

ON THE LARGEST PRIME FACTOR OF INTEGERS IN SHORT INTERVALS III

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ABSTRACT. Using Watt's mean value theorem and a delicate sieve decomposition, the author shows that the interval $[x, x + x^{\frac{1}{2}+\varepsilon}]$ contains an integer with a prime factor larger than $x^{\frac{35}{36}-\varepsilon}$ for sufficiently large x . This gives a solution with $\gamma = \frac{1}{36}$ to the Exercise 5.1 in Harman's monograph and improves the previous record of the author proved in 2024, where $\gamma = \frac{1}{26.5}$ is obtained.

If you are feeling brave, try to bring in Watt's mean value theorem.

—Glyn Harman

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1. INTRODUCTION

The Legendre's conjecture, which states that there is always a prime number between consecutive squares, is one of Landau's problems on prime numbers. Clearly this means that there is always a prime number in the interval $[x, x + x^{\frac{1}{2}}]$. However, we cannot prove it even on the Riemann Hypothesis. Assuming RH, one can only show that there is always a prime number in the interval $[x, x + x^{\frac{1}{2} \log x}]$. The best unconditional result is due to the author [16], where he showed the interval $[x, x + x^{0.52}]$ contains primes. Under a special condition concerning the existence of Siegel zeros, he [12] also showed that the interval $[x, x + x^{0.4923}]$ contains primes for long ranges of x .

Instead of relaxing the length of the short interval, one can attack this conjecture by relaxing our restriction of primes. A number with a large prime factor is a good approximation of prime numbers. Thus, we can try to find numbers with a large prime factor in the interval $[x, x + x^{\frac{1}{2}}]$ or $[x, x + x^{\frac{1}{2}+\varepsilon}]$. In [13] and [14] we considered the former and latter intervals respectively, and here we shall return to the former one again.

In this direction, Jutila [11] showed in 1973 that this interval contains a number with a prime factor larger than $x^{\frac{2}{3}-\varepsilon}$. The exponent $\frac{2}{3}$ has been improved to

$$0.73, 0.7338, 0.772, 0.82, \frac{11}{12}, \frac{17}{18}, \frac{19}{20}, \frac{24}{25}, \frac{25}{26} \text{ and } \frac{25.5}{26.5}$$

by Balog [1] [2], Balog, Harman and Pintz [3], Heath-Brown [7], Heath-Brown and Jia [8], Harman [5], Chapter 5], Haugland [6], Jia and Liu [10] and the author [13] respectively. In his monograph, Harman [5], Exercise 5.1 encouraged us to reduce this exponent as much as we can. In this paper, by applying Watt's mean value theorem [17] together with a delicate sieve decomposition and the arguments in [13], we give a large improvement over previous result by obtaining the following result.

Theorem 1. *For sufficiently large x , the interval $[x, x + x^{\frac{1}{2}+\varepsilon}]$ contains an integer with a prime factor larger than $x^{\frac{35}{36}-\varepsilon}$.*

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Throughout this paper, we always suppose that ε is a sufficiently small positive constant and $B = B(\varepsilon)$ is a sufficiently large positive constant. We choose ε such that $K = \frac{8}{\varepsilon}(\frac{1}{36} + \frac{\varepsilon}{2})$ is an integer. The letters p and β , with or without subscript, are reserved for primes and almost-primes respectively. Let c_1 and c_2 denote positive constants which may have different values at different places, and we write $m \sim M$ to mean that $c_1 M < m \leq c_2 M$. Let $b = 1 + \frac{1}{\log x}$, $v = x^{\frac{35}{36} - \frac{\varepsilon}{2}}$ and $P = x^{\frac{\varepsilon}{8}}$. We use $M(s)$, $N(s)$ and some other capital letters (except L and P) to denote 1-bounded Dirichlet polynomials

$$M(s) = \sum_{m \sim M} a_m m^{-s}, \quad N(s) = \sum_{n \sim N} b_n n^{-s}.$$

We use $L(s)$ to denote a “zeta-factor”

$$L(s) = \sum_{l \sim L} l^{-s}.$$

We also use $P(s)$ to denote

$$P(s) = \sum_{P < p \leq 2P} p^{-s}.$$

2. ARITHMETIC INFORMATION

In this section we provide some arithmetic information (i.e. mean value bounds for some Dirichlet polynomials) which will help us prove the asymptotic formulas for the sieve functions in next section. The first two lemmas come from the work of Heath-Brown and Jia [8], and they are also the main arithmetic information inputs used in [[5], Chapter 5] and [6].

Lemma 2.1. ([8], Lemma 1). *Suppose that $MN = v$ and $v^{\frac{17}{35}} \ll M \ll v^{\frac{18}{35}}$. Then for $(\log x)^{2B} \leq T \leq x^{\frac{1}{2} - \frac{\varepsilon}{6}}$, we have*

$$\int_T^{2T} |M(b+it)N(b+it)P^K(b+it)| dt \ll (\log x)^{-B}.$$

Lemma 2.2. ([8], Lemma 2). *Suppose that $MNL = v$, $M \ll v^{\frac{18}{35}}$ and $N \ll v^{\frac{9}{35}}$. Then for $\sqrt{L} \leq T \leq x^{\frac{1}{2} - \frac{\varepsilon}{6}}$, we have*

$$\int_T^{2T} |M(b+it)N(b+it)L(b+it)P^K(b+it)| dt \ll (\log x)^{-B}.$$

In Exercise 5.1 in his monograph [[5], Chapter 5], Harman encouraged the readers to “reduce the value of γ (the exponent) as much as you can.” He also said that “if you are feeling brave, try to bring in Watt’s mean value theorem [17].” Here we shall use another form of Watt’s theorem given in [15].

Lemma 2.3. ([15], Lemma 4.1). *Let $T \geq 1$, then*

$$\int_T^{2T} \left| L\left(\frac{1}{2} + it\right) \right|^4 \left| N\left(\frac{1}{2} + it\right) \right|^2 dt \ll \left(T + N^2 T^{\frac{1}{2}} + L^2 T^{-4} (N + T) \right) T^{\frac{\varepsilon}{2}}.$$

We shall use Lemma 2.3 to derive the following important lemma, which gives more Type-I information than in [10] and [13]. This new arithmetic information, together with a more delicate sieving process similar to that in [16], is the key of our improvement over [13].

Lemma 2.4. *Suppose that $MNHL = v$, and M , N and H satisfy one of the following 3 conditions:*

- (1). $M \ll v^{\frac{18}{35}}$, $N \gg H$, $N^{\frac{3}{4}} H \ll v^{\frac{9}{35}}$, $NH^{\frac{1}{2}} \ll v^{\frac{9}{35}}$, $N^{\frac{7}{4}} H^{\frac{3}{2}} \ll v^{\frac{18}{35}}$;
- (2). $M \ll v^{\frac{18}{35}}$, $N \ll v^{\frac{9}{35}}$, $H \ll v^{\frac{9}{140}}$;
- (3). $M \ll v^{\frac{18}{35}}$, $N^2 H \ll v^{\frac{18}{35}}$, $H \ll v^{\frac{9}{70}}$.

Then for $\sqrt{L} \leq T \leq x^{\frac{1}{2} - \frac{\varepsilon}{6}}$, we have

$$\int_T^{2T} |M(b+it)N(b+it)H(b+it)L(b+it)P^K(b+it)| dt \ll (\log x)^{-B}.$$

Proof. We only need to prove that

$$\int_T^{2T} \left| M\left(\frac{1}{2} + it\right) N\left(\frac{1}{2} + it\right) H\left(\frac{1}{2} + it\right) L\left(\frac{1}{2} + it\right) P^K\left(\frac{1}{2} + it\right) \right| dt \ll x^{\frac{1}{2}} (\log x)^{-B}$$

holds true under each condition above.

Assume that Condition (1) holds. The proof of this case is similar to that of [[10], Lemma 3] where [[4], Theorem 2] is used.

Assume that Condition (2) holds. Using Hölder's inequality and the mean value theorem for Dirichlet polynomials, we have

$$\begin{aligned} & \int_T^{2T} \left| M\left(\frac{1}{2} + it\right) N\left(\frac{1}{2} + it\right) H\left(\frac{1}{2} + it\right) L\left(\frac{1}{2} + it\right) P^K\left(\frac{1}{2} + it\right) \right| dt \\ & \ll \left(\int_T^{2T} \left| M\left(\frac{1}{2} + it\right) \right|^2 \left| P^K\left(\frac{1}{2} + it\right) \right|^2 dt \right)^{\frac{1}{2}} \left(\int_T^{2T} \left| N\left(\frac{1}{2} + it\right) \right|^4 dt \right)^{\frac{1}{4}} \\ & \quad \left(\int_T^{2T} \left| L\left(\frac{1}{2} + it\right) \right|^4 \left| H\left(\frac{1}{2} + it\right) \right|^4 dt \right)^{\frac{1}{4}} \\ & \ll x^{\frac{1}{4}} \left(\int_T^{2T} \left| N\left(\frac{1}{2} + it\right) \right|^4 dt \right)^{\frac{1}{4}} \left(\int_T^{2T} \left| L\left(\frac{1}{2} + it\right) \right|^4 \left| H\left(\frac{1}{2} + it\right) \right|^4 dt \right)^{\frac{1}{4}}. \end{aligned}$$

By the mean value theorem for Dirichlet polynomials, we have

$$\left(\int_T^{2T} \left| N\left(\frac{1}{2} + it\right) \right|^4 dt \right)^{\frac{1}{4}} \ll (N^2 + T)^{\frac{1}{4}}.$$

Note that $\sqrt{L} \leq T$, we have

$$L^2 T^{-4} (N + T) \ll N + T.$$

Now, using Lemma 2.3 with $N = H^2$, we get

$$\left(\int_T^{2T} \left| L\left(\frac{1}{2} + it\right) \right|^4 \left| H\left(\frac{1}{2} + it\right) \right|^4 dt \right)^{\frac{1}{4}} \ll (T + H^4 T^{\frac{1}{2}} + H^2)^{\frac{1}{4}} T^{\frac{\varepsilon}{8}}.$$

Combining the bounds above with Condition (2), we have

$$\begin{aligned} & \int_T^{2T} \left| M\left(\frac{1}{2} + it\right) N\left(\frac{1}{2} + it\right) H\left(\frac{1}{2} + it\right) L\left(\frac{1}{2} + it\right) P^K\left(\frac{1}{2} + it\right) \right| dt \\ & \ll x^{\frac{1}{4}} (N^2 + T)^{\frac{1}{4}} (T + H^4 T^{\frac{1}{2}} + H^2)^{\frac{1}{4}} T^{\frac{\varepsilon}{8}} \\ & \ll x^{\frac{1}{2}} (\log x)^{-B}. \end{aligned}$$

Note that this case is an improvement over Lemma 2.2.

