

ON PRIME-PRODUCING SIEVES AND DISTRIBUTION OF $\alpha p - \beta \pmod{1}$

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ABSTRACT. The author proves that there are infinitely many primes p such that $\|\alpha p - \beta\| < p^{-\frac{28}{87}}$, where α is an irrational number and β is a real number. This sharpens a result of Jia (2000) and provides a new triple $(\gamma, \theta, \nu) = (\frac{59}{87}, \frac{28}{87}, \frac{1}{29})$ that can produce special primes in Ford and Maynard's work on prime-producing sieves. Our minimum amount of Type-II information required ($\nu = \frac{1}{29}$) is less than any previous work on this topic using only traditional Type-I and Type-II information.

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1. INTRODUCTION

Let α be an irrational number and $\|y\|$ denote the smallest distance from y to integers. Earlier work on this problem was done by Vinogradov [14] in 1954, who showed that for any real number β , there are infinitely many prime numbers p such that if $\tau = \frac{1}{5} - \varepsilon$, then

$$\|\alpha p - \beta\| < p^{-\tau}. \quad (1)$$

In 1977, Vaughan [13] got $\tau = \frac{1}{4} - \varepsilon$ using his identity. In 1983, Harman [3] introduced a new sieve method to this topic and got $\tau = \frac{3}{10}$. Jia [7] improved it to $\tau = \frac{4}{13}$ in 1993. In 1996, Harman [4] further improved it to $\tau = \frac{7}{22}$ by applying a new technique (the variable role-reversal) in his sieve. In 2000, Jia [9] got $\theta = \frac{9}{28}$. It is worth to mention that Balog [1] also got the same result in 1986 under the condition that $\|\alpha n\| < n^{-\frac{43}{31}-\varepsilon}$ holds for infinitely many integers n . If we only focus on the special case $\beta = 0$, then even better exponents $\frac{16}{49}$ and $\frac{1}{3} - \varepsilon$ were obtained by Heath-Brown and Jia [6] and Matomäki [12] respectively. In a personal communication, Matomäki mentioned that Maynard has got some $\tau > \frac{1}{3}$. Note that the Riemann Hypothesis implies that (1) holds for $\tau = \frac{1}{3} - \varepsilon$. In this paper, we show that (1) holds for $\tau = \frac{28}{87}$.

Theorem 1.1. *Suppose that α is an irrational number, then for any real number β , there are infinitely many prime numbers p such that*

$$\|\alpha p - \beta\| < p^{-\frac{28}{87}}.$$

A direct corollary of our Theorem 1.1 is the distribution of $p^\theta - \beta \pmod{1}$ for some $\theta < 1$.

Theorem 1.2. *For $\frac{31}{87} \leq \theta < 1$ and any real number β , there are infinitely many prime numbers p such that*

$$\|p^\theta - \beta\| < p^{-\frac{1-\theta}{2}+\varepsilon}.$$

Another corollary of our Theorem 1.1 is the following result focusing on Diophantine approximation with Gaussian primes, which improves Harman's exponent $\frac{7}{22}$ [5].

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Theorem 1.3. *Let $0 \leq \omega_1 < \omega_2 \leq 2\pi$. Then, given $\alpha \in \mathbb{C} \setminus \mathbb{Q}[i], \beta \in \mathbb{C}$, there are infinitely many Gaussian primes \mathfrak{p} such that*

$$\|\alpha\mathfrak{p} - \beta\| < |\mathfrak{p}|^{-\frac{28}{87}}, \quad \omega_1 \leq \arg \mathfrak{p} \leq \omega_2.$$

In 2024, Ford and Maynard [2] considered a series of general problems on prime-producing sieves. For those sieves that only use Type-I and Type-II information inputs, they defined a triple (γ, θ, ν) and considered various values of γ, θ and ν that can produce primes. In their notation, our Theorem 1.1 implies the following result:

Theorem 1.4. *The triple $(\gamma, \theta, \nu) = (\frac{59}{87}, \frac{28}{87}, \frac{1}{29})$ can produce primes with the property $\|\alpha p - \beta\| < p^{-\frac{28}{87}}$.*

Throughout this paper, we suppose that $\frac{a}{q}$ is a convergent to the continued fraction for α and ε is a sufficiently small positive constant. The letter p , with or without subscript, is reserved for prime numbers. Let $\tau = \frac{28}{87}$, $x = q^{\frac{2}{1+\tau}}$ and $\delta = (2x)^{-\tau}$. We define the boolean function as

$$\text{Boole}[\mathbf{X}] = \begin{cases} 1 & \text{if } \mathbf{X} \text{ is true,} \\ 0 & \text{if } \mathbf{X} \text{ is false.} \end{cases}$$

2. ASYMPTOTIC FORMULAS

Now we follow the discussion in [9]. Let $p_j = x^{t_j}$ and put

$$\mathcal{B} = \{n : x < n \leq 2x\}, \quad \mathcal{A} = \{n : x < n \leq 2x, \|\alpha n - \beta\| < \delta\},$$

$$\mathcal{A}_d = \{a : ad \in \mathcal{A}\}, \quad P(z) = \prod_{p < z} p, \quad S(\mathcal{A}, z) = \sum_{\substack{a \in \mathcal{A} \\ (a, P(z))=1}} 1.$$

Then we only need to show that $S(\mathcal{A}, (2x)^{\frac{1}{2}}) > 0$. Our aim is to show that the sparser set \mathcal{A} contains the expected proportion of primes compared to the bigger set \mathcal{B} , which requires us to decompose $S(\mathcal{A}, (2v)^{\frac{1}{2}})$ and prove asymptotic formulas of the form

$$S(\mathcal{A}, z) = 2\delta(1 + o(1))S(\mathcal{B}, z) \tag{2}$$

for some parts of it, and drop the other positive parts.

Here are some known asymptotic formulas for types of sieve functions, which were first proved by Harman [3] using traditional Type-I and Type-II arithmetic information.

Lemma 2.1. *(Type-I) Suppose that $M \ll x^{\frac{59}{87}}$ and $a(m) = O(1)$. Then we have*

$$\sum_{m \sim M} a(m)S(\mathcal{A}_m, x^{\frac{1}{29}}) = 2\delta(1 + o(1)) \sum_{m \sim M} a(m)S(\mathcal{B}_m, x^{\frac{1}{29}}).$$

Proof. The proof is similar to that of [3], Lemma 7. □

Lemma 2.2. *(Type-II) Suppose that $x^{\frac{28}{87}} \ll M \ll x^{\frac{31}{87}}$ or $x^{\frac{56}{87}} \ll M \ll x^{\frac{59}{87}}$ and that $a(m), b(n) = O(1)$. Then we have*

$$\sum_{m \sim M} \sum_n a(m)b(n)S(\mathcal{A}_{mn}, v(m, n)) = 2\delta(1 + o(1)) \sum_{m \sim M} \sum_n a(m)b(n)S(\mathcal{B}_{mn}, v(m, n)).$$

Proof. The proof is similar to that of [3], Lemma 6. □

Many authors, such as Heath-Brown and Jia [6] and Matomäki [12], have found more arithmetic information by using estimations of Kloosterman sums. However, their methods usually require $\beta = 0$, which is not applicable on the present paper.

