ON THE PIATETSKI–SHAPIRO PRIME NUMBER THEOREM II

RUNBO LI

Abstract. The author prove that there are infinitely many primes of the form $[n^c]$ for $1 < c < \frac{919}{775}$. Using the theory of exponent pairs, the author also show that there are infinitely many almost primes of the form $[n^c]$ with some larger c.

Contents

1. Introduction

The Euler's conjecture, which states that there are infinitely many primes of the form $n^2 + 1$, is one of Landau's problems on prime numbers. There are several ways to attack this conjecture. One way is to consider the degree of the polynomial. In 1953, Piatetski–Shapiro [12] has proposed to investigate the prime numbers of the form $[n^c]$, where $c > 1$ and $[n^c]$ denotes the integer part of n^c. Clearly $[n^c]$ can be regarded as "polynomials of degree c". Define

$$\pi_c(x) := \left|\{n \leqslant x : [n^c] \text{ is a prime number}\}\right|,$$

then he has shown that $\pi_c(x) \sim x(c \log x)^{-1}$ holds for any $1 < c < \frac{12}{11} \approx 1.0909$ as $x \to \infty$. This range has been improved by many authors, and the best record now is due to Rivat and Sargos [14], where they proved the above asymptotic formula holds for any $1 < c < \frac{2817}{2426} \approx 1.1612$.

In 1992, Rivat [13] first introduced a sieve method into this problem. He established a lower bound with correct order (instead of an asymptotic formula) with $1 < c < \frac{7}{6} \approx 1.1616$. After this, many improvements were made and the range of c was enlarged successively to

$$1 < c < \frac{20}{17} \approx 1.1765,\ 1 < c < \frac{13}{11} \approx 1.1818,\ 1 < c < \frac{45}{38} \approx 1.1842,$$
$$1 < c < \frac{243}{205} \approx 1.18536 \quad \text{and} \quad 1 < c < \frac{211}{178} \approx 1.18539$$

by Jia [6] (and Baker, Harman and Rivat [1]), Jia [5], Kumchev [8], Rivat and Wu [15] and Li [9] respectively. In this paper, we obtain the following result.

Theorem 1.1. *For sufficiently large x and $1 < c < \frac{919}{775} \approx 1.1858$, we have $\pi_c(x) \gg x(\log x)^{-1}$.*

In 1992, Balog and Friedlander [2] considered a hybrid of the Three Primes Theorem and the Piatetski–Shapiro prime number theorem. They proved that every sufficiently large odd integer can be written as the sum of three primes of the form $[n^{c_0}]$ for any fixed $1 < c_0 < \frac{21}{20}$, and every sufficiently large odd integer can be written as the sum of two normal primes and another prime of the form $[n^{c_1}]$ for any fixed $1 < c_1 < \frac{9}{8}$. Their result has been improved by many authors. Using the same method as in [3] but with our Theorem 1.1, we can easily deduce the following.

2020 Mathematics Subject Classification. 11N35, 11N36.

Key words and phrases. prime, sieve methods, Piatetski–Shapiro sequences.

Theorem 1.2. *Every sufficiently large odd integer can be written as the sum of two normal primes and another prime of the form* $[n^{c_1}]$ *for any fixed* $1 < c_1 < \frac{919}{775}$.

However, if we consider the almost primes instead of primes, the results will be much better. Let P_r denotes an integer with at most r prime factors counted with multiplicity. In 2021, Guo [4] proved that there are infinitely many almost primes P_r of the form $[n^c]$ with

$$1 < c < \begin{cases} \frac{889}{741} \approx 1.1997, & r = 3, \\ \frac{25882}{16071} \approx 1.6104, & r = 4, \\ 3 - \frac{128}{3(8r-1)}, & r \geqslant 5. \end{cases}$$

In this paper, we shall use the exponent pair processes of Sargos [17] [16] together with the traditional processes to produce more efficient exponent pairs and improve the above result when $r = 3, 4$.

Theorem 1.3. *Let*

$$\pi_{c,r}(x) := \left|\{n \leqslant x : [n^c] = P_r\}\right|.$$

Then for sufficiently large x *and*

$$1 < c < \begin{cases} \frac{281563}{234507} \approx 1.2006, & r = 3, \\ \frac{51409}{31655} \approx 1.624, & r = 4, \end{cases}$$

we have $\pi_{c,r}(x) \gg x(\log x)^{-1}$.

