots, t_n)} S\left(\mathcal{A}_{p_1 p_2 \cdots p_n}, X^\delta\right) - \sum_{\substack{(t_1, \dots, t_n) \\ \delta \leq t_{n+1} < \frac{79}{817}}} S\left(\mathcal{A}_{p_1 p_2 \cdots p_n p_{n+1}}, p_{n+1}\right).$$

We use Iwaniec's linear sieve (taking $D = X^{10^{-1000}}$, $z = X^\delta$) together with Lemma 5.1 or Lemma 5.2 (depending on the region that $P_1, P_2, \dots, P_n, X^{10^{-1000}}$ are partitioned into) to deal with the first term on the right-hand side. The functions $F(s)$ and $f(s)$ defined by the following differential-difference equation

$$\begin{cases} F(s) = \frac{2e^\gamma}{s}, & f(s) = 0, & 0 < s \leq 2, \\ (sF(s))' = f(s-1), & (sf(s))' = F(s-1), & s \geq 2, \end{cases}$$

give the same value 1 with negligible error $O(\varepsilon^2)$, where γ is the Euler's constant. Hence we have

$$\sum_{(t_1, \dots, t_n)} S\left(\mathcal{A}_{p_1 p_2 \cdots p_n}, X^\delta\right) = \eta \sum_{(t_1, \dots, t_n)} S\left(\mathcal{B}_{p_1 p_2 \cdots p_n}, X^\delta\right) + O\left(\frac{\varepsilon \eta x}{\log x}\right).$$

Let $M = \frac{X}{P_{n+1}}$. Clearly we have $X^{\frac{738}{817}} \ll M \ll X^{1-\delta}$. By Lemma 5.3, we have

$$\sum_{\substack{(t_1, \dots, t_n) \\ \delta \leq t_{n+1} < \frac{79}{817}}} S(\mathcal{A}_{p_1 p_2 \dots p_n p_{n+1}}, p_{n+1}) = \eta \sum_{\substack{(t_1, \dots, t_n) \\ \delta \leq t_{n+1} < \frac{79}{817}}} S(\mathcal{B}_{p_1 p_2 \dots p_n p_{n+1}}, p_{n+1}) + O\left(\frac{\varepsilon \eta x}{\log x}\right).$$

Now, Lemma 5.6 is proved by another application of Buchstab's identity

$$\sum_{(t_1, \dots, t_n)} S(\mathcal{B}_{p_1 p_2 \dots p_n}, X^\delta) - \sum_{\substack{(t_1, \dots, t_n) \\ \delta \leq t_{n+1} < \frac{79}{817}}} S(\mathcal{B}_{p_1 p_2 \dots p_n p_{n+1}}, p_{n+1}) = \sum_{(t_1, \dots, t_n)} S(\mathcal{B}_{p_1 p_2 \dots p_n}, X^{\frac{79}{817}}).$$

We remark that in many places, we ignore the additional variable $X^{10^{-1000}}$ when applying this lemma since it only produces an extremely small difference (less than 10^{-10}). \square

6. THE FINAL DECOMPOSITION

In this section, sets \mathcal{A} and \mathcal{B} are defined respectively. Let $\omega(u)$ denote the Buchstab function determined by the following differential-difference equation

$$\begin{cases} \omega(u) = \frac{1}{u}, & 1 \leq u \leq 2, \\ (u\omega(u))' = \omega(u-1), & u \geq 2. \end{cases}$$

Moreover, we have the upper and lower bounds for $\omega(u)$:

$$\begin{aligned} \omega(u) \geq \omega_0(u) &= \begin{cases} \frac{1}{u}, & 1 \leq u < 2, \\ \frac{1}{1+\log(u-1)}, & 2 \leq u < 3, \\ \frac{1+\log(u-1)}{u} + \frac{1}{u} \int_2^{u-1} \frac{\log(t-1)}{t} dt, & 3 \leq u < 4, \\ 0.5612, & u \geq 4, \end{cases} \\ \omega(u) \leq \omega_1(u) &= \begin{cases} \frac{1}{u}, & 1 \leq u < 2, \\ \frac{1}{1+\log(u-1)}, & 2 \leq u < 3, \\ \frac{1+\log(u-1)}{u} + \frac{1}{u} \int_2^{u-1} \frac{\log(t-1)}{t} dt, & 3 \leq u < 4, \\ 0.5617, & u \geq 4. \end{cases} \end{aligned}$$

We shall use $\omega_0(u)$ and $\omega_1(u)$ to give numerical bounds for some sieve functions discussed below.

By [[26], Lemma 21], we know that, for sufficiently large z ,

$$S(\mathcal{B}, z) = \sum_{\substack{a \in \mathcal{B} \\ (a, P(z))=1}} 1 = (1 + o(1)) \frac{x}{\log z} \omega\left(\frac{\log x}{\log z}\right), \quad (4)$$

and we expect that the similar relation also holds for $S(\mathcal{A}, z)$:

$$S(\mathcal{A}, z) = \sum_{\substack{a \in \mathcal{A} \\ (a, P(z))=1}} 1 = (1 + o(1)) \frac{\eta x}{\log z} \omega\left(\frac{\log x}{\log z}\right). \quad (5)$$

If (3) holds for $S(\mathcal{A}, z)$, then we can deduce (5) easily from (3) and (4). Otherwise we must drop this $S(\mathcal{A}, z)$. We define the *loss* from this term by the size of corresponding $S(\mathcal{B}, z)$:

$$S(\mathcal{B}, z) = (\text{loss} + o(1)) \frac{\eta x}{\log x}. \quad (6)$$

We note that we can only drop positive parts and the total loss of the dropped parts must be less than 1.

Beginning with (2), we can easily give asymptotic formulas for S_1 and S_2 by Lemma 5.6. Before estimating S_3 , we first define the disjoint regions $U_{01}-U_{03}$ as

$$\begin{aligned} U_{01}(t_1, t_2) &:= \{(t_1, t_2) \in U_{0101} \cup U_{0102} \cup \dots \cup U_{0109}, \\ &\quad \frac{79}{817} \leq t_1 < \frac{1}{2}, \frac{79}{817} \leq t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\}, \\ U_{02}(t_1, t_2) &:= \{(t_1, t_2) \in U_{0201} \cup U_{0202} \cup \dots \cup U_{0237}, (t_1, t_2) \notin U_{01}, \\ &\quad \frac{79}{817} \leq t_1 < \frac{1}{2}, \frac{79}{817} \leq t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\}, \\ U_{03}(t_1, t_2) &:= \{(t_1, t_2) \in U_{0301} \cup U_{0302} \cup \dots \cup U_{0318}, (t_1, t_2) \notin U_{01} \cup U_{02}, \\ &\quad \frac{79}{817} \leq t_1 < \frac{1}{2}, \frac{79}{817} \leq t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\}, \end{aligned}$$

