ON PRIME-PRODUCING SIEVES AND DISTRIBUTION OF $\alpha p - \beta$ MOD 1

RUNBO LI

Abstract. The author proves that there are infinitely many primes p such that $\|\alpha p - \beta\| < p^{-\frac{28}{87}}$, where α is an irrational number and β is a real number. This sharpens a result of Jia (2000) and provides a new triple $(\gamma, \theta, \nu) = (\frac{59}{87}, \frac{28}{87}, \frac{1}{29})$ that can produce special primes in Ford and Maynard's work on prime-producing sieves. Our minimum amount of Type-II information required ($\nu = \frac{1}{29}$) is less than any previous work on this topic using only traditional Type-I and Type-II information.

Contents

1. Introduction

Let α be an irrational number and $\|y\|$ denote the smallest distance from y to integers. Earlier work on this problem was done by Vinogradov [14] in 1954, who showed that for any real number β, there are infinitely many prime numbers p such that if $\tau = \frac{1}{5} - \varepsilon$, then

$$\|\alpha p - \beta\| < p^{-\tau}. \tag{1}$$

In 1977, Vaughan [13] got $\tau = \frac{1}{4} - \varepsilon$ using his identity. In 1983, Harman [3] introduced a new sieve method to this topic and got $\tau = \frac{3}{10}$. Jia [7] improved it to $\tau = \frac{4}{13}$ in 1993. In 1996, Harman [4] further improved it to $\tau = \frac{7}{22}$ by applying a new technique (the variable role-reversal) in his sieve. In 2000, Jia [9] got $\theta = \frac{9}{28}$. It is worth to mention that Balog [1] also got the same result in 1986 under the condition that $\|\alpha n\| < n^{-\frac{43}{31}-\varepsilon}$ holds for infinitely many integers n. If we only focus on the special case $\beta = 0$, then even better exponents $\frac{16}{49}$ and $\frac{1}{3} - \varepsilon$ were obtained by Heath-Brown and Jia [6] and Matomäki [12] respectively. In a personal communication, Matomäki mentioned that Maynard has got some $\tau > \frac{1}{3}$. Note that the Riemann Hypothesis implies that (1) holds for $\tau = \frac{1}{3} - \varepsilon$. In this paper, we show that (1) holds for $\tau = \frac{28}{87}$.

Theorem 1.1. *Suppose that α is an irrational number, then for any real number β, there are infinitely many prime numbers p such that*

$$\|\alpha p - \beta\| < p^{-\frac{28}{87}}.$$

A direct corollary of our Theorem 1.1 is the distribution of $p^\theta - \beta$ mod 1 for some $\theta < 1$.

Theorem 1.2. *For $\frac{31}{87} \leqslant \theta < 1$ and any real number β, there are infinitely many prime numbers p such that*

$$\|p^\theta - \beta\| < p^{-\frac{1-\theta}{2}+\varepsilon}.$$

Another corollary of our Theorem 1.1 is the following result focusing on Diophantine approximation with Gaussian primes, which improves Harman's exponent $\frac{7}{22}$ [5].

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Theorem 1.3. *Let $0 \leqslant \omega_1 < \omega_2 \leqslant 2\pi$. Then, given $\alpha \in \mathbb{C}\backslash\mathbb{Q}[i], \beta \in \mathbb{C}$, there are infinitely many Gaussian primes $\mathfrak{p}$ such that*

$$\|\alpha\mathfrak{p} - \beta\| < |\mathfrak{p}|^{-\frac{28}{87}}, \quad \omega_1 \leqslant \arg \mathfrak{p} \leqslant \omega_2.$$

In 2024, Ford and Maynard [2] considered a series of general problems on prime-producing sieves. For those sieves that only use Type-I and Type-II information inputs, they defined a triple (γ, θ, ν) and considered various values of γ, θ and ν that can produce primes. In their notation, our Theorem 1.1 implies the following result:

Theorem 1.4. *The triple $(\gamma, \theta, \nu) = (\frac{59}{87}, \frac{28}{87}, \frac{1}{29})$ can produce primes with the property $\|\alpha p - \beta\| < p^{-\frac{28}{87}}$.*

Throughout this paper, we suppose that $\frac{a}{q}$ is a convergent to the continued fraction for α and ε is a sufficiently small positive constant. The letter p, with or without subscript, is reserved for prime numbers. Let $\tau = \frac{28}{87}$, $x = q^{\frac{2}{1+\tau}}$ and $\delta = (2x)^{-\tau}$.

2. Asymptotic formulas

Now we follow the discussion in [9]. Let $p_j = x^{t_j}$ and put

$$\mathcal{B} = \{n : x < n \leqslant 2x\}, \quad \mathcal{A} = \{n : x < n \leqslant 2x, \ \|\alpha n - \beta\| < \delta\},$$

$$\mathcal{A}_d = \{a : ad \in \mathcal{A}\}, \quad P(z) = \prod_{p<z} p, \quad S(\mathcal{A}, z) = \sum_{\substack{a \in \mathcal{A} \\ (a, P(z))=1}} 1.$$

Then we only need to show that $S\left(\mathcal{A}, (2x)^{\frac{1}{2}}\right) > 0$. Our aim is to show that the sparser set $\mathcal{A}$ contains the expected proportion of primes compared to the bigger set $\mathcal{B}$, which requires us to decompose $S\left(\mathcal{A}, (2v)^{\frac{1}{2}}\right)$ and prove asymptotic formulas of the form

$$S(\mathcal{A}, z) = 2\delta(1 + o(1)) S(\mathcal{B}, z) \tag{2}$$

for some parts of it, and drop the other positive parts.

Here are some known asymptotic formulas for types of sieve functions, which were first proved by Harman [3] using traditional Type-I and Type-II arithmetic information.