Throughout this paper, we always suppose that x is a sufficiently large integer, γ and θ_0–θ_6 are positive numbers which will be fixed later. Let $\frac{37}{44} < \gamma < \frac{28}{33}$ and $c = \frac{1}{\gamma}$. The letter p, with or without subscript, is reserved for prime numbers. We define the sets $\mathcal{A}$ and $\mathcal{B}$ as

$$\mathcal{A} = \{m : m = [n^c],\ x \leqslant n^c < 2x\}, \quad \mathcal{B} = \{n : x \leqslant n < 2x\},$$

and we put

$$\mathcal{A}_d = \{a : ad \in \mathcal{A}\}, \quad \mathcal{B}_d = \{b : bd \in \mathcal{A}\}, \quad P(z) = \prod_{p<z} p, \quad S(\mathcal{A}, z) = \sum_{\substack{a \in \mathcal{A} \\ (a, P(z))=1}} 1, \quad S(\mathcal{B}, z) = \sum_{\substack{b \in \mathcal{B} \\ (b, P(z))=1}} 1.$$

Then we only need to show that $S\left(\mathcal{A}, (2x)^{\frac{1}{2}}\right) > 0$. Our aim is to show that the sparser set $\mathcal{A}$ contains the expected proportion of primes compared to the bigger set $\mathcal{B}$, which requires us to decompose $S\left(\mathcal{A}, (2x)^{\frac{1}{2}}\right)$ and prove asymptotic formulas of the form

$$S(\mathcal{A}, z) = (1 + o(1)) x^{\gamma - 1} \left(2^\gamma - 1\right) S(\mathcal{B}, z) \tag{1}$$

for some parts of it, and drop the other positive parts. The asymptotic formulas will be given in the next section.

2. Sieve asymptotic formulas

In this section we provide some asymptotic formulas for sieve functions. Let $\omega(u)$ denote the Buchstab function determined by the following differential–difference equation

$$\begin{cases} \omega(u) = \frac{1}{u}, & 1 \leqslant u \leqslant 2, \\ (u\omega(u))' = \omega(u-1), & u \geqslant 2. \end{cases}$$

Following [15] directly, we set $\gamma = \frac{775}{919}$, $\theta_0 = 6\gamma - 5$, $\theta_1 = 1 - \gamma$, $\theta_2 = \frac{61\gamma - 49}{11}$, $\theta_3 = 3 - 3\gamma$, $\theta_4 = 3\gamma - 2$, $\theta_5 = \frac{60 - 61\gamma}{11}$, $\theta_6 = \gamma$ and let $p_j = x^{t_j}$. We define the asymptotic region I as

$$\begin{aligned} I(m, n) := \{&\theta_1 \leqslant m < \theta_2 \text{ or } \theta_3 \leqslant m < \theta_4 \text{ or } \theta_5 \leqslant m < \theta_6 \text{ or} \\ &\theta_1 \leqslant m + n < \theta_2 \text{ or } \theta_3 \leqslant m + n < \theta_4 \text{ or } \theta_5 \leqslant m + n < \theta_6\}. \end{aligned}$$

We also define a new region I_2 as

$$\begin{aligned} I_2(m, n) := \{&m + n < \theta_4, \\ &\theta_1 \leqslant m + \theta_0 < m + n < \theta_2 \text{ or } \theta_3 \leqslant m + \theta_0 < m + n < \theta_4 \text{ or } \theta_5 \leqslant m + \theta_0 < m + n < \theta_6 \text{ or} \end{aligned}$$

$\theta_1 \leqslant n+\theta_0 < n+n < \theta_2$ or $\theta_3 \leqslant n+\theta_0 < n+n < \theta_4$ or $\theta_5 \leqslant n+\theta_0 < n+n < \theta_6$ or
$\theta_1 \leqslant m+n+\theta_0 < m+n+n < \theta_2$ or $\theta_3 \leqslant m+n+\theta_0 < m+n+n < \theta_4$ or
$\theta_5 \leqslant m+n+\theta_0 < m+n+n < \theta_6\}$.