3. THE FINAL DECOMPOSITION

Let $\omega(u)$ denote the Buchstab function determined by the following differential–difference equation

$$\begin{cases} \omega(u) = \frac{1}{u}, & 1 \leq u \leq 2, \\ (u\omega(u))' = \omega(u-1), & u \geq 2. \end{cases}$$

Moreover, we have the upper and lower bounds for $\omega(u)$:

$$\omega(u) \geq \omega_0(u) = \begin{cases} \frac{1}{u}, & 1 \leq u < 2, \\ \frac{\frac{1}{1+\log(u-1)}}{\frac{1+\log(u-1)}{u}} + \frac{1}{u} \int_2^{u-1} \frac{\log(t-1)}{t} dt, & 2 \leq u < 3, \\ 0.5612, & u \geq 4, \end{cases}$$

$$\omega(u) \leq \omega_1(u) = \begin{cases} \frac{1}{u}, & 1 \leq u < 2, \\ \frac{\frac{1}{1+\log(u-1)}}{\frac{1+\log(u-1)}{u}} + \frac{1}{u} \int_2^{u-1} \frac{\log(t-1)}{t} dt, & 2 \leq u < 3, \\ 0.5617, & u \geq 4. \end{cases}$$

We shall use $\omega_0(u)$ and $\omega_1(u)$ to give numerical bounds for some sieve functions discussed below. We shall also use the simple upper bound $\omega(u) \leq \max(\frac{1}{u}, 0.5672)$ (see Lemma 8(iii) of [8]) to estimate high–dimensional integrals.

Before decomposing, we define the asymptotic region I as

$$I(m, n) := \left\{ \begin{aligned} & \frac{28}{87} \leq m \leq \frac{31}{87} \text{ or } \frac{56}{87} \leq m \leq \frac{59}{87} \text{ or } \frac{28}{87} \leq n \leq \frac{31}{87} \text{ or } \frac{56}{87} \leq n \leq \frac{59}{87} \\ & \text{or } \frac{28}{87} \leq m+n \leq \frac{31}{87} \text{ or } \frac{56}{87} \leq m+n \leq \frac{59}{87} \end{aligned} \right\}.$$

By Buchstab's identity, we have

$$\begin{aligned} S\left(\mathcal{A}, (2x)^{\frac{1}{2}}\right) &= S\left(\mathcal{A}, x^{\frac{1}{29}}\right) - \sum_{\frac{1}{29} \leq t_1 < \frac{1}{2}} S\left(\mathcal{A}_{p_1}, x^{\frac{1}{29}}\right) + \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1))}} S(\mathcal{A}_{p_1 p_2}, p_2) \\ &= S_1 - S_2 + S_3. \end{aligned} \tag{3}$$

By Lemma 2.1, we can give asymptotic formulas for S_1 and S_2 . Before estimating S_3 , we first split it into six parts:

$$\begin{aligned} S_3 &= \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1))}} S(\mathcal{A}_{p_1 p_2}, p_2) \\ &= \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in I}} S(\mathcal{A}_{p_1 p_2}, p_2) + \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \notin I \\ t_1 < \frac{28}{87}, t_1 + 2t_2 > \frac{59}{87} \text{ or } t_1 + t_2 > \frac{59}{87}}} S(\mathcal{A}_{p_1 p_2}, p_2) \\ &+ \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \notin I \\ t_1 \leq \frac{31}{87}, t_1 + 2t_2 \leq \frac{59}{87}}} S(\mathcal{A}_{p_1 p_2}, p_2) + \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \notin I \\ t_1 > \frac{31}{87}, t_1 + 2t_2 \leq \frac{59}{87}}} S(\mathcal{A}_{p_1 p_2}, p_2) \end{aligned}$$

$$\begin{aligned}
& + \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \notin I \\ t_1 > \frac{31}{87}, t_1 + 2t_2 > \frac{59}{87} \\ t_2 \leq \max(\frac{1-t_1}{3}, \frac{t_1}{2})}} S(\mathcal{A}_{p_1 p_2}, p_2) + \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \notin I \\ t_1 > \frac{31}{87}, t_1 + 2t_2 > \frac{59}{87} \\ t_2 > \max(\frac{1-t_1}{3}, \frac{t_1}{2})}} S(\mathcal{A}_{p_1 p_2}, p_2) \\
& = \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in I}} S(\mathcal{A}_{p_1 p_2}, p_2) + \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_1}} S(\mathcal{A}_{p_1 p_2}, p_2) \\
& + \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_2}} S(\mathcal{A}_{p_1 p_2}, p_2) + \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_3}} S(\mathcal{A}_{p_1 p_2}, p_2) \\
& + \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_4}} S(\mathcal{A}_{p_1 p_2}, p_2) + \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_5}} S(\mathcal{A}_{p_1 p_2}, p_2) \\
& = S_{31} + S_{32} + S_{33} + S_{34} + S_{35} + S_{36}. \tag{4}
\end{aligned}$$

S_{31} has an asymptotic formula. For S_{32} , we cannot decompose further but have to discard the whole region giving the loss

$$L_{32} := \int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min(t_1, \frac{1-t_1}{2})} \text{Boolean}[(t_1, t_2) \in J_1] \frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1 t_2^2} dt_2 dt_1 < 0.397685. \tag{5}$$

For S_{33} we can use Buchstab's identity to get

$$\begin{aligned}
S_{33} & = \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_2}} S(\mathcal{A}_{p_1 p_2}, p_2) \\
& = \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_2}} S(\mathcal{A}_{p_1 p_2}, x^{\frac{1}{29}}) - \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_2 \\ \frac{1}{29} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2))}} S(\mathcal{A}_{p_1 p_2 p_3}, x^{\frac{1}{29}}) \\
& + \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_2 \\ \frac{1}{29} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ \frac{1}{29} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \text{ can be partitioned into } (m, n) \in I}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4) \\
& + \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_2 \\ \frac{1}{29} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ \frac{1}{29} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4) \\
& = S_{331} - S_{332} + S_{333} + S_{334}. \tag{6}
\end{aligned}$$

We have asymptotic formulas for S_{331} – S_{333} . For the remaining S_{334} , we have two ways to get more possible savings: One way is to use Buchstab's identity twice more for some parts if we have $t_1 + t_2 + t_3 + 2t_4 \leq \frac{59}{87}$.

Another way is to use Buchstab's identity in reverse to make almost-primes visible. The details of further decompositions are similar to those in [11]. Combining the cases above we get a loss from S_{33} of

$$\begin{aligned}
& \left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min(t_1, \frac{1-t_1}{2})} \int_{\frac{1}{29}}^{\min(t_2, \frac{1-t_1-t_2}{2})} \int_{\frac{1}{29}}^{\min(t_3, \frac{1-t_1-t_2-t_3}{2})} \right. \\
& \quad \text{Boole}[(t_1, t_2, t_3, t_4) \in J_{331}] \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1 \Bigg) \\
& - \left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min(t_1, \frac{1-t_1}{2})} \int_{\frac{1}{29}}^{\min(t_2, \frac{1-t_1-t_2}{2})} \int_{\frac{1}{29}}^{\min(t_3, \frac{1-t_1-t_2-t_3}{2})} \int_{t_4}^{\frac{1-t_1-t_2-t_3-t_4}{2}} \right. \\
& \quad \text{Boole}[(t_1, t_2, t_3, t_4, t_5) \in J_{332}] \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1 \Bigg) \\
& + \left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min(t_1, \frac{1-t_1}{2})} \int_{\frac{1}{29}}^{\min(t_2, \frac{1-t_1-t_2}{2})} \int_{\frac{1}{29}}^{\min(t_3, \frac{1-t_1-t_2-t_3}{2})} \right. \\
& \quad \int_{\frac{1}{29}}^{\min(t_4, \frac{1-t_1-t_2-t_3-t_4}{2})} \int_{\frac{1}{29}}^{\min(t_5, \frac{1-t_1-t_2-t_3-t_4-t_5}{2})} \\
& \quad \text{Boole}[(t_1, t_2, t_3, t_4, t_5, t_6) \in J_{333}] \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{t_6}\right)}{t_1 t_2 t_3 t_4 t_5 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \Bigg) \\
& + \left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min(t_1, \frac{1-t_1}{2})} \int_{\frac{1}{29}}^{\min(t_2, \frac{1-t_1-t_2}{2})} \int_{\frac{1}{29}}^{\min(t_3, \frac{1-t_1-t_2-t_3}{2})} \int_{\frac{1}{29}}^{\min(t_4, \frac{1-t_1-t_2-t_3-t_4}{2})} \right. \\
& \quad \int_{\frac{1}{29}}^{\min(t_5, \frac{1-t_1-t_2-t_3-t_4-t_5}{2})} \int_{\frac{1}{29}}^{\min(t_6, \frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{2})} \\
& \quad \int_{\frac{1}{29}}^{\min(t_7, \frac{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7}{2})} \text{Boole}[(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \in J_{334}] \times \\
& \quad \frac{\max\left(\frac{t_8}{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7-t_8}, 0.5672\right)}{t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \Bigg) \\
& = L_{331} - L_{332} + L_{333} + L_{334}, \\
& < 0.111391 - 0.021501 + 0.001491 + 0.000002 \\
& = 0.091383,
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
J_{331}(t_1, t_2, t_3, t_4) &:= \{(t_1, t_2) \in J_2, \\
& \frac{1}{29} \leq t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \\
& \frac{1}{29} \leq t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\
& (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I, \\
& t_1 + t_2 + t_3 + 2t_4 > \frac{59}{87}, \\
& \frac{1}{29} \leq t_1 < \frac{1}{2}, \frac{1}{29} \leq t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\} ,
\end{aligned}$$

