ON PRIME–PRODUCING SIEVES AND DISTRIBUTION OF $\alpha p-\beta$ MOD 1

RUNBO LI

Abstract. The author proves that there are infinitely many primes p such that $\|\alpha p-\beta\|<p^{-\frac{28}{87}}$, where α is an irrational number and β is a real number. This sharpens a result of Jia (2000) and provides a new triple $(\gamma,\theta,\nu)=(\frac{59}{87},\frac{28}{87},\frac{1}{29})$ that can produce primes in Ford and Maynard's work on prime–producing sieves. Our minimum amount of Type–II information required ($\nu=\frac{1}{29}$) is less than any previous work on this topic using only traditional Type–I and Type–II information.

Contents

1. Introduction

Let α be an irrational number and $\|y\|$ denote the smallest distance from y to integers. Earlier work on this problem was done by Vinogradov [14] in 1954, who showed that for any real number β, there are infinitely many prime numbers p such that if $\tau=\frac{1}{5}-\varepsilon$, then

$$\|\alpha p-\beta\|<p^{-\tau}. \tag{1}$$

In 1977, Vaughan [13] got $\tau=\frac{1}{4}-\varepsilon$ using his identity. In 1983, Harman [3] introduced a new sieve method to this topic and got $\tau=\frac{3}{10}$. Jia [7] improved it to $\tau=\frac{4}{13}$ in 1993. In 1996, Harman [4] further improved it to $\tau=\frac{7}{22}$ by applying a new technique (the variable role–reversal) in his sieve. In 2000, Jia [9] got $\theta=\frac{9}{28}$. It is worth to mention that Balog [1] also got the same result in 1986 under the condition that $\|\alpha n\|<n^{-\frac{43}{31}-\varepsilon}$ holds for infinitely many integers n. If we only focus on the special case $\beta=0$, then even better exponents $\frac{16}{49}$ and $\frac{1}{3}-\varepsilon$ were obtained by Heath–Brown and Jia [6] and Matomäki [12] respectively. In a personal communication, Matomäki mentioned that Maynard has got some $\tau>\frac{1}{3}$. Note that the Riemann Hypothesis implies that (1) holds for $\tau=\frac{1}{3}-\varepsilon$. In this paper, we show that (1) holds for $\tau=\frac{28}{87}$.

Theorem 1.1. *Suppose that α is an irrational number, then for any real number β, there are infinitely many prime numbers p such that*

$$\|\alpha p-\beta\|<p^{-\frac{28}{87}}.$$

A direct corollary of our Theorem 1.1 is the distribution of $p^{\theta}-\beta$ mod 1 for some $\theta<1$.

Theorem 1.2. *For $\frac{31}{87}\leqslant\theta<1$ and any real number β, there are infinitely many prime numbers p such that*

$$\|p^{\theta}-\beta\|<p^{-\frac{1-\theta}{2}+\varepsilon}.$$

Another corollary of our Theorem 1.1 is the following result focusing on Diophantine approximation with Gaussian primes, which improves Harman's exponent $\frac{7}{22}$ [5].

2020 Mathematics Subject Classification. 11N35, 11N36.
Key words and phrases. prime, sieve methods, asymptotic formula.

Theorem 1.3. *Let $0 \leqslant \omega_1 < \omega_2 \leqslant 2\pi$. Then, given $\alpha \in \mathbb{C}\backslash\mathbb{Q}[i], \beta \in \mathbb{C}$, there are infinitely many Gaussian primes $\mathfrak{p}$ such that*

$$\|\alpha\mathfrak{p} - \beta\| < |\mathfrak{p}|^{-\frac{28}{87}}, \quad \omega_1 \leqslant \arg \mathfrak{p} \leqslant \omega_2.$$

In 2024, Ford and Maynard [2] considered a series of general problems on prime–producing sieves. For those sieves that only use Type–I and Type–II information inputs, they defined a triple (γ, θ, ν) and considered various values of γ, θ and ν that can produce primes. In their notation, our Theorem 1.1 implies the following result:

Theorem 1.4. *$C^{-}(\gamma, \theta, \nu) = C^{-}(\frac{59}{87}, \frac{28}{87}, \frac{1}{29}) > 0$. That is, $(\gamma, \theta, \nu) = (\frac{59}{87}, \frac{28}{87}, \frac{1}{29})$ is a prime–producing triple.*

Throughout this paper, we suppose that $\frac{a}{q}$ is a convergent to the continued fraction for α and ε is a sufficiently small positive constant. The letter p, with or without subscript, is reserved for prime numbers. Let $\tau = \frac{28}{87}$, $x = q^{\frac{2}{1+\tau}}$ and $\delta = (2x)^{-\tau}$. We define the boolean function as

$$\texttt{Boole}[\mathbf{X}] = \begin{cases} 1 & \text{if } \mathbf{X} \text{ is true,} \\ 0 & \text{if } \mathbf{X} \text{ is false.} \end{cases}$$

2. Asymptotic formulas

Now we follow the discussion in [9]. Let $p_j = x^{t_j}$ and put

$$\mathcal{B} = \{n : x < n \leqslant 2x\}, \quad \mathcal{A} = \{n : x < n \leqslant 2x, \ \|\alpha n - \beta\| < \delta\},$$

$$\mathcal{A}_d = \{a : ad \in \mathcal{A}\}, \quad P(z) = \prod_{p<z} p, \quad S(\mathcal{A}, z) = \sum_{\substack{a \in \mathcal{A} \\ (a, P(z))=1}} 1.$$

Then we only need to show that $S\left(\mathcal{A}, (2x)^{\frac{1}{2}}\right) > 0$. Our aim is to show that the sparser set $\mathcal{A}$ contains the expected proportion of primes compared to the bigger set $\mathcal{B}$, which requires us to decompose $S\left(\mathcal{A}, (2v)^{\frac{1}{2}}\right)$ and prove asymptotic formulas of the form

$$S\left(\mathcal{A}, z\right) = 2\delta(1 + o(1)) S\left(\mathcal{B}, z\right) \tag{2}$$

for some parts of it, and drop the other positive parts.

Here are some known asymptotic formulas for types of sieve functions, which were first proved by Harman [3] using traditional Type–I and Type–II arithmetic information.