where

$$U_{0101}(t_1, t_2) := \left\{ t_1 + t_2 \leq \frac{673}{1247}, \frac{82}{473} \leq t_2, 29t_1 - t_2 \leq \frac{427}{43}, \frac{246}{817} \leq t_1, 29t_2 - t_1 \leq \frac{263}{43}, \frac{246}{43} \leq 12t_1 + 11t_2 \right\},$$

$$\begin{aligned}
U_{0102}(t_1, t_2) &:= \left\{ t_1 + t_2 \leq \frac{223}{387}, 29t_1 + 19t_2 \leq \frac{632}{43}, \frac{328}{387} \leq 2t_1 + t_2, 2t_1 + 11t_2 \leq \frac{120}{43}, \frac{1230}{43} \leq 58t_1 + 49t_2 \right\}, \\
U_{0103}(t_1, t_2) &:= \left\{ t_1 + t_2 \leq \frac{268}{473}, \frac{41}{215} \leq t_2, 6t_1 + t_2 \leq \frac{94}{43}, \frac{82}{301} \leq t_1 \leq \frac{16}{43}, t_1 + 8t_2 \leq \frac{98}{43}, \frac{123}{43} \leq 6t_1 + 5t_2 \right\}, \\
U_{0104}(t_1, t_2) &:= \left\{ t_1 + t_2 \leq \frac{571}{817}, \frac{82}{473} \leq t_2, 12t_1 + t_2 \leq \frac{270}{43}, \frac{574}{1247} \leq t_1, 19t_2 - t_1 \leq \frac{161}{43}, \frac{328}{43} \leq 12t_1 + 11t_2 \right\}, \\
U_{0105}(t_1, t_2) &:= \left\{ 2t_1 + t_2 \leq \frac{446}{387}, 58t_1 + 9t_2 \leq \frac{1264}{43}, \frac{164}{387} \leq t_1, t_1 + 5t_2 \leq \frac{60}{43}, \frac{615}{43} \leq 29t_1 + 10t_2 \right\}, \\
U_{0106}(t_1, t_2) &:= \left\{ \frac{27}{43} \leq t_1 + t_2 \leq \frac{219}{301}, \frac{41}{215} \leq t_2, 6t_1 + t_2 \leq \frac{135}{43}, \frac{205}{473} \leq t_1, 7t_2 - t_1 \leq \frac{55}{43}, \frac{164}{43} \leq 6t_1 + 5t_2 \right\}, \\
U_{0107}(t_1, t_2) &:= \left\{ t_1 + t_2 \leq \frac{268}{473}, 35t_1 + 23t_2 \leq \frac{767}{43}, \frac{410}{473} \leq 2t_1 + t_2, 2t_1 + 13t_2 \leq \frac{130}{43}, \frac{1476}{43} \leq 70t_1 + 59t_2 \right\}, \\
U_{0108}(t_1, t_2) &:= \left\{ t_1 + t_2 \leq \frac{313}{559}, 41t_1 + 27t_2 \leq \frac{902}{43}, \frac{492}{559} \leq 2t_1 + t_2, 2t_1 + 15t_2 \leq \frac{138}{43}, \frac{1722}{43} \leq 82t_1 + 69t_2 \right\}, \\
U_{0109}(t_1, t_2) &:= \left\{ 2t_1 + t_2 \leq \frac{536}{473}, 70t_1 + 11t_2 \leq \frac{1534}{43}, \frac{205}{473} \leq t_1, t_1 + 6t_2 \leq \frac{65}{43}, \frac{738}{43} \leq 35t_1 + 12t_2 \right\}, \\
U_{0201}(t_1, t_2) &:= \left\{ \frac{27267}{66994} \leq t_1 + t_2 \leq \frac{231}{559}, \frac{79}{817} \leq t_2 \leq -\frac{35}{23}(t_1 + t_2) + \frac{767}{989} \right\}, \\
U_{0202}(t_1, t_2) &:= \left\{ \frac{231}{559} \leq t_1 + t_2 \leq \frac{3275}{7826}, \frac{79}{817} \leq t_2 \leq -(t_1 + t_2) + \frac{313}{559} \right\}, \\
U_{0203}(t_1, t_2) &:= \left\{ \frac{3275}{7826} \leq t_1 + t_2 \leq \frac{15005}{33497}, \frac{79}{817} \leq t_2 \leq -\frac{41}{27}(t_1 + t_2) + \frac{902}{1161} \right\}, \\
U_{0204}(t_1, t_2) &:= \left\{ \frac{13074}{28595} \leq t_1 + t_2 \leq \frac{20}{43}, \frac{79}{817} \leq t_2 \leq -\frac{1}{5}(t_1 + t_2) + \frac{12}{43} \right\}, \\
U_{0205}(t_1, t_2) &:= \left\{ \frac{20}{43} \leq t_1 + t_2 \leq \frac{41}{86}, \frac{79}{817} \leq t_2 \leq \frac{1}{7}(t_1 + t_2) + \frac{55}{301} \right\}, \\
U_{0206}(t_1, t_2) &:= \left\{ \frac{41}{86} \leq t_1 + t_2 \leq \frac{227}{473}, \frac{79}{817} \leq t_2 \leq -(t_1 + t_2) + \frac{219}{301} \right\}, \\
U_{0207}(t_1, t_2) &:= \left\{ \frac{227}{473} \leq t_1 + t_2 \leq \frac{2333}{4859}, \frac{79}{817} \leq t_2 \leq -\frac{58}{9}(t_1 + t_2) + \frac{1264}{387} \right\}, \\
U_{0208}(t_1, t_2) &:= \left\{ \frac{2333}{4859} \leq t_1 + t_2 \leq \frac{2501}{5203}, \frac{79}{817} \leq t_2 \leq -\frac{1}{6}(t_1 + t_2) + \frac{65}{258} \right\}, \\
U_{0209}(t_1, t_2) &:= \left\{ \frac{2501}{5203} \leq t_1 + t_2 \leq \frac{499}{1032}, \frac{79}{817} \leq t_2 \leq -2(t_1 + t_2) + \frac{536}{473} \right\}, \\
U_{0210}(t_1, t_2) &:= \left\{ \frac{499}{1032} \leq t_1 + t_2 \leq \frac{28277}{57190}, \frac{79}{817} \leq t_2 \leq -\frac{70}{11}(t_1 + t_2) + \frac{1534}{473} \right\}, \\
U_{0211}(t_1, t_2) &:= \left\{ (t_1 + t_2) + t_2 \leq \frac{673}{1247}, \frac{82}{473} \leq t_2, 29(t_1 + t_2) - t_2 \leq \frac{427}{43}, \frac{246}{817} \leq \left(t_1 + \frac{79}{817} \right), \right. \\
&\quad \left. 29t_2 - \left(t_1 + \frac{79}{817} \right) \leq \frac{263}{43}, \frac{246}{43} \leq 12 \left(t_1 + \frac{79}{817} \right) + 11t_2 \right\}, \\
U_{0212}(t_1, t_2) &:= \left\{ (t_1 + t_2) + t_2 \leq \frac{223}{387}, 29(t_1 + t_2) + 19t_2 \leq \frac{632}{43}, \frac{328}{387} \leq 2 \left(t_1 + \frac{79}{817} \right) + t_2, \right. \\
&\quad \left. 2(t_1 + t_2) + 11t_2 \leq \frac{120}{43}, \frac{1230}{43} \leq 58 \left(t_1 + \frac{79}{817} \right) + 49t_2 \right\}, \\
U_{0213}(t_1, t_2) &:= \left\{ (t_1 + t_2) + t_2 \leq \frac{268}{473}, \frac{41}{215} \leq t_2, 6(t_1 + t_2) + t_2 \leq \frac{94}{43}, \frac{82}{301} \leq \left(t_1 + \frac{79}{817} \right), \right. \\
&\quad \left. (t_1 + t_2) \leq \frac{16}{43}, (t_1 + t_2) + 8t_2 \leq \frac{98}{43}, \frac{123}{43} \leq 6 \left(t_1 + \frac{79}{817} \right) + 5t_2 \right\}, \\
U_{0214}(t_1, t_2) &:= \left\{ (t_1 + t_2) + t_2 \leq \frac{571}{817}, \frac{82}{473} \leq t_2, 12(t_1 + t_2) + t_2 \leq \frac{270}{43}, \frac{574}{1247} \leq \left(t_1 + \frac{79}{817} \right), \right. \\
&\quad \left. 19t_2 - \left(t_1 + \frac{79}{817} \right) \leq \frac{161}{43}, \frac{328}{43} \leq 12 \left(t_1 + \frac{79}{817} \right) + 11t_2 \right\}, \\
U_{0215}(t_1, t_2) &:= \left\{ 2(t_1 + t_2) + t_2 \leq \frac{446}{387}, 58(t_1 + t_2) + 9t_2 \leq \frac{1264}{43}, \frac{164}{387} \leq \left(t_1 + \frac{79}{817} \right), \right. \\
&\quad \left. (t_1 + t_2) + 5t_2 \leq \frac{60}{43}, \frac{615}{43} \leq 29 \left(t_1 + \frac{79}{817} \right) + 10t_2 \right\},
\end{aligned}$$

