A REFINEMENT OF IMO 2020 PROBLEM 2

RUNBO LI

Abstract. In this short note, we refine the Problem 2 of the 2020 IMO by using improved versions of the weighted AM–GM inequality.

Contents

1. Introduction

The Problem 2 of the 2020 International Mathematical Olympiad (IMO) is proposed by Stijn Cambie in Belgium. The original problem is to prove the inequality

$$(a+2b+3c+4d)a^a b^b c^c d^d \leqslant 1 \tag{1}$$

for any real $a \geqslant b \geqslant c \geqslant d > 0$ satisfying $a+b+c+d=1$. There are many different proofs of this problem, and one can see [1] as a reference.

In 2020, user bel.jad5 on AoPS improved inequality (1) by showing a stronger version:

$$(a+2b+3c+4d)a^a b^b c^c d^d \leqslant 1 - 32(1-a)bcd. \tag{2}$$

In this short note, we further refine his proof and insert some improved versions of the weighted AM–GM inequality to prove the following Theorem.

Theorem 1.1. *Let $a \geqslant b \geqslant c \geqslant d > 0$ be real numbers satisfying $a+b+c+d=1$. Then we have*

$$(a+2b+3c+4d)a^a b^b c^c d^d \leqslant 1 - K,$$

where

$$\begin{aligned}
K &= 32(1-a)bcd + (b-d)a^a b^b c^c d^d + (a+3b+3c+3d)\max(I_1, I_2, I_3, I_{41}, I_{42}, I_{43}, I_{44}, I_{45}, I_{46}),\\
I_1 &= \frac{1}{2a}\left(a^3+b^3+c^3+d^3+a^{2a}b^{2b}c^{2c}d^{2d} - 2a^a b^b c^c d^d(a^2+b^2+c^2+d^2)\right),\\
I_2 &= \frac{1}{1-d}\left(a^2+b^2+c^2+d^2 - \left(a^{\frac{3}{2}}+b^{\frac{3}{2}}+c^{\frac{3}{2}}+d^{\frac{3}{2}}\right)^2\right),\\
I_3 &= 4d\left(\frac{1}{4} - (abcd)^{\frac{1}{4}}\right),\\
I_{41} &= \frac{3d(a+b+c)(c-d)^2}{(4d+2c+2b+2a)c + (4c+4b+4a+2d)d},\\
I_{42} &= \frac{3d(a+b)(b-d)^2}{(4d+2b+2a)b + (4b+4a+2d)d},
\end{aligned}$$

2020 Mathematics Subject Classification. 26D15.

Key words and phrases. weighted AM–GM inequality.

$$
\begin{aligned}
I_{43} &= \frac{3da(a-d)^2}{(4d+2a)a+(4a+2d)d}, \\
I_{44} &= \frac{3c(a+b)(b-c)^2}{(4d+4c+2b+2a)b+(4b+4a+2d+2c)c}, \\
I_{45} &= \frac{3ca(a-c)^2}{(4d+4c+2a)a+(4a+2d+2c)c}, \\
I_{46} &= \frac{3ba(a-b)^2}{(4d+4c+4b+2a)a+(4a+2c+2d+2b)b}.
\end{aligned}
$$

Note that each of the nine I_j may be the largest in some special cases. Here we list some values of K with fixed a, b, c, d, and the readers can compare the sizes of the "proved" savings and the "real" savings over the upper bound 1. We remark that we write

$$
\begin{aligned}
K_R &= 1-(a+2b+3c+4d)a^a b^b c^c d^d, \\
K_1 &= 32(1-a)bcd+(b-d)a^a b^b c^c d^d+(a+3b+3c+3d)I_1, \\
K_2 &= 32(1-a)bcd+(b-d)a^a b^b c^c d^d+(a+3b+3c+3d)I_2, \\
K_3 &= 32(1-a)bcd+(b-d)a^a b^b c^c d^d+(a+3b+3c+3d)I_3, \\
K_{41} &= 32(1-a)bcd+(b-d)a^a b^b c^c d^d+(a+3b+3c+3d)I_{41}, \\
K_{42} &= 32(1-a)bcd+(b-d)a^a b^b c^c d^d+(a+3b+3c+3d)I_{42}, \\
K_{43} &= 32(1-a)bcd+(b-d)a^a b^b c^c d^d+(a+3b+3c+3d)I_{43}, \\
K_{44} &= 32(1-a)bcd+(b-d)a^a b^b c^c d^d+(a+3b+3c+3d)I_{44}, \\
K_{45} &= 32(1-a)bcd+(b-d)a^a b^b c^c d^d+(a+3b+3c+3d)I_{45}, \\
K_{46} &= 32(1-a)bcd+(b-d)a^a b^b c^c d^d+(a+3b+3c+3d)I_{46}.
\end{aligned}
$$

