SUMS OF TWO PRIME SQUARES IN SHORT INTERVALS

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Abstract. Let N be a sufficiently large integer. It is proved that the inequality

$$\left|N-p^2-q^2\right|<H$$

is solvable in primes p,q providing $H>N^{\frac{151}{320}+\varepsilon}$. This improves previous results of Naumenko.

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1. Introduction

An important topic in the study of a sparse set of integers is the study of the gaps between consecutive elements. Suppose that the set we want to study contains and only contains positive integers that are expressible as the sum of two squares of integers. Let a,b denote integers and N is sufficiently large. Then, we want to show the inequality

$$\left|N-a^2-b^2\right|<H \tag{1}$$

is solvable for some $H \ll N$. A classical and elementary result is $H \gg N^{\frac{1}{4}}$: Let $f(N)=N-\left[\sqrt{N}\right]^2$ where $[x]$ denote the integer part of x. Then we know that $f(x)$ is an integer for an integer x,

$$0 \leqslant f(N) \ll N^{\frac{1}{2}}, \tag{2}$$

and thus

$$0 \leqslant N-(N-f(N))-(f(N)-f(f(N)))=f(f(N)) \ll N^{\frac{1}{4}}. \tag{3}$$

Since $x-f(x)=[x]^2$ is a square for $x=N$ and $x=f(N)$, taking $H \gg N^{\frac{1}{4}}$ ensures at least one pair of acceptable (a,b).

If we consider positive integers that are expressible as the sum of two squares of *primes* instead of *integers*, then the problem becomes much more difficult. In 2012, Gritsenko and Cha [2] first obtained a result of this type: if $H>N^{\frac{1}{2}}\exp\left(-(\log N)^{0.1}\right)$, then the inequality

$$\left|N-p^2-q^2\right|<H \tag{4}$$

is solvable in primes p,q. In 2018, Naumenko [6] improved this result by showing (4) is solvable in primes p,q providing only $H>N^{\frac{31}{64}+\varepsilon}$. In 2019 he [7] further improved this to $H>N^{\frac{2309}{4800}+\varepsilon}$.

In Naumenko's 2019 paper [7], two important tools were used in his proof: the result of Baker, Harman and Pintz [1] on primes in short intervals, and the zero-density estimates proved by Ivić [3]. In 2025, the former result was improved by the author [5], while the latter result was refined by Tao, Trudgian and Yang [8] in their ANTEDB project. Thus, using both improved results leads to a better answer to this problem. In the present paper, we shall prove that

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Theorem 1.1. *(4) is solvable in primes p, q providing $H > N^{\frac{151}{320}+\varepsilon}$.*

Note that
$$\frac{2309}{4800} = \frac{31}{64} - \frac{1}{300} \qquad \text{and} \qquad \frac{151}{320} = \frac{31}{64} - \frac{1}{80}.$$

2. Preliminary lemmas

In this section we shall list several lemmas that will be used in the next section. The first lemma is the result regarding primes in short intervals, proved by the author [5] in 2025.

Lemma 2.1. *([5], Theorem 1]). For $\frac{13}{25} \leqslant \theta < 1$ and sufficiently large n, the interval $\left[n - n^{\theta}, n\right]$ contains $\gg \frac{n^{\theta}}{\log n}$ prime numbers.*

The second lemma is the "current best" zero-density estimate (except for the last interval) in the ANTEDB project [8].