Assume that Condition (3) holds. Using Hölder's inequality and the mean value theorem for Dirichlet polynomials, we have

$$\begin{aligned} & \int_T^{2T} \left| M\left(\frac{1}{2} + it\right) N\left(\frac{1}{2} + it\right) H\left(\frac{1}{2} + it\right) L\left(\frac{1}{2} + it\right) P^K\left(\frac{1}{2} + it\right) \right| dt \\ & \ll \left(\int_T^{2T} \left| M\left(\frac{1}{2} + it\right) \right|^2 \left| P^K\left(\frac{1}{2} + it\right) \right|^2 dt \right)^{\frac{1}{2}} \left(\int_T^{2T} \left| N\left(\frac{1}{2} + it\right) \right|^4 \left| H\left(\frac{1}{2} + it\right) \right|^2 dt \right)^{\frac{1}{4}} \\ & \quad \left(\int_T^{2T} \left| L\left(\frac{1}{2} + it\right) \right|^4 \left| H\left(\frac{1}{2} + it\right) \right|^2 dt \right)^{\frac{1}{4}} \end{aligned}$$

$$\ll x^{\frac{1}{4}} \left(\int_T^{2T} \left| N\left(\frac{1}{2} + it\right) \right|^4 \left| H\left(\frac{1}{2} + it\right) \right|^2 dt \right)^{\frac{1}{4}} \left(\int_T^{2T} \left| L\left(\frac{1}{2} + it\right) \right|^4 \left| H\left(\frac{1}{2} + it\right) \right|^2 dt \right)^{\frac{1}{4}}.$$

By Condition (3), Lemma 2.3 with $N = H$ and the mean value theorem for Dirichlet polynomials, we have

$$\begin{aligned} & \int_T^{2T} \left| M\left(\frac{1}{2} + it\right) N\left(\frac{1}{2} + it\right) H\left(\frac{1}{2} + it\right) L\left(\frac{1}{2} + it\right) P^K\left(\frac{1}{2} + it\right) \right| dt \\ & \ll x^{\frac{1}{4}} (N^2 H + T)^{\frac{1}{4}} \left(T + H^2 T^{\frac{1}{2}} + H \right)^{\frac{1}{4}} T^{\frac{\varepsilon}{8}} \\ & \ll x^{\frac{1}{2}} (\log x)^{-B}. \end{aligned}$$

Now the proof of Lemma 2.4 is completed. \square

3. THE FINAL DECOMPOSITION

Let \mathcal{C} denote a finite set of positive integers, $p_j \sim v^{t_j}$ and put

$$\begin{aligned} N(d) &= \sum_{\substack{x < pp_1 \dots p_K \leq x + x^{\frac{1}{2} + \varepsilon} \\ P < p_i \leq 2P}} 1, \quad \mathcal{A} = \{n : 2^{-K}v < n \leq 2v, \text{ } n \text{ repeats } N(n) \text{ times}\}, \\ \mathcal{B} &= \{n : v < n \leq 2v\}, \quad \mathcal{C}_d = \{a : ad \in \mathcal{C}\}, \quad P(z) = \prod_{p < z} p, \quad S(\mathcal{C}, z) = \sum_{\substack{a \in \mathcal{C} \\ (a, P(z)) = 1}} 1. \end{aligned}$$

Then we only need to show that $S(\mathcal{A}, (2v)^{\frac{1}{2}}) > 0$.

Buchstab's identity is the equation

$$S(\mathcal{C}, z) = S(\mathcal{C}, w) - \sum_{w \leq p < z} S(\mathcal{C}_p, p),$$

where $2 \leq w < z$.

Our aim is to show that the sparser set \mathcal{A} contains the expected proportion of primes compared to the bigger set \mathcal{B} , which requires us to decompose $S(\mathcal{A}, (2v)^{\frac{1}{2}})$ and prove asymptotic formulas of the form

$$S(\mathcal{A}, z) = v^{-1} x^{\frac{1}{2} + \varepsilon} \left(\sum_{P < p \leq 2P} \frac{1}{p} \right)^K (1 + o(1)) S(\mathcal{B}, z) \quad (1)$$

for some parts of it, and drop the other positive parts.

Let $\omega(u)$ denote the Buchstab function determined by the following differential-difference equation

$$\begin{cases} \omega(u) = \frac{1}{u}, & 1 \leq u \leq 2, \\ (u\omega(u))' = \omega(u-1), & u \geq 2. \end{cases}$$

Moreover, we have the upper and lower bounds for $\omega(u)$:

$$\begin{aligned} \omega(u) \geq \omega_0(u) &= \begin{cases} \frac{1}{u}, & 1 \leq u < 2, \\ \frac{1 + \log(u-1)}{u}, & 2 \leq u < 3, \\ \frac{1 + \log(u-1)}{u} + \frac{1}{u} \int_2^{u-1} \frac{\log(t-1)}{t} dt \geq 0.5607, & 3 \leq u < 4, \\ 0.5612, & u \geq 4, \end{cases} \\ \omega(u) \leq \omega_1(u) &= \begin{cases} \frac{1}{u}, & 1 \leq u < 2, \\ \frac{1 + \log(u-1)}{u}, & 2 \leq u < 3, \\ \frac{1 + \log(u-1)}{u} + \frac{1}{u} \int_2^{u-1} \frac{\log(t-1)}{t} dt \leq 0.5644, & 3 \leq u < 4, \\ 0.5617, & u \geq 4. \end{cases} \end{aligned}$$

We shall use $\omega_0(u)$ and $\omega_1(u)$ to give numerical bounds for some sieve functions discussed below. We shall also use the simple upper bound $\omega(u) \leq \max(\frac{1}{u}, 0.5672)$ (see Lemma 8(iii) of [9]) to estimate high-dimensional integrals.

By Prime Number Theorem with Vinogradov's error term and the inductive arguments in [[5], Chapter A.2], we know that, for sufficiently large z ,

$$S(\mathcal{B}, z) = \sum_{\substack{a \in \mathcal{B} \\ (a, P(z))=1}} 1 = (1 + o(1)) \frac{v}{\log z} \omega\left(\frac{\log v}{\log z}\right), \quad (2)$$

and we expect that the similar relation also holds for $S(\mathcal{A}, z)$:

$$S(\mathcal{A}, z) = \sum_{\substack{a \in \mathcal{A} \\ (a, P(z))=1}} 1 = (1 + o(1)) \left(\sum_{P < p \leq 2P} \frac{1}{p} \right)^K \frac{x^{\frac{1}{2} + \varepsilon}}{\log z} \omega\left(\frac{\log v}{\log z}\right). \quad (3)$$

If (1) holds for $S(\mathcal{A}, z)$, then we can deduce (3) easily from (1) and (2). Otherwise we must drop this $S(\mathcal{A}, z)$. We define the *loss* from this term by the size of corresponding $S(\mathcal{B}, z)$:

$$S(\mathcal{B}, z) = (\text{loss} + o(1)) \frac{v}{\log x}. \quad (4)$$

We note that for the lower bound problem, we can only drop positive parts and the total loss of the dropped parts must be less than 1.

Before decomposing, we define the asymptotic regions \mathcal{T}_0 - \mathcal{T}_2 , \mathcal{I}_n \mathcal{J}_n and a two-dimensional region \mathcal{L} as

$$\begin{aligned} \mathcal{T}_0(m_1, m_2) &:= \left\{ \frac{17}{35} \leq m_1 \leq \frac{18}{35} \text{ or } \frac{17}{35} \leq m_2 \leq \frac{18}{35} \text{ or } \frac{17}{35} \leq m_1 + m_2 \leq \frac{18}{35} \right\}, \\ \mathcal{T}_1(m_1, m_2) &:= \left\{ m_1 \leq \frac{18}{35}, m_2 \leq \frac{9}{35} \right\}, \\ \mathcal{T}_{21}(m_1, m_2, m_3) &:= \left\{ m_1 \leq \frac{18}{35}, m_2 \geq m_3, \frac{3}{4}m_2 + m_3 \leq \frac{9}{35}, m_2 + \frac{1}{2}m_3 \leq \frac{9}{35}, \frac{7}{4}m_2 + \frac{3}{2}m_3 \leq \frac{18}{35} \right\}, \\ \mathcal{T}_{22}(m_1, m_2, m_3) &:= \left\{ m_1 \leq \frac{18}{35}, m_2 \leq \frac{9}{35}, m_3 \leq \frac{9}{140} \right\}, \\ \mathcal{T}_{23}(m_1, m_2, m_3) &:= \left\{ m_1 \leq \frac{18}{35}, 2m_2 + m_3 \leq \frac{18}{35}, m_3 \leq \frac{9}{70} \right\}, \\ \mathcal{T}_2(m_1, m_2, m_3) &:= \mathcal{T}_{21} \cup \mathcal{T}_{22} \cup \mathcal{T}_{23}, \\ \mathcal{I}_i(m_1, m_2, \dots, m_i) &:= \{(m_1, m_2, \dots, m_i) \text{ can be partitioned into } (\alpha_1, \alpha_2) \in \mathcal{T}_0\}, \\ \mathcal{J}_i(m_1, m_2, \dots, m_i) &:= \{(m_1, m_2, \dots, m_i) \text{ can be partitioned into } (\alpha_1, \alpha_2) \in \mathcal{T}_1 \text{ or } (\alpha_1, \alpha_2, \alpha_3) \in \mathcal{T}_2\}, \\ \mathcal{L}(m, n) &:= \left\{ (m, n) \notin \mathcal{T}_0, (m, n, n) \notin \mathcal{J}_3, n \geq \frac{53}{255} \text{ or } m \geq \frac{1129}{2448} \text{ or } \frac{1}{2}m + n \geq \frac{9361}{24480} \right\}. \end{aligned}$$

By the definition of \mathcal{J}_i , if $(m_1, m_2, \dots, m_i) \in \mathcal{J}_i$, then $(m_1, m_2, \dots, m_{i-1}, l) \in \mathcal{J}_i$ for all $l \leq m_i$. The cases of reducing other variables are similar.

Now we state the general sieve asymptotic formulas we need in our final decomposition.

Lemma 3.1. *We can give an asymptotic formula of the form (1) for*

$$\sum_{t_1 \dots t_n} S\left(\mathcal{A}_{p_1 \dots p_n}, v^{\frac{1}{35}}\right)$$

and similar sums involving role-reversals if $(t_1, \dots, t_n) \in \mathcal{J}_n$.

Proof. This lemma can be proved by Lemma 2.4 together with the discussions in [[8], Lemmas 7,9]. \square

Lemma 3.2. *We can give an asymptotic formula of the form (1) for*

$$\sum_{t_1 \dots t_n} S(\mathcal{A}_{p_1 \dots p_n}, p_n)$$

and similar sums involving role-reversals if $(t_1, \dots, t_n) \in \mathcal{I}_n$.

Proof. This lemma can be proved by Lemma 2.1 together with the discussions in [[8], Lemma 8]. \square

We start our final decomposition of $S\left(\mathcal{A}, (2v)^{\frac{1}{2}}\right)$. By Buchstab's identity, we have

$$\begin{aligned}
S\left(\mathcal{A}, (2v)^{\frac{1}{2}}\right) &= S\left(\mathcal{A}, v^{\frac{1}{35}}\right) - \sum_{\frac{1}{35} \leq t_1 < \frac{17}{35}} S(\mathcal{A}_{p_1}, p_1) - \sum_{\frac{17}{35} \leq t_1 < \frac{1}{2}} S(\mathcal{A}_{p_1}, p_1) \\
&= S\left(\mathcal{A}, v^{\frac{1}{35}}\right) - \sum_{\frac{1}{35} \leq t_1 < \frac{17}{35}} S\left(\mathcal{A}_{p_1}, v^{\frac{1}{35}}\right) - \sum_{\frac{17}{35} \leq t_1 < \frac{1}{2}} S(\mathcal{A}_{p_1}, p_1) \\
&\quad + \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1))}} S(\mathcal{A}_{p_1 p_2}, p_2) \\
&= S_1 - S_2 - S_3 + S_4.
\end{aligned} \tag{5}$$

By Lemma 2.1 and Lemma 2.2, we can give asymptotic formulas for S_1 , S_2 and S_3 .

Before estimating S_4 , we first split it into three parts:

$$\begin{aligned}
S_4 &= \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1))}} S(\mathcal{A}_{p_1 p_2}, p_2) \\
&= \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{T}_0}} S(\mathcal{A}_{p_1 p_2}, p_2) + \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{L}}} S(\mathcal{A}_{p_1 p_2}, p_2) \\
&\quad + \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \notin \mathcal{T}_0 \\ (t_1, t_2, t_2) \in \mathcal{J}_3}} S(\mathcal{A}_{p_1 p_2}, p_2) + \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \notin \mathcal{T}_0 \cup \mathcal{L} \\ (t_1, t_2, t_2) \notin \mathcal{J}_3}} S(\mathcal{A}_{p_1 p_2}, p_2) \\
&= \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{T}_0}} S(\mathcal{A}_{p_1 p_2}, p_2) + \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{L}}} S(\mathcal{A}_{p_1 p_2}, p_2) \\
&\quad + \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M}}} S(\mathcal{A}_{p_1 p_2}, p_2) + \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N}}} S(\mathcal{A}_{p_1 p_2}, p_2) \\
&= S_{41} + S_{42} + S_{43} + S_{44},
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
\mathcal{M}(m, n) &:= \{(m, n) \notin \mathcal{T}_0, (m, n, n) \in \mathcal{J}_3\}, \\
\mathcal{N}(m, n) &:= \{(m, n) \notin \mathcal{T}_0 \cup \mathcal{L}, (m, n, n) \notin \mathcal{J}_3\}.
\end{aligned}$$

By Lemma 3.2, S_{41} has an asymptotic formula. For S_{42} , we cannot decompose further but have to discard the whole region giving the loss

$$\int_{\frac{1}{35}}^{\frac{17}{35}} \int_{\frac{1}{35}}^{\min(t_1, \frac{1-t_1}{2})} \mathbb{1}_{(t_1, t_2) \in \mathcal{L}} \frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1 t_2^2} dt_2 dt_1 < 0.7226. \tag{7}$$

For S_{43} we can use Buchstab's identity to get

$$\begin{aligned}
S_{43} &= \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M}}} S(\mathcal{A}_{p_1 p_2}, p_2) \\
&= \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M}}} S(\mathcal{A}_{p_1 p_2}, v^{\frac{1}{35}}) - \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \in \mathcal{I}_3}} S(\mathcal{A}_{p_1 p_2 p_3}, p_3) \\
&\quad - \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3}} S(\mathcal{A}_{p_1 p_2 p_3}, p_3) \\
&= \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M}}} S(\mathcal{A}_{p_1 p_2}, v^{\frac{1}{35}}) - \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \in \mathcal{I}_3}} S(\mathcal{A}_{p_1 p_2 p_3}, p_3) \\
&\quad - \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3}} S(\mathcal{A}_{p_1 p_2 p_3}, v^{\frac{1}{35}}) + \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \in \mathcal{I}_4}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4) \\
&\quad + \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4) \\
&= S_{431} - S_{432} - S_{433} + S_{434} + S_{435}. \tag{8}
\end{aligned}$$

Since $(t_1, t_2) \in \mathcal{M}$, we have asymptotic formulas for S_{431} and S_{433} by Lemma 3.1. We also have asymptotic formulas for S_{432} and S_{434} by Lemma 3.2.