Lemma 2.1. *(Type–I) Suppose that $M \ll x^{\frac{59}{87}}$ and $a(m) = O(1)$. Then we have*

$$\sum_{m \sim M} a(m) S\left(\mathcal{A}_m, x^{\frac{1}{29}}\right) = 2\delta(1 + o(1)) \sum_{m \sim M} a(m) S\left(\mathcal{B}_m, x^{\frac{1}{29}}\right).$$

Proof. The proof is similar to that of [[3], Lemma 7]. □

Lemma 2.2. *(Type–II) Suppose that $x^{\frac{28}{87}} \ll M \ll x^{\frac{31}{87}}$ or $x^{\frac{56}{87}} \ll M \ll x^{\frac{59}{87}}$ and that $a(m), b(n) = O(1)$. Then we have*

$$\sum_{m \sim M} \sum_{n} a(m) b(n) S\left(\mathcal{A}_{mn}, v(m, n)\right) = 2\delta(1 + o(1)) \sum_{m \sim M} \sum_{n} a(m) b(n) S\left(\mathcal{B}_{mn}, v(m, n)\right).$$

Proof. The proof is similar to that of [[3], Lemma 6]. □

Many authors, such as Heath–Brown and Jia [6] and Matomäki [12], have found more arithmetic information by using estimations of Kloosterman sums. However, their methods usually require $\beta = 0$, which is not applicable on the present paper.

3. The final decomposition

Let $\omega(u)$ denote the Buchstab function determined by the following differential–difference equation

$$\begin{cases}\omega(u)=\frac{1}{u}, & 1\leqslant u\leqslant 2,\\ (u\omega(u))'=\omega(u-1), & u\geqslant 2.\end{cases}$$

Moreover, we have the upper and lower bounds for $\omega(u)$:

$$\omega(u)\geqslant\omega_0(u)=\begin{cases}\frac{1}{u}, & 1\leqslant u<2,\\ \frac{1+\log(u-1)}{u}, & 2\leqslant u<3,\\ \frac{1+\log(u-1)}{u}+\frac{1}{u}\int_2^{u-1}\frac{\log(t-1)}{t}dt, & 3\leqslant u<4,\\ 0.5612, & u\geqslant 4,\end{cases}$$

$$\omega(u)\leqslant\omega_1(u)=\begin{cases}\frac{1}{u}, & 1\leqslant u<2,\\ \frac{1+\log(u-1)}{u}, & 2\leqslant u<3,\\ \frac{1+\log(u-1)}{u}+\frac{1}{u}\int_2^{u-1}\frac{\log(t-1)}{t}dt, & 3\leqslant u<4,\\ 0.5617, & u\geqslant 4.\end{cases}$$

We shall use $\omega_0(u)$ and $\omega_1(u)$ to give numerical bounds for some sieve functions discussed below. We shall also use the simple upper bound $\omega(u)\leqslant\max(\frac{1}{u},0.5672)$ (see Lemma 8(iii) of [8]) to estimate high–dimensional integrals.

Before decomposing, we define the asymptotic region I as

$$I(m,n):=\left\{\frac{28}{87}\leqslant m\leqslant\frac{31}{87}\text{ or }\frac{56}{87}\leqslant m\leqslant\frac{59}{87}\text{ or }\frac{28}{87}\leqslant n\leqslant\frac{31}{87}\text{ or }\frac{56}{87}\leqslant n\leqslant\frac{59}{87}\right.$$
$$\left.\text{or }\frac{28}{87}\leqslant m+n\leqslant\frac{31}{87}\text{ or }\frac{56}{87}\leqslant m+n\leqslant\frac{59}{87}\right\}.$$

By Buchstab's identity, we have

$$\begin{aligned}S\left(\mathcal{A},(2x)^{\frac{1}{2}}\right)&=S\left(\mathcal{A},x^{\frac{1}{29}}\right)-\sum_{\frac{1}{29}\leqslant t_1<\frac{1}{2}}S\left(\mathcal{A}_{p_1},x^{\frac{1}{29}}\right)+\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)}}S\left(\mathcal{A}_{p_1p_2},p_2\right)\\ &=S_1-S_2+S_3.\end{aligned}\tag{3}$$

By Lemma 2.1, we can give asymptotic formulas for S_1 and S_2. Before estimating S_3, we first split it into six parts:

$$\begin{aligned}S_3=&\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)}}S\left(\mathcal{A}_{p_1p_2},p_2\right)\\ =&\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in I}}S\left(\mathcal{A}_{p_1p_2},p_2\right)+\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\notin I\\ t_1<\frac{28}{87},\ t_1+2t_2>\frac{59}{87}\text{ or }t_1+t_2>\frac{59}{87}}}S\left(\mathcal{A}_{p_1p_2},p_2\right)\\ &+\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\notin I\\ t_1\leqslant\frac{31}{87},\ t_1+2t_2\leqslant\frac{59}{87}}}S\left(\mathcal{A}_{p_1p_2},p_2\right)+\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\notin I\\ t_1>\frac{31}{87},\ t_1+2t_2\leqslant\frac{59}{87}}}S\left(\mathcal{A}_{p_1p_2},p_2\right)\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \notin I \\ t_1 > \frac{31}{87},\ t_1+2t_2 > \frac{59}{87} \\ t_2 \leqslant \max\left(\frac{1-t_1}{3}, \frac{t_1}{2}\right)}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) + \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \notin I \\ t_1 > \frac{31}{87},\ t_1+2t_2 > \frac{59}{87} \\ t_2 > \max\left(\frac{1-t_1}{3}, \frac{t_1}{2}\right)}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
&= \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in I}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) + \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_1}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
&+ \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_2}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) + \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_3}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
&+ \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_4}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) + \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_5}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
&= S_{31} + S_{32} + S_{33} + S_{34} + S_{35} + S_{36}.
\end{aligned} \tag{4}
$$

S_{31} has an asymptotic formula. For S_{32}, we cannot decompose further but have to discard the whole region giving the loss

$$
L_{32} := \int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \texttt{Boole}[(t_1, t_2) \in J_1] \frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1 t_2^2} dt_2 dt_1 < 0.397685. \tag{5}
$$

For S_{33} we can use Buchstab's identity to get

$$
\begin{aligned}
S_{33} &= \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_2}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
&= \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_2}} S\left(\mathcal{A}_{p_1 p_2}, x^{\frac{1}{29}}\right) - \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_2 \\ \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{A}_{p_1 p_2 p_3}, x^{\frac{1}{29}}\right) \\
&+ \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_2 \\ \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ can be partitioned into } (m,n) \in I}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right) \\
&+ \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_2 \\ \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right) \\
&= S_{331} - S_{332} + S_{333} + S_{334}.
\end{aligned} \tag{6}
$$

We have asymptotic formulas for S_{331}–S_{333}. For the remaining S_{334}, we have two ways to get more possible savings: One way is to use Buchstab's identity twice more for some parts if we have $t_1 + t_2 + t_3 + 2t_4 \leqslant \frac{59}{87}$.

