ON A CONJECTURE INVOLVING TWIN PRACTICAL NUMBERS

RUNBO LI

Abstract. In this paper, we state a conjecture that for any positive integer p, there are infinitely many practical numbers q such that both q^p and q^p+2 are practical numbers. We prove the conjecture when p is a prime number between 7 and 100, refining a result involving twin practical numbers of Wang and Sun. We also prove several results related to this conjecture.

1. Introduction

A positive integer m is called a practical number if each $n = 1, \ldots, m$ can be written as the sum of some distinct divisors of m. Clearly all practical numbers are even except 1, and all even perfect numbers and powers of 2 are practical. Let $P(x)$ denote the number of practical numbers less than x. In 1984, Hausman and Shapiro [1] showed that

$$P(x) \ll \frac{x}{(\log x)^{\beta}} \tag{1}$$

for any $\beta < \frac{1}{2}(1 - \frac{1}{\log 2})^2 \approx 0.0979$. Margenstern [2] showed in 1991 that

$$P(x) \gg \frac{x}{\exp\left(\frac{1}{2\log 2}(\log\log x)^2 + 3\log\log x\right)}. \tag{2}$$

Tenenbaum [6] [7] improved the bounds (1) and (2) to

$$\frac{x}{\log x}(\log\log x)^{-\frac{5}{3}-\varepsilon} \ll P(x) \ll \frac{x}{\log x}\log\log x \log\log\log x. \tag{3}$$

Using a sieve method, Saias [5] improved Tenenbaum's result to

$$\frac{x}{\log x} \ll P(x) \ll \frac{x}{\log x}. \tag{4}$$

Margenstern conjectured that

$$P(x) \sim \lambda_1 \frac{x}{\log x} \tag{5}$$

with $\lambda_1 \approx 1.341$.

Like the famous twin prime conjecture, Margenstern [2] established that there are infinitely many twin practical numbers m and $m+2$, and another detailed proof was given by Melfi [3] in 1996. Let $P_2(x)$ denote the number of practical number pairs $(m, m+2)$ less than x. Margenstern's proof leads to a lower bound $P_2(x) \gg \log\log x$, and Melfi's proof leads to a better lower bound $P_2(x) \gg \log x$. In 2002, Melfi [4] improved the lower bound to

$$P_2(x) \gg \frac{x}{\exp\left(k(\log x)^{\frac{1}{2}}\right)} \tag{6}$$

Key words and phrases. practical numbers, cyclotomic polynomials, additive combinatorics.

with $k > 2 + \log\frac{3}{2}$. Margenstern also conjectured that

$$P_2(x) \sim \lambda_2 \frac{x}{\log x} \tag{7}$$

with $\lambda_1 \approx 1.436$.

In 2022, Wang and Sun [8] generalized Margenstern's result and showed that there are infinitely many practical numbers q such that q^4 and q^4+2 are also practical numbers. They showed their result by modifying Melfi's cyclotomic method.

In this paper, we further state a conjecture and show some new results.

Conjecture 1.1. *For any positive integer C, there are infinitely many practical numbers q such that q^C and q^C+2 are also practical numbers.*

Clearly the results of Margenstern [2] and Melfi [3] [4] imply the case $C = 1$ of Conjecture 1.1, and Wang and Sun [8] solved the cases $C = 2$ and $C = 4$. Wang and Sun [8] solved the cases $C = 2$ and $C = 4$ by proving that the sequence $2^{35\times 3^k+1} + 2$ are practical for every integer $k \geqslant 0$. In this paper, we find four new sequences that generate infinitely many practical numbers.

Theorem 1.2. *$2^{135\times 7^k+1} + 2$, $2^{165\times 3^k+1} + 2$, $2^{175\times 3^k+1} + 2$ and $2^{189\times 5^k+1} + 2$ are practical for every integer $k \geqslant 0$.*

Using Theorem 1.2 we can easily deduce the following three Theorems.

Theorem 1.3. *Let p denotes a prime number, $7 \leqslant p < 100$. Then there are infinitely many practical numbers q such that q^p and q^p+2 are also practical numbers.*

Theorem 1.4. *There are infinitely many practical numbers q such that:*
(1). q^3 and q^3+8 are also practical numbers.
(2). q^5 and q^5+32 are also practical numbers.

