ON THE LARGEST PRIME FACTOR OF INTEGERS IN SHORT INTERVALS

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ABSTRACT. The author sharpens a result of Jia and Liu (2000), showing that for sufficiently large x, the interval $[x, x+x^{\frac{1}{2}+\varepsilon}]$ contains an integer with a prime factor larger than $x^{\frac{51}{53}-\varepsilon}$. This gives a solution with $\gamma = \frac{2}{53}$ to the Exercise 5.1 in Harman's monograph.

CONTENTS

1. INTRODUCTION

The Legendre's conjecture, which states that there is always a prime number between consecutive squares, is one of Landau's problems on prime numbers. Clearly this means that there is always a prime number in the interval $[x, x+x^{\frac{1}{2}}]$. However, we cannot prove it even on the Riemann Hypothesis. Assuming RH, one can only show that there is always a prime number in the interval $[x, x+x^{\frac{1}{2}}\log x]$. The best unconditional result is due to Li [22], where he showed the interval $[x, x+x^{0.52}]$ contains primes.

Instead of relaxing the length of the short interval, one can attack this conjecture by relaxing our restriction of primes. A number with a large prime factor is a good approximation of prime numbers. Thus, we can try to find numbers with a large prime factor in three intervals $[x, x+x^{\frac{1}{2}}]$, $[x, x+x^{\frac{1}{2}}(\log x)^A]$ and $[x, x+x^{\frac{1}{2}+\varepsilon}]$.

For the first interval, Ramachandra [27] showed in 1969 that this interval contains a number with a prime factor larger than $x^{0.576}$. The exponent 0.576 has been improved to

$$0.625,\ 0.662,\ 0.675225,\ 0.692,\ 0.7,\ 0.71,\ 0.723,\ 0.728,\ 0.732,\ 0.738,\ 0.74 \text{ and } 0.7428$$

by Ramachandra [28], Graham [10], Zhu [29], Jia [15], Baker [1], Jia [16], Jia [17] (and Liu [23]), Jia [18], Baker and Harman [2], Liu and Wu [24], Harman [[11], Chapter 6] and Baker and Harman [3] respectively. For the second interval, Balog, Harman and Pintz [7] showed that this interval contains a number with a prime factor larger than $x^{0.712}$, and the exponent 0.712 has been improved to $\frac{5}{6}$ by Lou [25] and $\frac{18}{19}$ by Merikoski [26].

In this paper we shall focus on the third interval. In 1973, Jutila [21] showed that this interval contains a number with a prime factor larger than $x^{\frac{2}{3}-\varepsilon}$. The exponent $\frac{2}{3}$ has been improved to

$$0.73,\ 0.7338,\ 0.772,\ 0.82,\ \frac{11}{12},\ \frac{17}{18},\ \frac{19}{20},\ \frac{24}{25} \text{ and } \frac{25}{26}$$

by Balog [5] [6], Balog, Harman and Pintz [8], Heath-Brown [13], Heath-Brown and Jia [14], Harman [[11], Chapter 5], Haugland [12] and Jia and Liu [20] respectively. In his monograph, Harman [[11], Exercise 5.1] encouraged us to reduce this exponent as much as we can. In this paper, we obtain the following result.

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Theorem 1.1. *For sufficiently large x, the interval $[x, x+x^{\frac{1}{2}+\varepsilon}]$ contains an integer with a prime factor larger than $x^{\frac{51}{53}-\varepsilon}$.*

Of course, our proof is much simpler than the similar arguments used in [14], [12] and [20]. Throughout this paper, we always suppose that ε is a sufficiently small positive constant and $B = B(\varepsilon)$ is a sufficiently large positive constant. We choose ε such that $K = \frac{8}{\varepsilon}(\frac{1}{26.5} + \frac{\varepsilon}{2})$ is an integer. The letter p, with or without subscript, is reserved for prime numbers. Let $v = x^{\frac{51}{53}-\frac{\varepsilon}{2}}$, $P = x^{\frac{\varepsilon}{8}}$ and $T_0 = x^{\frac{1}{2}-\frac{\varepsilon}{6}}$. Let c_0, c_1 and c_2 denote positive constants which may have different values at different places, and we write $m \sim M$ to mean that $c_1 M < m \leqslant c_2 M$. We use $M(s)$, $N(s)$ and some other capital letters to denote the Dirichlet polynomials

