BOMBIERI–VINOGRADOV THEOREM IN SHORTER INTERVALS

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Abstract. Let $y = x^{\theta}$ and $Q = x^{\psi}(\log x)^{-B}$ where $B = B(A)$. Using a recent large value estimate for Dirichlet L-functions proved by Chen, the author proves that

$$\sum_{q \leqslant Q} \max_{(a,q)=1} \max_{h \leqslant y} \max_{\frac{x}{2} \leqslant z \leqslant x} \left| \pi(z+h; q, a) - \pi(z; q, a) - \frac{\mathrm{Li}(z+h) - \mathrm{Li}(z)}{\varphi(q)} \right| \ll \frac{y}{(\log x)^A}$$

holds true for $\theta > \frac{4}{7}$ and $\psi < 2\theta - \frac{8}{7}$. The "interval length" $x^{\frac{4}{7}+\varepsilon}$ is shorter than any previous results of this type.

Contents

1. Introduction

Let x denote a sufficiently large integer and p denote prime numbers. Let

$$\pi(x) = \sum_{p \leqslant x} 1 \quad \text{and} \quad \pi(x; q, a) = \sum_{\substack{p \leqslant x \\ p \equiv a (\bmod q)}} 1.$$

Prime Number Theorem tells us that $\pi(x) \sim \mathrm{Li}(x)$, the logarithmic integral function. The well-known Bombieri–Vinogradov Theorem, proved independently by Bombieri [1] and Vinogradov [17] in 1965, states that

$$\sum_{q \leqslant Q} \max_{(a,q)=1} \max_{z \leqslant x} \left| \pi(z; q, a) - \frac{\mathrm{Li}(z)}{\varphi(q)} \right| \ll \frac{x}{(\log x)^A},$$

where A is a large positive constant, $Q = x^{\frac{1}{2}}(\log x)^{-B}$ and $B = B(A) > 0$.

In 1969, Jutila [8] first considered the analogous result for short intervals. By using zero-density method, he established a result of the following form:

$$\sum_{q \leqslant Q} \max_{(a,q)=1} \max_{h \leqslant y} \max_{\frac{x}{2} \leqslant z \leqslant x} \left| \pi(z+h; q, a) - \pi(z; q, a) - \frac{\mathrm{Li}(z+h) - \mathrm{Li}(z)}{\varphi(q)} \right| \ll \frac{y}{(\log x)^A}, \tag{1}$$

where $y = x^{\theta}$ and $\theta \leqslant 1$. Write $Q = x^{\psi}(\log x)^{-B}$, Jutila showed that (1) holds for

$$\psi < \frac{4c\theta + 2\theta - 1 - 4c}{6 + 4c}, \quad \text{if} \quad \zeta\left(\frac{1}{2} + it\right) \ll t^{c}.$$

After Jutila, many mathematicians improved this result. In 1971, Motohashi [12] showed that (1) holds for

$$\psi \leqslant \frac{8}{26}\theta - \frac{5}{26}, \qquad \frac{5}{8} < \theta \leqslant 1.$$

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In 1975, Huxley and Iwaniec [7] showed that (1) holds for

$$\psi \leqslant \theta - \frac{1}{2}, \qquad \frac{3}{4} < \theta \leqslant 1;$$

$$\psi < \left(\frac{1}{5} + \sqrt{\frac{3}{5}\left(\theta - \frac{3}{5}\right)}\right)^2, \qquad \frac{29}{48} < \theta \leqslant \frac{3}{4};$$

$$\psi < 3\theta - \frac{7}{4}, \qquad \frac{7}{12} < \theta \leqslant \frac{29}{48}.$$

In 1978, Ricci [15] showed that (1) holds for

$$\psi < \min\left(\theta - \frac{1}{2}, \frac{5}{2}\theta - \frac{3}{2}\right), \qquad \frac{3}{5} < \theta \leqslant 1.$$

In 1984, Perelli, Pintz and Salerno [13] showed that (1) holds for

$$\psi \leqslant \theta - \frac{1}{2}, \qquad \frac{3}{5} < \theta \leqslant 1.$$

In 1985, Perelli, Pintz and Salerno [14] showed that (1) holds for

$$\psi \leqslant \frac{1}{40}, \qquad \frac{7}{12} < \theta \leqslant 1.$$

In 1989, Zhan [19] showed that (1) holds for

$$\psi \leqslant \frac{1}{38.5}, \qquad \frac{7}{12} < \theta \leqslant 1.$$

In 1988, Timofeev [16] showed that (1) holds for

$$\psi \leqslant \theta - \frac{1}{2}, \qquad \frac{3}{5} < \theta \leqslant 1;$$

$$\psi \leqslant \theta - \frac{11}{20}, \qquad \frac{7}{12} < \theta \leqslant 1.$$

The Zero-Density Hypothesis implies that (1) holds for

$$\psi \leqslant \theta - \frac{1}{2}, \qquad \frac{1}{2} < \theta \leqslant 1.$$

In 2012, under the assumption of sixth power large sieve mean-value of Dirichlet L-function, Lao [9] showed that (1) holds for

$$\psi \leqslant \theta - \frac{1}{2}, \qquad \frac{7}{12} < \theta \leqslant 1.$$

Lou and Yao [10] and Wu [18] proved that a generalized version of (1) holds under some conditions. The range of θ is $\frac{7}{12} < \theta \leqslant 1$ in [10] and $\frac{3}{5} < \theta \leqslant 1$ in [18].