Another way is to use Buchstab's identity in reverse to make almost–primes visible. The details of further decompositions are similar to those in [11]. Combining the cases above we get a loss from S_{33} of

$$\begin{aligned}
&\left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\frac{1}{29}}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\frac{1}{29}}^{\min\left(t_3, \frac{1-t_1-t_2-t_3}{2}\right)} \right.\\
&\qquad\left. \texttt{Boole}[(t_1,t_2,t_3,t_4) \in J_{331}] \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1 \right)\\
&- \left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\frac{1}{29}}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\frac{1}{29}}^{\min\left(t_3, \frac{1-t_1-t_2-t_3}{2}\right)} \int_{t_4}^{\frac{1-t_1-t_2-t_3-t_4}{2}} \right.\\
&\qquad\left. \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5) \in J_{332}] \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1 \right)\\
&+ \left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\frac{1}{29}}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\frac{1}{29}}^{\min\left(t_3, \frac{1-t_1-t_2-t_3}{2}\right)} \right.\\
&\qquad \int_{\frac{1}{29}}^{\min\left(t_4, \frac{1-t_1-t_2-t_3-t_4}{2}\right)} \int_{\frac{1}{29}}^{\min\left(t_5, \frac{1-t_1-t_2-t_3-t_4-t_5}{2}\right)}\\
&\qquad\left. \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6) \in J_{333}] \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{t_6}\right)}{t_1 t_2 t_3 t_4 t_5 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right)\\
&+ \left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\frac{1}{29}}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\frac{1}{29}}^{\min\left(t_3, \frac{1-t_1-t_2-t_3}{2}\right)} \int_{\frac{1}{29}}^{\min\left(t_4, \frac{1-t_1-t_2-t_3-t_4}{2}\right)} \right.\\
&\qquad \int_{\frac{1}{29}}^{\min\left(t_5, \frac{1-t_1-t_2-t_3-t_4-t_5}{2}\right)} \int_{\frac{1}{29}}^{\min\left(t_6, \frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{2}\right)}\\
&\qquad \int_{\frac{1}{29}}^{\min\left(t_7, \frac{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7}{2}\right)} \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) \in J_{334}] \times\\
&\qquad\left. \frac{\max\left(\frac{t_8}{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7-t_8}, 0.5672\right)}{t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right)\\
&= L_{331} - L_{332} + L_{333} + L_{334},\\
&< 0.111391 - 0.021501 + 0.001491 + 0.000002\\
&= 0.091383,
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
J_{331}(t_1,t_2,t_3,t_4) := \ &\{(t_1,t_2) \in J_2,\\
&\frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\\
&\frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I,\\
&t_1+t_2+t_3+2t_4 > \frac{59}{87},\\
&\frac{1}{29} \leqslant t_1 < \frac{1}{2}, \ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Bigg\},
\end{aligned}$$

$$
\begin{aligned}
J_{332}(t_1,t_2,t_3,t_4,t_5) := \ & \{(t_1,t_2) \in J_2, \\
& \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \\
& \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\
& (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I, \\
& t_1+t_2+t_3+2t_4 > \frac{59}{87}, \\
& t_4 < t_5 < \frac{1}{2}(1-t_1-t_2-t_3-t_4), \\
& (t_1,t_2,t_3,t_4,t_5) \text{ can be partitioned into } (m,n) \in I, \\
& \frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\}, \\
J_{333}(t_1,t_2,t_3,t_4,t_5,t_6) := \ & \{(t_1,t_2) \in J_2, \\
& \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \\
& \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\
& (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I, \\
& t_1+t_2+t_3+2t_4 \leqslant \frac{59}{87}, \\
& \frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right), \\
& \frac{1}{29} \leqslant t_6 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right), \\
& (t_1,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n) \in I, \\
& t_1+t_2+t_3+t_4+t_5+2t_6 > \frac{59}{87}, \\
& \frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\}, \\
J_{334}(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) := \ & \{(t_1,t_2) \in J_2, \\
& \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \\
& \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\
& (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I, \\
& t_1+t_2+t_3+2t_4 \leqslant \frac{59}{87}, \\
& \frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right), \\
& \frac{1}{29} \leqslant t_6 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right), \\
& (t_1,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n) \in I, \\
& t_1+t_2+t_3+t_4+t_5+2t_6 \leqslant \frac{59}{87},
\end{aligned}
$$

$$\frac{1}{29} \leqslant t_7 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6)\right),$$
$$\frac{1}{29} \leqslant t_8 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6-t_7)\right),$$
$$(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) \text{ cannot be partitioned into } (m,n) \in I,$$
$$\left.\frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\}.$$

Next we shall decompose S_{34}. By the same process as the decomposition of S_{33} above, we can reach a four–dimentional sum

$$\sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_2 \\ \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right).$$

We divide it into three parts:

$$\begin{aligned}
&\sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_2 \\ \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right) \\
=& \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_2 \\ \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I \\ t_1+t_2+t_3+t_4 > \frac{59}{87}}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right) \\
&+ \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_2 \\ \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I \\ t_1+t_2+t_3+2t_4 \leqslant \frac{59}{87}}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right) \\
&+ \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_2 \\ \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I \\ t_1+t_2+t_3+2t_4 > \frac{59}{87} \\ t_1+t_2+t_3+t_4 \leqslant \frac{59}{87}}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right). \qquad (8)
\end{aligned}$$

We can only use Buchstab's identity in reverse to make more savings for the first sum on the right–hand side. For the second sum in (8), we can perform a straightforward decomposition to get a six–dimensional sum

$$\sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_3 \\ \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I \\ t_1+t_2+t_3+2t_4 \leqslant \frac{59}{87} \\ \frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right) \\ \frac{1}{29} \leqslant t_6 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right) \\ (t_1,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n) \in I \\ t_1+t_2+t_3+t_4+t_5+2t_6 > \frac{59}{87}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5p_6}, p_6\right) \tag{9}$$

and an eight–dimensional sum

$$\sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_3 \\ \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I \\ t_1+t_2+t_3+2t_4 \leqslant \frac{59}{87} \\ \frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right) \\ \frac{1}{29} \leqslant t_6 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right) \\ (t_1,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n) \in I \\ t_1+t_2+t_3+t_4+t_5+2t_6 \leqslant \frac{59}{87} \\ \frac{1}{29} \leqslant t_7 < \min\left(t_6, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6)\right) \\ \frac{1}{29} \leqslant t_8 < \min\left(t_7, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6-t_7)\right) \\ (t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) \text{ cannot be partitioned into } (m,n) \in I}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5p_6p_7p_8}, p_8\right). \tag{10}$$