$$\begin{aligned}
U_{0216}(t_1, t_2) &:= \left\{ \frac{27}{43} \leq \left(t_1 + \frac{79}{817} \right) + t_2, (t_1 + t_2) + t_2 \leq \frac{219}{301}, \frac{41}{215} \leq t_2, 6(t_1 + t_2) + t_2 \leq \frac{135}{43}, \right. \\
&\quad \left. \frac{205}{473} \leq \left(t_1 + \frac{79}{817} \right), 7t_2 - \left(t_1 + \frac{79}{817} \right) \leq \frac{55}{43}, \frac{164}{43} \leq 6 \left(t_1 + \frac{79}{817} \right) + 5t_2 \right\}, \\
U_{0217}(t_1, t_2) &:= \left\{ (t_1 + t_2) + t_2 \leq \frac{268}{473}, 35(t_1 + t_2) + 23t_2 \leq \frac{767}{43}, \frac{410}{473} \leq 2 \left(t_1 + \frac{79}{817} \right) + t_2, \right. \\
&\quad \left. 2(t_1 + t_2) + 13t_2 \leq \frac{130}{43}, \frac{1476}{43} \leq 70 \left(t_1 + \frac{79}{817} \right) + 59t_2 \right\}, \\
U_{0218}(t_1, t_2) &:= \left\{ (t_1 + t_2) + t_2 \leq \frac{313}{559}, 41(t_1 + t_2) + 27t_2 \leq \frac{902}{43}, \frac{492}{559} \leq 2 \left(t_1 + \frac{79}{817} \right) + t_2, \right. \\
&\quad \left. 2(t_1 + t_2) + 15t_2 \leq \frac{138}{43}, \frac{1722}{43} \leq 82 \left(t_1 + \frac{79}{817} \right) + 69t_2 \right\}, \\
U_{0219}(t_1, t_2) &:= \left\{ 2(t_1 + t_2) + t_2 \leq \frac{536}{473}, 70(t_1 + t_2) + 11t_2 \leq \frac{1534}{43}, \frac{205}{473} \leq \left(t_1 + \frac{79}{817} \right), \right. \\
&\quad \left. (t_1 + t_2) + 6t_2 \leq \frac{65}{43}, \frac{738}{43} \leq 35 \left(t_1 + \frac{79}{817} \right) + 12t_2 \right\}, \\
U_{0220}(t_1, t_2) &:= \left\{ (t_2 + t_2) + t_1 \leq \frac{673}{1247}, \frac{82}{473} \leq t_1, 29(t_2 + t_2) - t_1 \leq \frac{427}{43}, \frac{246}{817} \leq \left(t_2 + \frac{79}{817} \right), \right. \\
&\quad \left. 29t_1 - \left(t_2 + \frac{79}{817} \right) \leq \frac{263}{43}, \frac{246}{43} \leq 12 \left(t_2 + \frac{79}{817} \right) + 11t_1 \right\}, \\
U_{0221}(t_1, t_2) &:= \left\{ (t_2 + t_2) + t_1 \leq \frac{223}{387}, 29(t_2 + t_2) + 19t_1 \leq \frac{632}{43}, \frac{328}{387} \leq 2 \left(t_2 + \frac{79}{817} \right) + t_1, \right. \\
&\quad \left. 2(t_2 + t_2) + 11t_1 \leq \frac{120}{43}, \frac{1230}{43} \leq 58 \left(t_2 + \frac{79}{817} \right) + 49t_1 \right\}, \\
U_{0222}(t_1, t_2) &:= \left\{ (t_2 + t_2) + t_1 \leq \frac{268}{473}, \frac{41}{215} \leq t_1, 6(t_2 + t_2) + t_1 \leq \frac{94}{43}, \frac{82}{301} \leq \left(t_2 + \frac{79}{817} \right), \right. \\
&\quad \left. (t_2 + t_2) \leq \frac{16}{43}, (t_2 + t_2) + 8t_1 \leq \frac{98}{43}, \frac{123}{43} \leq 6 \left(t_2 + \frac{79}{817} \right) + 5t_1 \right\}, \\
U_{0223}(t_1, t_2) &:= \left\{ (t_2 + t_2) + t_1 \leq \frac{571}{817}, \frac{82}{473} \leq t_1, 12(t_2 + t_2) + t_1 \leq \frac{270}{43}, \frac{574}{1247} \leq \left(t_2 + \frac{79}{817} \right), \right. \\
&\quad \left. 19t_1 - \left(t_2 + \frac{79}{817} \right) \leq \frac{161}{43}, \frac{328}{43} \leq 12 \left(t_2 + \frac{79}{817} \right) + 11t_1 \right\}, \\
U_{0224}(t_1, t_2) &:= \left\{ 2(t_2 + t_2) + t_1 \leq \frac{446}{387}, 58(t_2 + t_2) + 9t_1 \leq \frac{1264}{43}, \frac{164}{387} \leq \left(t_2 + \frac{79}{817} \right), \right. \\
&\quad \left. (t_2 + t_2) + 5t_1 \leq \frac{60}{43}, \frac{615}{43} \leq 29 \left(t_2 + \frac{79}{817} \right) + 10t_1 \right\}, \\
U_{0225}(t_1, t_2) &:= \left\{ \frac{27}{43} \leq \left(t_2 + \frac{79}{817} \right) + t_1, (t_2 + t_2) + t_1 \leq \frac{219}{301}, \frac{41}{215} \leq t_1, 6(t_2 + t_2) + t_1 \leq \frac{135}{43}, \right. \\
&\quad \left. \frac{205}{473} \leq \left(t_2 + \frac{79}{817} \right), 7t_1 - \left(t_2 + \frac{79}{817} \right) \leq \frac{55}{43}, \frac{164}{43} \leq 6 \left(t_2 + \frac{79}{817} \right) + 5t_1 \right\}, \\
U_{0226}(t_1, t_2) &:= \left\{ (t_2 + t_2) + t_1 \leq \frac{268}{473}, 35(t_2 + t_2) + 23t_1 \leq \frac{767}{43}, \frac{410}{473} \leq 2 \left(t_2 + \frac{79}{817} \right) + t_1, \right. \\
&\quad \left. 2(t_2 + t_2) + 13t_1 \leq \frac{130}{43}, \frac{1476}{43} \leq 70 \left(t_2 + \frac{79}{817} \right) + 59t_1 \right\}, \\
U_{0227}(t_1, t_2) &:= \left\{ (t_2 + t_2) + t_1 \leq \frac{313}{559}, 41(t_2 + t_2) + 27t_1 \leq \frac{902}{43}, \frac{492}{559} \leq 2 \left(t_2 + \frac{79}{817} \right) + t_1, \right. \\
&\quad \left. 2(t_2 + t_2) + 15t_1 \leq \frac{138}{43}, \frac{1722}{43} \leq 82 \left(t_2 + \frac{79}{817} \right) + 69t_1 \right\}, \\
U_{0228}(t_1, t_2) &:= \left\{ 2(t_2 + t_2) + t_1 \leq \frac{536}{473}, 70(t_2 + t_2) + 11t_1 \leq \frac{1534}{43}, \frac{205}{473} \leq \left(t_2 + \frac{79}{817} \right), \right. \\
&\quad \left. (t_2 + t_2) + 6t_1 \leq \frac{65}{43}, \frac{738}{43} \leq 35 \left(t_2 + \frac{79}{817} \right) + 12t_1 \right\}, \\
U_{0229}(t_1, t_2) &:= \left\{ t_1 + (t_2 + t_2) \leq \frac{673}{1247}, \frac{82}{473} \leq \left(t_2 + \frac{79}{817} \right), 29t_1 - \left(t_2 + \frac{79}{817} \right) \leq \frac{427}{43}, \frac{246}{817} \leq t_1, \right. \\
&\quad \left. 29(t_2 + t_2) - t_1 \leq \frac{263}{43}, \frac{246}{43} \leq 12t_1 + 11 \left(t_2 + \frac{79}{817} \right) \right\},
\end{aligned}$$