Lemma 2.1. *(Type-I) Suppose that $M \ll x^{\frac{59}{87}}$ and $a(m) = O(1)$. Then we have*

$$\sum_{m \sim M} a(m) S\left(\mathcal{A}_m, x^{\frac{1}{29}}\right) = 2\delta(1 + o(1)) \sum_{m \sim M} a(m) S\left(\mathcal{B}_m, x^{\frac{1}{29}}\right).$$

Proof. The proof is similar to that of [[3], Lemma 7]. □

Lemma 2.2. *(Type-II) Suppose that $x^{\frac{28}{87}} \ll M \ll x^{\frac{31}{87}}$ or $x^{\frac{56}{87}} \ll M \ll x^{\frac{59}{87}}$ and that $a(m), b(n) = O(1)$. Then we have*

$$\sum_{m \sim M} \sum_{n} a(m) b(n) S(\mathcal{A}_{mn}, v(m, n)) = 2\delta(1 + o(1)) \sum_{m \sim M} \sum_{n} a(m) b(n) S(\mathcal{B}_{mn}, v(m, n)).$$

Proof. The proof is similar to that of [[3], Lemma 6]. □

Many authors, such as Heath-Brown and Jia [6] and Matomäki [12], have found more arithmetic information by using estimations of Kloosterman sums. However, their methods usually require $\beta = 0$, which is not applicable on the present paper.

3. The final decomposition

Let $\omega(u)$ denote the Buchstab function determined by the following differential-difference equation

$$\begin{cases}\omega(u)=\frac{1}{u}, & 1\leqslant u\leqslant 2,\\ (u\omega(u))'=\omega(u-1), & u\geqslant 2.\end{cases}$$

Moreover, we have the upper and lower bounds for $\omega(u)$:

$$\omega(u)\geqslant\omega_0(u)=\begin{cases}\frac{1}{u}, & 1\leqslant u<2,\\ \frac{1+\log(u-1)}{u}, & 2\leqslant u<3,\\ \frac{1+\log(u-1)}{u}+\frac{1}{u}\int_2^{u-1}\frac{\log(t-1)}{t}dt\geqslant 0.5607, & 3\leqslant u<4,\\ 0.5612, & u\geqslant 4,\end{cases}$$

$$\omega(u)\leqslant\omega_1(u)=\begin{cases}\frac{1}{u}, & 1\leqslant u<2,\\ \frac{1+\log(u-1)}{u}, & 2\leqslant u<3,\\ \frac{1+\log(u-1)}{u}+\frac{1}{u}\int_2^{u-1}\frac{\log(t-1)}{t}dt\leqslant 0.5644, & 3\leqslant u<4,\\ 0.5617, & u\geqslant 4.\end{cases}$$

We shall use $\omega_0(u)$ and $\omega_1(u)$ to give numerical bounds for some sieve functions discussed below. We shall also use the simple upper bound $\omega(u)\leqslant\max(\frac{1}{u},0.5672)$ (see Lemma 8(iii) of [8]) to estimate high-dimensional integrals.

Before decomposing, we define the asymptotic region I as

$$I(m,n):=\left\{\frac{28}{87}\leqslant m\leqslant\frac{31}{87}\text{ or }\frac{56}{87}\leqslant m\leqslant\frac{59}{87}\text{ or }\frac{28}{87}\leqslant n\leqslant\frac{31}{87}\text{ or }\frac{56}{87}\leqslant n\leqslant\frac{59}{87}\right.$$
$$\left.\text{or }\frac{28}{87}\leqslant m+n\leqslant\frac{31}{87}\text{ or }\frac{56}{87}\leqslant m+n\leqslant\frac{59}{87}\right\}.$$

By Buchstab's identity, we have

$$\begin{aligned}S\left(\mathcal{A},(2x)^{\frac{1}{2}}\right)&=S\left(\mathcal{A},x^{\frac{1}{29}}\right)-\sum_{\frac{1}{29}\leqslant t_1<\frac{1}{2}}S\left(\mathcal{A}_{p_1},x^{\frac{1}{29}}\right)+\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)}}S\left(\mathcal{A}_{p_1p_2},p_2\right)\\ &=S_1-S_2+S_3.\end{aligned}\tag{3}$$

By Lemma 2.1, we can give asymptotic formulas for S_1 and S_2. Before estimating S_3, we first split it into six parts:

$$\begin{aligned}S_3=&\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)}}S\left(\mathcal{A}_{p_1p_2},p_2\right)\\ =&\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in I}}S\left(\mathcal{A}_{p_1p_2},p_2\right)+\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\notin I\\ t_1<\frac{28}{87},\ t_1+2t_2>\frac{59}{87}\text{ or }t_1+t_2>\frac{59}{87}}}S\left(\mathcal{A}_{p_1p_2},p_2\right)\\ &+\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\notin I\\ t_1\leqslant\frac{31}{87},\ t_1+2t_2\leqslant\frac{59}{87}}}S\left(\mathcal{A}_{p_1p_2},p_2\right)+\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\notin I\\ t_1>\frac{31}{87},\ t_1+2t_2\leqslant\frac{59}{87}}}S\left(\mathcal{A}_{p_1p_2},p_2\right)\end{aligned}$$

$$+ \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \notin I \\ t_1 > \frac{31}{87},\ t_1+2t_2 > \frac{59}{87} \\ t_2 \leqslant \max\left(\frac{1-t_1}{3}, \frac{t_1}{2}\right)}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) + \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \notin I \\ t_1 > \frac{31}{87},\ t_1+2t_2 > \frac{59}{87} \\ t_2 > \max\left(\frac{1-t_1}{3}, \frac{t_1}{2}\right)}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right)$$

$$= \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in I}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) + \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_1}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right)$$

$$+ \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_2}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) + \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_3}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right)$$

$$+ \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_4}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) + \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_5}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right)$$

$$= S_{31} + S_{32} + S_{33} + S_{34} + S_{35} + S_{36}. \tag{4}$$

S_{31} has an asymptotic formula. For S_{32}, we cannot decompose further but have to discard the whole region giving the loss

$$L_{32} := \int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \mathbb{1}_{(t_1,t_2) \in J_1} \frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1 t_2^2} dt_2 dt_1 < 0.397685. \tag{5}$$