We can also use reversed Buchstab's identity to gain a five–dimensional saving. For the last sum, we cannot decompose it in a straightforward way. However, we can perform a role–reversal to get

$$\begin{aligned} & \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_2 \\ \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I \\ t_1+t_2+t_3+2t_4 > \frac{59}{87} \\ t_1+t_2+t_3+t_4 \leqslant \frac{59}{87}}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right) \\ = & \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_2 \\ \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I \\ t_1+t_2+t_3+2t_4 > \frac{59}{87} \\ t_1+t_2+t_3+t_4 \leqslant \frac{59}{87}}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, x^{\frac{1}{29}}\right) \end{aligned}$$

$$
\begin{aligned}
&- \sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in J_2\\ \frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{1}{29}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)\\ (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in I\\ t_1+t_2+t_3+2t_4>\frac{59}{87}\\ t_1+t_2+t_3+t_4\leqslant\frac{59}{87}\\ \frac{1}{29}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5) \text{ can be partitioned into } (m,n)\in I}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5}, p_5\right)\\
&- \sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in J_2\\ \frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{1}{29}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)\\ (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in I\\ t_1+t_2+t_3+2t_4>\frac{59}{87}\\ t_1+t_2+t_3+t_4\leqslant\frac{59}{87}\\ \frac{1}{29}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (m,n)\in I}} S\left(\mathcal{A}_{\gamma p_2p_3p_4p_5}, x^{\frac{1}{29}}\right)\\
&+ \sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in J_2\\ \frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{1}{29}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)\\ (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in I\\ t_1+t_2+t_3+2t_4>\frac{59}{87}\\ t_1+t_2+t_3+t_4\leqslant\frac{59}{87}\\ \frac{1}{29}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (m,n)\in I\\ \frac{1}{29}\leqslant t_6<\frac{1}{2}t_1\\ (1-t_1-t_2-t_3-t_4-t_5,t_2,t_3,t_4,t_5,t_6) \text{ can be partitioned into } (m,n)\in I}} S\left(\mathcal{A}_{\gamma p_2p_3p_4p_5p_6}, p_6\right),\\
&+ \sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in J_2\\ \frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{1}{29}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)\\ (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in I\\ t_1+t_2+t_3+2t_4>\frac{59}{87}\\ t_1+t_2+t_3+t_4\leqslant\frac{59}{87}\\ \frac{1}{29}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (m,n)\in I\\ \frac{1}{29}\leqslant t_6<\frac{1}{2}t_1\\ (1-t_1-t_2-t_3-t_4-t_5,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n)\in I}} S\left(\mathcal{A}_{\gamma p_2p_3p_4p_5p_6}, p_6\right), \qquad (11)
\end{aligned}
$$

where $\gamma \sim x^{1-t_1-t_2-t_3-t_4-t_5}$ and $(\gamma, P(p_5)) = 1$. Since $t_1+t_2+t_3+t_4 \leqslant \frac{59}{87}$ and $(1-t_1-t_2-t_3-t_4-t_5)+t_2+t_3+t_4+t_5 = (1-t_1) \leqslant (1-\frac{31}{87}) = \frac{56}{87}$, we can give asymptotic formulas for all sums on the right hand side except for the last sum. Note that the last sum in (11) counts numbers with two almost–prime variables, we can make further decompositions on either variable, leading to two eight–dimensional sums

$$\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in J_2\\ \frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{1}{29}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)\\ (t_1,t_2,t_3,t_4)\text{ cannot be partitioned into }(m,n)\in I\\ t_1+t_2+t_3+2t_4>\frac{59}{87}\\ t_1+t_2+t_3+t_4\leqslant\frac{59}{87}\\ \frac{1}{29}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5)\text{ cannot be partitioned into }(m,n)\in I\\ \frac{1}{29}\leqslant t_6<\frac{1}{2}t_1\\ (1-t_1-t_2-t_3-t_4-t_5,t_2,t_3,t_4,t_5,t_6)\text{ cannot be partitioned into }(m,n)\in I\\ (1-t_1-t_2-t_3-t_4-t_5)+t_2+t_3+t_4+t_5+2t_6\leqslant\frac{59}{87}\\ \frac{1}{29}\leqslant t_7<\min\left(t_6,\frac{1}{2}(t_1-t_6)\right)\\ (1-t_1-t_2-t_3-t_4-t_5,t_2,t_3,t_4,t_5,t_6,t_7)\text{ cannot be partitioned into }(m,n)\in I\\ \frac{1}{29}\leqslant t_8<\min\left(t_7,\frac{1}{2}(t_1-t_6-t_7)\right)\\ (1-t_1-t_2-t_3-t_4-t_5,t_2,t_3,t_4,t_5,t_6,t_7,t_8)\text{ cannot be partitioned into }(m,n)\in I}} S\left(\mathcal{A}_{\gamma p_2 p_3 p_4 p_5 p_6 p_7 p_8}, p_8\right) \tag{12}$$

and

$$\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in J_2\\ \frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{1}{29}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)\\ (t_1,t_2,t_3,t_4)\text{ cannot be partitioned into }(m,n)\in I\\ t_1+t_2+t_3+2t_4>\frac{59}{87}\\ t_1+t_2+t_3+t_4\leqslant\frac{59}{87}\\ \frac{1}{29}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5)\text{ cannot be partitioned into }(m,n)\in I\\ \frac{1}{29}\leqslant t_6<\frac{1}{2}t_1\\ (1-t_1-t_2-t_3-t_4-t_5,t_2,t_3,t_4,t_5,t_6)\text{ cannot be partitioned into }(m,n)\in I\\ (1-t_1-t_2-t_3-t_4-t_5)+t_2+t_3+t_4+t_5+2t_6>\frac{59}{87}\\ (t_1-t_6)+t_2+t_3+t_4+2t_5+t_6\leqslant\frac{59}{87}\\ \frac{1}{29}\leqslant t_7<\min\left(t_5,\frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right)\\ (t_1-t_6,t_2,t_3,t_4,t_5,t_6,t_7)\text{ cannot be partitioned into }(m,n)\in I\\ \frac{1}{29}\leqslant t_8<\min\left(t_7,\frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_7)\right)\\ (t_1-t_6,t_2,t_3,t_4,t_5,t_6,t_7,t_8)\text{ cannot be partitioned into }(m,n)\in I}} S\left(\mathcal{A}_{\gamma_1 p_2 p_3 p_4 p_5 p_6 p_7 p_8}, p_8\right), \tag{13}$$

where $\gamma_1 \sim x^{t_1-t_6}$ and $(\gamma_1, P(p_6)) = 1$. We refer the readers to [10] and [11] for more applications of role–reversals. Combining the cases above we get a loss from S_{34} of

$$\left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min\left(t_1,\frac{1-t_1}{2}\right)} \int_{\frac{1}{29}}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)} \int_{\frac{1}{29}}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)} \texttt{Boole}[(t_1,t_2,t_3,t_4)\in J_{341}] \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1\right)$$

$$-\left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min\left(t_1,\frac{1-t_1}{2}\right)} \int_{\frac{1}{29}}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)} \int_{\frac{1}{29}}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)} \int_{t_4}^{\frac{1-t_1-t_2-t_3-t_4}{2}} \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5)\in J_{342}] \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1\right)$$