Theorem 1.5. *For any positive integer C, there exists an integer h such that there are infinitely many practical numbers q such that q^C and q^C+2^h are also practical numbers.*

2. Proof of Theorem 1.2

In order to prove our Theorems, we need the following structure theorem:

Lemma 2.1. *[[3], Lemma 1]. Let m be any practical number. Then mn is practical for every $n = 1, \ldots, \sigma(m)+1$. In particular, mn is practical for every $1 \leqslant n \leqslant 2m$.*

Let $\Phi_m(x)$ denotes the mth cyclotomic polynomial, then we have

Lemma 2.2.

$$x^n - 1 = \prod_{d|n} \Phi_d(x) \quad \textit{for all } n = 1, 2, 3, \ldots$$

Now we start our proof of Theorem 1.2. Note that we may use m_k to represent different sequences in different subsection.

2.1. **Sequence 1.** We shall prove that $2^{135\times 7^k+1}+2$ are practical for every $k \geqslant 0$.

Write $m_k = 2^{135\times 7^k+1}+2$ for $k = 0, 1, 2, \ldots$, then we only need to show that m_k is practical for every $k = 0, 1, 2, \ldots$. By a computer calculation we find that

$$\begin{aligned} m_0 &= 87112285931760246646623899502532662132738 \\ &= 2 \times 3^4 \times 11 \times 19 \times 331 \times 811 \times 15121 \times 87211 \times 18837001 \times 385838642647891. \end{aligned}$$

By Lemma 2.1, we know that m_0 is practical. Let $x = 2^{7^k}$, then we have $m_k = 2(x^{135}+1)$ and $m_{k+1} = 2(x^{945}+1)$. Since $k \geqslant 0$, we have $x \geqslant 2$. By Lemma 2.2, we have

$$\frac{x^{1890}+1}{x^{945}+1} = \frac{x^{270}+1}{x^{135}+1}\Phi_{14}(x)\Phi_{42}(x)\Phi_{70}(x)\Phi_{126}(x)\Phi_{210}(x)\Phi_{378}(x)\Phi_{630}(x)\Phi_{1890}(x). \tag{8}$$

Assume that $2(x^{135}+1)$ is practical, then we only need to show that

$$2(x^{135}+1)\Phi_{14}(x)\Phi_{42}(x)\Phi_{70}(x)\Phi_{126}(x)\Phi_{210}(x)\Phi_{378}(x)\Phi_{630}(x)\Phi_{1890}(x)$$

is practical.

Note that

$$\begin{aligned} \Phi_{14}(x) &= x^6 - x^5 + x^4 - x^3 + x^2 - x + 1, \\ \Phi_{42}(x) &= x^{12} + x^{11} - x^9 - x^8 + x^6 - x^4 - x^3 + x + 1, \\ \Phi_{70}(x) &= x^{24} + x^{23} - x^{19} - x^{18} - x^{17} - x^{16} + x^{14} + x^{13} + x^{12} + x^{11} \\ &\quad + x^{10} - x^8 - x^7 - x^6 - x^5 + x + 1, \\ \Phi_{126}(x) &= x^{36} + x^{33} - x^{27} - x^{24} + x^{18} - x^{12} - x^9 + x^3 + 1, \\ \Phi_{210}(x) &= x^{48} - x^{47} + x^{46} + x^{43} - x^{42} + 2x^{41} - x^{40} + x^{39} + x^{36} - x^{35} \\ &\quad + x^{34} - x^{33} + x^{32} - x^{31} - x^{28} - x^{26} - x^{24} - x^{22} - x^{20} - x^{17} \\ &\quad + x^{16} - x^{15} + x^{14} - x^{13} + x^{12} + x^9 - x^8 \\ &\quad + 2x^7 - x^6 + x^5 + x^2 - x + 1, \\ \Phi_{378}(x) &= x^{108} + x^{99} - x^{81} - x^{72} + x^{54} - x^{36} - x^{27} + x^9 + 1, \\ \Phi_{630}(x) &= x^{144} - x^{141} + x^{138} + x^{129} - x^{126} + 2x^{123} - x^{120} \\ &\quad + x^{117} + x^{108} - x^{105} + x^{102} - x^{99} + x^{96} - x^{93} - x^{84} \\ &\quad - x^{78} - x^{72} - x^{66} - x^{60} - x^{51} + x^{48} - x^{45} + x^{42} \\ &\quad - x^{39} + x^{36} + x^{27} - x^{24} + 2x^{21} - x^{18} + x^{15} + x^6 - x^3 + 1, \\ \Phi_{1890}(x) &= x^{432} - x^{423} + x^{414} + x^{387} - x^{378} + 2x^{369} - x^{360} + x^{351} \\ &\quad + x^{324} - x^{315} + x^{306} - x^{297} + x^{288} - x^{279} - x^{252} - x^{234} \\ &\quad - x^{216} - x^{198} - x^{180} - x^{153} + x^{144} - x^{135} + x^{126} - x^{117} \\ &\quad + x^{108} + x^{81} - x^{72} + 2x^{63} - x^{54} + x^{45} + x^{18} - x^9 + 1. \end{aligned}$$