$$M(s) = \sum_{m \sim M} a(m) m^{-s}, \quad N(s) = \sum_{n \sim N} b(n) n^{-s}$$

where $a(m)$, $b(n)$ are complex numbers with $a(m) = O(1)$ and $b(n) = O(1)$. We also use $P(s)$ to denote

$$P(s) = \sum_{P < p \leqslant 2P} p^{-s}.$$

2. Arithmetic Information

In this section we provide some arithmetic information (i.e. mean value bounds for some Dirichlet polynomials) which will help us prove the asymptotic formulas for sieve functions.

Lemma 2.1. *Suppose that $MN = v$ where $M(s), N(s)$ are Dirichlet polynomials and $v^{\frac{49}{102}} \ll M \ll v^{\frac{53}{102}}$. Let $b = 1 + \frac{1}{\log x}$, $T_1 = (\log x)^{2B}$, then for $T_1 \leqslant T \leqslant T_0$ we have*

$$\int_T^{2T} |M(b+it)N(b+it)P^K(b+it)| dt \ll (\log x)^{-B}.$$

Proof. The proof is similar to that of [[20], Lemma 1]. □

Lemma 2.2. *Suppose that $MNL = v$ where $M(s), N(s)$ are Dirichlet polynomials and $L(s) = \sum_{l \sim L} l^{-s}$. Let $b = 1 + \frac{1}{\log x}$, $T_2 = \sqrt{L}$. Assume that $M \ll v^{\frac{53}{102}}$ and $N \ll v^{\frac{53}{204}}$, then for $T_2 \leqslant T \leqslant T_0$ we have*

$$\int_T^{2T} |M(b+it)N(b+it)L(b+it)P^K(b+it)| dt \ll (\log x)^{-B}.$$

Proof. The proof is similar to that of [[20], Lemma 2]. □

Lemma 2.3. *Suppose that $MNHL = v$ where $M(s), N(s), H(s)$ are Dirichlet polynomials and $L(s) = \sum_{l \sim L} l^{-s}$. Let $b = 1 + \frac{1}{\log x}$, $T_2 = \sqrt{L}$. Assume that M, N and H satisfy the following conditions:*
$M \ll v^{\frac{53}{102}}$, $N \gg H$, $N^{\frac{3}{4}} H \ll v^{\frac{53}{204}}$, $N H^{\frac{1}{2}} \ll v^{\frac{53}{204}}$, $N^{\frac{7}{4}} H^{\frac{3}{2}} \ll v^{\frac{53}{102}}$,
Then for $T_2 \leqslant T \leqslant T_0$ we have

$$\int_T^{2T} |M(b+it)N(b+it)H(b+it)L(b+it)P^K(b+it)| dt \ll (\log x)^{-B}.$$

Proof. The proof is similar to that of [[20], Lemma 3] where [[9], Theorem 2] is used. □

3. The final decomposition

Now we follow the discussion in [14] and [20]. Let $p_j = v^{t_j}$ and put

$$N(d) = \sum_{\substack{x < p p_1 \ldots p_K \leqslant x + x^{\frac{1}{2}+\varepsilon} \\ P < p_i \leqslant 2P}} 1, \quad \mathcal{A} = \{n : 2^{-K} v < n \leqslant 2v, \ n \text{ repeats } N(n) \text{ times}\},$$

$$\mathcal{B} = \{n : v < n \leqslant 2v\}, \quad \mathcal{A}_d = \{a : a \in \mathcal{A}, \ d \mid a\}, \quad P(z) = \prod_{p < z} p, \quad S(\mathcal{A}, z) = \sum_{\substack{a \in \mathcal{A} \\ (a, P(z)) = 1}} 1.$$