$$\begin{aligned}
J_{332}(t_1, t_2, t_3, t_4, t_5) := & \{(t_1, t_2) \in J_2, \\
& \frac{1}{29} \leq t_3 < \min\left(t_2, \frac{1}{2}(1 - t_1 - t_2)\right), \\
& \frac{1}{29} \leq t_4 < \min\left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3)\right), \\
& (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I, \\
& t_1 + t_2 + t_3 + 2t_4 > \frac{59}{87}, \\
& t_4 < t_5 < \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4), \\
& (t_1, t_2, t_3, t_4, t_5) \text{ can be partitioned into } (m, n) \in I, \\
& \left. \frac{1}{29} \leq t_1 < \frac{1}{2}, \frac{1}{29} \leq t_2 < \min\left(t_1, \frac{1}{2}(1 - t_1)\right) \right\},
\end{aligned}$$

$$\begin{aligned}
J_{333}(t_1, t_2, t_3, t_4, t_5, t_6) := & \{(t_1, t_2) \in J_2, \\
& \frac{1}{29} \leq t_3 < \min\left(t_2, \frac{1}{2}(1 - t_1 - t_2)\right), \\
& \frac{1}{29} \leq t_4 < \min\left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3)\right), \\
& (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I, \\
& t_1 + t_2 + t_3 + 2t_4 \leq \frac{59}{87}, \\
& \frac{1}{29} \leq t_5 < \min\left(t_4, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4)\right), \\
& \frac{1}{29} \leq t_6 < \min\left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5)\right), \\
& (t_1, t_2, t_3, t_4, t_5, t_6) \text{ cannot be partitioned into } (m, n) \in I, \\
& t_1 + t_2 + t_3 + t_4 + t_5 + 2t_6 > \frac{59}{87}, \\
& \left. \frac{1}{29} \leq t_1 < \frac{1}{2}, \frac{1}{29} \leq t_2 < \min\left(t_1, \frac{1}{2}(1 - t_1)\right) \right\},
\end{aligned}$$

$$\begin{aligned}
J_{334}(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) := & \{(t_1, t_2) \in J_2, \\
& \frac{1}{29} \leq t_3 < \min\left(t_2, \frac{1}{2}(1 - t_1 - t_2)\right), \\
& \frac{1}{29} \leq t_4 < \min\left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3)\right), \\
& (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I, \\
& t_1 + t_2 + t_3 + 2t_4 \leq \frac{59}{87}, \\
& \frac{1}{29} \leq t_5 < \min\left(t_4, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4)\right), \\
& \frac{1}{29} \leq t_6 < \min\left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5)\right), \\
& (t_1, t_2, t_3, t_4, t_5, t_6) \text{ cannot be partitioned into } (m, n) \in I, \\
& t_1 + t_2 + t_3 + t_4 + t_5 + 2t_6 \leq \frac{59}{87},
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{29} \leq t_7 < \min \left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5 - t_6) \right), \\
& \frac{1}{29} \leq t_8 < \min \left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5 - t_6 - t_7) \right), \\
& (t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \text{ cannot be partitioned into } (m, n) \in I, \\
& \frac{1}{29} \leq t_1 < \frac{1}{2}, \quad \frac{1}{29} \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \Big\}.
\end{aligned}$$

Next we shall decompose S_{34} . By the same process as the decomposition of S_{33} above, we can reach a four-dimensional sum

$$\sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_2 \\ \frac{1}{29} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ \frac{1}{29} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4).$$

We divide it into three parts:

$$\begin{aligned}
& \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_2 \\ \frac{1}{29} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ \frac{1}{29} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4) \\
= & \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_2 \\ \frac{1}{29} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ \frac{1}{29} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I \\ t_1 + t_2 + t_3 + t_4 > \frac{59}{87}}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4) \\
+ & \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_2 \\ \frac{1}{29} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ \frac{1}{29} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I \\ t_1 + t_2 + t_3 + 2t_4 \leq \frac{59}{87}}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4) \\
+ & \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_2 \\ \frac{1}{29} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ \frac{1}{29} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I \\ t_1 + t_2 + t_3 + 2t_4 > \frac{59}{87} \\ t_1 + t_2 + t_3 + t_4 \leq \frac{59}{87}}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4). \tag{8}
\end{aligned}$$

We can only use Buchstab's identity in reverse to make more savings for the first sum on the right-hand side. For the second sum in (8), we can perform a straightforward decomposition to get a six-dimensional

sum

$$\begin{aligned}
& \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_3 \\ \frac{1}{29} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ \frac{1}{29} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I \\ t_1+t_2+t_3+2t_4 \leq \frac{59}{87} \\ \frac{1}{29} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ \frac{1}{29} \leq t_6 < \min(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)) \\ (t_1, t_2, t_3, t_4, t_5, t_6) \text{ cannot be partitioned into } (m, n) \in I \\ t_1+t_2+t_3+t_4+t_5+2t_6 > \frac{59}{87}}} S(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5 p_6}, p_6) \quad (9)
\end{aligned}$$

and an eight-dimensional sum

$$\begin{aligned}
& \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_3 \\ \frac{1}{29} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ \frac{1}{29} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I \\ t_1+t_2+t_3+2t_4 \leq \frac{59}{87} \\ \frac{1}{29} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ \frac{1}{29} \leq t_6 < \min(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)) \\ (t_1, t_2, t_3, t_4, t_5, t_6) \text{ cannot be partitioned into } (m, n) \in I \\ t_1+t_2+t_3+t_4+t_5+2t_6 \leq \frac{59}{87} \\ \frac{1}{29} \leq t_7 < \min(t_6, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6)) \\ \frac{1}{29} \leq t_8 < \min(t_7, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6-t_7)) \\ (t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \text{ cannot be partitioned into } (m, n) \in I}}} S(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5 p_6 p_7 p_8}, p_8) \quad (10)
\end{aligned}$$

