PRIMES IN ARITHMETIC PROGRESSIONS TO LARGE MODULI AND REFINEMENTS OF HARMAN'S SIEVE

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Abstract. We study the average distribution of primes of size x in arithmetic progressions to moduli larger than $x^{\frac{1}{2}}$. Using arithmetic information from the works of many authors together with different variants of the original Harman's sieve, we construct suitable majorants and minorants for the prime indicator function $\mathbb{1}_p(n)$ that satisfy Bombieri–Vinogradov type mean value theorems with different types of moduli. Specifically, we obtain some mean value theorems for primes with bilinear forms of moduli up to $x^{\frac{9}{17}}$ or with trilinear forms of moduli up to $x^{\frac{17}{32}}$.

1. Introduction

One of the famous topics in prime number theory is the distribution of primes in arithmetic progressions. The Siegel–Walfisz Theorem states that

$$\left|\pi(x;q,a)-\frac{\pi(x)}{\varphi(q)}\right| \ll \frac{x}{(\log x)^A} \tag{1}$$

uniformly for $q \leqslant (\log x)^A$ and $(a,q)=1$. Under the Generalized Riemann Hypothesis, the range of q such that (1) holds can be extended to $q \leqslant x^{\frac{1}{2}}(\log x)^{-B}$, where $B=B(A)$. In [33], Montgomery conjectured that (1) should hold for all $q \leqslant x^{1-\varepsilon}$.

In many applications, however, mathematicians only need an average distribution result of primes in arithmetic progressions rather than an individual estimate (1). In this case, a substitute of the Generalized Riemann Hypothesis is the Bombieri–Vinogradov Theorem [5] [39]. This theorem states that

$$\sum_{q \leqslant x^{\frac{1}{2}}(\log x)^{-B}} \max_{(a,q)=1} \left|\pi(x;q,a)-\frac{\pi(x)}{\varphi(q)}\right| \ll \frac{x}{(\log x)^A}. \tag{2}$$

By (2), we also know that (1) holds for almost all $q \leqslant x^{\frac{1}{2}}(\log x)^{-B}$ with $(a,q)=1$. As well as (1), Elliott and Halberstam [9] conjectured that the range of q in (2) can be extended to $q \leqslant x^{1-\varepsilon}$. However, one still cannot prove (2) even with $q \leqslant x^{\frac{1}{2}+\varepsilon}$ now.

From here, we suppose that $a \in \mathbb{Z}\backslash\{0\}$ is fixed. In 2013, Zhang [40] proved that (2) is valid for $q \leqslant x^{\frac{293}{584}-\varepsilon}$ when the moduli q is square-free and only has small prime factors:

$$\sum_{\substack{q \leqslant x^{\frac{293}{584}-\varepsilon} \\ q \mid P(x^\delta) \\ (q,a)=1}} \left|\pi(x;q,a)-\frac{\pi(x)}{\varphi(q)}\right| \ll \frac{x}{(\log x)^A}, \tag{3}$$

where $\delta=\delta(\varepsilon)$ is a small positive number. Polymath8b [36] and Stadlmann [38] further extended the range of q in (3) to $q \leqslant x^{\frac{157}{300}-\varepsilon}$ and $q \leqslant x^{\frac{21}{40}-\varepsilon}$.

In 2025, Maynard [28] proved that

$$\sum_{\substack{q_1 \leqslant Q_1 \\ q_2 \leqslant Q_2 \\ (q_1q_2,a)=1}} \left|\pi(x;q_1q_2,a)-\frac{\pi(x)}{\varphi(q_1q_2)}\right| \ll \frac{x}{(\log x)^A} \tag{4}$$

if

$$Q_1^2Q_2 < x^{1-\varepsilon}, \quad Q_1^7Q_2^{12} < x^{4-\varepsilon}, \quad Q_1^{19}Q_2^{20} < x^{10-\varepsilon}.$$

In another paper, he [30] also proved that

$$\sum_{\substack{q_1 \leqslant Q_1 \\ q_2 \leqslant Q_2 \\ q_3 \leqslant Q_3 \\ (q_1q_2q_3,a)=1}} \left|\pi(x;q_1q_2q_3,a)-\frac{\pi(x)}{\varphi(q_1q_2q_3)}\right| \ll \frac{x}{(\log x)^A} \tag{5}$$

if $\delta \in (0, 0.001)$ and

$$Q_1Q_2Q_3 = x^{\frac{1}{2}+\delta}, \quad x^{40\delta} < Q_2 < x^{\frac{1}{20}-7\delta}, \quad x^{\frac{1}{10}+12\delta}Q_2^{-1} < Q_3 < x^{\frac{1}{10}-4\delta}Q_2^{-\frac{3}{5}}.$$

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The problem of bounding sums of the form

$$\sum_{q\leqslant Q}\left|\pi(x;q,a)-\frac{\pi(x)}{\varphi(q)}\right|$$

can be seen as a problem equivalent to bounding

$$\sum_{q\leqslant Q}\lambda_q\left(\pi(x;q,a)-\frac{\pi(x)}{\varphi(q)}\right),$$

for an arbitrary divisor-bounded weight λ_q. Naturally, we can try to make the weight λ_q more "flexible" to extend the range of Q beyond $x^{\frac{1}{2}}$. One example is to introduce the "bilinear weights" and prove results of the following form:

$$\sum_{\substack{q_1\leqslant Q_1\\ q_2\leqslant Q_2\\ (q_1q_2,a)=1}}\lambda_{1,q_1}\lambda_{2,q_2}\left(\pi(x;q_1q_2,a)-\frac{\pi(x)}{\varphi(q_1q_2)}\right)\ll\frac{x}{(\log x)^A}. \tag{6}$$

In 1986, Bombieri, Friedlander and Iwaniec [6] first showed that (6) holds if

$$Q_1<x^{\frac{1}{3}},\quad Q_2<x^{\frac{1}{5}},\quad Q_1^5Q_2^2<x^2,\quad Q_1Q_2<x^{\frac{29}{56}}.$$

In 1987, Fouvry [13] proved that (6) holds if either

$$Q_1Q_2^3<x,\quad Q_1Q_2<x^{\frac{29}{56}},\quad Q_1<\max\left(x^{\frac{1}{2}}Q_2^{-1},x^{\frac{2}{5}}Q_2^{-\frac{2}{5}}\right)$$

or

$$Q_1Q_2^3<x,\quad Q_1Q_2<x^{\frac{29}{56}},\quad Q_1^4Q_2<x^{\frac{403}{266}},\quad Q_1^{\frac{7}{4}}Q_2<x^{\frac{403}{532}}.$$

In 1998, Baker and Harman [3] improved the result of Bombieri, Friedlander and Iwaniec [6] by removing the condition $Q_1^5Q_2^2<x^2$ in their result above. They showed that (6) holds if

$$Q_1<x^{\frac{1}{3}},\quad Q_2<x^{\frac{1}{5}},\quad Q_1Q_2<x^{\frac{29}{56}}.$$

Those results extended the range of $Q=Q_1Q_2$ up to $x^{\frac{29}{56}}$ in some special cases. The main result of Maynard [28] can also be seen as a "bilinear" result, with $Q=Q_1Q_2$ up to $x^{\frac{11}{21}}$. In 2022, Lichtman [25] considered a more "flexible" case with quadrilinear weights and extended the moduli up to $x^{\frac{17}{32}}$. He showed that

$$\sum_{\substack{q_1\leqslant Q_1\\ q_2\leqslant Q_2\\ q_3\leqslant Q_3\\ q_4\leqslant Q_3\\ (q_1q_2q_3q_4,a)=1}}\lambda_{1,q_1}\lambda_{2,q_2}\lambda_{3,q_3}\lambda_{4,q_4}\left(\pi(x;q_1q_2q_3q_4,a)-\frac{\pi(x)}{\varphi(q_1q_2q_3q_4)}\right)\ll\frac{x}{(\log x)^A} \tag{7}$$

holds if

$$Q_1Q_2<x^{\frac{1}{2}+\varepsilon},\quad Q_1Q_3^2<x^{\frac{1}{2}-2\varepsilon},\quad Q_3^2<Q_2<x^{\frac{1}{32}-\varepsilon}.$$

Weights that are more "flexible" than above are the well-factorable weights, which means that for any Q_1,Q_2 such that $Q_1Q_2=Q$, we can "split" a well-factorable function λ_q to $\lambda_q=\lambda_{1,q_1}\lambda_{2,q_2}$ supported on $[1,Q_1]$ and $[1,Q_2]$ respectively. There are also lots of works on this topic, and we refer the readers to [14], [10], [6], [29], [27], [26], [35] and [41].

In 1996, Baker and Harman [2] considered a different variant of (2): They used Harman's sieve [17] to construct majorants and minorants for the prime indicator function $\mathbb{1}_p(n)$ and studied their distributions in fixed residue classes with moduli $q\geqslant x^{\frac{1}{2}}$. Write $q\asymp x^\theta$. Baker and Harman [2] constructed majorants $\rho_1(n)\geqslant\mathbb{1}_p(n)$ for $0.5\leqslant\theta\leqslant 0.56$ and minorants $\rho_0(n)\leqslant\mathbb{1}_p(n)$ for $0.5\leqslant\theta\leqslant 0.52$ that satisfy our Theorem 2.1 below. Similar results before them are obtained by Motohashi [34], Hooley [18] [19], Iwaniec [21], Deshouillers and Iwaniec [8], Fouvry [11] [12] and Rousselet [37] respectively. In 2001, Mikawa [32] further constructed minorants $\rho_0(n)\leqslant\mathbb{1}_p(n)$ for $0.5\leqslant\theta<\frac{17}{32}$ using a different sieve method.

In this paper, we refine the methods developed by Baker and Harman [2] and Mikawa [32] to construct majorants and minorants for the prime indicator function $\mathbb{1}_p(n)$, and we study the distributions of them in residue classes with different types of moduli. One result we obtain in this paper is the following theorem.

Theorem 1.1. *Let $Q_1=x^{\theta_1}$ and $Q_2=x^{\theta_2}$. Suppose that θ_1 and θ_2 satisfy any of the following conditions:*

(1). $\frac{1}{4}\leqslant\theta_1<\frac{3}{10},\ \frac{1-2\theta_1}{2}\leqslant\theta_2<\min\left(\frac{3-4\theta_1}{8},\frac{5-14\theta_1}{4}\right)$;

(2). $\frac{1}{3}<\theta_1<\frac{3}{8},\ \frac{1-2\theta_1}{2}\leqslant\theta_2<\min\left(\frac{5-8\theta_1}{14},\frac{7-16\theta_1}{8}\right)$;

(3). $2\theta_1+\theta_2<1,\ 7\theta_1+12\theta_2<4$.

Let λ_{1,q_1} and λ_{2,q_2} be divisor-bounded complex sequences. Then, for any fixed $a\in\mathbb{Z}\backslash\{0\}$ and any $A>0$, we have

$$\sum_{\substack{q_1\sim Q_1\\ q_2\sim Q_2\\ (q_1q_2,a)=1}}\lambda_{1,q_1}\lambda_{2,q_2}\left(\pi(x;q_1q_2,a)-\frac{\pi(x)}{\varphi(q_1q_2)}\right)\ll\frac{x}{(\log x)^A}.$$

Throughout this paper, we always suppose that ε is a sufficiently small positive constant, $A, B > 0$ (may depend on other variables) and x is sufficiently large. Let $\theta, \theta_i \in (0,1)$ and $\delta = 10^{-100}$. The letters p and β, with or without subscript, are reserved for primes and almost-primes respectively. We put $p_i \asymp x^{\alpha_i}$ and write $\boldsymbol{\alpha}_n$ to denote $(\alpha_1, \ldots, \alpha_n)$. We use $P^+(n)$ and $P^-(n)$ to denote the largest and smallest prime factor of n. We shall use the terms *partition* and *exactly partition* many times in the rest of our paper, and one can see [[17], Page 162] for a definition. Put

$$P(z) = \prod_{p<z} p, \quad \psi(n,z) = \mathbb{1}_{(n,P(z))=1}, \quad \Psi(n,z) = \mathbb{1}_{P^+(n)<z}.$$

Let $\mathcal{C}$ denote a finite set of positive integers and put

$$\mathcal{C}_d = \{n : nd \in \mathcal{C}\}, \quad S(\mathcal{C}, z) = \sum_{n \in \mathcal{C}} \psi(n,z) = \sum_{\substack{n \in \mathcal{C} \\ (n,P(z))=1}} 1.$$

Buchstab's identity is the equation

$$S(\mathcal{C}, z) = S(\mathcal{C}, w) - \sum_{w \leqslant p < z} S(\mathcal{C}_p, p),$$

where $2 \leqslant w < z$.

Let $\omega(u)$ denote the Buchstab function determined by the following differential-difference equation

$$\begin{cases} \omega(u) = \frac{1}{u}, & 1 \leqslant u \leqslant 2, \\ (u\omega(u))' = \omega(u-1), & u \geqslant 2. \end{cases}$$

Moreover, we have the upper and lower bounds for $\omega(u)$:

$$\omega(u) \geqslant \omega_0(u) = \begin{cases} \frac{1}{u}, & 1 \leqslant u < 2, \\ \frac{1+\log(u-1)}{u}, & 2 \leqslant u < 3, \\ \frac{1+\log(u-1)}{u} + \frac{1}{u}\int_2^{u-1} \frac{\log(t-1)}{t} dt \geqslant 0.5607, & 3 \leqslant u < 4, \\ 0.5612, & u \geqslant 4, \end{cases}$$

$$\omega(u) \leqslant \omega_1(u) = \begin{cases} \frac{1}{u}, & 1 \leqslant u < 2, \\ \frac{1+\log(u-1)}{u}, & 2 \leqslant u < 3, \\ \frac{1+\log(u-1)}{u} + \frac{1}{u}\int_2^{u-1} \frac{\log(t-1)}{t} dt \leqslant 0.5644, & 3 \leqslant u < 4, \\ 0.5617, & u \geqslant 4. \end{cases}$$

In this paper, a "Type-I$_j$" sum refers to a sum of type

$$\sum_{m_0, m_1, \ldots, m_j} a_{0,m_0},$$

and a "Type-II$_j$" sum refers to a sum of type

$$\sum_{m_1, \ldots, m_j} a_{1,m_1} \cdots a_{j,m_j},$$

where a_{i,m_i} $(0 \leqslant i \leqslant j)$ are divisor-bounded complex sequences. For the sake of simplicity, we often write "Type-I$_1$" as "Type-I" and "Type-II$_2$" as "Type-II".

2. General Moduli

In this section we focus on the general case, where the moduli $q \sim Q = x^\theta$. We put

$$\mathcal{A}^q = \{n : n \sim x, \ n \equiv a(\bmod q)\} \quad \text{and} \quad \mathcal{B}^q = \{n : n \sim x, \ (n,q) = 1\}.$$

By the definitions of the sieved set $\mathcal{C}^q$ and the sieve function $S(\mathcal{C}, z)$, and by Prime Number Theorem, we have

$$\pi(x;q,a) = \sum_{p \in \mathcal{A}^q} 1 = S\left(\mathcal{A}^q, (2x)^{\frac{1}{2}}\right) \quad \text{and} \quad S\left(\mathcal{B}^q, (2x)^{\frac{1}{2}}\right) = (1+o(1))\frac{x}{\log x}. \tag{8}$$

Our aim is to show that the sparser set $\mathcal{A}^q$ contains the expected proportion of primes compared to the larger set $\mathcal{B}^q$, which requires us to decompose $S\left(\mathcal{A}^q, (2x)^{\frac{1}{2}}\right)$ and prove "asymptotic formulas" for almost all $q \sim Q$ of the form

$$S(\mathcal{A}^q, z) = (1+o(1))\frac{1}{\varphi(q)} S(\mathcal{B}^q, z) \tag{9}$$

for some parts of it, and drop the remaining parts to construct a suitable majorant or minorant. For the majorant case we can only drop negative parts, while for the minorant case we can only drop positive parts. After the final decompositions, we can get the following result with some $0 < C_0(\theta) \leqslant 1$ and $C_1(\theta) \geqslant 1$:

Theorem 2.1. *There exist functions ρ_0 and ρ_1 which satisfies the following properties:*

(Majorant / Minorant). $\rho_0(n)$ is a minorant for the prime indicator function $\mathbb{1}_p(n)$, and $\rho_1(n)$ is a majorant for the prime indicator function $\mathbb{1}_p(n)$. That is, we have

$$\rho_0(n) \leqslant \mathbb{1}_p(n) \leqslant \rho_1(n).$$

(Upper and Lower bounds). We have

$$\sum_{n \leqslant x} \rho_0(n) \geqslant (1+o(1)) \frac{C_0(\theta)x}{\log x} \quad \text{and} \quad \sum_{n \leqslant x} \rho_1(n) \leqslant (1+o(1)) \frac{C_1(\theta)x}{\log x}$$

for two functions $C_0(\theta)$ and $C_1(\theta)$ satisfy $0 < C_0(\theta) \leqslant 1$ and $C_1(\theta) \geqslant 1$.

(Distributions in Arithmetic Progressions). For any $a \in \mathbb{Z}\backslash\{0\}$ and any $A > 0$, we have

$$\sum_{\substack{q \sim Q \\ (q,a)=1}} \left| \sum_{\substack{n \leqslant x \\ n \equiv a (\bmod q)}} \rho_j(n) - \frac{1}{\varphi(q)} \sum_{\substack{n \leqslant x \\ (n,q)=1}} \rho_j(n) \right| \ll \frac{x}{(\log x)^A}$$

for $j = 0, 1$.

In order to give asymptotic formulas (9) for sieve functions $S(\mathcal{A}^q, z)$, we need results of the form

$$\sum_{\substack{q \sim Q \\ (q,a)=1}} \left| \sum_{\substack{n \sim x \\ n \equiv a (\bmod q)}} f(n) - \frac{1}{\varphi(q)} \sum_{\substack{n \sim x \\ (n,q)=1}} f(n) \right| \ll \frac{x}{(\log x)^A}. \tag{10}$$

There are two conditions that we may want the coefficients to satisfy. We shall use a divisor-bounded coefficient sequence λ_l as an example. The first one is the Siegel–Walfisz condition, which demonstrate that at least one of the coefficient sequences is well-distributed in arithmetic progressions having small moduli. This condition is necessary in the dispersion estimates.

(**Condition A**: Siegel–Walfisz condition) For any $f \geqslant 1$, $k \geqslant 1$, $b \neq 0$ and $(k, b) = 1$, we have

$$\sum_{\substack{l \sim L \\ l \equiv b (\bmod k) \\ (l,f)=1}} \lambda_l = \frac{1}{\varphi(k)} \sum_{\substack{l \sim L \\ (l,fk)=1}} \lambda_l + O\left(\frac{L(d(f))^B}{(\log L)^A} \right).$$

We note that λ_l certainly satisfies the Siegel–Walfisz condition if $\lambda_l = 1$, if $\lambda_l = \mu(n)$, or if

$$\lambda_l = \sum_{\substack{p_1 \cdots p_j = l \\ p_j \sim P_j}} 1$$

by the Siegel–Walfisz theorem.

The next condition ensures that λ_l is supported on almost-primes: integers with all prime factors larger than $\exp\left(\log x (\log\log x)^{-2}\right)$.

(**Condition B**: No small prime factors) We have $\lambda_l = 0$ whenever l has a prime factor smaller than $\exp\left(\log x (\log\log x)^{-2}\right)$.

2.1. **Preliminary Lemmas.**

Before constructing the majorant and minorant, we need estimate results of the form (10). Note that many of them are still useful in the later sections.

2.1.1. *Type-II estimate.*

The first lemma comes from [10], and it served as one of the most important Type-II information inputs in previous works [2], [3], [23] and [32].

Lemma 2.2. *([[10], Théorème 1]). Let $M_1 M_2 \asymp x$ and $M_2 > x^{\varepsilon}$. Let a_{1,m_1} and a_{2,m_2} be divisor-bounded complex sequences. Suppose that a_{2,m_2} satisfies **Condition A**. If we have*

$$Q^2 x^{-1+\varepsilon} \leqslant M_2 \leqslant Q^{-\frac{4}{3}} x^{\frac{5}{6}-\varepsilon},$$

then

$$\sum_{\substack{q \sim Q \\ (q,a)=1}} \left| \sum_{\substack{m_1 \sim M_1 \\ m_2 \sim M_2 \\ m_1 m_2 \equiv a (\bmod q)}} a_{1,m_1} a_{2,m_2} - \frac{1}{\varphi(q)} \sum_{\substack{m_1 \sim M_1 \\ m_2 \sim M_2 \\ (m_1 m_2, q)=1}} a_{1,m_1} a_{2,m_2} \right| \ll \frac{x}{(\log x)^A}.$$

Note that this lemma is nontrivial when $Q < x^{\frac{11}{20}}$.

The second result comes from [[16], Corollary 1.1], and it will play a vital role in our final decomposition when role-reversals are applied.

Lemma 2.3. *([[16], Corollary 1.1(i)]). Let $M_1 M_2 \asymp x$ and $M_2 > x^{\varepsilon}$. Let a_{1,m_1} and a_{2,m_2} be divisor-bounded complex sequences. Suppose that a_{2,m_2} satisfies **Condition A**. If we have*

$$M_2 \leqslant Q^{-\frac{11}{12}} x^{\frac{17}{36}-\varepsilon},$$

then

$$\sum_{\substack{q\sim Q\\(q,a)=1}}\left|\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ m_1m_2\equiv a(\bmod q)}}a_{1,m_1}a_{2,m_2}-\frac{1}{\varphi(q)}\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ (m_1m_2,q)=1}}a_{1,m_1}a_{2,m_2}\right|\ll\frac{x}{(\log x)^A}.$$

Many other estimate results, such as [[13], Corollaire 1], [[15], Theorem 1.1] and [[11], Lemme 3], are also applicable in this problem. However, all of them can be deduced by Lemma 2.2 or Lemma 2.3 when a is a fixed nonzero integer. Combining Lemma 2.2 and Lemma 2.3, we can deduce [[16], Corollary 1.1(ii)(iii)] for a fixed nonzero integer a.

2.1.2. *Type-II$_3$ estimate.* Most of the next 5 lemmas were used in previous works [2], [3] and [23], and we still need them in this section and later sections. Note that Lemma 2.8 gives the main Type-I information in this and later (except for the last two) sections.

Lemma 2.4. *([7], Theorem 3]). Let $M_1M_2M_3\asymp x$, $\min(M_1,M_2,M_3)>x^{\varepsilon}$. Let a_{1,m_1}, a_{2,m_2} and a_{3,m_3} be divisor-bounded complex sequences. Suppose that a_{1,m_1}, a_{2,m_2} and a_{3,m_3} satisfy* ***Condition B****, and a_{2,m_2} also satisfies* ***Condition A****. If we have*

$$Qx^{\varepsilon}<M_1M_2,\quad M_1^2M_2^3<Qx^{1-\varepsilon},\quad M_1^5M_2^2<x^{2-\varepsilon},\quad M_1^4M_2^3<x^{2-\varepsilon},$$

then

$$\sum_{\substack{q\sim Q\\(q,a)=1}}\left|\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ m_3\sim M_3\\ m_1m_2m_3\equiv a(\bmod q)}}a_{1,m_1}a_{2,m_2}a_{3,m_3}-\frac{1}{\varphi(q)}\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ m_3\sim M_3\\ (m_1m_2m_3,q)=1}}a_{1,m_1}a_{2,m_2}a_{3,m_3}\right|\ll\frac{x}{(\log x)^A}.$$

Lemma 2.5. *([2], Lemma 5]). Let $M_1M_2M_3\asymp x$, $\min(M_1,M_2,M_3)>x^{\varepsilon}$. Let a_{1,m_1}, a_{2,m_2} and a_{3,m_3} be divisor-bounded complex sequences. Suppose that a_{1,m_1}, a_{2,m_2} and a_{3,m_3} satisfy* ***Condition B****, and a_{2,m_2} also satisfies* ***Condition A****. If we have*

$$Qx^{\varepsilon}<M_1M_2,\quad M_1M_2^2Q^2<x^{2-3\varepsilon},\quad M_1^5M_2^2<x^{2-3\varepsilon},$$

then

$$\sum_{\substack{q\sim Q\\(q,a)=1}}\left|\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ m_3\sim M_3\\ m_1m_2m_3\equiv a(\bmod q)}}a_{1,m_1}a_{2,m_2}a_{3,m_3}-\frac{1}{\varphi(q)}\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ m_3\sim M_3\\ (m_1m_2m_3,q)=1}}a_{1,m_1}a_{2,m_2}a_{3,m_3}\right|\ll\frac{x}{(\log x)^A}.$$

Lemma 2.6. *([28], Proposition 8.3]). Let $M_1M_2M_3\asymp x$, $\min(M_1,M_2,M_3)>x^{\varepsilon}$. Let a_{1,m_1}, a_{2,m_2} and a_{3,m_3} be divisor-bounded complex sequences. Suppose that a_{1,m_1}, a_{2,m_2} and a_{3,m_3} satisfy* ***Condition B****, and a_{2,m_2} also satisfies* ***Condition A****. If we have*

$$Q<x^{0.7-\varepsilon},\quad Qx^{\varepsilon}<M_1M_2,\quad M_2<Q^{-1}x^{1-2\varepsilon},\quad M_1M_2<Q^{-\frac{1}{7}}x^{\frac{153}{224}-10\varepsilon},\quad M_1^4M_2<Q^{-1}x^{\frac{57}{32}-10\varepsilon},$$

then

$$\sum_{\substack{q\sim Q\\(q,a)=1}}\left|\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ m_3\sim M_3\\ m_1m_2m_3\equiv a(\bmod q)}}a_{1,m_1}a_{2,m_2}a_{3,m_3}-\frac{1}{\varphi(q)}\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ m_3\sim M_3\\ (m_1m_2m_3,q)=1}}a_{1,m_1}a_{2,m_2}a_{3,m_3}\right|\ll\frac{x}{(\log x)^A}.$$

Lemma 2.7. *([28], Proposition 8.6]). Let $M_1M_2M_3\asymp x$, $\min(M_1,M_2,M_3)>x^{\varepsilon}$. Let a_{1,m_1}, a_{2,m_2} and a_{3,m_3} be divisor-bounded complex sequences. Suppose that a_{1,m_1}, a_{2,m_2} and a_{3,m_3} satisfy* ***Condition B****, and a_{2,m_2} also satisfies* ***Condition A****. If we have*

$$Q<x^{\frac{127}{224}-\varepsilon},\quad Qx^{\varepsilon}<M_1M_2\leqslant x^{\frac{4}{7}-\varepsilon},\quad M_1\leqslant M_2\leqslant(M_1M_2)^{\frac{3}{4}},$$

then

$$\sum_{\substack{q\sim Q\\(q,a)=1}}\left|\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ m_3\sim M_3\\ m_1m_2m_3\equiv a(\bmod q)}}a_{1,m_1}a_{2,m_2}a_{3,m_3}-\frac{1}{\varphi(q)}\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ m_3\sim M_3\\ (m_1m_2m_3,q)=1}}a_{1,m_1}a_{2,m_2}a_{3,m_3}\right|\ll\frac{x}{(\log x)^A}.$$

Note that this lemma can be deduced from Lemma 2.4 and Lemma 2.6.

2.1.3. *Type-I/II estimate.*

Lemma 2.8. *([7], Theorems 5 and 5*]). Let $M_1M_2M_3 \asymp x$, $\min(M_1, M_2, M_3) > x^{\varepsilon}$ and $z \ll \exp\left(\log x(\log\log x)^{-2}\right)$. Let a_{1,m_1}, a_{2,m_2} and a_{3,m_3} be divisor-bounded complex sequences. Suppose that*

$$a_{2,m_2} = \mathbb{1}_{m_2 \in \mathbf{M}} \qquad \text{or} \qquad a_{2,m_2} = \mathbb{1}_{\substack{m_2 \in \mathbf{M} \\ (m_2, P(z))=1}}$$

for some interval $\mathbf{M} \subseteq [M_2, 2M_2]$. If we have

$$M_3 Q < x^{1-\varepsilon}, \quad M_1^4 M_3 Q < x^{2-\varepsilon}, \quad M_1^2 M_3 Q^2 < x^{2-\varepsilon},$$

then

$$\sum_{\substack{q \sim Q \\ (q,a)=1}} \left| \sum_{\substack{m_1 \sim M_1 \\ m_2 \in \mathbf{M} \\ m_3 \sim M_3 \\ m_1 m_2 m_3 \equiv a (\bmod q)}} a_{1,m_1} a_{2,m_2} a_{3,m_3} - \frac{1}{\varphi(q)} \sum_{\substack{m_1 \sim M_1 \\ m_2 \in \mathbf{M} \\ m_3 \sim M_3 \\ (m_1 m_2 m_3, q)=1}} a_{1,m_1} a_{2,m_2} a_{3,m_3} \right| \ll \frac{x}{(\log x)^A}.$$

2.1.4. *Another Type-II estimate.* The last estimate is a new type of Type-II estimate for convolutions, and it will be useful in proving a better asymptotic formula than the corresponding result used in [2], [3] and [23]. The proof of this lemma requires Lemma 2.7 and Lemma 2.8.