Lemma 2.1. *We can give an asymptotic formula for*

$$\sum_{t_1\cdots t_n} S\left(\mathcal{A}_{p_1\cdots p_n}, x^{\theta_0}\right)$$

if we have $t_1+\ldots+t_n < \theta_4$.

Lemma 2.2. *We can give an asymptotic formula for*

$$\sum_{t_1\cdots t_n} S\left(\mathcal{A}_{p_1\cdots p_n}, p_n\right)$$

if we can group $(t_1,\ldots,t_n)$ into $(m,n)\in I$.

Lemma 2.3. *We can give an asymptotic formula for*

$$\sum_{t_1,t_2} S\left(\mathcal{A}_{p_1p_2}, p_2\right)$$

if we have $(t_1,t_2)\in I_2$.

Proof. By Buchstab's identity, we have

$$\sum_{t_1,t_2} S\left(\mathcal{A}_{p_1p_2}, p_2\right) = \sum_{t_1,t_2} S\left(\mathcal{A}_{p_1p_2}, x^{\theta_0}\right) - \sum_{\substack{t_1,t_2\\ \theta_0\leqslant t_3<\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right). \tag{2}$$

We can give asymptotic formulas for the first sum on the right hand side by Lemma 2.1 and the second sum on the right hand side by Lemma 2.2. Thus, Lemma 2.3 is proved. Note that this technique was also used in [7] and [10]. □

3. The final decomposition

Before decomposing, we define non–overlapping regions U_1–U_3 as

$$\begin{aligned}
U_1(m,n) &:= \ \{(m,n)\notin I\cup I_2,\ m+2n<\theta_4\}\\
U_2(m,n) &:= \ \left\{(m,n)\notin I\cup I_2,\ m+2n\geqslant\theta_4,\ \frac{1-m-n}{n}<2\right\},\\
U_3(m,n) &:= \ \left\{(m,n)\notin I\cup I_2,\ m+2n\geqslant\theta_4,\ \frac{1-m-n}{n}\geqslant 2\right\}.
\end{aligned}$$

We shall apply different techniques to the different regions above. By Buchstab's identity, we have

$$\begin{aligned}
S\left(\mathcal{A},(2x)^{\frac{1}{2}}\right) &= S\left(\mathcal{A},x^{\theta_0}\right) - \sum_{\theta_0\leqslant t_1<\frac{1}{2}} S\left(\mathcal{A}_{p_1},x^{\theta_0}\right) + \sum_{\substack{\theta_0\leqslant t_1<\frac{1}{2}\\ \theta_0\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)}} S\left(\mathcal{A}_{p_1p_2},p_2\right)\\
&= S\left(\mathcal{A},x^{\theta_0}\right) - \sum_{\theta_0\leqslant t_1<\frac{1}{2}} S\left(\mathcal{A}_{p_1},x^{\theta_0}\right) + \sum_{\substack{\theta_0\leqslant t_1<\frac{1}{2}\\ \theta_0\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in I\cup I_2}} S\left(\mathcal{A}_{p_1p_2},p_2\right)\\
&\quad + \sum_{\substack{\theta_0\leqslant t_1<\frac{1}{2}\\ \theta_0\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in U_1}} S\left(\mathcal{A}_{p_1p_2},p_2\right) + \sum_{\substack{\theta_0\leqslant t_1<\frac{1}{2}\\ \theta_0\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in U_2}} S\left(\mathcal{A}_{p_1p_2},p_2\right)\\
&\quad + \sum_{\substack{\theta_0\leqslant t_1<\frac{1}{2}\\ \theta_0\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\\ (t_1,t_2)\in U_3}} S\left(\mathcal{A}_{p_1p_2},p_2\right)
\end{aligned}$$

$$= S_1 - S_2 + S_I + S_{U1} + S_{U2} + S_{U3}. \tag{3}$$

By Lemma 2.1 and Lemma 2.2, we can give asymptotic formulas for S_1, S_2 and S_I. For S_{U1}, we can use Buchstab's identity twice more to get