We can also use reversed Buchstab's identity to gain a five-dimensional saving. For the last sum, we cannot decompose it in a straightforward way. However, we can perform a role-reversal to get

$$\begin{aligned}
& \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_2 \\ \frac{1}{29} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ \frac{1}{29} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I \\ t_1+t_2+t_3+2t_4 > \frac{59}{87} \\ t_1+t_2+t_3+t_4 \leq \frac{59}{87}}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4) \\
= & \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_2 \\ \frac{1}{29} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ \frac{1}{29} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I \\ t_1+t_2+t_3+2t_4 > \frac{59}{87} \\ t_1+t_2+t_3+t_4 \leq \frac{59}{87}}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, x^{\frac{1}{29}})
\end{aligned}$$

$$\begin{aligned}
& - \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_2 \\ \frac{1}{29} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ \frac{1}{29} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I \\ t_1+t_2+t_3+2t_4 > \frac{59}{87} \\ t_1+t_2+t_3+t_4 \leq \frac{59}{87} \\ \frac{1}{29} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \text{ can be partitioned into } (m, n) \in I}} S(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5) \\
& - \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_2 \\ \frac{1}{29} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ \frac{1}{29} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I \\ t_1+t_2+t_3+2t_4 > \frac{59}{87} \\ t_1+t_2+t_3+t_4 \leq \frac{59}{87} \\ \frac{1}{29} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \text{ cannot be partitioned into } (m, n) \in I}} S(\mathcal{A}_{\gamma p_2 p_3 p_4 p_5}, x^{\frac{1}{29}}) \\
& + \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_2 \\ \frac{1}{29} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ \frac{1}{29} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I \\ t_1+t_2+t_3+2t_4 > \frac{59}{87} \\ t_1+t_2+t_3+t_4 \leq \frac{59}{87} \\ \frac{1}{29} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \text{ cannot be partitioned into } (m, n) \in I \\ \frac{1}{29} \leq t_6 < \frac{1}{2} t_1 \\ (1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6) \text{ can be partitioned into } (m, n) \in I}} S(\mathcal{A}_{\gamma p_2 p_3 p_4 p_5 p_6}, p_6), \\
& + \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_2 \\ \frac{1}{29} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ \frac{1}{29} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I \\ t_1+t_2+t_3+2t_4 > \frac{59}{87} \\ t_1+t_2+t_3+t_4 \leq \frac{59}{87} \\ \frac{1}{29} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \text{ cannot be partitioned into } (m, n) \in I \\ \frac{1}{29} \leq t_6 < \frac{1}{2} t_1 \\ (1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6) \text{ cannot be partitioned into } (m, n) \in I}} S(\mathcal{A}_{\gamma p_2 p_3 p_4 p_5 p_6}, p_6), \tag{11}
\end{aligned}$$

where $\gamma \sim x^{1-t_1-t_2-t_3-t_4-t_5}$ and $(\gamma, P(p_5)) = 1$. Since $t_1 + t_2 + t_3 + t_4 \leq \frac{59}{87}$ and $(1 - t_1 - t_2 - t_3 - t_4 - t_5) + t_2 + t_3 + t_4 + t_5 = (1 - t_1) \leq (1 - \frac{31}{87}) = \frac{56}{87}$, we can give asymptotic formulas for all sums on the right hand side except for the last sum. Note that the last sum in (11) counts numbers with two almost-prime

variables, we can make further decompositions on either variable, leading to two eight-dimensional sums

$$\begin{aligned}
& \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_2 \\ \frac{1}{29} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ \frac{1}{29} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I \\ t_1+t_2+t_3+2t_4 > \frac{59}{87} \\ t_1+t_2+t_3+t_4 \leq \frac{59}{87} \\ \frac{1}{29} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \text{ cannot be partitioned into } (m, n) \in I \\ \frac{1}{29} \leq t_6 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6) \text{ cannot be partitioned into } (m, n) \in I \\ (1-t_1-t_2-t_3-t_4-t_5)+t_2+t_3+t_4+t_5+2t_6 \leq \frac{59}{87} \\ \frac{1}{29} \leq t_7 < \min(t_6, \frac{1}{2}(t_1-t_6)) \\ (1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6, t_7) \text{ cannot be partitioned into } (m, n) \in I \\ \frac{1}{29} \leq t_8 < \min(t_7, \frac{1}{2}(t_1-t_6-t_7)) \\ (1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \text{ cannot be partitioned into } (m, n) \in I}} S(\mathcal{A}_{\gamma p_2 p_3 p_4 p_5 p_6 p_7 p_8}, p_8) \quad (12)
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{\substack{\frac{1}{29} \leq t_1 < \frac{1}{2} \\ \frac{1}{29} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in J_2 \\ \frac{1}{29} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ \frac{1}{29} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I \\ t_1+t_2+t_3+2t_4 > \frac{59}{87} \\ t_1+t_2+t_3+t_4 \leq \frac{59}{87} \\ \frac{1}{29} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \text{ cannot be partitioned into } (m, n) \in I \\ \frac{1}{29} \leq t_6 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6) \text{ cannot be partitioned into } (m, n) \in I \\ (1-t_1-t_2-t_3-t_4-t_5)+t_2+t_3+t_4+t_5+2t_6 > \frac{59}{87} \\ (t_1-t_6)+t_2+t_3+t_4+2t_5+t_6 \leq \frac{59}{87} \\ \frac{1}{29} \leq t_7 < \min(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)) \\ (t_1-t_6, t_2, t_3, t_4, t_5, t_6, t_7) \text{ cannot be partitioned into } (m, n) \in I \\ \frac{1}{29} \leq t_8 < \min(t_7, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_7)) \\ (t_1-t_6, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \text{ cannot be partitioned into } (m, n) \in I}} S(\mathcal{A}_{\gamma_1 p_2 p_3 p_4 p_5 p_6 p_7 p_8}, p_8), \quad (13)
\end{aligned}$$

where $\gamma_1 \sim x^{t_1-t_6}$ and $(\gamma_1, P(p_6)) = 1$. We refer the readers to [10] and [11] for more applications of role-reversals. Combining the cases above we get a loss from S_{34} of

$$\begin{aligned}
& \left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min(t_1, \frac{1-t_1}{2})} \int_{\frac{1}{29}}^{\min(t_2, \frac{1-t_1-t_2}{2})} \int_{\frac{1}{29}}^{\min(t_3, \frac{1-t_1-t_2-t_3}{2})} \right. \\
& \quad \text{Boole}[(t_1, t_2, t_3, t_4) \in J_{341}] \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1 \Bigg) \\
& - \left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min(t_1, \frac{1-t_1}{2})} \int_{\frac{1}{29}}^{\min(t_2, \frac{1-t_1-t_2}{2})} \int_{\frac{1}{29}}^{\min(t_3, \frac{1-t_1-t_2-t_3}{2})} \int_{t_4}^{\frac{1-t_1-t_2-t_3-t_4}{2}} \right. \\
& \quad \text{Boole}[(t_1, t_2, t_3, t_4, t_5) \in J_{342}] \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1 \Bigg) \\
& + \left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min(t_1, \frac{1-t_1}{2})} \int_{\frac{1}{29}}^{\min(t_2, \frac{1-t_1-t_2}{2})} \int_{\frac{1}{29}}^{\min(t_3, \frac{1-t_1-t_2-t_3}{2})} \right.
\end{aligned}$$

