A REMARK ON THE DISTRIBUTION OF $\sqrt{p}$ MODULO ONE INVOLVING PRIMES OF SPECIAL TYPE II

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Abstract. Let P_r denote an integer with at most r prime factors counted with multiplicity. In this paper we prove that for some $\lambda < \frac{3}{28}$, the inequality $\{\sqrt{p}\} < p^{-\lambda}$ has infinitely many solutions in primes p such that $p + 2 = P_r$, where $r = 4, 5, 6, 7$. Specially, when $r = 4$ we obtain $\lambda = \frac{1}{15.1}$, which improves Cai's $\frac{1}{15.5}$.

Contents

1. Introduction

Beginning with Vinogradov [10], many mathematicians have studied the inequality $\{\sqrt{p}\} < p^{-\lambda}$ with prime solutions. Now the best result is due to Harman and Lewis [5]. In [5] they proved that there are infinitely many solutions in primes p to the inequality $\{\sqrt{p}\} < p^{-\lambda}$ with $\lambda = 0.262$, which improved the previous results of Vinogradov [10], Kaufman [7], Harman [4] and Balog [1].

On the other hand, one of the famous problems in prime number theory is the twin primes problem, which states that there are infinitely many primes p such that $p + 2$ is also a prime. Let P_r denote an integer with at most r prime factors counted with multiplicity. Now the best result in this aspect is due to Chen [3], who showed that there are infinitely many primes p such that $p + 2 = P_2$.

In 2013, Cai [2] combined those two problems and considered a mixed version.

Definition 1.1. Let $M(\lambda, r)$ denotes the following statement: The inequality

$$\{\sqrt{p}\} < p^{-\lambda}$$

holds for infinitely many primes p such that $p + 2 = P_r$.

In his paper [2], he also showed that

Theorem 1.2. *$M(\frac{1}{15.5}, 4)$ holds true.*

In 2024, Li [8] generalized Cai's result to a wider range of λ. He got

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Theorem 1.3. *$M(\lambda, \lfloor\frac{8}{1-4\lambda}\rfloor)$ holds true for all $0<\lambda<\frac{1}{4}$.*

In [8], Li mentioned that Cai [2] actually prove a new mean value theorem (see [[2], Lemma 5]) for this problem and it may be useful on improving the results. In the present paper, we shall make use of this mean value theorem and improve previous results.

Theorem 1.4. *$M(\frac{1}{15.1},4)$, $M(\frac{1}{9.83},5)$, $M(\frac{1}{9.38},6)$ and $M(\frac{1}{9.34},7)$ hold true.*

We mention that $\lambda=\frac{1}{9.34}$ is near the limit of our method that we will explain later.

2. Preliminary lemmas

Let $\mathcal{A}$ denote a finite set of positive integers and $z\geqslant 2$. Suppose that $|\mathcal{A}|\sim X_{\mathcal{A}}$ and for square–free d, put

$$\mathcal{P}=\{p:(p,2)=1\},\quad \mathcal{P}(r)=\{p:p\in\mathcal{P},\ (p,r)=1\},$$

$$P(z)=\prod_{\substack{p\in\mathcal{P}\\ p<z}}p,\quad \mathcal{A}_d=\{a:a\in\mathcal{A},\ a\equiv 0(\bmod d)\},\quad S(\mathcal{A};\mathcal{P},z)=\sum_{\substack{a\in\mathcal{A}\\ (a,P(z))=1}}1.$$

Lemma 2.1. *([[6], Lemma 2]). If*

$$\sum_{z_1\leqslant p<z_2}\frac{\omega(p)}{p}=\log\frac{\log z_2}{\log z_1}+O\left(\frac{1}{\log z_1}\right),\quad z_2>z_1\geqslant 2,$$

where $\omega(d)$ is a multiplicative function, $0\leqslant\omega(p)<p$, $X_{\mathcal{A}}>1$ is independent of d. Then

$$S(\mathcal{A};\mathcal{P},z)\geqslant X_{\mathcal{A}}W(z)\left\{f\left(\frac{\log D}{\log z}\right)+O\left(\frac{1}{\log^{\frac{1}{3}}D}\right)\right\}-\sum_{\substack{n\leqslant D\\ n|P(z)}}|\eta(X_{\mathcal{A}},n)|,$$

$$S(\mathcal{A};\mathcal{P},z)\leqslant X_{\mathcal{A}}W(z)\left\{F\left(\frac{\log D}{\log z}\right)+O\left(\frac{1}{\log^{\frac{1}{3}}D}\right)\right\}+\sum_{\substack{n\leqslant D\\ n|P(z)}}|\eta(X_{\mathcal{A}},n)|,$$

where D is a power of z,

$$W(z)=\prod_{\substack{p<z\\ (p,2)=1}}\left(1-\frac{\omega(p)}{p}\right),$$

$$\eta(X_{\mathcal{A}},n)=|\mathcal{A}_n|-\frac{\omega(n)}{n}X_{\mathcal{A}}=\sum_{\substack{a\in\mathcal{A}\\ a\equiv 0(\bmod n)}}1-\frac{\omega(n)}{n}X_{\mathcal{A}}$$

and $f(s)$ and $F(s)$ are determined by the following differential–difference equation

$$\begin{cases} F(s)=\frac{2e^{\gamma}}{s},\quad f(s)=0, & 0<s\leqslant 2,\\ (sF(s))'=f(s-1),\quad (sf(s))'=F(s-1), & s\geqslant 2.\end{cases}$$