For the remaining S_{435} , we first split it into three parts:

$$\begin{aligned}
S_{435} &= \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_5}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4) \\
&+ \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4) \notin \mathcal{J}_5 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \in \mathcal{J}_4}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4) \\
&+ \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4) \notin \mathcal{J}_5 \\ \text{either } (t_1, t_2, t_3, t_4) \notin \mathcal{J}_4 \text{ or } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \notin \mathcal{J}_4}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4) \\
&= S_{4351} + S_{4352} + S_{4353}. \tag{9}
\end{aligned}$$

We have three ways to get more possible savings corresponding to each sum above: For S_{4351} we can use Buchstab's identity twice more in a straightforward manner to get

$$\begin{aligned}
S_{4351} &= \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_5}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4) \\
&= \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_5}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, v^{\frac{1}{35}}) - \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_5 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \in \mathcal{I}_5}} S(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5)
\end{aligned}$$

$$\begin{aligned}
& - \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4, t_5) \in \mathcal{J}_5 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5}} S(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5) \\
& = \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4, t_5) \in \mathcal{J}_5}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, v^{\frac{1}{35}}) - \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4, t_5) \in \mathcal{J}_5 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \in \mathcal{I}_5}} S(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5) \\
& - \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4, t_5) \in \mathcal{J}_5 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5}} S(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, v^{\frac{1}{35}}) \\
& + \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4, t_5) \in \mathcal{J}_5 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5}} S(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5 p_6}, p_6) \\
& + \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4, t_5) \in \mathcal{J}_5 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5}} S(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5 p_6}, p_6) \\
& = S_{43511} - S_{43512} - S_{43513} + S_{43514} + S_{43515}.
\end{aligned} \tag{10}$$

We have asymptotic formulas for S_{43512} and S_{43514} by Lemma 3.2. Since we have $(t_1, t_2, t_3, t_4, t_4) \in \mathcal{J}_5$ in all five sums above, we also have $(t_1, t_2, t_3, t_4) \in \mathcal{J}_4$ and $(t_1, t_2, t_3, t_4, t_5) \in \mathcal{J}_5$. Now we have asymptotic formulas for S_{43511} and S_{43513} by Lemma 3.1. For S_{43515} we can decompose those parts with $(t_1, t_2, t_3, t_4, t_5, t_6, t_6) \in \mathcal{J}_7$ in a similar way to reach an eight-dimensional sum

$$\sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4, t_4) \in \mathcal{J}_5 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5 \\ \frac{1}{35} \leq t_6 < \min(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)) \\ (t_1, t_2, t_3, t_4, t_5, t_6) \notin \mathcal{I}_6 \\ (t_1, t_2, t_3, t_4, t_5, t_6, t_6) \in \mathcal{J}_7 \\ \frac{1}{35} \leq t_7 < \min(t_6, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6)) \\ (t_1, t_2, t_3, t_4, t_5, t_6, t_7) \notin \mathcal{I}_7 \\ \frac{1}{35} \leq t_8 < \min(t_7, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6-t_7)) \\ (t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \notin \mathcal{I}_8}} S(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5 p_6 p_7 p_8}, p_8), \quad (11)$$

and we discard the remaining parts where $(t_1, t_2, t_3, t_4, t_5, t_6, t_6) \notin \mathcal{J}_7$:

$$\sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4, t_4) \in \mathcal{J}_5 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5 \\ \frac{1}{35} \leq t_6 < \min(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)) \\ (t_1, t_2, t_3, t_4, t_5, t_6) \notin \mathcal{I}_6 \\ (t_1, t_2, t_3, t_4, t_5, t_6, t_6) \notin \mathcal{J}_7}} S(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5 p_6}, p_6). \quad (12)$$

The corresponding loss from S_{4351} are two sums in (11) and (12).

For S_{4352} we cannot use Buchstab's identity twice more in a straightforward manner, but we can use Buchstab's identity twice more with a variable role-reversal, which can be seen as "an application of Buchstab's identity on large prime variables". We first apply Buchstab's identity once to get

$$S_{4352} = \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4, t_4) \notin \mathcal{J}_5 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \in \mathcal{J}_4}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4)$$

$$\begin{aligned}
&= \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4) \notin \mathcal{J}_5 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \in \mathcal{J}_4}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, v^{\frac{1}{35}}\right) \\
&- \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4) \notin \mathcal{J}_5 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \in \mathcal{J}_4 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \in \mathcal{I}_5}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5\right) \\
&- \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4) \notin \mathcal{J}_5 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \in \mathcal{J}_4 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5\right). \tag{13}
\end{aligned}$$

We have asymptotic formulas for the first and second sum on the right-hand side of (13) by Lemma 3.1 and Lemma 3.2 respectively. The last sum on the right-hand side of (13) counts numbers of the form $p_1 p_2 p_3 p_4 p_5 \beta_1$, where $\beta_1 \sim v^{1-t_1-t_2-t_3-t_4-t_5}$ and $(\beta_1, P(p_5)) = 1$. Now we reverse the roles of p_1 and β_1 :

$$\begin{aligned}
&\sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4) \notin \mathcal{J}_5 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \in \mathcal{J}_4 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5\right) \\
&= \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4) \notin \mathcal{J}_5 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \in \mathcal{J}_4 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5}} S\left(\mathcal{A}_{\beta_1 p_2 p_3 p_4 p_5}, \left(\frac{2v}{\beta_1 p_2 p_3 p_4 p_5}\right)^{\frac{1}{2}}\right). \tag{14}
\end{aligned}$$

Note that the sum on the right-hand side of (14) counts numbers of the form $\beta_1 p_2 p_3 p_4 p_5 p_1$. Now we perform a Buchstab iteration on the variable p_1 to get

$$\begin{aligned}
& \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{J}_5 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \in \mathcal{J}_4 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5}} S\left(\mathcal{A}_{\beta_1 p_2 p_3 p_4 p_5}, \left(\frac{2v}{\beta_1 p_2 p_3 p_4 p_5}\right)^{\frac{1}{2}}\right) \\
= & \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{J}_5 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \in \mathcal{J}_4 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5}} S\left(\mathcal{A}_{\beta_1 p_2 p_3 p_4 p_5}, v^{\frac{1}{35}}\right) \\
- & \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{J}_5 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \in \mathcal{J}_4 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5 \\ \frac{1}{35} \leq t_6 < \frac{1}{2} t_1 \\ (1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6) \in \mathcal{I}_6}} S(\mathcal{A}_{\beta_1 p_2 p_3 p_4 p_5 p_6}, p_6) \\
- & \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{J}_5 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \in \mathcal{J}_4 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5 \\ \frac{1}{35} \leq t_6 < \frac{1}{2} t_1 \\ (1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6) \notin \mathcal{I}_6}} S(\mathcal{A}_{\beta_1 p_2 p_3 p_4 p_5 p_6}, p_6). \tag{15}
\end{aligned}$$

Combining (13)–(15), we have

$$\begin{aligned}
S_{4352} = & \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4) \notin \mathcal{J}_5 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \in \mathcal{J}_4}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4) \\
= & \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4) \notin \mathcal{J}_5 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \in \mathcal{J}_4}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, v^{\frac{1}{35}}) \\
- & \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4) \notin \mathcal{J}_5 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \in \mathcal{J}_4 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \in \mathcal{I}_5}} S(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5) \\
- & \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4) \notin \mathcal{J}_5 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \in \mathcal{J}_4 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5}} S(\mathcal{A}_{\beta_1 p_2 p_3 p_4 p_5}, v^{\frac{1}{35}}) \\
+ & \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4) \notin \mathcal{J}_5 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \in \mathcal{J}_4 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5 \\ \frac{1}{35} \leq t_6 < \frac{1}{2} t_1 \\ (1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6) \in \mathcal{I}_6}} S(\mathcal{A}_{\beta_1 p_2 p_3 p_4 p_5 p_6}, p_6)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \in \mathcal{J}_4 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5 \\ \frac{1}{35} \leq t_6 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6) \notin \mathcal{I}_6}} S(\mathcal{A}_{\beta_1 p_2 p_3 p_4 p_5 p_6}, p_6) \\
& = S_{43521} - S_{43522} - S_{43523} + S_{43524} + S_{43525}.
\end{aligned} \tag{16}$$

We have an asymptotic formula for S_{43521} by Lemma 3.1. We have asymptotic formulas for S_{43522} and S_{43524} by Lemma 3.2. Since we have $\beta_1 p_5 \sim v^{1-t_1-t_2-t_3-t_4}$, we have an asymptotic formula for S_{43523} . For S_{43525} we can decompose some parts in a way similar to that in decomposing S_{43515} , but now we have two almost-prime variables. Note that S_{43525} counts numbers of the form $\beta_1 p_2 p_3 p_4 p_5 p_6 \beta_2$, where $\beta_2 \sim v^{t_1-t_6}$ and $(\beta_2, P(p_6)) = 1$. Now we can perform Buchstab iterations on either β_1 or β_2 . For the case of decomposing β_2 , we need the condition $(1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6, t_6) \in \mathcal{J}_7$, and the corresponding eight-dimensional sum is

$$\begin{aligned}
& \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \in \mathcal{J}_4 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5 \\ \frac{1}{35} \leq t_6 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6) \notin \mathcal{I}_6 \\ (1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6, t_6) \in \mathcal{J}_7 \\ \frac{1}{35} \leq t_7 < \min(t_6, \frac{1}{2}(t_1-t_6)) \\ (1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6, t_7) \notin \mathcal{I}_7 \\ \frac{1}{35} \leq t_8 < \min(t_7, \frac{1}{2}(t_1-t_6-t_7)) \\ (1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \notin \mathcal{I}_8}} S(\mathcal{A}_{\beta_1 p_2 p_3 p_4 p_5 p_6 p_7 p_8}, p_8).
\end{aligned} \tag{17}$$

For the case of decomposing β_1 , we need the condition $(t_1-t_6, t_2, t_3, t_4, t_5, t_6, t_5) \in \mathcal{J}_7$, and the corresponding eight-dimensional sum is

$$\begin{aligned}
& \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \in \mathcal{J}_4 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5 \\ \frac{1}{35} \leq t_6 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6) \notin \mathcal{I}_6 \\ (t_1-t_6, t_2, t_3, t_4, t_5, t_6, t_5) \in \mathcal{J}_7 \\ \frac{1}{35} \leq t_7 < \min(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)) \\ (t_1-t_6, t_2, t_3, t_4, t_5, t_6, t_7) \notin \mathcal{I}_7 \\ \frac{1}{35} \leq t_8 < \min(t_7, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_7)) \\ (t_1-t_6, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \notin \mathcal{I}_8}} S(\mathcal{A}_{\beta_2 p_2 p_3 p_4 p_5 p_6 p_7 p_8}, p_8).
\end{aligned} \tag{18}$$

We also need to discard the remaining parts of S_{43525} :

$$\begin{aligned}
& \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{J}_5 \\ (t_1, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \in \mathcal{J}_4 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)) \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5 \\ \frac{1}{35} \leq t_6 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6) \notin \mathcal{I}_6 \\ (1-t_1-t_2-t_3-t_4-t_5, t_2, t_3, t_4, t_5, t_6, t_6) \notin \mathcal{J}_7 \\ (t_1-t_6, t_2, t_3, t_4, t_5, t_6, t_5) \notin \mathcal{J}_7}} S(\mathcal{A}_{\beta_1 p_2 p_3 p_4 p_5 p_6}, p_6). \quad (19)
\end{aligned}$$

The corresponding loss from S_{4352} are three sums in (17), (18) and (19).