$$\begin{aligned}
& \int_{\frac{1}{29}}^{\min(t_4, \frac{1-t_1-t_2-t_3-t_4}{2})} \int_{\frac{1}{29}}^{\min(t_5, \frac{1-t_1-t_2-t_3-t_4-t_5}{2})} \\
& \text{Boole}[(t_1, t_2, t_3, t_4, t_5, t_6) \in J_{343}] \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{t_6}\right)}{t_1 t_2 t_3 t_4 t_5 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \Bigg) \\
& + \left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min(t_1, \frac{1-t_1}{2})} \int_{\frac{1}{29}}^{\min(t_2, \frac{1-t_1-t_2}{2})} \int_{\frac{1}{29}}^{\min(t_3, \frac{1-t_1-t_2-t_3}{2})} \int_{\frac{1}{29}}^{\min(t_4, \frac{1-t_1-t_2-t_3-t_4}{2})} \right. \\
& \quad \int_{\frac{1}{29}}^{\min(t_5, \frac{1-t_1-t_2-t_3-t_4-t_5}{2})} \int_{\frac{1}{29}}^{\min(t_6, \frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{2})} \\
& \quad \left. \int_{\frac{1}{29}}^{\min(t_7, \frac{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7}{2})} \text{Boole}[(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \in J_{344}] \times \right. \\
& \quad \left. \frac{\max\left(\frac{t_8}{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7-t_8}, 0.5672\right)}{t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
& + \left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min(t_1, \frac{1-t_1}{2})} \int_{\frac{1}{29}}^{\min(t_2, \frac{1-t_1-t_2}{2})} \int_{\frac{1}{29}}^{\min(t_3, \frac{1-t_1-t_2-t_3}{2})} \int_{\frac{1}{29}}^{\min(t_4, \frac{1-t_1-t_2-t_3-t_4}{2})} \int_{\frac{1}{29}}^{\frac{1}{2} t_1} \right. \\
& \quad \text{Boole}[(t_1, t_2, t_3, t_4, t_5, t_6) \in J_{345}] \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right) \omega_1\left(\frac{t_1-t_6}{t_6}\right)}{t_2 t_3 t_4 t_5^2 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \Bigg) \\
& + \left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min(t_1, \frac{1-t_1}{2})} \int_{\frac{1}{29}}^{\min(t_2, \frac{1-t_1-t_2}{2})} \int_{\frac{1}{29}}^{\min(t_3, \frac{1-t_1-t_2-t_3}{2})} \int_{\frac{1}{29}}^{\min(t_4, \frac{1-t_1-t_2-t_3-t_4}{2})} \right. \\
& \quad \int_{\frac{1}{29}}^{\frac{1}{2} t_1} \int_{\frac{1}{29}}^{\min(t_6, \frac{t_1-t_6}{2})} \int_{\frac{1}{29}}^{\min(t_7, \frac{t_1-t_6-t_7}{2})} \text{Boole}[(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \in J_{346}] \times \\
& \quad \left. \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right) \omega_1\left(\frac{t_1-t_6-t_7-t_8}{t_8}\right)}{t_2 t_3 t_4 t_5^2 t_6^2 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
& + \left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min(t_1, \frac{1-t_1}{2})} \int_{\frac{1}{29}}^{\min(t_2, \frac{1-t_1-t_2}{2})} \int_{\frac{1}{29}}^{\min(t_3, \frac{1-t_1-t_2-t_3}{2})} \int_{\frac{1}{29}}^{\min(t_4, \frac{1-t_1-t_2-t_3-t_4}{2})} \int_{\frac{1}{29}}^{\frac{1}{2} t_1} \right. \\
& \quad \int_{\frac{1}{29}}^{\min(t_5, \frac{1-t_1-t_2-t_3-t_4-t_5}{2})} \int_{\frac{1}{29}}^{\min(t_7, \frac{1-t_1-t_2-t_3-t_4-t_5-t_7}{2})} \text{Boole}[(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \in J_{347}] \times \\
& \quad \left. \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_7-t_8}{t_8}\right) \omega_1\left(\frac{t_1-t_6}{t_6}\right)}{t_2 t_3 t_4 t_5 t_6^2 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
& = L_{341} - L_{342} + L_{343} + L_{344} + L_{345} + L_{346} + L_{347}, \\
& < 0.103669 - 0.03892 + 0.001712 + 0.000001 + 0.007242 + 0.000056 + 0 \\
& = 0.07376,
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
J_{341}(t_1, t_2, t_3, t_4) &:= \{(t_1, t_2) \in J_3, \\
&\quad \frac{1}{29} \leq t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{29} \leq t_4 < \min \left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3) \right), \\
& (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I, \\
& t_1 + t_2 + t_3 + t_4 > \frac{59}{87}, \\
& \frac{1}{29} \leq t_1 < \frac{1}{2}, \quad \frac{1}{29} \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \Big\}, \\
J_{342}(t_1, t_2, t_3, t_4, t_5) &:= \{(t_1, t_2) \in J_3, \\
& \frac{1}{29} \leq t_3 < \min \left(t_2, \frac{1}{2}(1 - t_1 - t_2) \right), \\
& \frac{1}{29} \leq t_4 < \min \left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3) \right), \\
& (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I, \\
& t_1 + t_2 + t_3 + t_4 > \frac{59}{87}, \\
& t_4 < t_5 < \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4), \\
& (t_1, t_2, t_3, t_4, t_5) \text{ can be partitioned into } (m, n) \in I, \\
& \frac{1}{29} \leq t_1 < \frac{1}{2}, \quad \frac{1}{29} \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \Big\}, \\
J_{343}(t_1, t_2, t_3, t_4, t_5, t_6) &:= \{(t_1, t_2) \in J_3, \\
& \frac{1}{29} \leq t_3 < \min \left(t_2, \frac{1}{2}(1 - t_1 - t_2) \right), \\
& \frac{1}{29} \leq t_4 < \min \left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3) \right), \\
& (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I, \\
& t_1 + t_2 + t_3 + 2t_4 \leq \frac{59}{87}, \\
& \frac{1}{29} \leq t_5 < \min \left(t_4, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4) \right), \\
& \frac{1}{29} \leq t_6 < \min \left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5) \right), \\
& (t_1, t_2, t_3, t_4, t_5, t_6) \text{ cannot be partitioned into } (m, n) \in I, \\
& t_1 + t_2 + t_3 + t_4 + t_5 + 2t_6 > \frac{59}{87}, \\
& \frac{1}{29} \leq t_1 < \frac{1}{2}, \quad \frac{1}{29} \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \Big\}, \\
J_{344}(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) &:= \{(t_1, t_2) \in J_3, \\
& \frac{1}{29} \leq t_3 < \min \left(t_2, \frac{1}{2}(1 - t_1 - t_2) \right), \\
& \frac{1}{29} \leq t_4 < \min \left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3) \right), \\
& (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I, \\
& t_1 + t_2 + t_3 + 2t_4 \leq \frac{59}{87},
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{29} \leq t_5 < \min \left(t_4, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4) \right), \\
& \frac{1}{29} \leq t_6 < \min \left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5) \right), \\
& (t_1, t_2, t_3, t_4, t_5, t_6) \text{ cannot be partitioned into } (m, n) \in I, \\
& t_1 + t_2 + t_3 + t_4 + t_5 + 2t_6 \leq \frac{59}{87}, \\
& \frac{1}{29} \leq t_7 < \min \left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5 - t_6) \right), \\
& \frac{1}{29} \leq t_8 < \min \left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5 - t_6 - t_7) \right), \\
& (t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \text{ cannot be partitioned into } (m, n) \in I, \\
& \frac{1}{29} \leq t_1 < \frac{1}{2}, \quad \frac{1}{29} \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \Big\}, \\
J_{345}(t_1, t_2, t_3, t_4, t_5, t_6) := & \{(t_1, t_2) \in J_3, \\
& \frac{1}{29} \leq t_3 < \min \left(t_2, \frac{1}{2}(1 - t_1 - t_2) \right), \\
& \frac{1}{29} \leq t_4 < \min \left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3) \right), \\
& (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I, \\
& t_1 + t_2 + t_3 + t_4 \leq \frac{59}{87}, \quad t_1 + t_2 + t_3 + 2t_4 > \frac{59}{87}, \\
& \frac{1}{29} \leq t_5 < \min \left(t_4, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4) \right), \\
& (t_1, t_2, t_3, t_4, t_5) \text{ cannot be partitioned into } (m, n) \in I, \\
& \frac{1}{29} \leq t_6 < \frac{1}{2}t_1, \\
& (1 - t_1 - t_2 - t_3 - t_4 - t_5, t_2, t_3, t_4, t_5, t_6) \\
& \quad \text{cannot be partitioned into } (m, n) \in I, \\
& \frac{1}{29} \leq t_1 < \frac{1}{2}, \quad \frac{1}{29} \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \Big\}, \\
J_{346}(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) := & \{(t_1, t_2) \in J_3, \\
& \frac{1}{29} \leq t_3 < \min \left(t_2, \frac{1}{2}(1 - t_1 - t_2) \right), \\
& \frac{1}{29} \leq t_4 < \min \left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3) \right), \\
& (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I, \\
& t_1 + t_2 + t_3 + t_4 \leq \frac{59}{87}, \quad t_1 + t_2 + t_3 + 2t_4 > \frac{59}{87}, \\
& \frac{1}{29} \leq t_5 < \min \left(t_4, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4) \right), \\
& (t_1, t_2, t_3, t_4, t_5) \text{ cannot be partitioned into } (m, n) \in I, \\
& \frac{1}{29} \leq t_6 < \frac{1}{2}t_1, \\
& (1 - t_1 - t_2 - t_3 - t_4 - t_5, t_2, t_3, t_4, t_5, t_6) \\
& \quad \text{cannot be partitioned into } (m, n) \in I,
\end{aligned}$$