For S_{33} we can use Buchstab's identity to get

$$S_{33} = \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_2}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right)$$

$$= \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_2}} S\left(\mathcal{A}_{p_1 p_2}, x^{\frac{1}{29}}\right) - \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_2 \\ \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{A}_{p_1 p_2 p_3}, x^{\frac{1}{29}}\right)$$

$$+ \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_2 \\ \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ can be partitioned into } (m,n) \in I}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right)$$

$$+ \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_2 \\ \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right)$$

$$= S_{331} - S_{332} + S_{333} + S_{334}. \tag{6}$$

We have asymptotic formulas for S_{331}–S_{333}. For the remaining S_{334}, we have two ways to get more possible savings: One way is to use Buchstab's identity twice more for some parts if we have $t_1 + t_2 + t_3 + 2t_4 \leqslant \frac{59}{87}$.

Another way is to use Buchstab's identity in reverse to make almost-primes visible. The details of further decompositions are similar to those in [11]. Combining the cases above we get a loss from S_{33} of

$$
\begin{aligned}
&\left(\int_{(t_1,t_2,t_3,t_4)\in J_{331}} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1t_2t_3t_4^2} dt_4dt_3dt_2dt_1\right)\\
&-\left(\int_{(t_1,t_2,t_3,t_4,t_5)\in J_{332}} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1t_2t_3t_4t_5^2} dt_5dt_4dt_3dt_2dt_1\right)\\
&+\left(\int_{(t_1,t_2,t_3,t_4,t_5,t_6)\in J_{333}} \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{t_6}\right)}{t_1t_2t_3t_4t_5t_6^2} dt_6dt_5dt_4dt_3dt_2dt_1\right)\\
&+\left(\int_{(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8)\in J_{334}} \frac{\max\left(\frac{t_8}{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7-t_8}, 0.5672\right)}{t_1t_2t_3t_4t_5t_6t_7t_8^2} dt_8dt_7dt_6dt_5dt_4dt_3dt_2dt_1\right)\\
&= L_{331} - L_{332} + L_{333} + L_{334},\\
&< 0.111391 - 0.021501 + 0.001491 + 0.000002\\
&= 0.091383, \qquad (7)
\end{aligned}
$$

where

$$
\begin{aligned}
J_{331}(t_1,t_2,t_3,t_4) := \Big\{ & (t_1,t_2)\in J_2,\\
& \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\\
& \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
& (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in I,\\
& t_1+t_2+t_3+2t_4 > \frac{59}{87},\\
& \frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Big\},\\
J_{332}(t_1,t_2,t_3,t_4,t_5) := \Big\{ & (t_1,t_2)\in J_2,\\
& \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\\
& \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
& (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in I,\\
& t_1+t_2+t_3+2t_4 > \frac{59}{87},\\
& t_4 < t_5 < \frac{1}{2}(1-t_1-t_2-t_3-t_4),\\
& (t_1,t_2,t_3,t_4,t_5) \text{ can be partitioned into } (m,n)\in I,\\
& \frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Big\},\\
J_{333}(t_1,t_2,t_3,t_4,t_5,t_6) := \Big\{ & (t_1,t_2)\in J_2,\\
& \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I, \\
&t_1+t_2+t_3+2t_4 \leqslant \frac{59}{87}, \\
&\frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right), \\
&\frac{1}{29} \leqslant t_6 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right), \\
&(t_1,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n) \in I, \\
&t_1+t_2+t_3+t_4+t_5+2t_6 > \frac{59}{87}, \\
&\left.\frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
J_{334}(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) := &\ \{(t_1,t_2) \in J_2, \\
&\frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \\
&\frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I, \\
&t_1+t_2+t_3+2t_4 \leqslant \frac{59}{87}, \\
&\frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right), \\
&\frac{1}{29} \leqslant t_6 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right), \\
&(t_1,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n) \in I, \\
&t_1+t_2+t_3+t_4+t_5+2t_6 \leqslant \frac{59}{87}, \\
&\frac{1}{29} \leqslant t_7 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6)\right), \\
&\frac{1}{29} \leqslant t_8 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6-t_7)\right), \\
&(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) \text{ cannot be partitioned into } (m,n) \in I, \\
&\left.\frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\}.
\end{aligned}
$$

Next we shall decompose S_{34}. By the same process as the decomposition of S_{33} above, we can reach a four-dimentional sum

$$
\sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_2 \\ \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right).
$$

We divide it into three parts:

$$
\begin{aligned}
&\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in J_2\\ \frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{1}{29}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)\\ (t_1,t_2,t_3,t_4)\text{ cannot be partitioned into }(m,n)\in I}} S\left(\mathcal{A}_{p_1p_2p_3p_4},p_4\right)\\
=&\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in J_2\\ \frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{1}{29}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)\\ (t_1,t_2,t_3,t_4)\text{ cannot be partitioned into }(m,n)\in I\\ t_1+t_2+t_3+t_4>\frac{59}{87}}} S\left(\mathcal{A}_{p_1p_2p_3p_4},p_4\right)\\
&+\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in J_2\\ \frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{1}{29}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)\\ (t_1,t_2,t_3,t_4)\text{ cannot be partitioned into }(m,n)\in I\\ t_1+t_2+t_3+2t_4\leqslant\frac{59}{87}}} S\left(\mathcal{A}_{p_1p_2p_3p_4},p_4\right)\\
&+\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in J_2\\ \frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{1}{29}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)\\ (t_1,t_2,t_3,t_4)\text{ cannot be partitioned into }(m,n)\in I\\ t_1+t_2+t_3+2t_4>\frac{59}{87}\\ t_1+t_2+t_3+t_4\leqslant\frac{59}{87}}} S\left(\mathcal{A}_{p_1p_2p_3p_4},p_4\right). \qquad (8)
\end{aligned}
$$