For the remaining S_{4353} we cannot use straightforward decompositions and role-reversals, but we can use Buchstab's identity in another way to get possible savings. Note that the numbers that S_{4353} counts is of the form $p_1 p_2 p_3 p_4 m_0$, where the smallest prime factor of m_0 is larger than p_4 . If $t_4 > \frac{1}{2}(1-t_1-t_2-t_3-t_4)$, we know that m_0 cannot have two or more prime factors and therefore must be a prime. Otherwise m_0 may have more than one prime factor, which means that we can “split” the smallest prime factor of m_0 and consider it separately:

$$\begin{aligned}
& \sum_{\substack{t_1, t_2, t_3, t_4 \\ t_4 < \frac{1}{2}(1-t_1-t_2-t_3-t_4)}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4) \\
&= \sum_{\substack{t_1, t_2, t_3, t_4 \\ t_4 < \frac{1}{2}(1-t_1-t_2-t_3-t_4)}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, \left(\frac{x}{p_1 p_2 p_3 p_4}\right)^{\frac{1}{2}}\right) + \sum_{\substack{t_1, t_2, t_3, t_4 \\ t_4 < t_5 < \frac{1}{2}(1-t_1-t_2-t_3-t_4)}} S(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5). \quad (20)
\end{aligned}$$

The sum $S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, \left(\frac{x}{p_1 p_2 p_3 p_4}\right)^{\frac{1}{2}}\right)$ counts numbers of the form $p_1 p_2 p_3 p_4 p'$, where $p' = m_0$. The sum $S(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5)$ counts numbers of the form $p_1 p_2 p_3 p_4 (p_5 m_1)$, where $p_5 m_1 = m_0$, $p_5 > p_4$ and all prime factors of m_1 are larger than p_5 . In this sum we have a new variable t_5 , which means that part of this sum may have an asymptotic formula. In this situation, we can give an asymptotic formula for part of the last sum on the right-hand side of (20), hence we can subtract its contribution from the loss from S_{4353} . Again, for the remaining part of $S(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5)$, we can do a similar process as (20) to subtract the contribution of the numbers $p_1 p_2 p_3 p_4 p_5 (p_6 m_2)$ that have asymptotic formulas. The above process can be rewritten as

$$\begin{aligned}
& \sum_{\substack{t_1, t_2, t_3, t_4 \\ t_4 < \frac{1}{2}(1-t_1-t_2-t_3-t_4)}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, \left(\frac{x}{p_1 p_2 p_3 p_4}\right)^{\frac{1}{2}}\right) \\
&= \sum_{\substack{t_1, t_2, t_3, t_4 \\ t_4 < \frac{1}{2}(1-t_1-t_2-t_3-t_4)}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4) - \sum_{\substack{t_1, t_2, t_3, t_4 \\ t_4 < t_5 < \frac{1}{2}(1-t_1-t_2-t_3-t_4)}} S(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5). \quad (21)
\end{aligned}$$

Since (21) is a direct application of Buchstab's identity, we shall call (20) “Buchstab's identity in reverse” or “reversed Buchstab's identity” in the rest of our paper. Since applying reversed Buchstab's identity twice or more times gives only a small saving over high-dimensional sums, we only apply it once on S_{4353} and

subtract the contribution of the sum

$$\sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{M} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5 \\ \text{either } (t_1, t_2, t_3, t_4) \notin \mathcal{J}_4 \text{ or } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \notin \mathcal{J}_4 \\ t_4 < t_5 < \frac{1}{2}(1-t_1-t_2-t_3-t_4) \\ (t_1, t_2, t_3, t_4, t_5) \in \mathcal{I}_5}} S(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5) \quad (22)$$

from S_{4353} . We discard other parts of S_{4353} .

Combining all the three cases above, we get a loss from S_{435} and also a loss from S_{43} of

$$\begin{aligned} & \left(\int_{(t_1, t_2, t_3, t_4) \in U_{\mathcal{M}1}} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1 \right) \\ & - \left(\int_{(t_1, t_2, t_3, t_4, t_5) \in U_{\mathcal{M}2}} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\ & + \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6) \in U_{\mathcal{M}3}} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{t_6}\right)}{t_1 t_2 t_3 t_4 t_5 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\ & + \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6) \in U_{\mathcal{M}4}} \frac{\omega\left(\frac{t_1-t_6}{t_6}\right) \omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_2 t_3 t_4 t_5^2 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\ & + \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \in U_{\mathcal{M}5}} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7-t_8}{t_8}\right)}{t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\ & + \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \in U_{\mathcal{M}6}} \frac{\omega\left(\frac{t_1-t_6-t_7-t_8}{t_8}\right) \omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_2 t_3 t_4 t_5^2 t_6 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\ & + \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \in U_{\mathcal{M}7}} \frac{\omega\left(\frac{t_1-t_6}{t_6}\right) \omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_7-t_8}{t_8}\right)}{t_2 t_3 t_4 t_5 t_6^2 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\ & \leq \left(\int_{(t_1, t_2, t_3, t_4) \in U_{\mathcal{M}1}} \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1 \right) \\ & - \left(\int_{(t_1, t_2, t_3, t_4, t_5) \in U_{\mathcal{M}2}} \frac{\omega_0\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\ & + \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6) \in U_{\mathcal{M}3}} \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{t_6}\right)}{t_1 t_2 t_3 t_4 t_5 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\ & + \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6) \in U_{\mathcal{M}4}} \frac{\omega_1\left(\frac{t_1-t_6}{t_6}\right) \omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_2 t_3 t_4 t_5^2 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \end{aligned}$$

$$\begin{aligned}
& + \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \in U_{\mathcal{M}5}} \frac{\max\left(\frac{t_8}{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7-t_8}, 0.5672\right)}{t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
& + \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \in U_{\mathcal{M}6}} \frac{\max\left(\frac{t_8}{t_1-t_6-t_7-t_8}, 0.5672\right) \max\left(\frac{t_5}{1-t_1-t_2-t_3-t_4-t_5}, 0.5672\right)}{t_2 t_3 t_4 t_5^2 t_6 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
& + \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \in U_{\mathcal{M}7}} \frac{\max\left(\frac{t_6}{t_1-t_6}, 0.5672\right) \max\left(\frac{t_8}{1-t_1-t_2-t_3-t_4-t_5-t_7-t_8}, 0.5672\right)}{t_2 t_3 t_4 t_5 t_6^2 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
& \leq (0.081234 - 0.002216 + 0.030572 + 0.061125 + 0.005742 + 0.000001 + 0.000001) \\
& = 0.176459,
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
U_{\mathcal{M}1}(t_1, t_2, t_3, t_4) &:= \left\{ (t_1, t_2) \in \mathcal{M}, \frac{1}{35} \leq t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), (t_1, t_2, t_3) \notin \mathcal{I}_3, \right. \\
&\quad \frac{1}{35} \leq t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\
&\quad (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4, (t_1, t_2, t_3, t_4, t_4) \notin \mathcal{J}_5, \\
&\quad \text{either } (t_1, t_2, t_3, t_4) \notin \mathcal{J}_4 \text{ or } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \notin \mathcal{J}_4, \\
&\quad \left. \frac{1}{35} \leq t_1 < \frac{17}{35}, \frac{1}{35} \leq t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \right\}, \\
U_{\mathcal{M}2}(t_1, t_2, t_3, t_4, t_5) &:= \left\{ (t_1, t_2) \in \mathcal{M}, \frac{1}{35} \leq t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), (t_1, t_2, t_3) \notin \mathcal{I}_3, \right. \\
&\quad \frac{1}{35} \leq t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\
&\quad (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4, (t_1, t_2, t_3, t_4, t_4) \notin \mathcal{J}_5, \\
&\quad \text{either } (t_1, t_2, t_3, t_4) \notin \mathcal{J}_4 \text{ or } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \notin \mathcal{J}_4, \\
&\quad t_4 < t_5 < \frac{1}{2}(1-t_1-t_2-t_3-t_4), (t_1, t_2, t_3, t_4, t_5) \in \mathcal{I}_5, \\
&\quad \left. \frac{1}{35} \leq t_1 < \frac{17}{35}, \frac{1}{35} \leq t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \right\}, \\
U_{\mathcal{M}3}(t_1, t_2, t_3, t_4, t_5, t_6) &:= \left\{ (t_1, t_2) \in \mathcal{M}, \frac{1}{35} \leq t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), (t_1, t_2, t_3) \notin \mathcal{I}_3, \right. \\
&\quad \frac{1}{35} \leq t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\
&\quad (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4, (t_1, t_2, t_3, t_4, t_4) \in \mathcal{J}_5, \\
&\quad \frac{1}{35} \leq t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right), (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5, \\
&\quad \frac{1}{35} \leq t_6 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right), \\
&\quad (t_1, t_2, t_3, t_4, t_5, t_6) \notin \mathcal{I}_6, (t_1, t_2, t_3, t_4, t_5, t_6, t_6) \notin \mathcal{J}_7, \\
&\quad \left. \frac{1}{35} \leq t_1 < \frac{17}{35}, \frac{1}{35} \leq t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \right\}, \\
U_{\mathcal{M}4}(t_1, t_2, t_3, t_4, t_5, t_6) &:= \left\{ (t_1, t_2) \in \mathcal{M}, \frac{1}{35} \leq t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), (t_1, t_2, t_3) \notin \mathcal{I}_3, \right.
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{35} \leq t_4 < \min \left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3) \right), \\
& (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4, (t_1, t_2, t_3, t_4, t_4) \notin \mathcal{J}_5, \\
& (t_1, t_2, t_3, t_4) \in \mathcal{J}_4, (1 - t_1 - t_2 - t_3 - t_4, t_2, t_3, t_4) \in \mathcal{J}_4, \\
& \frac{1}{35} \leq t_5 < \min \left(t_4, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4) \right), (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5, \\
& \frac{1}{35} \leq t_6 < \frac{1}{2}t_1, (1 - t_1 - t_2 - t_3 - t_4 - t_5, t_2, t_3, t_4, t_5, t_6) \notin \mathcal{I}_6, \\
& (1 - t_1 - t_2 - t_3 - t_4 - t_5, t_2, t_3, t_4, t_5, t_6, t_6) \notin \mathcal{J}_7, \\
& (t_1 - t_6, t_2, t_3, t_4, t_5, t_6, t_5) \notin \mathcal{J}_7, \\
& \frac{1}{35} \leq t_1 < \frac{17}{35}, \frac{1}{35} \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \Big\}, \\
U_{\mathcal{M}5}(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) := & \left\{ (t_1, t_2) \in \mathcal{M}, \frac{1}{35} \leq t_3 < \min \left(t_2, \frac{1}{2}(1 - t_1 - t_2) \right), (t_1, t_2, t_3) \notin \mathcal{I}_3, \right. \\
& \frac{1}{35} \leq t_4 < \min \left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3) \right), \\
& (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4, (t_1, t_2, t_3, t_4, t_4) \in \mathcal{J}_5, \\
& \frac{1}{35} \leq t_5 < \min \left(t_4, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4) \right), (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5, \\
& \frac{1}{35} \leq t_6 < \min \left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5) \right), \\
& (t_1, t_2, t_3, t_4, t_5, t_6) \notin \mathcal{I}_6, (t_1, t_2, t_3, t_4, t_5, t_6, t_6) \in \mathcal{J}_7, \\
& \frac{1}{35} \leq t_7 < \min \left(t_6, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5 - t_6) \right), \\
& (t_1, t_2, t_3, t_4, t_5, t_6, t_7) \notin \mathcal{I}_7, \\
& \frac{1}{35} \leq t_8 < \min \left(t_7, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5 - t_6 - t_7) \right), \\
& (t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \notin \mathcal{I}_8, \\
& \frac{1}{35} \leq t_1 < \frac{17}{35}, \frac{1}{35} \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \Big\}, \\
U_{\mathcal{M}6}(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) := & \left\{ (t_1, t_2) \in \mathcal{M}, \frac{1}{35} \leq t_3 < \min \left(t_2, \frac{1}{2}(1 - t_1 - t_2) \right), (t_1, t_2, t_3) \notin \mathcal{I}_3, \right. \\
& \frac{1}{35} \leq t_4 < \min \left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3) \right), \\
& (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4, (t_1, t_2, t_3, t_4, t_4) \notin \mathcal{J}_5, \\
& (t_1, t_2, t_3, t_4) \in \mathcal{J}_4, (1 - t_1 - t_2 - t_3 - t_4, t_2, t_3, t_4) \in \mathcal{J}_4, \\
& \frac{1}{35} \leq t_5 < \min \left(t_4, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4) \right), (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5, \\
& \frac{1}{35} \leq t_6 < \frac{1}{2}t_1, (1 - t_1 - t_2 - t_3 - t_4 - t_5, t_2, t_3, t_4, t_5, t_6) \notin \mathcal{I}_6, \\
& (1 - t_1 - t_2 - t_3 - t_4 - t_5, t_2, t_3, t_4, t_5, t_6, t_6) \in \mathcal{J}_7, \\
& \frac{1}{35} \leq t_7 < \min \left(t_6, \frac{1}{2}(t_1 - t_6) \right), \\
& (1 - t_1 - t_2 - t_3 - t_4 - t_5, t_2, t_3, t_4, t_5, t_6, t_7) \notin \mathcal{I}_7,
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{35} \leq t_8 < \min \left(t_7, \frac{1}{2}(t_1 - t_6 - t_7) \right), \\
& (1 - t_1 - t_2 - t_3 - t_4 - t_5, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \notin \mathcal{I}_8, \\
& \frac{1}{35} \leq t_1 < \frac{17}{35}, \frac{1}{35} \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \Bigg\}, \\
U_{\mathcal{M}7}(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) := & \left\{ (t_1, t_2) \in \mathcal{M}, \frac{1}{35} \leq t_3 < \min \left(t_2, \frac{1}{2}(1 - t_1 - t_2) \right), (t_1, t_2, t_3) \notin \mathcal{I}_3, \right. \\
& \frac{1}{35} \leq t_4 < \min \left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3) \right), \\
& (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4, (t_1, t_2, t_3, t_4, t_4) \notin \mathcal{J}_5, \\
& (t_1, t_2, t_3, t_4) \in \mathcal{J}_4, (1 - t_1 - t_2 - t_3 - t_4, t_2, t_3, t_4) \in \mathcal{J}_4, \\
& \frac{1}{35} \leq t_5 < \min \left(t_4, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4) \right), (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5, \\
& \frac{1}{35} \leq t_6 < \frac{1}{2}t_1, (1 - t_1 - t_2 - t_3 - t_4 - t_5, t_2, t_3, t_4, t_5, t_6) \notin \mathcal{I}_6, \\
& (1 - t_1 - t_2 - t_3 - t_4 - t_5, t_2, t_3, t_4, t_5, t_6, t_6) \notin \mathcal{J}_7, \\
& (t_1 - t_6, t_2, t_3, t_4, t_5, t_6, t_5) \in \mathcal{J}_7, \\
& \frac{1}{35} \leq t_7 < \min \left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5) \right), \\
& (t_1 - t_6, t_2, t_3, t_4, t_5, t_6, t_7) \notin \mathcal{I}_7, \\
& \frac{1}{35} \leq t_8 < \min \left(t_7, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5 - t_7) \right), \\
& (t_1 - t_6, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \notin \mathcal{I}_8, \\
& \left. \frac{1}{35} \leq t_1 < \frac{17}{35}, \frac{1}{35} \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \right\}.
\end{aligned}$$