$$\begin{aligned}
& (1 - t_1 - t_2 - t_3 - t_4 - t_5) + t_2 + t_3 + t_4 + t_5 + 2t_6 \leq \frac{59}{87}, \\
& \frac{1}{29} \leq t_7 < \min \left(t_6, \frac{1}{2}(t_1 - t_6) \right), \\
& (1 - t_1 - t_2 - t_3 - t_4 - t_5, t_2, t_3, t_4, t_5, t_6, t_7) \\
& \quad \text{cannot be partitioned into } (m, n) \in I, \\
& \frac{1}{29} \leq t_8 < \min \left(t_7, \frac{1}{2}(t_1 - t_6 - t_7) \right), \\
& (1 - t_1 - t_2 - t_3 - t_4 - t_5, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \\
& \quad \text{cannot be partitioned into } (m, n) \in I, \\
& \left. \frac{1}{29} \leq t_1 < \frac{1}{2}, \frac{1}{29} \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \right\}, \\
J_{347}(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) := & \{(t_1, t_2) \in J_3, \\
& \frac{1}{29} \leq t_3 < \min \left(t_2, \frac{1}{2}(1 - t_1 - t_2) \right), \\
& \frac{1}{29} \leq t_4 < \min \left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3) \right), \\
& (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I, \\
& t_1 + t_2 + t_3 + t_4 \leq \frac{59}{87}, t_1 + t_2 + t_3 + 2t_4 > \frac{59}{87}, \\
& \frac{1}{29} \leq t_5 < \min \left(t_4, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4) \right), \\
& (t_1, t_2, t_3, t_4, t_5) \text{ cannot be partitioned into } (m, n) \in I, \\
& \frac{1}{29} \leq t_6 < \frac{1}{2}t_1, \\
& (1 - t_1 - t_2 - t_3 - t_4 - t_5, t_2, t_3, t_4, t_5, t_6) \\
& \quad \text{cannot be partitioned into } (m, n) \in I, \\
& (1 - t_1 - t_2 - t_3 - t_4 - t_5) + t_2 + t_3 + t_4 + t_5 + 2t_6 > \frac{59}{87}, \\
& (t_1 - t_6) + t_2 + t_3 + t_4 + 2t_5 + t_6 \leq \frac{59}{87}, \\
& \frac{1}{29} \leq t_7 < \min \left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5) \right), \\
& (t_1 - t_6, t_2, t_3, t_4, t_5, t_6, t_7) \\
& \quad \text{cannot be partitioned into } (m, n) \in I, \\
& \frac{1}{29} \leq t_8 < \min \left(t_7, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5 - t_7) \right), \\
& (t_1 - t_6, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \\
& \quad \text{cannot be partitioned into } (m, n) \in I, \\
& \left. \frac{1}{29} \leq t_1 < \frac{1}{2}, \frac{1}{29} \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \right\}.
\end{aligned}$$

For S_{35} we can also use the devices mentioned earlier, but in this case we will perform a role-reversal on the triple sum if $t_1 + t_2 + t_3 > \frac{59}{87}$. In this way we get a loss of

$$\left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min(t_1, \frac{1-t_1}{2})} \int_{\frac{1}{29}}^{\min(t_2, \frac{1-t_1-t_2}{2})} \int_{\frac{1}{29}}^{\min(t_3, \frac{1-t_1-t_2-t_3}{2})} \right)$$

$$\begin{aligned}
& \text{Boole}[(t_1, t_2, t_3, t_4) \in J_{3501}] \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1 \Bigg) \\
& - \left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min(t_1, \frac{1-t_1}{2})} \int_{\frac{1}{29}}^{\min(t_2, \frac{1-t_1-t_2}{2})} \int_{\frac{1}{29}}^{\min(t_3, \frac{1-t_1-t_2-t_3}{2})} \int_{t_4}^{\frac{1-t_1-t_2-t_3-t_4}{2}} \right. \\
& \quad \text{Boole}[(t_1, t_2, t_3, t_4, t_5) \in J_{3502}] \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1 \Bigg) \\
& + \left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min(t_1, \frac{1-t_1}{2})} \int_{\frac{1}{29}}^{\min(t_2, \frac{1-t_1-t_2}{2})} \int_{\frac{1}{29}}^{\min(t_3, \frac{1-t_1-t_2-t_3}{2})} \right. \\
& \quad \int_{\frac{1}{29}}^{\min(t_4, \frac{1-t_1-t_2-t_3-t_4}{2})} \int_{\frac{1}{29}}^{\min(t_5, \frac{1-t_1-t_2-t_3-t_4-t_5}{2})} \\
& \quad \text{Boole}[(t_1, t_2, t_3, t_4, t_5, t_6) \in J_{3503}] \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{t_6}\right)}{t_1 t_2 t_3 t_4 t_5 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \Bigg) \\
& + \left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min(t_1, \frac{1-t_1}{2})} \int_{\frac{1}{29}}^{\min(t_2, \frac{1-t_1-t_2}{2})} \int_{\frac{1}{29}}^{\min(t_3, \frac{1-t_1-t_2-t_3}{2})} \int_{\frac{1}{29}}^{\min(t_4, \frac{1-t_1-t_2-t_3-t_4}{2})} \right. \\
& \quad \int_{\frac{1}{29}}^{\min(t_5, \frac{1-t_1-t_2-t_3-t_4-t_5}{2})} \int_{\frac{1}{29}}^{\min(t_6, \frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{2})} \\
& \quad \int_{\frac{1}{29}}^{\min(t_7, \frac{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7}{2})} \text{Boole}[(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \in J_{3504}] \times \\
& \quad \frac{\max\left(\frac{t_8}{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7-t_8}, 0.5672\right)}{t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \Bigg) \\
& + \left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min(t_1, \frac{1-t_1}{2})} \int_{\frac{1}{29}}^{\min(t_2, \frac{1-t_1-t_2}{2})} \int_{\frac{1}{29}}^{\frac{1}{2} t_1} \right. \\
& \quad \text{Boole}[(t_1, t_2, t_3, t_4) \in J_{3505}] \frac{\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right) \omega\left(\frac{t_1-t_4}{t_4}\right)}{t_2 t_3^2 t_4^2} dt_4 dt_3 dt_2 dt_1 \Bigg) \\
& - \left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min(t_1, \frac{1-t_1}{2})} \int_{\frac{1}{29}}^{\min(t_2, \frac{1-t_1-t_2}{2})} \int_{\frac{1}{29}}^{\frac{1}{2} t_1} \int_{t_4}^{\frac{t_1-t_4}{2}} \right. \\
& \quad \text{Boole}[(t_1, t_2, t_3, t_4, t_5) \in J_{3506}] \frac{\omega_0\left(\frac{1-t_1-t_2-t_3}{t_3}\right) \omega_0\left(\frac{t_1-t_4-t_5}{t_5}\right)}{t_2 t_3^2 t_4^2 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1 \Bigg) \\
& - \left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min(t_1, \frac{1-t_1}{2})} \int_{\frac{1}{29}}^{\min(t_2, \frac{1-t_1-t_2}{2})} \int_{\frac{1}{29}}^{\frac{1}{2} t_1} \int_{t_3}^{\frac{1-t_1-t_2-t_3}{2}} \right. \\
& \quad \text{Boole}[(t_1, t_2, t_3, t_4, t_5) \in J_{3507}] \frac{\omega_0\left(\frac{1-t_1-t_2-t_3-t_5}{t_5}\right) \omega_0\left(\frac{t_1-t_4}{t_4}\right)}{t_2 t_3 t_4^2 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1 \Bigg) \\
& + \left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min(t_1, \frac{1-t_1}{2})} \int_{\frac{1}{29}}^{\min(t_2, \frac{1-t_1-t_2}{2})} \int_{\frac{1}{29}}^{\frac{1}{2} t_1} \int_{t_4}^{\frac{t_1-t_4}{2}} \int_{t_3}^{\frac{1-t_1-t_2-t_3}{2}} \right.
\end{aligned}$$