$$+\left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min\left(t_1,\frac{1-t_1}{2}\right)} \int_{\frac{1}{29}}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)} \int_{\frac{1}{29}}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\right.$$

$$
\begin{aligned}
&\int_{\frac{1}{29}}^{\min\left(t_4,\frac{1-t_1-t_2-t_3-t_4}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_5,\frac{1-t_1-t_2-t_3-t_4-t_5}{2}\right)} \\
&\qquad \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6)\in J_{343}]\frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{t_6}\right)}{t_1t_2t_3t_4t_5t_6^2}dt_6dt_5dt_4dt_3dt_2dt_1\Bigg) \\
&+\Bigg(\int_{\frac{1}{29}}^{\frac{1}{2}}\int_{\frac{1}{29}}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_4,\frac{1-t_1-t_2-t_3-t_4}{2}\right)} \\
&\qquad \int_{\frac{1}{29}}^{\min\left(t_5,\frac{1-t_1-t_2-t_3-t_4-t_5}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_6,\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{2}\right)} \\
&\qquad \int_{\frac{1}{29}}^{\min\left(t_7,\frac{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7}{2}\right)}\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8)\in J_{344}]\times \\
&\qquad \frac{\max\left(\frac{t_8}{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7-t_8},0.5672\right)}{t_1t_2t_3t_4t_5t_6t_7t_8^2}dt_8dt_7dt_6dt_5dt_4dt_3dt_2dt_1\Bigg) \\
&+\Bigg(\int_{\frac{1}{29}}^{\frac{1}{2}}\int_{\frac{1}{29}}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_4,\frac{1-t_1-t_2-t_3-t_4}{2}\right)}\int_{\frac{1}{29}}^{\frac{1}{2}t_1} \\
&\qquad \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6)\in J_{345}]\frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)\omega_1\left(\frac{t_1-t_6}{t_6}\right)}{t_2t_3t_4t_5^2t_6^2}dt_6dt_5dt_4dt_3dt_2dt_1\Bigg) \\
&+\Bigg(\int_{\frac{1}{29}}^{\frac{1}{2}}\int_{\frac{1}{29}}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_4,\frac{1-t_1-t_2-t_3-t_4}{2}\right)} \\
&\qquad \int_{\frac{1}{29}}^{\frac{1}{2}t_1}\int_{\frac{1}{29}}^{\min\left(t_6,\frac{t_1-t_6}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_7,\frac{t_1-t_6-t_7}{2}\right)}\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8)\in J_{346}]\times \\
&\qquad \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)\omega_1\left(\frac{t_1-t_6-t_7-t_8}{t_8}\right)}{t_2t_3t_4t_5^2t_6t_7t_8^2}dt_8dt_7dt_6dt_5dt_4dt_3dt_2dt_1\Bigg) \\
&+\Bigg(\int_{\frac{1}{29}}^{\frac{1}{2}}\int_{\frac{1}{29}}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_4,\frac{1-t_1-t_2-t_3-t_4}{2}\right)}\int_{\frac{1}{29}}^{\frac{1}{2}t_1} \\
&\qquad \int_{\frac{1}{29}}^{\min\left(t_5,\frac{1-t_1-t_2-t_3-t_4-t_5}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_7,\frac{1-t_1-t_2-t_3-t_4-t_5-t_7}{2}\right)}\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8)\in J_{347}]\times \\
&\qquad \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_7-t_8}{t_8}\right)\omega_1\left(\frac{t_1-t_6}{t_6}\right)}{t_2t_3t_4t_5t_6^2t_7t_8^2}dt_8dt_7dt_6dt_5dt_4dt_3dt_2dt_1\Bigg) \\
&= L_{341}-L_{342}+L_{343}+L_{344}+L_{345}+L_{346}+L_{347}, \\
&< 0.103669-0.03892+0.001712+0.000001+0.007242+0.000056+0 \\
&= 0.07376, \qquad (14)
\end{aligned}
$$

where

$$
\begin{aligned}
J_{341}(t_1,t_2,t_3,t_4) := \ &\{(t_1,t_2)\in J_3, \\
&\frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I,\\
&t_1+t_2+t_3+t_4 > \frac{59}{87},\\
&\left.\frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
J_{342}(t_1,t_2,t_3,t_4,t_5) := \ &\{(t_1,t_2) \in J_3,\\
&\frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\\
&\frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I,\\
&t_1+t_2+t_3+t_4 > \frac{59}{87},\\
&t_4 < t_5 < \frac{1}{2}(1-t_1-t_2-t_3-t_4),\\
&(t_1,t_2,t_3,t_4,t_5) \text{ can be partitioned into } (m,n) \in I,\\
&\left.\frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
J_{343}(t_1,t_2,t_3,t_4,t_5,t_6) := \ &\{(t_1,t_2) \in J_3,\\
&\frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\\
&\frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I,\\
&t_1+t_2+t_3+2t_4 \leqslant \frac{59}{87},\\
&\frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right),\\
&\frac{1}{29} \leqslant t_6 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right),\\
&(t_1,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n) \in I,\\
&t_1+t_2+t_3+t_4+t_5+2t_6 > \frac{59}{87},\\
&\left.\frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
J_{344}(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) := \ &\{(t_1,t_2) \in J_3,\\
&\frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\\
&\frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I,\\
&t_1+t_2+t_3+2t_4 \leqslant \frac{59}{87},
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right),\\
&\frac{1}{29} \leqslant t_6 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right),\\
&(t_1,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n)\in I,\\
&t_1+t_2+t_3+t_4+t_5+2t_6 \leqslant \frac{59}{87},\\
&\frac{1}{29} \leqslant t_7 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6)\right),\\
&\frac{1}{29} \leqslant t_8 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6-t_7)\right),\\
&(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) \text{ cannot be partitioned into } (m,n)\in I,\\
&\left.\frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\},\\
J_{345}(t_1,t_2,t_3,t_4,t_5,t_6) :=\ &\{(t_1,t_2)\in J_3,\\
&\frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\\
&\frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in I,\\
&t_1+t_2+t_3+t_4 \leqslant \frac{59}{87},\ t_1+t_2+t_3+2t_4 > \frac{59}{87},\\
&\frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right),\\
&(t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (m,n)\in I,\\
&\frac{1}{29} \leqslant t_6 < \frac{1}{2}t_1,\\
&(1-t_1-t_2-t_3-t_4-t_5,t_2,t_3,t_4,t_5,t_6)\\
&\qquad \text{cannot be partitioned into } (m,n)\in I,\\
&\left.\frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\},\\
J_{346}(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) :=\ &\{(t_1,t_2)\in J_3,\\
&\frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\\
&\frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in I,\\
&t_1+t_2+t_3+t_4 \leqslant \frac{59}{87},\ t_1+t_2+t_3+2t_4 > \frac{59}{87},\\
&\frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right),\\
&(t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (m,n)\in I,\\
&\frac{1}{29} \leqslant t_6 < \frac{1}{2}t_1,\\
&(1-t_1-t_2-t_3-t_4-t_5,t_2,t_3,t_4,t_5,t_6)\\
&\qquad \text{cannot be partitioned into } (m,n)\in I,
\end{aligned}
$$