Lemma 2.9. *([28], Lemma 8.11]). Let $Q \leqslant x^{\frac{127}{224}-\varepsilon}$ and $M_1, M_2, \ldots, M_r \geqslant 1$ be such that $\prod_{1 \leqslant i \leqslant r} M_i \asymp x$. Let a_{j,m_j} $(1 \leqslant j \leqslant r)$ be divisor-bounded complex sequences. Suppose that a_{j,m_j} $(1 \leqslant j \leqslant r)$ satisfy **Conditions A and B** and*

$$a_{j,m_j} = \mathbb{1}_{(m_j, P(z))=1}, \qquad z = \exp\left(\log x(\log\log x)^{-2}\right)$$

for all $m_j \geqslant x^{\frac{1}{15}}$. Let $\mathbf{M}_i$ $(1 \leqslant i \leqslant r)$ be intervals such that $\mathbf{M}_i \subseteq [M_i, 2M_i]$. If we have

$$x^{\frac{3}{7}+\varepsilon} < \prod_{j \in \mathcal{J}} M_j < Q^{-1} x^{1-\varepsilon}$$

for some set $\mathcal{J} \subseteq \{1, \ldots, r\}$, then

$$\sum_{\substack{q \sim Q \\ (q,a)=1}} \left| \sum_{\substack{m_i \sim \mathbf{M}_i \\ 1 \leqslant i \leqslant r}} \left(\prod_{1 \leqslant j \leqslant r} a_{j,m_j} \right) \left(\mathbb{1}_{m_1 \cdots m_r \equiv a (\bmod q)} - \frac{\mathbb{1}_{(m_1 \cdots m_r, q)=1}}{\varphi(q)} \right) \right| \ll \frac{x}{(\log x)^A}.$$

2.2. **Sieve Asymptotic Formulas.** In this subsection we give asymptotic formulas for sums of sieve functions $S\left(\mathcal{A}^q_{p_1 \cdots p_n}, p_n\right)$ and $S\left(\mathcal{A}^q_{p_1 \cdots p_n}, x^{\kappa_0}\right)$ with $\kappa_0 = \kappa$ or κ' or other values, where

$$\kappa = \kappa(\theta) = \begin{cases} \frac{5-8\theta}{6} - \varepsilon, & \theta \leqslant \frac{17}{32} - \varepsilon, \\ \frac{5-8\theta}{12} - 3\varepsilon, & \frac{17}{32} - \varepsilon < \theta \leqslant \frac{7}{13} - \varepsilon, \\ \frac{3-5\theta}{7} - 2\varepsilon, & \frac{7}{13} - \varepsilon < \theta \leqslant \frac{4}{7} - \varepsilon, \end{cases}$$

and

$$\kappa' = \kappa'(\theta) = \begin{cases} \frac{11-20\theta}{6} - 2\varepsilon, & \frac{7}{13} - \varepsilon < \theta \leqslant \frac{11}{20} - \varepsilon, \\ \kappa, & \text{otherwise}. \end{cases}$$

We also write

$$\tau = \tau(\theta) = \begin{cases} \frac{3(1-\theta)}{5} - \varepsilon, & \theta \leqslant \frac{11}{21}, \\ \frac{2}{7} - \varepsilon, & \frac{11}{21} < \theta \leqslant \frac{6}{11} - \varepsilon, \\ \frac{5-6\theta}{7} - \varepsilon, & \frac{6}{11} - \varepsilon < \theta, \end{cases}$$

and

$$\tau' = \tau'(\theta) = \begin{cases} \frac{5-6\theta}{7}, & \frac{7}{13} - \varepsilon < \theta \leqslant \frac{11}{20} - \varepsilon, \\ \tau, & \text{otherwise}. \end{cases}$$

Lemma 2.10. *Let $\frac{1}{2} \leqslant \theta < \frac{4}{7}$. Define*

$$\begin{aligned}
\boldsymbol{g}_1 = \boldsymbol{g}_1(\theta) &= \left\{(s,t) : 2\theta - 1 < s < \frac{5-8\theta}{6}\right\}, \\
\boldsymbol{g}_2 = \boldsymbol{g}_2(\theta) &= \left\{(s,t) : s < \frac{17-33\theta}{36}\right\}, \\
\boldsymbol{g}_3 = \boldsymbol{g}_3(\theta) &= \left\{(s,t) : s+t > \theta,\ 2s+3t < 1+\theta,\ 5s+2t < 2,\ 4s+3t < 2\right\}, \\
\boldsymbol{g}_4 = \boldsymbol{g}_4(\theta) &= \left\{(s,t) : s+t > \theta,\ s+2t < 2-2\theta,\ 5s+2t < 2\right\}, \\
\boldsymbol{g}_5 = \boldsymbol{g}_5(\theta) &= \left\{(s,t) : s+t > \theta,\ t < 1-\theta,\ s+t < \frac{153}{224} - \frac{1}{7}\theta,\ 4s+t < \frac{57}{32} - \theta\right\},
\end{aligned}$$

$$\boldsymbol{\mathcal{G}}_j = \boldsymbol{\mathcal{G}}_j(\theta) = \{\boldsymbol{\alpha}_j : \boldsymbol{\alpha}_j \textit{ partitions exactly into } \boldsymbol{g}_1 \cup \boldsymbol{g}_2 \cup \boldsymbol{g}_3 \cup \boldsymbol{g}_4 \cup \boldsymbol{g}_5\}.$$

Suppose that $\min \boldsymbol{\alpha}_j \geqslant \varepsilon^2$*. Then we have, for almost all* $q \sim x^\theta$*,*

$$\sum_{\boldsymbol{\alpha}_j \in \boldsymbol{\mathcal{G}}_j} S\left(\mathcal{A}^q_{p_1 \cdots p_j}, p_j\right)$$

has an asymptotic formula of the form (9).

Proof. This lemma can be proved by the same method used in the proof of [[2], Lemma 7]. Note that the sets $\boldsymbol{g}_1$–$\boldsymbol{g}_5$ correspond to the conditions in Lemmas 2.2–2.6. □

Lemma 2.11. *([17], Lemmas 8.10 and 8.15]). Let* $\frac{1}{2} \leqslant \theta < \frac{17}{32}$ *and* $M < x^{2-\varepsilon} Q^{-3}$*. Then we have, for almost all* $q \sim x^\theta$ *and a divisor-bounded sequence* a_m*,*

$$\sum_{m \sim M} a_m S\left(\mathcal{A}^q_m, x^\kappa\right)$$

has an asymptotic formula of the form (9). Note that we have $x^{\frac{3}{7}} < x^{2-\varepsilon} Q^{-3}$ *when* $\theta < \frac{11}{21} - \varepsilon$*.*

Let $\frac{1}{2} \leqslant \theta < \frac{11}{20}$ *and* $M < x^{2-\varepsilon} Q^{-3}$*. Then we have, for almost all* $q \sim x^\theta$ *and a divisor-bounded sequence* a_m*,*

$$\sum_{m \sim M} a_m S\left(\mathcal{A}^q_m, x^{\frac{11-20\theta}{6} - \varepsilon}\right)$$

has an asymptotic formula of the form (9).

Remark. The condition $M < x^{2-\varepsilon} Q^{-3}$ in this lemma can be relaxed to $M < x^{1-\varepsilon} Q^{-1}$, but this brings no useful improvement in our final decompositions. We give a sketch of the proof: using Buchstab's identity many times (or Möbius inversion), we can replace the sum

$$\sum_{m \sim M} a_m S\left(\mathcal{A}^q_m, x^{\frac{11-20\theta}{6} - \varepsilon}\right)$$

with $\ll \log x (\log \log x)^{-1}$ sums of the form

$$\sum_{m \sim M} a_m \sum_{\substack{p_1 p_2 \cdots p_k m n \in \mathcal{A}^q \\ p_k < \ldots < p_1 < x^{\frac{11-20\theta}{6} - \varepsilon}}} 1$$

and an outer summation $\sum_{k \geqslant 0} (-1)^k$. We write $p_1 p_2 \cdots p_k = d \asymp D$. When $D > x^{2\theta - 1}$, we know that there must be a product of some prime variables that lies in $\left(2\theta - 1, \frac{5-8\theta}{6}\right)$ since $p_k < \ldots < p_1 < x^{\frac{11-20\theta}{6} - \varepsilon}$. We split the prime variables, starting from the largest p_1, and group the remaining ones together to form a Type-II sum after removing cross conditions.

When $D \leqslant x^{2\theta - 1}$, we can use Lemma 2.8 with $M_1 = D$ and $M_3 = M$. The conditions $D \leqslant x^{2\theta - 1}$ and $M < x^{1-\theta-\varepsilon}$ ensure the required conditions in Lemma 2.8: we have $M_3 < x^{1-\theta-\varepsilon}$, $M_1^4 M_3 < x^{7\theta - 3 - \varepsilon} < x^{2-\theta-\varepsilon}$ (since $\theta < \frac{5}{8}$) and $M_1^2 M_3 < x^{3\theta - 1 - \varepsilon} < x^{2 - 2\theta - \varepsilon}$ (since $\theta < \frac{3}{5}$).

Now we can give an asymptotic formula for the whole sum

$$\sum_{m \sim M} a_m S\left(\mathcal{A}^q_m, x^{\frac{11-20\theta}{6} - \varepsilon}\right).$$

The proof of an asymptotic formula for the sum

$$\sum_{m \sim M} a_m S\left(\mathcal{A}^q_m, x^\kappa\right)$$

can thus be done by the exact same process as in [[17], Lemma 8.15]. Note that the two ranges $\frac{1}{2} \leqslant \theta < \frac{11}{20}$ and $\frac{1}{2} \leqslant \theta < \frac{17}{32}$ cannot be enlarged because of the restriction on the width of the Type-II range $\left(2\theta - 1, \frac{5-8\theta}{6}\right)$ here.

Lemma 2.12. *([2], Lemma 14]). Let* $\frac{1}{2} \leqslant \theta < \frac{4}{7}$*. Define*

$$\boldsymbol{S} = \boldsymbol{S}(\theta) = \{(s, t) : s < 1 - \theta,\ s + 2t < 2 - 2\theta,\ s + 4t < 2 - \theta\},$$
$$\boldsymbol{S}_j = \boldsymbol{S}_j(\theta) = \{\boldsymbol{\alpha}_j : \boldsymbol{\alpha}_j \textit{ partitions exactly into } \boldsymbol{S}\}.$$

Then we have, for almost all $q \sim x^\theta$*,*

$$\sum_{\boldsymbol{\alpha}_j \in \boldsymbol{S}_j} S\left(\mathcal{A}^q_{p_1 \cdots p_j}, x^{\varepsilon^2}\right)$$

has an asymptotic formula of the form (9). Note that the set $\boldsymbol{S}$ *corresponds to the conditions in Lemma 2.8.*

Using Lemma 2.12, Lemma 2.10($\boldsymbol{g}_1$) and combinatorial arguments as in [2], the following lemma can be deduced.

Lemma 2.13. *([2], Lemmas 15 and 16]). Define*

$$\begin{aligned}
\boldsymbol{A}_j &= \boldsymbol{A}_j(\theta) = \left\{\boldsymbol{\alpha}_j : \varepsilon^2 \leqslant \alpha_j < \cdots < \alpha_1 < \tau,\ \alpha_1 + \cdots + \alpha_j \leqslant 1\right\},\\
\boldsymbol{T}^* &= \boldsymbol{T}^*(\theta) = \left\{(s,t) : 0 \leqslant s \leqslant \frac{8\theta - 2}{7},\ 0 \leqslant t \leqslant \frac{5 - 6\theta}{7}\right\},\\
\boldsymbol{T}_j^* &= \boldsymbol{T}_j^*(\theta) = \{\boldsymbol{\alpha}_j : \boldsymbol{\alpha}_j \text{ partitions exactly into } \boldsymbol{T}^*\},\\
\boldsymbol{U}_j' &= \boldsymbol{U}_j'(\theta) = \{\boldsymbol{\alpha}_j : \boldsymbol{\alpha}_j \in \boldsymbol{A}_j,\ (\alpha_1, \ldots, \alpha_j, 2\theta - 1 + \varepsilon) \in \boldsymbol{S}_{j+1}\},\\
\boldsymbol{U}_j &= \boldsymbol{U}_j(\theta) = \begin{cases} \boldsymbol{U}_j'(\theta), & \theta < \frac{7}{13},\\ \left\{\boldsymbol{\alpha}_j : \boldsymbol{\alpha}_j \in \boldsymbol{A}_j,\ \boldsymbol{\alpha}_j \in \boldsymbol{T}_j^*\right\}, & \theta \geqslant \frac{7}{13}. \end{cases}
\end{aligned}$$

Let $\frac{1}{2} \leqslant \theta < \frac{4}{7}$. Then we have, for almost all $q \sim x^\theta$,

$$\sum_{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j} S\left(\mathcal{A}_{p_1 \cdots p_j}^q, x^\kappa\right)$$

has an asymptotic formula of the form (9).

Let $\frac{1}{2} \leqslant \theta < \frac{11}{20}$. Then we have, for almost all $q \sim x^\theta$,

$$\sum_{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j'} S\left(\mathcal{A}_{p_1 \cdots p_j}^q, x^{\kappa'}\right)$$

has an asymptotic formula of the form (9).

Next, we shall use Lemma 2.12 and Lemma 2.10 to deduce new asymptotic formulas, which can be seen as an alternative of Lemma 2.11 and Lemma 2.13 when θ lies in a short interval.

Lemma 2.14. *Let $\frac{1}{2} \leqslant \theta < \frac{53}{105}$. Then we have, for almost all $q \sim x^\theta$,*

$$\sum_{\boldsymbol{\alpha}_j \in \boldsymbol{S}_j} S\left(\mathcal{A}_{p_1 \cdots p_j}^q, x^\kappa\right)$$

has an asymptotic formula of the form (9).

Let $\frac{1}{2} \leqslant \theta < \frac{17}{33}$. Then we have, for almost all $q \sim x^\theta$,

$$\sum_{\boldsymbol{\alpha}_j \in \boldsymbol{S}_j} S\left(\mathcal{A}_{p_1 \cdots p_j}^q, x^{\frac{17 - 33\theta}{36} - \varepsilon}\right)$$

has an asymptotic formula of the form (9).

Proof. We first prove the case 1 of Lemma 2.14. Using Buchstab's identity, we have

$$\sum_{\boldsymbol{\alpha}_j \in \boldsymbol{S}_j} S\left(\mathcal{A}_{p_1 \cdots p_j}^q, x^\kappa\right) = \sum_{\boldsymbol{\alpha}_j \in \boldsymbol{S}_j} S\left(\mathcal{A}_{p_1 \cdots p_j}^q, x^{\varepsilon^2}\right) - \sum_{\substack{\boldsymbol{\alpha}_j \in \boldsymbol{S}_j \\ \varepsilon^2 \leqslant \alpha_{j+1} < \kappa}} S\left(\mathcal{A}_{p_1 \cdots p_j p_{j+1}}^q, p_{j+1}\right).$$

By Lemma 2.12, the first sum on the right-hand side has an asymptotic formula of the form (9). Note that $2\theta - 1 < \frac{17 - 33\theta}{36}$ when $\theta < \frac{53}{105}$, the second sum on the right-hand side has an asymptotic formula of the form (9) by Lemma 2.10($\boldsymbol{g}_1$ and $\boldsymbol{g}_2$). Now the case 1 of Lemma 2.14 is proved.

The case 2 of Lemma 2.14 can be proved in the same way, using Lemma 2.12 and Lemma 2.10($\boldsymbol{g}_2$). □

The next lemma is one of the most important asymptotic formulas used in [2] and [23].

Lemma 2.15. *([17], Lemma 8.14]). Let $\frac{1}{2} \leqslant \theta < \frac{4}{7}$. Define*

$$\begin{aligned}
\boldsymbol{T}^{**} &= \boldsymbol{T}^{**}(\theta) = \left\{(s,t) : \frac{3}{7} < s < 1 - \theta,\ 0 \leqslant t \leqslant \frac{1 - s}{2}\right\},\\
\boldsymbol{U}_j^* &= \boldsymbol{U}_j^*(\theta) = \{\boldsymbol{\alpha}_j : \boldsymbol{\alpha}_j \text{ partitions exactly into } \boldsymbol{T}^{**}\}.
\end{aligned}$$

Let $\xi(\boldsymbol{\alpha_j})$ be a continuous function with $\varepsilon^2 \leqslant \xi(\boldsymbol{\alpha_j}) \leqslant \frac{1}{2}$. Then we have, for almost all $q \sim x^\theta$,

$$\sum_{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j^*} S\left(\mathcal{A}_{p_1 \cdots p_j}^q, x^{\xi(\boldsymbol{\alpha_j})}\right)$$

has an asymptotic formula of the form (9).

Lemma 2.15 gives asymptotic formulas for sums of sieve functions $S\left(\mathcal{A}_{p_1 \cdots p_n}^q, p_n\right)$ when some of the variables can be grouped to lie in the interval $\left(\frac{3}{7}, 1 - \theta\right)$. However, we will not use this lemma in the final decompositions. Instead, we will prove a stronger result that does not require the condition $t \leqslant \frac{1}{2}(1 - s)$.

Lemma 2.16. *Let $\frac{1}{2} \leqslant \theta < \frac{127}{224}$. Define*

$$\boldsymbol{T}^{***} = \boldsymbol{T}^{***}(\theta) = \left\{(s,t) : \frac{3}{7} < s < 1-\theta \text{ or } \theta < s < \frac{4}{7}\right\},$$
$$\boldsymbol{V}_j = \boldsymbol{V}_j(\theta) = \left\{\boldsymbol{\alpha}_j : \boldsymbol{\alpha}_j \text{ partitions exactly into } \boldsymbol{T}^{***}\right\}.$$

Suppose that $\min \boldsymbol{\alpha}_j \geqslant \frac{1}{1000} + 10\varepsilon$. Then we have, for almost all $q \sim x^\theta$,

$$\sum_{\boldsymbol{\alpha}_j \in \boldsymbol{V}_j} S\left(\mathcal{A}^q_{p_1 \cdots p_j}, p_j\right)$$

has an asymptotic formula of the form (9).

Proof. This lemma can be proved by the same method used in the proof of [[28], Lemma 8.12]. The proof requires Heath-Brown identity, Lemma 2.9, a discussion on the contribution from higher prime-powers counted in the von Mangoldt function (see [[28], Lemma 8.9]), and a removal of the dependencies between variables (see [[28], Lemma 8.10]). □

Finally, we define the whole "Type-II" region $\boldsymbol{G}_j$ and the two-dimensional region U_2. We split $U_2 \backslash \boldsymbol{G}_2$ into three parts A, B and C. Regions A, B, C and $\boldsymbol{G}_j$ will also be used in the next sections.

Lemma 2.17. *([17], Lemma 8.16]). Define*

$$\boldsymbol{G}_j = \boldsymbol{G}_j(\theta) = \left\{\boldsymbol{\alpha}_j : \boldsymbol{\alpha}_j \in \boldsymbol{\mathcal{G}}_j \cup \boldsymbol{V}_j\right\},$$
$$U_2 = U_2(\theta) = \left\{\boldsymbol{\alpha}_2 : \kappa \leqslant \alpha_1 \leqslant \frac{3}{7},\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)\right\},$$
$$A = A(\theta) = \left\{\boldsymbol{\alpha}_2 : \boldsymbol{\alpha}_2 \in U_2,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2,\ \alpha_1 + \alpha_2 < \theta\right\},$$
$$B = B(\theta) = \left\{\boldsymbol{\alpha}_2 : \boldsymbol{\alpha}_2 \in U_2,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \cup A,\ \alpha_1 + 4\alpha_2 < 3 - 3\theta\right\},$$
$$C = C(\theta) = \left\{\boldsymbol{\alpha}_2 : \boldsymbol{\alpha}_2 \in U_2,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \cup A \cup B\right\}.$$

Let $\frac{1}{2} \leqslant \theta < \frac{11}{21}$. Then we have, for almost all $q \sim x^\theta$,

$$\sum_{\boldsymbol{\alpha}_2 \in A \cup B} S\left(\mathcal{A}^q_{p_1 p_2}, x^\kappa\right)$$

has an asymptotic formula of the form (9).

2.3. **High-dimensional Sieves.** In this section, we mention several results regarding the upper and lower bounds for some sieve functions. The first two lemmas are proved using a two-dimensional Harman's sieve, and we shall use them in the final decomposition for both the majorant and the minorant.

Lemma 2.18. *([2], Lemma 20]). Let $\frac{1}{2} \leqslant \theta < \frac{17}{32}$. Then we have, for almost all $q \sim x^\theta$,*

$$-\sum_{1-\theta \leqslant \alpha_1 < \frac{1}{2}} S\left(\mathcal{A}^q_{p_1}, p_1\right) \leqslant (1+o(1)) \frac{1}{\varphi(q)} \left(-\sum_{1-\theta \leqslant \alpha_1 < \frac{1}{2}} S\left(\mathcal{B}^q_{p_1}, p_1\right) + \sum_{\substack{\kappa \leqslant \alpha_3 < \alpha_1 < \frac{\theta}{2} \\ 1-\theta \leqslant \alpha_1 + v \leqslant \theta \\ (\alpha_1, v) \notin \boldsymbol{G}_2 \\ \alpha_3 < \min\left(v, \frac{1}{2}(1-\alpha_1-v)\right) \\ (\alpha_1, v, \alpha_3) \notin \boldsymbol{G}_3}} S\left(\mathcal{B}^q_{p_1 m p_3}, p_3\right) \right),$$

where $m = x^v$ and $(m, P(p_1)) = 1$.

Lemma 2.19. *([2], Lemma 21]). Let $\frac{1}{2} \leqslant \theta < \frac{17}{32}$. Then we have, for almost all $q \sim x^\theta$,*

$$-\sum_{1-\theta \leqslant \alpha_1 < \frac{1}{2}} S\left(\mathcal{A}^q_{p_1}, p_1\right) \geqslant (1+o(1)) \frac{1}{\varphi(q)} \left(-\sum_{1-\theta \leqslant \alpha_1 < \frac{1}{2}} S\left(\mathcal{B}^q_{p_1}, p_1\right) - \sum_{\substack{\kappa \leqslant \alpha_1 < v \\ 1-\theta \leqslant \alpha_1 + v \leqslant \theta \\ (\alpha_1, v) \notin \boldsymbol{G}_2}} S\left(\mathcal{B}^q_{p_1 m}, p_1\right) \right),$$

where $m = x^v$ and $(m, P(p_1)) = 1$.

The next two lemmas are proved using a three-dimensional Harman's sieve, and we shall use them in the final decomposition for the majorant.

Lemma 2.20. *([2], Lemma 24]). Define*

$$\boldsymbol{R} = \boldsymbol{R}(\theta) = \left\{\boldsymbol{\alpha}_2 : \alpha_2 \leqslant \alpha_1,\ \alpha_1 + 2\alpha_2 \leqslant 1,\ \alpha_1 + 4\alpha_2 \geqslant 3 - 3\theta,\ 3\alpha_2 \geqslant 2\alpha_1, \right.$$
$$\left. \max\left(\tau, \frac{31\theta - 15}{3}\right) \leqslant \alpha_1 \leqslant \min\left(\frac{3}{7}, 4 - 7\theta\right)\right\},$$

$$\begin{aligned}
\boldsymbol{D}_1 = \boldsymbol{D}_1(\theta) = \{ & \boldsymbol{\alpha}_3 : \alpha_1 \geqslant \kappa,\ \alpha_2 \geqslant \kappa,\ \alpha_3 \geqslant \kappa,\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3, \\
& (\alpha_1, \alpha_2+\alpha_3) \in \boldsymbol{R} \text{ with } \alpha_2 \geqslant \alpha_3 \text{ or } (\alpha_1+\alpha_2, \alpha_3) \in \boldsymbol{R} \text{ with } \alpha_1 \geqslant \alpha_2 \}, \\
\boldsymbol{D}_2 = \boldsymbol{D}_2(\theta) = \{ & \boldsymbol{\alpha}_4 : \alpha_1 \geqslant \kappa,\ \alpha_2 \geqslant \kappa,\ \alpha_3 \geqslant \kappa,\ \alpha_4 \geqslant \kappa, \\
& \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ (\alpha_1+\alpha_2, \alpha_3+\alpha_4) \in \boldsymbol{R} \}.
\end{aligned}$$

Let $\frac{1}{2} \leqslant \theta < \frac{16}{31}$. *Then we have, for almost all* $q \sim x^\theta$,

$$\sum_{\boldsymbol{\alpha}_2 \in \boldsymbol{R}} S\left(\mathcal{A}^q_{p_1 p_2}, x^\kappa\right) \leqslant (1+o(1)) \frac{1}{\varphi(q)} \left(\sum_{\boldsymbol{\alpha}_2 \in \boldsymbol{R}} S\left(\mathcal{B}^q_{p_1 p_2}, x^\kappa\right) + \frac{x}{\log x}(I_1 + I_2) \right),$$

where

$$I_1 = \frac{1}{\kappa} \int_{(t_1,t_2,t_3) \in \boldsymbol{D}_1} \frac{\omega\left(\frac{1-t_1-t_2-t_3}{\kappa}\right)}{t_1 t_2 t_3} dt_3 dt_2 dt_1,$$

$$I_2 = \frac{1}{\kappa} \int_{(t_1,t_2,t_3,t_4) \in \boldsymbol{D}_2} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{\kappa}\right)}{t_1 t_2 t_3 t_4} dt_4 dt_3 dt_2 dt_1.$$

Lemma 2.21. *([3], Lemma 24]). Define*

$$\begin{aligned}
\boldsymbol{R}_0 = \boldsymbol{R}_0(\theta) = \{ & \boldsymbol{\alpha}_2 : \alpha_1 + 2\alpha_2 \leqslant 1,\ \alpha_1 + 4\alpha_2 \geqslant 3 - 3\theta,\ 3\alpha_2 \geqslant 2\alpha_1, \\
& \max\left(\frac{19\theta - 7}{7},\ \frac{50\theta - 19}{17} \right) \leqslant \alpha_1 \leqslant \frac{3}{7} \}, \\
\boldsymbol{D}_3 = \boldsymbol{D}_3(\theta) = \{ & \boldsymbol{\alpha}_3 : \alpha_1 \geqslant \kappa,\ \alpha_2 \geqslant \kappa,\ \alpha_3 \geqslant \kappa,\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3, \\
& (\alpha_1, \alpha_2+\alpha_3) \in \boldsymbol{R}_0 \text{ with } \alpha_2 \geqslant \alpha_3 \text{ or } (\alpha_1+\alpha_2, \alpha_3) \in \boldsymbol{R}_0 \text{ with } \alpha_1 \geqslant \alpha_2 \}, \\
\boldsymbol{D}_4 = \boldsymbol{D}_4(\theta) = \{ & \boldsymbol{\alpha}_4 : \alpha_1 \geqslant \kappa,\ \alpha_2 \geqslant \kappa,\ \alpha_3 \geqslant \kappa,\ \alpha_4 \geqslant \kappa, \\
& \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ (\alpha_1+\alpha_2, \alpha_3+\alpha_4) \in \boldsymbol{R}_0 \}.
\end{aligned}$$

Let $\frac{25}{49} \leqslant \theta \leqslant \frac{92}{175}$. *Then we have, for almost all* $q \sim x^\theta$,

$$\sum_{\boldsymbol{\alpha}_2 \in \boldsymbol{R}_0} S\left(\mathcal{A}^q_{p_1 p_2}, x^\kappa\right) \leqslant (1+o(1)) \frac{1}{\varphi(q)} \left(\sum_{\boldsymbol{\alpha}_2 \in \boldsymbol{R}_0} S\left(\mathcal{B}^q_{p_1 p_2}, x^\kappa\right) + \frac{x}{\log x}(I_3 + I_4) \right),$$

where

$$I_3 = \frac{1}{\kappa} \int_{(t_1,t_2,t_3) \in \boldsymbol{D}_3} \frac{\omega\left(\frac{1-t_1-t_2-t_3}{\kappa}\right)}{t_1 t_2 t_3} dt_3 dt_2 dt_1,$$

$$I_4 = \frac{1}{\kappa} \int_{(t_1,t_2,t_3,t_4) \in \boldsymbol{D}_4} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{\kappa}\right)}{t_1 t_2 t_3 t_4} dt_4 dt_3 dt_2 dt_1.$$

2.4. **Upper Bounds.** We shall construct the majorant $\rho_1(n)$ in this subsection. Before constructing, we first mention some existing results of $C_1(\theta)$.

Theorem 2.22. *The function* $C_1(\theta)$ *satisfies the following conditions:*
(1). $C_1(\theta) = 1$ *for all* $\theta < 0.5$ *and* $C_1(\theta) \leqslant 1 + \varepsilon$ *for* $\theta = 0.5$*;*
(2). $C_1(\theta)$ *is monotonic increasing for* $0.5 \leqslant \theta \leqslant 0.6$.

Proof. This theorem follows easily from the Bombieri–Vinogradov Theorem and [[2], Theorem 1(i)(ii)]. □

Now we split the range $\theta \in (0.5, 1)$ to several subranges and use different methods to treat them and obtain good bounds for $C_1(\theta)$. We shall split the range of θ based on the work done in [23]. Note that when $\theta \in (0.5, 0.565]$ we can remove the condition $t \leqslant \frac{1}{2}(1-s)$ when applying the Type-II information on $(\frac{3}{7}, 1-\theta)$.

