ON THE GENERALIZED DIRICHLET DIVISOR PROBLEM

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Abstract. Using more advanced results on the growth exponent for Riemann zeta–function and accurate numerical estimations, we obtain better upper bounds for α_k $(9 \leqslant k \leqslant 20)$ on the generalized Dirichlet divisor problem. This gives a minor improvement upon the recent result of Trudgian and Yang.

Contents

1. Introduction

Let $k \geqslant 2$ denotes an integer and $d_k(n)$ is the divisor function that represents the number of ways n may be written as a product of exactly k factors. The generalized Dirichlet divisor problem consists of the estimation of the function

$$\Delta_k(x) = \sum_{n \leqslant x} d_k(n) - xP_{k-1}(\log x), \tag{1}$$

where P_{k-1} is an explicit polynomial of degree $k-1$. Clearly we have $\Delta_k(x) = o(x)$. We then define α_k as the least exponent for which

$$\Delta_k(x) \ll x^{\alpha_k+\varepsilon}. \tag{2}$$

In 1916, Hardy [2] first proved a lower bound that $\alpha_k \geqslant \frac{1}{2} - \frac{1}{2k}$ for all $k \geqslant 2$. The generalized Dirichlet divisor problem conjecture states that $\alpha_k = \frac{1}{2} - \frac{1}{2k}$ holds for all $k \geqslant 2$, and this conjecture implies the Lindelöf hypothesis. Now, the best upper bounds for α_k $(k \leqslant 8)$ are

$$\alpha_2 \leqslant 0.3144831759741, \qquad \alpha_3 \leqslant \frac{43}{96}, \qquad \alpha_k \leqslant \frac{3k-4}{4k} \text{ for } 4 \leqslant k \leqslant 8$$

by Li and Yang [8], Kolesnik [7] and Heath–Brown [3] (and Ivić [5]) respectively. Ivić also gave upper bounds with $k \geqslant 9$ in his book. For results with large k, one can see works of Heath–Brown [4] and Bellotti and Yang [1]. We also refer the readers to the blueprint of the new project ANTEDB organized by Tao, Trudgian and Yang [9].

In 1989, Ivić and Ouellet [6] refined the technique used in and gave better bounds for α_k with $k \geqslant 9$. In [5], Ivić connected this problem with the function $m(\sigma)$ defined as follows: For any fixed $\frac{1}{2} < \sigma < 1$ we define $m(\sigma)$ as the supremum of all numbers $m \geqslant 4$ such that

$$\int_1^T |\zeta(\sigma + it)|^m \, dt \ll T^{1+\varepsilon}. \tag{3}$$

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In order to obtain good bounds for α_k, one need to get lower bounds for $m(\sigma)$. Ivić and Ouellet [6] used a large value theorem and growth exponents for Riemann zeta–function to bound $m(\sigma)$. Specially, for $10 \leqslant k \leqslant 20$ they got

$$\begin{array}{llll} \alpha_{10} \leqslant 0.675, & \alpha_{11} \leqslant 0.6957, & \alpha_{12} \leqslant 0.7130, & \alpha_{13} \leqslant 0.7306, \\ \alpha_{14} \leqslant 0.7461, & \alpha_{15} \leqslant 0.75851, & \alpha_{16} \leqslant 0.7691, & \alpha_{17} \leqslant 0.7785, \\ \alpha_{18} \leqslant 0.7868, & \alpha_{19} \leqslant 0.7942, & \alpha_{20} \leqslant 0.8009. & \end{array}$$

In 2024, Trudgian and Yang [10] mentioned a series of new bounds for α_k. They combined the method of Ivić and Ouellet [6] with their new growth exponents for Riemann zeta–function to obtain those bounds.

Theorem 1.1. *([10], Theorem 2.9]). We have*

$$\begin{array}{lllll} \alpha_9 \leqslant 0.64720, & \alpha_{10} \leqslant 0.67173, & \alpha_{11} \leqslant 0.69156, & \alpha_{12} \leqslant 0.70818, & \\ \alpha_{13} \leqslant 0.72350, & \alpha_{14} \leqslant 0.73696, & \alpha_{15} \leqslant 0.74886, & \alpha_{16} \leqslant 0.75952, & \\ \alpha_{17} \leqslant 0.76920, & \alpha_{18} \leqslant 0.77792, & \alpha_{19} \leqslant 0.78581, & \alpha_{20} \leqslant 0.79297, & \alpha_{21} \leqslant 0.79951. \end{array}$$

In this paper, we use the essentially same methods to give a very minor improvement on their results.