We can only use Buchstab's identity in reverse to make more savings for the first sum on the right-hand side. For the second sum in (8), we can perform a straightforward decomposition to get a six-dimensional sum

$$
\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in J_3\\ \frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{1}{29}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)\\ (t_1,t_2,t_3,t_4)\text{ cannot be partitioned into }(m,n)\in I\\ t_1+t_2+t_3+2t_4\leqslant\frac{59}{87}\\ \frac{1}{29}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ \frac{1}{29}\leqslant t_6<\min\left(t_5,\frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right)\\ (t_1,t_2,t_3,t_4,t_5,t_6)\text{ cannot be partitioned into }(m,n)\in I\\ t_1+t_2+t_3+t_4+t_5+2t_6>\frac{59}{87}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5p_6},p_6\right) \qquad (9)
$$

and an eight-dimensional sum

$$\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in J_3\\ \frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{1}{29}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)\\ (t_1,t_2,t_3,t_4)\text{ cannot be partitioned into }(m,n)\in I\\ t_1+t_2+t_3+2t_4\leqslant\frac{59}{87}\\ \frac{1}{29}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ \frac{1}{29}\leqslant t_6<\min\left(t_5,\frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right)\\ (t_1,t_2,t_3,t_4,t_5,t_6)\text{ cannot be partitioned into }(m,n)\in I\\ t_1+t_2+t_3+t_4+t_5+2t_6\leqslant\frac{59}{87}\\ \frac{1}{29}\leqslant t_7<\min\left(t_6,\frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6)\right)\\ \frac{1}{29}\leqslant t_8<\min\left(t_7,\frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6-t_7)\right)\\ (t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8)\text{ cannot be partitioned into }(m,n)\in I}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5p_6p_7p_8},p_8\right). \tag{10}$$

We can also use reversed Buchstab's identity to gain a five-dimensional saving. For the last sum, we cannot decompose it in a straightforward way. However, we can perform a role-reversal to get

$$\begin{aligned}
&\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in J_2\\ \frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{1}{29}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)\\ (t_1,t_2,t_3,t_4)\text{ cannot be partitioned into }(m,n)\in I\\ t_1+t_2+t_3+2t_4>\frac{59}{87}\\ t_1+t_2+t_3+t_4\leqslant\frac{59}{87}}} S\left(\mathcal{A}_{p_1p_2p_3p_4},p_4\right)\\
=&\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in J_2\\ \frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{1}{29}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)\\ (t_1,t_2,t_3,t_4)\text{ cannot be partitioned into }(m,n)\in I\\ t_1+t_2+t_3+2t_4>\frac{59}{87}\\ t_1+t_2+t_3+t_4\leqslant\frac{59}{87}}} S\left(\mathcal{A}_{p_1p_2p_3p_4},x^{\frac{1}{29}}\right)\\
-&\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in J_2\\ \frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{1}{29}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)\\ (t_1,t_2,t_3,t_4)\text{ cannot be partitioned into }(m,n)\in I\\ t_1+t_2+t_3+2t_4>\frac{59}{87}\\ t_1+t_2+t_3+t_4\leqslant\frac{59}{87}\\ \frac{1}{29}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5)\text{ can be partitioned into }(m,n)\in I}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5},p_5\right)\\
-&\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in J_2\\ \frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{1}{29}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)\\ (t_1,t_2,t_3,t_4)\text{ cannot be partitioned into }(m,n)\in I\\ t_1+t_2+t_3+2t_4>\frac{59}{87}\\ t_1+t_2+t_3+t_4\leqslant\frac{59}{87}\\ \frac{1}{29}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5)\text{ cannot be partitioned into }(m,n)\in I}} S\left(\mathcal{A}_{\gamma p_2p_3p_4p_5},x^{\frac{1}{29}}\right)
\end{aligned}$$

$$+ \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_2 \\ \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I \\ t_1+t_2+t_3+2t_4 > \frac{59}{87} \\ t_1+t_2+t_3+t_4 \leqslant \frac{59}{87} \\ \frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right) \\ (t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (m,n) \in I \\ \frac{1}{29} \leqslant t_6 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3-t_4-t_5,t_2,t_3,t_4,t_5,t_6) \text{ can be partitioned into } (m,n) \in I}} S\left(\mathcal{A}_{\gamma p_2 p_3 p_4 p_5 p_6}, p_6\right),$$

$$+ \sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_2 \\ \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I \\ t_1+t_2+t_3+2t_4 > \frac{59}{87} \\ t_1+t_2+t_3+t_4 \leqslant \frac{59}{87} \\ \frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right) \\ (t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (m,n) \in I \\ \frac{1}{29} \leqslant t_6 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3-t_4-t_5,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n) \in I}} S\left(\mathcal{A}_{\gamma p_2 p_3 p_4 p_5 p_6}, p_6\right), \tag{11}$$

where $\gamma \sim x^{1-t_1-t_2-t_3-t_4-t_5}$ and $(\gamma, P(p_5)) = 1$. Since $t_1 + t_2 + t_3 + t_4 \leqslant \frac{59}{87}$ and $(1 - t_1 - t_2 - t_3 - t_4 - t_5) + t_2 + t_3 + t_4 + t_5 = (1 - t_1) \leqslant (1 - \frac{31}{87}) = \frac{56}{87}$, we can give asymptotic formulas for all sums on the right hand side except for the last sum. Note that the last sum in (11) counts numbers with two almost-prime variables, we can make further decompositions on either variable, leading to two eight-dimensional sums