2.4.1. *Case 1.* $\frac{1}{2} < \theta < \frac{53}{105}$. In this case, our Type-II range becomes

$$\left(0,\ \frac{1}{6}(5 - 8\theta)\right). \tag{11}$$

Compare the definitions of $\boldsymbol{S}_j$ and $\boldsymbol{U}_j$, we can easily show that $\boldsymbol{\alpha}_j \in \boldsymbol{U}_j$ implies $\boldsymbol{\alpha}_j \in \boldsymbol{S}_j$ when $\frac{1}{2} < \theta < \frac{53}{105}$. Hence we can use Lemma 2.14 instead of Lemma 2.13 to give more asymptotic formulas. Using Buchstab's identity, we have

$$\begin{aligned}
S\left(\mathcal{A}^q, (2x)^{\frac{1}{2}}\right) &= S\left(\mathcal{A}^q, x^\kappa\right) - \sum_{\kappa \leqslant \alpha_1 < \frac{1}{2}} S\left(\mathcal{A}^q_{p_1}, p_1\right) \\
&= S\left(\mathcal{A}^q, x^\kappa\right) - \sum_{\kappa \leqslant \alpha_1 \leqslant \frac{3}{7}} S\left(\mathcal{A}^q_{p_1}, p_1\right) - \sum_{\frac{3}{7} < \alpha_1 < 1-\theta} S\left(\mathcal{A}^q_{p_1}, p_1\right) - \sum_{1-\theta \leqslant \alpha_1 < \frac{1}{2}} S\left(\mathcal{A}^q_{p_1}, p_1\right) \\
&= S^q_{211} - S^q_{212} - S^q_{213} - S^q_{214}.
\end{aligned} \tag{12}$$

By Lemma 2.11 and Lemma 2.16, we can give asymptotic formulas for S_{211}^{q} and S_{213}^{q}. For S_{214}^{q}, we can use Lemma 2.18 to give an upper bound with a loss of

$$\sum_{\substack{\kappa \leqslant \alpha_3 < \alpha_1 < \frac{\theta}{2} \\ 1-\theta \leqslant \alpha_1 + v \leqslant \theta \\ (\alpha_1, v) \notin \boldsymbol{G}_2 \\ \alpha_3 < \min\left(v, \frac{1}{2}(1-\alpha_1-v)\right) \\ (\alpha_1, v, \alpha_3) \notin \boldsymbol{G}_3}} S\left(\mathcal{A}_{p_1 m p_3}^{q}, p_3\right). \tag{13}$$

For S_{212}^{q}, we use Buchstab's identity again and split the resulting sum into three subsums corresponding to $\boldsymbol{\alpha}_2 \in \boldsymbol{G}_2$, $\boldsymbol{\alpha}_2 \in A \cup B$ and $\boldsymbol{\alpha}_2 \in C$. The next steps are almost same as the decomposing procedure in [[23], Section 6.1]: we give asymptotic formulas when $\boldsymbol{\alpha}_2 \in \boldsymbol{G}_2$, perform a straightforward decomposition when $\boldsymbol{\alpha}_2 \in A \cup B$, and use Lemma 2.20 when $\boldsymbol{\alpha}_2 \in C$. The only difference here is that we can replace the required condition $(\alpha_1, \alpha_2, \alpha_3, \alpha_3) \in \boldsymbol{U}_4$ with $(\alpha_1, \alpha_2, \alpha_3, \alpha_3) \in \boldsymbol{S}_4$ when decomposing the two-dimensional sum with $\boldsymbol{\alpha}_2 \in A \cup B$. Again, the condition $t \leqslant \frac{1}{2}(1-s)$ in the Type-II range $(\frac{3}{7}, 1-\theta)$ can be removed.

Another important device that can be used here is the *role-reversal*, which can be seen as a trick to "use Type-I information more effective" by changing the roles of a larger explicit variable and a smaller implicit variable in the sum. The definition of a role-reversal can be found in [[17], Chapter 5]. In this case, we can perform a role-reversal on the three-dimensional sum

$$\sum_{\substack{\boldsymbol{\alpha}_2 \in A \cup B \\ \kappa \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ (\alpha_1, \alpha_2, \alpha_3, \alpha_3) \notin \boldsymbol{S}_4}} S\left(\mathcal{A}_{p_1 p_2 p_3}^{q}, p_3\right)$$

if $\boldsymbol{\alpha}_3 \in \boldsymbol{S}_3$ and $(1-\alpha_1-\alpha_2-\alpha_3, \alpha_2, \alpha_3) \in \boldsymbol{S}_3$. That is, we have

$$\begin{aligned}
&\sum_{\substack{\boldsymbol{\alpha}_2 \in A \cup B \\ \kappa \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ (\alpha_1, \alpha_2, \alpha_3, \alpha_3) \notin \boldsymbol{S}_4 \\ \boldsymbol{\alpha}_3 \in \boldsymbol{S}_3 \\ (1-\alpha_1-\alpha_2-\alpha_3, \alpha_2, \alpha_3) \in \boldsymbol{S}_3}} S\left(\mathcal{A}_{p_1 p_2 p_3}^{q}, p_3\right) \\
=& \sum_{\substack{\boldsymbol{\alpha}_2 \in A \cup B \\ \kappa \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ (\alpha_1, \alpha_2, \alpha_3, \alpha_3) \notin \boldsymbol{S}_4 \\ \boldsymbol{\alpha}_3 \in \boldsymbol{S}_3 \\ (1-\alpha_1-\alpha_2-\alpha_3, \alpha_2, \alpha_3) \in \boldsymbol{S}_3}} S\left(\mathcal{A}_{p_1 p_2 p_3}^{q}, x^{\kappa}\right) - \sum_{\substack{\boldsymbol{\alpha}_2 \in A \cup B \\ \kappa \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ (\alpha_1, \alpha_2, \alpha_3, \alpha_3) \notin \boldsymbol{S}_4 \\ \boldsymbol{\alpha}_3 \in \boldsymbol{S}_3 \\ (1-\alpha_1-\alpha_2-\alpha_3, \alpha_2, \alpha_3) \in \boldsymbol{S}_3 \\ \kappa \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \in \boldsymbol{G}_4}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}^{q}, p_4\right) \\
&- \sum_{\substack{\boldsymbol{\alpha}_2 \in A \cup B \\ \kappa \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ (\alpha_1, \alpha_2, \alpha_3, \alpha_3) \notin \boldsymbol{S}_4 \\ \boldsymbol{\alpha}_3 \in \boldsymbol{S}_3 \\ (1-\alpha_1-\alpha_2-\alpha_3, \alpha_2, \alpha_3) \in \boldsymbol{S}_3 \\ \kappa \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}^{q}, p_4\right) \\
=& \sum_{\substack{\boldsymbol{\alpha}_2 \in A \cup B \\ \kappa \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ (\alpha_1, \alpha_2, \alpha_3, \alpha_3) \notin \boldsymbol{S}_4 \\ \boldsymbol{\alpha}_3 \in \boldsymbol{S}_3 \\ (1-\alpha_1-\alpha_2-\alpha_3, \alpha_2, \alpha_3) \in \boldsymbol{S}_3}} S\left(\mathcal{A}_{p_1 p_2 p_3}^{q}, x^{\kappa}\right) - \sum_{\substack{\boldsymbol{\alpha}_2 \in A \cup B \\ \kappa \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ (\alpha_1, \alpha_2, \alpha_3, \alpha_3) \notin \boldsymbol{S}_4 \\ \boldsymbol{\alpha}_3 \in \boldsymbol{S}_3 \\ (1-\alpha_1-\alpha_2-\alpha_3, \alpha_2, \alpha_3) \in \boldsymbol{S}_3 \\ \kappa \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \in \boldsymbol{G}_4}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}^{q}, p_4\right) \\
&- \sum_{\substack{\boldsymbol{\alpha}_2 \in A \cup B \\ \kappa \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ (\alpha_1, \alpha_2, \alpha_3, \alpha_3) \notin \boldsymbol{S}_4 \\ \boldsymbol{\alpha}_3 \in \boldsymbol{S}_3 \\ (1-\alpha_1-\alpha_2-\alpha_3, \alpha_2, \alpha_3) \in \boldsymbol{S}_3 \\ \kappa \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4}} S\left(\mathcal{A}_{\beta_1 p_2 p_3 p_4}^{q}, \left(\frac{2x}{\beta_1 p_2 p_3 p_4}\right)^{\frac{1}{2}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{\boldsymbol{\alpha}_2 \in A\cup B \\ \kappa \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ (\alpha_1,\alpha_2,\alpha_3,\alpha_3) \notin \boldsymbol{S}_4 \\ \boldsymbol{\alpha}_3 \in \boldsymbol{S}_3 \\ (1-\alpha_1-\alpha_2-\alpha_3,\alpha_2,\alpha_3) \in \boldsymbol{S}_3}} S\left(\mathcal{A}^q_{p_1p_2p_3}, x^{\kappa}\right) - \sum_{\substack{\boldsymbol{\alpha}_2 \in A\cup B \\ \kappa \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ (\alpha_1,\alpha_2,\alpha_3,\alpha_3) \notin \boldsymbol{S}_4 \\ \boldsymbol{\alpha}_3 \in \boldsymbol{S}_3 \\ (1-\alpha_1-\alpha_2-\alpha_3,\alpha_2,\alpha_3) \in \boldsymbol{S}_3 \\ \kappa \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \in \boldsymbol{G}_4}} S\left(\mathcal{A}^q_{p_1p_2p_3p_4}, p_4\right) \\
&\quad - \sum_{\substack{\boldsymbol{\alpha}_2 \in A\cup B \\ \kappa \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ (\alpha_1,\alpha_2,\alpha_3,\alpha_3) \notin \boldsymbol{S}_4 \\ \boldsymbol{\alpha}_3 \in \boldsymbol{S}_3 \\ (1-\alpha_1-\alpha_2-\alpha_3,\alpha_2,\alpha_3) \in \boldsymbol{S}_3 \\ \kappa \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4}} S\left(\mathcal{A}^q_{\beta_1p_2p_3p_4}, x^{\kappa}\right) + \sum_{\substack{\boldsymbol{\alpha}_2 \in A\cup B \\ \kappa \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ (\alpha_1,\alpha_2,\alpha_3,\alpha_3) \notin \boldsymbol{S}_4 \\ \boldsymbol{\alpha}_3 \in \boldsymbol{S}_3 \\ (1-\alpha_1-\alpha_2-\alpha_3,\alpha_2,\alpha_3) \in \boldsymbol{S}_3 \\ \kappa \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4 \\ \kappa \leqslant \alpha_5 < \frac{1}{2}\alpha_1 \\ (1-\alpha_1-\alpha_2-\alpha_3-\alpha_4,\alpha_2,\alpha_3,\alpha_4,\alpha_5) \in \boldsymbol{G}_5}} S\left(\mathcal{A}^q_{\beta_1p_2p_3p_4p_5}, p_5\right) \\
&\quad + \sum_{\substack{\boldsymbol{\alpha}_2 \in A\cup B \\ \kappa \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ (\alpha_1,\alpha_2,\alpha_3,\alpha_3) \notin \boldsymbol{S}_4 \\ \boldsymbol{\alpha}_3 \in \boldsymbol{S}_3 \\ (1-\alpha_1-\alpha_2-\alpha_3,\alpha_2,\alpha_3) \in \boldsymbol{S}_3 \\ \kappa \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4 \\ \kappa \leqslant \alpha_5 < \frac{1}{2}\alpha_1 \\ (1-\alpha_1-\alpha_2-\alpha_3-\alpha_4,\alpha_2,\alpha_3,\alpha_4,\alpha_5) \notin \boldsymbol{G}_5}} S\left(\mathcal{A}^q_{\beta_1p_2p_3p_4p_5}, p_5\right) \\
&= S^q_{2121} - S^q_{2122} - S^q_{2123} + S^q_{2124} + S^q_{2125}. \qquad (14)
\end{aligned}
$$

We can give asymptotic formulas for S^q_{2121}–S^q_{2124} by Lemmas 2.14, 2.10 and 2.16, and we simply discard S^q_{2125} with a small five-dimensional loss. When $\boldsymbol{\alpha}_3 \in \boldsymbol{U}_3$ and $(1-\alpha_1-\alpha_2-\alpha_3,\alpha_2,\alpha_3) \in \boldsymbol{U}_3$ (or only one $\boldsymbol{S}_3$ is replaced by $\boldsymbol{U}_3$ and another one remains unchanged), we can still use the above role-reversal device (with a rearrangement of the subscripts of prime variables, sometimes split β_1).

2.4.2. *Case 2.* $\frac{53}{105} \leqslant \theta < \frac{17}{33}$. In this range, the only new arithmetic information input is that we can estimate

$$
\sum_{\boldsymbol{\alpha}_j \in \boldsymbol{S}_j} S\left(\mathcal{A}^q_{p_1\cdots p_j}, x^{\frac{17-33\theta}{36}-\varepsilon}\right).
$$

However, we must handle sums with some prime variables lie in $\left[x^{\frac{17-33\theta}{36}}, x^{2\theta-1}\right]$ if we perform a straightforward decomposition or a role-reversal to get the above sum. When $(\alpha_1,\alpha_2,\alpha_3,\alpha_3) \in \boldsymbol{U}_4$, performing a straightforward decomposition is good. But when $(\alpha_1,\alpha_2,\alpha_3,\alpha_3) \in \boldsymbol{S}_4$ and $(\alpha_1,\alpha_2,\alpha_3,\alpha_3) \notin \boldsymbol{U}_4$, doing a straightforward decomposition is not that good anymore. When $\theta = \frac{53}{105}$, we have $\frac{17-33\theta}{36} = \frac{1}{105}$, which means that the corresponding sums may count numbers with about 100 prime factors. After numerical calculation we found that the sizes of resulting sums exceed the sizes of the original ones, hence we decide not to use this new arithmetic information. The remaining things here are the same as in [23], with a removal of the restriction $t \leqslant \frac{1}{2}(1-s)$ in Lemma 2.15.

2.4.3. *Case 3.* $\frac{17}{33} \leqslant \theta < 0.6$. In this range we do not have new asymptotic formulas outside of those in [23] except Lemma 2.16. Since $\frac{127}{224} \approx 0.5669 > 0.565$, for $\frac{17}{33} \leqslant \theta \leqslant 0.565$ we only need to remove the restriction $t \leqslant \frac{1}{2}(1-s)$ in Lemma 2.15 and do the same decomposing process as in [23]. For $0.565 < \theta < 0.6$, we use the existing upper bound for $C_1(\theta)$ proved by Baker and Harman [2]:

$$
C_1(\theta) \leqslant \begin{cases} \frac{14}{12-13\theta} - \log\left(\frac{4(1-\theta)}{3\theta}\right), & 0.565 < \theta < \frac{4}{7}, \\ \frac{14}{12-13\theta}, & \frac{4}{7} \leqslant \theta < 0.6. \end{cases} \qquad (15)
$$

2.4.4. *Case 4.* $0.6 \leqslant \theta < 1$. From here, we use the results of Fouvry [12] and Fouvry–Radziwiłł [16]:

$$
C_1(\theta) \leqslant \begin{cases} \frac{8}{3-\theta}, & 0.6 \leqslant \theta < \frac{5}{7}, \\ \frac{6}{1+\theta}, & \frac{5}{7} \leqslant \theta < \frac{3}{4}, \\ \frac{12}{5-2\theta}, & \frac{3}{4} \leqslant \theta < \frac{5}{6}, \\ \frac{48}{15-2\theta}, & \frac{5}{6} \leqslant \theta < \frac{9}{10}, \\ \frac{210}{53}, & \frac{104}{105} \leqslant \theta \leqslant \frac{209}{210}, \\ \frac{66}{36\theta-19}, & \frac{209}{210} < \theta < 1. \end{cases} \qquad (16)
$$

Note that the bounds in the last two ranges of θ come from [16], and the remaining bounds come from [12].

The following table gives the values of the upper bounds for $C_1(\theta)$ when $0.5 < \theta < 0.565$.

$\boldsymbol{\theta}$	$\boldsymbol{C_1(\theta)}$	$\boldsymbol{\theta}$	$\boldsymbol{C_1(\theta)}$	$\boldsymbol{\theta}$	$\boldsymbol{C_1(\theta)}$
0.501	1.0001	0.523	1.7138	0.545	2.2179
0.502	1.0003	0.524	1.7382	0.546	2.3122
0.503	1.0013	0.525	1.7500	0.547	2.3310
0.504	1.0019	0.526	1.7644	0.548	2.3725
0.505	1.0030	0.527	1.7775	0.549	2.3738
0.506	1.0043	0.528	1.7907	0.550	2.3996
0.507	1.0071	0.529	1.8038	0.551	2.4318
0.508	1.0091	0.530	1.8169	0.552	2.4393
0.509	1.0111	0.531	1.8300	0.553	2.4404
0.510	1.0150	0.532	1.8619	0.554	2.4426
0.511	1.0237	0.533	1.8658	0.555	2.4540
0.512	1.0315	0.534	1.8780	0.556	2.4776
0.513	1.0789	0.535	1.9028	0.557	2.4987
0.514	1.1431	0.536	1.9200	0.558	2.5120
0.515	1.2199	0.537	1.9277	0.559	2.5327
0.516	1.3199	0.538	1.9466	0.560	2.5345
0.517	1.5740	0.539	1.9568	0.561	2.6895
0.518	1.5948	0.540	1.9993	0.562	2.7144
0.519	1.6136	0.541	2.0582	0.563	2.8054
0.520	1.6320	0.542	2.0793	0.564	2.8788
0.521	1.6591	0.543	2.1223	0.565	2.9548
0.522	1.6873	0.544	2.1604		

Table 2.1: Upper Bounds for $\boldsymbol{C_1(\theta)}$ $\boldsymbol{(0.5 < \theta \leqslant 0.565)}$

Combining various bounds in this subsection, we can recover the estimate

$$\int_{0.5}^{0.679} C_1(\theta) d\theta < 0.5 \tag{17}$$

proved in [23]. However, we are still unable to show that

$$\int_{0.5}^{0.68} C_1(\theta) d\theta < 0.5. \tag{18}$$

2.5. **Lower Bounds.** We shall construct the minorant $\rho_0(n)$ in this subsection. Before constructing, we first mention some existing results of $C_0(\theta)$.

Theorem 2.23. *The function $C_0(\theta)$ satisfies the following conditions:*
(1). $C_0(\theta) = 1$ for all $\theta < 0.5$ and $C_0(\theta) \geqslant 1 - \varepsilon$ for $\theta = 0.5$;
(2). $C_0(\theta)$ is monotonic decreasing for $0.5 \leqslant \theta \leqslant 0.6$.

Proof. This theorem follows easily from the Bombieri–Vinogradov Theorem and [[2], Theorem 1(i)(iii)]. □

We shall use two different methods to construct $\rho_0(n)$. The first method comes from Harman's sieve (see [2]), while the second method comes from the idea of Mikawa [32].

2.5.1. *First Method.* The first method is to use Harman's sieve as in [2]. Unlike the upper bound case, in this case we can only discard positive terms that do not have asymptotic formulas. Now we focus on the range $\theta \in (\frac{1}{2}, \frac{17}{32})$ and split it into two subranges.

2.5.1.1. *Case 1.* $\frac{1}{2} < \theta < \frac{53}{105}$. Just as in Subsection 2.4.1, our Type-II range in this case is still (11), and we can also replace $\boldsymbol{U}_j$ with $\boldsymbol{S}_j$ in many places. By Buchstab's identity, we have

$$\begin{aligned}
S\left(\mathcal{A}^q, (2x)^{\frac{1}{2}}\right) &= S\left(\mathcal{A}^q, x^{\kappa}\right) - \sum_{\kappa \leqslant \alpha_1 < \frac{1}{2}} S\left(\mathcal{A}^q_{p_1}, p_1\right) \\
&= S\left(\mathcal{A}^q, x^{\kappa}\right) - \sum_{\kappa \leqslant \alpha_1 \leqslant \frac{3}{7}} S\left(\mathcal{A}^q_{p_1}, p_1\right) - \sum_{\frac{3}{7} < \alpha_1 < 1-\theta} S\left(\mathcal{A}^q_{p_1}, p_1\right) - \sum_{1-\theta \leqslant \alpha_1 < \frac{1}{2}} S\left(\mathcal{A}^q_{p_1}, p_1\right) \\
&= S\left(\mathcal{A}^q, x^{\kappa}\right) - \sum_{\kappa \leqslant \alpha_1 \leqslant \frac{3}{7}} S\left(\mathcal{A}^q_{p_1}, x^{\kappa}\right) - \sum_{\frac{3}{7} < \alpha_1 < 1-\theta} S\left(\mathcal{A}^q_{p_1}, p_1\right) - \sum_{1-\theta \leqslant \alpha_1 < \frac{1}{2}} S\left(\mathcal{A}^q_{p_1}, p_1\right) \\
&\quad + \sum_{\substack{\kappa \leqslant \alpha_1 \leqslant \frac{3}{7} \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)}} S\left(\mathcal{A}^q_{p_1 p_2}, p_2\right) \\
&= S^q_{221} - S^q_{222} - S^q_{223} - S^q_{224} + S^q_{225}.
\end{aligned} \tag{19}$$

We can give asymptotic formulas for S_{221}^{q}–S_{223}^{q} by Lemma 2.11 and Lemma 2.16. We use Lemma 2.19 to give a lower bound for $-S_{224}^{q}$ with a two-dimensional loss. For S_{225}^{q} we split it into three subsums:

$$\begin{aligned} S_{225}^{q} =& \sum_{\substack{\kappa \leqslant \alpha_1 \leqslant \frac{3}{7} \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)}} S\left(\mathcal{A}_{p_1 p_2}^{q}, p_2\right) \\ =& \sum_{\substack{\kappa \leqslant \alpha_1 \leqslant \frac{3}{7} \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in \boldsymbol{G}_2}} S\left(\mathcal{A}_{p_1 p_2}^{q}, p_2\right) + \sum_{\substack{\kappa \leqslant \alpha_1 \leqslant \frac{3}{7} \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in A \cup B}} S\left(\mathcal{A}_{p_1 p_2}^{q}, p_2\right) + \sum_{\substack{\kappa \leqslant \alpha_1 \leqslant \frac{3}{7} \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in C}} S\left(\mathcal{A}_{p_1 p_2}^{q}, p_2\right) \\ =& S_{2251}^{q} + S_{2252}^{q} + S_{2253}^{q}. \end{aligned} \tag{20}$$

For S_{2251}^{q} we can use Lemma 2.10 or Lemma 2.16 to give an asymptotic formula. We discard the whole of S_{2253}^{q}, leading to a two-dimensional loss. For the remaining S_{2252}^{q}, we can perform further straightforward decompositions if $(\alpha_1, \alpha_2, \alpha_2) \in \boldsymbol{S}_3$. Note that the conditions and details of further decompositions are similar to the upper bound case. Also, role-reversals can be applied if $\boldsymbol{\alpha}_2 \in \boldsymbol{S}_2$ and $(1-\alpha_1-\alpha_2, \alpha_2) \in \boldsymbol{S}_2$. In this way we obtain the following bounds of $C_0(\theta)$:

$\boldsymbol{\theta}$	$\boldsymbol{C_0(\theta)}$
0.501	0.8654
0.502	0.8456
0.503	0.8303
0.504	0.8129

Table 2.2: Lower Bounds for $\boldsymbol{C_0(\theta)}$ (First Method, $\boldsymbol{0.5 < \theta < \frac{53}{105}}$)

2.5.1.2. *Case 2.* $\frac{53}{105} \leqslant \theta < \frac{17}{32}$. The decompositions in this case are similar to the first case, and the only difference is to replace $\boldsymbol{S}_j$ with $\boldsymbol{U}_j$ in various sums. Since we do not need to consider the estimate of $S\left(\mathcal{A}_{p_1 p_2}^{q}, x^{\kappa}\right)$ as in the upper bound case, we can remove the condition $\alpha_1 < \tau$ in the definition of $\boldsymbol{U}_j$. Role-reversals can still be applied, but we need to add two extra $2\theta - 1$ in the conditions (note that $\boldsymbol{\alpha}_2 \in \boldsymbol{U}_2$ implies $(\alpha_1, \alpha_2, 2\theta-1) \in \boldsymbol{S}_3$ and similarly for $(1-\alpha_1-\alpha_2, \alpha_2)$). Working like the above case we get

$\boldsymbol{\theta}$	$\boldsymbol{C_0(\theta)}$	$\boldsymbol{\theta}$	$\boldsymbol{C_0(\theta)}$
0.505	0.7935	0.518	0.4624
0.506	0.7730	0.519	0.4261
0.507	0.7505	0.520	0.3890
0.508	0.7334	0.521	0.3510
0.509	0.7045	0.522	0.3113
0.510	0.6808	0.523	0.2714
0.511	0.6551	0.524	0.2277
0.512	0.6373	0.525	0.1785
0.513	0.6033	0.526	0.1324
0.514	0.5796	0.527	0.0779
0.515	0.5496	0.528	0.0256
0.516	0.5245	0.529	-0.04
0.517	0.4936	0.530	-0.10

Table 2.3: Lower Bounds for $\boldsymbol{C_0(\theta)}$ (First Method, $\boldsymbol{\frac{53}{105} \leqslant \theta < \frac{17}{32}}$)

Note that the lower bound becomes trivial when $\theta \geqslant 0.529$.

2.5.2. *Second Method.* The second method is to use a modified Eratosthenes sieve (or a modified Harman's sieve) developed by Mikawa [32]. The main idea of Mikawa [32] is to introduce a generalized sieve function that involves a sum over Möbius function $\mu(d)$, "split" this sieve function in two different ways to get two sums

$$\sum_{p \sim x} 1 \quad \text{and} \quad \sum_{p \in \mathcal{A}^{q}} 1$$

together with "other sums" that count almost primes, and use a combinatorial way to measure the contribution from "other sums". Before the decomposition, we need the following 3 combinatorial lemmas. The first 2 lemmas focus on the case $\Omega(n) \neq 5$:

Lemma 2.24. *([32], Lemma 4]). For square-free n, we have*

$$\sum_{\substack{d \mid n \\ d < \sqrt{n}}} \mu(d) = \begin{cases} 0, & \mu(n) = 1, \\ 1, & n = p, \\ 0, & p \mid n \text{ with } \sqrt{n} < p < n, \\ -2, & n = p_1 p_2 p_3 \text{ with } p_3 < p_2 < p_1 < \sqrt{n}, \\ -20, & \Omega(n) = 7 \text{ with } P^{-}(n) > n^{\frac{1}{8}}. \end{cases}$$

Proof. Mikawa did not give a proof of this lemma in [32], so we provide a proof here for a reference. Let $n = p_1 p_2 \cdots p_k$ where $p_1 > p_2 > \ldots > p_k$, and write $\mathscr{K} = \{1, 2, \ldots, k\}$.

If $\mu(n)=1$, then k is even. For any $\mathscr{I}\subseteq\mathscr{K}$, we know that $|\mathscr{I}|$ and $\left|\mathscr{I}^{\complement}\right|$ must have the same parity. Thus, we have

$$\mu\left(\prod_{i\in\mathscr{I}}p_i\right)=\mu\left(\prod_{i\in\mathscr{I}^{\complement}}p_i\right).$$

Let $\mathscr{I}$ be a set that satisfies

$$\prod_{i\in\mathscr{I}}p_i<\sqrt{n},$$

then we must have

$$\prod_{i\in\mathscr{I}^{\complement}}p_i>\sqrt{n}.$$

Now, we have

$$\sum_{\substack{d\mid n\\ d<\sqrt{n}}}\mu(d)=\sum_{\mathscr{I}\subseteq\mathscr{K}}\mu\left(\prod_{i\in\mathscr{I}}p_i\right)=\sum_{\mathscr{I}\subseteq\mathscr{K}}\mu\left(\prod_{i\in\mathscr{I}^{\complement}}p_i\right)=\sum_{\substack{d\mid n\\ d>\sqrt{n}}}\mu(d)$$

and

$$\sum_{\mathscr{I}\subseteq\mathscr{K}}\mu\left(\prod_{i\in\mathscr{I}}p_i\right)+\sum_{\mathscr{I}\subseteq\mathscr{K}}\mu\left(\prod_{i\in\mathscr{I}^{\complement}}p_i\right)=\sum_{\substack{d\mid n\\ d<\sqrt{n}}}\mu(d)+\sum_{\substack{d\mid n\\ d>\sqrt{n}}}\mu(d)=\sum_{d\mid n}\mu(d)=0.$$

Of course these imply

$$\sum_{\substack{d\mid n\\ d<\sqrt{n}}}\mu(d)=\sum_{\substack{d\mid n\\ d>\sqrt{n}}}\mu(d)=0.$$

If $n=p_1$, then the only d that satisfies the condition $d<\sqrt{n}=\sqrt{p_1}$ is $d=1$. Hence

$$\sum_{\substack{d\mid n\\ d<\sqrt{n}}}\mu(d)=\mu(1)=1.$$

If we have $p_1\mid n$ with $\sqrt{n}<p_1<n$, we must need $p_1\nmid d$ in the sum to ensure $d<\sqrt{n}$. Since $p_2p_3\cdots p_k=\frac{n}{p_1}<\sqrt{n}$, any possible choices of prime factors of d among $p_2,p_3,\cdots,p_k$ are acceptable. Write $\mathscr{K}'=\{2,\dots,k\}$ and $\mathscr{I}'\subseteq\mathscr{K}'$, we know that

$$\sum_{\substack{d\mid n\\ d<\sqrt{n}}}\mu(d)=\sum_{\mathscr{I}'\subseteq\mathscr{K}'}\mu\left(\prod_{i\in\mathscr{I}'}p_i\right)=\sum_{d\mid p_2p_3\cdots p_k}\mu(d)=0.$$

If $n=p_1p_2p_3$ with $p_3<p_2<p_1<\sqrt{n}$, we have $p_1p_2>p_1p_3>p_2p_3=\frac{n}{p_1}>\sqrt{n}$. Thus, the only possible choices of d are 1, p_1, p_2 and p_3. Now,

$$\sum_{\substack{d\mid n\\ d<\sqrt{n}}}\mu(d)=\mu(1)+\mu(p_1)+\mu(p_2)+\mu(p_3)=1-3=-2.$$

If $\Omega(n)=7$ with $P^-(n)>n^{\frac{1}{8}}$, we know that $p_4p_5p_6p_7>\left(n^{\frac{1}{8}}\right)^4=n^{\frac{1}{2}}$ and $p_1p_2p_3=\frac{n}{p_4p_5p_6p_7}<n^{\frac{1}{2}}$. Since we have $p_1>p_2>\ldots>p_k$, we know that $d<\sqrt{n}$ implies $\Omega(d)<3$. Now,

$$\sum_{\substack{d\mid n\\ d<\sqrt{n}}}\mu(d)=\sum_{0\leqslant i\leqslant 3}(-1)^i\binom{7}{i}=-20.$$

Combining the above 5 cases, the proof of Lemma 2.24 is completed. $\square$

Lemma 2.25. *([32], Lemma 6]). For square-free n, we have*

$$\sum_{\substack{d\mid n\\ \mu^2(d)=1\\ \Omega(d)=3\\ \sqrt{n}<d<\sqrt{nP^-(d)}}}2=\begin{cases}2, & n=p_1p_2p_3p_4(p_4<p_3<p_2<p_1)\ \text{with}\ p_1<p_2p_3p_4\ \text{and}\ p_2p_3<p_1,\\ 0, & \Omega(n)\leqslant 3,\\ 0, & n=p_1p_2p_3p_4(p_4<p_3<p_2<p_1)\ \text{with}\ p_1>p_2p_3p_4\ \text{or}\ p_2p_3>p_1,\\ \leqslant 20, & \Omega(n)=6,\\ 0, & \Omega(n)=7\ \text{with}\ P^-(n)>n^{\frac{1}{8}}.\end{cases}$$

Remark. The conditions $p_1<p_2p_3p_4$ and $p_2p_3<p_1$ correspond to the condition $\sqrt{n}<d<\sqrt{nP^-(d)}$ with $d=p_2p_3p_4$, since

$$\sqrt{p_1p_2p_3p_4}<p_2p_3p_4\Longrightarrow p_1p_2p_3p_4<p_2^2p_3^2p_4^2\Longrightarrow p_1<p_2p_3p_4$$

and

$$p_2p_3p_4<\sqrt{p_1p_2p_3p_4^2}\Longrightarrow p_2^2p_3^2<p_1p_2p_3\Longrightarrow p_2p_3<p_1.$$

For the case $\Omega(n) = 6$, we can improve the upper bound 20 when $\boldsymbol{\alpha}_6$ lies in some special regions. We shall explain this improvement in the end of this subsubsection.