$$\begin{aligned}
S_{U1} = \sum_{\substack{\theta_0 \leqslant t_1 < \frac{1}{2} \\ \theta_0 \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in U_1}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) = & \sum_{\substack{\theta_0 \leqslant t_1 < \frac{1}{2} \\ \theta_0 \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in U_1}} S\left(\mathcal{A}_{p_1 p_2}, x^{\theta_0}\right) \\
& - \sum_{\substack{\theta_0 \leqslant t_1 < \frac{1}{2} \\ \theta_0 \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in U_1 \\ \theta_0 \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ can be partitioned into } (m,n) \in I}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right) \\
& - \sum_{\substack{\theta_0 \leqslant t_1 < \frac{1}{2} \\ \theta_0 \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in U_1 \\ \theta_0 \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n) \in I}} S\left(\mathcal{A}_{p_1 p_2 p_3}, x^{\theta_0}\right) \\
& + \sum_{\substack{\theta_0 \leqslant t_1 < \frac{1}{2} \\ \theta_0 \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in U_1 \\ \theta_0 \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n) \in I \\ \theta_0 \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ can be partitioned into } (m,n) \in I}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right) \\
& + \sum_{\substack{\theta_0 \leqslant t_1 < \frac{1}{2} \\ \theta_0 \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2) \in U_1 \\ \theta_0 \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n) \in I \\ \theta_0 \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right) \\
= & S_{U11} - S_{U12} - S_{U13} + S_{U14} + S_{U15}.
\end{aligned} \tag{4}$$

We can give asymptotic formulas for S_{U11}–S_{U14}. For S_{U15} we can perform Buchstab's identity more times to make savings, but we choose to discard all of it for the sake of simplicity. Combining the above cases, we get a loss from S_{U1} of

$$\begin{aligned}
& \int_{\theta_0}^{\frac{1}{2}} \int_{\theta_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\theta_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\theta_0}^{\min\left(t_3, \frac{1-t_1-t_2-t_3}{2}\right)} \mathbb{1}_{(t_1,t_2,t_3,t_4) \in U_{15}} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1 \\
& < 0.001993,
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
U_{15}(t_1,t_2,t_3,t_4) := \Big\{ & (t_1,t_2) \in U_1,\ \theta_0 \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \\
& (t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n) \in I, \\
& \theta_0 \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\
& (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in I \Big\}.
\end{aligned}$$

For S_{U2}, we cannot decompose further but have to discard the whole region giving the loss

$$\int_{\theta_0}^{\frac{1}{2}} \int_{\theta_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \mathbb{1}_{(t_1,t_2)\in U_2} \frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1 t_2^2} dt_2 dt_1 < 0.421388. \tag{6}$$

For S_{U3} we cannot use Buchstab's identity in a straightforward manner, but we can use Buchstab's identity in reverse to make almost–primes visible. The details of using Buchstab's identity in reverse are similar to those in [10] and [11]. By using Buchstab's identity in reverse twice, we have

$$\begin{aligned}
S_{U3} =& \sum_{\substack{\theta_0 \leqslant t_1 < \frac{1}{2} \\ \theta_0 \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2)\in U_3}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
=& \sum_{\substack{\theta_0 \leqslant t_1 < \frac{1}{2} \\ \theta_0 \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2)\in U_3}} S\left(\mathcal{A}_{p_1 p_2}, \left(\frac{2x}{p_1 p_2}\right)^{\frac{1}{2}}\right) \\
&+ \sum_{\substack{\theta_0 \leqslant t_1 < \frac{1}{2} \\ \theta_0 \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2)\in U_3 \\ t_2 < t_3 < \frac{1}{2}(1-t_1-t_2)}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right) \\
=& \sum_{\substack{\theta_0 \leqslant t_1 < \frac{1}{2} \\ \theta_0 \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2)\in U_3}} S\left(\mathcal{A}_{p_1 p_2}, \left(\frac{2x}{p_1 p_2}\right)^{\frac{1}{2}}\right) \\
&+ \sum_{\substack{\theta_0 \leqslant t_1 < \frac{1}{2} \\ \theta_0 \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2)\in U_3 \\ t_2 < t_3 < \frac{1}{2}(1-t_1-t_2) \\ (t_1,t_2,t_3) \text{ can be partitioned into } (m,n)\in I}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right) \\
&+ \sum_{\substack{\theta_0 \leqslant t_1 < \frac{1}{2} \\ \theta_0 \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2)\in U_3 \\ t_2 < t_3 < \frac{1}{2}(1-t_1-t_2) \\ (t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in I}} S\left(\mathcal{A}_{p_1 p_2 p_3}, \left(\frac{2x}{p_1 p_2 p_3}\right)^{\frac{1}{2}}\right) \\
&+ \sum_{\substack{\theta_0 \leqslant t_1 < \frac{1}{2} \\ \theta_0 \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2)\in U_3 \\ t_2 < t_3 < \frac{1}{2}(1-t_1-t_2) \\ (t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in I \\ t_3 < t_4 < \frac{1}{2}(1-t_1-t_2-t_3) \\ (t_1,t_2,t_3,t_4) \text{ can be partitioned into } (m,n)\in I}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right) \\
&+ \sum_{\substack{\theta_0 \leqslant t_1 < \frac{1}{2} \\ \theta_0 \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \\ (t_1,t_2)\in U_3 \\ t_2 < t_3 < \frac{1}{2}(1-t_1-t_2) \\ (t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in I \\ t_3 < t_4 < \frac{1}{2}(1-t_1-t_2-t_3) \\ (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in I}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right) \\
=& S_{U31} + S_{U32} + S_{U33} + S_{U34} + S_{U35}.
\end{aligned} \tag{7}$$