Lemma 2.2. *([8], Tables 11.1 and 11.2]). For sufficiently large T, we have*
$$N(\sigma, T) \ll \begin{cases} T^{\frac{3(1-\sigma)}{2-\sigma}+\varepsilon}, & \frac{1}{2} \leqslant \sigma < \frac{7}{10}, \\ T^{\frac{15(1-\sigma)}{5\sigma+3}+\varepsilon}, & \frac{7}{10} \leqslant \sigma < \frac{19}{25}, \\ T^{\frac{9(1-\sigma)}{8\sigma-2}+\varepsilon}, & \frac{19}{25} \leqslant \sigma < \frac{127}{167}, \\ T^{\frac{15(1-\sigma)}{13\sigma-3}+\varepsilon}, & \frac{127}{167} \leqslant \sigma < \frac{13}{17}, \\ T^{\frac{6(1-\sigma)}{5\sigma-1}+\varepsilon}, & \frac{13}{17} \leqslant \sigma < \frac{17}{22}, \\ T^{\frac{2(1-\sigma)}{9\sigma-6}+\varepsilon}, & \frac{17}{22} \leqslant \sigma < \frac{41}{53}, \\ T^{\frac{9(1-\sigma)}{7\sigma-1}+\varepsilon}, & \frac{41}{53} \leqslant \sigma < \frac{7}{9}, \\ T^{\frac{9(1-\sigma)}{16\sigma-8}+\varepsilon}, & \frac{7}{9} \leqslant \sigma < \frac{1867}{2347}, \\ T^{\frac{3(1-\sigma)}{2\sigma}+\varepsilon}, & \frac{1867}{2347} \leqslant \sigma < \frac{7}{8}, \\ T^{\frac{3(1-\sigma)}{10\sigma-7}+\varepsilon}, & \frac{7}{8} \leqslant \sigma < \frac{279}{314}, \\ T^{\frac{24(1-\sigma)}{30\sigma-11}+\varepsilon}, & \frac{279}{314} \leqslant \sigma < \frac{9}{10}, \\ T^{\frac{3(1-\sigma)}{10\sigma-7}+\varepsilon}, & \frac{9}{10} \leqslant \sigma < \frac{31}{34}, \\ T^{\frac{11(1-\sigma)}{48\sigma-36}+\varepsilon}, & \frac{31}{34} \leqslant \sigma < \frac{14}{15}, \\ T^{\frac{391(1-\sigma)}{2493\sigma-2014}+\varepsilon}, & \frac{14}{15} \leqslant \sigma < \frac{2841}{3016}, \\ T^{\frac{22232(1-\sigma)}{163248\sigma-134765}+\varepsilon}, & \frac{2841}{3016} \leqslant \sigma < \frac{859}{908}, \\ T^{\frac{356(1-\sigma)}{2742\sigma-2279}+\varepsilon}, & \frac{859}{908} \leqslant \sigma < \frac{23}{24}, \\ T^{\frac{3(1-\sigma)}{24\sigma-20}+\varepsilon}, & \frac{23}{24} \leqslant \sigma < \frac{2211487}{2274732}, \\ T^{\frac{86152(1-\sigma)}{1447460\sigma-1311509}+\varepsilon}, & \frac{2211487}{2274732} \leqslant \sigma < \frac{39}{40}, \\ T^{\frac{2(1-\sigma)}{15\sigma-12}+\varepsilon}, & \frac{39}{40} \leqslant \sigma < \frac{41}{42}, \\ T^{\frac{3(1-\sigma)}{40\sigma-35}+\varepsilon}, & \frac{41}{42} \leqslant \sigma < \frac{59}{60}, \\ T^{6.42(1-\sigma)^{3/2}+\varepsilon}, & \frac{59}{60} \leqslant \sigma \leqslant 1. \end{cases}$$
Moreover, write
$$N(\sigma, T) \ll T^{(1-\sigma)A(\sigma)+\varepsilon}$$
for $\frac{1}{2} \leqslant \sigma \leqslant 1$, where $A(\sigma)$ is the piecewise function defined by the above bounds.

The last two lemmas come from the book of Karatsuba [4].

Lemma 2.3. *([4], Page 58, Corollary 1]). The number of zeros $\rho = \beta + i\gamma$ of the ζ-function that satisfy $T \leqslant |\gamma| \leqslant T + 1$ does not exceed $c \log T$.*

Lemma 2.4. *([4], Page 58, Corollary 2]). For $T \geqslant 2$, we have*
$$\sum_{|T-\gamma|>1} \frac{1}{|T-\gamma|^2} \ll \log T.$$

3. Proof of Theorem 1.1

Let $N_1 = N^{\frac{19}{25}+\varepsilon}$ and $N^{\frac{151}{320}+\varepsilon} < H < \frac{N^{\frac{1}{2}}}{2}$. Consider the sum

$$S = \sum_{\substack{N-2N_1<p^2\leqslant N-N_1\\ \sqrt{N-p^2-H}<k\leqslant\sqrt{N-p^2+H}}} \Lambda(k).$$

Note that S counts not only primes q but also prime powers q^r satisfy (4). The contribution of those prime powers to S can be simply bounded by

$$U = \frac{HN_1^{\frac{1}{4}+\varepsilon}}{N^{\frac{1}{2}}}. \tag{5}$$

Clearly Theorem 1.1 holds if $S \gg U\log N$. In this paper we want to show that

$$S \gg \frac{HN_1^{\frac{1}{2}}}{N^{\frac{1}{2}}\log N}. \tag{6}$$

Because of the conditions in the summation S, we know that $N-p^2 \asymp N_1$. By the Explicit Formula, we know that

$$\begin{aligned} S = &\sum_{N-2N_1<p^2\leqslant N-N_1} \left(\sqrt{N-p^2+H}-\sqrt{N-p^2-H}\right)\\ &-\sum_{N-2N_1<p^2\leqslant N-N_1}\left(\sum_{\substack{\rho=\beta+i\gamma\\ |\gamma|<T}}\int_{\sqrt{N-p^2-H}}^{\sqrt{N-p^2+H}} x^{\rho-1}dx + O\left(\frac{N_1^{\frac{1}{2}}(\log N)^2}{T}\right)\right)\\ = &\, S_1 - S_2. \end{aligned} \tag{7}$$