For the case $\Omega(n) = 5$, we use the following third lemma instead:

Lemma 2.26. *([32], Lemma 5]). For $n = p_1p_2p_3p_4p_5$ with $p_5 < p_4 < p_3 < p_2 < p_1$, we have*

$$0 \leqslant \sum_{\substack{d \mid n \\ \mu^2(d)=1 \\ \Omega(d)=3 \\ \sqrt{n}<d<\sqrt{nP^-(d)}}} 2 - \sum_{\substack{d \mid n \\ d<\sqrt{n}}} \mu(d) \leqslant \begin{cases} 2, & p_2p_3 < p_1p_5, \\ 0, & \text{otherwise.} \end{cases}$$

Now we start our final decomposition. Let $0.5 < \theta < \frac{17}{32}$. We begin with the generalized sieve functions

$$S_{23A} = \sum_{n \in \mathcal{A}} \left(\sum_{\substack{d \mid n \\ d<\sqrt{x}}} \mu(d) \right) \psi\left(n, x^{\kappa}\right) \quad \text{and} \quad S_{23B} = \sum_{n \in \mathcal{B}} \left(\sum_{\substack{d \mid n \\ d<\sqrt{x}}} \mu(d) \right) \psi\left(n, x^{\kappa}\right). \tag{21}$$

For S_{23A}, we restrict n to square-free integers and replace the condition $d < \sqrt{x}$ with $d < \sqrt{n}$, leading to a cost error term of

$$\sum_{n \in \mathcal{A}} \left(\sum_{\substack{p^2 \mid n \\ p \geqslant x^{\kappa}}} 1 + \sum_{\substack{d \mid n \\ d^2 \sim x}} 1 \right). \tag{22}$$

The first term can be bounded by

$$\sum_{n \in \mathcal{A}} \sum_{\substack{p^2 \mid n \\ p \geqslant x^{\kappa}}} 1 \ll \sum_{n \sim x} \sum_{\substack{p^2 \mid n \\ p \geqslant x^{\kappa}}} d(n-a) \ll x^{\varepsilon} \sum_{x^{\kappa} \leqslant p \leqslant \sqrt{x}} \left(\frac{x}{p^2} + 1 \right) \ll x^{\frac{7}{8}+\varepsilon} \tag{23}$$

since $\theta < \frac{17}{32}$. By the method of Hooley [[20], Page 18] and bounds for incomplete Kloosterman sums, we get

$$\sum_{n \in \mathcal{A}} \sum_{\substack{d \mid n \\ d^2 \sim x}} 1 \ll \sum_{d^2 \sim x} \frac{x}{qd} + q^{\frac{1}{2}+\varepsilon} + \frac{x^{1-\varepsilon}}{q} \ll \frac{x}{q (\log x)^4}. \tag{24}$$

Combining (21)–(24) and by Lemma 2.24, we have

$$\begin{aligned}
S_{23A} =& \sum_{n \in \mathcal{A}} \left(\sum_{\substack{d \mid n \\ d<\sqrt{x}}} \mu(d) \right) \psi\left(n, x^{\kappa}\right) \\
=& \sum_{n \in \mathcal{A}} \mu^2(n) \left(\sum_{\substack{d \mid n \\ d<\sqrt{n}}} \mu(d) \right) \psi\left(n, x^{\kappa}\right) + O\left(\frac{x}{q (\log x)^4} \right) \\
=& \sum_{p \in \mathcal{A}} 1 - \sum_{\substack{p_1p_2p_3 \in \mathcal{A} \\ x^{\kappa}<p_3<p_2<p_1<p_2p_3}} 2 - \sum_{\substack{p_1 \cdots p_7 \in \mathcal{A} \\ x^{\kappa}<p_7<\cdots<p_1}} 20 + \sum_{\substack{p_1 \cdots p_5 \in \mathcal{A} \\ x^{\kappa}<p_5<\cdots<p_1}} \mu^2(n) \left(\sum_{\substack{d \mid n \\ d<\sqrt{n}}} \mu(d) \right) \psi\left(n, x^{\kappa}\right) + O\left(\frac{x}{q (\log x)^4} \right) \\
=& \sum_{p \in \mathcal{A}} 1 - S_{231} - S_{232} + \sum_{\substack{n=p_1 \cdots p_5 \in \mathcal{A} \\ x^{\kappa}<p_5<\cdots<p_1}} \mu^2(n) \left(\sum_{\substack{d \mid n \\ d<\sqrt{n}}} \mu(d) \right) \psi\left(n, x^{\kappa}\right) + O\left(\frac{x}{q (\log x)^4} \right).
\end{aligned} \tag{25}$$

Similarly, we can decompose S_{23B} as

$$\begin{aligned}
S_{23B} =& \sum_{n \in \mathcal{B}} \left(\sum_{\substack{d \mid n \\ d<\sqrt{x}}} \mu(d) \right) \psi\left(n, x^{\kappa}\right) \\
=& \sum_{n \sim x} \mu^2(n) \left(\sum_{\substack{d \mid n \\ d<\sqrt{n}}} \mu(d) \right) \psi\left(n, x^{\kappa}\right) + O\left(\frac{x}{(\log x)^4} \right)
\end{aligned}$$

$$= \sum_{p\sim x} 1 - \sum_{\substack{p_1p_2p_3\sim x\\ x^\kappa<p_3<p_2<p_1<p_2p_3}} 2 - \sum_{\substack{p_1\cdots p_7\sim x\\ x^\kappa<p_7<\cdots<p_1}} 20 + \sum_{\substack{p_1\cdots p_5\sim x\\ x^\kappa<p_5<\cdots<p_1}} \mu^2(n)\left(\sum_{\substack{d\mid n\\ d<\sqrt{n}}}\mu(d)\right)\psi\left(n,x^\kappa\right) + O\left(\frac{x}{(\log x)^4}\right)$$

$$= \sum_{p\sim x} 1 - S_{233} - S_{234} + \sum_{\substack{n=p_1\cdots p_5\sim x\\ x^\kappa<p_5<\cdots<p_1}} \mu^2(n)\left(\sum_{\substack{d\mid n\\ d<\sqrt{n}}}\mu(d)\right)\psi\left(n,x^\kappa\right) + O\left(\frac{x}{(\log x)^4}\right). \tag{26}$$

Now, if we have

$$\sum_{n\in\mathcal{A}}\left(\sum_{\substack{d\mid n\\ d<\sqrt{x}}}\mu(d)\right)\psi\left(n,x^\kappa\right) = \frac{1}{\varphi(q)}\sum_{n\in\mathcal{B}}\left(\sum_{\substack{d\mid n\\ d<\sqrt{x}}}\mu(d)\right)\psi\left(n,x^\kappa\right) + O\left(\frac{x}{q(\log x)^3}\right), \tag{27}$$

we can put (25) and (26) together. For the sum

$$\sum_{\substack{n=p_1\cdots p_5\in\mathcal{A}\\ x^\kappa<p_5<\cdots<p_1}} \mu^2(n)\left(\sum_{\substack{d\mid n\\ d<\sqrt{n}}}\mu(d)\right)\psi\left(n,x^\kappa\right),$$

we can use Lemma 2.26 to get

$$\sum_{\substack{n=p_1\cdots p_5\in\mathcal{A}\\ x^\kappa<p_5<\cdots<p_1}} \mu^2(n)\left(\sum_{\substack{d\mid n\\ d<\sqrt{n}}}\mu(d)\right)\psi\left(n,x^\kappa\right) \leqslant \sum_{\substack{n=p_1\cdots p_5\in\mathcal{A}\\ x^\kappa<p_5<\cdots<p_1}} \mu^2(n)\left(\sum_{\substack{d\mid n\\ \mu^2(d)=1\\ \Omega(d)=3\\ \sqrt{n}<d<\sqrt{nP^-(d)}}} 2\right)\psi\left(n,x^\kappa\right). \tag{28}$$

Similar to (22)–(24), we have

$$\begin{aligned}
&\sum_{\substack{n=p_1\cdots p_5\in\mathcal{A}\\ x^\kappa<p_5<\cdots<p_1}} \mu^2(n)\left(\sum_{\substack{d\mid n\\ d<\sqrt{n}}}\mu(d)\right)\psi\left(n,x^\kappa\right)\\
\leqslant &\sum_{\substack{n=p_1\cdots p_5\in\mathcal{A}\\ x^\kappa<p_5<\cdots<p_1}} \mu^2(n)\left(\sum_{\substack{d\mid n\\ \mu^2(d)=1\\ \Omega(d)=3\\ \sqrt{n}<d<\sqrt{nP^-(d)}}} 2\right)\psi\left(n,x^\kappa\right)\\
\leqslant &\sum_{n\in\mathcal{A}}\left(\sum_{\substack{d\mid n\\ \mu^2(d)=1\\ \Omega(d)=3\\ \sqrt{x}<d<\sqrt{xP^-(d)}}} 2\right)\psi\left(n,x^\kappa\right) + O\left(\sum_{n\in\mathcal{A}}\sum_{\substack{d\mid n\\ d^2\sim x}} 1\right)\\
= &\sum_{n\in\mathcal{A}}\left(\sum_{\substack{d\mid n\\ \mu^2(d)=1\\ \Omega(d)=3\\ \sqrt{x}<d<\sqrt{xP^-(d)}}} 2\right)\psi\left(n,x^\kappa\right) + O\left(\frac{x}{q(\log x)^4}\right).
\end{aligned} \tag{29}$$

Now we consider the above main term with condition $p_1 \cdots p_5 \in \mathcal{A}$ replaced by $p_1 \cdots p_5 \in \mathcal{B}$. Revisiting the above process (29) but with opposite direction and by Lemma 2.25, we get

$$
\begin{aligned}
&\sum_{n \in \mathcal{B}} \left(\sum_{\substack{d \mid n \\ \mu^2(d)=1 \\ \Omega(d)=3 \\ \sqrt{x}<d<\sqrt{xP^-(d)}}} 2 \right) \psi\left(n, x^{\kappa}\right) \\
&= \sum_{n \in \mathcal{B}} \left(\sum_{\substack{d \mid n \\ \mu^2(d)=1 \\ \Omega(d)=3 \\ \sqrt{n}<d<\sqrt{nP^-(d)}}} 2 \right) \psi\left(n, x^{\kappa}\right) + O\left(\frac{x}{(\log x)^3}\right) \\
&= \sum_{n \sim x} \mu^2(n) \left(\sum_{\substack{d \mid n \\ \mu^2(d)=1 \\ \Omega(d)=3 \\ \sqrt{n}<d<\sqrt{nP^-(d)}}} 2 \right) \psi\left(n, x^{\kappa}\right) + O\left(\frac{x}{(\log x)^3}\right) \\
&\leqslant \sum_{\substack{p_1 p_2 p_3 p_4 \sim x \\ x^{\kappa}<p_4<p_3<p_2<p_1 \\ p_1<p_2 p_3 p_4 \\ p_2 p_3<p_1}} 2 + \sum_{\substack{p_1 \cdots p_5 \sim x \\ x^{\kappa}<p_5<p_4<p_3<p_2<p_1 \\ p_2 p_3<p_1 p_5}} 2 + \sum_{\substack{p_1 \cdots p_6 \sim x \\ x^{\kappa}<p_6<p_5<p_4<p_3<p_2<p_1}} 20 \\
&\quad + \sum_{\substack{p_1 \cdots p_5 \sim x \\ x^{\kappa}<p_5<\cdots<p_1}} \mu^2(n) \left(\sum_{\substack{d \mid n \\ d<\sqrt{n}}} \mu(d) \right) \psi\left(n, x^{\kappa}\right) + O\left(\frac{x}{(\log x)^3}\right) \\
&= S_{235} + S_{236} + S_{237} + \sum_{\substack{n=p_1 \cdots p_5 \sim x \\ x^{\kappa}<p_5<\cdots<p_1}} \mu^2(n) \left(\sum_{\substack{d \mid n \\ d<\sqrt{n}}} \mu(d) \right) \psi\left(n, x^{\kappa}\right) + O\left(\frac{x}{(\log x)^3}\right). \qquad (30)
\end{aligned}
$$

Now we suppose that (27) holds true. Suppose also that we have

$$
\sum_{n \in \mathcal{A}} \left(\sum_{\substack{d \mid n \\ \mu^2(d)=1 \\ \Omega(d)=3 \\ \sqrt{x}<d<\sqrt{xP^-(d)}}} 2 \right) \psi\left(n, x^{\kappa}\right) = \frac{1}{\varphi(q)} \sum_{n \in \mathcal{B}} \left(\sum_{\substack{d \mid n \\ \mu^2(d)=1 \\ \Omega(d)=3 \\ \sqrt{x}<d<\sqrt{xP^-(d)}}} 2 \right) \psi\left(n, x^{\kappa}\right) + O\left(\frac{x}{q(\log x)^3}\right). \qquad (31)
$$

By (29)–(31), we get

$$
\begin{aligned}
&\sum_{\substack{n=p_1 \cdots p_5 \in \mathcal{A} \\ x^{\kappa}<p_5<\cdots<p_1}} \mu^2(n) \left(\sum_{\substack{d \mid n \\ d<\sqrt{n}}} \mu(d) \right) \psi\left(n, x^{\kappa}\right) \\
&\leqslant \frac{1}{\varphi(q)} \sum_{\substack{n=p_1 \cdots p_5 \sim x \\ x^{\kappa}<p_5<\cdots<p_1}} \mu^2(n) \left(\sum_{\substack{d \mid n \\ d<\sqrt{n}}} \mu(d) \right) \psi\left(n, x^{\kappa}\right) + \frac{1}{\varphi(q)} \left(S_{235} + S_{236} + S_{237}\right) + O\left(\frac{x}{q(\log x)^3}\right) \qquad (32)
\end{aligned}
$$

and thus

$$\sum_{p\in\mathcal{A}} 1 - S_{231} - S_{232} + \sum_{\substack{n=p_1\cdots p_5\in\mathcal{A}\\ x^\kappa<p_5<\cdots<p_1}} \mu^2(n)\left(\sum_{\substack{d\mid n\\ d<\sqrt{n}}}\mu(d)\right)\psi\left(n,x^\kappa\right)$$

$$\leqslant \frac{1}{\varphi(q)}\sum_{\substack{n=p_1\cdots p_5\sim x\\ x^\kappa<p_5<\cdots<p_1}} \mu^2(n)\left(\sum_{\substack{d\mid n\\ d<\sqrt{n}}}\mu(d)\right)\psi\left(n,x^\kappa\right)$$

$$+\sum_{p\in\mathcal{A}} 1 - S_{231} - S_{232} + \frac{1}{\varphi(q)}\left(S_{235}+S_{236}+S_{237}\right) + O\left(\frac{x}{q(\log x)^3}\right). \tag{33}$$

By (25)–(27), we also have

$$\sum_{p\in\mathcal{A}} 1 - S_{231} - S_{232} + \sum_{\substack{n=p_1\cdots p_5\in\mathcal{A}\\ x^\kappa<p_5<\cdots<p_1}} \mu^2(n)\left(\sum_{\substack{d\mid n\\ d<\sqrt{n}}}\mu(d)\right)\psi\left(n,x^\kappa\right)$$

$$= \frac{1}{\varphi(q)}\sum_{\substack{n=p_1\cdots p_5\sim x\\ x^\kappa<p_5<\cdots<p_1}} \mu^2(n)\left(\sum_{\substack{d\mid n\\ d<\sqrt{n}}}\mu(d)\right)\psi\left(n,x^\kappa\right) + \frac{1}{\varphi(q)}\left(\sum_{p\sim x} 1 - S_{233} - S_{234}\right) + O\left(\frac{x}{q(\log x)^3}\right). \tag{34}$$

Combining (33) and (34) we get

$$\sum_{p\in\mathcal{A}} 1 \geqslant S_{231} + S_{232} + \frac{1}{\varphi(q)}\left(\sum_{p\sim x} 1 - S_{233} - S_{234} - S_{235} - S_{236} - S_{237}\right) + O\left(\frac{x}{q(\log x)^3}\right)$$

$$= \frac{1}{\varphi(q)}\sum_{p\sim x} 1 + \left(S_{231} - \frac{1}{\varphi(q)}S_{233}\right) + \left(S_{232} - \frac{1}{\varphi(q)}S_{234}\right) - \frac{1}{\varphi(q)}\left(S_{235}+S_{236}+S_{237}\right) + O\left(\frac{x}{q(\log x)^3}\right). \tag{35}$$

Note that by Prime Number Theorem and partial summation, we can calculate the loss from S_{235}, S_{236} and S_{237}:

$$S_{235} = 2(1+o(1))\frac{x}{\log x}\left(\int_{(t_1,t_2,t_3)\in U_{235}} \frac{1}{t_1t_2t_3(1-t_1-t_2-t_3)}dt_3dt_2dt_1\right), \tag{36}$$

$$S_{236} = 2(1+o(1))\frac{x}{\log x}\left(\int_{(t_1,t_2,t_3,t_4)\in U_{236}} \frac{1}{t_1t_2t_3t_4(1-t_1-t_2-t_3-t_4)}dt_4dt_3dt_2dt_1\right), \tag{37}$$

$$S_{237} = 20(1+o(1))\frac{x}{\log x}\left(\int_{(t_1,t_2,t_3,t_4,t_5)\in U_{237}} \frac{1}{t_1t_2t_3t_4t_5(1-t_1-t_2-t_3-t_4-t_5)}dt_5dt_4dt_3dt_2dt_1\right), \tag{38}$$

where

$$U_{235}(\boldsymbol{\alpha}_3) := \left\{\kappa<\alpha_3<\alpha_2<\alpha_1,\ \alpha_1+\alpha_2+\alpha_3>\frac{1}{2},\ 2\alpha_1+2\alpha_2+\alpha_3<1\right\},$$
$$U_{236}(\boldsymbol{\alpha}_4) := \left\{\kappa<\alpha_4<\alpha_3<\alpha_2<\alpha_1,\ 2\alpha_1+2\alpha_2+\alpha_3<1\right\},$$
$$U_{237}(\boldsymbol{\alpha}_5) := \left\{\kappa<\alpha_5<\alpha_4<\alpha_3<\alpha_2<\alpha_1,\ 2\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5<1\right\}.$$

Now we only need to give lower bounds for

$$\sum_{\substack{p_1p_2p_3\in\mathcal{A}\\ x^\kappa<p_3<p_2<p_1<p_2p_3}} 2 - \frac{1}{\varphi(q)}\sum_{\substack{p_1p_2p_3\sim x\\ x^\kappa<p_3<p_2<p_1<p_2p_3}} 2 \tag{39}$$

and

$$\sum_{\substack{p_1\cdots p_7\in\mathcal{A}\\ x^\kappa<p_7<\cdots<p_1}} 20 - \frac{1}{\varphi(q)}\sum_{\substack{p_1\cdots p_7\sim x\\ x^\kappa<p_7<\cdots<p_1}} 20. \tag{40}$$

Here, Lemmas 2.4–2.7 and 2.16 (see the proof of [[28], Lemma 8.12]) are applicable for parts of (39) and (40). We discard the remaining parts of S_{231} and S_{232} since they are positive. Note that when $\theta<\frac{29}{56}$, we have $\kappa\geqslant\frac{1}{7}$ and (40) equals zero. The loss from (39) and (40) is

$$2(1+o(1))\frac{x}{\log x}\left(\int_{(t_1,t_2)\in U_{233}} \frac{1}{t_1t_2(1-t_1-t_2)}dt_2dt_1\right) \tag{41}$$

$$+\ 20(1+o(1))\frac{x}{\log x}\left(\int_{(t_1,t_2,t_3,t_4,t_5,t_6)\in U_{234}} \frac{1}{t_1t_2t_3t_4t_5t_6(1-t_1-t_2-t_3-t_4-t_5-t_6)}dt_6dt_5dt_4dt_3dt_2dt_1\right), \tag{42}$$

where

$$U_{233}(\boldsymbol{\alpha}_2) := \left\{\kappa < \alpha_2 < \alpha_1,\ \alpha_1 + \alpha_2 > \frac{1}{2},\ 2\alpha_1 + \alpha_2 < 1,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2\right\},$$
$$U_{234}(\boldsymbol{\alpha}_6) := \{\kappa < \alpha_6 < \alpha_5 < \alpha_4 < \alpha_3 < \alpha_2 < \alpha_1,\ 2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 < 1,\ \boldsymbol{\alpha}_6 \notin \boldsymbol{G}_6\}.$$

Finally, we can prove lower bounds for $C_0(\theta)$ in the range $\theta \in (\frac{1}{2}, \frac{17}{32})$ by subtracting the values of the 5 integrals in (36)–(38) and (41)–(42) from 1, under our assumptions (27) and (31). The proof of (27) and (31) for the case $\theta = \frac{17}{32} - \varepsilon$ was given in [[32], Propositions 1 and 2]. In the proof,

$$\sum_{n \in \mathcal{A}} \left(\sum_{\substack{d \mid n \\ d < \sqrt{x}}} \mu(d) \right) \psi\left(n, x^{\kappa}\right) \quad \text{and} \quad \sum_{n \in \mathcal{A}} \left(\sum_{\substack{d \mid n \\ \mu^2(d)=1 \\ \Omega(d)=3 \\ \sqrt{x} < d < \sqrt{x P^{-}(d)}}} 2 \right) \psi\left(n, x^{\kappa}\right) \tag{43}$$

was decomposed into Type-I and Type-II sums, where the coefficients are convolutions of the form, say,

$$\psi * \psi * \psi * \Psi * \Psi$$

and similar convolutions. After that, Lemma 2.8 and Lemma 2.2 are applied to estimate the Type-I case and the Type-II case. Note that Lemma 2.8 does not require any non-fixed coefficient to satisfy **Condition A** (the Siegel–Walfisz condition). This fact is important in this part of the next section.

For $\theta < \frac{17}{32} - \varepsilon$, similar decompositions of (21) are still valid (the readers can follow the proof in [32] and check all the conditions we need, especially some upper bounds). After decompositions we get

θ	$C_0(\theta)$	θ	$C_0(\theta)$
0.501	0.8025	0.517	0.6033
0.502	0.8003	0.518	0.5851
0.503	0.7981	0.519	0.5689
0.504	0.7859	0.520	0.5487
0.505	0.7699	0.521	0.5323
0.506	0.7579	0.522	0.5139
0.507	0.7439	0.523	0.4919
0.508	0.7339	0.524	0.4699
0.509	0.7199	0.525	0.4499
0.510	0.7079	0.526	0.4299
0.511	0.6919	0.527	0.4079
0.512	0.6838	0.528	0.3839
0.513	0.6658	0.529	0.3579
0.514	0.6517	0.530	0.3339
0.515	0.6355	0.531	0.3099
0.516	0.6215		

Table 2.4: Lower Bounds for $C_0(\theta)$ (Second Method, $\frac{53}{105} \leqslant \theta < \frac{17}{32}$)

Note that the lower bound does not become trivial even when $\theta = \frac{17}{32} - \varepsilon$ using this method. However, this method collapses when $\theta \geqslant \frac{17}{32}$ and we cannot get any nontrivial result using this method. There is no "grey area" between a lower bound > 0.3 and no result at all.

θ	$C_0(\theta)$ (Method 1)	$C_0(\theta)$ (Method 2)	θ	$C_0(\theta)$ (Method 1)	$C_0(\theta)$ (Method 2)
0.501	**0.8654**	0.8025	0.517	0.4936	**0.6033**
0.502	**0.8456**	0.8003	0.518	0.4624	**0.5851**
0.503	**0.8303**	0.7981	0.519	0.4261	**0.5689**
0.504	**0.8129**	0.7859	0.520	0.3890	**0.5487**
0.505	**0.7935**	0.7699	0.521	0.3510	**0.5323**
0.506	**0.7730**	0.7579	0.522	0.3113	**0.5139**
0.507	**0.7505**	0.7439	0.523	0.2714	**0.4919**
0.508	0.7334	**0.7339**	0.524	0.2277	**0.4699**
0.509	0.7045	**0.7199**	0.525	0.1785	**0.4499**
0.510	0.6808	**0.7079**	0.526	0.1324	**0.4299**
0.511	0.6551	**0.6919**	0.527	0.0779	**0.4079**
0.512	0.6373	**0.6838**	0.528	0.0256	**0.3839**
0.513	0.6033	**0.6658**	0.529	0	**0.3579**
0.514	0.5796	**0.6517**	0.530	0	**0.3339**
0.515	0.5496	**0.6355**	0.531	0	**0.3099**
0.516	0.5245	**0.6215**			

Table 2.5: A Comparison of Two Methods on the Lower Bounds for $C_0(\theta)$ ($\frac{53}{105} \leqslant \theta < \frac{17}{32}$)

In the end of this subsection, we mention an improvement over Lemma 2.25 on the case $\Omega(n) = 6$. In this case, Lemma 2.25 gives that

$$\sum_{\substack{\Omega(n)=6 \\ \mu^2(n)=1 \\ d|n \\ \mu^2(d)=1 \\ \Omega(d)=3 \\ \sqrt{n}<d<\sqrt{nP^-(d)}}} 1 \leqslant 10. \tag{44}$$

We shall prove that the upper bound 10 can be reduced under some conditions on the prime factors of d. Let $n = p_1p_2p_3p_4p_5p_6$ with $p_1 > p_2 > p_3 > p_4 > p_5 > p_6$. The upper bound 10 can be easily obtained: Let $d = p_ip_jp_k$ with $1 \leqslant i < j < k \leqslant 6$, we have $\binom{6}{3} = 20$ choices for (i, j, k). Since only one of choices d and $\frac{n}{d}$ is larger than $\sqrt{n}$, the number of d counted is no more than $\frac{1}{2}\binom{6}{3} = 10$.

However, this upper bound ignores another restriction $d < \sqrt{nP^-(d)} = \sqrt{np_k}$ in the sum. Taking this condition into our consideration, we can show that some of the 10 possible combinations of d are not acceptable when $\boldsymbol{\alpha}_6$ lies in some special regions. We give a table in the end of this section to reveal one possible $\boldsymbol{\alpha}_6$ for each (i, j, k) such that $d = p_ip_jp_k$ will not be counted in the sum. Trivially, we know that the following 5 choices of (i, j, k) are impossible for d since $p_ip_jp_k < \frac{n}{p_ip_jp_k}$:

$$(2, 4, 6),\ (2, 5, 6),\ (3, 4, 6),\ (3, 5, 6),\ (4, 5, 6).$$

Thus, we only need to consider the remaining 15 choices.