Then we only need to show that $S\left(\mathcal{A},(2v)^{\frac{1}{2}}\right)>0$. Our aim is to show that the sparser set $\mathcal{A}$ contains the expected proportion of primes compared to the bigger set $\mathcal{B}$, which requires us to decompose $S\left(\mathcal{A},(2v)^{\frac{1}{2}}\right)$ and prove asymptotic formulas of the form

$$S(\mathcal{A},z)=v^{-1}x^{\frac{1}{2}+\varepsilon}\left(\sum_{P<p\leqslant 2P}\frac{1}{p}\right)^{K}(1+o(1))S(\mathcal{B},z) \tag{1}$$

for some parts of it, and drop the other positive parts.

Let $\omega(u)$ denote the Buchstab function determined by the following differential-difference equation

$$\begin{cases}\omega(u)=\frac{1}{u}, & 1\leqslant u\leqslant 2,\\ (u\omega(u))'=\omega(u-1), & u\geqslant 2.\end{cases}$$

Moreover, we have the upper and lower bounds for $\omega(u)$:

$$\omega(u)\geqslant\omega_0(u)=\begin{cases}\frac{1}{u}, & 1\leqslant u<2,\\ \frac{1+\log(u-1)}{u}, & 2\leqslant u<3,\\ \frac{1+\log(u-1)}{u}+\frac{1}{u}\int_2^{u-1}\frac{\log(t-1)}{t}dt\geqslant 0.5607, & 3\leqslant u<4,\\ 0.5612, & u\geqslant 4,\end{cases}$$

$$\omega(u)\leqslant\omega_1(u)=\begin{cases}\frac{1}{u}, & 1\leqslant u<2,\\ \frac{1+\log(u-1)}{u}, & 2\leqslant u<3,\\ \frac{1+\log(u-1)}{u}+\frac{1}{u}\int_2^{u-1}\frac{\log(t-1)}{t}dt\leqslant 0.5644, & 3\leqslant u<4,\\ 0.5617, & u\geqslant 4.\end{cases}$$

We shall use $\omega_0(u)$ and $\omega_1(u)$ to give numerical bounds for some sieve functions discussed below.

Before decomposing, we define the asymptotic regions T_1–T_3 and L as

$$\begin{aligned}
T_1(m,n) &:= \left\{m\leqslant\frac{53}{102},\ n\leqslant\frac{53}{204}\right\}\\
T_2(m,n,h) &:= \left\{m\leqslant\frac{53}{102},\ n\geqslant h,\ \frac{3}{4}n+h\leqslant\frac{53}{204},\ n+\frac{1}{2}h\leqslant\frac{53}{204},\ \frac{7}{4}n+\frac{3}{2}h\leqslant\frac{53}{102}\right\},\\
T_3(m,n) &:= \left\{\frac{49}{102}\leqslant m\leqslant\frac{53}{102}\text{ or }\frac{49}{102}\leqslant m+n\leqslant\frac{53}{102}\right\},\\
L(m,n) &:= \{(m,n)\notin T_3,\ (m,n,n)\text{ cannot be partitioned into }(\alpha,\eta)\in T_1\text{ or }(\alpha,\eta,\gamma)\in T_2,\\
&\qquad n\geqslant\frac{53}{255}\text{ or }m\geqslant\frac{1129}{2448}\text{ or }\frac{1}{2}m+n\geqslant\frac{9361}{24480}\right\}.
\end{aligned}$$

Lemma 3.1. *We can give an asymptotic formula for*

$$\sum_{t_1\cdots t_n}S\left(\mathcal{A}_{p_1\cdots p_n},v^{\frac{2}{51}}\right)$$

if we can group $(t_1,\ldots,t_n)$ into $(m,n)\in T_1$ or $(m,n,h)\in T_2$.

Lemma 3.2. *We can give an asymptotic formula for*

$$\sum_{t_1\cdots t_n}S\left(\mathcal{A}_{p_1\cdots p_n},p_n\right)$$

if we can group $(t_1,\ldots,t_n)$ into $(m,n)\in T_3$.