We can choose

$$T = \frac{N_1(\log N)^3}{H}. \tag{8}$$

For S_1, by Lemma 2.1 we know that the interval $\left[\sqrt{N-2N_1}, \sqrt{N-N_1}\right]$ contains $\gg \frac{N_1}{N^{\frac{1}{2}}\log N}$ prime numbers. Thus,

$$S_1 = \sum_{N-2N_1<p^2\leqslant N-N_1} \left(\sqrt{N-p^2+H}-\sqrt{N-p^2-H}\right) \gg \frac{HN_1^{\frac{1}{2}}}{N^{\frac{1}{2}}\log N}. \tag{9}$$

Now we only need to show that

$$S_2 \ll \frac{HN_1^{\frac{1}{2}}}{N^{\frac{1}{2}}(\log N)^2}. \tag{10}$$

We split the sum S_2 into two parts:

$$\begin{aligned} S_2 = &\sum_{N-2N_1<p^2\leqslant N-N_1}\left(\sum_{\substack{\rho=\beta+i\gamma\\ |\gamma|<T}}\int_{\sqrt{N-p^2-H}}^{\sqrt{N-p^2+H}} x^{\rho-1}dx + O\left(\frac{N_1^{\frac{1}{2}}(\log N)^2}{T}\right)\right)\\ \leqslant &\sum_{N-2N_1<n^2\leqslant N-N_1}\left|\sum_{\substack{\rho=\beta+i\gamma\\ |\gamma|<T}}\int_{\sqrt{N-n^2-H}}^{\sqrt{N-n^2+H}} x^{\rho-1}dx\right|\\ = &\sum_{N-2N_1<n^2\leqslant N-N_1}\left|\sum_{\substack{\rho=\beta+i\gamma\\ 0.92376\leqslant\beta\leqslant 1\\ |\gamma|<T}}\int_{\sqrt{N-n^2-H}}^{\sqrt{N-n^2+H}} x^{\rho-1}dx\right| \end{aligned}$$

$$
\begin{aligned}
&+\sum_{N-2N_1<n^2\leqslant N-N_1}\left|\sum_{\substack{\rho=\beta+i\gamma\\ \frac{1}{2}\leqslant\beta<0.92376\\ |\gamma|<T}}\int_{\sqrt{N-n^2-H}}^{\sqrt{N-n^2+H}}x^{\rho-1}dx\right|\\
&=S_{21}+S_{22}. \qquad (11)
\end{aligned}
$$

For S_{21} we have

$$
\begin{aligned}
S_{21}=&\sum_{N-2N_1<n^2\leqslant N-N_1}\left|\sum_{\substack{\rho=\beta+i\gamma\\ 0.92376\leqslant\beta\leqslant1\\ |\gamma|<T}}\int_{\sqrt{N-n^2-H}}^{\sqrt{N-n^2+H}}x^{\rho-1}dx\right|\\
\ll&\sum_{N-2N_1<n^2\leqslant N-N_1}\sum_{\substack{\rho=\beta+i\gamma\\ 0.92376\leqslant\beta\leqslant1\\ |\gamma|<T}}\int_{\sqrt{N-n^2-H}}^{\sqrt{N-n^2+H}}x^{\beta-1}dx\\
\ll&\frac{HN_1^{\frac{1}{2}}\log N}{N^{\frac{1}{2}}}\max_{0.92376\leqslant\sigma\leqslant1}\left(N_1^{\frac{\sigma-1}{2}}N(\sigma,T)\right). \qquad (12)
\end{aligned}
$$

Then we only need to show that

$$
N_1^{\frac{\sigma-1}{2}}N(\sigma,T)\ll(\log N)^{-3} \qquad (13)
$$

for any $0.92736\leqslant\sigma\leqslant1$. We divide this range of σ into 3 subranges:

1. $1-\delta(T)<\sigma\leqslant1$, where $\delta(T)=\frac{c}{(\log|T|)^{2/3}(\log\log|T|)^{1/3}}$: By the Vinogradov–Korobov zero-free region, we know that $N(\sigma,T)=0$ in this case.