(i,j,k)	$\boldsymbol{\alpha}_6$
$(1, 2, 3)$	$(0.295, 0.143, 0.142, 0.141, 0.140, 0.139)$
$(1, 2, 4)$	$(0.295, 0.143, 0.142, 0.141, 0.140, 0.139)$
$(1, 2, 5)$	$(0.295, 0.143, 0.142, 0.141, 0.140, 0.139)$
$(1, 2, 6)$	$(0.295, 0.143, 0.142, 0.141, 0.140, 0.139)$
$(1, 3, 4)$	$(0.295, 0.143, 0.142, 0.141, 0.140, 0.139)$
$(1, 3, 5)$	$(0.295, 0.143, 0.142, 0.141, 0.140, 0.139)$
$(1, 3, 6)$	$(0.295, 0.143, 0.142, 0.141, 0.140, 0.139)$
$(1, 4, 5)$	$(0.295, 0.143, 0.142, 0.141, 0.140, 0.139)$
$(1, 4, 6)$	$(0.295, 0.143, 0.142, 0.141, 0.140, 0.139)$
$(1, 5, 6)$	$(0.295, 0.143, 0.142, 0.141, 0.140, 0.139)$
$(2, 3, 4)$	Do Not Exist
$(2, 3, 5)$	Do Not Exist
$(2, 3, 6)$	Do Not Exist
$(2, 4, 5)$	Do Not Exist
$(3, 4, 5)$	Do Not Exist

Table 2.6: Examples of $d = p_ip_jp_k$ such that $d > \sqrt{nP^-(d)}$

By this table, we know that d will always be counted in the sum if $i \neq 1$. For a comparison, we also give a table that shows one possible $\boldsymbol{\alpha}_6$ for each (i, j, k) such that $d = p_ip_jp_k$ will be counted in the sum.

(i,j,k)	α_6
$(1,2,3)$	$(0.279, 0.147, 0.146, 0.145, 0.143, 0.140)$
$(1,2,4)$	$(0.280, 0.147, 0.146, 0.145, 0.143, 0.139)$
$(1,2,5)$	$(0.283, 0.146, 0.145, 0.143, 0.142, 0.141)$
$(1,2,6)$	$(0.284, 0.146, 0.145, 0.143, 0.142, 0.140)$
$(1,3,4)$	$(0.287, 0.151, 0.142, 0.141, 0.140, 0.139)$
$(1,3,5)$	$(0.287, 0.151, 0.142, 0.141, 0.140, 0.139)$
$(1,3,6)$	$(0.287, 0.151, 0.142, 0.141, 0.140, 0.139)$
$(1,4,5)$	$(0.288, 0.150, 0.142, 0.141, 0.140, 0.139)$
$(1,4,6)$	$(0.288, 0.150, 0.142, 0.141, 0.140, 0.139)$
$(1,5,6)$	$(0.289, 0.149, 0.142, 0.141, 0.140, 0.139)$
$(2,3,4)$	$(0.220, 0.217, 0.143, 0.141, 0.140, 0.139)$
$(2,3,5)$	$(0.219, 0.217, 0.144, 0.141, 0.140, 0.139)$
$(2,3,6)$	$(0.217, 0.216, 0.147, 0.141, 0.140, 0.139)$
$(2,4,5)$	$(0.214, 0.213, 0.146, 0.145, 0.143, 0.139)$
$(3,4,5)$	$(0.190, 0.170, 0.169, 0.168, 0.164, 0.139)$

Table 2.7: Examples of $d = p_i p_j p_k$ such that $\sqrt{n} < d < \sqrt{nP^-(d)}$

3. 2-factored Moduli, 1

In this section we focus on the first 2-factored case, where the moduli $q = q_1 q_2$ with $q_1 \sim Q_1 = x^{\theta_1}$ and $q_2 \sim Q_2 = x^{\theta_2}$. By the definitions of the sieved set $\mathcal{C}^q$ and the sieve function $S(\mathcal{C}, z)$, and by Prime Number Theorem, we have

$$\pi(x; q_1 q_2, a) = \sum_{p \in \mathcal{A}^{q_1 q_2}} 1 = S\left(\mathcal{A}^{q_1 q_2}, (2x)^{\frac{1}{2}}\right) \quad \text{and} \quad S\left(\mathcal{B}^{q_1 q_2}, (2x)^{\frac{1}{2}}\right) = (1+o(1)) \frac{x}{\log x}. \tag{45}$$

Our aim is again to show that the sparser set $\mathcal{A}^{q_1 q_2}$ contains the expected proportion of primes compared to the larger set $\mathcal{B}^{q_1 q_2}$, which requires us to decompose $S\left(\mathcal{A}^{q_1 q_2}, (2x)^{\frac{1}{2}}\right)$ and prove "asymptotic formulas" for almost all moduli q_1, q_2 of the form

$$S\left(\mathcal{A}^{q_1 q_2}, z\right) = (1+o(1)) \frac{1}{\varphi(q_1 q_2)} S\left(\mathcal{B}^{q_1 q_2}, z\right) \tag{46}$$

for some parts of it, and drop the remaining parts to construct a suitable majorant or minorant. For the majorant case we can only drop negative parts, while for the minorant case we can only drop positive parts. After the final decompositions, we can get the following result with some $0 < C_0(\theta_1, \theta_2) \leqslant 1$ and $C_1(\theta_1, \theta_2) \geqslant 1$:

Theorem 3.1. *There exist functions ρ_0 and ρ_1 which satisfies the following properties:*

(Majorant / Minorant). $\rho_0(n)$ is a minorant for the prime indicator function $\mathbb{1}_p(n)$, and $\rho_1(n)$ is a majorant for the prime indicator function $\mathbb{1}_p(n)$. That is, we have

$$\rho_0(n) \leqslant \mathbb{1}_p(n) \leqslant \rho_1(n).$$

(Upper and Lower bounds). We have

$$\sum_{n \leqslant x} \rho_0(n) \geqslant (1+o(1)) \frac{C_0(\theta_1, \theta_2) x}{\log x} \quad \text{and} \quad \sum_{n \leqslant x} \rho_1(n) \leqslant (1+o(1)) \frac{C_1(\theta_1, \theta_2) x}{\log x}$$

for two functions $C_0(\theta_1, \theta_2)$ and $C_1(\theta_1, \theta_2)$ satisfy $0 < C_0(\theta_1, \theta_2) \leqslant 1$ and $C_1(\theta_1, \theta_2) \geqslant 1$.

(Distributions in Arithmetic Progressions). For any $a \in \mathbb{Z} \backslash \{0\}$ and any $A > 0$, we have

$$\sum_{\substack{q_1 \sim Q_1 \\ q_2 \sim Q_2 \\ (q_1 q_2, a) = 1}} \left| \sum_{\substack{n \leqslant x \\ n \equiv a (\bmod q_1 q_2)}} \rho_j(n) - \frac{1}{\varphi(q_1 q_2)} \sum_{\substack{n \leqslant x \\ (n, q_1 q_2) = 1}} \rho_j(n) \right| \ll \frac{x}{(\log x)^A}$$

for $j = 0, 1$.

In order to give asymptotic formulas (46) for sieve functions $S\left(\mathcal{A}^{q_1 q_2}, z\right)$, we need results of the form

$$\sum_{\substack{q_1 \sim Q_1 \\ q_2 \sim Q_2 \\ (q_1 q_2, a) = 1}} \left| \sum_{\substack{n \sim x \\ n \equiv a (\bmod q_1 q_2)}} f(n) - \frac{1}{\varphi(q_1 q_2)} \sum_{\substack{n \sim x \\ (n, q_1 q_2) = 1}} f(n) \right| \ll \frac{x}{(\log x)^A}. \tag{47}$$

As in Section 2, we may want the coefficients to satisfy **Conditions A and B**.

3.1. Preliminary Lemmas.

Before constructing the majorant and minorant, we need estimate results of the form (47). Note that the results from Section 2 are still applicable in the final decomposition, and the results here are still useful in the later sections.

3.1.1. *Type-II estimate.* All of the following three lemmas come from [28].

Lemma 3.2. *([28], Proposition 12.1]). Let $M_1M_2 \asymp x$. Let a_{1,m_1} and a_{2,m_2} be divisor-bounded complex sequences. Suppose that a_{2,m_2} satisfies **Conditions A and B**. If we have*

$$Q_2x^{\varepsilon} < M_2,\ Q_1^2Q_2^3M_2^6 < x^{2-15\varepsilon},\ Q_1^3Q_2^3M_2^3 < x^{2-15\varepsilon},\ Q_1^4Q_2^3M_2^3 < x^{\frac{5}{2}-15\varepsilon},\ Q_1^2Q_2^2 < M_2x^{1-4\varepsilon},$$

then

$$\sum_{\substack{q_1\sim Q_1\\ q_2\sim Q_2\\ (q_1q_2,a)=1}}\left|\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ m_1m_2\equiv a(\bmod q_1q_2)}} a_{1,m_1}a_{2,m_2}-\frac{1}{\varphi(q_1q_2)}\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ (m_1m_2,q_1q_2)=1}} a_{1,m_1}a_{2,m_2}\right| \ll \frac{x}{(\log x)^A}.$$

Lemma 3.3. *([28], Proposition 12.2]). Let $M_1M_2 \asymp x$. Let a_{1,m_1} and a_{2,m_2} be divisor-bounded complex sequences. Suppose that a_{2,m_2} satisfies **Conditions A and B**. If we have*

$$Q_1Q_2^2 < M_2x^{1-7\varepsilon},\ Q_1^8Q_2^7M_2^6 < x^{4-13\varepsilon},$$

then

$$\sum_{\substack{q_1\sim Q_1\\ q_2\sim Q_2\\ (q_1q_2,a)=1}}\left|\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ m_1m_2\equiv a(\bmod q_1q_2)}} a_{1,m_1}a_{2,m_2}-\frac{1}{\varphi(q_1q_2)}\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ (m_1m_2,q_1q_2)=1}} a_{1,m_1}a_{2,m_2}\right| \ll \frac{x}{(\log x)^A}.$$

Lemma 3.4. *([28], Proposition 8.2]). Let $M_1M_2 \asymp x$. Let a_{1,m_1} and a_{2,m_2} be divisor-bounded complex sequences. Suppose that a_{2,m_2} satisfies **Conditions A and B**. If we have*

$$Q_1^7Q_2^{12} < x^{4-19\varepsilon},\ Q_1x^{\varepsilon} < M_2 < Q_1^{-1}x^{1-6\varepsilon},$$

then

$$\sum_{\substack{q_1\sim Q_1\\ q_2\sim Q_2\\ (q_1q_2,a)=1}}\left|\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ m_1m_2\equiv a(\bmod q_1q_2)}} a_{1,m_1}a_{2,m_2}-\frac{1}{\varphi(q_1q_2)}\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ (m_1m_2,q_1q_2)=1}} a_{1,m_1}a_{2,m_2}\right| \ll \frac{x}{(\log x)^A}.$$

Other results from [30], such as [[30], Proposition 5.1] and [[30], Proposition 5.3], can also be used here; however, we decide not to use them since they are only valid for $Q_1Q_2 < x^{0.501}$.

3.1.2. *Type-I_3 estimate.* Now we provide some estimates for the triple divisor function, which will be useful when dealing with sieve functions that count products of three large variables. These can be seen as variants of the three-dimensional Harman's sieve in Section 2.

Lemma 3.5. *([28], Lemma 20.7]). Let $x^{2\varepsilon} \leqslant M_3 \leqslant M_2 \leqslant M_1$, $x^{\varepsilon} \leqslant M_0$ and $M_0M_1M_2M_3 \asymp x$. Let a_{m_0} be a divisor-bounded complex sequence, $z = \exp\left(\log x(\log\log x)^{-2}\right)$. Let $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$ be intervals such that $\mathbf{M}_i \subseteq [M_i, 2M_i]$. If we have*

$$Q_1Q_2 \leqslant x^{1-\varepsilon},\ \frac{M_0Q_1^{\frac{5}{2}}Q_2^3}{x^{1-15\varepsilon}} \leqslant M_1 \leqslant \frac{x^{2-\varepsilon}}{Q_1^3Q_2^2M_0},$$

then

$$\sum_{\substack{q_1\sim Q_1\\ q_2\sim Q_2\\ (q_1q_2,a)=1}}\left|\sum_{\substack{m_0\sim M_0\\ m_1\in\mathbf{M}_1\\ m_2\in\mathbf{M}_2\\ m_3\in\mathbf{M}_3\\ (m_1m_2m_3,P(z))=1\\ m_0m_1m_2m_3\equiv a(\bmod q_1q_2)}} a_{m_0}-\frac{1}{\varphi(q_1q_2)}\sum_{\substack{m_0\sim M_0\\ m_1\in\mathbf{M}_1\\ m_2\in\mathbf{M}_2\\ m_3\in\mathbf{M}_3\\ (m_1m_2m_3,P(z))=1\\ (m_0m_1m_2m_3,q_1q_2)=1}} a_{m_0}\right| \ll \frac{x}{(\log x)^A}.$$

Lemma 3.6. *([28], Proposition 11.1]). Let $x^{\varepsilon} \leqslant M_3 \leqslant M_2 \leqslant M_1 \leqslant x^{\frac{3}{7}}$, $x^{\varepsilon} \leqslant M_0$ and $M_0M_1M_2M_3 \asymp x$. Let a_{m_0} be a divisor-bounded complex sequence, $z = \exp\left(\log x(\log\log x)^{-2}\right)$. Let $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$ be intervals such that $\mathbf{M}_i \subseteq [M_i, 2M_i]$. If we have*

$$Q_1^7Q_2^9 < x^4,\ Q_1^9Q_2 < x^{\frac{32}{7}},\ Q_1Q_2 < x^{\frac{11}{21}},\ Q_1^{\frac{15}{8}}Q_2^{\frac{15}{8}}M_0 < x^{1-20\varepsilon},$$

then

$$\sum_{\substack{q_1\sim Q_1\\ q_2\sim Q_2\\ (q_1q_2,a)=1}}\left|\sum_{\substack{m_0\sim M_0\\ m_1\in\mathbf{M}_1\\ m_2\in\mathbf{M}_2\\ m_3\in\mathbf{M}_3\\ (m_1m_2m_3,P(z))=1\\ m_0m_1m_2m_3\equiv a(\bmod q_1q_2)}} a_{m_0}-\frac{1}{\varphi(q_1q_2)}\sum_{\substack{m_0\sim M_0\\ m_1\in\mathbf{M}_1\\ m_2\in\mathbf{M}_2\\ m_3\in\mathbf{M}_3\\ (m_1m_2m_3,P(z))=1\\ (m_0m_1m_2m_3,q_1q_2)=1}} a_{m_0}\right| \ll \frac{x}{(\log x)^A}.$$

Lemma 3.7. *([25], Proposition 12.2]). Let $x^{\varepsilon} \leqslant M_3 \leqslant M_2 \leqslant M_1 \leqslant x^{\frac{3}{7}}$, $M_0 = x^{\varepsilon}$ and $M_0M_1M_2M_3 \asymp x$. Let a_{m_0} be a divisor-bounded complex sequence, $z = \exp\left(\log x(\log\log x)^{-2}\right)$. Let $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$ be intervals such that $\mathbf{M}_i \subseteq [M_i, 2M_i]$. If we have*

$$Q_1^3Q_2^2 < x^{\frac{11}{7}-30\varepsilon},\ Q_1^{11}Q_2^{12} < x^{6-30\varepsilon},\ Q_1Q_2 < x^{\frac{8}{15}-30\varepsilon},$$

then

$$\sum_{\substack{q_1\sim Q_1\\ q_2\sim Q_2\\ (q_1q_2,a)=1}}\left|\sum_{\substack{m_0\sim M_0\\ m_1\in\mathbf{M}_1\\ m_2\in\mathbf{M}_2\\ m_3\in\mathbf{M}_3\\ (m_1m_2m_3,P(z))=1\\ m_0m_1m_2m_3\equiv a(\bmod q_1q_2)}} a_{m_0}-\frac{1}{\varphi(q_1q_2)}\sum_{\substack{m_0\sim M_0\\ m_1\in\mathbf{M}_1\\ m_2\in\mathbf{M}_2\\ m_3\in\mathbf{M}_3\\ (m_1m_2m_3,P(z))=1\\ (m_0m_1m_2m_3,q_1q_2)=1}} a_{m_0}\right| \ll \frac{x}{(\log x)^A}.$$

3.2. **Sieve Asymptotic Formulas.** In this section, many asymptotic formulas used in the decompositions will be adopted from Section 2. We shall also use the following powerful lemma, which comes from [28] and gives asymptotic formulas for all sums that count numbers with 4 or more prime factors, all larger than $x^{\frac{1}{7}}$.

Lemma 3.8. *([28], Proposition 7.3]). Let $j \geqslant 4$, $P_1P_2\cdots P_j \asymp x$ and $P_1 \geqslant P_2 \geqslant \cdots \geqslant P_j \geqslant x^{\frac{1}{7}+10\varepsilon}$. Suppose that*

$$2\theta_1+\theta_2<1 \quad \text{and} \quad 7\theta_1+12\theta_2<4.$$

Then we have, for almost all q_1, q_2,

$$\sum_{\substack{p_1,\ldots,p_j\\ p_i\sim P_i,\ 1\leqslant i\leqslant j\\ p_1\cdots p_j\equiv a(\bmod q_1q_2)}} 1$$

has an asymptotic formula of the form (46).

We can use Lemma 3.6 and Lemma 3.7 to construct a new three-dimensional Harman's sieve similar to Lemma 2.20 and Lemma 2.21, and we will discuss the construction later.

3.3. **Upper Bounds.** We shall construct the majorant $\rho_1(n)$ in this subsection. Before constructing, we first mention some existing results of $C_1(\theta_1,\theta_2)$.

Theorem 3.9. *The function $C_1(\theta_1,\theta_2)$ satisfies the following conditions:*
(1). $C_1(\theta_1,\theta_2)=C_1(\theta_2,\theta_1)$;
(2). $C_1(\theta_1,\theta_2)=1$ for all θ_1,θ_2 satisfy $\theta_1+\theta_2<0.5$;
(3). $C_1(\theta_1,\theta_2)=1$ for all θ_1,θ_2 satisfy $2\theta_1+\theta_2<1$, $7\theta_1+12\theta_2<4$ and $19\theta_1+20\theta_2<10$;
(4). $C_1(\theta_1,\theta_2)\leqslant C_1(\theta_1+\theta_2)$ for $0.5\leqslant\theta_1+\theta_2\leqslant 1$;
(5). $C_1(\theta_1,\theta_2)\leqslant 1+\varepsilon$ for all θ_1,θ_2 satisfy $\theta_1+\theta_2=0.5$;
(6). $C_1(\theta_1,\theta_2)\leqslant 1+\varepsilon$ for all θ_1,θ_2 satisfy $\theta_1\leqslant 0.5$, $\theta_1\neq\frac{10}{21}$ and $\theta_2=\min\left(1-2\theta_1,\frac{4-7\theta_1}{12},\frac{10-19\theta_1}{20}\right)$.

Proof. The first statement is obvious. The second and third statements follow easily from the Bombieri–Vinogradov Theorem and [[28], Theorem 1.1]. The fourth statement holds trivially by the work done in Section 2. When there are no new arithmetic information inputs outside of those in Section 2, we use $C_1(\theta_1+\theta_2)$ as an upper bound for $C_1(\theta_1,\theta_2)$. The fifth statement holds from the fourth statement and statement (1) of Theorem 2.22. The sixth statement holds from similar arguments as in [[28], Theorem 1.1] (with a "loss" of size $O(\varepsilon)$) and a fact that a "three-dimensional Harman's sieve"(which will be explained later) is still applicable on the boundary unless $(\theta_1,\theta_2)=\left(\frac{10}{21},\frac{1}{21}\right)$. □

From here to the end of this section, we assume that $\theta_1\geqslant\theta_2$ to simplify the conditions. We also write $\theta=\theta_1+\theta_2$. Before performing our final decompositions, we define several regions of the pair (θ_1,θ_2) based on various arithmetic information inputs.

$$\boldsymbol{U} = \{(\theta_1,\theta_2): 0\leqslant\theta_2\leqslant\theta_1<1,\ \theta_1+\theta_2<1\},$$
$$\boldsymbol{I} = \left\{(\theta_1,\theta_2): (\theta_1,\theta_2)\in\boldsymbol{U};\ \theta_1+\theta_2<\frac{1}{2}\right.$$

$$
\begin{aligned}
&\qquad\qquad \text{or } 2\theta_1+\theta_2<1,\ 7\theta_1+12\theta_2<4,\ 19\theta_1+20\theta_2<10\},\\
\boldsymbol{T}_1 =& \left\{(\theta_1,\theta_2):(\theta_1,\theta_2)\in \boldsymbol{U};\ 7\theta_1+9\theta_2<4,\ 9\theta_1+\theta_2<\frac{32}{7},\ \theta_1+\theta_2<\frac{11}{21}\right\},\\
\boldsymbol{T}_2 =& \left\{(\theta_1,\theta_2):(\theta_1,\theta_2)\in \boldsymbol{U}\backslash \boldsymbol{T}_1;\ 3\theta_1+2\theta_2<\frac{11}{7},\ 11\theta_1+12\theta_2<6,\ \theta_1+\theta_2<\frac{8}{15}\right\},\\
\boldsymbol{T} =& \{(\theta_1,\theta_2):(\theta_1,\theta_2)\in \boldsymbol{U};\ (\theta_1,\theta_2)\in \boldsymbol{T}_1\cup \boldsymbol{T}_2\},\\
\boldsymbol{A} =& \left\{(\theta_1,\theta_2):(\theta_1,\theta_2)\in \boldsymbol{U}\backslash \boldsymbol{I};\ \frac{5}{14}<\theta_1\leqslant \frac{2}{5},\ \frac{1}{2}(1-2\theta_1)<\theta_2<\frac{1}{9}(2-2\theta_1)\right.\\
&\qquad\qquad \text{or } \frac{2}{5}<\theta_1\leqslant\frac{4}{9},\ \frac{1}{2}(1-2\theta_1)<\theta_2<\frac{1}{6}(2-3\theta_1)\\
&\qquad\qquad \text{or } \frac{4}{9}<\theta_1\leqslant\frac{1}{2},\ \frac{1}{2}(1-2\theta_1)<\theta_2<\frac{1}{9}(5-9\theta_1)\\
&\qquad\qquad \left.\text{or } \frac{1}{2}<\theta_1<\frac{11}{20},\ 0<\theta_2<\frac{1}{18}(11-20\theta_1)\right\},\\
\boldsymbol{B} =& \left\{(\theta_1,\theta_2):(\theta_1,\theta_2)\in \boldsymbol{U}\backslash \boldsymbol{I};\ \frac{1}{4}<\theta_1\leqslant \frac{10}{33},\ \frac{1}{2}(1-2\theta_1)<\theta_2<\theta_1\right.\\
&\qquad\qquad \text{or } \frac{10}{33}<\theta_1\leqslant\frac{1}{2},\ \frac{1}{2}(1-2\theta_1)<\theta_2<\frac{1}{19}(10-14\theta_1)\\
&\qquad\qquad \left.\text{or } \frac{1}{2}<\theta_1<\frac{5}{7},\ 0<\theta_2<\frac{1}{19}(10-14\theta_1)\right\},\\
\boldsymbol{C} =& \left\{(\theta_1,\theta_2):(\theta_1,\theta_2)\in \boldsymbol{U}\backslash \boldsymbol{I};\ \frac{2}{5}<\theta_1<\frac{1}{2},\ \frac{1}{2}(1-2\theta_1)<\theta_2<\frac{1}{12}(4-7\theta_1)\right\}.
\end{aligned}
$$

Here, $\boldsymbol{U}$ denote all possible pairs (θ_1,θ_2) in our problem, $\boldsymbol{I}$ denote the region that $C_1(\theta_1,\theta_2)=1$ follows by the Bombieri–Vinogradov Theorem or [[28], Theorem 1.1], and $\boldsymbol{T}$ denote a new "three-dimensional Harman's sieve" region corresponding to Lemma 3.6 and Lemma 3.7. Region $\boldsymbol{A}$ corresponds to Lemma 3.2, region $\boldsymbol{B}$ corresponds to Lemma 3.3, and region $\boldsymbol{C}$ corresponds to Lemma 3.4. Region $\boldsymbol{B}$ covers both region $\boldsymbol{A}$ and region $\boldsymbol{C}$.

Before our discussions on each region, we first give a result of the three-dimensional Harman's sieve. We shall implicitly use this result in many decompositions below. Assume that $\theta<\frac{17}{32}$. Put

$$
\mathcal{F}^q=\left\{m_1m_2m_3: m_1m_2m_3\sim x,\ m_1m_2m_3\equiv a(\bmod q),\ x^{\varepsilon}\leqslant m_3\leqslant m_2\leqslant m_1\leqslant x^{\frac{3}{7}}\right\}.
$$

Similar to Lemma 2.20 and Lemma 2.21, we want to give an asymptotic formula for

$$
S\left(\mathcal{F}^{q_1q_2},x^{\kappa}\right) \tag{48}
$$

with $\kappa=\frac{5-8\theta}{6}-\varepsilon$ or $\frac{2-3\theta}{3}-\varepsilon$ or some other values. In order to give an asymptotic formula for (48), we need to use Lemma 3.6 and Lemma 3.7 to give asymptotic formulas for

$$
\sum_{m_0\leqslant x^{v}} a_{m_0} S\left(\mathcal{F}^{q_1q_2},x^{\varepsilon^2}\right), \tag{49}
$$

where $a_{m_0}=0$ unless $P^{-}(m_0)>x^{\varepsilon^2}$, and v is the "left endpoint" of the corresponding available Type-II interval, usually equals $2\theta-1+\varepsilon$ or $2\theta_1+\theta_2-1+\varepsilon$ or zero. When using Lemma 3.6, we can deduce an asymptotic formula for (49) when $v<1-\frac{15}{8}\theta$. We can use this three-dimensional Harman's sieve for $\theta<\frac{16}{31}\approx 0.5161$ (note that Lemma 2.20 needs the same upper bound for θ) when the Type-II range is $(2\theta-1,\cdots)$, and for all $\theta<\frac{17}{32}$ when the Type-II range is $(0,\cdots)$ since $1-\frac{15}{8}\cdot\frac{17}{32}=\frac{1}{256}>0$. When using Lemma 3.7, however, we need the Type-II range to be $(0,\cdots)$ since $M_0=x^{\varepsilon}$.

When applying this device, one can follow the process in [2]: Suppose that $\kappa>\frac{1}{7}$. For three "large" variables $p_1p_2m_3\sim x$ such that $x^{\kappa}\leqslant p_1,p_2,m_3\leqslant x^{\frac{3}{7}}$, we have

$$
\begin{aligned}
\sum_{\kappa\leqslant\alpha_1,\alpha_2\leqslant\frac{3}{7}} S\left(\mathcal{A}^{q_1q_2}_{p_1p_2},x^{\kappa}\right) =& \sum_{\substack{x^{\kappa}\leqslant p_2<p_1<x^{\frac{3}{7}}\\ p_1p_2m_3\in\mathcal{A}^{q_1q_2}\\ (p_1p_2m_3,P(x^{\kappa}))=1}} 1\\
=& \sum_{\substack{x^{\kappa}\leqslant m_2<m_1<x^{\frac{3}{7}}\\ m_1m_2m_3\in\mathcal{A}^{q_1q_2}\\ (m_1m_2m_3,P(x^{\kappa}))=1}} 1 - \sum_{\substack{x^{\kappa}\leqslant m_2<m_1<x^{\frac{3}{7}}\\ m_1m_2m_3\in\mathcal{A}^{q_1q_2}\\ (m_1m_2m_3,P(x^{\kappa}))=1\\ \Omega(m_1m_2)\geqslant 3}} 1\\
=& S\left(\mathcal{F}^{q_1q_2},x^{\kappa}\right) - \sum_{\substack{x^{\kappa}\leqslant m_2<m_1<x^{\frac{3}{7}}\\ m_1m_2m_3\in\mathcal{A}^{q_1q_2}\\ (m_1m_2m_3,P(x^{\kappa}))=1\\ \Omega(m_1m_2)\geqslant 3}} 1.
\end{aligned} \tag{50}
$$

We can give an asymptotic formula for the first sum on the right-hand side of (50). For the second sum, since $\kappa > \frac{1}{7}$ and $m_1, m_2 \leqslant x^{\frac{3}{7}}$, we know that $\Omega(m_1), \Omega(m_2) \leqslant 2$. Thus, the loss come from this sum is similar to I_1 ($\Omega(m_1 m_2) = 3$) and I_2 ($\Omega(m_1 m_2) = 4$) in Lemma 2.20 with modified integration regions. We note that in many applications of this device, we can use it on the whole of the two-dimensional region C defined in Section 2 since it only requires the variables are smaller than $x^{\frac{3}{7}}$, which holds naturally since we have $\alpha_2 < \frac{1}{3}$, $\alpha_1 < \frac{3}{7}$ and $\alpha_1 + \alpha_2 > \frac{4}{7}$ (note that $\left(\theta, \frac{4}{7}\right)$ is a Type-II range and $\alpha_1 + \alpha_2 < \theta \Rightarrow \boldsymbol{\alpha}_2 \in A$) when $\boldsymbol{\alpha}_2 \in C$, $\theta < \frac{17}{32}$. In addition, the above process is still applicable in some cases with $\kappa \leqslant \frac{1}{7}$ in Section 4, and we shall discuss them in the next section.