We can give asymptotic formulas for S_{U32} and S_{U34}, hence we can subtract them from the loss. In this way we obtain a loss from S_{U3} of

$$
\begin{aligned}
&\left(\int_{\theta_0}^{\frac{1}{2}}\int_{\theta_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\mathbb{1}_{(t_1,t_2)\in U_3}\frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1t_2^2}dt_2dt_1\right)\\
&-\left(\int_{\theta_0}^{\frac{1}{2}}\int_{\theta_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{t_2}^{\frac{1-t_1-t_2}{2}}\mathbb{1}_{(t_1,t_2,t_3)\in U_{32}}\frac{\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_1t_2t_3^2}dt_3dt_2dt_1\right)\\
&-\left(\int_{\theta_0}^{\frac{1}{2}}\int_{\theta_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{t_2}^{\frac{1-t_1-t_2}{2}}\int_{t_3}^{\frac{1-t_1-t_2-t_3}{2}}\mathbb{1}_{(t_1,t_2,t_3,t_4)\in U_{34}}\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1t_2t_3t_4^2}dt_4dt_3dt_2dt_1\right)\\
&<(0.954145-0.363595-0.019119)=0.571431, \qquad (8)
\end{aligned}
$$

where

$$
\begin{aligned}
U_{32}(t_1,t_2,t_3):=\Big\{&(t_1,t_2)\in U_3,\ t_2<t_3<\frac{1}{2}(1-t_1-t_2),\\
&(t_1,t_2,t_3)\text{ can be partitioned into }(m,n)\in I\Big\},\\
U_{34}(t_1,t_2,t_3,t_4):=\Big\{&(t_1,t_2)\in U_3,\ t_2<t_3<\frac{1}{2}(1-t_1-t_2),\\
&(t_1,t_2,t_3)\text{ cannot be partitioned into }(m,n)\in I,\\
&t_3<t_4<\frac{1}{2}(1-t_1-t_2-t_3),\\
&(t_1,t_2,t_3,t_4)\text{ can be partitioned into }(m,n)\in I\Big\}.
\end{aligned}
$$

Finally, by (3)–(8), the total loss is less than

$$0.001993+0.421388+0.571431<0.995<1$$

and the proof of Theorem 1.1 is completed.

4. Exponent pairs

In this section we shall give a proof of Theorem 1.3. Using the same arguments as in [4], we only need to find an exponent pair (k,l) to give an upper bound for c. The corresponding upper bounds when $r=3,4$ are

$$c<\frac{70-117l}{117k}+1\text{ when }t=3\quad\text{and}\quad c<\frac{362-487l}{487k}+1\text{ when }t=4. \qquad (9)$$

For the definition of exponent pairs, one can see [[18], Definition 5.1]. We know that if (k,l) is an exponent pair, then both

$$A(k,l)=\left(\frac{k}{2k+2},\frac{l}{2k+2}+\frac{1}{2}\right)$$

and

$$B(k,l)=\left(l-\frac{1}{2},k+\frac{1}{2}\right)$$

are exponent pairs.