Since $x \geqslant 2$, we have

$$\frac{1}{2}x^6 \leqslant \Phi_{14}(x) \leqslant x^6, \tag{9}$$

$$x^{12} \leqslant \Phi_{42}(x) \leqslant 2x^{12}, \tag{10}$$

$$x^{24} \leqslant \Phi_{70}(x) \leqslant 2x^{24}, \tag{11}$$
$$x^{36} \leqslant \Phi_{126}(x) \leqslant 2x^{36}, \tag{12}$$
$$\frac{1}{2}x^{48} \leqslant \Phi_{210}(x) \leqslant x^{48}, \tag{13}$$
$$x^{108} \leqslant \Phi_{378}(x) \leqslant 2x^{108}, \tag{14}$$
$$\frac{1}{2}x^{144} \leqslant \Phi_{630}(x) \leqslant x^{144}, \tag{15}$$
$$\frac{1}{2}x^{432} \leqslant \Phi_{1890}(x) \leqslant x^{432}. \tag{16}$$

By (9)–(13), we have

$$\Phi_{14}(x)\Phi_{42}(x)\Phi_{70}(x)\Phi_{126}(x)\Phi_{210}(x) \leqslant 8x^{126} \leqslant x^{129} \leqslant 2(x^{135}+1) \tag{17}$$

and

$$2(x^{135}+1)\Phi_{14}(x)\Phi_{42}(x)\Phi_{70}(x)\Phi_{126}(x)\Phi_{210}(x)$$

is practical by Lemma 2.1.

By (9)–(13), we have

$$2(x^{135}+1)\Phi_{14}(x)\Phi_{42}(x)\Phi_{70}(x)\Phi_{126}(x)\Phi_{210}(x) \geqslant \frac{1}{2}x^{261} \geqslant x^{260}. \tag{18}$$

By (14)–(15), we have

$$\Phi_{378}(x)\Phi_{630}(x) \leqslant 2x^{252} \leqslant x^{253}. \tag{19}$$

By (18)–(19), we have

$$\Phi_{378}(x)\Phi_{630}(x) \leqslant x^{253} \leqslant x^{260} \leqslant 2(x^{135}+1)\Phi_{14}(x)\Phi_{42}(x)\Phi_{70}(x)\Phi_{126}(x)\Phi_{210}(x) \tag{20}$$

and

$$2(x^{135}+1)\Phi_{14}(x)\Phi_{42}(x)\Phi_{70}(x)\Phi_{126}(x)\Phi_{210}(x)\Phi_{378}(x)\Phi_{630}(x)$$

is practical by Lemma 2.1.

By (9)–(15), we have

$$2(x^{135}+1)\Phi_{14}(x)\Phi_{42}(x)\Phi_{70}(x)\Phi_{126}(x)\Phi_{210}(x)\Phi_{378}(x)\Phi_{630}(x) \geqslant \frac{1}{4}x^{513} \geqslant x^{511}. \tag{21}$$

By (16) and (21), we have

$$\Phi_{1890}(x) \leqslant x^{432} \leqslant x^{511} \leqslant 2(x^{135}+1)\Phi_{14}(x)\Phi_{42}(x)\Phi_{70}(x)\Phi_{126}(x)\Phi_{210}(x)\Phi_{378}(x)\Phi_{630}(x) \tag{22}$$

and

$$2(x^{135}+1)\Phi_{14}(x)\Phi_{42}(x)\Phi_{70}(x)\Phi_{126}(x)\Phi_{210}(x)\Phi_{378}(x)\Phi_{630}(x)\Phi_{1890}(x)$$

is practical by Lemma 2.1.

Since m_0 is practical, we can show that m_k is practical for every $k = 0, 1, 2, \ldots$ by an inductive process. Now the proof of sequence $2^{135\times 7^k+1}+2$ is completed.

2.2. **Sequence 2.** We shall prove that $2^{165\times 3^k+1}+2$ are practical for every $k\geqslant 0$.