Note that the seven integrals above correspond to S_{4353} and sums in (22), (12), (19), (11), (17) and (18) respectively.

Next we shall decompose S_{44} . By Buchstab's identity, we have

$$\begin{aligned}
S_{44} &= \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N}}} S(\mathcal{A}_{p_1 p_2}, p_2) \\
&= \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N}}} S(\mathcal{A}_{p_1 p_2}, v^{\frac{1}{35}}) - \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \in \mathcal{I}_3}} S(\mathcal{A}_{p_1 p_2 p_3}, p_3) \\
&\quad - \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \in \mathcal{J}_3}} S(\mathcal{A}_{p_1 p_2 p_3}, p_3) - \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3}} S(\mathcal{A}_{p_1 p_2 p_3}, p_3) \\
&= S_{441} - S_{442} - S_{443} - S_{444}.
\end{aligned} \tag{24}$$

By Lemma 3.1 and Lemma 3.2, we have asymptotic formulas for S_{441} and S_{442} .

For S_{443} , we can use Buchstab's identity again to get

$$\begin{aligned}
S_{443} &= \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \in \mathcal{J}_3}} S(\mathcal{A}_{p_1 p_2 p_3}, p_3) \\
&= \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \in \mathcal{J}_3}} S(\mathcal{A}_{p_1 p_2 p_3}, v^{\frac{1}{35}}) - \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \in \mathcal{J}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \in \mathcal{I}_4}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4) \\
&\quad - \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \in \mathcal{J}_3 \\ \frac{1}{35} \leq t_4 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4) \\
&= S_{4431} - S_{4432} - S_{4433}. \tag{25}
\end{aligned}$$

By Lemma 3.1 and Lemma 3.2, we have asymptotic formulas for S_{4431} and S_{4432} . For S_{4433} , we can perform the whole process of decomposing S_{435} to find the loss. The resulting integrals are very similar to those in (23), and the only change we need to do is to replace the condition $(t_1, t_2) \in \mathcal{M}$ in those integrals by $(t_1, t_2) \in \mathcal{N}$ and $(t_1, t_2, t_3) \in \mathcal{J}_3$. Thus, the loss from S_{443} can be bounded by

$$\begin{aligned}
&\left(\int_{(t_1, t_2, t_3, t_4) \in U_{N01}} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1 \right) \\
&- \left(\int_{(t_1, t_2, t_3, t_4, t_5) \in U_{N02}} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
&+ \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6) \in U_{N03}} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{t_6}\right)}{t_1 t_2 t_3 t_4 t_5 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
&+ \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6) \in U_{N04}} \frac{\omega\left(\frac{t_1-t_6}{t_6}\right) \omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_2 t_3 t_4 t_5^2 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
&+ \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \in U_{N05}} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7-t_8}{t_8}\right)}{t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
&+ \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \in U_{N06}} \frac{\omega\left(\frac{t_1-t_6-t_7-t_8}{t_8}\right) \omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_2 t_3 t_4 t_5^2 t_6 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
&+ \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \in U_{N07}} \frac{\omega\left(\frac{t_1-t_6}{t_6}\right) \omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_7-t_8}{t_8}\right)}{t_2 t_3 t_4 t_5 t_6^2 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right)
\end{aligned}$$

$$\begin{aligned}
&\leq \left(\int_{(t_1, t_2, t_3, t_4) \in U_{\mathcal{N}01}} \frac{\omega_1 \left(\frac{1-t_1-t_2-t_3-t_4}{t_4} \right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1 \right) \\
&- \left(\int_{(t_1, t_2, t_3, t_4, t_5) \in U_{\mathcal{N}02}} \frac{\omega_0 \left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5} \right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
&+ \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6) \in U_{\mathcal{N}03}} \frac{\omega_1 \left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{t_6} \right)}{t_1 t_2 t_3 t_4 t_5 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
&+ \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6) \in U_{\mathcal{N}04}} \frac{\omega_1 \left(\frac{t_1-t_6}{t_6} \right) \omega_1 \left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5} \right)}{t_2 t_3 t_4 t_5^2 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
&+ \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \in U_{\mathcal{N}05}} \frac{\max \left(\frac{t_8}{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7-t_8}, 0.5672 \right)}{t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
&+ \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \in U_{\mathcal{N}06}} \frac{\max \left(\frac{t_8}{t_1-t_6-t_7-t_8}, 0.5672 \right) \max \left(\frac{t_5}{1-t_1-t_2-t_3-t_4-t_5}, 0.5672 \right)}{t_2 t_3 t_4 t_5^2 t_6 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
&+ \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \in U_{\mathcal{N}07}} \frac{\max \left(\frac{t_6}{t_1-t_6}, 0.5672 \right) \max \left(\frac{t_8}{1-t_1-t_2-t_3-t_4-t_5-t_7-t_8}, 0.5672 \right)}{t_2 t_3 t_4 t_5 t_6^2 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
&\leq (0.016780 - 0 + 0.007814 + 0.015516 + 0.000001 + 0.000001 + 0.000001) \\
&= 0.040113,
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
U_{\mathcal{N}01}(t_1, t_2, t_3, t_4) &:= \left\{ (t_1, t_2) \in \mathcal{N}, \frac{1}{35} \leq t_3 < \min \left(t_2, \frac{1}{2}(1-t_1-t_2) \right), \right. \\
&\quad (t_1, t_2, t_3) \notin \mathcal{I}_3, (t_1, t_2, t_3) \in \mathcal{J}_3, \\
&\quad \frac{1}{35} \leq t_4 < \min \left(t_3, \frac{1}{2}(1-t_1-t_2-t_3) \right), \\
&\quad (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4, (t_1, t_2, t_3, t_4, t_4) \notin \mathcal{J}_5, \\
&\quad \text{either } (t_1, t_2, t_3, t_4) \notin \mathcal{J}_4 \text{ or } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \notin \mathcal{J}_4, \\
&\quad \left. \frac{1}{35} \leq t_1 < \frac{17}{35}, \frac{1}{35} \leq t_2 < \min \left(t_1, \frac{1}{2}(1-t_1) \right) \right\}, \\
U_{\mathcal{N}02}(t_1, t_2, t_3, t_4, t_5) &:= \left\{ (t_1, t_2) \in \mathcal{N}, \frac{1}{35} \leq t_3 < \min \left(t_2, \frac{1}{2}(1-t_1-t_2) \right), \right. \\
&\quad (t_1, t_2, t_3) \notin \mathcal{I}_3, (t_1, t_2, t_3) \in \mathcal{J}_3, \\
&\quad \frac{1}{35} \leq t_4 < \min \left(t_3, \frac{1}{2}(1-t_1-t_2-t_3) \right), \\
&\quad (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4, (t_1, t_2, t_3, t_4, t_4) \notin \mathcal{J}_5, \\
&\quad \text{either } (t_1, t_2, t_3, t_4) \notin \mathcal{J}_4 \text{ or } (1-t_1-t_2-t_3-t_4, t_2, t_3, t_4) \notin \mathcal{J}_4, \\
&\quad t_4 < t_5 < \frac{1}{2}(1-t_1-t_2-t_3-t_4), (t_1, t_2, t_3, t_4, t_5) \in \mathcal{I}_5, \\
&\quad \left. \frac{1}{35} \leq t_1 < \frac{17}{35}, \frac{1}{35} \leq t_2 < \min \left(t_1, \frac{1}{2}(1-t_1) \right) \right\},
\end{aligned}$$