$\boldsymbol{(a,b,c,d)}$	$\boldsymbol{K_R}$	$\boldsymbol{K_1}$	$\boldsymbol{K_2}$	$\boldsymbol{K_3}$	$\boldsymbol{K_{41}}$	$\boldsymbol{K_{42}}$	$\boldsymbol{K_{43}}$	$\boldsymbol{K_{44}}$	$\boldsymbol{K_{45}}$	$\boldsymbol{K_{46}}$
$\mathbf{(0.5, 0.2, 0.2, 0.1)}$	0.43940	**0.15034**	0.14202	0.12432	0.10009	0.09986	0.14567	0.09350	0.12683	0.11921
$\mathbf{(0.92, 0.04, 0.03, 0.01)}$	0.20903	0.07158	**0.07382**	0.02995	0.02116	0.02128	0.03603	0.02107	0.05998	0.06735
$\mathbf{(0.34, 0.33, 0.32, 0.01)}$	0.36248	0.12822	0.12964	**0.13473**	0.13395	0.13420	0.13427	0.12438	0.12453	0.12434
$\mathbf{(0.31, 0.3, 0.3, 0.09)}$	0.40988	0.25092	0.24947	0.25835	**0.26180**	0.26031	0.25829	0.23595	0.23600	0.23599
$\mathbf{(0.31, 0.31, 0.3, 0.08)}$	0.40780	0.24289	0.24165	0.25112	0.25425	**0.25459**	0.25084	0.22769	0.22767	0.22762
$\mathbf{(0.34, 0.31, 0.3, 0.05)}$	0.40436	0.18672	0.18730	0.19686	0.19843	0.19887	**0.20012**	0.17345	0.17424	0.17372
$\mathbf{(0.422, 0.42, 0.113, 0.045)}$	0.41561	0.20367	0.20004	0.19232	0.16566	0.19927	0.19663	**0.20430**	0.19722	0.16255
$\mathbf{(0.55, 0.35, 0.09, 0.01)}$	0.40209	0.17770	0.17286	0.14512	0.13647	0.14356	0.14892	0.16291	**0.19219**	0.15122
$\mathbf{(0.75, 0.21, 0.03, 0.01)}$	0.35103	0.17408	0.16424	0.11037	0.10052	0.10418	0.11593	0.10791	0.13959	**0.17952**

2. Refinements of the weighted AM–GM inequality

In this section we shall list several variants of the weighted AM–GM inequality. First, recall that the classical weighted AM–GM inequality states that

Lemma 2.1. *If real numbers $\alpha_1, \ldots, \alpha_n > 0$ satisfy $\alpha_1 + \cdots + \alpha_n = 1$, then for $x_1, \ldots, x_n \geqslant 0$, we have*

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} \leqslant \alpha_1 x_1 + \cdots + \alpha_n x_n.$$

Next, we mention some results that yield improvements on Lemma 2.1. We will use them to prove our Theorem 1.1 in the next section. Note that in [7] the author also used those inequalities to prove some new inequalities between arithmetic functions.

Lemma 2.2. *([4], Theorem]). Let $n \geqslant 2$, $x_1, \ldots, x_n \geqslant 0$ and $\alpha_1, \ldots, \alpha_n > 0$. Suppose that $\alpha_1 + \cdots + \alpha_n = 1$. Then we have*

$$\frac{1}{2\max(x_1, \cdots, x_n)} \sum_{1 \leqslant i \leqslant n} \alpha_i \left(x_i - \prod_{1 \leqslant j \leqslant n} x_j^{\alpha_j} \right)^2 \leqslant \sum_{1 \leqslant i \leqslant n} \alpha_i x_i - \prod_{1 \leqslant i \leqslant n} x_i^{\alpha_i}.$$