$$\begin{aligned}
& \text{Bool e}[(t_1, t_2, t_3, t_4, t_5, t_6) \in J_{3508}] \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_6}{t_6}\right) \omega_1\left(\frac{t_1-t_4-t_5}{t_5}\right)}{t_2 t_3 t_4 t_5^2 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \\
& + \left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min(t_1, \frac{1-t_1}{2})} \int_{\frac{1}{29}}^{\min(t_2, \frac{1-t_1-t_2}{2})} \int_{\frac{1}{29}}^{\frac{1}{2} t_1} \int_{\frac{1}{29}}^{\min(t_4, \frac{t_1-t_4}{2})} \int_{\frac{1}{29}}^{\min(t_5, \frac{t_1-t_4-t_5}{2})} \right. \\
& \quad \left. \text{Bool e}[(t_1, t_2, t_3, t_4, t_5, t_6) \in J_{3509}] \frac{\omega_1\left(\frac{1-t_1-t_2-t_3}{t_3}\right) \omega_1\left(\frac{t_1-t_4-t_5-t_6}{t_6}\right)}{t_2 t_3^2 t_4 t_5 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
& + \left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min(t_1, \frac{1-t_1}{2})} \int_{\frac{1}{29}}^{\min(t_2, \frac{1-t_1-t_2}{2})} \int_{\frac{1}{29}}^{\frac{1}{2} t_1} \int_{\frac{1}{29}}^{\min(t_3, \frac{1-t_1-t_2-t_3}{2})} \int_{\frac{1}{29}}^{\min(t_5, \frac{1-t_1-t_2-t_3-t_5}{2})} \right. \\
& \quad \left. \text{Bool e}[(t_1, t_2, t_3, t_4, t_5, t_6) \in J_{3510}] \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_5-t_6}{t_6}\right) \omega_1\left(\frac{t_1-t_4}{t_4}\right)}{t_2 t_3 t_4^2 t_5 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
& = L_{3501} - L_{3502} + L_{3503} + L_{3504} + L_{3505} - L_{3506} - L_{3507} + L_{3508} + L_{3509} + L_{3510}, \\
& < 0.079609 - 0.02541 + 0.000001 + 0 + 0.469283 - 0.100821 - 0.124982 + 0.035428 + 0.006114 + 0 \\
& = 0.339222, \tag{15}
\end{aligned}$$

where

$$\begin{aligned}
J_{3501}(t_1, t_2, t_3, t_4) &:= \{(t_1, t_2) \in J_4, \\
&\quad \frac{1}{29} \leq t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \quad t_1+t_2+t_3 \leq \frac{59}{87}, \\
&\quad \frac{1}{29} \leq t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\
&\quad (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I, \\
&\quad t_1+t_2+t_3+2t_4 > \frac{59}{87}, \\
&\quad \left. \frac{1}{29} \leq t_1 < \frac{1}{2}, \quad \frac{1}{29} \leq t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \right\}, \\
J_{3502}(t_1, t_2, t_3, t_4, t_5) &:= \{(t_1, t_2) \in J_4, \\
&\quad \frac{1}{29} \leq t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \quad t_1+t_2+t_3 \leq \frac{59}{87}, \\
&\quad \frac{1}{29} \leq t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\
&\quad (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I, \\
&\quad t_1+t_2+t_3+2t_4 > \frac{59}{87}, \\
&\quad t_4 < t_5 < \frac{1}{2}(1-t_1-t_2-t_3-t_4), \\
&\quad (t_1, t_2, t_3, t_4, t_5) \text{ can be partitioned into } (m, n) \in I, \\
&\quad \left. \frac{1}{29} \leq t_1 < \frac{1}{2}, \quad \frac{1}{29} \leq t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \right\}, \\
J_{3503}(t_1, t_2, t_3, t_4, t_5, t_6) &:= \{(t_1, t_2) \in J_4, \\
&\quad \frac{1}{29} \leq t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \quad t_1+t_2+t_3 \leq \frac{59}{87}, \\
&\quad \frac{1}{29} \leq t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),
\end{aligned}$$

(t_1, t_2, t_3, t_4) cannot be partitioned into $(m, n) \in I$,

$$t_1 + t_2 + t_3 + 2t_4 \leq \frac{59}{87},$$

$$\frac{1}{29} \leq t_5 < \min \left(t_4, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4) \right),$$

$$\frac{1}{29} \leq t_6 < \min \left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5) \right),$$

$(t_1, t_2, t_3, t_4, t_5, t_6)$ cannot be partitioned into $(m, n) \in I$,

$$t_1 + t_2 + t_3 + t_4 + t_5 + 2t_6 > \frac{59}{87},$$

$$\frac{1}{29} \leq t_1 < \frac{1}{2}, \quad \frac{1}{29} \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \Big\},$$

$$J_{3504}(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) := \{(t_1, t_2) \in J_4,$$

$$\frac{1}{29} \leq t_3 < \min \left(t_2, \frac{1}{2}(1 - t_1 - t_2) \right), \quad t_1 + t_2 + t_3 \leq \frac{59}{87},$$

$$\frac{1}{29} \leq t_4 < \min \left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3) \right),$$

(t_1, t_2, t_3, t_4) cannot be partitioned into $(m, n) \in I$,

$$t_1 + t_2 + t_3 + 2t_4 \leq \frac{59}{87},$$

$$\frac{1}{29} \leq t_5 < \min \left(t_4, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4) \right),$$

$$\frac{1}{29} \leq t_6 < \min \left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5) \right),$$

$(t_1, t_2, t_3, t_4, t_5, t_6)$ cannot be partitioned into $(m, n) \in I$,

$$t_1 + t_2 + t_3 + t_4 + t_5 + 2t_6 \leq \frac{59}{87},$$

$$\frac{1}{29} \leq t_7 < \min \left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5 - t_6) \right),$$

$$\frac{1}{29} \leq t_8 < \min \left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5 - t_6 - t_7) \right),$$