Lemma 2.2. *([2], Lemma 4]). For any given constant $A>0$ and $0<\lambda<\frac{1}{4}, 0<\theta<\frac{1}{4}-\lambda$ we have*

$$\sum_{d\leqslant x^{\theta}}\max_{(l,d)=1}\left|\sum_{\substack{x<p\leqslant 2x\\ \{\sqrt{p}\}<p^{-\lambda}\\ p\equiv l(\bmod d)}}1-\frac{(2x)^{1-\lambda}-x^{1-\lambda}}{\varphi(d)(1-\lambda)\log x}\right|\ll\frac{x^{1-\lambda}}{\log^{A}x}.$$

Lemma 2.3. *([2], Lemma 5]). Let*

$$\mathcal{N}=\left\{p_1p_2p_3p_4m: x^{\frac{1}{14}}\leqslant p_1<p_2<p_3<p_4,\ x<p_1p_2p_3p_4m\leqslant 2x,\ (m,P(p_4))=1\right\}.$$

Then for any given constant $A>0$ and $0<\lambda<\frac{1}{8}, 0<\theta<\frac{1}{4}-\lambda$ we have

$$\sum_{d\leqslant x^{\theta}}\max_{(l,d)=1}\left|\sum_{\substack{n\in\mathcal{N}\\ n\equiv l(\bmod d)\\ \{\sqrt{n-2}\}<(n-2)^{-\lambda}}}1-\frac{1}{\varphi(d)}\sum_{\substack{n\in\mathcal{N}\\ (n,d)=1}}n^{-\lambda}\right|\ll\frac{x^{1-\lambda}}{\log^{A}x}.$$

3. Proof of Theorem 1.4

In this section, we define the function ω as $\omega(p)=0$ for $p=2$ and $\omega(p)=\frac{p}{p-1}$ for other primes. Put

$$D=x^{\frac{1}{4}-\lambda-\varepsilon},\quad \mathcal{A}=\left\{p+2: x<p\leqslant 2x,\ \{\sqrt{p}\}<p^{-\lambda}\right\},$$
$$\mathcal{B}=\left\{n-2: n\in\mathcal{N},\ \{\sqrt{n-2}\}<(n-2)^{-\lambda}\right\}.$$

Let γ denotes the Euler's constant, $4\leqslant r\leqslant 7$ and S_r denote the number of prime solutions to the inequality (1) such that $p+2=P_r$, then we have

$$\begin{aligned}S_r&\geqslant S\left(\mathcal{A};\mathcal{P},x^{\frac{1}{14}}\right)-\sum_{x^{\frac{1}{14}}\leqslant p_1<\cdots<p_r<\left(\frac{2x}{p_1\cdots p_{r-1}}\right)^{\frac{1}{2}}}S\left(\mathcal{A}_{p_1\cdots p_r};\mathcal{P}(p_1\cdots p_{r-1}),p_r\right)+O\left(x^{\frac{13}{14}}\right)\\&=S_{r,1}-S_{r,2}+O\left(x^{\frac{13}{14}}\right).\end{aligned}\tag{1}$$

Now we ignore the presence of ε for clarity. By similar arguments as in [2] we can take

$$X_{\mathcal{A}}=\frac{(2x)^{1-\lambda}-x^{1-\lambda}}{(1-\lambda)\log x}.\tag{2}$$

And by the similar arguments as in [6] we know that

$$W\left(x^{\frac{1}{14}}\right)=(1+o(1))2C_2\frac{e^{-\gamma}}{\frac{1}{14}\log x},\tag{3}$$

where

$$C_2:=\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right).\tag{4}$$

To deal with the error terms, by Lemma 2.2 we can easily show that

$$\sum_{\substack{n\leqslant D\\ n|P(x^{\frac{1}{14}})}} |\eta(X_{\mathcal{A}},n)| \ll \sum_{n\leqslant D} \mu^2(n)\,|\eta(X_{\mathcal{A}},n)| \ll x^{1-\lambda}(\log x)^{-5}. \tag{5}$$