$$\begin{aligned}
U_{0230}(t_1, t_2) &:= \left\{ t_1 + (t_2 + t_2) \leq \frac{223}{387}, 29t_1 + 19(t_2 + t_2) \leq \frac{632}{43}, \frac{328}{387} \leq 2t_1 + \left(t_2 + \frac{79}{817}\right), \right. \\
&\quad \left. 2t_1 + 11(t_2 + t_2) \leq \frac{120}{43}, \frac{1230}{43} \leq 58t_1 + 49 \left(t_2 + \frac{79}{817}\right) \right\}, \\
U_{0231}(t_1, t_2) &:= \left\{ t_1 + (t_2 + t_2) \leq \frac{268}{473}, \frac{41}{215} \leq \left(t_2 + \frac{79}{817}\right), 6t_1 + (t_2 + t_2) \leq \frac{94}{43}, \frac{82}{301} \leq t_1 \leq \frac{16}{43}, \right. \\
&\quad \left. t_1 + 8(t_2 + t_2) \leq \frac{98}{43}, \frac{123}{43} \leq 6t_1 + 5 \left(t_2 + \frac{79}{817}\right) \right\}, \\
U_{0232}(t_1, t_2) &:= \left\{ t_1 + (t_2 + t_2) \leq \frac{571}{817}, \frac{82}{473} \leq \left(t_2 + \frac{79}{817}\right), 12t_1 + (t_2 + t_2) \leq \frac{270}{43}, \frac{574}{1247} \leq t_1, \right. \\
&\quad \left. 19(t_2 + t_2) - t_1 \leq \frac{161}{43}, \frac{328}{43} \leq 12t_1 + 11 \left(t_2 + \frac{79}{817}\right) \right\}, \\
U_{0233}(t_1, t_2) &:= \left\{ 2t_1 + (t_2 + t_2) \leq \frac{446}{387}, 58t_1 + 9(t_2 + t_2) \leq \frac{1264}{43}, \frac{164}{387} \leq t_1, \right. \\
&\quad \left. t_1 + 5(t_2 + t_2) \leq \frac{60}{43}, \frac{615}{43} \leq 29t_1 + 10 \left(t_2 + \frac{79}{817}\right) \right\}, \\
U_{0234}(t_1, t_2) &:= \left\{ \frac{27}{43} \leq t_1 + \left(t_2 + \frac{79}{817}\right), t_1 + (t_2 + t_2) \leq \frac{219}{301}, \frac{41}{215} \leq \left(t_2 + \frac{79}{817}\right), 6t_1 + (t_2 + t_2) \leq \frac{135}{43}, \right. \\
&\quad \left. \frac{205}{473} \leq t_1, 7(t_2 + t_2) - t_1 \leq \frac{55}{43}, \frac{164}{43} \leq 6t_1 + 5 \left(t_2 + \frac{79}{817}\right) \right\}, \\
U_{0235}(t_1, t_2) &:= \left\{ t_1 + (t_2 + t_2) \leq \frac{268}{473}, 35t_1 + 23(t_2 + t_2) \leq \frac{767}{43}, \frac{410}{473} \leq 2t_1 + \left(t_2 + \frac{79}{817}\right), \right. \\
&\quad \left. 2t_1 + 13(t_2 + t_2) \leq \frac{130}{43}, \frac{1476}{43} \leq 70t_1 + 59 \left(t_2 + \frac{79}{817}\right) \right\}, \\
U_{0236}(t_1, t_2) &:= \left\{ t_1 + (t_2 + t_2) \leq \frac{313}{559}, 41t_1 + 27(t_2 + t_2) \leq \frac{902}{43}, \frac{492}{559} \leq 2t_1 + \left(t_2 + \frac{79}{817}\right), \right. \\
&\quad \left. 2t_1 + 15(t_2 + t_2) \leq \frac{138}{43}, \frac{1722}{43} \leq 82t_1 + 69 \left(t_2 + \frac{79}{817}\right) \right\}, \\
U_{0237}(t_1, t_2) &:= \left\{ 2t_1 + (t_2 + t_2) \leq \frac{536}{473}, 70t_1 + 11(t_2 + t_2) \leq \frac{1534}{43}, \frac{205}{473} \leq t_1, \right. \\
&\quad \left. t_1 + 6(t_2 + t_2) \leq \frac{65}{43}, \frac{738}{43} \leq 35t_1 + 12 \left(t_2 + \frac{79}{817}\right) \right\}, \\
U_{0301}(t_1, t_2) &:= \left\{ 2(t_1 + t_2) + t_2 \leq 2 - \frac{20.5}{21.5}, 4(t_1 + t_2) + 6t_2 \leq 4 - \frac{20.5}{21.5}, 4t_2 \leq 4 - \frac{61.5}{21.5} \right\}, \\
U_{0302}(t_1, t_2) &:= \left\{ (t_1 + t_2) \leq \frac{45}{86}, t_2 \leq \frac{41}{344} \right\}, \\
U_{0303}(t_1, t_2) &:= \left\{ \frac{55}{129} \leq (t_1 + t_2) \leq \frac{20}{43}, t_2 \leq -\frac{1}{5}(t_1 + t_2) + \frac{12}{43} \right\}, \\
U_{0304}(t_1, t_2) &:= \left\{ \frac{20}{43} \leq (t_1 + t_2) \leq \frac{41}{86}, t_2 \leq \frac{1}{7}(t_1 + t_2) + \frac{55}{301} \right\}, \\
U_{0305}(t_1, t_2) &:= \left\{ \frac{41}{86} \leq (t_1 + t_2) \leq \frac{227}{473}, t_2 \leq -(t_1 + t_2) + \frac{219}{301} \right\}, \\
U_{0306}(t_1, t_2) &:= \left\{ \frac{227}{473} \leq (t_1 + t_2) \leq \frac{2333}{4859}, t_2 \leq -\frac{58}{9}(t_1 + t_2) + \frac{1264}{387} \right\}, \\
U_{0307}(t_1, t_2) &:= \left\{ \frac{2333}{4859} \leq (t_1 + t_2) \leq \frac{2501}{5203}, t_2 \leq -\frac{1}{6}(t_1 + t_2) + \frac{65}{258} \right\}, \\
U_{0308}(t_1, t_2) &:= \left\{ \frac{2501}{5203} \leq (t_1 + t_2) \leq \frac{499}{1032}, t_2 \leq -2(t_1 + t_2) + \frac{536}{473} \right\}, \\
U_{0309}(t_1, t_2) &:= \left\{ \frac{499}{1032} \leq (t_1 + t_2) \leq \frac{28277}{57190}, t_2 \leq -\frac{70}{11}(t_1 + t_2) + \frac{1534}{473} \right\}, \\
U_{0310}(t_1, t_2) &:= \left\{ 2(t_2 + t_2) + t_1 \leq 2 - \frac{20.5}{21.5}, 4(t_2 + t_2) + 6t_1 \leq 4 - \frac{20.5}{21.5}, 4t_1 \leq 4 - \frac{61.5}{21.5} \right\}, \\
U_{0311}(t_1, t_2) &:= \left\{ (t_2 + t_2) \leq \frac{45}{86}, t_1 \leq \frac{41}{344} \right\}, \\
U_{0312}(t_1, t_2) &:= \left\{ \frac{55}{129} \leq \left(t_2 + \frac{79}{817}\right), (t_2 + t_2) \leq \frac{20}{43}, t_1 \leq -\frac{1}{5}(t_2 + t_2) + \frac{12}{43} \right\},
\end{aligned}$$