$$\begin{aligned}
U_{N03}(t_1, t_2, t_3, t_4, t_5, t_6) := & \left\{ (t_1, t_2) \in \mathcal{N}, \frac{1}{35} \leq t_3 < \min\left(t_2, \frac{1}{2}(1 - t_1 - t_2)\right), \right. \\
& (t_1, t_2, t_3) \notin \mathcal{I}_3, (t_1, t_2, t_3) \in \mathcal{J}_3, \\
& \frac{1}{35} \leq t_4 < \min\left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3)\right), \\
& (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4, (t_1, t_2, t_3, t_4) \in \mathcal{J}_5, \\
& \frac{1}{35} \leq t_5 < \min\left(t_4, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4)\right), (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5, \\
& \frac{1}{35} \leq t_6 < \min\left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5)\right), \\
& (t_1, t_2, t_3, t_4, t_5, t_6) \notin \mathcal{I}_6, (t_1, t_2, t_3, t_4, t_5, t_6, t_6) \notin \mathcal{J}_7, \\
& \left. \frac{1}{35} \leq t_1 < \frac{17}{35}, \frac{1}{35} \leq t_2 < \min\left(t_1, \frac{1}{2}(1 - t_1)\right) \right\}, \\
U_{N04}(t_1, t_2, t_3, t_4, t_5, t_6) := & \left\{ (t_1, t_2) \in \mathcal{N}, \frac{1}{35} \leq t_3 < \min\left(t_2, \frac{1}{2}(1 - t_1 - t_2)\right), \right. \\
& (t_1, t_2, t_3) \notin \mathcal{I}_3, (t_1, t_2, t_3) \in \mathcal{J}_3, \\
& \frac{1}{35} \leq t_4 < \min\left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3)\right), \\
& (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4, (t_1, t_2, t_3, t_4, t_4) \notin \mathcal{J}_5, \\
& (t_1, t_2, t_3, t_4) \in \mathcal{J}_4, (1 - t_1 - t_2 - t_3 - t_4, t_2, t_3, t_4) \in \mathcal{J}_4, \\
& \frac{1}{35} \leq t_5 < \min\left(t_4, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4)\right), (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5, \\
& \frac{1}{35} \leq t_6 < \frac{1}{2}t_1, (1 - t_1 - t_2 - t_3 - t_4 - t_5, t_2, t_3, t_4, t_5, t_6) \notin \mathcal{I}_6, \\
& (1 - t_1 - t_2 - t_3 - t_4 - t_5, t_2, t_3, t_4, t_5, t_6, t_6) \notin \mathcal{J}_7, \\
& (t_1 - t_6, t_2, t_3, t_4, t_5, t_6, t_5) \notin \mathcal{J}_7, \\
& \left. \frac{1}{35} \leq t_1 < \frac{17}{35}, \frac{1}{35} \leq t_2 < \min\left(t_1, \frac{1}{2}(1 - t_1)\right) \right\}, \\
U_{N05}(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) := & \left\{ (t_1, t_2) \in \mathcal{N}, \frac{1}{35} \leq t_3 < \min\left(t_2, \frac{1}{2}(1 - t_1 - t_2)\right), \right. \\
& (t_1, t_2, t_3) \notin \mathcal{I}_3, (t_1, t_2, t_3) \in \mathcal{J}_3, \\
& \frac{1}{35} \leq t_4 < \min\left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3)\right), \\
& (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4, (t_1, t_2, t_3, t_4, t_4) \in \mathcal{J}_5, \\
& \frac{1}{35} \leq t_5 < \min\left(t_4, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4)\right), (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5, \\
& \frac{1}{35} \leq t_6 < \min\left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5)\right), \\
& (t_1, t_2, t_3, t_4, t_5, t_6) \notin \mathcal{I}_6, (t_1, t_2, t_3, t_4, t_5, t_6, t_6) \in \mathcal{J}_7, \\
& \frac{1}{35} \leq t_7 < \min\left(t_6, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5 - t_6)\right), \\
& (t_1, t_2, t_3, t_4, t_5, t_6, t_7) \notin \mathcal{I}_7, \\
& \left. \frac{1}{35} \leq t_8 < \min\left(t_7, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5 - t_6 - t_7)\right) \right\},
\end{aligned}$$

$$\begin{aligned}
& (t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \notin \mathcal{I}_8, \\
& \left. \frac{1}{35} \leq t_1 < \frac{17}{35}, \frac{1}{35} \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \right\}, \\
U_{N06}(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) := & \left\{ (t_1, t_2) \in \mathcal{N}, \frac{1}{35} \leq t_3 < \min \left(t_2, \frac{1}{2}(1 - t_1 - t_2) \right), \right. \\
& (t_1, t_2, t_3) \notin \mathcal{I}_3, (t_1, t_2, t_3) \in \mathcal{J}_3, \\
& \frac{1}{35} \leq t_4 < \min \left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3) \right), \\
& (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4, (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{J}_5, \\
& (t_1, t_2, t_3, t_4) \in \mathcal{J}_4, (1 - t_1 - t_2 - t_3 - t_4, t_2, t_3, t_4) \in \mathcal{J}_4, \\
& \frac{1}{35} \leq t_5 < \min \left(t_4, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4) \right), (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5, \\
& \frac{1}{35} \leq t_6 < \frac{1}{2}t_1, (1 - t_1 - t_2 - t_3 - t_4 - t_5, t_2, t_3, t_4, t_5, t_6) \notin \mathcal{I}_6, \\
& (1 - t_1 - t_2 - t_3 - t_4 - t_5, t_2, t_3, t_4, t_5, t_6, t_7) \in \mathcal{J}_7, \\
& \frac{1}{35} \leq t_7 < \min \left(t_6, \frac{1}{2}(t_1 - t_6) \right), \\
& (1 - t_1 - t_2 - t_3 - t_4 - t_5, t_2, t_3, t_4, t_5, t_6, t_7) \notin \mathcal{I}_7, \\
& \frac{1}{35} \leq t_8 < \min \left(t_7, \frac{1}{2}(t_1 - t_6 - t_7) \right), \\
& (1 - t_1 - t_2 - t_3 - t_4 - t_5, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \notin \mathcal{I}_8, \\
& \left. \frac{1}{35} \leq t_1 < \frac{17}{35}, \frac{1}{35} \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \right\}, \\
U_{N07}(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) := & \left\{ (t_1, t_2) \in \mathcal{N}, \frac{1}{35} \leq t_3 < \min \left(t_2, \frac{1}{2}(1 - t_1 - t_2) \right), \right. \\
& (t_1, t_2, t_3) \notin \mathcal{I}_3, (t_1, t_2, t_3) \in \mathcal{J}_3, \\
& \frac{1}{35} \leq t_4 < \min \left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3) \right), \\
& (t_1, t_2, t_3, t_4) \notin \mathcal{I}_4, (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{J}_5, \\
& (t_1, t_2, t_3, t_4) \in \mathcal{J}_4, (1 - t_1 - t_2 - t_3 - t_4, t_2, t_3, t_4) \in \mathcal{J}_4, \\
& \frac{1}{35} \leq t_5 < \min \left(t_4, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4) \right), (t_1, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5, \\
& \frac{1}{35} \leq t_6 < \frac{1}{2}t_1, (1 - t_1 - t_2 - t_3 - t_4 - t_5, t_2, t_3, t_4, t_5, t_6) \notin \mathcal{I}_6, \\
& (1 - t_1 - t_2 - t_3 - t_4 - t_5, t_2, t_3, t_4, t_5, t_6, t_7) \notin \mathcal{J}_7, \\
& (t_1 - t_6, t_2, t_3, t_4, t_5, t_6, t_7) \in \mathcal{J}_7, \\
& \frac{1}{35} \leq t_7 < \min \left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5) \right), \\
& (t_1 - t_6, t_2, t_3, t_4, t_5, t_6, t_7) \notin \mathcal{I}_7, \\
& \frac{1}{35} \leq t_8 < \min \left(t_7, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5 - t_7) \right), \\
& (t_1 - t_6, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \notin \mathcal{I}_8, \\
& \left. \frac{1}{35} \leq t_1 < \frac{17}{35}, \frac{1}{35} \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \right\}.
\end{aligned}$$

For S_{444} we cannot perform Buchstab iteration directly as in (25) since we have $(t_1, t_2, t_3) \notin \mathcal{J}_3$ in this sum. However, we can use role-reversals together with Buchstab's identity to decompose this sum further. Note that S_{444} counts numbers of the form $p_1 p_2 p_3 \beta_3$, where $\beta_3 \sim v^{1-t_1-t_2-t_3}$ and $(\beta_3, P(p_3)) = 1$. By reversing the roles of p_1 and β_3 , together with Buchstab's identity, we have

$$\begin{aligned}
S_{444} &= \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3}} S(\mathcal{A}_{p_1 p_2 p_3}, p_3) \\
&= \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3}} S\left(\mathcal{A}_{\beta_3 p_2 p_3}, \left(\frac{2v}{\beta_3 p_2 p_3}\right)^{\frac{1}{2}}\right) \\
&= \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3}} S\left(\mathcal{A}_{\beta_3 p_2 p_3}, v^{\frac{1}{35}}\right) - \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3 \\ \frac{1}{35} \leq t_4 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \in \mathcal{I}_4}} S(\mathcal{A}_{\beta_3 p_2 p_3 p_4}, p_4) \\
&\quad - \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3 \\ \frac{1}{35} \leq t_4 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{I}_4}} S(\mathcal{A}_{\beta_3 p_2 p_3 p_4}, p_4) \\
&= S_{4441} - S_{4442} - S_{4443}. \tag{27}
\end{aligned}$$

We have an asymptotic formula for S_{4442} by Lemma 3.2. Since we have $t_2 < \frac{53}{255} < \frac{9}{35}$ in region \mathcal{N} , we must have $t_1 + t_2 > \frac{18}{35}$. Otherwise we have $(t_1, t_2) \in \mathcal{M}$ by Lemma 2.2 and the definition of \mathcal{M} with $m = t_1$ and $n = t_2$. Now we have $1 - t_1 - t_2 < \frac{17}{35}$ for $(t_1, t_2) \in \mathcal{N}$. Since $\beta_3 p_3 \sim x^{1-t_1-t_2}$, we have $(1 - t_1 - t_2 - t_3, t_2, t_3) \in \mathcal{J}_3$. By Lemma 3.1, we have an asymptotic formula for S_{4441} .

For the remaining S_{4443} , we first split it into four parts:

$$\begin{aligned}
S_{4443} &= \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3 \\ \frac{1}{35} \leq t_4 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{I}_4}} S(\mathcal{A}_{\beta_3 p_2 p_3 p_4}, p_4)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3 \\ \frac{1}{35} \leq t_4 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_4) \in \mathcal{J}_5}} S(\mathcal{A}_{\beta_3 p_2 p_3 p_4}, p_4) + \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3 \\ \frac{1}{35} \leq t_4 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_4) \notin \mathcal{J}_5 \\ (t_1-t_4, t_2, t_3, t_4, t_3) \in \mathcal{J}_5}} S(\mathcal{A}_{\beta_3 p_2 p_3 p_4}, p_4) \\
&+ \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3 \\ \frac{1}{35} \leq t_4 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_4) \notin \mathcal{J}_5 \\ (t_1-t_4, t_2, t_3, t_4, t_3) \notin \mathcal{J}_5 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3, t_1-t_4, t_3, \min(t_2, t_4)) \in \mathcal{J}_4}} S(\mathcal{A}_{\beta_3 p_2 p_3 p_4}, p_4) \\
&+ \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3 \\ \frac{1}{35} \leq t_4 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_4) \notin \mathcal{J}_5 \\ (t_1-t_4, t_2, t_3, t_4, t_3) \notin \mathcal{J}_5 \\ \text{either } (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{J}_4 \text{ or } (1-t_1-t_2-t_3, t_1-t_4, t_3, \min(t_2, t_4)) \notin \mathcal{J}_4}} S(\mathcal{A}_{\beta_3 p_2 p_3 p_4}, p_4) \\
&= S_{44431} + S_{44432} + S_{44433} + S_{44434}. \tag{28}
\end{aligned}$$

Similar to the sum S_{435} , we can decompose each part of S_{4443} using different ways. For S_{44431} we can apply Buchstab's identity twice directly, and the corresponding six-dimensional sum that will be discarded is

$$\begin{aligned}
&\sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3 \\ \frac{1}{35} \leq t_4 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_4) \in \mathcal{J}_5 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(t_1-t_4)) \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5 \\ \frac{1}{35} \leq t_6 < \min(t_5, \frac{1}{2}(t_1-t_4-t_5)) \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_5, t_6) \notin \mathcal{I}_6}} S(\mathcal{A}_{\beta_3 p_2 p_3 p_4 p_5 p_6}, p_6). \tag{29}
\end{aligned}$$

Note that S_{4443} counts numbers of the form $\beta_3 p_2 p_3 p_4 \beta_4$, where $\beta_4 \sim v^{t_1-t_4}$ and $(\beta_4, P(p_4)) = 1$. For S_{44432} we can rewrite it as

$$\sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3 \\ \frac{1}{35} \leq t_4 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_4) \notin \mathcal{J}_5 \\ (t_1-t_4, t_2, t_3, t_4, t_3) \in \mathcal{J}_5}} S(\mathcal{A}_{\beta_3 p_2 p_3 p_4}, p_4) = \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3 \\ \frac{1}{35} \leq t_4 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_4) \notin \mathcal{J}_5 \\ (t_1-t_4, t_2, t_3, t_4, t_3) \in \mathcal{J}_5}} S(\mathcal{A}_{\beta_4 p_2 p_3 p_4}, p_3) \quad (30)$$

and use Buchstab's identity twice on β_3 . The corresponding six-dimensional sum that will be discarded is

$$\sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3 \\ \frac{1}{35} \leq t_4 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_4) \notin \mathcal{J}_5 \\ (t_1-t_4, t_2, t_3, t_4, t_3) \in \mathcal{J}_5 \\ \frac{1}{35} \leq t_5 < \min(t_3, \frac{1}{2}(1-t_1-t_2-t_3)) \\ (t_1-t_4, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5 \\ \frac{1}{35} \leq t_6 < \min(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_5)) \\ (t_1-t_4, t_2, t_3, t_4, t_5, t_6) \notin \mathcal{I}_6}} S(\mathcal{A}_{\beta_4 p_2 p_3 p_4 p_5 p_6}, p_6). \quad (31)$$