Theorem 1.2. *We have*

$$\begin{array}{llll} \alpha_9 \leqslant 0.638889, & \alpha_{10} \leqslant 0.663329, & \alpha_{11} \leqslant 0.684349, & \alpha_{12} \leqslant 0.701768, \\ \alpha_{13} \leqslant 0.717523, & \alpha_{14} \leqslant 0.731898, & \alpha_{15} \leqslant 0.744898, & \alpha_{16} \leqslant 0.756380, \\ \alpha_{17} \leqslant 0.766588, & \alpha_{18} \leqslant 0.775721, & \alpha_{19} \leqslant 0.783939, & \alpha_{20} \leqslant 0.791374. \end{array}$$

2. Growth exponents for Riemann zeta–function

In this section we list the new growth exponents for Riemann zeta–function proved by Trudgian and Yang [10], which is the most powerful and important input in the proof of Theorem 1.2. In the proof of Theorem 1.1 a weaker version of this was used.

Lemma 2.1. *([9], Table 6.2]). We have*

$$\mu(\sigma) \leqslant \begin{cases} \frac{31}{36} - \frac{3}{7}\sigma, & \frac{1}{2} \leqslant \sigma \leqslant \frac{88225}{153852}, \\ \frac{220633}{620612} - \frac{62831}{155153}\sigma, & \frac{88225}{153852} \leqslant \sigma \leqslant \frac{521}{796}, \\ \frac{1333}{3825} - \frac{1508}{3825}\sigma, & \frac{521}{796} \leqslant \sigma \leqslant \frac{53141}{76066}, \\ \frac{405}{1202} - \frac{227}{601}\sigma, & \frac{53141}{76066} \leqslant \sigma \leqslant \frac{454}{641}, \\ \frac{779}{2590} - \frac{423}{1295}\sigma, & \frac{454}{641} \leqslant \sigma \leqslant \frac{1744}{2411}, \\ \frac{179}{622} - \frac{96}{311}\sigma, & \frac{1744}{2411} \leqslant \sigma \leqslant \frac{951057}{1298878}, \\ \frac{157319}{560830} - \frac{251324}{841245}\sigma, & \frac{951057}{1298878} \leqslant \sigma \leqslant \frac{1389}{1736}, \\ \frac{2841}{10316} - \frac{754}{2579}\sigma, & \frac{1389}{1736} \leqslant \sigma \leqslant \frac{587779}{702192}, \\ \frac{1691}{6554} - \frac{890}{3277}\sigma, & \frac{587779}{702192} \leqslant \sigma \leqslant \frac{7441}{8695}, \\ \frac{29}{130} - \frac{3}{13}\sigma, & \frac{7441}{8695} \leqslant \sigma \leqslant \frac{277}{300}, \\ \frac{3}{23} - \frac{3}{23}\sigma, & \frac{277}{300} \leqslant \sigma < 1. \end{cases}$$

In order to use the large value theorem in the next section, we also need the following two results, which give the upper bounds for $\mu(\sigma)$ when $\sigma < \frac{1}{2}$:

Lemma 2.2. *([9], Lemma 6.4]). We have*

$$\mu(\sigma) = \mu(1-\sigma) + \frac{1}{2} - \sigma$$

for all $0 < \sigma \leqslant \frac{1}{2}$.

Lemma 2.3. *([9], Lemma 6.5]). We have*

$$\mu(\sigma) = \frac{1}{2} - \sigma$$

for all $\sigma \leqslant 0$.

3. A large value theorem

Now we provide a new large value theorem, which is a refined version of Ivić's large value theorem [[5], Lemma 8.2]. Note that Ivić's version was used by Ivić and Ouellet [6].

Lemma 3.1. *Let $t_1, \ldots, t_R$ be real numbers such that $T \leqslant t_r \leqslant 2T$ for $r = 1, \ldots, R$ and $|t_r - t_s| \geqslant (\log T)^4$ for $1 \leqslant r \neq s \leqslant R$. If*

$$T^{\varepsilon} < V \leqslant \left| \sum_{m \sim M} a_m m^{-\sigma - it_r} \right|$$