$(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8)$ cannot be partitioned into $(m, n) \in I$,

$$\frac{1}{29} \leq t_1 < \frac{1}{2}, \quad \frac{1}{29} \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \Big\},$$

$$J_{3505}(t_1, t_2, t_3, t_4) := \{(t_1, t_2) \in J_4,$$

$$\frac{1}{29} \leq t_3 < \min \left(t_2, \frac{1}{2}(1 - t_1 - t_2) \right), \quad t_1 + t_2 + t_3 > \frac{59}{87},$$

$$\frac{1}{29} \leq t_4 < \frac{1}{2}t_1,$$

$(1 - t_1 - t_2 - t_3, t_2, t_3, t_4)$ cannot be partitioned into $(m, n) \in I$,

$$(1 - t_1 - t_2 - t_3) + t_2 + t_3 + 2t_4 > \frac{59}{87},$$

$$(t_1 - t_4) + t_2 + 2t_3 + t_4 > \frac{59}{87},$$

$$\frac{1}{29} \leq t_1 < \frac{1}{2}, \quad \frac{1}{29} \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \Big\},$$

$$\begin{aligned}
J_{3506}(t_1, t_2, t_3, t_4, t_5) := & \{(t_1, t_2) \in J_4, \\
& \frac{1}{29} \leq t_3 < \min\left(t_2, \frac{1}{2}(1 - t_1 - t_2)\right), \quad t_1 + t_2 + t_3 > \frac{59}{87}, \\
& \frac{1}{29} \leq t_4 < \frac{1}{2}t_1, \\
& (1 - t_1 - t_2 - t_3, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I, \\
& (1 - t_1 - t_2 - t_3) + t_2 + t_3 + 2t_4 > \frac{59}{87}, \\
& (t_1 - t_4) + t_2 + 2t_3 + t_4 > \frac{59}{87}, \\
& t_4 < t_5 < \frac{1}{2}(t_1 - t_4), \\
& (1 - t_1 - t_2 - t_3, t_2, t_3, t_4, t_5) \text{ can be partitioned into } (m, n) \in I, \\
& \left. \frac{1}{29} \leq t_1 < \frac{1}{2}, \quad \frac{1}{29} \leq t_2 < \min\left(t_1, \frac{1}{2}(1 - t_1)\right) \right\},
\end{aligned}$$

$$\begin{aligned}
J_{3507}(t_1, t_2, t_3, t_4, t_5) := & \{(t_1, t_2) \in J_4, \\
& \frac{1}{29} \leq t_3 < \min\left(t_2, \frac{1}{2}(1 - t_1 - t_2)\right), \quad t_1 + t_2 + t_3 > \frac{59}{87}, \\
& \frac{1}{29} \leq t_4 < \frac{1}{2}t_1, \\
& (1 - t_1 - t_2 - t_3, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I, \\
& (1 - t_1 - t_2 - t_3) + t_2 + t_3 + 2t_4 > \frac{59}{87}, \\
& (t_1 - t_4) + t_2 + 2t_3 + t_4 > \frac{59}{87}, \\
& t_3 < t_5 < \frac{1}{2}(1 - t_1 - t_2 - t_3), \\
& (t_1 - t_4, t_2, t_3, t_4, t_5) \text{ can be partitioned into } (m, n) \in I, \\
& \left. \frac{1}{29} \leq t_1 < \frac{1}{2}, \quad \frac{1}{29} \leq t_2 < \min\left(t_1, \frac{1}{2}(1 - t_1)\right) \right\},
\end{aligned}$$

$$\begin{aligned}
J_{3508}(t_1, t_2, t_3, t_4, t_5, t_6) := & \{(t_1, t_2) \in J_4, \\
& \frac{1}{29} \leq t_3 < \min\left(t_2, \frac{1}{2}(1 - t_1 - t_2)\right), \quad t_1 + t_2 + t_3 > \frac{59}{87}, \\
& \frac{1}{29} \leq t_4 < \frac{1}{2}t_1, \\
& (1 - t_1 - t_2 - t_3, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I, \\
& (1 - t_1 - t_2 - t_3) + t_2 + t_3 + 2t_4 > \frac{59}{87}, \\
& (t_1 - t_4) + t_2 + 2t_3 + t_4 > \frac{59}{87}, \\
& t_4 < t_5 < \frac{1}{2}(t_1 - t_4), \\
& (1 - t_1 - t_2 - t_3, t_2, t_3, t_4, t_5) \text{ can be partitioned into } (m, n) \in I, \\
& t_3 < t_6 < \frac{1}{2}(1 - t_1 - t_2 - t_3), \\
& (t_1 - t_4, t_2, t_3, t_4, t_6) \text{ can be partitioned into } (m, n) \in I, \\
& \left. \frac{1}{29} \leq t_1 < \frac{1}{2}, \quad \frac{1}{29} \leq t_2 < \min\left(t_1, \frac{1}{2}(1 - t_1)\right) \right\},
\end{aligned}$$

$$\begin{aligned}
J_{3509}(t_1, t_2, t_3, t_4, t_5, t_6) := & \{(t_1, t_2) \in J_4, \\
& \frac{1}{29} \leq t_3 < \min\left(t_2, \frac{1}{2}(1 - t_1 - t_2)\right), \quad t_1 + t_2 + t_3 > \frac{59}{87}, \\
& \frac{1}{29} \leq t_4 < \frac{1}{2}t_1, \\
& (1 - t_1 - t_2 - t_3, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I, \\
& (1 - t_1 - t_2 - t_3) + t_2 + t_3 + 2t_4 \leq \frac{59}{87}, \\
& \frac{1}{29} \leq t_5 < \min\left(t_4, \frac{1}{2}(t_1 - t_4)\right), \\
& (1 - t_1 - t_2 - t_3, t_2, t_3, t_4, t_5) \text{ cannot be partitioned into } (m, n) \in I, \\
& \frac{1}{29} \leq t_6 < \min\left(t_5, \frac{1}{2}(t_1 - t_4 - t_5)\right), \\
& (1 - t_1 - t_2 - t_3, t_2, t_3, t_4, t_5, t_6) \text{ cannot be partitioned into } (m, n) \in I, \\
& \left. \frac{1}{29} \leq t_1 < \frac{1}{2}, \quad \frac{1}{29} \leq t_2 < \min\left(t_1, \frac{1}{2}(1 - t_1)\right) \right\}, \\
J_{3510}(t_1, t_2, t_3, t_4, t_5, t_6) := & \{(t_1, t_2) \in J_4, \\
& \frac{1}{29} \leq t_3 < \min\left(t_2, \frac{1}{2}(1 - t_1 - t_2)\right), \quad t_1 + t_2 + t_3 > \frac{59}{87}, \\
& \frac{1}{29} \leq t_4 < \frac{1}{2}t_1, \\
& (1 - t_1 - t_2 - t_3, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in I, \\
& (1 - t_1 - t_2 - t_3) + t_2 + t_3 + 2t_4 > \frac{59}{87}, \\
& (t_1 - t_4) + t_2 + 2t_3 + t_4 \leq \frac{59}{87}, \\
& \frac{1}{29} \leq t_5 < \min\left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3)\right), \\
& (t_1 - t_4, t_2, t_3, t_4, t_5) \text{ cannot be partitioned into } (m, n) \in I, \\
& \frac{1}{29} \leq t_6 < \min\left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_5)\right), \\
& (t_1 - t_4, t_2, t_3, t_4, t_5, t_6) \text{ cannot be partitioned into } (m, n) \in I, \\
& \left. \frac{1}{29} \leq t_1 < \frac{1}{2}, \quad \frac{1}{29} \leq t_2 < \min\left(t_1, \frac{1}{2}(1 - t_1)\right) \right\}.
\end{aligned}$$

For the remaining S_{36} , we choose to discard the whole region giving the loss

$$L_{36} := \int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min(t_1, \frac{1-t_1}{2})} \text{Bool}[(t_1, t_2) \in J_5] \frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1 t_2^2} dt_2 dt_1 < 0.093181. \quad (16)$$

Note that in this region only products of three primes are counted.

Finally, by (3)–(16), the total loss is less than

$$\begin{aligned}
& L_{32} + (L_{331} + L_{332} + L_{333} + L_{334}) + (L_{341} + L_{342} + L_{343} + L_{344} + L_{345} + L_{346} + L_{347}) \\
& + (L_{3501} + L_{3502} + L_{3503} + L_{3504} + L_{3505} + L_{3506} + L_{3507} + L_{3508} + L_{3509} + L_{3510}) + L_{36} \\
& < 0.397685 + 0.091383 + 0.07376 + 0.339222 + 0.093181 \\
& < 0.996
\end{aligned}$$

and the proof of Theorem 1.1 is completed.

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