$$\begin{aligned}
U_{0313}(t_1, t_2) &:= \left\{ \frac{20}{43} \leq \left(t_2 + \frac{79}{817} \right), (t_2 + t_2) \leq \frac{41}{86}, t_1 \leq \frac{1}{7} \left(t_2 + \frac{79}{817} \right) + \frac{55}{301} \right\}, \\
U_{0314}(t_1, t_2) &:= \left\{ \frac{41}{86} \leq \left(t_2 + \frac{79}{817} \right), (t_2 + t_2) \leq \frac{227}{473}, t_1 \leq -(t_2 + t_2) + \frac{219}{301} \right\}, \\
U_{0315}(t_1, t_2) &:= \left\{ \frac{227}{473} \leq \left(t_2 + \frac{79}{817} \right), (t_2 + t_2) \leq \frac{2333}{4859}, t_1 \leq -\frac{58}{9}(t_2 + t_2) + \frac{1264}{387} \right\}, \\
U_{0316}(t_1, t_2) &:= \left\{ \frac{2333}{4859} \leq \left(t_2 + \frac{79}{817} \right), (t_2 + t_2) \leq \frac{2501}{5203}, t_1 \leq -\frac{1}{6}(t_2 + t_2) + \frac{65}{258} \right\}, \\
U_{0317}(t_1, t_2) &:= \left\{ \frac{2501}{5203} \leq \left(t_2 + \frac{79}{817} \right), (t_2 + t_2) \leq \frac{499}{1032}, t_1 \leq -2(t_2 + t_2) + \frac{536}{473} \right\}, \\
U_{0318}(t_1, t_2) &:= \left\{ \frac{499}{1032} \leq \left(t_2 + \frac{79}{817} \right), (t_2 + t_2) \leq \frac{28277}{57190}, t_1 \leq -\frac{70}{11}(t_2 + t_2) + \frac{1534}{473} \right\},
\end{aligned}$$

and let $U_0 = U_{01} \cup U_{02} \cup U_{03}$. Clearly U_{01} corresponds to Lemma 3.2, U_{02} corresponds to Lemmas 3.3–3.6, and U_{03} corresponds to Lemmas 4.3–4.4. Then we have

$$\begin{aligned}
S_3 &= \sum_{(t_1, t_2) \in U_{01}} S(\mathcal{A}_{p_1 p_2}, p_2) + \sum_{(t_1, t_2) \in U_{02}} S(\mathcal{A}_{p_1 p_2}, p_2) \\
&\quad + \sum_{(t_1, t_2) \in U_{03}} S(\mathcal{A}_{p_1 p_2}, p_2) + \sum_{(t_1, t_2) \notin U_0} S(\mathcal{A}_{p_1 p_2}, p_2) \\
&= S_{301} + S_{302} + \Sigma_{303} + \Sigma_0.
\end{aligned} \tag{7}$$

Note that U_{01} corresponds to Lemma 3.2, U_{02} corresponds to Lemmas 3.3–3.6, and U_{03} corresponds to Lemmas 4.3–4.4.