Now we assume that $(\theta_1, \theta_2) \in \boldsymbol{A}$. We divide $\boldsymbol{A}$ into 17 subregions:

$$\boldsymbol{A} = \boldsymbol{A}_{01} \cup \boldsymbol{A}_{02} \cup \boldsymbol{A}_{03} \cup \boldsymbol{A}_{04} \cup \boldsymbol{A}_{05} \cup \boldsymbol{A}_{06} \cup \boldsymbol{A}_{07} \cup \boldsymbol{A}_{08} \cup \boldsymbol{A}_{09} \cup \boldsymbol{A}_{10} \cup \boldsymbol{A}_{11} \cup \boldsymbol{A}_{12} \cup \boldsymbol{A}_{13} \cup \boldsymbol{A}_{14} \cup \boldsymbol{A}_{15} \cup \boldsymbol{A}_{16} \cup \boldsymbol{A}_{17},$$

where

$$\begin{aligned}
\boldsymbol{A}_{01} &= \left\{(\theta_1, \theta_2) : \frac{5}{14} < \theta_1 \leqslant \frac{2}{5},\ \frac{1}{2}(1 - 2\theta_1) < \theta_2 < \frac{1}{9}(2 - 2\theta_1)\right\},\\
\boldsymbol{A}_{02} &= \left\{(\theta_1, \theta_2) : \frac{2}{5} < \theta_1 \leqslant \frac{4}{9},\ \frac{1}{12}(4 - 7\theta_1) < \theta_2 < \frac{1}{3}(2 - 4\theta_1)\right\},\\
\boldsymbol{A}_{03} &= \left\{(\theta_1, \theta_2) : \frac{2}{5} < \theta_1 \leqslant \frac{4}{9},\ \frac{1}{3}(2 - 4\theta_1) \leqslant \theta_2 < \frac{1}{6}(2 - 3\theta_1),\ \theta_2 < \frac{1}{13}(10 - 20\theta_1)\right\},\\
\boldsymbol{A}_{04} &= \left\{(\theta_1, \theta_2) : \frac{2}{5} < \theta_1 \leqslant \frac{4}{9},\ \frac{1}{13}(10 - 20\theta_1) \leqslant \theta_2 < \frac{1}{6}(2 - 3\theta_1),\ \theta_1 + \theta_2 < \frac{11}{20}\right\},\\
\boldsymbol{A}_{05} &= \left\{(\theta_1, \theta_2) : \frac{2}{5} < \theta_1 \leqslant \frac{4}{9},\ \frac{11}{20} - \theta_1 \leqslant \theta_2 < \frac{1}{6}(2 - 3\theta_1)\right\},\\
\boldsymbol{A}_{06} &= \left\{(\theta_1, \theta_2) : \frac{4}{9} < \theta_1 \leqslant \frac{5}{11},\ \frac{1}{12}(4 - 7\theta_1) < \theta_2 < \frac{1}{13}(10 - 20\theta_1)\right\},\\
\boldsymbol{A}_{07} &= \left\{(\theta_1, \theta_2) : \frac{4}{9} < \theta_1 \leqslant \frac{5}{11},\ \frac{1}{13}(10 - 20\theta_1) \leqslant \theta_2 < 1 - 2\theta_1,\ \theta_1 + \theta_2 < \frac{11}{20}\right\},\\
\boldsymbol{A}_{08} &= \left\{(\theta_1, \theta_2) : \frac{4}{9} < \theta_1 \leqslant \frac{5}{11},\ \frac{11}{20} - \theta_1 \leqslant \theta_2 < 1 - 2\theta_1\right\},\\
\boldsymbol{A}_{09} &= \left\{(\theta_1, \theta_2) : \frac{5}{11} < \theta_1 < \frac{1}{2},\ \frac{1}{20}(10 - 19\theta_1) < \theta_2 < \frac{1}{13}(10 - 20\theta_1)\right\},\\
\boldsymbol{A}_{10} &= \left\{(\theta_1, \theta_2) : \frac{5}{11} < \theta_1 < \frac{1}{2},\ \frac{1}{20}(10 - 19\theta_1) < \theta_2 < 1 - 2\theta_1,\ \frac{1}{13}(10 - 20\theta_1) \leqslant \theta_2\right\},\\
\boldsymbol{A}_{11} &= \left\{(\theta_1, \theta_2) : \frac{4}{9} < \theta_1 < \frac{1}{2},\ 1 - 2\theta_1 \leqslant \theta_2 < \frac{1}{9}(5 - 9\theta_1),\ \theta_1 + \theta_2 < \frac{11}{20},\ \theta_2 < \frac{1}{14}(10 - 19\theta_1)\right\},\\
\boldsymbol{A}_{12} &= \left\{(\theta_1, \theta_2) : \frac{4}{9} < \theta_1 < \frac{1}{2},\ 1 - 2\theta_1 \leqslant \theta_2 < \frac{1}{9}(5 - 9\theta_1),\ \theta_1 + \theta_2 < \frac{11}{20},\ \theta_2 \geqslant \frac{1}{14}(10 - 19\theta_1)\right\},\\
\boldsymbol{A}_{13} &= \left\{(\theta_1, \theta_2) : \frac{4}{9} < \theta_1 < \frac{1}{2},\ 1 - 2\theta_1 \leqslant \theta_2 < \frac{1}{9}(5 - 9\theta_1),\ \theta_1 + \theta_2 \geqslant \frac{11}{20},\ \theta_2 < \frac{1}{14}(10 - 19\theta_1)\right\},\\
\boldsymbol{A}_{14} &= \left\{(\theta_1, \theta_2) : \frac{4}{9} < \theta_1 < \frac{1}{2},\ 1 - 2\theta_1 \leqslant \theta_2 < \frac{1}{9}(5 - 9\theta_1),\ \theta_1 + \theta_2 \geqslant \frac{11}{20},\ \theta_2 \geqslant \frac{1}{14}(10 - 19\theta_1)\right\},\\
\boldsymbol{A}_{15} &= \left\{(\theta_1, \theta_2) : \frac{1}{2} \leqslant \theta_1 < \frac{10}{19},\ 0 < \theta_2 < \frac{1}{14}(10 - 19\theta_1)\right\},\\
\boldsymbol{A}_{16} &= \left\{(\theta_1, \theta_2) : \frac{1}{2} \leqslant \theta_1 < \frac{11}{20},\ \frac{1}{14}(10 - 19\theta_1) \leqslant \theta_2 < \frac{11}{20} - \theta_1\right\},\\
\boldsymbol{A}_{17} &= \left\{(\theta_1, \theta_2) : \frac{1}{2} \leqslant \theta_1 < \frac{11}{20},\ \frac{11}{20} - \theta_1 \leqslant \theta_2 < \frac{1}{18}(11 - 20\theta_1)\right\}.
\end{aligned}$$

3.3.1. $\boldsymbol{A}_{01}$. For $(\theta_1, \theta_2) \in \boldsymbol{A}_{01}$ we have 3 available Type-II information ranges:

$$\left(2\theta_1 + 2\theta_2 - 1,\ \frac{1}{6}(5 - 8\theta_1 - 8\theta_2)\right), \quad \left(\theta_2,\ \frac{1}{6}(2 - 2\theta_1 - 3\theta_2)\right) \quad \text{and} \quad \left(0,\ \frac{1}{6}(4 - 7\theta_1 - 8\theta_2)\right). \tag{51}$$

The first range comes from Lemma 2.2, the second comes from Lemma 3.2, and the third comes from Lemma 3.3. We divide $\boldsymbol{A}_{01}$ into 4 subregions based on the overlapping conditions of these ranges.

$$\boldsymbol{A}_{01} = \boldsymbol{A}_{0101} \cup \boldsymbol{A}_{0102} \cup \boldsymbol{A}_{0103} \cup \boldsymbol{A}_{0104},$$

where

$$\begin{aligned}
\boldsymbol{A}_{0101} = \Bigg\{(\theta_1, \theta_2) : &\ \frac{5}{14} < \theta_1 \leqslant \frac{50}{131},\ \frac{1}{2}(1 - 2\theta_1) < \theta_2 < \frac{1}{9}(2 - 2\theta_1)\\
&\text{or } \frac{50}{131} < \theta_1 \leqslant \frac{2}{5},\ \frac{1}{2}(1 - 2\theta_1) < \theta_2 < \frac{1}{20}(10 - 19\theta_1)\Bigg\},
\end{aligned}$$

$$\boldsymbol{A}_{0102} = \left\{(\theta_1, \theta_2) : \frac{50}{131} < \theta_1 \leqslant \frac{17}{44},\ \frac{1}{20}(10 - 19\theta_1) < \theta_2 < \frac{1}{9}(2 - 2\theta_1) \right.$$
$$\left. \text{or } \frac{17}{44} < \theta_1 \leqslant \frac{2}{5},\ \frac{1}{20}(10 - 19\theta_1) < \theta_2 < \frac{1}{5}(3 - 6\theta_1) \right\},$$
$$\boldsymbol{A}_{0103} = \left\{(\theta_1, \theta_2) : \frac{17}{44} < \theta_1 \leqslant \frac{2}{5},\ \frac{1}{5}(3 - 6\theta_1) < \theta_2 < \frac{1}{14}(5 - 8\theta_1) \right\},$$
$$\boldsymbol{A}_{0104} = \left\{(\theta_1, \theta_2) : \frac{17}{44} < \theta_1 \leqslant \frac{2}{5},\ \frac{1}{14}(5 - 8\theta_1) < \theta_2 < \frac{1}{9}(2 - 2\theta_1) \right\}.$$

Note that we have $\theta < \frac{7}{13}$ for $(\theta_1, \theta_2) \in \boldsymbol{A}_{01}$, and $\theta < \frac{17}{32}$ for $(\theta_1, \theta_2) \in \boldsymbol{A}_{0101} \cup \boldsymbol{A}_{0102} \cup \boldsymbol{A}_{0103}$.

In $\boldsymbol{A}_{0101}$ we have

$$\frac{1}{6}(4 - 7\theta_1 - 8\theta_2) > 2\theta_1 + 2\theta_2 - 1 \quad \text{and} \quad \frac{1}{6}(5 - 8\theta_1 - 8\theta_2) > \frac{1}{6}(2 - 2\theta_1 - 3\theta_2).$$

Hence, the Type-II range for $\boldsymbol{A}_{0101}$ is

$$\left(0,\ \frac{1}{6}(5 - 8\theta_1 - 8\theta_2)\right). \tag{52}$$

Similar to Lemma 2.14, we can prove that for $(\theta_1, \theta_2) \in \boldsymbol{A}_{0101}$,

$$\sum_{\boldsymbol{\alpha}_j \in \boldsymbol{S}_j} S\left(\mathcal{A}_{p_1 \cdots p_j}^{q_1 q_2}, x^{\kappa}\right)$$

has an asymptotic formula of the form (46). Now, the decompositions in this case are very similar to the case $\frac{1}{2} < \theta < \frac{53}{105}$ in Subsection 2.4. We split the range of $\theta_1 + \theta_2$ as in [23], and replace the conditions $\boldsymbol{\alpha}_j \in \boldsymbol{U}_j$ with $\boldsymbol{\alpha}_j \in \boldsymbol{S}_j$. Role-reversals are also applicable here.

In $\boldsymbol{A}_{0102}$ we have

$$\frac{1}{6}(4 - 7\theta_1 - 8\theta_2) < 2\theta_1 + 2\theta_2 - 1 \quad \text{and} \quad \frac{1}{6}(5 - 8\theta_1 - 8\theta_2) > \frac{1}{6}(2 - 2\theta_1 - 3\theta_2).$$

Hence, the Type-II range for $\boldsymbol{A}_{0102}$ is

$$\left(0,\ \frac{1}{6}(4 - 7\theta_1 - 8\theta_2)\right) \cup \left(2\theta_1 + 2\theta_2 - 1,\ \frac{1}{6}(5 - 8\theta_1 - 8\theta_2)\right). \tag{53}$$

This case is similar to the case $\frac{53}{105} \leqslant \theta < \frac{17}{33}$ in Subsection 2.4. Performing role-reversals here is not so efficient, hence we ignore the first range $\left(0,\ \frac{1}{6}(4 - 7\theta_1 - 8\theta_2)\right)$ and use only the range comes from Lemma 2.2. The decompositions in this case are very similar to parts of the work done in [23], where $\theta_1 + \theta_2 = \theta$ lies in some subranges.

In $\boldsymbol{A}_{0103}$ we have

$$\frac{1}{6}(4 - 7\theta_1 - 8\theta_2) < 2\theta_1 + 2\theta_2 - 1 \quad \text{and} \quad \theta_2 < \frac{1}{6}(5 - 8\theta_1 - 8\theta_2) < \frac{1}{6}(2 - 2\theta_1 - 3\theta_2).$$

Hence, the Type-II range for $\boldsymbol{A}_{0103}$ is

$$\left(0,\ \frac{1}{6}(4 - 7\theta_1 - 8\theta_2)\right) \cup \left(2\theta_1 + 2\theta_2 - 1,\ \frac{1}{6}(2 - 2\theta_1 - 3\theta_2)\right). \tag{54}$$

This case is almost the same as the above case. The only difference here is that we replace the value of $\kappa = \frac{5-8\theta}{6} - \varepsilon$ with $\frac{2-2\theta_1-3\theta_2}{6} - \varepsilon$. Since we have $\frac{5-8\theta}{6} < \frac{2-2\theta_1-3\theta_2}{6}$ in this case, we can simply use Buchstab's identity to get

$$\sum_{\boldsymbol{\alpha}_j} S\left(\mathcal{A}_{p_1 \cdots p_j}^{q_1 q_2}, x^{\frac{2-2\theta_1-3\theta_2}{6} - \varepsilon}\right) = \sum_{\boldsymbol{\alpha}_j} S\left(\mathcal{A}_{p_1 \cdots p_j}^{q_1 q_2}, x^{\kappa}\right) - \sum_{\substack{\boldsymbol{\alpha}_j \\ \kappa \leqslant \alpha_{j+1} < \frac{2-2\theta_1-3\theta_2}{6} - \varepsilon}} S\left(\mathcal{A}_{p_1 \cdots p_j p_{j+1}}^{q_1 q_2}, p_{j+1}\right). \tag{55}$$

We can give an asymptotic formula for the last sum in (55) by our Type-II information in this case. By applying (55), we know that for $(\theta_1, \theta_2) \in \boldsymbol{A}_{0103}$, any $\boldsymbol{\alpha}_j$ such that

$$\sum_{\boldsymbol{\alpha}_j} S\left(\mathcal{A}_{p_1 \cdots p_j}^{q_1 q_2}, x^{\kappa}\right)$$

has an asymptotic formula also yields an asymptotic formula for the sum

$$\sum_{\boldsymbol{\alpha}_j} S\left(\mathcal{A}_{p_1 \cdots p_j}^{q_1 q_2}, x^{\frac{2-2\theta_1-3\theta_2}{6} - \varepsilon}\right).$$

Hence, we can change our "starting point" from κ to $\frac{2-2\theta_1-3\theta_2}{6} - \varepsilon$. In the applications of two-dimensional and three-dimensional sieves, the details are similar. Now the decompositions can be easily performed as in the case $\boldsymbol{A}_{0102}$.

In $\boldsymbol{A}_{0104}$ we have

$$\frac{1}{6}(4 - 7\theta_1 - 8\theta_2) < 2\theta_1 + 2\theta_2 - 1 \quad \text{and} \quad \frac{1}{6}(5 - 8\theta_1 - 8\theta_2) < \theta_2.$$

Hence, the Type-II range for $\boldsymbol{A}_{0104}$ is

$$\left(0,\ \frac{1}{6}(4 - 7\theta_1 - 8\theta_2)\right) \cup \left(2\theta_1 + 2\theta_2 - 1,\ \frac{1}{6}(5 - 8\theta_1 - 8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(2 - 2\theta_1 - 3\theta_2)\right). \tag{56}$$

Assume that $\theta < \frac{17}{32}$. We use the middle Type-II range $\left(2\theta_1 + 2\theta_2 - 1,\ \frac{1}{6}(5 - 8\theta_1 - 8\theta_2)\right)$ to give a "starting point" $\kappa = \frac{5-8\theta}{6} - \varepsilon$, and the third Type-II range $\left(\theta_2,\ \frac{1}{6}(2 - 2\theta_1 - 3\theta_2)\right)$ is used to subtract the contributions of those sums with products of variables lie in this range. The decompositions are also similar to which in the case $\boldsymbol{A}_{0102}$.

When $\theta \geqslant \frac{17}{32}$, we follow the decompositions in [23], and use our Type-II range to give asymptotic formulas for sums with products of variables lie in the range.

3.3.2. $\boldsymbol{A}_{02}$. For $(\theta_1, \theta_2) \in \boldsymbol{A}_{02}$ we have 3 available Type-II information ranges:

$$\left(2\theta_1 + 2\theta_2 - 1,\ \frac{1}{6}(5 - 8\theta_1 - 8\theta_2)\right), \quad \left(\theta_2,\ \frac{1}{6}(2 - 2\theta_1 - 3\theta_2)\right) \quad \text{and} \quad \left(0,\ \frac{1}{6}(4 - 7\theta_1 - 8\theta_2)\right). \tag{57}$$

The first range comes from Lemma 2.2, the second comes from Lemma 3.2, and the third comes from Lemma 3.3. We divide $\boldsymbol{A}_{02}$ into 4 subregions based on the overlapping conditions of these ranges.

$$\boldsymbol{A}_{02} = \boldsymbol{A}_{0201} \cup \boldsymbol{A}_{0202} \cup \boldsymbol{A}_{0203} \cup \boldsymbol{A}_{0204},$$

where

$$\begin{aligned}
\boldsymbol{A}_{0201} = &\left\{(\theta_1, \theta_2) : \frac{2}{5} < \theta_1 < \frac{16}{37},\ \frac{1}{12}(4 - 7\theta_1) < \theta_2 < \frac{1}{5}(3 - 6\theta_1)\right\},\\
\boldsymbol{A}_{0202} = &\left\{(\theta_1, \theta_2) : \frac{2}{5} < \theta_1 \leqslant \frac{16}{37},\ \frac{1}{5}(3 - 6\theta_1) \leqslant \theta_2 < \frac{1}{20}(10 - 19\theta_1)\right.\\
&\quad \text{or } \frac{16}{37} < \theta_1 \leqslant \frac{10}{23},\ \frac{1}{12}(4 - 7\theta_1) < \theta_2 < \frac{1}{20}(10 - 19\theta_1)\\
&\quad \left.\text{or } \frac{10}{23} < \theta_1 < \frac{4}{9},\ \frac{1}{12}(4 - 7\theta_1) < \theta_2 < \frac{1}{3}(2 - 4\theta_1)\right\},\\
\boldsymbol{A}_{0203} = &\left\{(\theta_1, \theta_2) : \frac{2}{5} < \theta_1 \leqslant \frac{13}{32},\ \frac{1}{20}(10 - 19\theta_1) \leqslant \theta_2 < \frac{1}{14}(5 - 8\theta_1)\right.\\
&\quad \left.\text{or } \frac{13}{32} < \theta_1 < \frac{10}{23},\ \frac{1}{20}(10 - 19\theta_1) \leqslant \theta_2 < \frac{1}{3}(2 - 4\theta_1)\right\},\\
\boldsymbol{A}_{0204} = &\left\{(\theta_1, \theta_2) : \frac{2}{5} < \theta_1 < \frac{13}{32},\ \frac{1}{14}(5 - 8\theta_1) \leqslant \theta_2 < \frac{1}{3}(2 - 4\theta_1)\right\}.
\end{aligned}$$

Note that we have $\theta < \frac{7}{13}$ for $(\theta_1, \theta_2) \in \boldsymbol{A}_{02}$, and $\theta < \frac{17}{32}$ for $(\theta_1, \theta_2) \in \boldsymbol{A}_{0201} \cup \boldsymbol{A}_{0202} \cup \boldsymbol{A}_{0203}$.

In $\boldsymbol{A}_{0201}$ we have

$$\frac{1}{6}(4 - 7\theta_1 - 8\theta_2) > 2\theta_1 + 2\theta_2 - 1, \quad \theta_2 < \frac{1}{6}(5 - 8\theta_1 - 8\theta_2) \quad \text{and} \quad \frac{1}{6}(5 - 8\theta_1 - 8\theta_2) > \frac{1}{6}(2 - 2\theta_1 - 3\theta_2).$$

Hence, the Type-II range for $\boldsymbol{A}_{0201}$ is

$$\left(0,\ \frac{1}{6}(5 - 8\theta_1 - 8\theta_2)\right). \tag{58}$$

The decompositions are similar to which in the case $\boldsymbol{A}_{0101}$.

In $\boldsymbol{A}_{0202}$ we have

$$\frac{1}{6}(4 - 7\theta_1 - 8\theta_2) > 2\theta_1 + 2\theta_2 - 1 \quad \text{and} \quad \theta_2 < \frac{1}{6}(5 - 8\theta_1 - 8\theta_2) < \frac{1}{6}(2 - 2\theta_1 - 3\theta_2).$$

Hence, the Type-II range for $\boldsymbol{A}_{0202}$ is

$$\left(0,\ \frac{1}{6}(2 - 2\theta_1 - 3\theta_2)\right). \tag{59}$$

Using (55), one can replace the value of $\kappa = \frac{5-8\theta}{6} - \varepsilon$ with $\frac{2-2\theta_1-3\theta_2}{6} - \varepsilon$. Again, the decompositions are similar to which in the case $\boldsymbol{A}_{0101}$.

In $\boldsymbol{A}_{0203}$ we have

$$\frac{1}{6}(4 - 7\theta_1 - 8\theta_2) < 2\theta_1 + 2\theta_2 - 1 \quad \text{and} \quad \theta_2 < \frac{1}{6}(5 - 8\theta_1 - 8\theta_2) < \frac{1}{6}(2 - 2\theta_1 - 3\theta_2).$$

Hence, the Type-II range for $\boldsymbol{A}_{0203}$ is

$$\left(0,\ \frac{1}{6}(4 - 7\theta_1 - 8\theta_2)\right) \cup \left(2\theta_1 + 2\theta_2 - 1,\ \frac{1}{6}(2 - 2\theta_1 - 3\theta_2)\right). \tag{60}$$

The decompositions are similar to which in the case $\boldsymbol{A}_{0103}$.

In $\boldsymbol{A}_{0204}$ we have

$$\frac{1}{6}(4 - 7\theta_1 - 8\theta_2) < 2\theta_1 + 2\theta_2 - 1 \quad \text{and} \quad \frac{1}{6}(5 - 8\theta_1 - 8\theta_2) < \theta_2.$$

Hence, the Type-II range for $\boldsymbol{A}_{0204}$ is

$$\left(0,\ \frac{1}{6}(4 - 7\theta_1 - 8\theta_2)\right) \cup \left(2\theta_1 + 2\theta_2 - 1,\ \frac{1}{6}(5 - 8\theta_1 - 8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(2 - 2\theta_1 - 3\theta_2)\right). \tag{61}$$

The decompositions are similar to which in the case $\boldsymbol{A}_{0104}$.

3.3.3. $\boldsymbol{A}_{03}$. For $(\theta_1,\theta_2)\in\boldsymbol{A}_{03}$ we have 3 available Type-II information ranges:

$$\left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right),\quad \left(\theta_2,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right)\quad\text{and}\quad\left(0,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right). \tag{62}$$

The first range comes from Lemma 2.2, the second comes from Lemma 3.2, and the third comes from Lemma 3.3. We divide $\boldsymbol{A}_{03}$ into 3 subregions based on the overlapping conditions of these ranges.

$$\boldsymbol{A}_{03}=\boldsymbol{A}_{0301}\cup\boldsymbol{A}_{0302}\cup\boldsymbol{A}_{0303},$$

where

$$\begin{aligned}
\boldsymbol{A}_{0301}=\ &\left\{(\theta_1,\theta_2):\frac{10}{23}<\theta_1\leqslant\frac{4}{9},\ \frac{1}{3}(2-4\theta_1)\leqslant\theta_2<\frac{1}{20}(10-19\theta_1)\right\},\\
\boldsymbol{A}_{0302}=\ &\left\{(\theta_1,\theta_2):\frac{13}{32}<\theta_1\leqslant\frac{75}{176},\ \frac{1}{3}(2-4\theta_1)\leqslant\theta_2<\frac{1}{14}(5-8\theta_1)\right.\\
&\quad\text{or }\frac{75}{176}<\theta_1<\frac{10}{23},\ \frac{1}{3}(2-4\theta_1)<\theta_2<\frac{1}{13}(10-20\theta_1)\\
&\quad\left.\text{or }\frac{10}{23}\leqslant\theta_1\leqslant\frac{4}{9},\ \frac{1}{20}(10-19\theta_1)<\theta_2<\frac{1}{13}(10-20\theta_1)\right\},\\
\boldsymbol{A}_{0303}=\ &\left\{(\theta_1,\theta_2):\frac{2}{5}<\theta_1<\frac{13}{32},\ \frac{1}{3}(2-4\theta_1)\leqslant\theta_2<\frac{1}{6}(2-3\theta_1)\right.\\
&\quad\text{or }\frac{13}{32}\leqslant\theta_1\leqslant\frac{34}{81},\ \frac{1}{14}(5-8\theta_1)<\theta_2<\frac{1}{6}(2-3\theta_1)\\
&\quad\left.\text{or }\frac{34}{81}<\theta_1<\frac{75}{176},\ \frac{1}{14}(5-8\theta_1)<\theta_2<\frac{1}{13}(10-20\theta_1)\right\}.
\end{aligned}$$

Note that we have $\theta<\frac{17}{32}$ for $(\theta_1,\theta_2)\in\boldsymbol{A}_{0301}$.

In $\boldsymbol{A}_{0301}$ we have

$$\frac{1}{6}(4-7\theta_1-8\theta_2)>2\theta_1+2\theta_2-1\quad\text{and}\quad\frac{1}{6}(5-8\theta_1-8\theta_2)>\theta_2.$$

Hence, the Type-II range for $\boldsymbol{A}_{0301}$ is

$$\left(0,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right). \tag{63}$$

Similar to (55), one can replace the value of $\kappa=\frac{5-8\theta}{6}-\varepsilon$ with $\frac{2-3\theta}{3}-\varepsilon$ by applying Buchstab's identity as in (55). The decompositions are similar to which in the case $\boldsymbol{A}_{0101}$.

In $\boldsymbol{A}_{0302}$ we have

$$\frac{1}{6}(4-7\theta_1-8\theta_2)<2\theta_1+2\theta_2-1\quad\text{and}\quad\frac{1}{6}(5-8\theta_1-8\theta_2)>\theta_2.$$

Hence, the Type-II range for $\boldsymbol{A}_{0302}$ is

$$\left(0,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right)\cup\left(2\theta_1+2\theta_2-1,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right). \tag{64}$$

For $\theta<\frac{17}{32}$, we replace κ with $\frac{2-3\theta}{3}-\varepsilon$, and the decompositions are similar to which in the case $\boldsymbol{A}_{0103}$. For $\theta\geqslant\frac{17}{32}$, we follow the decompositions in [23] and use our Type-II information to give extra asymptotic formulas for sums with products of variables lie in the range $\left(\kappa,\frac{2-3\theta}{3}-\varepsilon\right)$.

In $\boldsymbol{A}_{0303}$ we have

$$\frac{1}{6}(4-7\theta_1-8\theta_2)<2\theta_1+2\theta_2-1\quad\text{and}\quad\frac{1}{6}(5-8\theta_1-8\theta_2)<\theta_2.$$

Hence, the Type-II range for $\boldsymbol{A}_{0303}$ is

$$\left(0,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right)\cup\left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right)\cup\left(\theta_2,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right). \tag{65}$$

The decompositions are similar to which in the case $\boldsymbol{A}_{0302}$, but now we cannot replace κ when $\theta<\frac{17}{32}$, and we can only give asymptotic formulas for sums with products of variables lie in the range $\left(\theta_2,\frac{2-3\theta}{3}-\varepsilon\right)$.

3.3.4. $\boldsymbol{A}_{04}$. For $(\theta_1,\theta_2)\in\boldsymbol{A}_{04}$ we have 4 available Type-II information ranges:

$$\left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right),\quad\left(\theta_2,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right), \tag{66}$$

$$\left(0,\ \frac{1}{6}(4-8\theta_1-7\theta_2)\right)\quad\text{and}\quad\left(2\theta_1+\theta_2-1,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right). \tag{67}$$

The first range comes from Lemma 2.2, the second comes from Lemma 3.2, and the third and fourth come from Lemma 3.3. Since we have

$$\frac{1}{6}(4-8\theta_1-7\theta_2)<0\quad\text{and}\quad 2\theta_1+\theta_2-1<0$$

in this region, (66)–(67) are equivalent to 3 Type-II information ranges

$$\left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right),\quad \left(\theta_2,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right)\quad \text{and}\quad \left(0,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right). \tag{68}$$

We divide $\boldsymbol{A}_{04}$ into 2 subregions based on the overlapping conditions of these ranges.

$$\boldsymbol{A}_{04}=\boldsymbol{A}_{0401}\cup\boldsymbol{A}_{0402},$$

where

$$\begin{aligned}
\boldsymbol{A}_{0401}=&\left\{(\theta_1,\theta_2): \frac{75}{176}<\theta_1\leqslant\frac{4}{9},\ \frac{1}{13}(10-20\theta_1)\leqslant\theta_2<\frac{1}{14}(5-8\theta_1)\right\},\\
\boldsymbol{A}_{0402}=&\left\{(\theta_1,\theta_2): \frac{34}{81}<\theta_1<\frac{75}{176},\ \frac{1}{13}(10-20\theta_1)\leqslant\theta_2<\frac{1}{6}(2-3\theta_1)\right.\\
&\quad\text{or } \frac{75}{176}\leqslant\theta_1\leqslant\frac{13}{30},\ \frac{1}{14}(5-8\theta_1)<\theta_2<\frac{1}{6}(2-3\theta_1)\\
&\quad\left.\text{or } \frac{13}{30}<\theta_1\leqslant\frac{4}{9},\ \frac{1}{14}(5-8\theta_1)<\theta_2<\frac{1}{20}(11-20\theta_1)\right\}.
\end{aligned}$$

Note that we have $\theta>\frac{7}{13}$ for $(\theta_1,\theta_2)\in\boldsymbol{A}_{0402}$.