By Buchstab's identity, we have

$$S\left(\mathcal{A},(2v)^{\frac{1}{2}}\right)=S\left(\mathcal{A},v^{\frac{2}{51}}\right)-\sum_{\frac{2}{51}\leqslant t_1<\frac{49}{102}}S(\mathcal{A}_{p_1},p_1)-\sum_{\frac{49}{102}\leqslant t_1<\frac{1}{2}}S(\mathcal{A}_{p_1},p_1)$$

$$
\begin{aligned}
&= S\left(\mathcal{A}, v^{\frac{2}{51}}\right) - \sum_{\frac{2}{51} \leqslant t_1 < \frac{49}{102}} S\left(\mathcal{A}_{p_1}, v^{\frac{2}{51}}\right) - \sum_{\frac{49}{102} \leqslant t_1 < \frac{1}{2}} S\left(\mathcal{A}_{p_1}, p_1\right) \\
&\quad + \sum_{\substack{\frac{2}{51} \leqslant t_1 < \frac{49}{102} \\ \frac{2}{51} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
&= S_1 - S_2 - S_3 + S_4. \qquad (2)
\end{aligned}
$$

By Lemma 2.1 and Lemma 2.2, we can give asymptotic formulas for S_1, S_2 and S_3. Before estimating S_4, we first split it into three parts:

$$
\begin{aligned}
S_4 =& \sum_{\substack{\frac{2}{51} \leqslant t_1 < \frac{49}{102} \\ \frac{2}{51} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
=& \sum_{\substack{\frac{2}{51} \leqslant t_1 < \frac{49}{102} \\ \frac{2}{51} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1, t_2) \in T_3}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) + \sum_{\substack{\frac{2}{51} \leqslant t_1 < \frac{49}{102} \\ \frac{2}{51} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1, t_2) \in L}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
&+ \sum_{\substack{\frac{2}{51} \leqslant t_1 < \frac{49}{102} \\ \frac{2}{51} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1, t_2) \notin T_3 \\ (t_1, t_2, t_2) \text{ can be partitioned into } (m,n) \in T_1 \text{ or } (m,n,h) \in T_2}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
&+ \sum_{\substack{\frac{2}{51} \leqslant t_1 < \frac{49}{102} \\ \frac{2}{51} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1, t_2) \notin T_3 \\ (t_1, t_2) \notin L \\ (t_1, t_2, t_2) \text{ cannot be partitioned into } (m,n) \in T_1 \text{ or } (m,n,h) \in T_2}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
=& S_{41} + S_{42} + S_{43} + S_{44}. \qquad (3)
\end{aligned}
$$

S_{41} has an asymptotic formula. For S_{42}, we cannot decompose further but have to discard the whole region giving the loss

$$
\int_{\frac{2}{51}}^{\frac{49}{102}} \int_{\frac{2}{51}}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \mathbb{1}_{(t_1, t_2) \in L} \frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1 t_2^2} dt_2 dt_1 < 0.687415. \qquad (4)
$$