For S_{301} , by Lemma 3.2 and Lemma 5.4, we can give an asymptotic formula. For S_{302} , we can apply Buchstab's identity again to get

$$\begin{aligned}
S_{302} &= \sum_{(t_1, t_2) \in U_{02}} S(\mathcal{A}_{p_1 p_2}, p_2) \\
&= \sum_{(t_1, t_2) \in U_{02}} S\left(\mathcal{A}_{p_1 p_2}, X^{\frac{79}{817}}\right) - \sum_{\substack{(t_1, t_2) \in U_{02} \\ \frac{79}{817} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2))}} S(\mathcal{A}_{p_1 p_2 p_3}, p_3).
\end{aligned} \tag{8}$$

By Lemma 5.6 we have

$$\sum_{(t_1, t_2) \in U_{02}} S\left(\mathcal{A}_{p_1 p_2}, X^{\frac{79}{817}}\right) = \eta \sum_{(t_1, t_2) \in U_{02}} S\left(\mathcal{B}_{p_1 p_2}, X^{\frac{79}{817}}\right) + O\left(\frac{\varepsilon \eta x}{\log x}\right). \tag{9}$$

By Lemmas 3.3–3.6 and Lemma 5.5 we have

$$\sum_{\substack{(t_1, t_2) \in U_{02} \\ \frac{79}{817} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2))}} S(\mathcal{A}_{p_1 p_2 p_3}, p_3) = \eta \sum_{\substack{(t_1, t_2) \in U_{02} \\ \frac{79}{817} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2))}} S(\mathcal{B}_{p_1 p_2 p_3}, p_3) + O\left(\frac{\varepsilon \eta x}{\log x}\right). \tag{10}$$

Combining (8)–(10), we have

$$S_{302} = \sum_{(t_1, t_2) \in U_{02}} S(\mathcal{A}_{p_1 p_2}, p_2) = \eta \sum_{(t_1, t_2) \in U_{02}} S(\mathcal{B}_{p_1 p_2}, p_2) + O\left(\frac{\varepsilon \eta x}{\log x}\right). \tag{11}$$

Before decomposing Σ_{303} , we first make a definition: we say a set of variables (t_1, \dots, t_n) is *good* if the variables can be partitioned into $(t_i, h) \in U_{01}$ or $(h, t_i) \in U_{01}$. Now we can apply Buchstab's identity twice more to get

$$\begin{aligned}
\Sigma_{303} &= \sum_{(t_1, t_2) \in U_{03}} S(\mathcal{A}_{p_1 p_2}, p_2) \\
&= \sum_{(t_1, t_2) \in U_{03}} S\left(\mathcal{A}_{p_1 p_2}, X^{\frac{79}{817}}\right) - \sum_{\substack{(t_1, t_2) \in U_{03} \\ \frac{79}{817} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \text{ is good}}} S(\mathcal{A}_{p_1 p_2 p_3}, p_3) \\
&\quad - \sum_{\substack{(t_1, t_2) \in U_{03} \\ \frac{79}{817} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \text{ is not good}}} S\left(\mathcal{A}_{p_1 p_2 p_3}, X^{\frac{79}{817}}\right) + \sum_{\substack{(t_1, t_2) \in U_{03} \\ \frac{79}{817} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \text{ is not good} \\ \frac{79}{817} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \text{ is good}}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{\substack{(t_1, t_2) \in U_{03} \\ \frac{79}{817} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \text{ is not good} \\ \frac{79}{817} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \text{ is not good}}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4) \\
& = S_{3031} - S_{3032} - S_{3033} + S_{3034} + S_{3035}.
\end{aligned} \tag{12}$$

By Lemma 5.6 we have

$$S_{3031} = \sum_{(t_1, t_2) \in U_{03}} S(\mathcal{A}_{p_1 p_2}, p_2) = \eta \sum_{(t_1, t_2) \in U_{03}} S(\mathcal{B}_{p_1 p_2}, p_2) + O\left(\frac{\varepsilon \eta x}{\log x}\right) \tag{13}$$

and

$$S_{3033} = \sum_{\substack{(t_1, t_2) \in U_{03} \\ \frac{79}{817} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \text{ is not good}}} S(\mathcal{A}_{p_1 p_2 p_3}, X^{\frac{79}{817}}) = \eta \sum_{\substack{(t_1, t_2) \in U_{03} \\ \frac{79}{817} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \text{ is not good}}} S(\mathcal{B}_{p_1 p_2 p_3}, X^{\frac{79}{817}}) + O\left(\frac{\varepsilon \eta x}{\log x}\right). \tag{14}$$

By Lemma 3.2 and a variant of Lemma 5.5 we have

$$S_{3032} = \sum_{\substack{(t_1, t_2) \in U_{03} \\ \frac{79}{817} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \text{ is good}}} S(\mathcal{A}_{p_1 p_2 p_3}, p_3) = \eta \sum_{\substack{(t_1, t_2) \in U_{03} \\ \frac{79}{817} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \text{ is good}}} S(\mathcal{B}_{p_1 p_2 p_3}, p_3) + O\left(\frac{\varepsilon \eta x}{\log x}\right) \tag{15}$$

and

$$\begin{aligned}
S_{3034} & = \sum_{\substack{(t_1, t_2) \in U_{03} \\ \frac{79}{817} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \text{ is not good} \\ \frac{79}{817} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \text{ is good}}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4) \\
& = \eta \sum_{\substack{(t_1, t_2) \in U_{03} \\ \frac{79}{817} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \text{ is not good} \\ \frac{79}{817} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \text{ is good}}} S(\mathcal{B}_{p_1 p_2 p_3 p_4}, p_4) + O\left(\frac{\varepsilon \eta x}{\log x}\right).
\end{aligned} \tag{16}$$

For the remaining S_{3035} , we can simply discard the whole sum, leading to a loss of size

$$\int_{(t_1, t_2, t_3, t_4) \in V_1} \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1 < 0.03, \tag{17}$$

where

$$\begin{aligned}
V_1(t_1, t_2, t_3, t_4) & := \left\{ (t_1, t_2) \in U_{03}, \frac{79}{817} \leq t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \right. \\
& \quad (t_1, t_2, t_3) \text{ is not good,} \\
& \quad \frac{79}{817} \leq t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\
& \quad (t_1, t_2, t_3, t_4) \text{ is not good,} \\
& \quad \left. \frac{79}{817} \leq t_1 < \frac{1}{2}, \frac{79}{817} \leq t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \right\}.
\end{aligned}$$

We note that further decompositions on S_{3035} are possible, but we do not consider them here.