Lemma 2.3. *([3], Theorem 2.2]). Let $n \geqslant 2$, $x_1, \ldots, x_n \geqslant 0$ and $\alpha_1, \ldots, \alpha_n > 0$. Suppose that $\alpha_1 + \cdots + \alpha_n = 1$. Then we have*

$$\begin{aligned} &\frac{1}{1 - \min(\alpha_1, \cdots, \alpha_n)} \sum_{1 \leqslant i \leqslant n} \alpha_i \left(x_i^{\frac{1}{2}} - \sum_{1 \leqslant j \leqslant n} \alpha_j x_j^{\frac{1}{2}} \right)^2 \\ \leqslant &\sum_{1 \leqslant i \leqslant n} \alpha_i x_i - \prod_{1 \leqslant i \leqslant n} x_i^{\alpha_i} \\ \leqslant &\frac{1}{\min(\alpha_1, \cdots, \alpha_n)} \sum_{1 \leqslant i \leqslant n} \alpha_i \left(x_i^{\frac{1}{2}} - \sum_{1 \leqslant j \leqslant n} \alpha_j x_j^{\frac{1}{2}} \right)^2. \end{aligned}$$

Lemma 2.4. *([2], Corollary 2.3]). Let $n \geqslant 2$, $x_1, \ldots, x_n \geqslant 0$ and $\alpha_1, \ldots, \alpha_n > 0$. Suppose that $\alpha_1 + \cdots + \alpha_n = 1$. Then we have*

$$\begin{aligned} &n \min(\alpha_1, \cdots, \alpha_n) \left(\frac{1}{n} \sum_{1 \leqslant i \leqslant n} x_i - \prod_{1 \leqslant i \leqslant n} x_i^{\frac{1}{n}} \right) \\ \leqslant &\sum_{1 \leqslant i \leqslant n} \alpha_i x_i - \prod_{1 \leqslant i \leqslant n} x_i^{\alpha_i} \\ \leqslant &\, n \max(\alpha_1, \cdots, \alpha_n) \left(\frac{1}{n} \sum_{1 \leqslant i \leqslant n} x_i - \prod_{1 \leqslant i \leqslant n} x_i^{\frac{1}{n}} \right). \end{aligned}$$

Lemma 2.5. *([6], Proposition 2.7]). Let $n \geqslant 2$, $0 \leqslant x_1 \leqslant \ldots \leqslant x_j \leqslant \ldots \leqslant x_k \leqslant \ldots \leqslant x_n$ and $\alpha_1, \ldots, \alpha_n > 0$. Suppose that $\alpha_1 + \cdots + \alpha_n = 1$. Then we have*

$$\frac{3(\alpha_1 + \cdots + \alpha_j)(\alpha_k + \cdots + \alpha_n)(x_k - x_j)^2}{(4(\alpha_1 + \cdots + \alpha_j) + 2(\alpha_k + \cdots + \alpha_n))x_k + (4(\alpha_k + \cdots + \alpha_n) + 2(\alpha_1 + \cdots + \alpha_j))x_j} \leqslant \sum_{1 \leqslant i \leqslant n} \alpha_i x_i - \prod_{1 \leqslant i \leqslant n} x_i^{\alpha_i}.$$

3. Proof of Theorem 1.1

We first prove the inequality (which was first proved by user bel.jad5 on AoPS)

$$(a + 3b + 3c + 3d)(a^2 + b^2 + c^2 + d^2) + 32(1 - a)bcd \leqslant 1. \tag{3}$$

We have (see [5])

$$\begin{aligned} &(a + 3b + 3c + 3d)(a^2 + b^2 + c^2 + d^2) + 32(1 - a)bcd \\ \leqslant &(a + 3b + 3c + 3d) \left(a^2 + \frac{1}{3}(b + c + d)^2 \right) + \frac{32}{27}(1 - a)(b + c + d)^3 \\ = &(1 + 6t) \left((1 - 3t)^2 + 3t^2 \right) + 96t^4 \leqslant 1, \end{aligned} \tag{4}$$

where $t = \frac{1}{3}(b + c + d)$. Since (4) holds when $0 \leqslant t \leqslant \frac{1}{4}$, (3) is proved.