$$
\begin{aligned}
&(1-t_1-t_2-t_3-t_4-t_5)+t_2+t_3+t_4+t_5+2t_6 \leqslant \frac{59}{87},\\
&\frac{1}{29} \leqslant t_7 < \min\left(t_6, \frac{1}{2}(t_1-t_6)\right),\\
&(1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6, t_7)\\
&\quad \text{cannot be partitioned into } (m,n) \in I,\\
&\frac{1}{29} \leqslant t_8 < \min\left(t_7, \frac{1}{2}(t_1-t_6-t_7)\right),\\
&(1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6, t_7, t_8)\\
&\quad \text{cannot be partitioned into } (m,n) \in I,\\
&\left.\frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
J_{347}(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) := \ &\{(t_1,t_2) \in J_3,\\
&\frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\\
&\frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I,\\
&t_1+t_2+t_3+t_4 \leqslant \frac{59}{87},\ t_1+t_2+t_3+2t_4 > \frac{59}{87},\\
&\frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right),\\
&(t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (m,n) \in I,\\
&\frac{1}{29} \leqslant t_6 < \frac{1}{2}t_1,\\
&(1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6)\\
&\quad \text{cannot be partitioned into } (m,n) \in I,\\
&(1-t_1-t_2-t_3-t_4-t_5)+t_2+t_3+t_4+t_5+2t_6 > \frac{59}{87},\\
&(t_1-t_6)+t_2+t_3+t_4+2t_5+t_6 \leqslant \frac{59}{87},\\
&\frac{1}{29} \leqslant t_7 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right),\\
&(t_1-t_6, t_2, t_3, t_4, t_5, t_6, t_7)\\
&\quad \text{cannot be partitioned into } (m,n) \in I,\\
&\frac{1}{29} \leqslant t_8 < \min\left(t_7, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_7)\right),\\
&(t_1-t_6, t_2, t_3, t_4, t_5, t_6, t_7, t_8)\\
&\quad \text{cannot be partitioned into } (m,n) \in I,\\
&\left.\frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\}.
\end{aligned}
$$

For S_{35} we can also use the devices mentioned earlier, but in this case we will perform a role–reversal on the triple sum if $t_1+t_2+t_3 > \frac{59}{87}$. In this way we get a loss of

$$
\left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\frac{1}{29}}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\frac{1}{29}}^{\min\left(t_3, \frac{1-t_1-t_2-t_3}{2}\right)}
$$

$$\texttt{Boole}[(t_1,t_2,t_3,t_4)\in J_{3501}]\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1t_2t_3t_4^2}dt_4dt_3dt_2dt_1\Bigg)$$

$$-\Bigg(\int_{\frac{1}{29}}^{\frac{1}{2}}\int_{\frac{1}{29}}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\int_{t_4}^{\frac{1-t_1-t_2-t_3-t_4}{2}}$$

$$\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5)\in J_{3502}]\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1t_2t_3t_4t_5^2}dt_5dt_4dt_3dt_2dt_1\Bigg)$$

$$+\Bigg(\int_{\frac{1}{29}}^{\frac{1}{2}}\int_{\frac{1}{29}}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}$$

$$\int_{\frac{1}{29}}^{\min\left(t_4,\frac{1-t_1-t_2-t_3-t_4}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_5,\frac{1-t_1-t_2-t_3-t_4-t_5}{2}\right)}$$

$$\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6)\in J_{3503}]\frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{t_6}\right)}{t_1t_2t_3t_4t_5t_6^2}dt_6dt_5dt_4dt_3dt_2dt_1\Bigg)$$

$$+\Bigg(\int_{\frac{1}{29}}^{\frac{1}{2}}\int_{\frac{1}{29}}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_4,\frac{1-t_1-t_2-t_3-t_4}{2}\right)}$$

$$\int_{\frac{1}{29}}^{\min\left(t_5,\frac{1-t_1-t_2-t_3-t_4-t_5}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_6,\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{2}\right)}$$

$$\int_{\frac{1}{29}}^{\min\left(t_7,\frac{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7}{2}\right)}\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8)\in J_{3504}]\times$$

$$\frac{\max\left(\frac{t_8}{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7-t_8},0.5672\right)}{t_1t_2t_3t_4t_5t_6t_7t_8^2}dt_8dt_7dt_6dt_5dt_4dt_3dt_2dt_1\Bigg)$$

$$+\Bigg(\int_{\frac{1}{29}}^{\frac{1}{2}}\int_{\frac{1}{29}}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\frac{1}{29}}^{\frac{1}{2}t_1}$$

$$\texttt{Boole}[(t_1,t_2,t_3,t_4)\in J_{3505}]\frac{\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)\omega\left(\frac{t_1-t_4}{t_4}\right)}{t_2t_3^2t_4^2}dt_4dt_3dt_2dt_1\Bigg)$$

$$-\Bigg(\int_{\frac{1}{29}}^{\frac{1}{2}}\int_{\frac{1}{29}}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\frac{1}{29}}^{\frac{1}{2}t_1}\int_{t_4}^{\frac{t_1-t_4}{2}}$$

$$\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5)\in J_{3506}]\frac{\omega_0\left(\frac{1-t_1-t_2-t_3}{t_3}\right)\omega_0\left(\frac{t_1-t_4-t_5}{t_5}\right)}{t_2t_3^2t_4t_5^2}dt_5dt_4dt_3dt_2dt_1\Bigg)$$

$$-\Bigg(\int_{\frac{1}{29}}^{\frac{1}{2}}\int_{\frac{1}{29}}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\frac{1}{29}}^{\frac{1}{2}t_1}\int_{t_3}^{\frac{1-t_1-t_2-t_3}{2}}$$

$$\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5)\in J_{3507}]\frac{\omega_0\left(\frac{1-t_1-t_2-t_3-t_5}{t_5}\right)\omega_0\left(\frac{t_1-t_4}{t_4}\right)}{t_2t_3t_4^2t_5^2}dt_5dt_4dt_3dt_2dt_1\Bigg)$$

$$+\Bigg(\int_{\frac{1}{29}}^{\frac{1}{2}}\int_{\frac{1}{29}}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\frac{1}{29}}^{\frac{1}{2}t_1}\int_{t_4}^{\frac{t_1-t_4}{2}}\int_{t_3}^{\frac{1-t_1-t_2-t_3}{2}}$$