For the remaining Σ_0 , we cannot give asymptotic formulas using Lemmas 5.3-5.5. Hence we must discard this sum, leading to a much bigger two-dimensional loss of

$$\int_{\frac{79}{817}}^{\frac{1}{2}} \int_{\frac{79}{817}}^{\min(t_1, \frac{1-t_1}{2})} \mathbb{1}_{(t_1, t_2) \notin U_0} \frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1 t_2^2} dt_2 dt_1 < 0.992, \tag{18}$$

However, we can use Buchstab's identity in reverse to get asymptotic formulas for some almost-primes counted. When $t_2 < \frac{1}{2}(1-t_1-t_2)$, we have

$$\sum_{\substack{(t_1, t_2) \notin U_0 \\ t_2 < \frac{1}{2}(1-t_1-t_2)}} S(\mathcal{A}_{p_1 p_2}, p_2) = \sum_{\substack{(t_1, t_2) \notin U_0 \\ t_2 < \frac{1}{2}(1-t_1-t_2)}} S\left(\mathcal{A}_{p_1 p_2}, \left(\frac{2X}{p_1 p_2}\right)^{\frac{1}{2}}\right) + \sum_{\substack{(t_1, t_2) \notin U_0 \\ t_2 < t_3 < \frac{1}{2}(1-t_1-t_2)}} S(\mathcal{A}_{p_1 p_2 p_3}, p_3). \tag{19}$$

This can be seen as a "reversed" application of Buchstab's identity. The first term on the right-hand side counts primes, while the second term counts almost-primes. We cannot give an asymptotic formula for the first term on the right-hand side, but we can give asymptotic formulas for part of the second term if (t_1, t_2, t_3) is good. We can then subtract the size of this part from the loss of Σ_0 . The savings from this part are larger than

$$\int_{(t_1, t_2, t_3) \in V_3} \frac{\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 dt_2 dt_1 > 0.027, \quad (20)$$

where

$$V_3(t_1, t_2, t_3) := \left\{ (t_1, t_2) \notin U_0, t_2 < t_3 < \frac{1}{2}(1 - t_1 - t_2), (t_1, t_2, t_3) \text{ is good}, \right. \\ \left. \frac{79}{817} \leq t_1 < \frac{1}{2}, \frac{79}{817} \leq t_2 < \min\left(t_1, \frac{1}{2}(1 - t_1)\right) \right\}.$$

Finally, by (1)–(20) we have

$$\pi(x + \eta x) - \pi(x) = S\left(\mathcal{A}, (2X)^{\frac{1}{2}}\right) \geq (1 - 0.03 - 0.992 + 0.027) \frac{\eta x}{\log x} = 0.005 \frac{\eta x}{\log x},$$

and the proof of Theorem 1.1 is completed. The exponent $\frac{1}{21.5}$ is rather near to the limit obtained by the method employed in this paper, and something like $\frac{1}{22}$ is far out of reach. With the exponent $\frac{1}{22}$, the above method only gives a lower bound constant less than -0.1 , which is even worse than the trivial lower bound. We remark that the above method gives a lower bound constant > 0.04 when the exponent is $\frac{1}{21}$, which is useful for an application in the next section.

7. APPLICATIONS OF THEOREM 1.1

Clearly our Theorem 1.1 has many interesting applications. The following application of our Theorem 1.1 is about Goldbach numbers (sum of two primes) in short intervals. By combining our Theorem 1.1 with the main theorem proved in [4], we can easily deduce the following theorem.

Theorem 7.1. *The interval $[X, X + X^{\frac{21}{860}}]$ contains Goldbach numbers. That is,*

$$g_{n+1} - g_n \ll g_n^{\frac{21}{860}},$$

where g_n denote the n -th Goldbach number.

Previous exponent $\frac{21}{800}$ (see either [39] or [41]) comes from Jia's exponent $\frac{1}{20}$ [26]. We remark that if we focus on Maillet numbers (difference of two primes) instead of Goldbach numbers in short intervals, Pintz [40] improved this exponent to any $\varepsilon > 0$.

Another application of our Theorem 1.1 is about prime values of the integer parts of real sequences, improving the previous result of Harman [13] by adding one more term on both of the sequences $[p^k \alpha]$ and $[(p\alpha)^k]$.

Theorem 7.2. *For almost all $\alpha > 0$, both of the following statements are true:*

- (1) *Infinitely often $p, [p\alpha], [p^2\alpha], \dots, [p^{21}\alpha]$ are all prime.*
- (2) *Infinitely often $p, [p\alpha], [(p\alpha)^2], \dots, [(p\alpha)^{21}]$ are all prime.*

The proof of Theorem 7.2 can be done by replacing [[13], Lemma 4] by a variant of our Theorem 1.1. In this way, the ratio $\frac{20}{19}$ in [[13], Theorem 4] can be improved to $\frac{21.5}{20.5}$. However, due to the lack of a positive lower bound when the exponent is $\frac{1}{22}$, we cannot add even one more term on our Theorem 7.2.

The last application of our Theorem 1.1 is about the Three Primes Theorem with small prime solutions, improving the previous result of Cai [5] by reducing the exponent $\frac{11}{400}$ to $\frac{11}{420}$.

Theorem 7.3. *Let Y denote a sufficiently large odd integer. The equation*

$$Y = p_1 + p_2 + p_3, \quad p_1 \leq Y^{\frac{11}{420}}$$

is solvable.

The exponent $\frac{11}{420}$ comes from a weaker version of our Theorem 1.1 with exponent $\frac{1}{21}$. Note that we have $u_0 > 0.04$ in this case, and we can show that $u_1 < 2.88$ by similar arguments as in [5]. By using the vector sieve together with the same bounds for v_0 and v_1 as in [5], the proof of our Theorem 7.3 is essentially the same as the proof of [[5], Theorem]. However, our sieve machinery is not very numerically significant when the exponent drops to $\frac{1}{21.5}$ (that is, we only have $u_0 > 0.005$ by our Theorem 1.1), we cannot use vector sieve to prove Theorem 7.3 with condition $p_1 \leq Y^{\frac{11}{430}}$.