For S_{43} we can use Buchstab's identity to get

$$
\begin{aligned}
S_{43} =& \sum_{\substack{\frac{2}{51} \leqslant t_1 < \frac{49}{102} \\ \frac{2}{51} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1, t_2) \notin T_3 \\ (t_1, t_2, t_2) \text{ can be partitioned into } (m,n) \in T_1 \text{ or } (m,n,h) \in T_2}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
=& \sum_{\substack{\frac{2}{51} \leqslant t_1 < \frac{49}{102} \\ \frac{2}{51} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1, t_2) \notin T_3 \\ (t_1, t_2, t_2) \text{ can be partitioned into } (m,n) \in T_1 \text{ or } (m,n,h) \in T_2}} S\left(\mathcal{A}_{p_1 p_2}, v^{\frac{2}{51}}\right) \\
&- \sum_{\substack{\frac{2}{51} \leqslant t_1 < \frac{49}{102} \\ \frac{2}{51} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1, t_2) \notin T_3 \\ (t_1, t_2, t_2) \text{ can be partitioned into } (m,n) \in T_1 \text{ or } (m,n,h) \in T_2 \\ \frac{2}{51} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ (t_1, t_2, t_3) \text{ can be partitioned into } (m,n) \in T_3}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{\substack{\frac{2}{51} \leqslant t_1 < \frac{49}{102} \\ \frac{2}{51} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1, t_2) \notin T_3 \\ (t_1, t_2, t_2) \text{ can be partitioned into } (m,n) \in T_1 \text{ or } (m,n,h) \in T_2 \\ \frac{2}{51} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ (t_1, t_2, t_3) \text{ cannot be partitioned into } (m,n) \in T_3}} S\left(\mathcal{A}_{p_1 p_2 p_3}, v^{\frac{2}{51}}\right) \\
&+ \sum_{\substack{\frac{2}{51} \leqslant t_1 < \frac{49}{102} \\ \frac{2}{51} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1, t_2) \notin T_3 \\ (t_1, t_2, t_2) \text{ can be partitioned into } (m,n) \in T_1 \text{ or } (m,n,h) \in T_2 \\ \frac{2}{51} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ (t_1, t_2, t_3) \text{ cannot be partitioned into } (m,n) \in T_3 \\ \frac{2}{51} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1, t_2, t_3, t_4) \text{ can be partitioned into } (m,n) \in T_3}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right) \\
&+ \sum_{\substack{\frac{2}{51} \leqslant t_1 < \frac{49}{102} \\ \frac{2}{51} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1, t_2) \notin T_3 \\ (t_1, t_2, t_2) \text{ can be partitioned into } (m,n) \in T_1 \text{ or } (m,n,h) \in T_2 \\ \frac{2}{51} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ (t_1, t_2, t_3) \text{ cannot be partitioned into } (m,n) \in T_3 \\ \frac{2}{51} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m,n) \in T_3}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right) \\
&= S_{431} - S_{432} - S_{433} + S_{434} + S_{435}. \qquad (5)
\end{aligned}
$$

We have asymptotic formulas for S_{431}–S_{434}. For the remaining S_{435}, we have two ways to get more possible savings: One way is to use Buchstab's identity twice more for some parts if we can group $(t_1, t_2, t_3, t_4, t_4)$ into $(m, n) \in T_1$ or $(m, n, h) \in T_2$. Another way is to use Buchstab's identity in reverse to make almost-primes visible. The details of further decompositions are similar to those in [22]. Combining the cases above we get a loss from S_{43} of

$$
\begin{aligned}
&\left(\int_{(t_1,t_2,t_3,t_4) \in U_1} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1 \right) \\
&+ \left(\int_{(t_1,t_2,t_3,t_4,t_5,t_6) \in U_2} \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{t_6}\right)}{t_1 t_2 t_3 t_4 t_5 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
&- \left(\int_{(t_1,t_2,t_3,t_4,t_5) \in U_3} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
&\leqslant (0.161005 + 0.072354 - 0.009022) = 0.224337, \qquad (6)
\end{aligned}
$$