Then by Lemma 2.1 we have

$$\begin{aligned}
S_{r,1} &\geqslant X_{\mathcal{A}} W\left(x^{\frac{1}{14}}\right)\left\{f\left(\frac{\log D}{\log x^{\frac{1}{14}}}\right)+O\left(\frac{1}{\log^{\frac{1}{3}} D}\right)\right\} - \sum_{\substack{n\leqslant D\\ n|P(x^{\frac{1}{14}})}} |\eta(X_{\mathcal{A}},n)| \\
&\geqslant (1+o(1))\frac{2C_2X_{\mathcal{A}}}{\log D}\frac{e^{-\gamma}}{\left(\frac{1}{14}/\left(\frac{1}{4}-\lambda\right)\right)} f\left(\frac{\frac{1}{4}-\lambda}{\frac{1}{14}}\right).
\end{aligned} \tag{6}$$

For $S_{r,2}$, by Chen's switching principle [3], Lemma 2.1, Lemma 2.3 and similar arguments in [2], we have

$$\begin{aligned}
S_{r,2} &= S\left(\mathcal{B};\mathcal{P},(2x)^{\frac{1}{2}}\right) \\
&\leqslant S\left(\mathcal{B};\mathcal{P},D^{\frac{1}{2}}\right) \\
&\leqslant (1+o(1))\frac{4C_2X_{\mathcal{A}}}{\log D} e^{-\gamma}F(2)T_r,
\end{aligned}$$

where

$$T_r = \int_{\frac{1}{14}}^{\frac{1}{r+1}} \int_{t_1}^{\frac{1-t_1}{r}} \cdots \int_{t_{r-1}}^{\frac{1-t_1-\cdots-t_{r-1}}{2}} \frac{\omega\left(\frac{1-t_1-\cdots-t_r}{t_r}\right)}{t_1t_2\cdots t_{r-1}t_r^2} dt_r\cdots dt_1, \tag{7}$$

where $\omega(u)$ is the Buchstab function determined by the following differential–difference equation

$$\begin{cases} \omega(u)=\frac{1}{u}, & 1\leqslant u\leqslant 2,\\ (u\omega(u))'=\omega(u-1), & u\geqslant 2.\end{cases}$$

Note that for the error term, we take $m=p_5\cdots p_r m_1$ with $p_4<p_5<\cdots<p_r$ and $(m_1,P(p_r))=1$ in Lemma 2.3 when $r\geqslant 5$. Clearly we have $p_1p_2p_3p_4p_5\cdots p_rm_1\in\mathcal{N}$.

By [[9], (48)] we know that $F(2)=e^{\gamma}$. Combining (1), (6) and (7), we have

$$S_r \geqslant (1+o(1))\frac{2C_2X_{\mathcal{A}}}{\log D}\left(\frac{e^{-\gamma}}{\left(\frac{1}{14}/\left(\frac{1}{4}-\lambda\right)\right)} f\left(\frac{\frac{1}{4}-\lambda}{\frac{1}{14}}\right) - 2T_r\right). \tag{8}$$

Hence we only need

$$\frac{e^{-\gamma}}{\left(\frac{1}{14}/\left(\frac{1}{4}-\lambda\right)\right)} f\left(\frac{\frac{1}{4}-\lambda}{\frac{1}{14}}\right) - 2T_r > 0. \tag{9}$$

When $r=4,5,6,7$, numerical calculation shows that

$$\frac{e^{-\gamma}}{\left(\frac{1}{14}/\left(\frac{1}{4}-\frac{1}{15.1}\right)\right)} f\left(\frac{\frac{1}{4}-\frac{1}{15.1}}{\frac{1}{14}}\right) - 2T_4 > 0,$$

$$\frac{e^{-\gamma}}{\left(\frac{1}{14}/\left(\frac{1}{4}-\frac{1}{9.83}\right)\right)} f\left(\frac{\frac{1}{4}-\frac{1}{9.83}}{\frac{1}{14}}\right) - 2T_5 > 0,$$

$$\frac{e^{-\gamma}}{\left(\frac{1}{14}/\left(\frac{1}{4}-\frac{1}{9.38}\right)\right)} f\left(\frac{\frac{1}{4}-\frac{1}{9.38}}{\frac{1}{14}}\right) - 2T_6 > 0$$

and

$$\frac{e^{-\gamma}}{\left(\frac{1}{14}/\left(\frac{1}{4}-\frac{1}{9.34}\right)\right)} f\left(\frac{\frac{1}{4}-\frac{1}{9.34}}{\frac{1}{14}}\right) - 2T_7 > 0.$$

Now Theorem 1.4 is proved. We remark that for positive λ, we have

$$f\left(\frac{\frac{1}{4}-\lambda}{\frac{1}{14}}\right) > 0 \tag{10}$$

only when $\lambda > \frac{3}{28} = \frac{1}{9.333\ldots}$, so $\lambda = \frac{1}{9.34}$ is rather near the limit point.

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