where $a_m \ll M^{\varepsilon}$ for $m \sim M$, $1 \ll M \ll T^C$, then

$$R \ll T^{\varepsilon} \left(M^{2-2\sigma} V^{-2} + TV^{-f(\sigma)} \right),$$

where

$$f(\sigma) = \begin{cases} \frac{2}{3-4\sigma}, & \frac{1}{2} < \sigma \leqslant \frac{2}{3}, \\ \frac{58}{63-80\sigma}, & \frac{2}{3} < \sigma \leqslant \frac{583}{860}, \\ \frac{47}{41-50\sigma}, & \frac{583}{860} < \sigma \leqslant \frac{16581}{24022}, \\ \frac{4968}{3981-4774\sigma}, & \frac{16581}{24022} < \sigma \leqslant \frac{1333047}{1920826}, \\ \frac{15998}{12283-14600\sigma}, & \frac{1333047}{1920826} < \sigma \leqslant \frac{3269}{4658}, \\ \frac{656601}{497599-589921\sigma}, & \frac{3269}{4658} < \sigma \leqslant \frac{2410373}{3361430}, \\ \frac{245}{182-215\sigma}, & \frac{2410373}{3361430} < \sigma \leqslant \frac{2233}{3105}, \\ \frac{1037}{743-872\sigma}, & \frac{2233}{3105} < \sigma \leqslant \frac{592}{819}, \\ \frac{503}{325-374\sigma}, & \frac{592}{819} < \sigma \leqslant \frac{140323}{193464}, \\ \frac{12982}{8109-9268\sigma}, & \frac{140323}{193464} < \sigma \leqslant \frac{1461}{1982}, \\ \frac{1061878}{648903-738576\sigma}, & \frac{1461}{1982} < \sigma \leqslant \frac{1960121}{2577906}, \\ \frac{146}{85-96\sigma}, & \frac{1960121}{2577906} < \sigma \leqslant \frac{76}{97}, \\ \frac{158}{67-72\sigma}, & \frac{76}{97} < \sigma \leqslant \frac{1960121}{2419300}. \end{cases}$$

Proof. We follow the arguments in [[5], Lemma 8.2]. Let $c(\sigma)$ be an upper bound for $\mu(\sigma)$. By Lemmas 2.1–2.3, we know that $c(\sigma)$ can be written in the form $A - B\sigma$ with potisive A, B when $\sigma \leqslant 0.8$. We choose 0.8 as the end point because it is enough for our proof of Theorem 1.2. Let $\theta = \theta(\sigma)$ be implicitly defined by

$$2c(\theta) + 1 + \theta - 2(1 + c(\theta))\sigma = 0. \tag{4}$$

Suppose that for some σ, the value of θ lies in an interval $[\sigma_1, \sigma_2]$ with fixed A, B. Then by (4) we have

$$\theta = \frac{2(1+A)\sigma - (2A+1)}{2B\sigma + (1-2B)}. \tag{5}$$

Furthermore, let

$$f(\sigma) = \frac{2(1 + c(\theta))}{c(\theta)}. \tag{6}$$

The values of $f(\sigma)$ when $\frac{1}{2} \leqslant \sigma \leqslant 0.8$ are listed above. Now by similar arguments as in the proof of [[5], Lemma 8.2], Lemma 3.1 is proved. □

4. Proof of Theorem 1.2

We shall use the method of Ivić and Ouellet [6] to prove Theorem 1.2. It was shown in [[5], Chapter 8] that to obtain bounds for $m(\sigma)$ it suffices to obtain bounds of the form

$$R \ll T^{1+\varepsilon} V^{-m(\sigma)}, \tag{7}$$

where R is the number of points $t_r (1 \leqslant r \leqslant R)$ such that $|t_r| \leqslant T$, $|t_r - t_s| \geqslant (\log T)^4$ for $1 \leqslant r \neq s \leqslant R$ and $|\zeta(\sigma + it_r)| \geqslant V > 0$ for any given V. Moreover, by [[5], (8.97)] we know that

$$R \ll T^{\varepsilon} \left(TV^{-2f(\sigma)} + T^{\frac{4-4\sigma}{1+2\sigma}} V^{\frac{-12}{1+2\sigma}} + T^{\frac{4(1-\sigma)(\kappa+\lambda)}{((2-4\lambda)\sigma - 1 + 2\kappa - 2\lambda)}} V^{\frac{-4(1+2\kappa+2\lambda)}{((2-4\lambda)\sigma-1+2\kappa-2\lambda)}} \right), \tag{8}$$

where (κ, λ) is an exponent pair. We shall use $(\kappa, \lambda) = \left(\frac{3}{40}, \frac{31}{40}\right)$ in the rest of our paper for the sake of convenience.

Note that $c(\sigma)$ is an upper bound for $\mu(\sigma)$ given by Lemmas 2.1–2.3. By (8) and the definitions of $f(\sigma)$ and $c(\sigma)$, we can easily calculate the corresponding $m(\sigma)$ for σ between $\frac{1}{2}$ and 0.8. Numerical calculation gives that

$$\begin{aligned} &m(0.638889) > 9, && m(0.663329) > 10, && m(0.684349) > 11, && m(0.701768) > 12, \\ &m(0.717523) > 13, && m(0.731898) > 14, && m(0.744898) > 15, && m(0.756380) > 16, \\ &m(0.766588) > 17, && m(0.775721) > 18, && m(0.783939) > 19, && m(0.791374) > 20, \end{aligned}$$

and Theorem 1.2 is now proved.

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