where

$$
\begin{aligned}
U_1(t_1, t_2, t_3, t_4) := \{ & (t_1, t_2) \notin T_3,\ (t_1, t_2, t_2) \text{ can be partitioned into } (m, n) \in T_1 \text{ or } (m, n, h) \in T_2, \\
& \frac{2}{51} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1 - t_1 - t_2)\right), \\
& (t_1, t_2, t_3) \text{ cannot be partitioned into } (m, n) \in T_3, \\
& \frac{2}{51} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3)\right), \\
& (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in T_3, \\
& (t_1, t_2, t_3, t_4, t_4) \text{ cannot be partitioned into } (m, n) \in T_1 \text{ or } (m, n, h) \in T_2,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{2}{51} \leqslant t_1 < \frac{49}{102},\ \frac{2}{51} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Bigg\},\\
U_2(t_1,t_2,t_3,t_4,t_5,t_6) := \ &\{(t_1,t_2) \notin T_3,\ (t_1,t_2,t_2) \text{ can be partitioned into } (m,n) \in T_1 \text{ or } (m,n,h) \in T_2,\\
&\frac{2}{51} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\\
&(t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n) \in T_3,\\
&\frac{2}{51} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in T_3,\\
&(t_1,t_2,t_3,t_4,t_4) \text{ can be partitioned into } (m,n) \in T_1 \text{ or } (m,n,h) \in T_2,\\
&\frac{2}{51} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right),\\
&(t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (m,n) \in T_3,\\
&\frac{2}{51} \leqslant t_6 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right),\\
&(t_1,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n) \in T_3,\\
&\frac{2}{51} \leqslant t_1 < \frac{49}{102},\ \frac{2}{51} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Bigg\},\\
U_3(t_1,t_2,t_3,t_4,t_5) := \ &\{(t_1,t_2) \notin T_3,\ (t_1,t_2,t_2) \text{ can be partitioned into } (m,n) \in T_1 \text{ or } (m,n,h) \in T_2,\\
&\frac{2}{51} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\\
&(t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n) \in T_3,\\
&\frac{2}{51} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in T_3,\\
&(t_1,t_2,t_3,t_4,t_4) \text{ cannot be partitioned into } (m,n) \in T_1 \text{ or } (m,n,h) \in T_2,\\
&t_4 < t_5 < \frac{1}{2}(1-t_1-t_2-t_3-t_4),\\
&(t_1,t_2,t_3,t_4,t_5) \text{ can be partitioned into } (m,n) \in T_3,\\
&\frac{2}{51} \leqslant t_1 < \frac{49}{102},\ \frac{2}{51} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Bigg\}.
\end{aligned}
$$

Next we shall decompose S_{44}. By Buchstab's identity, we have

$$
\begin{aligned}
S_{44} = &\sum_{\substack{\frac{2}{51} \leqslant t_1 < \frac{49}{102} \\ \frac{2}{51} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \notin T_3 \\ (t_1,t_2) \notin L \\ (t_1,t_2,t_2) \text{ cannot be partitioned into } (m,n) \in T_1 \text{ or } (m,n,h) \in T_2}} S\left(\mathcal{A}_{p_1p_2}, p_2\right)\\
= &\sum_{\substack{\frac{2}{51} \leqslant t_1 < \frac{49}{102} \\ \frac{2}{51} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \notin T_3 \\ (t_1,t_2) \notin L \\ (t_1,t_2,t_2) \text{ cannot be partitioned into } (m,n) \in T_1 \text{ or } (m,n,h) \in T_2}} S\left(\mathcal{A}_{p_1p_2}, v^{\frac{2}{51}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{\substack{\frac{2}{51}\leqslant t_1<\frac{49}{102} \\ \frac{2}{51}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right) \\ (t_1,t_2)\notin T_3 \\ (t_1,t_2)\notin L \\ (t_1,t_2,t_2) \text{ cannot be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2 \\ \frac{2}{51}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right) \\
=& \sum_{\substack{\frac{2}{51}\leqslant t_1<\frac{49}{102} \\ \frac{2}{51}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right) \\ (t_1,t_2)\notin T_3 \\ (t_1,t_2)\notin L \\ (t_1,t_2,t_2) \text{ cannot be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2}} S\left(\mathcal{A}_{p_1p_2},v^{\frac{2}{51}}\right) \\
&- \sum_{\substack{\frac{2}{51}\leqslant t_1<\frac{49}{102} \\ \frac{2}{51}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right) \\ (t_1,t_2)\notin T_3 \\ (t_1,t_2)\notin L \\ (t_1,t_2,t_2) \text{ cannot be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2 \\ \frac{2}{51}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ can be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right) \\
&- \sum_{\substack{\frac{2}{51}\leqslant t_1<\frac{49}{102} \\ \frac{2}{51}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right) \\ (t_1,t_2)\notin T_3 \\ (t_1,t_2)\notin L \\ (t_1,t_2,t_2) \text{ cannot be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2 \\ \frac{2}{51}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right) \\
=& S_{441} - S_{442} - S_{443}. \qquad (7)
\end{aligned}
$$