Now, by (3), we only need to prove that

$$\max(I_1, I_2, I_3, I_{41}, I_{42}, I_{43}, I_{44}, I_{45}, I_{46}) \leqslant a^2 + b^2 + c^2 + d^2 - a^a b^b c^c d^d. \tag{5}$$

Taking $n = 4$, $x_1 = \alpha_1 = d$, $x_2 = \alpha_2 = c$, $x_3 = \alpha_3 = b$ and $x_4 = \alpha_4 = a$, Lemma 2.2 yields

$$I_1 \leqslant a^2 + b^2 + c^2 + d^2 - a^a b^b c^c d^d. \tag{6}$$

Taking $n = 4$, $x_1 = \alpha_1 = d$, $x_2 = \alpha_2 = c$, $x_3 = \alpha_3 = b$ and $x_4 = \alpha_4 = a$, Lemma 2.3 yields

$$I_2 \leqslant a^2 + b^2 + c^2 + d^2 - a^a b^b c^c d^d. \tag{7}$$

Taking $n = 4$, $x_1 = \alpha_1 = d$, $x_2 = \alpha_2 = c$, $x_3 = \alpha_3 = b$ and $x_4 = \alpha_4 = a$, Lemma 2.4 yields

$$I_3 \leqslant a^2 + b^2 + c^2 + d^2 - a^a b^b c^c d^d. \tag{8}$$

Taking $n = 4$, $x_1 = \alpha_1 = d$, $x_2 = \alpha_2 = c$, $x_3 = \alpha_3 = b$, $x_4 = \alpha_4 = a$, $j = 1$ and $k = 2$, Lemma 2.5 yields

$$I_{41} \leqslant a^2 + b^2 + c^2 + d^2 - a^a b^b c^c d^d. \tag{9}$$

Taking $n = 4$, $x_1 = \alpha_1 = d$, $x_2 = \alpha_2 = c$, $x_3 = \alpha_3 = b$, $x_4 = \alpha_4 = a$, $j = 1$ and $k = 3$, Lemma 2.5 yields

$$I_{42} \leqslant a^2 + b^2 + c^2 + d^2 - a^a b^b c^c d^d. \tag{10}$$

Taking $n = 4$, $x_1 = \alpha_1 = d$, $x_2 = \alpha_2 = c$, $x_3 = \alpha_3 = b$, $x_4 = \alpha_4 = a$, $j = 1$ and $k = 4$, Lemma 2.5 yields

$$I_{43} \leqslant a^2 + b^2 + c^2 + d^2 - a^a b^b c^c d^d. \tag{11}$$

Taking $n = 4$, $x_1 = \alpha_1 = d$, $x_2 = \alpha_2 = c$, $x_3 = \alpha_3 = b$, $x_4 = \alpha_4 = a$, $j = 2$ and $k = 3$, Lemma 2.5 yields

$$I_{44} \leqslant a^2 + b^2 + c^2 + d^2 - a^a b^b c^c d^d. \tag{12}$$

Taking $n = 4$, $x_1 = \alpha_1 = d$, $x_2 = \alpha_2 = c$, $x_3 = \alpha_3 = b$, $x_4 = \alpha_4 = a$, $j = 2$ and $k = 4$, Lemma 2.5 yields

$$I_{45} \leqslant a^2 + b^2 + c^2 + d^2 - a^a b^b c^c d^d. \tag{13}$$

Taking $n = 4$, $x_1 = \alpha_1 = d$, $x_2 = \alpha_2 = c$, $x_3 = \alpha_3 = b$, $x_4 = \alpha_4 = a$, $j = 3$ and $k = 4$, Lemma 2.5 yields

$$I_{46} \leqslant a^2 + b^2 + c^2 + d^2 - a^a b^b c^c d^d. \tag{14}$$

Now we can prove (5) by combining (6)–(14), and the proof of Theorem 1.1 is completed.

References

[1] IMO2020 Shortlisted Problems with Solutions. 2020.

[2] J. M. Aldaz. Comparison of differences between arithmetic and geometric means. *Tamkang J. Math.*, 43(4):453–462, 2011.

[3] J. M. Aldaz. Sharp bounds for the difference between the arithmetic and geometric means. *Arch. Math.*, 99:393–399, 2012.

[4] H. Alzer. A new refinement of the arithmetic mean–geometric mean inequality. *Rocky Mountain J. Math.*, 27(3):663–667, 1997.

[5] bel.jad5. Stronger than P2 IMO2020. *https://artofproblemsolving.com/community/c6h2283130p17899737*, 2020.

[6] V. Cîrtoaje. The best lower bound depended on two fixed variables for Jensen's inequality with ordered variables. *J. Inequal. Appl.*, 128258, 2010.

[7] R. Li. On the weighted AM–GM inequality and refined inequalities between arithmetic functions. *preprint*, 2025.

International Curriculum Center, The High School Affiliated to Renmin University of China, Beijing, China

Email address: carey.lee.0433@gmail.com