Huxley and Iwaniec [7], and several results before, used only zero-density methods. After Perelli, Pintz and Salerno [13], Heath-Brown's "generalized Vaughan's identity" [6] was used in the proof of many results on this topic. However, all unconditional results above stop at $\theta = \frac{7}{12}$ or larger values. Recently, using the new method of Guth and Maynard [4], Chen [2] announced a better large value estimate for Dirichlet L-functions, which brings the possibility of obtaining new results of type (1). In the persent paper, instead of using Heath-Brown's identity, we follow the zero-density method used by Huxley and Iwaniec [7] to show that (1) holds for a wider range of θ.

Theorem 1.1. *The estimate (1) holds true for*

$$\psi < 2\theta - \frac{8}{7}, \qquad \frac{4}{7} < \theta \leqslant \frac{7}{12}.$$

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Chen's new zero-density estimate is also applicable for a variant of Bombieri–Vinogradov Theorem whose moduli can be divisible by powers of a given integer. Using similar arguments, one can also show that

$$\sum_{\substack{q\leqslant x^{\frac{9}{20}}l^{-1}(\log x)^{-B}\\(q,l)=1}} \max_{(a,ql)=1}\max_{z\leqslant x}\left|\pi(z;ql,a)-\frac{\mathrm{Li}(z)}{\varphi(ql)}\right| \ll \frac{x}{\varphi(l)(\log x)^A} \tag{2}$$

holds for $l\leqslant x^{\frac{3}{7}}\exp\left(-(\log\log x)^3\right)$ that are powers of a given integer. This gives an improvement of Theorem 1.2 of Guo [3]. One can also see another application of Chen's estimate due to Harm [5].

2. Proof of Theorem 1.1

Now we follow the steps in [7]. Instead of showing (1) directly, we are going to prove an equivalent form of (1):

$$\sum_{q\leqslant Q}\max_{(a,q)=1}\max_{h\leqslant y}\max_{\frac{x}{2}\leqslant z\leqslant x}\left|\sum_{\substack{z<n\leqslant z+h\\ n\equiv a(\bmod q)}}\Lambda(n)-\frac{h}{\varphi(q)}\right| \ll \frac{y}{(\log x)^A} \tag{3}$$

holds for $\frac{4}{7}<\theta\leqslant\frac{7}{12}$ and $\psi=\frac{7}{2}\theta-2$, where $\Lambda(x)$ denote the von Mangoldt function. We have

$$\sum_{\substack{z<n\leqslant z+h\\ n\equiv a(\bmod q)}}\Lambda(n)-\frac{h}{\varphi(q)}=\frac{1}{\varphi(q)}\sum_{\chi(\bmod q)}\bar{\chi}(a)E(z,h;\chi), \tag{4}$$

where

$$E(z,h;\chi)=\begin{cases}\sum_{z<n\leqslant z+h}\Lambda(n)\chi(n), & \chi \text{ is not principal;}\\ \sum_{z<n\leqslant z+h}\Lambda(n)\chi(n)-h, & \chi \text{ is principal.}\end{cases} \tag{5}$$

If the character χ_1, proper mod f, induces χ mod q, then

$$E(z,h;\chi_1)=E(z,h;\chi)+O(\log q\log z). \tag{6}$$

Since we have

$$\sum_{\substack{q\leqslant Q\\ f|q}}\frac{1}{\varphi(q)}\ll\frac{\log Q}{\varphi(f)}, \tag{7}$$

we can estimate the left-hand side of (3) as

$$\begin{aligned}&\sum_{q\leqslant Q}\max_{(a,q)=1}\max_{h\leqslant y}\max_{\frac{x}{2}\leqslant z\leqslant x}\left|\sum_{\substack{z<n\leqslant z+h\\ n\equiv a(\bmod q)}}\Lambda(n)-\frac{h}{\varphi(q)}\right|\\ \ll&\sum_{f\leqslant Q}\frac{\log Q}{\varphi(f)}\sideset{}{^*}\sum_{\chi(\bmod f)}\max_{h\leqslant y}\max_{\frac{x}{2}\leqslant z\leqslant x}|E(z,h;\chi)|+Q(\log x)^2,\end{aligned} \tag{8}$$

where $\sum^*$ denote sums over proper characters. In order to deal with the term $Q(\log x)^2$, we need to assume that $Q\ll y(\log x)^{-C}$, where C is a large constant that may have different values at different places. We recall the Explicit Formula:

$$E(z,h;\chi)=-\sum_{\substack{\rho=\beta+i\gamma\\ |\gamma|<T}}\frac{(z+h)^\rho-z^\rho}{\rho}+O\left(\frac{x(\log x)^2}{T}\right)\quad\text{for } z\leqslant x,\ T\ll x \text{ and proper } \chi. \tag{9}$$