We have an asymptotic formula for S_{441}. For S_{442} we can use the same methods as above (i.e. using Buchstab's identity twice more and making almost-primes visible) to get a loss of

$$
\begin{aligned}
&\left(\int_{(t_1,t_2,t_3,t_4)\in U_4} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1t_2t_3t_4^2} dt_4dt_3dt_2dt_1\right) \\
+&\left(\int_{(t_1,t_2,t_3,t_4,t_5,t_6)\in U_5} \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{t_6}\right)}{t_1t_2t_3t_4t_5t_6^2} dt_6dt_5dt_4dt_3dt_2dt_1\right) \\
-&\left(\int_{(t_1,t_2,t_3,t_4,t_5)\in U_6} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1t_2t_3t_4t_5^2} dt_5dt_4dt_3dt_2dt_1\right) \\
\leqslant& (0.038404 + 0.002564 - 0.005445) = 0.035523, \qquad (8)
\end{aligned}
$$

where

$$
\begin{aligned}
U_4(t_1,t_2,t_3,t_4) := \{&(t_1,t_2)\notin T_3,\ (t_1,t_2,t_2) \text{ cannot be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2, \\
&\frac{2}{51}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right), \\
&(t_1,t_2,t_3) \text{ can be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2, \\
&(t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in T_3, \\
&\frac{2}{51}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right), \\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in T_3,
\end{aligned}
$$

$$
\begin{aligned}
&\quad (t_1,t_2,t_3,t_4,t_4) \text{ cannot be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2,\\
&\quad \frac{2}{51}\leqslant t_1<\frac{49}{102},\ \frac{2}{51}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\Bigg\},\\
U_5(t_1,t_2,t_3,t_4,t_5,t_6) :=\ &\{(t_1,t_2)\notin T_3,\ (t_1,t_2,t_2) \text{ cannot be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2,\\
&\quad \frac{2}{51}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right),\\
&\quad (t_1,t_2,t_3) \text{ can be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2,\\
&\quad (t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in T_3,\\
&\quad \frac{2}{51}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right),\\
&\quad (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in T_3,\\
&\quad (t_1,t_2,t_3,t_4,t_4) \text{ can be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2,\\
&\quad \frac{2}{51}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right),\\
&\quad (t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (m,n)\in T_3,\\
&\quad \frac{2}{51}\leqslant t_6<\min\left(t_5,\frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right),\\
&\quad (t_1,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n)\in T_3,\\
&\quad \frac{2}{51}\leqslant t_1<\frac{49}{102},\ \frac{2}{51}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\Bigg\},\\
U_6(t_1,t_2,t_3,t_4,t_5) :=\ &\{(t_1,t_2)\notin T_3,\ (t_1,t_2,t_2) \text{ cannot be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2,\\
&\quad \frac{2}{51}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right),\\
&\quad (t_1,t_2,t_3) \text{ can be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2,\\
&\quad (t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in T_3,\\
&\quad \frac{2}{51}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right),\\
&\quad (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in T_3,\\
&\quad (t_1,t_2,t_3,t_4,t_4) \text{ cannot be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2,\\
&\quad t_4<t_5<\frac{1}{2}(1-t_1-t_2-t_3-t_4),\\
&\quad (t_1,t_2,t_3,t_4,t_5) \text{ can be partitioned into } (m,n)\in T_3,\\
&\quad \frac{2}{51}\leqslant t_1<\frac{49}{102},\ \frac{2}{51}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\Bigg\}.
\end{aligned}
$$