REFERENCES

- [1] R. Alweiss and S. Luo. Bounded gaps between primes in short intervals. *Res. Number Theory*, 4(2):No. 15, 2018.
- [2] R. C. Baker and G. Harman. The difference between consecutive primes. *Proc. London Math. Soc.*, 72(3):261–280, 1996.
- [3] R. C. Baker, G. Harman, and J. Pintz. The exceptional set for Goldbach's problem in short intervals. In *Sieve methods, exponential sums and their applications in number theory*, ed. G. R. H. Greaves, G. Harman and M. N. Huxley, Cambridge University Press, pages 1–54. Cambridge University Press, Cambridge, 1997.
- [4] R. C. Baker, G. Harman, and J. Pintz. The difference between consecutive primes, II. *Proc. London Math. Soc.*, 83(3):532–562, 2001.
- [5] Y. Cai. A remark on the Goldbach–Vinogradov theorem. *Funct. Approx. Comment. Math.*, 48(1):123–131, 2013.
- [6] N. G. Chudakov. On the difference between two neighboring prime numbers. *Mat. Sb.*, 43(1):799–813, 1936.
- [7] H. Cramér. On the order of magnitude of the difference between consecutive prime numbers. *Acta Arith.*, 2:23–46, 1937.
- [8] J.-M. Deshouillers and H. Iwaniec. Power mean values of the Riemann zeta-function. *Mathematika*, 29:202–212, 1982.
- [9] L. Guth and J. Maynard. New large value estimates for Dirichlet polynomials. *Ann. of Math.*, to appear. *arXiv e-prints*, page arXiv:2405.20552v1, 2024.
- [10] G. Harman. Primes in short intervals. *Math. Z.*, 180:335–348, 1982.
- [11] G. Harman. On the distribution of ap modulo one. *J. London Math. Soc.*, 27(2):9–18, 1983.
- [12] G. Harman. On the distribution of ap modulo one II. *Proc. London Math. Soc.*, 72(3):241–260, 1996.
- [13] G. Harman. Metrical theorems on prime values of the integer parts of real sequences II. *J. London Math. Soc.*, 64(2):287–298, 2001.
- [14] G. Harman. *Prime-detecting Sieves*, volume 33 of *London Mathematical Society Monographs (New Series)*. Princeton University Press, Princeton, NJ, 2007.
- [15] D. R. Heath-Brown. Finding primes by sieve methods. In *Proceedings of the International Congress of Mathematicians August 16–24, 1983, Warszawa*, pages 487–492. PWN–Polish Sci. Publ., Warszawa, 1984.
- [16] D. R. Heath-Brown and H. Iwaniec. On the difference between consecutive primes. *Invent. Math.*, 55:49–69, 1979.
- [17] H. Heilbronn. Über den Primzahlsatz von Herrn Hoheisel. *Math. Z.*, 36:394–423, 1933.
- [18] G. Hoheisel. Primzahlprobleme in der analysis. *Sitz. Preuss. Akad. Wiss.*, 2:1–13, 1930.
- [19] M. N. Huxley. On the difference between consecutive primes. *Invent. Math.*, 15:164–170, 1972.
- [20] A. E. Ingham. On the difference between consecutive primes. *Q. J. Math.*, 8:255–266, 1936.
- [21] H. Iwaniec and M. Jutila. Primes in short intervals. *Ark. Mat.*, 17:167–176, 1979.
- [22] H. Iwaniec and J. Pintz. Primes in short intervals. *Monatsh. Math.*, 98:115–143, 1984.
- [23] C. Jia. Difference between consecutive primes. *Sci. China Ser. A*, 38(10):1163–1186, 1995.
- [24] C. Jia. Goldbach numbers in short interval (I). *Sci. China Ser. A*, 38(4):385–406, 1995.
- [25] C. Jia. Goldbach numbers in short interval (II). *Sci. China Ser. A*, 38(5):513–523, 1995.
- [26] C. Jia. Almost all short intervals containing prime numbers. *Acta Arith.*, 76(1):21–84, 1996.
- [27] C. Jia. On the exceptional set of Goldbach numbers in a short interval. *Acta Arith.*, 77(3):207–287, 1996.
- [28] H. Li. Goldbach numbers in short intervals. *Sci. China Ser. A*, 38(6):641–652, 1995.
- [29] H. Li. Primes in short intervals. *Math. Proc. Camb. Phil. Soc.*, 122(2):193–205, 1997.
- [30] S. Lou and Q. Yao. Difference between consecutive primes. *Chinese Journal of Nature (Nature Journal)*, 7(9):713, 1984.
- [31] S. Lou and Q. Yao. On the upper bound of difference between consecutive primes. *Kexue Tongbao*, 30(8):1127–1128, 1985.
- [32] S. Lou and Q. Yao. A Chebyshev's type of prime number theorem in a short interval–II. *Hardy-Ramanujan J.*, 15:1–33, 1992.
- [33] S. Lou and Q. Yao. The number of primes in a short interval. *Hardy-Ramanujan J.*, 16:21–43, 1993.
- [34] S. Lou and Q. Yao. Estimate of sums of Dirichlet series. *Hardy-Ramanujan J.*, 17:1–31, 1994.
- [35] H. L. Montgomery. *Topics in Multiplicative Number Theory*. Lecture Notes in Math. 227. Springer, Berlin, 1971.
- [36] C. J. Mozzochi. On the difference between consecutive primes. *J. Number Theory*, 24:181–187, 1986.
- [37] J. Pintz. On primes in short intervals I. *Studia Sci. Math. Hungar.*, 16:395–414, 1981.
- [38] J. Pintz. On primes in short intervals II. *Studia Sci. Math. Hungar.*, 19:89–96, 1984.
- [39] J. Pintz. The Bounded Gap Conjecture and bounds between consecutive Goldbach numbers. *Acta Arith.*, 155(4):397–405, 2012.
- [40] J. Pintz. On the difference of primes. *arXiv e-prints*, page arXiv:1206.0149v1, 2012.
- [41] J. Pintz. *A Goldbach-sejtésről*. Székfoglaló előadások a Magyar Tudományos Akadémián. Magyar Tudományos Akadémia, Budapest, 2014.
- [42] A. Selberg. On the normal density of primes in short intervals, and the difference between consecutive primes. *Arch. Math. Naturvid.*, 47:87–105, 1943.
- [43] N. Watt. Kloosterman sums and a mean value for Dirichlet polynomials. *J. Number Theory*, 53:179–210, 1995.
- [44] N. Watt. Short intervals almost all containing primes. *Acta Arith.*, 72(2):131–167, 1995.
- [45] K. C. Wong. Contribution to analytic number theory. *Ph.D. Thesis*, Cardiff, 1996.

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