For S_{44433} we use Buchstab's identity once:

$$\begin{aligned} S_{44433} &= \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3 \\ \frac{1}{35} \leq t_4 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_4) \notin \mathcal{J}_5 \\ (t_1-t_4, t_2, t_3, t_4, t_3) \notin \mathcal{J}_5 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3, t_1-t_4, t_3, \min(t_2, t_4)) \in \mathcal{J}_4}} S(\mathcal{A}_{\beta_3 p_2 p_3 p_4}, p_4) \\ &= \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3 \\ \frac{1}{35} \leq t_4 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_4) \notin \mathcal{J}_5 \\ (t_1-t_4, t_2, t_3, t_4, t_3) \notin \mathcal{J}_5 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3, t_1-t_4, t_3, \min(t_2, t_4)) \in \mathcal{J}_4}} S\left(\mathcal{A}_{\beta_3 p_2 p_3 p_4}, v^{\frac{1}{35}}\right) \end{aligned}$$

$$\begin{aligned}
& - \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3 \\ \frac{1}{35} \leq t_4 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_5) \notin \mathcal{J}_5 \\ (t_1-t_4, t_2, t_3, t_4, t_5) \notin \mathcal{J}_5 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3, t_1-t_4, t_3, \min(t_2, t_4)) \in \mathcal{J}_4 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(t_1-t_4)) \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_5) \in \mathcal{I}_5}} S(\mathcal{A}_{\beta_3 p_2 p_3 p_4 p_5}, p_5) \\
& - \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3 \\ \frac{1}{35} \leq t_4 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_5) \notin \mathcal{J}_5 \\ (t_1-t_4, t_2, t_3, t_4, t_5) \notin \mathcal{J}_5 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3, t_1-t_4, t_3, \min(t_2, t_4)) \in \mathcal{J}_4 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(t_1-t_4)) \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5}} S(\mathcal{A}_{\beta_3 p_2 p_3 p_4 p_5}, p_5) \\
& = S_{444331} - S_{444332} + S_{444333}. \tag{32}
\end{aligned}$$

We have asymptotic formulas for S_{444331} and S_{444332} by Lemma 3.1 and Lemma 3.2 respectively, and we want to perform a role-reversal on S_{444333} . Note that S_{444333} counts numbers of the form $\beta_3 p_2 p_3 p_4 p_5 \beta_5$, where $\beta_5 \sim v^{t_1-t_4-t_5}$ and $(\beta_5, P(p_5)) = 1$. Now we can either reverse the roles of β_5 and p_2 or reverse the roles of β_5 and p_4 . Since we do not know which prime variable is larger, we choose to reverse the roles of β_5 and $\min(p_2, p_4)$. We can write this process as:

$$\begin{aligned}
& S_{444333} = \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3 \\ \frac{1}{35} \leq t_4 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_5) \notin \mathcal{J}_5 \\ (t_1-t_4, t_2, t_3, t_4, t_5) \notin \mathcal{J}_5 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3, t_1-t_4, t_3, \min(t_2, t_4)) \in \mathcal{J}_4 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(t_1-t_4)) \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5}} S(\mathcal{A}_{\beta_3 p_2 p_3 p_4 p_5}, p_5) \\
& = \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3 \\ \frac{1}{35} \leq t_4 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_5) \notin \mathcal{J}_5 \\ (t_1-t_4, t_2, t_3, t_4, t_5) \notin \mathcal{J}_5 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \in \mathcal{J}_4 \\ (1-t_1-t_2-t_3, t_1-t_4, t_3, \min(t_2, t_4)) \in \mathcal{J}_4 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(t_1-t_4)) \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5}} S\left(\mathcal{A}_{\beta_3 \beta_5 p_3 p_5 \min(p_2, p_4)}, \left(\frac{2v}{\beta_3 \beta_5 p_3 p_5 \min(p_2, p_4)}\right)^{\frac{1}{2}}\right). \tag{33}
\end{aligned}$$

And, by an application of Buchstab's identity, the six-dimensional sum that we need to discard is

$$\begin{aligned}
& \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3 \\ \frac{1}{35} \leq t_4 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{J}_5 \\ (t_1-t_4, t_2, t_3, t_4, t_3) \notin \mathcal{J}_5 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \in \mathcal{J}_4 \text{ and } (1-t_1-t_2-t_3, t_1-t_4, t_3, \min(t_2, t_4)) \in \mathcal{J}_4 \\ \frac{1}{35} \leq t_5 < \min(t_4, \frac{1}{2}(t_1-t_4)) \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5 \\ \frac{1}{35} \leq t_4 < \frac{1}{2} \max(t_2, t_4) \\ (1-t_1-t_2-t_3, t_1-t_4-t_5, t_3, t_5, t_6, \min(t_2, t_4)) \notin \mathcal{I}_6}} S(\mathcal{A}_{\beta_3 \beta_5 p_3 p_5 p_6 \min(p_2, p_4)}, p_6). \quad (34)
\end{aligned}$$

For the remaining S_{44434} , we can use reversed Buchstab's identity to subtract the contribution of the almost-primes counted. Since we have two almost-prime variables β_3 and β_4 , we can subtract two sums

$$\begin{aligned}
& \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3 \\ \frac{1}{35} \leq t_4 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{J}_5 \\ (t_1-t_4, t_2, t_3, t_4, t_3) \notin \mathcal{J}_5 \\ \text{either } (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{J}_4 \text{ or } (1-t_1-t_2-t_3, t_1-t_4, t_3, \min(t_2, t_4)) \notin \mathcal{J}_4 \\ t_4 < t_5 < \frac{1}{2}(t_1-t_4) \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_5) \in \mathcal{I}_5}} S(\mathcal{A}_{\beta_3 p_2 p_3 p_4 p_5}, p_5) \quad (35)
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3 \\ \frac{1}{35} \leq t_4 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{J}_5 \\ (t_1-t_4, t_2, t_3, t_4, t_3) \notin \mathcal{J}_5 \\ \text{either } (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{J}_4 \text{ or } (1-t_1-t_2-t_3, t_1-t_4, t_3, \min(t_2, t_4)) \notin \mathcal{J}_4 \\ t_3 < t_5 < \frac{1}{2}(1-t_1-t_2-t_3) \\ (t_1-t_4, t_2, t_3, t_4, t_5) \in \mathcal{I}_5}} S(\mathcal{A}_{\beta_4 p_2 p_3 p_4 p_5}, p_5). \quad (36)
\end{aligned}$$

The sum (35) counts numbers of the form $\beta_3 p_2 p_3 p_4 (p_5 \beta_6)$, where $\beta_6 \sim v^{t_1-t_4-t_5}$ and $(\beta_6, P(p_5)) = 1$. The sum (36) counts numbers of the form $(\beta_7 p_6) p_2 p_3 p_4 \beta_4$, where $\beta_7 \sim v^{1-t_1-t_2-t_3-t_6}$ and $(\beta_7, P(p_6)) = 1$. Now, one can easily find that numbers of the form $(\beta_7 p_6) p_2 p_3 p_4 (p_5 \beta_6)$ are counted in both sum, and we must

subtract the following sum from the savings:

$$\begin{aligned}
& \sum_{\substack{\frac{1}{35} \leq t_1 < \frac{17}{35} \\ \frac{1}{35} \leq t_2 < \min(t_1, \frac{1}{2}(1-t_1)) \\ (t_1, t_2) \in \mathcal{N} \\ \frac{1}{35} \leq t_3 < \min(t_2, \frac{1}{2}(1-t_1-t_2)) \\ (t_1, t_2, t_3) \notin \mathcal{I}_3 \\ (t_1, t_2, t_3) \notin \mathcal{J}_3 \\ \frac{1}{35} \leq t_4 < \frac{1}{2}t_1 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{I}_4 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_5) \notin \mathcal{J}_5 \\ (t_1-t_4, t_2, t_3, t_4, t_5) \notin \mathcal{J}_5 \\ \text{either } (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{J}_4 \text{ or } (1-t_1-t_2-t_3, t_1-t_4, t_3, \min(t_2, t_4)) \notin \mathcal{J}_4 \\ t_4 < t_5 < \frac{1}{2}(t_1-t_4) \\ (1-t_1-t_2-t_3, t_2, t_3, t_4, t_5) \in \mathcal{I}_5 \\ t_3 < t_6 < \frac{1}{2}(1-t_1-t_2-t_3) \\ (t_1-t_4, t_2, t_3, t_4, t_6) \in \mathcal{I}_5}}
\end{aligned}$$

$$S(\mathcal{A}_{\beta_6 p_2 p_3 p_4 p_5 p_6, p_6}). \quad (37)$$

Combining all the four cases above, we get a loss from S_{4443} and also a loss from S_{444} of

$$\begin{aligned}
& \left(\int_{(t_1, t_2, t_3, t_4) \in U_{\mathcal{N}08}} \frac{\omega\left(\frac{t_1-t_4}{t_4}\right) \omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2 t_3^2 t_4^2} dt_4 dt_3 dt_2 dt_1 \right) \\
& - \left(\int_{(t_1, t_2, t_3, t_4, t_5) \in U_{\mathcal{N}09}} \frac{\omega\left(\frac{t_1-t_4-t_5}{t_5}\right) \omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2 t_3^2 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
& - \left(\int_{(t_1, t_2, t_3, t_4, t_5) \in U_{\mathcal{N}10}} \frac{\omega\left(\frac{t_1-t_4}{t_4}\right) \omega\left(\frac{1-t_1-t_2-t_3-t_5}{t_5}\right)}{t_2 t_3 t_4^2 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
& + \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6) \in U_{\mathcal{N}11}} \frac{\omega\left(\frac{t_1-t_4-t_5}{t_5}\right) \omega\left(\frac{1-t_1-t_2-t_3-t_6}{t_6}\right)}{t_2 t_3 t_4 t_5^2 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
& + \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6) \in U_{\mathcal{N}12}} \frac{\omega\left(\frac{t_1-t_4-t_5-t_6}{t_6}\right) \omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2 t_3^2 t_4 t_5 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
& + \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6) \in U_{\mathcal{N}13}} \frac{\omega\left(\frac{t_1-t_4}{t_4}\right) \omega\left(\frac{1-t_1-t_2-t_3-t_5-t_6}{t_6}\right)}{t_2 t_3 t_4^2 t_5 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
& + \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6) \in U_{\mathcal{N}14}} \frac{\omega\left(\frac{\max(t_2, t_4)-t_6}{t_6}\right) \omega\left(\frac{t_1-t_4-t_5}{t_5}\right) \omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{\min(t_2, t_4) t_3^2 t_5^2 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
& \leq \left(\int_{(t_1, t_2, t_3, t_4) \in U_{\mathcal{N}08}} \frac{\omega_1\left(\frac{t_1-t_4}{t_4}\right) \omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2 t_3^2 t_4^2} dt_4 dt_3 dt_2 dt_1 \right) \\
& - \left(\int_{(t_1, t_2, t_3, t_4, t_5) \in U_{\mathcal{N}09}} \frac{\omega_0\left(\frac{t_1-t_4-t_5}{t_5}\right) \omega_0\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2 t_3^2 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
& - \left(\int_{(t_1, t_2, t_3, t_4, t_5) \in U_{\mathcal{N}10}} \frac{\omega_0\left(\frac{t_1-t_4}{t_4}\right) \omega_0\left(\frac{1-t_1-t_2-t_3-t_5}{t_5}\right)}{t_2 t_3 t_4^2 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
& + \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6) \in U_{\mathcal{N}11}} \frac{\omega_1\left(\frac{t_1-t_4-t_5}{t_5}\right) \omega_1\left(\frac{1-t_1-t_2-t_3-t_6}{t_6}\right)}{t_2 t_3 t_4 t_5^2 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right)
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6) \in U_{\mathcal{N}12}} \frac{\omega_1 \left(\frac{t_1 - t_4 - t_5 - t_6}{t_6} \right) \omega_1 \left(\frac{1 - t_1 - t_2 - t_3}{t_3} \right)}{t_2 t_3^2 t_4 t_5 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
& + \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6) \in U_{\mathcal{N}13}} \frac{\omega_1 \left(\frac{t_1 - t_4}{t_4} \right) \omega_1 \left(\frac{1 - t_1 - t_2 - t_3 - t_5 - t_6}{t_6} \right)}{t_2 t_3 t_4^2 t_5 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
& + \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6) \in U_{\mathcal{N}14}} \frac{\omega_1 \left(\frac{\max(t_2, t_4) - t_6}{t_6} \right) \omega_1 \left(\frac{t_1 - t_4 - t_5}{t_5} \right) \omega_1 \left(\frac{1 - t_1 - t_2 - t_3}{t_3} \right)}{\min(t_2, t_4) t_3^2 t_5^2 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\
& \leq (0.057182 - 0.004301 - 0.000022 + 0.000001 + 0.002635 + 0.000001 + 0.000001) \\
& = 0.055497,
\end{aligned} \tag{38}$$