In $\boldsymbol{A}_{0401}$ we have

$$\frac{1}{6}(5-8\theta_1-8\theta_2)>\theta_2.$$

Hence, the Type-II range for $\boldsymbol{A}_{0401}$ is

$$\left(0,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right)\cup\left(2\theta_1+2\theta_2-1,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right). \tag{69}$$

The decompositions are similar to which in the case $\boldsymbol{A}_{0302}$.

In $\boldsymbol{A}_{0402}$ we have

$$\frac{1}{6}(5-8\theta_1-8\theta_2)<\theta_2.$$

Hence, the Type-II range for $\boldsymbol{A}_{0402}$ is

$$\left(0,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right)\cup\left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right)\cup\left(\theta_2,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right). \tag{70}$$

The decompositions are similar to which in the case $\boldsymbol{A}_{0303}$.

3.3.5. $\boldsymbol{A}_{05}$. For $(\theta_1,\theta_2)\in\boldsymbol{A}_{05}$ we have 3 available Type-II information ranges:

$$\left(\theta_2,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right),\quad \left(0,\ \frac{1}{6}(4-8\theta_1-7\theta_2)\right)\quad \text{and}\quad \left(2\theta_1+\theta_2-1,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right). \tag{71}$$

The first range comes from Lemma 3.2, and the second and third come from Lemma 3.3. Note that Lemma 2.2 becomes trivial in this region since $\theta_1+\theta_2\geqslant\frac{11}{20}$. Since we have

$$\frac{1}{6}(4-8\theta_1-7\theta_2)<0\quad\text{and}\quad 2\theta_1+\theta_2-1<0$$

in this region, (71) is equivalent to 2 Type-II information ranges

$$\left(\theta_2,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right)\quad\text{and}\quad\left(0,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right). \tag{72}$$

Hence, the Type-II range for $\boldsymbol{A}_{05}$ is

$$\left(0,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right)\cup\left(\theta_2,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right). \tag{73}$$

We follow the decomposing process in [23], and use our Type-II information to give asymptotic formulas for sums with products of variables lie in the range $\left(\theta_2,\frac{2-3\theta}{3}-\varepsilon\right)$.

3.3.6. $\boldsymbol{A}_{06}$. For $(\theta_1,\theta_2)\in\boldsymbol{A}_{06}$ we have 3 available Type-II information ranges:

$$\left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right),\quad \left(\theta_2,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right)\quad \text{and}\quad \left(0,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right). \tag{74}$$

The first range comes from Lemma 2.2, the second comes from Lemma 3.2, and the third comes from Lemma 3.3. We divide $\boldsymbol{A}_{06}$ into 2 subregions based on the overlapping conditions of these ranges.

$$\boldsymbol{A}_{06}=\boldsymbol{A}_{0601}\cup\boldsymbol{A}_{0602},$$

where

$$\begin{aligned}
\boldsymbol{A}_{0601}=&\left\{(\theta_1,\theta_2): \frac{4}{9}<\theta_1<\frac{5}{11},\ \frac{1}{12}(4-7\theta_1)<\theta_2<\frac{1}{20}(10-19\theta_1)\right\},\\
\boldsymbol{A}_{0602}=&\left\{(\theta_1,\theta_2): \frac{4}{9}<\theta_1\leqslant\frac{5}{11},\ \frac{1}{20}(10-19\theta_1)\leqslant\theta_2<\frac{1}{13}(10-20\theta_1)\right\}.
\end{aligned}$$

In $\boldsymbol{A}_{0601}$ we have

$$\frac{1}{6}(4-7\theta_1-8\theta_2) > 2\theta_1+2\theta_2-1.$$

Hence, the Type-II range for $\boldsymbol{A}_{0601}$ is

$$\left(0,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right). \tag{75}$$

The decompositions are similar to which in the case $\boldsymbol{A}_{0301}$.

In $\boldsymbol{A}_{0602}$ we have

$$\frac{1}{6}(4-7\theta_1-8\theta_2) < 2\theta_1+2\theta_2-1.$$

Hence, the Type-II range for $\boldsymbol{A}_{0602}$ is

$$\left(0,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right) \cup \left(2\theta_1+2\theta_2-1,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right). \tag{76}$$

The decompositions are similar to which in the case $\boldsymbol{A}_{0302}$.

3.3.7. $\boldsymbol{A}_{07}$. For $(\theta_1,\theta_2)\in\boldsymbol{A}_{07}$ we have 4 available Type-II information ranges:

$$\left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right),\quad \left(\theta_2,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right), \tag{77}$$

$$\left(0,\ \frac{1}{6}(4-8\theta_1-7\theta_2)\right)\quad\text{and}\quad\left(2\theta_1+\theta_2-1,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right). \tag{78}$$

The first range comes from Lemma 2.2, the second comes from Lemma 3.2, and the third and fourth come from Lemma 3.3. Since we have

$$\frac{1}{6}(4-8\theta_1-7\theta_2)<0\quad\text{and}\quad 2\theta_1+\theta_2-1<0$$

in this region, (77)–(78) are equivalent to 3 Type-II information ranges

$$\left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right),\quad \left(\theta_2,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right)\quad\text{and}\quad\left(0,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right). \tag{79}$$

We divide $\boldsymbol{A}_{07}$ into 2 subregions based on the overlapping conditions of these ranges.

$$\boldsymbol{A}_{07}=\boldsymbol{A}_{0701}\cup\boldsymbol{A}_{0702},$$

where

$$\begin{aligned}
\boldsymbol{A}_{0701} = &\left\{(\theta_1,\theta_2): \frac{4}{9}<\theta_1\leqslant\frac{9}{20},\ \frac{1}{13}(10-20\theta_1)\leqslant\theta_2<\frac{1}{14}(5-8\theta_1)\right.\\
&\qquad\left.\text{or } \frac{9}{20}<\theta_1\leqslant\frac{5}{11},\ \frac{1}{13}(10-20\theta_1)\leqslant\theta_2<1-2\theta_1\right\},\\
\boldsymbol{A}_{0702} = &\left\{(\theta_1,\theta_2): \frac{4}{9}<\theta_1<\frac{9}{20},\ \frac{1}{14}(5-8\theta_1)\leqslant\theta_2<\frac{1}{20}(11-20\theta_1)\right\}.
\end{aligned}$$

In $\boldsymbol{A}_{0701}$ we have

$$\frac{1}{6}(5-8\theta_1-8\theta_2)>\theta_2.$$

Hence, the Type-II range for $\boldsymbol{A}_{0701}$ is

$$\left(0,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right) \cup \left(2\theta_1+2\theta_2-1,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right). \tag{80}$$

The decompositions are similar to which in the case $\boldsymbol{A}_{0302}$.

In $\boldsymbol{A}_{0702}$ we have

$$\frac{1}{6}(5-8\theta_1-8\theta_2)<\theta_2.$$

Hence, the Type-II range for $\boldsymbol{A}_{0702}$ is

$$\left(0,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right) \cup \left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right). \tag{81}$$

The decompositions are similar to which in the case $\boldsymbol{A}_{0303}$.

3.3.8. $\boldsymbol{A}_{08}$. For $(\theta_1,\theta_2)\in\boldsymbol{A}_{08}$ we have 3 available Type-II information ranges:

$$\left(\theta_2,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right),\quad \left(0,\ \frac{1}{6}(4-8\theta_1-7\theta_2)\right)\quad\text{and}\quad \left(2\theta_1+\theta_2-1,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right). \tag{82}$$

The first range comes from Lemma 3.2, and the second and third come from Lemma 3.3. Note that Lemma 2.2 becomes trivial in this region since $\theta_1+\theta_2\geqslant\frac{11}{20}$. Since we have

$$\frac{1}{6}(4-8\theta_1-7\theta_2)<0\quad\text{and}\quad 2\theta_1+\theta_2-1<0$$

in this region, (82) is equivalent to 2 Type-II information ranges

$$\left(\theta_2,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right)\quad\text{and}\quad \left(0,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right). \tag{83}$$

Hence, the Type-II range for $\boldsymbol{A}_{08}$ is

$$\left(0,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right)\cup\left(\theta_2,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right). \tag{84}$$

The decompositions are similar to which in the case $\boldsymbol{A}_{05}$.

3.3.9. $\boldsymbol{A}_{09}$. For $(\theta_1,\theta_2)\in\boldsymbol{A}_{09}$ we have 3 available Type-II information ranges:

$$\left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right),\quad \left(\theta_2,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right)\quad\text{and}\quad \left(0,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right). \tag{85}$$

The first range comes from Lemma 2.2, the second comes from Lemma 3.2, and the third comes from Lemma 3.3. Note that in $\boldsymbol{A}_{09}$ we have

$$2\theta_1+2\theta_2-1<\theta_2<\frac{1}{6}(5-8\theta_1-8\theta_2)<\frac{1}{3}(2-3\theta_1-3\theta_2).$$

Hence, the Type-II range for $\boldsymbol{A}_{09}$ is

$$\left(0,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right)\cup\left(2\theta_1+2\theta_2-1,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right). \tag{86}$$

We shall discuss the decompositions on this interval in next subsubsection, together with the case $(\theta_1,\theta_2)\in\boldsymbol{A}_{10}$.

3.3.10. $\boldsymbol{A}_{10}$. For $(\theta_1,\theta_2)\in\boldsymbol{A}_{10}$ we have 4 available Type-II information ranges:

$$\left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right),\quad \left(\theta_2,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right), \tag{87}$$

$$\left(0,\ \frac{1}{6}(4-8\theta_1-7\theta_2)\right)\quad\text{and}\quad \left(2\theta_1+\theta_2-1,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right). \tag{88}$$

The first range comes from Lemma 2.2, the second comes from Lemma 3.2, and the third and fourth come from Lemma 3.3. Since we have

$$\frac{1}{6}(4-8\theta_1-7\theta_2)<0\quad\text{and}\quad 2\theta_1+\theta_2-1<0$$

in this region, (87)–(88) are equivalent to 3 Type-II information ranges

$$\left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right),\quad \left(\theta_2,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right)\quad\text{and}\quad \left(0,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right). \tag{89}$$

Note that in $\boldsymbol{A}_{10}$ we have

$$2\theta_1+2\theta_2-1<\theta_2<\frac{1}{6}(5-8\theta_1-8\theta_2)<\frac{1}{3}(2-3\theta_1-3\theta_2).$$

Hence, the Type-II range for $\boldsymbol{A}_{10}$ is

$$\left(0,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right)\cup\left(2\theta_1+2\theta_2-1,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right). \tag{90}$$

In $\boldsymbol{A}_{09}\cup\boldsymbol{A}_{10}$, we can replace the value of $\kappa=\frac{5-8\theta}{6}-\varepsilon$ with $\frac{2-3\theta}{3}-\varepsilon$, and thus $\kappa>\frac{1}{7}$ when $\theta<\frac{11}{21}$.

Comparing to previous cases, another region $\boldsymbol{C}$ covers $\boldsymbol{A}_{09}\cup\boldsymbol{A}_{10}$ when $\theta_2<\frac{4-7\theta_1}{12}$. Thus, in $\boldsymbol{A}_{09}\cup\boldsymbol{A}_{10}$ we can use an extra Type-II range

$$(\theta_1,\ 1-\theta_1) \tag{91}$$

comes from Lemma 3.4 on those parts covered by $\boldsymbol{C}$. Specifically, this Type-II range (91) is helpful to give asymptotic formulas for parts of the "loss term"

$$\sum_{\substack{\kappa\leqslant\alpha_3<\alpha_1<\frac{\theta}{2}\\ 1-\theta\leqslant\alpha_1+v\leqslant\theta\\ (\alpha_1,v)\notin\boldsymbol{G}_2\\ \alpha_3<\min\left(v,\frac{1}{2}(1-\alpha_1-v)\right)\\ (\alpha_1,v,\alpha_3)\notin\boldsymbol{G}_3}} S\left(\mathcal{A}_{p_1mp_3}^{q_1q_2},p_3\right) \tag{92}$$

in Lemma 2.18 (with $\alpha_1+v\in(\theta_1,\ 1-\theta_1)$, for example) and parts of the high-dimensional sums with $\boldsymbol{\alpha}_2\in A\cup B$.

Another important tool we can use in $\boldsymbol{A}_{09}\cup\boldsymbol{A}_{10}$ is Lemma 3.8, which gives asymptotic formulas for any sum that counts numbers with more than 4 prime factors and all prime factors are larger than $x^{\frac{1}{7}}$. This lemma is very useful on estimating high-dimensional sums with $\boldsymbol{\alpha}_2 \in A \cup B$. Suppose that $7\theta_1 + 12\theta_2 < 4$ and $\theta < \frac{11}{21}$. After using Buchstab's identity twice on the sum

$$\sum_{\kappa \leqslant \alpha_1 \leqslant \frac{3}{7}} S\left(\mathcal{A}_{p_1}^{q_1 q_2}, p_1\right), \tag{93}$$

we get

$$\sum_{\substack{\kappa \leqslant \alpha_1 \leqslant \frac{3}{7} \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \kappa \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right)}} S\left(\mathcal{A}_{p_1 p_2 p_3}^{q_1 q_2}, p_3\right). \tag{94}$$

We know that sum (94) only counts numbers with 4 or more prime factors. Since $\kappa > \frac{1}{7}$, we can use Lemma 3.8 to give an asymptotic formula. This process helps us reduce the total loss a lot.

3.3.11. $\boldsymbol{A}_{11}$. For $(\theta_1, \theta_2) \in \boldsymbol{A}_{11}$ we have 3 available Type-II information ranges:

$$\left(2\theta_1 + 2\theta_2 - 1,\ \frac{1}{6}(5 - 8\theta_1 - 8\theta_2)\right),\ \left(2\theta_1 + 2\theta_2 - 1,\ \frac{1}{3}(2 - 3\theta_1 - 3\theta_2)\right) \quad \text{and} \quad \left(2\theta_1 + \theta_2 - 1,\ \frac{1}{6}(4 - 7\theta_1 - 8\theta_2)\right). \tag{95}$$

The first range comes from Lemma 2.2, the second comes from Lemma 3.2, and the third comes from Lemma 3.3. We divide $\boldsymbol{A}_{11}$ into 2 subregions based on the overlapping conditions of these ranges.

$$\boldsymbol{A}_{11} = \boldsymbol{A}_{1101} \cup \boldsymbol{A}_{1102},$$

where

$$\begin{aligned}
\boldsymbol{A}_{1101} &= \left\{(\theta_1, \theta_2) : \frac{10}{21} < \theta_1 < \frac{1}{2},\ 1 - 2\theta_1 \leqslant \theta_2 < \frac{1}{20}(10 - 19\theta_1)\right\}, \\
\boldsymbol{A}_{1102} &= \left\{(\theta_1, \theta_2) : \frac{9}{20} < \theta_1 \leqslant \frac{23}{50},\ 1 - 2\theta_1 \leqslant \theta_2 < \frac{1}{20}(11 - 20\theta_1)\right. \\
&\qquad \text{or } \frac{23}{50} < \theta_1 < \frac{10}{21},\ 1 - 2\theta_1 \leqslant \theta_2 < \frac{1}{14}(10 - 19\theta_1) \\
&\qquad \left. \text{or } \frac{10}{21} \leqslant \theta_1 < \frac{1}{2},\ \frac{1}{20}(10 - 19\theta_1) < \theta_2 < \frac{1}{14}(10 - 19\theta_1)\right\}.
\end{aligned}$$

Note that we have $\frac{1}{6}(5 - 8\theta_1 - 8\theta_2) < \frac{1}{3}(2 - 3\theta_1 - 3\theta_2)$ for $(\theta_1, \theta_2) \in \boldsymbol{A}_{11}$ and $\theta < \frac{10}{19} < \frac{17}{32}$ for $(\theta_1, \theta_2) \in \boldsymbol{A}_{1101}$.

In $\boldsymbol{A}_{1101}$ we have

$$\frac{1}{6}(4 - 7\theta_1 - 8\theta_2) > 2\theta_1 + 2\theta_2 - 1.$$

Hence, the Type-II range for $\boldsymbol{A}_{1101}$ is

$$\left(2\theta_1 + \theta_2 - 1,\ \frac{1}{3}(2 - 3\theta_1 - 3\theta_2)\right). \tag{96}$$

The decompositions are similar to which in the case $\boldsymbol{A}_{10}$. A modification on the decompositions in this case is that we want to relax the condition $(\alpha_1, \ldots, \alpha_j, 2\theta - 1 + \varepsilon) \in \boldsymbol{S}_{j+1}$ in the definition of $\boldsymbol{U}_j$. Since the "left endpoint" of our Type-II interval now becomes $2\theta_1 + \theta_2 - 1$, we want to prove a variant of Lemma 2.13, where $(\alpha_1, \ldots, \alpha_j, 2\theta - 1 + \varepsilon) \in \boldsymbol{S}_{j+1}$ was replaced by a new condition $(\alpha_1, \ldots, \alpha_j, 2\theta_1 + \theta_2 - 1 + \varepsilon) \in \boldsymbol{S}_{j+1}$ when $(\theta_1, \theta_2) \in \boldsymbol{A}_{1101}$. Note that we have done similar things in case $\boldsymbol{A}_{01}$, where that condition was replaced by $(\alpha_1, \ldots, \alpha_j, 0) \in \boldsymbol{S}_{j+1}$ (equivalent to $(\alpha_1, \ldots, \alpha_j) \in \boldsymbol{S}_j$). Define

$$\boldsymbol{U}_j''(\theta_1, \theta_2) = \left\{\boldsymbol{\alpha}_j : \boldsymbol{\alpha}_j \in \boldsymbol{A}_j,\ (\alpha_1, \ldots, \alpha_j, 2\theta_1 + \theta_2 - 1 + \varepsilon) \in \boldsymbol{S}_{j+1}\right\}. \tag{97}$$

Now we prove that for almost all q_1, q_2,

$$\sum_{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j''} S\left(\mathcal{A}_{p_1 \cdots p_j}^{q_1 q_2}, x^{\frac{2-3\theta}{3} - \varepsilon}\right)$$

has an asymptotic formula of the form (46) when $(\theta_1, \theta_2) \in \boldsymbol{A}_{1101}$. Following the steps in the proof of [[2], Lemma 15], we have

$$\begin{aligned}
\sum_{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j''} S\left(\mathcal{A}_{p_1 \cdots p_j}^{q_1 q_2}, x^{\frac{2-3\theta}{3} - \varepsilon}\right) &= \sum_{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j''} S\left(\mathcal{A}_{p_1 \cdots p_j}^{q_1 q_2}, x^{\varepsilon^2}\right) - \sum_{\substack{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j'' \\ \boldsymbol{\alpha}_{j+1} \in \boldsymbol{A}_{j+1} \\ 2\theta_1 + \theta_2 - 1 + \varepsilon \leqslant \alpha_{j+1} < \frac{2-3\theta}{3} - \varepsilon}} S\left(\mathcal{A}_{p_1 \cdots p_{j+1}}^{q_1 q_2}, p_{j+1}\right) \\
&\quad - \sum_{\substack{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j'' \\ \boldsymbol{\alpha}_{j+1} \in \boldsymbol{A}_{j+1} \\ \alpha_{j+1} < 2\theta_1 + \theta_2 - 1 + \varepsilon}} S\left(\mathcal{A}_{p_1 \cdots p_{j+1}}^{q_1 q_2}, p_{j+1}\right) \\
&= \sum_{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j''} S\left(\mathcal{A}_{p_1 \cdots p_j}^{q_1 q_2}, x^{\varepsilon^2}\right) - \sum_{\substack{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j'' \\ \boldsymbol{\alpha}_{j+1} \in \boldsymbol{A}_{j+1} \\ 2\theta_1 + \theta_2 - 1 + \varepsilon \leqslant \alpha_{j+1} < \frac{2-3\theta}{3} - \varepsilon}} S\left(\mathcal{A}_{p_1 \cdots p_{j+1}}^{q_1 q_2}, p_{j+1}\right)
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\substack{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j'' \\ \boldsymbol{\alpha}_{j+1} \in \boldsymbol{A}_{j+1} \\ \alpha_{j+1} < 2\theta_1+\theta_2-1+\varepsilon}} S\left(\mathcal{A}_{p_1 \cdots p_{j+1}}^{q_1 q_2}, x^{\varepsilon^2}\right) \\
&+ \sum_{\substack{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j'' \\ \boldsymbol{\alpha}_{j+2} \in \boldsymbol{A}_{j+2} \\ \alpha_{j+1} < \frac{2-3\theta}{3}-\varepsilon \\ \alpha_{j+1} < 2\theta_1+\theta_2-1+\varepsilon \leqslant \alpha_{j+1}+\alpha_{j+2}}} S\left(\mathcal{A}_{p_1 \cdots p_{j+2}}^{q_1 q_2}, p_{j+2}\right) \\
&+ \sum_{\substack{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j'' \\ \boldsymbol{\alpha}_{j+2} \in \boldsymbol{A}_{j+2} \\ \alpha_{j+1} < \frac{2-3\theta}{3}-\varepsilon \\ \alpha_{j+1}+\alpha_{j+2} < 2\theta_1+\theta_2-1+\varepsilon}} S\left(\mathcal{A}_{p_1 \cdots p_{j+2}}^{q_1 q_2}, p_{j+2}\right) \\
= &\sum_{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j''} S\left(\mathcal{A}_{p_1 \cdots p_j}^{q_1 q_2}, x^{\varepsilon^2}\right) - \sum_{\substack{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j'' \\ \boldsymbol{\alpha}_{j+1} \in \boldsymbol{A}_{j+1} \\ 2\theta_1+\theta_2-1+\varepsilon \leqslant \alpha_{j+1} < \frac{2-3\theta}{3}-\varepsilon}} S\left(\mathcal{A}_{p_1 \cdots p_{j+1}}^{q_1 q_2}, p_{j+1}\right) \\
&- \sum_{\substack{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j'' \\ \boldsymbol{\alpha}_{j+1} \in \boldsymbol{A}_{j+1} \\ \alpha_{j+1} < 2\theta_1+\theta_2-1+\varepsilon}} S\left(\mathcal{A}_{p_1 \cdots p_{j+1}}^{q_1 q_2}, x^{\varepsilon^2}\right) \\
&+ \sum_{\substack{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j'' \\ \boldsymbol{\alpha}_{j+2} \in \boldsymbol{A}_{j+2} \\ \alpha_{j+1} < \frac{2-3\theta}{3}-\varepsilon \\ \alpha_{j+1} < 2\theta_1+\theta_2-1+\varepsilon \leqslant \alpha_{j+1}+\alpha_{j+2}}} S\left(\mathcal{A}_{p_1 \cdots p_{j+2}}^{q_1 q_2}, p_{j+2}\right) \\
&+ \sum_{\substack{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j'' \\ \boldsymbol{\alpha}_{j+2} \in \boldsymbol{A}_{j+2} \\ \alpha_{j+1} < \frac{2-3\theta}{3}-\varepsilon \\ \alpha_{j+1}+\alpha_{j+2} < 2\theta_1+\theta_2-1+\varepsilon}} S\left(\mathcal{A}_{p_1 \cdots p_{j+2}}^{q_1 q_2}, x^{\varepsilon^2}\right) \\
&- \sum_{\substack{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j'' \\ \boldsymbol{\alpha}_{j+3} \in \boldsymbol{A}_{j+3} \\ \alpha_{j+1} < \frac{2-3\theta}{3}-\varepsilon \\ \alpha_{j+1}+\alpha_{j+2} < 2\theta_1+\theta_2-1+\varepsilon \leqslant \alpha_{j+1}+\alpha_{j+2}+\alpha_{j+3}}} S\left(\mathcal{A}_{p_1 \cdots p_{j+3}}^{q_1 q_2}, p_{j+3}\right) - \cdots \\
= &\sum_{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j''} S\left(\mathcal{A}_{p_1 \cdots p_j}^{q_1 q_2}, x^{\varepsilon^2}\right) - \sum_{\substack{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j'' \\ \boldsymbol{\alpha}_{j+1} \in \boldsymbol{A}_{j+1} \\ 2\theta_1+\theta_2-1+\varepsilon \leqslant \alpha_{j+1} < \frac{2-3\theta}{3}-\varepsilon}} S\left(\mathcal{A}_{p_1 \cdots p_{j+1}}^{q_1 q_2}, p_{j+1}\right) \\
&+ \sum_{k \geqslant j+1} (-1)^{k-j} \sum_{\substack{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j'' \\ \boldsymbol{\alpha}_k \in \boldsymbol{A}_k \\ \alpha_{j+1} < \frac{2-3\theta}{3}-\varepsilon \\ \alpha_{j+1}+\cdots+\alpha_k < 2\theta_1+\theta_2-1+\varepsilon}} S\left(\mathcal{A}_{p_1 \cdots p_k}^{q_1 q_2}, x^{\varepsilon^2}\right) \\
&+ \sum_{k \geqslant j+2} (-1)^{k-j} \sum_{\substack{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j'' \\ \boldsymbol{\alpha}_k \in \boldsymbol{A}_k \\ \alpha_{j+1} < \frac{2-3\theta}{3}-\varepsilon \\ \alpha_{j+1}+\cdots+\alpha_{k-1} < 2\theta_1+\theta_2-1+\varepsilon \leqslant \alpha_{j+1}+\cdots+\alpha_k}} S\left(\mathcal{A}_{p_1 \cdots p_k}^{q_1 q_2}, p_k\right) \\
= &S_{33111} - S_{33112} + S_{33113} + S_{33114}. \qquad (98)
\end{aligned}
$$

Our Type-II information yields an asymptotic formula for S_{33112}. We can use Lemma 2.12 to give asymptotic formulas for S_{33111} and S_{33113}. For S_{33114}, we need to prove a variant of [[2], Lemma 11]: Let $\theta < \frac{17}{32}$ and $\boldsymbol{\alpha}_k \in \boldsymbol{A}_k$. Suppose that

$$\alpha_{j+1} + \cdots + \alpha_{k-1} < 2\theta_1 + \theta_2 - 1 + \varepsilon \leqslant \alpha_{j+1} + \cdots + \alpha_k$$

and

$$\alpha_{j+1} < \frac{2-3\theta}{3} - \varepsilon$$

for some j $(0 \leqslant j \leqslant k-1)$, then $\boldsymbol{\alpha}_k \in \boldsymbol{G}_k$.

When $\alpha_k < \frac{5-9\theta_1-6\theta_2}{3} - 2\varepsilon$, then

$$2\theta_1+\theta_2-1+\varepsilon \leqslant \alpha_{j+1}+\cdots+\alpha_k < (2\theta_1+\theta_2-1+\varepsilon)+\frac{5-9\theta_1-6\theta_2}{3}-2\varepsilon = \frac{2-3\theta}{3}-\varepsilon$$

and $\boldsymbol{\alpha}_k \in \boldsymbol{G}_k$.

Suppose that $\alpha_k \geqslant \frac{5-9\theta_1-6\theta_2}{3} - 2\varepsilon$. Since $\boldsymbol{\alpha}_k \in \boldsymbol{A}_k$, we have $\alpha_k < \alpha_{j+1} < \frac{2-3\theta}{3} - \varepsilon$. Now we only need to prove that

$$\frac{5-9\theta_1-6\theta_2}{3}-2\varepsilon \geqslant 2\theta_1+\theta_2-1+\varepsilon,$$

or

$$15\theta_1+9\theta_2 \leqslant 8-9\varepsilon$$

when $(\theta_1,\theta_2) \in \boldsymbol{A}_{1101}$. A simple verification then completes the proof.

Now, we can replace all $\boldsymbol{U}_j$ with $\boldsymbol{U}_j''$ in the decompositions. Since the whole $\boldsymbol{A}_{1101}$ is covered by $\boldsymbol{C}$, we can use the Type-II range $(\theta_1,\ 1-\theta_1)$ from Lemma 3.4 to give extra asymptotic formulas. The new three-dimensional Harman's sieve is also applicable here if we have $(\theta_1,\theta_2) \in \boldsymbol{T}$.

One difference between the decompositions for $(\theta_1,\theta_2) \in \boldsymbol{A}_{1101}$ and $(\theta_1,\theta_2) \in \boldsymbol{A}_{10}$ is that we can use Lemma 3.8 to give lots of asymptotic formulas when $(\theta_1,\theta_2) \in \boldsymbol{A}_{10}$, but we cannot use Lemma 3.8 when $(\theta_1,\theta_2) \in \boldsymbol{A}_{1101}$ since $2\theta_1+\theta_2 \geqslant 1$.