Write $m_k=2^{165\times 3^k+1}+2$ for $k=0,1,2,\ldots$, then we only need to show that m_k is practical for every $k=0,1,2,\ldots$. By a computer calculation we find that

$$\begin{aligned} m_0 &= 93536104789177786765035829293842113257979682750466 \\ &= 2\times 3^2\times 11^2\times 67\times 331\times 683\times 2971\times 20857\times 48912491 \\ &\quad \times 415365721\times 2252127523412251. \end{aligned}$$

By Lemma 2.1, we know that m_0 is practical. Let $x=2^{3^k}$, then we have $m_k=2(x^{165}+1)$ and $m_{k+1}=2(x^{495}+1)$. Since $k\geqslant 0$, we have $x\geqslant 2$. By Lemma 2.2, we have

$$\frac{x^{990}+1}{x^{495}+1}=\frac{x^{330}+1}{x^{165}+1}\Phi_{18}(x)\Phi_{90}(x)\Phi_{198}(x)\Phi_{990}(x). \tag{23}$$

Assume that $2(x^{165}+1)$ is practical, then we only need to show that

$$2(x^{165}+1)\Phi_{18}(x)\Phi_{90}(x)\Phi_{198}(x)\Phi_{990}(x)$$

is practical.

Note that

$$\begin{aligned} \Phi_{18}(x) &= x^6-x^3+1, \\ \Phi_{90}(x) &= x^{24}+x^{21}-x^{15}-x^{12}-x^9+x^3+1, \\ \Phi_{198}(x) &= x^{60}+x^{57}-x^{51}-x^{48}+x^{42}+x^{39}-x^{33} \\ &\quad -x^{30}-x^{27}+x^{21}+x^{18}-x^{12}-x^9+x^3+1, \\ \Phi_{990}(x) &= x^{240}-x^{237}+x^{234}+x^{225}-x^{222}+x^{219} \\ &\quad +x^{207}-x^{204}+x^{201}-x^{195}+2x^{192}-2x^{189} \\ &\quad +x^{186}-x^{180}+x^{177}-x^{174}-x^{162}+x^{159} \\ &\quad -x^{156}+x^{150}-2x^{147}+2x^{144}-2x^{141}+x^{138} \\ &\quad -x^{132}+x^{129}-x^{126}+x^{123}-x^{120}+x^{117} \\ &\quad -x^{114}+x^{111}-x^{108}+x^{102}-2x^{99}+2x^{96} \\ &\quad -2x^{93}+x^{90}-x^{84}+x^{81}-x^{78}-x^{66}+x^{63} \\ &\quad -x^{60}+x^{54}-2x^{51}+2x^{48}-x^{45}+x^{39}-x^{36} \\ &\quad +x^{33}+x^{21}-x^{18}+x^{15}+x^6-x^3+1. \end{aligned}$$

Since $x\geqslant 2$, we have

$$\frac{1}{2}x^6\leqslant \Phi_{18}(x)\leqslant x^6, \tag{24}$$

$$x^{24}\leqslant \Phi_{90}(x)\leqslant 2x^{24}, \tag{25}$$

$$x^{60}\leqslant \Phi_{198}(x)\leqslant 2x^{60}, \tag{26}$$

$$\frac{1}{2}x^{240}\leqslant \Phi_{990}(x)\leqslant x^{240}. \tag{27}$$

By (24)–(26), we have

$$\Phi_{18}(x)\Phi_{90}(x)\Phi_{198}(x)\leqslant 4x^{90}\leqslant x^{92}\leqslant 2(x^{165}+1) \tag{28}$$

and

$$2(x^{165}+1)\Phi_{18}(x)\Phi_{90}(x)\Phi_{198}(x)$$

is practical by Lemma 2.1.

By (24)–(26), we have

$$2(x^{165}+1)\Phi_{18}(x)\Phi_{90}(x)\Phi_{198}(x) \geqslant x^{255}. \tag{29}$$

By (27) and (29), we have

$$\Phi_{990}(x) \leqslant x^{240} \leqslant x^{255} \leqslant 2(x^{165}+1)\Phi_{18}(x)\Phi_{90}(x)\Phi_{198}(x) \tag{30}$$

and

$$2(x^{165}+1)\Phi_{18}(x)\Phi_{90}(x)\Phi_{198}(x)\Phi_{990}(x)$$

is practical by Lemma 2.1.

Since m_0 is practical, we can show that m_k is practical for every $k = 0, 1, 2, \ldots$ by an inductive process. Now the proof of sequence $2^{165\times 3^k+1}+2$ is completed.

2.3. **Sequence 3.** We shall prove that $2^{175\times 3^k+1}+2$ are practical for every $k \geqslant 0$.