where

$$\begin{aligned}
U_{\mathcal{N}08}(t_1, t_2, t_3, t_4) &:= \left\{ (t_1, t_2) \in \mathcal{N}, \frac{1}{35} \leq t_3 < \min \left(t_2, \frac{1}{2}(1 - t_1 - t_2) \right), \right. \\
&\quad (t_1, t_2, t_3) \notin \mathcal{I}_3, (t_1, t_2, t_3) \notin \mathcal{J}_3, \\
&\quad \frac{1}{35} \leq t_4 < \frac{1}{2}t_1, (1 - t_1 - t_2 - t_3, t_2, t_3, t_4) \notin \mathcal{I}_4, \\
&\quad (1 - t_1 - t_2 - t_3, t_2, t_3, t_4, t_4) \notin \mathcal{J}_5, (t_1 - t_4, t_2, t_3, t_4, t_3) \notin \mathcal{J}_5, \\
&\quad \text{either } (1 - t_1 - t_2 - t_3, t_2, t_3, t_4) \notin \mathcal{J}_4 \\
&\quad \text{or } (1 - t_1 - t_2 - t_3, t_1 - t_4, t_3, \min(t_2, t_4)) \notin \mathcal{J}_4, \\
&\quad \left. \frac{1}{35} \leq t_1 < \frac{17}{35}, \frac{1}{35} \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \right\}, \\
U_{\mathcal{N}09}(t_1, t_2, t_3, t_4, t_5) &:= \left\{ (t_1, t_2) \in \mathcal{N}, \frac{1}{35} \leq t_3 < \min \left(t_2, \frac{1}{2}(1 - t_1 - t_2) \right), \right. \\
&\quad (t_1, t_2, t_3) \notin \mathcal{I}_3, (t_1, t_2, t_3) \notin \mathcal{J}_3, \\
&\quad \frac{1}{35} \leq t_4 < \frac{1}{2}t_1, (1 - t_1 - t_2 - t_3, t_2, t_3, t_4) \notin \mathcal{I}_4, \\
&\quad (1 - t_1 - t_2 - t_3, t_2, t_3, t_4, t_4) \notin \mathcal{J}_5, (t_1 - t_4, t_2, t_3, t_4, t_3) \notin \mathcal{J}_5, \\
&\quad \text{either } (1 - t_1 - t_2 - t_3, t_2, t_3, t_4) \notin \mathcal{J}_4 \\
&\quad \text{or } (1 - t_1 - t_2 - t_3, t_1 - t_4, t_3, \min(t_2, t_4)) \notin \mathcal{J}_4, \\
&\quad t_4 < t_5 < \frac{1}{2}(t_1 - t_4), (1 - t_1 - t_2 - t_3, t_2, t_3, t_4, t_5) \in \mathcal{I}_5, \\
&\quad \left. \frac{1}{35} \leq t_1 < \frac{17}{35}, \frac{1}{35} \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \right\}, \\
U_{\mathcal{N}10}(t_1, t_2, t_3, t_4, t_5) &:= \left\{ (t_1, t_2) \in \mathcal{N}, \frac{1}{35} \leq t_3 < \min \left(t_2, \frac{1}{2}(1 - t_1 - t_2) \right), \right. \\
&\quad (t_1, t_2, t_3) \notin \mathcal{I}_3, (t_1, t_2, t_3) \notin \mathcal{J}_3, \\
&\quad \frac{1}{35} \leq t_4 < \frac{1}{2}t_1, (1 - t_1 - t_2 - t_3, t_2, t_3, t_4) \notin \mathcal{I}_4, \\
&\quad (1 - t_1 - t_2 - t_3, t_2, t_3, t_4, t_4) \notin \mathcal{J}_5, (t_1 - t_4, t_2, t_3, t_4, t_3) \notin \mathcal{J}_5, \\
&\quad \text{either } (1 - t_1 - t_2 - t_3, t_2, t_3, t_4) \notin \mathcal{J}_4 \\
&\quad \text{or } (1 - t_1 - t_2 - t_3, t_1 - t_4, t_3, \min(t_2, t_4)) \notin \mathcal{J}_4, \\
&\quad t_3 < t_5 < \frac{1}{2}(1 - t_1 - t_2 - t_3), (t_1 - t_4, t_2, t_3, t_4, t_5) \in \mathcal{I}_5, \\
&\quad \left. \frac{1}{35} \leq t_1 < \frac{17}{35}, \frac{1}{35} \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \right\},
\end{aligned}$$

$$\begin{aligned}
U_{N11}(t_1, t_2, t_3, t_4, t_5, t_6) := & \left\{ (t_1, t_2) \in \mathcal{N}, \frac{1}{35} \leq t_3 < \min\left(t_2, \frac{1}{2}(1 - t_1 - t_2)\right), \right. \\
& (t_1, t_2, t_3) \notin \mathcal{I}_3, (t_1, t_2, t_3) \notin \mathcal{J}_3, \\
& \frac{1}{35} \leq t_4 < \frac{1}{2}t_1, (1 - t_1 - t_2 - t_3, t_2, t_3, t_4) \notin \mathcal{I}_4, \\
& (1 - t_1 - t_2 - t_3, t_2, t_3, t_4, t_4) \notin \mathcal{J}_5, (t_1 - t_4, t_2, t_3, t_4, t_3) \notin \mathcal{J}_5, \\
& \text{either } (1 - t_1 - t_2 - t_3, t_2, t_3, t_4) \notin \mathcal{J}_4 \\
& \quad \text{or } (1 - t_1 - t_2 - t_3, t_1 - t_4, t_3, \min(t_2, t_4)) \notin \mathcal{J}_4, \\
& t_4 < t_5 < \frac{1}{2}(t_1 - t_4), (1 - t_1 - t_2 - t_3, t_2, t_3, t_4, t_5) \in \mathcal{I}_5, \\
& t_3 < t_6 < \frac{1}{2}(1 - t_1 - t_2 - t_3), (t_1 - t_4, t_2, t_3, t_4, t_6) \in \mathcal{I}_5, \\
& \left. \frac{1}{35} \leq t_1 < \frac{17}{35}, \frac{1}{35} \leq t_2 < \min\left(t_1, \frac{1}{2}(1 - t_1)\right) \right\}, \\
U_{N12}(t_1, t_2, t_3, t_4, t_5, t_6) := & \left\{ (t_1, t_2) \in \mathcal{N}, \frac{1}{35} \leq t_3 < \min\left(t_2, \frac{1}{2}(1 - t_1 - t_2)\right), \right. \\
& (t_1, t_2, t_3) \notin \mathcal{I}_3, (t_1, t_2, t_3) \notin \mathcal{J}_3, \\
& \frac{1}{35} \leq t_4 < \frac{1}{2}t_1, (1 - t_1 - t_2 - t_3, t_2, t_3, t_4) \notin \mathcal{I}_4, \\
& (1 - t_1 - t_2 - t_3, t_2, t_3, t_4, t_4) \in \mathcal{J}_5, \\
& \frac{1}{35} \leq t_5 < \min\left(t_4, \frac{1}{2}(t_1 - t_4)\right), \\
& (1 - t_1 - t_2 - t_3, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5, \\
& \frac{1}{35} \leq t_6 < \min\left(t_5, \frac{1}{2}(t_1 - t_4 - t_5)\right), \\
& (1 - t_1 - t_2 - t_3, t_2, t_3, t_4, t_5, t_6) \notin \mathcal{I}_6, \\
& \left. \frac{1}{35} \leq t_1 < \frac{17}{35}, \frac{1}{35} \leq t_2 < \min\left(t_1, \frac{1}{2}(1 - t_1)\right) \right\}, \\
U_{N13}(t_1, t_2, t_3, t_4, t_5, t_6) := & \left\{ (t_1, t_2) \in \mathcal{N}, \frac{1}{35} \leq t_3 < \min\left(t_2, \frac{1}{2}(1 - t_1 - t_2)\right), \right. \\
& (t_1, t_2, t_3) \notin \mathcal{I}_3, (t_1, t_2, t_3) \notin \mathcal{J}_3, \\
& \frac{1}{35} \leq t_4 < \frac{1}{2}t_1, (1 - t_1 - t_2 - t_3, t_2, t_3, t_4) \notin \mathcal{I}_4, \\
& (1 - t_1 - t_2 - t_3, t_2, t_3, t_4, t_4) \notin \mathcal{J}_5, (t_1 - t_4, t_2, t_3, t_4, t_3) \in \mathcal{J}_5, \\
& \frac{1}{35} \leq t_5 < \min\left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3)\right), \\
& (t_1 - t_4, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5, \\
& \frac{1}{35} \leq t_6 < \min\left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_5)\right), \\
& (t_1 - t_4, t_2, t_3, t_4, t_5, t_6) \notin \mathcal{I}_6, \\
& \left. \frac{1}{35} \leq t_1 < \frac{17}{35}, \frac{1}{35} \leq t_2 < \min\left(t_1, \frac{1}{2}(1 - t_1)\right) \right\}, \\
U_{N14}(t_1, t_2, t_3, t_4, t_5, t_6) := & \left\{ (t_1, t_2) \in \mathcal{N}, \frac{1}{35} \leq t_3 < \min\left(t_2, \frac{1}{2}(1 - t_1 - t_2)\right), \right. \\
& (t_1, t_2, t_3) \notin \mathcal{I}_3, (t_1, t_2, t_3) \notin \mathcal{J}_3,
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{35} \leq t_4 < \frac{1}{2}t_1, (1-t_1-t_2-t_3, t_2, t_3, t_4) \notin \mathcal{I}_4, \\
& (1-t_1-t_2-t_3, t_2, t_3, t_4, t_4) \notin \mathcal{J}_5, (t_1-t_4, t_2, t_3, t_4, t_3) \notin \mathcal{J}_5, \\
& (1-t_1-t_2-t_3, t_2, t_3, t_4) \in \mathcal{J}_4 \\
& \text{and } (1-t_1-t_2-t_3, t_1-t_4, t_3, \min(t_2, t_4)) \in \mathcal{J}_4, \\
& \frac{1}{35} \leq t_5 < \min\left(t_4, \frac{1}{2}(t_1-t_4)\right), \\
& (1-t_1-t_2-t_3, t_2, t_3, t_4, t_5) \notin \mathcal{I}_5, \\
& \frac{1}{35} \leq t_6 < \frac{1}{2} \max(t_2, t_4), \\
& (1-t_1-t_2-t_3, t_1-t_4-t_5, t_3, t_5, t_6, \min(t_2, t_4)) \notin \mathcal{I}_6, \\
& \frac{1}{35} \leq t_1 < \frac{17}{35}, \frac{1}{35} \leq t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \Big\}.
\end{aligned}$$

Note that the seven integrals above correspond to S_{44434} and sums in (35), (36), (37), (29), (31) and (34) respectively.

Finally, by (7), (23), (26) and (38), the total loss is less than

$$0.7226 + 0.176459 + 0.040113 + 0.055497 < 0.995,$$

and we have

$$S\left(\mathcal{A}, (2v)^{\frac{1}{2}}\right) > 0.005 \left(\sum_{P < p \leq 2P} \frac{1}{p} \right)^K \frac{x^{\frac{1}{2}+\varepsilon}}{\log v}.$$

Now the proof of Theorem 1 is completed.

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