Sargos [17] mentioned a different transformation process. If (k,l) is an exponent pair, then

$$C(k,l)=\left(\frac{k}{12(1+4k)},\frac{11(1+4k)+l}{12(1+4k)}\right)$$

is also an exponent pair. These three processes can also be found in [18]. Note that in [4] only processes A and B are used.

Now we shall complete our proof of Theorem 1.3. By [[19], Lemma 1.1] we know that $\left(\frac{18}{199}, \frac{593}{796}\right)$ is an exponent pair (actually it comes from Bourgain's pair $\left(\frac{13}{84}, \frac{55}{84}\right)$ and another transformation process of Sargos [16]), and we shall start our transforming process from this pair. For $r = 3$ we take the exponent pair

$$BC\left(\frac{18}{199}, \frac{593}{796}\right) = \left(\frac{6013}{13008}, \frac{137}{271}\right),$$

and for $r = 4$ we take the exponent pair

$$B\left(\frac{18}{199}, \frac{593}{796}\right) = \left(\frac{195}{796}, \frac{235}{398}\right).$$

By (9) and the arguments in [4], the proof of Theorem 1.3 is complete.

References

[1] R. C. Baker, G. Harman, and J. Rivat. Primes of the form $[n^c]$. *J. Number Theory*, 50:261–277, 1995.

[2] A. Balog and J. Friedlander. A hybrid of theorems of Vinogradov and Piatetski–Shapiro. *Pacific J. Math.*, 156(1):45–62, 1992.

[3] Y. Cai. A remark on the Piatetski–Shapiro–Vinogradov theorem. *Acta Arith.*, 110(1):73–75, 2003.

[4] V. Z. Guo. Almost primes in Piatetski–Shapiro sequences. *AIMS Mathematics*, 6(9):9536–9546, 2021.

[5] C. Jia. On Piatetski–Shapiro prime number theorem II. *Science in China*, 36:913–926, 1993.

[6] C. Jia. On Piatetski–Shapiro prime number theorem. *Chinese Ann. Math. Ser. B.*, 15:9–22, 1994.

[7] C. Jia. Almost all short intervals containing prime numbers. *Acta Arith.*, 76(1):21–84, 1996.

[8] A. Kumchev. On the distribution of prime numbers of the form $[n^c]$. *Glasgow Math. J.*, 41:85–102, 1999.

[9] R. Li. On the Piatetski–Shapiro prime number theorem. *preprint*, 2024.

[10] R. Li. Primes in almost all short intervals. *arXiv e-prints*, page arXiv:2407.05651v5, 2024.

[11] R. Li. The number of primes in short intervals and numerical calculations for Harman's sieve. *arXiv e-prints*, page arXiv:2308.04458v8, 2025.

[12] I. I. Piatetski-Shapiro. On the distribution of prime numbers in sequences of the form $[f(n)]$. *Mat. Sb.*, 33:559–566, 1953.

[13] J. Rivat. Autour d'un théorème de Piatetski–Shapiro (nombres premiers dans la suite $[n^c]$). *Thèse de Doctorat*, Université de Paris–Sud, 1992.

[14] J. Rivat and P. Sargos. Nombres premiers de la forme $\lfloor n^c \rfloor$. *Canad. J. Math. Vol.*, 53(2):414–433, 2001.

[15] J. Rivat and J. Wu. Prime numbers of the form $[n^c]$. *Glasgow Math. J.*, 43:237–254, 2001.

[16] P. Sargos. Points entiers au voisinage d'une courbe, sommes trigonométriques courtes et paires d'exposants. *Proc. London Math. Soc.*, s3–70(2):285–312, 1995.

[17] P. Sargos. An analog of van der Corput's A^4–process for exponential sums. *Acta Arith.*, 110(3):219–231, 2003.

[18] T. Tao, T. Trudgian, and A. Yang. Database of known results on analytic number theory exponents.

[19] T. S. Trudgian and A. Yang. Toward optimal exponent pairs. *arXiv e-prints*, page arXiv:2306.05599v3, 2024.

International Curriculum Center, The High School Affiliated to Renmin University of China, Beijing, China
Email address: runbo.li.carey@gmail.com