Now, for $\frac{x}{2}\leqslant z\leqslant x$ we have

$$\frac{(z+h)^\rho-z^\rho}{\rho}\ll\begin{cases}yx^{\beta-1}, & |\gamma|\leqslant\frac{x}{y};\\ \frac{x^\beta}{|\gamma|}, & |\gamma|>\frac{x}{y}.\end{cases} \tag{10}$$

By a standard dyadic division technique ($F \leqslant f < 2F$), we only need to show that

$$\sum_{F \leqslant f < 2F} \sideset{}{^*}\sum_{\chi (\bmod f)} \max_{h \leqslant y} \max_{\frac{x}{2} \leqslant z \leqslant x} |E(z,h;\chi)| \ll Fy(\log x)^{-C}. \tag{11}$$

By (9) and (10), we have

$$\begin{aligned}
&\sum_{F \leqslant f < 2F} \sideset{}{^*}\sum_{\chi (\bmod f)} \max_{h \leqslant y} \max_{\frac{x}{2} \leqslant z \leqslant x} |E(z,h;\chi)| \\
\ll &\sum_{F \leqslant f < 2F} \sideset{}{^*}\sum_{\chi (\bmod f)} \left(\sum_{\substack{\rho=\beta+i\gamma \\ |\gamma| \leqslant \frac{x}{y}}} yx^{\beta-1} + \sum_{\substack{\rho=\beta+i\gamma \\ \frac{x}{y} < |\gamma| < T}} \frac{x^\beta}{|\gamma|} \right) + F^2T^{-1}x(\log x)^2 \\
\ll &\log x \max_{\frac{1}{2} \leqslant \alpha \leqslant 1} \sum_{F \leqslant f < 2F} \sideset{}{^*}\sum_{\chi (\bmod f)} yx^{\alpha-1} N\left(\alpha, \frac{x}{y}, \chi\right) \\
&+ (\log x)^2 \max_{\frac{1}{2} \leqslant \alpha \leqslant 1} \max_{\frac{x}{y} < U < T} \sum_{F \leqslant f < 2F} \sideset{}{^*}\sum_{\chi (\bmod f)} U^{-1}x^\alpha N(\alpha, U, \chi) + F^2T^{-1}x(\log x)^2,
\end{aligned} \tag{12}$$

where

$$N(\sigma, T, \chi) = \#\{\text{zeros of } L(s,\chi) : \beta > \sigma,\ |\gamma| < T\}.$$

We can deal with the last term on the right-hand side of (12) by letting $T = Fxy(\log x)^C$. Clearly this choice satisfies $T \ll x$.

Now, we need several bounds for

$$M(F,U) = \sum_{f < 2F} \sideset{}{^*}\sum_{\chi (\bmod f)} N(\alpha, U, \chi).$$

We start from (12). If $\alpha \geqslant 1 - c\left(\max\left(\log F, (\log x)^{4/5}\right)\right)^{-1}$ for some constant c, then $M(F,U)$ is 0 or 1 and the only possible zero is an exceptional zero. By Siegel's Theorem, its contribution to (12) can be bounded by $y(\log x)^{-C}$.

If $\frac{6}{7} \leqslant \alpha < 1 - c\left(\max\left(\log F, (\log x)^{4/5}\right)\right)^{-1}$, we can use the zero-density estimate of Montgomery [[11], Theorem 12.2, (12.14)]:

$$M(F,U) \ll \left(F^2U\right)^{\frac{2(1-\alpha)}{\alpha}} (\log x)^{14}. \tag{13}$$

Since $y = x^\theta$, the terms of (12) is $\ll Fy(\log x)^{-C}$ if

$$\theta > \frac{4}{7}. \tag{14}$$

If $\frac{5}{7} \leqslant \alpha \leqslant \frac{6}{7}$, an application of the new result of Chen [[2], Theorem 1.3] tells us that

$$M(F,U) \ll \left(F^2U\right)^{\frac{7(1-\alpha)}{3}+\varepsilon}. \tag{15}$$

In this case, the terms of (12) is $\ll Fy(\log x)^{-C}$ if

$$\psi < \frac{7}{2}\theta - 2. \tag{16}$$

Finally, if $\frac{1}{2} \leqslant \alpha \leqslant \frac{5}{7}$, the arguments in [7] shows that the terms of (12) is $\ll Fy(\log x)^{-C}$ if

$$\psi < \min_{\frac{1}{2} \leqslant \alpha \leqslant \frac{5}{7}} \frac{(1-\alpha)(3\theta - 1 - \alpha)}{4 - 5\alpha}, \tag{17}$$

and we know that

$$\min_{\frac{1}{2} \leqslant \alpha \leqslant \frac{5}{7}} \frac{(1-\alpha)(3\theta - 1 - \alpha)}{4 - 5\alpha} = 2\theta - \frac{8}{7} \tag{18}$$

for $\theta \leqslant \frac{30}{49}$. Now since

$$0 < 2\theta - \frac{8}{7} < \frac{7}{2}\theta - 2 \tag{19}$$

for $\theta > \frac{4}{7}$, the proof of Theorem 1.1 is completed.

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