$$\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6)\in J_{3508}]\frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_6}{t_6}\right)\omega_1\left(\frac{t_1-t_4-t_5}{t_5}\right)}{t_2t_3t_4t_5^2t_6^2}dt_6dt_5dt_4dt_3dt_2dt_1\Bigg)$$

$$+\Bigg(\int_{\frac{1}{29}}^{\frac{1}{2}}\int_{\frac{1}{29}}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\frac{1}{29}}^{\frac{1}{2}t_1}\int_{\frac{1}{29}}^{\min\left(t_4,\frac{t_1-t_4}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_5,\frac{t_1-t_4-t_5}{2}\right)}$$

$$\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6)\in J_{3509}]\frac{\omega_1\left(\frac{1-t_1-t_2-t_3}{t_3}\right)\omega_1\left(\frac{t_1-t_4-t_5-t_6}{t_6}\right)}{t_2t_3^2t_4t_5t_6^2}dt_6dt_5dt_4dt_3dt_2dt_1\Bigg)$$

$$+\Bigg(\int_{\frac{1}{29}}^{\frac{1}{2}}\int_{\frac{1}{29}}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\frac{1}{29}}^{\frac{1}{2}t_1}\int_{\frac{1}{29}}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\int_{\frac{1}{29}}^{\min\left(t_5,\frac{1-t_1-t_2-t_3-t_5}{2}\right)}$$

$$\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6)\in J_{3510}]\frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_5-t_6}{t_6}\right)\omega_1\left(\frac{t_1-t_4}{t_4}\right)}{t_2t_3t_4^2t_5t_6^2}dt_6dt_5dt_4dt_3dt_2dt_1\Bigg)$$

$$\begin{aligned}&= L_{3501}-L_{3502}+L_{3503}+L_{3504}+L_{3505}-L_{3506}-L_{3507}+L_{3508}+L_{3509}+L_{3510},\\&< 0.079609-0.02541+0.000001+0+0.469283-0.100821-0.124982+0.035428+0.006114+0\\&= 0.339222,\end{aligned}\tag{15}$$

where

$$\begin{aligned}J_{3501}(t_1,t_2,t_3,t_4):=\ &\{(t_1,t_2)\in J_4,\\&\frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right),\ t_1+t_2+t_3\leqslant\frac{59}{87},\\&\frac{1}{29}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right),\\&(t_1,t_2,t_3,t_4)\text{ cannot be partitioned into }(m,n)\in I,\\&t_1+t_2+t_3+2t_4>\frac{59}{87},\\&\frac{1}{29}\leqslant t_1<\frac{1}{2},\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\Bigg\},\\J_{3502}(t_1,t_2,t_3,t_4,t_5):=\ &\{(t_1,t_2)\in J_4,\\&\frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right),\ t_1+t_2+t_3\leqslant\frac{59}{87},\\&\frac{1}{29}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right),\\&(t_1,t_2,t_3,t_4)\text{ cannot be partitioned into }(m,n)\in I,\\&t_1+t_2+t_3+2t_4>\frac{59}{87},\\&t_4<t_5<\frac{1}{2}(1-t_1-t_2-t_3-t_4),\\&(t_1,t_2,t_3,t_4,t_5)\text{ can be partitioned into }(m,n)\in I,\\&\frac{1}{29}\leqslant t_1<\frac{1}{2},\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\Bigg\},\\J_{3503}(t_1,t_2,t_3,t_4,t_5,t_6):=\ &\{(t_1,t_2)\in J_4,\\&\frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right),\ t_1+t_2+t_3\leqslant\frac{59}{87},\\&\frac{1}{29}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right),\end{aligned}$$