Write $m_k = 2^{175\times 3^k+1}+2$ for $k = 0, 1, 2, \ldots$, then we only need to show that m_k is practical for every $k = 0, 1, 2, \ldots$. By a computer calculation we find that

$$\begin{aligned} m_0 &= 95780971304118053647396689196894323976171195136475138 \\ &= 2 \times 3 \times 11 \times 43 \times 251 \times 281 \times 1051 \times 4051 \times 86171 \times 110251 \\ &\quad \times 347833278451 \times 34010032331525251. \end{aligned}$$

By Lemma 2.1, we know that m_0 is practical. Let $x = 2^{3^k}$, then we have $m_k = 2(x^{175}+1)$ and $m_{k+1} = 2(x^{525}+1)$. Since $k \geqslant 0$, we have $x \geqslant 2$. By Lemma 2.2, we have

$$\frac{x^{1050}+1}{x^{525}+1} = \frac{x^{350}+1}{x^{175}+1}\Phi_6(x)\Phi_{30}(x)\Phi_{42}(x)\Phi_{150}(x)\Phi_{210}(x)\Phi_{1050}(x). \tag{31}$$

Assume that $2(x^{175}+1)$ is practical, then we only need to show that

$$2(x^{175}+1)\Phi_6(x)\Phi_{30}(x)\Phi_{42}(x)\Phi_{150}(x)\Phi_{210}(x)\Phi_{1050}(x)$$

is practical.

Note that

$$\begin{aligned} \Phi_6(x) &= x^2 - x + 1, \\ \Phi_{30}(x) &= x^8 + x^7 - x^5 - x^4 - x^3 + x + 1, \\ \Phi_{42}(x) &= x^{12} + x^{11} - x^9 - x^8 + x^6 - x^4 - x^3 + x + 1, \\ \Phi_{150}(x) &= x^{40} + x^{35} - x^{25} - x^{20} - x^{15} + x^5 + 1, \\ \Phi_{210}(x) &= x^{48} - x^{47} + x^{46} + x^{43} - x^{42} + 2x^{41} - x^{40} \\ &\quad + x^{39} + x^{36} - x^{35} + x^{34} - x^{33} + x^{32} - x^{31} \\ &\quad - x^{28} - x^{26} - x^{24} - x^{22} - x^{20} - x^{17} + x^{16} \\ &\quad - x^{15} + x^{14} - x^{13} + x^{12} + x^9 - x^8 \\ &\quad + 2x^7 - x^6 + x^5 + x^2 - x + 1, \\ \Phi_{1050}(x) &= x^{240} - x^{235} + x^{230} + x^{215} - x^{210} + 2x^{205} \end{aligned}$$

$$
\begin{aligned}
&- x^{200} + x^{195} + x^{180} - x^{175} + x^{170} - x^{165} \\
&+ x^{160} - x^{155} - x^{140} - x^{130} - x^{120} - x^{110} \\
&- x^{100} - x^{85} + x^{80} - x^{75} + x^{70} - x^{65} + x^{60} \\
&+ x^{45} - x^{40} + 2x^{35} - x^{30} + x^{25} + x^{10} - x^{5} + 1.
\end{aligned}
$$

Since $x \geqslant 2$, we have

$$x \leqslant \Phi_6(x) \leqslant x^2, \tag{32}$$
$$x^8 \leqslant \Phi_{30}(x) \leqslant 2x^8, \tag{33}$$
$$x^{12} \leqslant \Phi_{42}(x) \leqslant 2x^{12}, \tag{34}$$
$$x^{40} \leqslant \Phi_{150}(x) \leqslant 2x^{40}, \tag{35}$$
$$\frac{1}{2}x^{48} \leqslant \Phi_{210}(x) \leqslant x^{48}, \tag{36}$$
$$\frac{1}{2}x^{240} \leqslant \Phi_{1050}(x) \leqslant x^{240}. \tag{37}$$

By (32)–(36), we have

$$\Phi_6(x)\Phi_{30}(x)\Phi_{42}(x)\Phi_{150}(x)\Phi_{210}(x) \leqslant 8x^{110} \leqslant x^{113} \leqslant 2(x^{175}+1) \tag{38}$$

and

$$2(x^{175}+1)\Phi_6(x)\Phi_{30}(x)\Phi_{42}(x)\Phi_{150}(x)\Phi_{210}(x)$$

is practical by Lemma 2.1.