$$\sum_{\substack{\frac{1}{29} \leqslant t_1 < \frac{1}{2} \\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in J_2 \\ \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I \\ t_1+t_2+t_3+2t_4 > \frac{59}{87} \\ t_1+t_2+t_3+t_4 \leqslant \frac{59}{87} \\ \frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right) \\ (t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (m,n) \in I \\ \frac{1}{29} \leqslant t_6 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3-t_4-t_5,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n) \in I \\ (1-t_1-t_2-t_3-t_4-t_5)+t_2+t_3+t_4+t_5+2t_6 \leqslant \frac{59}{87} \\ \frac{1}{29} \leqslant t_7 < \min\left(t_6, \frac{1}{2}(t_1-t_6)\right) \\ (1-t_1-t_2-t_3-t_4-t_5,t_2,t_3,t_4,t_5,t_6,t_7) \text{ cannot be partitioned into } (m,n) \in I \\ \frac{1}{29} \leqslant t_8 < \min\left(t_7, \frac{1}{2}(t_1-t_6-t_7)\right) \\ (1-t_1-t_2-t_3-t_4-t_5,t_2,t_3,t_4,t_5,t_6,t_7,t_8) \text{ cannot be partitioned into } (m,n) \in I}} S\left(\mathcal{A}_{\gamma p_2 p_3 p_4 p_5 p_6 p_7 p_8}, p_8\right) \tag{12}$$

and

$$\sum_{\substack{\frac{1}{29}\leqslant t_1<\frac{1}{2}\\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in J_2\\ \frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{1}{29}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)\\ (t_1,t_2,t_3,t_4)\text{ cannot be partitioned into }(m,n)\in I\\ t_1+t_2+t_3+2t_4>\frac{59}{87}\\ t_1+t_2+t_3+t_4\leqslant\frac{59}{87}\\ \frac{1}{29}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5)\text{ cannot be partitioned into }(m,n)\in I\\ \frac{1}{29}\leqslant t_6<\frac{1}{2}t_1\\ (1-t_1-t_2-t_3-t_4-t_5,t_2,t_3,t_4,t_5,t_6)\text{ cannot be partitioned into }(m,n)\in I\\ (1-t_1-t_2-t_3-t_4-t_5)+t_2+t_3+t_4+t_5+2t_6>\frac{59}{87}\\ (t_1-t_6)+t_2+t_3+t_4+2t_5+t_6\leqslant\frac{59}{87}\\ \frac{1}{29}\leqslant t_7<\min\left(t_5,\frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right)\\ (t_1-t_6,t_2,t_3,t_4,t_5,t_6,t_7)\text{ cannot be partitioned into }(m,n)\in I\\ \frac{1}{29}\leqslant t_8<\min\left(t_7,\frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_7)\right)\\ (t_1-t_6,t_2,t_3,t_4,t_5,t_6,t_7,t_8)\text{ cannot be partitioned into }(m,n)\in I}} S\left(\mathcal{A}_{\gamma_1 p_2 p_3 p_4 p_5 p_6 p_7 p_8}, p_8\right), \tag{13}$$

where $\gamma_1 \sim x^{t_1-t_6}$ and $(\gamma_1, P(p_6)) = 1$. We refer the readers to [10] and [11] for more applications of role-reversals. Combining the cases above we get a loss from S_{34} of

$$\begin{aligned}
&\left(\int_{\frac{1}{29}}^{\frac{1}{2}} \int_{(t_1,t_2,t_3,t_4)\in J_{341}} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1\right)\\
&-\left(\int_{(t_1,t_2,t_3,t_4,t_5)\in J_{342}} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1\right)\\
&+\left(\int_{(t_1,t_2,t_3,t_4,t_5,t_6)\in J_{343}} \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{t_6}\right)}{t_1 t_2 t_3 t_4 t_5 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1\right)\\
&+\left(\int_{(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8)\in J_{344}} \frac{\max\left(\frac{t_8}{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7-t_8}, 0.5672\right)}{t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1\right)\\
&+\left(\int_{(t_1,t_2,t_3,t_4,t_5,t_6)\in J_{345}} \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)\omega_1\left(\frac{t_1-t_6}{t_6}\right)}{t_2 t_3 t_4 t_5^2 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1\right)\\
&+\left(\int_{(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8)\in J_{346}} \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)\omega_1\left(\frac{t_1-t_6-t_7-t_8}{t_8}\right)}{t_2 t_3 t_4 t_5^2 t_6 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1\right)\\
&+\left(\int_{(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8)\in J_{347}} \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_7-t_8}{t_8}\right)\omega_1\left(\frac{t_1-t_6}{t_6}\right)}{t_2 t_3 t_4 t_5 t_6^2 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1\right)\\
&= L_{341} - L_{342} + L_{343} + L_{344} + L_{345} + L_{346} + L_{347},\\
&< 0.103669 - 0.03892 + 0.001712 + 0.000001 + 0.007242 + 0.000056 + 0\\
&= 0.07376,
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
J_{341}(t_1,t_2,t_3,t_4) := \ &\{(t_1,t_2)\in J_3,\\
&\frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right),
\end{aligned}$$