2. $\frac{59}{60}\leqslant\sigma\leqslant1-\delta(T)$: By Lemma 2.2, for sufficiently large T and $\frac{59}{60}\leqslant\sigma\leqslant1$, we have

$$
N(\sigma,T)\ll T^{6.42(1-\sigma)^{3/2}+\varepsilon}. \qquad (14)
$$

Since $N_1=N^{\frac{19}{25}+\varepsilon}$, $N^{\frac{151}{320}+\varepsilon}<H<\frac{N^{\frac{1}{2}}}{2}$ and $T=N_1H^{-1}(\log N)^3$, we have

$$
N^{0.26+\varepsilon}(\log N)^3<T<N^{0.288125+\varepsilon}(\log N)^3 \qquad (15)
$$

Now we only need to show that

$$
\frac{19(\sigma-1)}{50}+0.288125(6.42(1-\sigma)^{3/2})<0. \qquad (16)
$$

Numerical calculations show that (16) holds for $\frac{59}{60}\leqslant\sigma\leqslant1-\delta(T)$.

3. $0.92736\leqslant\sigma<\frac{59}{60}$: Using Lemma 2.2 and similar arguments as above, we only need to check that whether

$$
\frac{461}{1216}A(\sigma)<\frac{1}{2} \qquad (17)
$$

holds for $0.92736\leqslant\sigma<\frac{59}{60}$. Numerical calculations show that (17) holds for all σ in this range.

For S_{22}, by Cauchy–Schwarz inequality we get

$$
S_{22}^2=\left(\sum_{N-2N_1<n^2\leqslant N-N_1}\left|\sum_{\substack{\rho=\beta+i\gamma\\ \frac{1}{2}\leqslant\beta<0.92376\\ |\gamma|<T}}\int_{\sqrt{N-n^2-H}}^{\sqrt{N-n^2+H}}x^{\rho-1}dx\right|\right)^2\ll\frac{HN_1^{\frac{1}{2}}}{N^{\frac{1}{2}}}\int_{\frac{1}{2}N_1^{\frac{1}{2}}}^{2N_1^{\frac{1}{2}}}\left|\sum_{\substack{\rho=\beta+i\gamma\\ |\gamma|<T}}x^{\rho-1}\right|^2dx. \qquad (18)
$$

Now we divide the rectangle $\frac{1}{2} \leqslant \beta < 0.92376$, $-T < \gamma < T$ into $O(\log N)$ small rectangles $\sigma \leqslant \beta < \sigma + \frac{1}{\log N}$, $-T < \gamma < T$. Then we have

$$\begin{aligned} S_{22}^2 \ll \frac{HN_1^{\frac{1}{2}}(\log N)^2}{N^{\frac{1}{2}}} \Bigg(&\sum_{|\gamma|<T} \sum_{\substack{|\gamma_1|<T \\ |\gamma-\gamma_1|\leqslant 1}} \int_{\frac{1}{2}N_1^{\frac{1}{2}}}^{2N_1^{\frac{1}{2}}} x^{2\sigma-2} dx \\ &+ \sum_{|\gamma|<T} \sum_{\substack{|\gamma_1|<T \\ |\gamma-\gamma_1|>1}} \frac{1}{|\gamma-\gamma_1|} \int_{\frac{1}{2}N_1^{\frac{1}{2}}}^{2N_1^{\frac{1}{2}}} x^{2\sigma-2} dx \Bigg), \end{aligned} \tag{19}$$

where the summations take zeros $\rho = \beta + i\gamma$ such that $\sigma \leqslant \beta < \sigma + \frac{1}{\log N}$, where $\frac{1}{2} \leqslant \sigma < 0.92376$. Now by Lemma 2.3 and Lemma 2.4, we can bound S_{22} by

$$S_{22} \ll \frac{HN_1^{\frac{1}{2}}(\log N)^2}{N^{\frac{1}{2}}} \max_{\frac{1}{2}\leqslant\sigma<0.92376} \left(N_1^{\frac{\sigma-1}{2}} N(\sigma,T)^{\frac{1}{2}} N^{\frac{1}{4}} H^{-\frac{1}{2}} \right). \tag{20}$$

Then we only need to show that

$$N_1^{\sigma-1} N(\sigma,T) N^{\frac{1}{2}} H^{-1} \ll (\log N)^{-8} \tag{21}$$

holds for $\frac{1}{2} \leqslant \sigma < 0.92376$. For this, we only need to check whether

$$(1-\sigma)\left(\frac{461}{1216}A(\sigma)-1\right) + \frac{45}{1216} < 0 \tag{22}$$

holds for $\frac{1}{2} \leqslant \sigma < 0.92376$. Numerical calculations finish the proof of Theorem 1.1.

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