By (32)–(36), we have

$$2(x^{175}+1)\Phi_6(x)\Phi_{30}(x)\Phi_{42}(x)\Phi_{150}(x)\Phi_{210}(x) \geqslant x^{284}. \tag{39}$$

By (37) and (39), we have

$$\Phi_{1050}(x) \leqslant x^{240} \leqslant x^{284} \leqslant 2(x^{175}+1)\Phi_6(x)\Phi_{30}(x)\Phi_{42}(x)\Phi_{150}(x)\Phi_{210}(x) \tag{40}$$

and

$$2(x^{175}+1)\Phi_6(x)\Phi_{30}(x)\Phi_{42}(x)\Phi_{150}(x)\Phi_{210}(x)\Phi_{1050}(x)$$

is practical by Lemma 2.1.

Since m_0 is practical, we can show that m_k is practical for every $k = 0, 1, 2, \dots$ by an inductive process. Now the proof of sequence $2^{175\times 3^k+1}+2$ is completed.

2.4. **Sequence 4.** We shall prove that $2^{189\times 5^k+1}+2$ are practical for every $k \geqslant 0$.

Write $m_k = 2^{189\times 5^k+1}+2$ for $k = 0, 1, 2, \dots$, then we only need to show that m_k is practical for every $k = 0, 1, 2, \dots$. By a computer calculation we find that

$$
\begin{aligned}
m_0 &= 1569275433846670190958947355801916604025588861116008628226 \\
&= 2 \times 3^4 \times 19 \times 43 \times 379 \times 5419 \times 87211 \times 119827 \times 77158673929 \\
&\quad \times 127391413339 \times 56202143607667.
\end{aligned}
$$

By Lemma 2.1, we know that m_0 is practical. Let $x = 2^{5^k}$, then we have $m_k = 2(x^{189}+1)$ and $m_{k+1} = 2(x^{945}+1)$. Since $k \geqslant 0$, we have $x \geqslant 2$. By Lemma 2.2, we have

$$\frac{x^{1890}+1}{x^{945}+1} = \frac{x^{378}+1}{x^{189}+1}\Phi_{10}(x)\Phi_{30}(x)\Phi_{70}(x)\Phi_{90}(x)\Phi_{210}(x)\Phi_{270}(x)\Phi_{630}(x)\Phi_{1890}(x). \tag{41}$$

Assume that $2(x^{189}+1)$ is practical, then we only need to show that

$$2(x^{189}+1)\Phi_{10}(x)\Phi_{30}(x)\Phi_{70}(x)\Phi_{90}(x)\Phi_{210}(x)\Phi_{270}(x)\Phi_{630}(x)\Phi_{1890}(x)$$

is practical.

Note that

$$\begin{aligned}
\Phi_{10}(x) &= x^4 - x^3 + x^2 - x + 1,\\
\Phi_{30}(x) &= x^8 + x^7 - x^5 - x^4 - x^3 + x + 1,\\
\Phi_{70}(x) &= x^{24} + x^{23} - x^{19} - x^{18} - x^{17} - x^{16} + x^{14} + x^{13}\\
&\quad + x^{12} + x^{11} + x^{10} - x^8 - x^7 - x^6 - x^5 + x + 1,\\
\Phi_{90}(x) &= x^{24} + x^{21} - x^{15} - x^{12} - x^9 + x^3 + 1,\\
\Phi_{210}(x) &= x^{48} - x^{47} + x^{46} + x^{43} - x^{42} + 2x^{41} - x^{40}\\
&\quad + x^{39} + x^{36} - x^{35} + x^{34} - x^{33} + x^{32} - x^{31}\\
&\quad - x^{28} - x^{26} - x^{24} - x^{22} - x^{20} - x^{17} + x^{16}\\
&\quad - x^{15} + x^{14} - x^{13} + x^{12} + x^9 - x^8\\
&\quad + 2x^7 - x^6 + x^5 + x^2 - x + 1,\\
\Phi_{270}(x) &= x^{72} + x^{63} - x^{45} - x^{36} - x^{27} + x^9 + 1,\\
\Phi_{630}(x) &= x^{144} - x^{141} + x^{138} + x^{129} - x^{126} + 2x^{123} - x^{120}\\
&\quad + x^{117} + x^{108} - x^{105} + x^{102} - x^{99} + x^{96} - x^{93} - x^{84}\\
&\quad - x^{78} - x^{72} - x^{66} - x^{60} - x^{51} + x^{48} - x^{45} + x^{42}\\
&\quad - x^{39} + x^{36} + x^{27} - x^{24} + 2x^{21} - x^{18} + x^{15} + x^6 - x^3 + 1,\\
\Phi_{1890}(x) &= x^{432} - x^{423} + x^{414} + x^{387} - x^{378} + 2x^{369} - x^{360} + x^{351}\\
&\quad + x^{324} - x^{315} + x^{306} - x^{297} + x^{288} - x^{279} - x^{252} - x^{234}\\
&\quad - x^{216} - x^{198} - x^{180} - x^{153} + x^{144} - x^{135} + x^{126} - x^{117}\\
&\quad + x^{108} + x^{81} - x^{72} + 2x^{63} - x^{54} + x^{45} + x^{18} - x^9 + 1.
\end{aligned}$$