$$
\begin{aligned}
&\frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in I,\\
&t_1+t_2+t_3+t_4 > \frac{59}{87},\\
&\frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Bigg\},\\
J_{342}(t_1,t_2,t_3,t_4,t_5) := &\ \{(t_1,t_2)\in J_3,\\
&\frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\\
&\frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in I,\\
&t_1+t_2+t_3+t_4 > \frac{59}{87},\\
&t_4 < t_5 < \frac{1}{2}(1-t_1-t_2-t_3-t_4),\\
&(t_1,t_2,t_3,t_4,t_5) \text{ can be partitioned into } (m,n)\in I,\\
&\frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Bigg\},\\
J_{343}(t_1,t_2,t_3,t_4,t_5,t_6) := &\ \{(t_1,t_2)\in J_3,\\
&\frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\\
&\frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in I,\\
&t_1+t_2+t_3+2t_4 \leqslant \frac{59}{87},\\
&\frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right),\\
&\frac{1}{29} \leqslant t_6 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right),\\
&(t_1,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n)\in I,\\
&t_1+t_2+t_3+t_4+t_5+2t_6 > \frac{59}{87},\\
&\frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Bigg\},\\
J_{344}(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) := &\ \{(t_1,t_2)\in J_3,\\
&\frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\\
&\frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in I,\\
&t_1+t_2+t_3+2t_4 \leqslant \frac{59}{87},
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right), \\
&\frac{1}{29} \leqslant t_6 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right), \\
&(t_1,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n) \in I, \\
&t_1+t_2+t_3+t_4+t_5+2t_6 \leqslant \frac{59}{87}, \\
&\frac{1}{29} \leqslant t_7 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6)\right), \\
&\frac{1}{29} \leqslant t_8 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6-t_7)\right), \\
&(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) \text{ cannot be partitioned into } (m,n) \in I, \\
&\left. \frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\}, \\
J_{345}(t_1,t_2,t_3,t_4,t_5,t_6) := &\ \{(t_1,t_2) \in J_3, \\
&\frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \\
&\frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I, \\
&t_1+t_2+t_3+t_4 \leqslant \frac{59}{87},\ t_1+t_2+t_3+2t_4 > \frac{59}{87}, \\
&\frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right), \\
&(t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (m,n) \in I, \\
&\frac{1}{29} \leqslant t_6 < \frac{1}{2}t_1, \\
&(1-t_1-t_2-t_3-t_4-t_5,t_2,t_3,t_4,t_5,t_6) \\
&\quad \text{cannot be partitioned into } (m,n) \in I, \\
&\left. \frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\}, \\
J_{346}(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) := &\ \{(t_1,t_2) \in J_3, \\
&\frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \\
&\frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I, \\
&t_1+t_2+t_3+t_4 \leqslant \frac{59}{87},\ t_1+t_2+t_3+2t_4 > \frac{59}{87}, \\
&\frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right), \\
&(t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (m,n) \in I, \\
&\frac{1}{29} \leqslant t_6 < \frac{1}{2}t_1, \\
&(1-t_1-t_2-t_3-t_4-t_5,t_2,t_3,t_4,t_5,t_6) \\
&\quad \text{cannot be partitioned into } (m,n) \in I,
\end{aligned}
$$

$$
\begin{aligned}
& (1-t_1-t_2-t_3-t_4-t_5)+t_2+t_3+t_4+t_5+2t_6 \leqslant \frac{59}{87}, \\
& \frac{1}{29} \leqslant t_7 < \min\left(t_6, \frac{1}{2}(t_1-t_6)\right), \\
& (1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6, t_7) \\
& \quad \text{cannot be partitioned into } (m,n) \in I, \\
& \frac{1}{29} \leqslant t_8 < \min\left(t_7, \frac{1}{2}(t_1-t_6-t_7)\right), \\
& (1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \\
& \quad \text{cannot be partitioned into } (m,n) \in I, \\
& \left. \frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \right\},
\end{aligned}
$$

$$
\begin{aligned}
J_{347}(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) := \ & \{(t_1,t_2) \in J_3, \\
& \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \\
& \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\
& (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I, \\
& t_1+t_2+t_3+t_4 \leqslant \frac{59}{87},\ t_1+t_2+t_3+2t_4 > \frac{59}{87}, \\
& \frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right), \\
& (t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (m,n) \in I, \\
& \frac{1}{29} \leqslant t_6 < \frac{1}{2}t_1, \\
& (1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6) \\
& \quad \text{cannot be partitioned into } (m,n) \in I, \\
& (1-t_1-t_2-t_3-t_4-t_5)+t_2+t_3+t_4+t_5+2t_6 > \frac{59}{87}, \\
& (t_1-t_6)+t_2+t_3+t_4+2t_5+t_6 \leqslant \frac{59}{87}, \\
& \frac{1}{29} \leqslant t_7 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right), \\
& (t_1-t_6, t_2, t_3, t_4, t_5, t_6, t_7) \\
& \quad \text{cannot be partitioned into } (m,n) \in I, \\
& \frac{1}{29} \leqslant t_8 < \min\left(t_7, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_7)\right), \\
& (t_1-t_6, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \\
& \quad \text{cannot be partitioned into } (m,n) \in I, \\
& \left. \frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \right\}.
\end{aligned}
$$

For S_{35} we can also use the devices mentioned earlier, but in this case we will perform a role-reversal on the triple sum if $t_1+t_2+t_3 > \frac{59}{87}$. In this way we get a loss of

$$
\left(\int_{(t_1,t_2,t_3,t_4) \in J_{3501}} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1 \right)
$$