$$
\begin{aligned}
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in I,\\
&t_1+t_2+t_3+2t_4 \leqslant \frac{59}{87},\\
&\frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right),\\
&\frac{1}{29} \leqslant t_6 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right),\\
&(t_1,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n)\in I,\\
&t_1+t_2+t_3+t_4+t_5+2t_6 > \frac{59}{87},\\
&\left.\frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
J_{3504}(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) := \ &\{(t_1,t_2)\in J_4,\\
&\frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\ t_1+t_2+t_3 \leqslant \frac{59}{87},\\
&\frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in I,\\
&t_1+t_2+t_3+2t_4 \leqslant \frac{59}{87},\\
&\frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right),\\
&\frac{1}{29} \leqslant t_6 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right),\\
&(t_1,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n)\in I,\\
&t_1+t_2+t_3+t_4+t_5+2t_6 \leqslant \frac{59}{87},\\
&\frac{1}{29} \leqslant t_7 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6)\right),\\
&\frac{1}{29} \leqslant t_8 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6-t_7)\right),\\
&(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) \text{ cannot be partitioned into } (m,n)\in I,\\
&\left.\frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
J_{3505}(t_1,t_2,t_3,t_4) := \ &\{(t_1,t_2)\in J_4,\\
&\frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\ t_1+t_2+t_3 > \frac{59}{87},\\
&\frac{1}{29} \leqslant t_4 < \frac{1}{2}t_1,\\
&(1-t_1-t_2-t_3,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in I,\\
&(1-t_1-t_2-t_3)+t_2+t_3+2t_4 > \frac{59}{87},\\
&(t_1-t_4)+t_2+2t_3+t_4 > \frac{59}{87},\\
&\left.\frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
J_{3506}(t_1,t_2,t_3,t_4,t_5) := \ & \{(t_1,t_2) \in J_4, \\
& \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\ t_1+t_2+t_3 > \frac{59}{87}, \\
& \frac{1}{29} \leqslant t_4 < \frac{1}{2}t_1, \\
& (1-t_1-t_2-t_3, t_2, t_3, t_4) \text{ cannot be partitioned into } (m,n) \in I, \\
& (1-t_1-t_2-t_3)+t_2+t_3+2t_4 > \frac{59}{87}, \\
& (t_1-t_4)+t_2+2t_3+t_4 > \frac{59}{87}, \\
& t_4 < t_5 < \frac{1}{2}(t_1-t_4), \\
& (1-t_1-t_2-t_3, t_2, t_3, t_4, t_5) \text{ can be partitioned into } (m,n) \in I, \\
& \frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\}, \\
J_{3507}(t_1,t_2,t_3,t_4,t_5) := \ & \{(t_1,t_2) \in J_4, \\
& \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\ t_1+t_2+t_3 > \frac{59}{87}, \\
& \frac{1}{29} \leqslant t_4 < \frac{1}{2}t_1, \\
& (1-t_1-t_2-t_3, t_2, t_3, t_4) \text{ cannot be partitioned into } (m,n) \in I, \\
& (1-t_1-t_2-t_3)+t_2+t_3+2t_4 > \frac{59}{87}, \\
& (t_1-t_4)+t_2+2t_3+t_4 > \frac{59}{87}, \\
& t_3 < t_5 < \frac{1}{2}(1-t_1-t_2-t_3), \\
& (t_1-t_4, t_2, t_3, t_4, t_5) \text{ can be partitioned into } (m,n) \in I, \\
& \frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\}, \\
J_{3508}(t_1,t_2,t_3,t_4,t_5,t_6) := \ & \{(t_1,t_2) \in J_4, \\
& \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\ t_1+t_2+t_3 > \frac{59}{87}, \\
& \frac{1}{29} \leqslant t_4 < \frac{1}{2}t_1, \\
& (1-t_1-t_2-t_3, t_2, t_3, t_4) \text{ cannot be partitioned into } (m,n) \in I, \\
& (1-t_1-t_2-t_3)+t_2+t_3+2t_4 > \frac{59}{87}, \\
& (t_1-t_4)+t_2+2t_3+t_4 > \frac{59}{87}, \\
& t_4 < t_5 < \frac{1}{2}(t_1-t_4), \\
& (1-t_1-t_2-t_3, t_2, t_3, t_4, t_5) \text{ can be partitioned into } (m,n) \in I, \\
& t_3 < t_6 < \frac{1}{2}(1-t_1-t_2-t_3), \\
& (t_1-t_4, t_2, t_3, t_4, t_6) \text{ can be partitioned into } (m,n) \in I, \\
& \frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
J_{3509}(t_1,t_2,t_3,t_4,t_5,t_6) := \ & \{(t_1,t_2) \in J_4, \\
& \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\ t_1+t_2+t_3 > \frac{59}{87}, \\
& \frac{1}{29} \leqslant t_4 < \frac{1}{2}t_1, \\
& (1-t_1-t_2-t_3, t_2, t_3, t_4) \text{ cannot be partitioned into } (m,n) \in I, \\
& (1-t_1-t_2-t_3) + t_2 + t_3 + 2t_4 \leqslant \frac{59}{87}, \\
& \frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(t_1-t_4)\right), \\
& (1-t_1-t_2-t_3, t_2, t_3, t_4, t_5) \text{ cannot be partitioned into } (m,n) \in I, \\
& \frac{1}{29} \leqslant t_6 < \min\left(t_5, \frac{1}{2}(t_1-t_4-t_5)\right), \\
& (1-t_1-t_2-t_3, t_2, t_3, t_4, t_5, t_6) \text{ cannot be partitioned into } (m,n) \in I, \\
& \frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\bigg\}, \\
J_{3510}(t_1,t_2,t_3,t_4,t_5,t_6) := \ & \{(t_1,t_2) \in J_4, \\
& \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\ t_1+t_2+t_3 > \frac{59}{87}, \\
& \frac{1}{29} \leqslant t_4 < \frac{1}{2}t_1, \\
& (1-t_1-t_2-t_3, t_2, t_3, t_4) \text{ cannot be partitioned into } (m,n) \in I, \\
& (1-t_1-t_2-t_3) + t_2 + t_3 + 2t_4 > \frac{59}{87}, \\
& (t_1-t_4) + t_2 + 2t_3 + t_4 \leqslant \frac{59}{87}, \\
& \frac{1}{29} \leqslant t_5 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\
& (t_1-t_4, t_2, t_3, t_4, t_5) \text{ cannot be partitioned into } (m,n) \in I, \\
& \frac{1}{29} \leqslant t_6 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_5)\right), \\
& (t_1-t_4, t_2, t_3, t_4, t_5, t_6) \text{ cannot be partitioned into } (m,n) \in I, \\
& \frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\bigg\}.
\end{aligned}
$$

For the remaining S_{36}, we choose to discard the whole region giving the loss

$$
L_{36} := \int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \texttt{Boole}[(t_1,t_2) \in J_5] \frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1 t_2^2} dt_2 dt_1 < 0.093181. \tag{16}
$$

Note that in this region only products of three primes are counted.

Finally, by (3)–(16), the total loss is less than

$$
\begin{aligned}
& L_{32} + (L_{331} + L_{332} + L_{333} + L_{334}) + (L_{341} + L_{342} + L_{343} + L_{344} + L_{345} + L_{346} + L_{347}) \\
& \quad + (L_{3501} + L_{3502} + L_{3503} + L_{3504} + L_{3505} + L_{3506} + L_{3507} + L_{3508} + L_{3509} + L_{3510}) + L_{36} \\
& < 0.397685 + 0.091383 + 0.07376 + 0.339222 + 0.093181 \\
& < 0.996
\end{aligned}
$$

and the proof of Theorem 1.1 is completed.

Acknowledgements

The author would like to thank Professor Chaohua Jia for his encouragement and some helpful discussions.

References

[1] A. Balog. A remark on the distribution of αp modulo one. In *Analytic and elementary number theory (Marseile, 1983)*, pages 6–24. Publ. Math, Orsay, 86-1, Univ. Paris Xl, Orsay, 1986.
[2] K. Ford and J. Maynard. On the theory of prime producing sieves. *arXiv e-prints*, page arXiv:2407.14368v1, 2024.
[3] G. Harman. On the distribution of αp modulo one. *J. London Math. Soc.*, 27(2):9–18, 1983.
[4] G. Harman. On the distribution of αp modulo one II. *Proc. London Math. Soc.*, 72(3):241–260, 1996.
[5] G. Harman. Diophantine approximation with gaussian primes. *Quart. J. Math*, 00:1–15, 2019.
[6] D. R. Heath-Brown and C. Jia. The distribution of αp modulo one. *Proc. London Math. Soc.*, 84(3):79–104, 2002.
[7] C. Jia. On the distribution of αp modulo one. *J. Number Theory*, 45:241–253, 1993.
[8] C. Jia. On the Piatetski–Shapiro–Vinogradov Theorem. *Acta Arith.*, 73(1):1–28, 1995.
[9] C. Jia. On the distribution of αp modulo one (II). *Sci. China Ser. A*, 43:703–721, 2000.
[10] R. Li. Primes in almost all short intervals. *arXiv e-prints*, page arXiv:2407.05651v5, 2024.
[11] R. Li. The number of primes in short intervals and numerical calculations for Harman's sieve. *arXiv e-prints*, page arXiv:2308.04458v6, 2025.
[12] K. Matomäki. The distribution of αp modulo one. *Math. Proc. Camb. Phil. Soc.*, 147:267–283, 2009.
[13] R. C. Vaughan. On the distribution of αp modulo 1. *Mathmatika*, 24(2):135–141, 1977.
[14] I. M. Vinogradov. *The Method of Trigonometric Sums in the Theory of Numbers.* Wiley–Interscience, London, 1954.

International Curriculum Center, The High School Affiliated to Renmin University of China, Beijing, China
Email address: runbo.li.carey@gmail.com