For S_{443} we can perform a role-reversal to get a small saving. For the definition of a role-reversal one can see [4] or [[11], Chapter 5], and we refer the readers to [19] and [22] for more applications of role-reversals. In this way we have

$$
S_{443} = \sum_{\substack{\frac{2}{51}\leqslant t_1<\frac{49}{102}\\ \frac{2}{51}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\notin T_3\\ (t_1,t_2)\notin L\\ (t_1,t_2,t_2) \text{ cannot be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2\\ \frac{2}{51}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ (t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right)
$$

$$= \sum_{\substack{\frac{2}{51} \leqslant t_1 < \frac{49}{102} \\ \frac{2}{51} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \notin T_3 \\ (t_1,t_2) \notin L \\ (t_1,t_2,t_2) \text{ cannot be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2 \\ \frac{2}{51} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2}} S\left(\mathcal{A}_{\beta p_2 p_3}, \left(\frac{2v}{\beta p_2 p_3}\right)^{\frac{1}{2}}\right)$$

$$= \sum_{\substack{\frac{2}{51} \leqslant t_1 < \frac{49}{102} \\ \frac{2}{51} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \notin T_3 \\ (t_1,t_2) \notin L \\ (t_1,t_2,t_2) \text{ cannot be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2 \\ \frac{2}{51} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2}} S\left(\mathcal{A}_{\beta p_2 p_3}, v^{\frac{2}{51}}\right)$$

$$- \sum_{\substack{\frac{2}{51} \leqslant t_1 < \frac{49}{102} \\ \frac{2}{51} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \notin T_3 \\ (t_1,t_2) \notin L \\ (t_1,t_2,t_2) \text{ cannot be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2 \\ \frac{2}{51} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2 \\ \frac{2}{51} \leqslant t_4 < \frac{1}{2}t_1}} S\left(\mathcal{A}_{\beta p_2 p_3 p_4}, p_4\right),$$

where $\beta \sim v^{1-t_1-t_2-t_3}$ and $(\beta, P(p_3)) = 1$. Again, we can use Buchstab's identity in reverse to gain a small saving on the last term. Altogether we get a loss from S_{443} of

$$\begin{aligned} &\left(\int_{(t_1,t_2,t_3,t_4)\in U_7} \frac{\omega\left(\frac{t_1-t_4}{t_4}\right)\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2 t_3^2 t_4^2} dt_4 dt_3 dt_2 dt_1\right) \\ &- \left(\int_{(t_1,t_2,t_3,t_4,t_5)\in U_8} \frac{\omega\left(\frac{t_1-t_4-t_5}{t_5}\right)\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2 t_3^2 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1\right) \\ &\leqslant (0.046566 - 0.007144) = 0.039422, \end{aligned} \tag{9}$$

where

$$\begin{aligned} U_7(t_1,t_2,t_3,t_4) := \ &\{(t_1,t_2) \notin T_3,\ (t_1,t_2,t_2) \text{ cannot be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2, \\ &\frac{2}{51} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \\ &(t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2, \\ &(t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in T_3, \\ &\frac{2}{51} \leqslant t_4 < \frac{1}{2}t_1, \\ &(1-t_1-t_2-t_3,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in T_3, \\ &\frac{2}{51} \leqslant t_1 < \frac{49}{102},\ \frac{2}{51} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\}, \\ U_8(t_1,t_2,t_3,t_4,t_5) := \ &\{(t_1,t_2) \notin T_3,\ (t_1,t_2,t_2) \text{ cannot be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2, \\ &\frac{2}{51} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \\ &(t_1,t_2,t_3) \text{ can be partitioned into } (m,n)\in T_1 \text{ or } (m,n,h)\in T_2, \\ &(t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in T_3, \end{aligned}$$

$$
\begin{aligned}
&\frac{2}{51} \leqslant t_4 < \frac{1}{2} t_1, \\
&(1-t_1-t_2-t_3, t_2, t_3, t_4) \text{ cannot be partitioned into } (m,n) \in T_3, \\
&t_4 < t_5 < \frac{1}{2}(t_1 - t_4), \\
&(1-t_1-t_2-t_3, t_2, t_3, t_4, t_5) \text{ can be partitioned into } (m,n) \in T_3, \\
&\frac{2}{51} \leqslant t_1 < \frac{49}{102},\ \frac{2}{51} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \Bigg\}.
\end{aligned}
$$

Finally, by (2)–(9), the total loss is less than

$$0.687415 + 0.224337 + 0.035523 + 0.039422 < 0.99$$

and the proof of Theorem 1.1 is completed.

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