$$
\begin{aligned}
&-\left(\int_{(t_1,t_2,t_3,t_4,t_5)\in J_{3502}} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1t_2t_3t_4t_5^2} dt_5dt_4dt_3dt_2dt_1\right)\\
&+\left(\int_{(t_1,t_2,t_3,t_4,t_5,t_6)\in J_{3503}} \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{t_6}\right)}{t_1t_2t_3t_4t_5t_6^2} dt_6dt_5dt_4dt_3dt_2dt_1\right)\\
&+\left(\int_{(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8)\in J_{3504}} \frac{\max\left(\frac{t_8}{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7-t_8}, 0.5672\right)}{t_1t_2t_3t_4t_5t_6t_7t_8^2} dt_8dt_7dt_6dt_5dt_4dt_3dt_2dt_1\right)\\
&+\left(\int_{(t_1,t_2,t_3,t_4)\in J_{3505}} \frac{\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)\omega\left(\frac{t_1-t_4}{t_4}\right)}{t_2t_3^2t_4^2} dt_4dt_3dt_2dt_1\right)\\
&-\left(\int_{(t_1,t_2,t_3,t_4,t_5)\in J_{3506}} \frac{\omega_0\left(\frac{1-t_1-t_2-t_3}{t_3}\right)\omega_0\left(\frac{t_1-t_4-t_5}{t_5}\right)}{t_2t_3^2t_4t_5^2} dt_5dt_4dt_3dt_2dt_1\right)\\
&-\left(\int_{(t_1,t_2,t_3,t_4,t_5)\in J_{3507}} \frac{\omega_0\left(\frac{1-t_1-t_2-t_3-t_5}{t_5}\right)\omega_0\left(\frac{t_1-t_4}{t_4}\right)}{t_2t_3t_4^2t_5^2} dt_5dt_4dt_3dt_2dt_1\right)\\
&+\left(\int_{(t_1,t_2,t_3,t_4,t_5,t_6)\in J_{3508}} \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_6}{t_6}\right)\omega_1\left(\frac{t_1-t_4-t_5}{t_5}\right)}{t_2t_3t_4t_5^2t_6^2} dt_6dt_5dt_4dt_3dt_2dt_1\right)\\
&+\left(\int_{(t_1,t_2,t_3,t_4,t_5,t_6)\in J_{3509}} \frac{\omega_1\left(\frac{1-t_1-t_2-t_3}{t_3}\right)\omega_1\left(\frac{t_1-t_4-t_5-t_6}{t_6}\right)}{t_2t_3^2t_4t_5t_6^2} dt_6dt_5dt_4dt_3dt_2dt_1\right)\\
&+\left(\int_{(t_1,t_2,t_3,t_4,t_5,t_6)\in J_{3510}} \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_5-t_6}{t_6}\right)\omega_1\left(\frac{t_1-t_4}{t_4}\right)}{t_2t_3t_4^2t_5t_6^2} dt_6dt_5dt_4dt_3dt_2dt_1\right)\\
&= L_{3501} - L_{3502} + L_{3503} + L_{3504} + L_{3505} - L_{3506} - L_{3507} + L_{3508} + L_{3509} + L_{3510},\\
&< 0.079609 - 0.02541 + 0.000001 + 0 + 0.469283 - 0.100821 - 0.124982 + 0.035428 + 0.006114 + 0\\
&= 0.339222, \qquad (15)
\end{aligned}
$$

where

$$
\begin{aligned}
J_{3501}(t_1,t_2,t_3,t_4) := \ &\{(t_1,t_2)\in J_4,\\
&\frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right),\ t_1+t_2+t_3\leqslant\frac{59}{87},\\
&\frac{1}{29}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right),\\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in I,\\
&t_1+t_2+t_3+2t_4>\frac{59}{87},\\
&\frac{1}{29}\leqslant t_1<\frac{1}{2},\ \frac{1}{29}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\bigg\},\\
J_{3502}(t_1,t_2,t_3,t_4,t_5) := \ &\{(t_1,t_2)\in J_4,\\
&\frac{1}{29}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right),\ t_1+t_2+t_3\leqslant\frac{59}{87},\\
&\frac{1}{29}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right),
\end{aligned}
$$

$$
\begin{aligned}
& (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in I,\\
& t_1+t_2+t_3+2t_4 > \frac{59}{87},\\
& t_4 < t_5 < \frac{1}{2}(1-t_1-t_2-t_3-t_4),\\
& (t_1,t_2,t_3,t_4,t_5) \text{ can be partitioned into } (m,n)\in I,\\
& \left.\frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\},\\
J_{3503}(t_1,t_2,t_3,t_4,t_5,t_6) :=\ & \{(t_1,t_2)\in J_4,\\
& \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\ t_1+t_2+t_3 \leqslant \frac{59}{87},\\
& \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
& (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in I,\\
& t_1+t_2+t_3+2t_4 \leqslant \frac{59}{87},\\
& \frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right),\\
& \frac{1}{29} \leqslant t_6 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right),\\
& (t_1,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n)\in I,\\
& t_1+t_2+t_3+t_4+t_5+2t_6 > \frac{59}{87},\\
& \left.\frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\},\\
J_{3504}(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) :=\ & \{(t_1,t_2)\in J_4,\\
& \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\ t_1+t_2+t_3 \leqslant \frac{59}{87},\\
& \frac{1}{29} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
& (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in I,\\
& t_1+t_2+t_3+2t_4 \leqslant \frac{59}{87},\\
& \frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right),\\
& \frac{1}{29} \leqslant t_6 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right),\\
& (t_1,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n)\in I,\\
& t_1+t_2+t_3+t_4+t_5+2t_6 \leqslant \frac{59}{87},\\
& \frac{1}{29} \leqslant t_7 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6)\right),\\
& \frac{1}{29} \leqslant t_8 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6-t_7)\right),\\
& (t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) \text{ cannot be partitioned into } (m,n)\in I,
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Bigg\}, \\
J_{3505}(t_1,t_2,t_3,t_4) := \ & \{(t_1,t_2) \in J_4, \\
& \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\ t_1+t_2+t_3 > \frac{59}{87}, \\
& \frac{1}{29} \leqslant t_4 < \frac{1}{2}t_1, \\
& (1-t_1-t_2-t_3, t_2, t_3, t_4) \text{ cannot be partitioned into } (m,n) \in I, \\
& (1-t_1-t_2-t_3) + t_2 + t_3 + 2t_4 > \frac{59}{87}, \\
& (t_1-t_4) + t_2 + 2t_3 + t_4 > \frac{59}{87}, \\
& \frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Bigg\}, \\
J_{3506}(t_1,t_2,t_3,t_4,t_5) := \ & \{(t_1,t_2) \in J_4, \\
& \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\ t_1+t_2+t_3 > \frac{59}{87}, \\
& \frac{1}{29} \leqslant t_4 < \frac{1}{2}t_1, \\
& (1-t_1-t_2-t_3, t_2, t_3, t_4) \text{ cannot be partitioned into } (m,n) \in I, \\
& (1-t_1-t_2-t_3) + t_2 + t_3 + 2t_4 > \frac{59}{87}, \\
& (t_1-t_4) + t_2 + 2t_3 + t_4 > \frac{59}{87}, \\
& t_4 < t_5 < \frac{1}{2}(t_1-t_4), \\
& (1-t_1-t_2-t_3, t_2, t_3, t_4, t_5) \text{ can be partitioned into } (m,n) \in I, \\
& \frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Bigg\}, \\
J_{3507}(t_1,t_2,t_3,t_4,t_5) := \ & \{(t_1,t_2) \in J_4, \\
& \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\ t_1+t_2+t_3 > \frac{59}{87}, \\
& \frac{1}{29} \leqslant t_4 < \frac{1}{2}t_1, \\
& (1-t_1-t_2-t_3, t_2, t_3, t_4) \text{ cannot be partitioned into } (m,n) \in I, \\
& (1-t_1-t_2-t_3) + t_2 + t_3 + 2t_4 > \frac{59}{87}, \\
& (t_1-t_4) + t_2 + 2t_3 + t_4 > \frac{59}{87}, \\
& t_3 < t_5 < \frac{1}{2}(1-t_1-t_2-t_3), \\
& (t_1-t_4, t_2, t_3, t_4, t_5) \text{ can be partitioned into } (m,n) \in I, \\
& \frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Bigg\}, \\
J_{3508}(t_1,t_2,t_3,t_4,t_5,t_6) := \ & \{(t_1,t_2) \in J_4, \\
& \frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\ t_1+t_2+t_3 > \frac{59}{87},
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{29} \leqslant t_4 < \frac{1}{2} t_1, \\
&(1-t_1-t_2-t_3, t_2, t_3, t_4) \text{ cannot be partitioned into } (m,n) \in I, \\
&(1-t_1-t_2-t_3) + t_2 + t_3 + 2t_4 > \frac{59}{87}, \\
&(t_1-t_4) + t_2 + 2t_3 + t_4 > \frac{59}{87}, \\
&t_4 < t_5 < \frac{1}{2}(t_1-t_4), \\
&(1-t_1-t_2-t_3, t_2, t_3, t_4, t_5) \text{ can be partitioned into } (m,n) \in I, \\
&t_3 < t_6 < \frac{1}{2}(1-t_1-t_2-t_3), \\
&(t_1-t_4, t_2, t_3, t_4, t_6) \text{ can be partitioned into } (m,n) \in I, \\
&\left. \frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \right\},
\end{aligned}
$$