Since $x \geqslant 2$, we have

$$\frac{1}{2}x^4 \leqslant \Phi_{10}(x) \leqslant x^4, \tag{42}$$

$$x^8 \leqslant \Phi_{30}(x) \leqslant 2x^8, \tag{43}$$

$$x^{24} \leqslant \Phi_{70}(x) \leqslant 2x^{24}, \tag{44}$$

$$x^{24} \leqslant \Phi_{90}(x) \leqslant 2x^{24}, \tag{45}$$

$$\frac{1}{2}x^{48} \leqslant \Phi_{210}(x) \leqslant x^{48}, \tag{46}$$

$$x^{72} \leqslant \Phi_{270}(x) \leqslant 2x^{72}, \tag{47}$$

$$\frac{1}{2}x^{144} \leqslant \Phi_{630}(x) \leqslant x^{144}, \tag{48}$$

$$\frac{1}{2}x^{432} \leqslant \Phi_{1890}(x) \leqslant x^{432}. \tag{49}$$

By (42)–(47), we have

$$\Phi_{10}(x)\Phi_{30}(x)\Phi_{70}(x)\Phi_{90}(x)\Phi_{210}(x)\Phi_{270}(x) \leqslant 16x^{180} \leqslant x^{184} \leqslant 2(x^{189}+1) \tag{50}$$

and

$$2(x^{189}+1)\Phi_{10}(x)\Phi_{30}(x)\Phi_{70}(x)\Phi_{90}(x)\Phi_{210}(x)\Phi_{270}(x)$$

is practical by Lemma 2.1.

By (42)–(47), we have

$$2(x^{189}+1)\Phi_{10}(x)\Phi_{30}(x)\Phi_{70}(x)\Phi_{90}(x)\Phi_{210}(x)\Phi_{270}(x) \geqslant \frac{1}{2}x^{369} \geqslant x^{368}. \tag{51}$$

By (48) and (51), we have

$$\Phi_{630}(x) \leqslant x^{144} \leqslant x^{368} \leqslant 2(x^{189}+1)\Phi_{10}(x)\Phi_{30}(x)\Phi_{70}(x)\Phi_{90}(x)\Phi_{210}(x)\Phi_{270}(x) \tag{52}$$

and

$$2(x^{189}+1)\Phi_{10}(x)\Phi_{30}(x)\Phi_{70}(x)\Phi_{90}(x)\Phi_{210}(x)\Phi_{270}(x)\Phi_{630}(x)$$

is practical by Lemma 2.1.

By (42)–(48), we have

$$2(x^{189}+1)\Phi_{10}(x)\Phi_{30}(x)\Phi_{70}(x)\Phi_{90}(x)\Phi_{210}(x)\Phi_{270}(x)\Phi_{630}(x) \geqslant \frac{1}{4}x^{513} \geqslant x^{511}. \tag{53}$$

By (49) and (53), we have

$$\Phi_{1890}(x) \leqslant x^{432} \leqslant x^{511} \leqslant 2(x^{189}+1)\Phi_{10}(x)\Phi_{30}(x)\Phi_{70}(x)\Phi_{90}(x)\Phi_{210}(x)\Phi_{270}(x)\Phi_{630}(x) \tag{54}$$

and

$$2(x^{189}+1)\Phi_{10}(x)\Phi_{30}(x)\Phi_{70}(x)\Phi_{90}(x)\Phi_{210}(x)\Phi_{270}(x)\Phi_{630}(x)\Phi_{1890}(x)$$

is practical by Lemma 2.1.

Since m_0 is practical, we can show that m_k is practical for every $k = 0, 1, 2, \ldots$ by an inductive process. Now the proof of sequence $2^{189\times 5^k+1} + 2$ is completed.

Finally, we get Theorem 1.2 by combining the above four cases.