$$
\begin{aligned}
J_{3509}(t_1,t_2,t_3,t_4,t_5,t_6) := \Big\{ &(t_1,t_2) \in J_4, \\
&\frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\ t_1+t_2+t_3 > \frac{59}{87}, \\
&\frac{1}{29} \leqslant t_4 < \frac{1}{2} t_1, \\
&(1-t_1-t_2-t_3, t_2, t_3, t_4) \text{ cannot be partitioned into } (m,n) \in I, \\
&(1-t_1-t_2-t_3) + t_2 + t_3 + 2t_4 \leqslant \frac{59}{87}, \\
&\frac{1}{29} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(t_1-t_4)\right), \\
&(1-t_1-t_2-t_3, t_2, t_3, t_4, t_5) \text{ cannot be partitioned into } (m,n) \in I, \\
&\frac{1}{29} \leqslant t_6 < \min\left(t_5, \frac{1}{2}(t_1-t_4-t_5)\right), \\
&(1-t_1-t_2-t_3, t_2, t_3, t_4, t_5, t_6) \text{ cannot be partitioned into } (m,n) \in I, \\
&\left. \frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \right\},
\end{aligned}
$$

$$
\begin{aligned}
J_{3510}(t_1,t_2,t_3,t_4,t_5,t_6) := \Big\{ &(t_1,t_2) \in J_4, \\
&\frac{1}{29} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\ t_1+t_2+t_3 > \frac{59}{87}, \\
&\frac{1}{29} \leqslant t_4 < \frac{1}{2} t_1, \\
&(1-t_1-t_2-t_3, t_2, t_3, t_4) \text{ cannot be partitioned into } (m,n) \in I, \\
&(1-t_1-t_2-t_3) + t_2 + t_3 + 2t_4 > \frac{59}{87}, \\
&(t_1-t_4) + t_2 + 2t_3 + t_4 \leqslant \frac{59}{87}, \\
&\frac{1}{29} \leqslant t_5 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\
&(t_1-t_4, t_2, t_3, t_4, t_5) \text{ cannot be partitioned into } (m,n) \in I, \\
&\frac{1}{29} \leqslant t_6 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_5)\right), \\
&(t_1-t_4, t_2, t_3, t_4, t_5, t_6) \text{ cannot be partitioned into } (m,n) \in I,
\end{aligned}
$$

$$\frac{1}{29} \leqslant t_1 < \frac{1}{2},\ \frac{1}{29} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Bigg\}.$$

For the remaining S_{36}, we choose to discard the whole region giving the loss

$$L_{36} := \int_{\frac{1}{29}}^{\frac{1}{2}} \int_{\frac{1}{29}}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \mathbb{1}_{(t_1,t_2)\in J_5} \frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1 t_2^2} dt_2 dt_1 < 0.093181. \tag{16}$$

Note that in this region only products of three primes are counted.

Finally, by (3)–(16), the total loss is less than

$$\begin{aligned}
&L_{32} + (L_{331} + L_{332} + L_{333} + L_{334}) + (L_{341} + L_{342} + L_{343} + L_{344} + L_{345} + L_{346} + L_{347}) \\
&\quad + (L_{3501} + L_{3502} + L_{3503} + L_{3504} + L_{3505} + L_{3506} + L_{3507} + L_{3508} + L_{3509} + L_{3510}) + L_{36} \\
&< 0.397685 + 0.091383 + 0.07376 + 0.339222 + 0.093181 \\
&< 0.996
\end{aligned}$$

and the proof of Theorem 1.1 is completed.

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International Curriculum Center, The High School Affiliated to Renmin University of China, Beijing, China
Email address: runbo.li.carey@gmail.com