3. Proof of Theorems 1.3, 1.4 and 1.5

In order to prove Theorem 1.3, we need to show that at least one of the following five sequences generate infinitely many multiples of p for all prime $7 \leqslant p < 100$:

$$35 \times 3^k + 1,\ 135 \times 7^k + 1,\ 165 \times 3^k + 1,\ 175 \times 3^k + 1,\ 189 \times 5^k + 1.$$

By the following identities (where m is any positive integer)

$$\begin{aligned}
165 \times 3^{5+6m} + 1 &\equiv 0(\mathrm{mod} 7),\\
35 \times 3^{3+10m} + 1 &\equiv 0(\mathrm{mod} 11),\\
135 \times 7^{3+12m} + 1 &\equiv 0(\mathrm{mod} 13),\\
35 \times 3^{8+16m} + 1 &\equiv 0(\mathrm{mod} 17),\\
35 \times 3^{17+18m} + 1 &\equiv 0(\mathrm{mod} 19),\\
135 \times 7^{20+22m} + 1 &\equiv 0(\mathrm{mod} 23),
\end{aligned}$$

$$
\begin{aligned}
35 \times 3^{24+28m} + 1 &\equiv 0 (\bmod 29),\\
35 \times 3^{27+30m} + 1 &\equiv 0 (\bmod 31),\\
189 \times 5^{32+36m} + 1 &\equiv 0 (\bmod 37),\\
135 \times 7^{7+40m} + 1 &\equiv 0 (\bmod 41),\\
35 \times 3^{3+42m} + 1 &\equiv 0 (\bmod 43),\\
35 \times 3^{11+46m} + 1 &\equiv 0 (\bmod 47),\\
35 \times 3^{1+52m} + 1 &\equiv 0 (\bmod 53),\\
165 \times 3^{21+58m} + 1 &\equiv 0 (\bmod 59),\\
135 \times 7^{50+60m} + 1 &\equiv 0 (\bmod 61),\\
135 \times 7^{33+66m} + 1 &\equiv 0 (\bmod 67),\\
35 \times 3^{11+70m} + 1 &\equiv 0 (\bmod 71),\\
189 \times 5^{57+72m} + 1 &\equiv 0 (\bmod 73),\\
35 \times 3^{2+78m} + 1 &\equiv 0 (\bmod 79),\\
35 \times 3^{24+82m} + 1 &\equiv 0 (\bmod 83),\\
35 \times 3^{69+88m} + 1 &\equiv 0 (\bmod 89),\\
35 \times 3^{40+96m} + 1 &\equiv 0 (\bmod 97),
\end{aligned}
$$

we complete the proof of Theorem 1.3. By combining those identities, we can also show that Conjecture 1.1 holds true for many other integers.

We can prove Theorem 1.4 by taking the sequences

$$4 \times \left(2^{35\times 3^k+1} + 2\right) = 2^{35\times 3^k+3} + 8$$

and

$$16 \times \left(2^{35\times 3^k+1} + 2\right) = 2^{35\times 3^k+5} + 32.$$

Similarly, for Theorem 1.5 we can take the sequence

$$2^{h-1} \times \left(2^{35\times 3^k+1} + 2\right) = 2^{35\times 3^k+h} + 2^h$$

with a suitable h such that $35 \times 3^k + h$ is a multiple of C.

References

[1] M. Hausman and H. N. Shapiro. On practical numbers. *Comm. Pure Appl. Math.*, 37:705–713, 1984.

[2] M. Margenstern. Les nombres pratiques: théorie, observations et conjectures. *J. Number Theory*, 37:1–36, 1991.

[3] G. Melfi. On two conjectures about practical numbers. *J. Number Theory*, 56:205–210, 1996.

[4] G. Melfi. A note on twin practical numbers. *Le Matematiche*, 57:111–117, 2002.

[5] E. Saias. Entiers à diviseurs denses 1. *J. Number Theory*, 62:163–191, 1997.

[6] G. Tenenbaum. Sur un problème de crible et ses applications. *Ann. Sci. École Norm. Sup.*, 19(4):1–30, 1986.

[7] G. Tenenbaum. Sur un problème de crible et ses applications, 2. corrigendum et étude du graphe divisoriel. *Ann. Sci. École Norm. Sup.*, 28(4):115–127, 1995.

[8] L.-Y. Wang and Z.-W. Sun. On practical numbers of some special forms. *Houston J. Math.*, 48(2):241–247, 2022.

International Curriculum Center, The High School Affiliated to Renmin University of China, Beijing, China

Email address: carey.lee.0433@gmail.com