If we can use Lemma 3.8, then we have

$$2\theta_1+\theta_2 < 1 \tag{99}$$

and

$$7\theta_1+12\theta_2 < 4. \tag{100}$$

When $\theta > \frac{1}{2}$, (100) also yields $(\theta_1,\theta_2) \in \boldsymbol{C}$, and we can use the extra Type-II information given by Lemma 3.4. Note that (99) means that $\theta_1 < 1-\theta$, the new Type-II range $(\theta_1,\ 1-\theta_1)$ covers the whole interval $\left(1-\theta,\ \frac{1}{2}\right)$. If we can use the new three-dimensional Harman's sieve with $\theta > \frac{16}{31}$, then the "left endpoint" of our Type-II range cannot be $2\theta-1+\varepsilon$, which means that we need Lemma 3.3 to enlarge our Type-II range. The requirement of that is

$$\frac{1}{6}(4-7\theta_1-8\theta_2) > 2\theta-1,$$

or

$$19\theta_1+20\theta_2 < 10. \tag{101}$$

The readers can find that (99)–(101) are the conditions (1.3)–(1.5) in [[28], Theorem 1.1]. In fact, the above process gives a proof of [[28], Theorem 1.1]. Since $\theta < \frac{11}{21}$ is a requirement of using the new three-dimensional Harman's sieve, we have $\frac{2-3\theta}{3} > \frac{1}{7}$ and the sums after using the new three-dimensional Harman's sieve will have more than 4 prime factors by earlier discussions. Lemma 3.4 gives an asymptotic formula for the case $1-\theta \leqslant \alpha_1 < \frac{1}{2}$, and the sums

$$S\left(\mathcal{A}^{q_1q_2}, x^{\frac{2-3\theta}{3}-\varepsilon}\right), \quad \sum_{\alpha_1 \leqslant \frac{3}{7}} S\left(\mathcal{A}^{q_1q_2}_{p_1}, x^{\frac{2-3\theta}{3}-\varepsilon}\right), \quad \sum_{\boldsymbol{\alpha}_2 \in A\cup B} S\left(\mathcal{A}^{q_1q_2}_{p_1p_2}, x^{\frac{2-3\theta}{3}-\varepsilon}\right)$$

after straightforward decompositions have asymptotic formulas by Lemma 2.12 and our Type-II range $\left(0,\ \frac{2-3\theta}{3}\right)$ under (99)–(101). The remaining sums (without the new three-dimensional Harman's sieve) can be dealt with Lemma 3.8 since they only count numbers with 4 or more prime factors.

In $\boldsymbol{A}_{1102}$ we have

$$\frac{1}{6}(4-7\theta_1-8\theta_2) < 2\theta_1+2\theta_2-1.$$

Hence, the Type-II range for $\boldsymbol{A}_{1102}$ is

$$\left(2\theta_1+\theta_2-1,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right) \cup \left(2\theta_1+2\theta_2-1,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right). \tag{102}$$

The decompositions are similar to which in the case $\boldsymbol{A}_{10}$. We can still use the Type-II range $(\theta_1,\ 1-\theta_1)$ from Lemma 3.4 to give extra asymptotic formulas for parts of $\boldsymbol{A}_{1102}$.

3.3.12. $\boldsymbol{A}_{12}$. For $(\theta_1,\theta_2) \in \boldsymbol{A}_{12}$ we have 2 available Type-II information ranges:

$$\left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right), \quad \text{and} \quad \left(2\theta_1+2\theta_2-1,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right). \tag{103}$$

The first range comes from Lemma 2.2, and the second comes from Lemma 3.2. Note that in $\boldsymbol{A}_{12}$ we have

$$\frac{1}{6}(5-8\theta_1-8\theta_2) < \frac{1}{3}(2-3\theta_1-3\theta_2).$$

Hence, the Type-II range for $\boldsymbol{A}_{12}$ is

$$\left(2\theta_1+2\theta_2-1,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right). \tag{104}$$

The decompositions are similar to which in the case $\boldsymbol{A}_{10}$. We can still use the Type-II range $(\theta_1,\ 1-\theta_1)$ from Lemma 3.4 to give extra asymptotic formulas for parts of $\boldsymbol{A}_{12}$.

3.3.13. $\boldsymbol{A}_{13}$. For $(\theta_1,\theta_2)\in\boldsymbol{A}_{13}$ we have 2 available Type-II information ranges:

$$\left(2\theta_1+2\theta_2-1,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right)\quad\text{and}\quad\left(2\theta_1+\theta_2-1,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right). \tag{105}$$

The first range comes from Lemma 3.2, and the second comes from Lemma 3.3. Note that Lemma 2.2 becomes trivial in this region since $\theta_1+\theta_2\geqslant\frac{11}{20}$. Hence, the Type-II range for $\boldsymbol{A}_{13}$ is

$$\left(2\theta_1+\theta_2-1,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right)\cup\left(2\theta_1+2\theta_2-1,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right). \tag{106}$$

The decompositions are similar to which in the case $\boldsymbol{A}_{05}$.

3.3.14. $\boldsymbol{A}_{14}$. For $(\theta_1,\theta_2)\in\boldsymbol{A}_{14}$, we only have 1 available Type-II information range comes from Lemma 3.2:

$$\left(2\theta_1+2\theta_2-1,\ \frac{1}{3}(2-3\theta_1-3\theta_2)\right). \tag{107}$$

Again, the decompositions are similar to which in the case $\boldsymbol{A}_{05}$.

3.3.15. $\boldsymbol{A}_{15}$. For $(\theta_1,\theta_2)\in\boldsymbol{A}_{15}$ we have 3 available Type-II information ranges:

$$\left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right),\ \left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-6\theta_2)\right)\quad\text{and}\quad\left(2\theta_1+\theta_2-1,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right). \tag{108}$$

The first range comes from Lemma 2.2, the second comes from Lemma 3.2, and the third comes from Lemma 3.3. We divide $\boldsymbol{A}_{15}$ into 2 subregions based on the overlapping conditions of these ranges.

$$\boldsymbol{A}_{15}=\boldsymbol{A}_{1501}\cup\boldsymbol{A}_{1502},$$

where

$$\begin{aligned}\boldsymbol{A}_{1501}&=\left\{(\theta_1,\theta_2):\frac{1}{2}\leqslant\theta_1<\frac{10}{19},\ 0<\theta_2<\frac{1}{20}(10-19\theta_1)\right\},\\ \boldsymbol{A}_{1502}&=\left\{(\theta_1,\theta_2):\frac{1}{2}\leqslant\theta_1<\frac{10}{19},\ \frac{1}{20}(10-19\theta_1)\leqslant\theta_2<\frac{1}{14}(10-19\theta_1)\right\}.\end{aligned}$$

Note that we have $\theta<\frac{10}{19}<\frac{17}{32}$ for $(\theta_1,\theta_2)\in\boldsymbol{A}_{1501}$.

In $\boldsymbol{A}_{1501}$ we have

$$\frac{1}{6}(4-7\theta_1-8\theta_2)>2\theta_1+2\theta_2-1.$$

Hence, the Type-II range for $\boldsymbol{A}_{1501}$ is

$$\left(2\theta_1+\theta_2-1,\ \frac{1}{6}(5-8\theta_1-6\theta_2)\right). \tag{109}$$

The decompositions are similar to which in the case $\boldsymbol{A}_{1101}$. We want to replace $\boldsymbol{U}_j$ with $\boldsymbol{U}_j''$ in the decompositions as in $\boldsymbol{A}_{1101}$. Thus, we need to prove that for almost all q_1,q_2,

$$\sum_{\boldsymbol{\alpha}_j\in\boldsymbol{U}_j''}S\left(\mathcal{A}_{p_1\cdots p_j}^{q_1q_2},x^{\frac{5-8\theta_1-6\theta_2}{6}-\varepsilon}\right)$$

has an asymptotic formula of the form (46) when $(\theta_1,\theta_2)\in\boldsymbol{A}_{1501}$. Similar to the case $(\theta_1,\theta_2)\in\boldsymbol{A}_{1101}$, we have

$$\begin{aligned}\sum_{\boldsymbol{\alpha}_j\in\boldsymbol{U}_j''}S\left(\mathcal{A}_{p_1\cdots p_j}^{q_1q_2},x^{\frac{5-8\theta_1-6\theta_2}{6}-\varepsilon}\right)=&\sum_{\boldsymbol{\alpha}_j\in\boldsymbol{U}_j''}S\left(\mathcal{A}_{p_1\cdots p_j}^{q_1q_2},x^{\varepsilon^2}\right)\\ &-\sum_{\substack{\boldsymbol{\alpha}_j\in\boldsymbol{U}_j''\\ \boldsymbol{\alpha}_{j+1}\in\boldsymbol{A}_{j+1}\\ 2\theta_1+\theta_2-1+\varepsilon\leqslant\alpha_{j+1}<\frac{5-8\theta_1-6\theta_2}{6}-\varepsilon}}S\left(\mathcal{A}_{p_1\cdots p_{j+1}}^{q_1q_2},p_{j+1}\right)\\ &+\sum_{k\geqslant j+1}(-1)^{k-j}\sum_{\substack{\boldsymbol{\alpha}_j\in\boldsymbol{U}_j''\\ \boldsymbol{\alpha}_k\in\boldsymbol{A}_k\\ \alpha_{j+1}<\frac{5-8\theta_1-6\theta_2}{6}-\varepsilon\\ \alpha_{j+1}+\cdots+\alpha_k<2\theta_1+\theta_2-1+\varepsilon}}S\left(\mathcal{A}_{p_1\cdots p_k}^{q_1q_2},x^{\varepsilon^2}\right)\\ &+\sum_{k\geqslant j+2}(-1)^{k-j}\sum_{\substack{\boldsymbol{\alpha}_j\in\boldsymbol{U}_j''\\ \boldsymbol{\alpha}_k\in\boldsymbol{A}_k\\ \alpha_{j+1}<\frac{5-8\theta_1-6\theta_2}{6}-\varepsilon\\ \alpha_{j+1}+\cdots+\alpha_{k-1}<2\theta_1+\theta_2-1+\varepsilon\leqslant\alpha_{j+1}+\cdots+\alpha_k}}S\left(\mathcal{A}_{p_1\cdots p_k}^{q_1q_2},p_k\right)\\ =&\ S_{33151}-S_{33152}+S_{33153}+S_{33154}.\end{aligned} \tag{110}$$

Our Type-II information yields an asymptotic formula for S_{33152}. We can use Lemma 2.12 to give asymptotic formulas for S_{33151} and S_{33153}. Now we only need to show that S_{33114} has an asymptotic formula. Let $\theta<\frac{17}{32}$ and $\boldsymbol{\alpha}_k\in\boldsymbol{A}_k$. Suppose that

$$\alpha_{j+1}+\cdots+\alpha_{k-1}<2\theta_1+\theta_2-1+\varepsilon\leqslant\alpha_{j+1}+\cdots+\alpha_k$$

and

$$\alpha_{j+1} < \frac{5-8\theta_1-6\theta_2}{6} - \varepsilon$$

for some j $(0 \leqslant j \leqslant k-1)$, then $\boldsymbol{\alpha}_k \in \boldsymbol{G}_k$.

When $\alpha_k < \frac{11-20\theta_1-12\theta_2}{6} - 2\varepsilon$, then

$$2\theta_1+\theta_2-1+\varepsilon \leqslant \alpha_{j+1}+\cdots+\alpha_k < (2\theta_1+\theta_2-1+\varepsilon) + \frac{11-20\theta_1-12\theta_2}{6} - 2\varepsilon = \frac{5-8\theta_1-6\theta_2}{6} - \varepsilon$$

and $\boldsymbol{\alpha}_k \in \boldsymbol{G}_k$.

Suppose that $\alpha_k \geqslant \frac{11-20\theta_1-12\theta_2}{6} - 2\varepsilon$. Since $\boldsymbol{\alpha}_k \in \boldsymbol{A}_k$, we have $\alpha_k < \alpha_{j+1} < \frac{5-8\theta_1-6\theta_2}{6} - \varepsilon$. Now we only need to prove that

$$\frac{11-20\theta_1-12\theta_2}{6} - 2\varepsilon \geqslant 2\theta_1+\theta_2-1+\varepsilon,$$

or

$$32\theta_1+18\theta_2 \leqslant 17-18\varepsilon$$

when $(\theta_1,\theta_2) \in \boldsymbol{A}_{1501}$. A simple verification then completes the proof.

Since $\theta_1 \geqslant \frac{1}{2}$, Lemma 3.4 is not applicable here. However, we can still use the new three-dimensional Harman's sieve for parts of $\boldsymbol{A}_{1501}$ covered by $\boldsymbol{T}$.

In $\boldsymbol{A}_{1502}$ we have

$$\frac{1}{6}(4-7\theta_1-8\theta_2) < 2\theta_1+2\theta_2-1.$$

Hence, the Type-II range for $\boldsymbol{A}_{1502}$ is

$$\left(2\theta_1+\theta_2-1,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right) \cup \left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-6\theta_2)\right). \tag{111}$$

The decompositions are similar to which in the case $\boldsymbol{A}_{0302}$.

3.3.16. $\boldsymbol{A}_{16}$. For $(\theta_1,\theta_2) \in \boldsymbol{A}_{16}$ we have 2 available Type-II information ranges:

$$\left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right) \quad \text{and} \quad \left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-6\theta_2)\right). \tag{112}$$

The first range comes from Lemma 2.2, and the second comes from Lemma 3.2. Since $\theta_2 > 0$, the Type-II range for $\boldsymbol{A}_{16}$ is

$$\left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-6\theta_2)\right). \tag{113}$$

The decompositions are similar to which in the case $\boldsymbol{A}_{1502}$.

3.3.17. $\boldsymbol{A}_{17}$. For $(\theta_1,\theta_2) \in \boldsymbol{A}_{17}$, we only have 1 available Type-II information range comes from Lemma 3.2:

$$\left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-6\theta_2)\right). \tag{114}$$

The decompositions are similar to which in the case $\boldsymbol{A}_{05}$.

Next we assume that $(\theta_1,\theta_2) \in \boldsymbol{B}\backslash\boldsymbol{A}$. From the condition $Q_1^8 Q_2^7 M_2^6 < x^{4-13\varepsilon}$ in Lemma 3.3, we can find that the Type-II ranges generated by Lemma 3.3 are of the form

$$\left(\cdots,\ \frac{1}{6}(4-7\theta_1-8\theta_2)\right) \quad \text{and} \quad \left(\cdots,\ \frac{1}{6}(4-8\theta_1-7\theta_2)\right), \tag{115}$$

where the "left endpoint" may be 0 or $2\theta_1+\theta_2-1$. Since Lemma 3.2 and Lemma 3.4 are not applicable in this case (we have $\boldsymbol{C} \subset \boldsymbol{A}$), the only situation that Lemma 3.3 brings improvements is that we have

$$\frac{1}{6}(4-7\theta_1-8\theta_2) > 2\theta-1 \quad \text{or} \quad \frac{1}{6}(4-8\theta_1-7\theta_2) > 2\theta-1. \tag{116}$$

In this case, we can make the "left endpoint" of our Type-II interval smaller, hence to relax the condition $(\alpha_1,\ldots,\alpha_j,2\theta-1+\varepsilon) \in \boldsymbol{S}_{j+1}$ in the definition of $\boldsymbol{U}_j$. We may replace $\boldsymbol{U}_j$ with $\boldsymbol{S}_j$ or $\boldsymbol{U}_j''$ when the "left endpoint" becomes 0 or $2\theta_1+\theta_2-1$. After checking the conditions, we find that a subregion of $\boldsymbol{B}\backslash\boldsymbol{A}$ that satisfies (116) is

$$\begin{aligned}\boldsymbol{B}_{01} = \Bigg\{(\theta_1,\theta_2) :\ & \frac{1}{4} < \theta_1 < \frac{10}{39},\ \frac{1}{2}(1-2\theta_1) < \theta_2 \leqslant \theta_1 \\ \text{or }\ & \frac{10}{39} \leqslant \theta_1 \leqslant \frac{5}{14},\ \frac{1}{2}(1-2\theta_1) < \theta_2 < \frac{1}{20}(10-19\theta_1) \\ \text{or }\ & \frac{5}{14} < \theta_1 < \frac{50}{131},\ \frac{1}{9}(2-2\theta_1) < \theta_2 < \frac{1}{20}(10-19\theta_1)\Bigg\}.\end{aligned}$$

In this region, we have a Type-II range

$$\left(0,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right). \tag{117}$$

The decompositions are similar to which in the case $\boldsymbol{A}_{0101}$. The decompositions in remaining parts of $\boldsymbol{B}\backslash\boldsymbol{A}$ stay the same as in [23].

Working on each case above, we can get the following upper bounds for $C_1(\theta_1,\theta_2)$ $(0.5 < \theta \leqslant 0.55)$:

0.25	1.0150	1.6320	1.8169	1.9993	2.3996	—	—	—	—	—	—	—	—	—
0.24	$1+\varepsilon$	1.0150	1.6320	1.8169	1.9993	2.3996	—	—	—	—	—	—	—	—
0.23	1	$1+\varepsilon$	1.0150	1.6320	1.8169	1.9993	2.3996	—	—	—	—	—	—	—
0.22	1	1	$1+\varepsilon$	1.0150	1.6320	1.8169	1.9993	2.3996	—	—	—	—	—	—
0.21	1	1	1	$1+\varepsilon$	1.0150	1.6320	1.8169	1.9993	2.3996	—	—	—	—	—
0.20	1	1	1	1	$1+\varepsilon$	1.0150	1.6320	1.8169	1.9993	2.3996	—	—	—	—
0.19	1	1	1	1	1	$1+\varepsilon$	1.0150	1.6320	1.8169	1.9993	2.3996	—	—	—
0.18	1	1	1	1	1	1	$1+\varepsilon$	1.0150	1.6320	1.8169	1.9993	2.3996	—	—
0.17	1	1	1	1	1	1	1	$1+\varepsilon$	1.0150	1.6320	1.8169	1.9993	2.3996	—
0.16	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.0150	1.6320	1.8169	1.9993	2.3996
0.15	1	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.0150	1.6320	1.8169	1.9993
0.14	1	1	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.0150	1.6320	1.8169
0.13	1	1	1	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.0150	1.6320
0.12	1	1	1	1	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.0150
$\theta_2\backslash\theta_1$	**0.26**	**0.27**	**0.28**	**0.29**	**0.30**	**0.31**	**0.32**	**0.33**	**0.34**	**0.35**	**0.36**	**0.37**	**0.38**	**0.39**

Table 3.1: Upper Bounds for $C_1(\theta_1, \theta_2)$ $(0.5 < \theta \leqslant 0.55)$ 1/2

0.15	2.3996	—	—	—	—	—	—	—	—	—	—	—	—	—	—
0.14	1.9993	2.3996	—	—	—	—	—	—	—	—	—	—	—	—	—
0.13	1.8169	1.9993	2.3996	—	—	—	—	—	—	—	—	—	—	—	—
0.12	1.6320	1.8120	1.9851	2.3996	—	—	—	—	—	—	—	—	—	—	—
0.11	1.0150	1.0790	1.8120	1.9486	2.3852	—	—	—	—	—	—	—	—	—	—
0.10	$1+\varepsilon$	1.0150	1.0764	1.8120	1.9486	2.3511	—	—	—	—	—	—	—	—	—
0.09	1	1	1.0150	1.0751	1.8120	1.9486	2.3511	—	—	—	—	—	—	—	—
0.08	1	1	1	1	1.0739	1.8120	1.9486	2.3511	—	—	—	—	—	—	—
0.07	1	1	1	1	1	1	1.8120	1.9486	2.3511	—	—	—	—	—	—
0.06	1	1	1	1	1	1	1	1.8120	1.9486	2.3511	—	—	—	—	—
0.05	1	1	1	1	1	1	1	1	1.6992	1.9486	2.3511	—	—	—	—
0.04	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.7406	1.9486	2.3549	—	—	—
0.03	1	1	1	1	1	1	1	1	1	1.0228	1.8120	1.9615	2.3648	—	—
0.02	1	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.0743	1.8169	1.9765	2.3928	—
0.01	1	1	1	1	1	1	1	1	1	1	1.0141	1.6301	1.8169	1.9893	2.3935
$\theta_2\backslash\theta_1$	**0.40**	**0.41**	**0.42**	**0.43**	**0.44**	**0.45**	**0.46**	**0.47**	**0.48**	**0.49**	**0.50**	**0.51**	**0.52**	**0.53**	**0.54**

Table 3.2: Upper Bounds for $C_1(\theta_1, \theta_2)$ $(0.5 < \theta \leqslant 0.55)$ 2/2

3.4. **Lower Bounds.** We shall construct the minorant $\rho_0(n)$ in this subsection. Before constructing, we first mention some existing results of $C_0(\theta_1, \theta_2)$.

Theorem 3.10. *The function $C_0(\theta_1, \theta_2)$ satisfies the following conditions:*

(1). $C_0(\theta_1, \theta_2) = C_0(\theta_2, \theta_1)$;
(2). $C_0(\theta_1, \theta_2) = 1$ for all θ_1, θ_2 satisfy $\theta_1 + \theta_2 < 0.5$;
(3). $C_0(\theta_1, \theta_2) = 1$ for all θ_1, θ_2 satisfy $2\theta_1 + \theta_2 < 1$, $7\theta_1 + 12\theta_2 < 4$ and $19\theta_1 + 20\theta_2 < 10$;
(4). $C_0(\theta_1, \theta_2) \geqslant C_0(\theta_1 + \theta_2)$ for $0.5 \leqslant \theta_1 + \theta_2 \leqslant 1$;
(5). $C_0(\theta_1, \theta_2) \geqslant 1 - \varepsilon$ for all θ_1, θ_2 satisfy $\theta_1 + \theta_2 = 0.5$.

Proof. The first statement is obvious. The second and third statements follow easily from the Bombieri–Vinogradov Theorem and [[28], Theorem 1.1]. The fourth statement holds trivially by the work done in Section 2. When there are no new arithmetic information inputs outside of those in Section 2, we use $C_0(\theta_1 + \theta_2)$ as a lower bound for $C_0(\theta_1, \theta_2)$. The fifth statement holds from the fourth statement and statement (1) of Theorem 2.23. ☐

In this subsection we assume that $\theta < \frac{17}{32}$. We still use two different methods to construct $\rho_0(n)$: The first is Harman's sieve, and the second is due to Mikawa [32].

3.4.1. *First Method.* The first method is to use Harman's sieve to construct $\rho_0(n)$. Again, we can only discard positive terms that do not have asymptotic formulas in this case. The main steps remain the same as in Subsubsection 2.5.1, but now we can use the new Type-II information corresponding to different (θ_1, θ_2) given in Subsection 3.3. Modifications need to do in the lower bound case are similar to those in the upper bound case; However, we do not need to consider the validity of three-dimensional sieves (Lemma 2.20, Lemma 2.21 and new three-dimensional Harman's sieve given in the upper bound subsection) since they only give upper bounds for positive terms. Lemma 3.8 is still applicable for parts of $\boldsymbol{A}_{09}$ and $\boldsymbol{A}_{10}$, and we can apply Lemma 3.4 in parts of $\boldsymbol{A}_{09}$–$\boldsymbol{A}_{12}$. Working on each region and subregion carefully, we can obtain the following lower bounds for $C_0(\theta_1, \theta_2)$ $(0.5 < \theta \leqslant 0.53)$:

0.25	0.6857	0.3890	-0.10	—	—	—	—	—	—	—	—	—	—	—
0.24	$1-\varepsilon$	0.6857	0.3890	-0.10	—	—	—	—	—	—	—	—	—	—
0.23	1	$1-\varepsilon$	0.6857	0.3890	-0.10	—	—	—	—	—	—	—	—	—
0.22	1	1	$1-\varepsilon$	0.6857	0.3890	-0.10	—	—	—	—	—	—	—	—
0.21	1	1	1	$1-\varepsilon$	0.6857	0.3890	-0.10	—	—	—	—	—	—	—
0.20	1	1	1	1	$1-\varepsilon$	0.6857	0.3890	-0.10	—	—	—	—	—	—
0.19	1	1	1	1	1	$1-\varepsilon$	0.6857	0.3890	-0.10	—	—	—	—	—
0.18	1	1	1	1	1	1	$1-\varepsilon$	0.6857	0.3890	-0.10	—	—	—	—
0.17	1	1	1	1	1	1	1	$1-\varepsilon$	0.6857	0.3890	-0.10	—	—	—
0.16	1	1	1	1	1	1	1	1	$1-\varepsilon$	0.6857	0.3890	-0.10	—	—
0.15	1	1	1	1	1	1	1	1	1	$1-\varepsilon$	0.6857	0.3890	-0.10	—
0.14	1	1	1	1	1	1	1	1	1	1	$1-\varepsilon$	0.6857	0.3890	-0.10
0.13	1	1	1	1	1	1	1	1	1	1	1	$1-\varepsilon$	0.6857	0.3890
0.12	1	1	1	1	1	1	1	1	1	1	1	1	$1-\varepsilon$	0.6857
$\boldsymbol{\theta_2\backslash\theta_1}$	**0.26**	**0.27**	**0.28**	**0.29**	**0.30**	**0.31**	**0.32**	**0.33**	**0.34**	**0.35**	**0.36**	**0.37**	**0.38**	**0.39**

Table 3.3: Lower Bounds for $C_0(\theta_1,\theta_2)$ (First Method, $0.5<\theta\leqslant 0.53$) 1/2

0.13	-0.08	—	—	—	—	—	—	—	—	—	—	—	—
0.12	0.3890	0.0450	—	—	—	—	—	—	—	—	—	—	—
0.11	0.6857	0.4072	0.0450	—	—	—	—	—	—	—	—	—	—
0.10	$1-\varepsilon$	0.6857	0.4192	0.0450	—	—	—	—	—	—	—	—	—
0.09	1	1	0.6857	0.4394	0.0450	—	—	—	—	—	—	—	—
0.08	1	1	1	1	0.4409	0.0450	—	—	—	—	—	—	—
0.07	1	1	1	1	1	1	0.0450	—	—	—	—	—	—
0.06	1	1	1	1	1	1	1	0.0450	—	—	—	—	—
0.05	1	1	1	1	1	1	1	1	0.2431	—	—	—	—
0.04	1	1	1	1	1	1	1	1	0.6042	0.1463	—	—	—
0.03	1	1	1	1	1	1	1	1	1	0.5227	0.0450	—	—
0.02	1	1	1	1	1	1	1	1	1	0.7732	0.4383	-0.01	—
0.01	1	1	1	1	1	1	1	1	1	1	0.6991	0.4145	-0.05
$\boldsymbol{\theta_2\backslash\theta_1}$	**0.40**	**0.41**	**0.42**	**0.43**	**0.44**	**0.45**	**0.46**	**0.47**	**0.48**	**0.49**	**0.50**	**0.51**	**0.52**

Table 3.4: Lower Bounds for $C_0(\theta_1,\theta_2)$ (First Method, $0.5<\theta\leqslant 0.53$) 2/2

3.4.2. *Second Method.* The second method is to use Mikawa's modified sieve developed in [32], and the whole process is discussed in Section 2. When doing the decomposing process in [32], we need the Type-II range $(r_0,\ r_1)$ satisfies $r_1 > 2r_0$. In the case $q \sim Q$ in Section 2, the Type-II range is fixed on $\left(2\theta-1,\ \frac{5-8\theta}{6}\right)$ (when $\theta < \frac{53}{105}$, the Type-II range $(0,\ 2\theta-1)$ would not bring useful improvements here), and that is why this method is not applicable when $\theta \geqslant \frac{17}{32}$. In the first 2-factored moduli case, we can enlarge r_1 to values like $\frac{2-3\theta}{3}$ and $\frac{5-8\theta_1-6\theta_2}{6}$ for some special (θ_1,θ_2). For example, when κ is replaced by $\frac{2-3\theta}{3}-\varepsilon$, then we only need

$$\frac{2-3\theta}{3} > 2(2\theta-1), \quad \text{or} \quad \theta < \frac{8}{15}$$

for the Type-II requirements. However, the method in [32] still becomes invalid when $\theta > \frac{17}{32}$ even in this 2-factored case. The real problem is not on the Type-II requirements but on the Type-I requirements. In the estimate of S_I'' in [32] (see [[32], Page 148]), we need the following three conditions corresponding to Lemma 2.8:

$$\frac{1}{2}-(2\theta-1) < 1-\theta, \quad \frac{1}{2}+15(2\theta-1) < 2-\theta \quad \text{and} \quad \frac{1}{2}+7(2\theta-1) < 2-2\theta.$$

The last condition above is equivalent to $\theta < \frac{17}{32}$. Replacing $2\theta-1$ with smaller $2\theta_1+\theta_2-1$ or 0 is meaningless here since these replacements also need the condition $\theta < \frac{17}{32}$ or stronger $\theta < \frac{10}{19}$. Another idea is to use Lemmas 2.4–2.7 to cover the region that Lemma 2.8 cannot cover; Indeed Lemma 2.5 covers the remaining region for $\theta < \frac{17}{30}$. However, the coefficients in the Type-I sum S_I'' in [32] are convolutions involving the Ψ function, which means that they do not satisfy **Condition B** (No small prime factors), and Lemmas 2.4–2.7 are not applicable here. If we can prove Lemma 2.5 without the **Condition B** on a_{1,m_1} and a_{3,m_3}, then we can enlarge the applicable θ to some value larger than $\frac{17}{32}$.

In the first 2-factored moduli case, we fail to extend the "Mikawa applicable range" of θ to $\geqslant \frac{17}{32}$. However, in this case we can still make some minor improvements over those results in Section 2. In many subregions we can use the Type-II range $\left(\frac{5-8\theta}{6},\ \frac{2-3\theta}{3}\right)$ or similar ranges to give more asymptotic formulas for (39) and (40), and we can also use Lemma 3.4 if $(\theta_1,\theta_2) \in \boldsymbol{C}$. Working on each region and subregion carefully, we can obtain the following lower bounds for $C_0(\theta_1,\theta_2)$ $(0.5<\theta\leqslant 0.53)$:

0.13	0.3477	—	—	—	—	—	—	—	—	—	—	—	—
0.12	0.5487	0.3625	—	—	—	—	—	—	—	—	—	—	—
0.11	0.7079	0.5519	0.3625	—	—	—	—	—	—	—	—	—	—
0.10	$1-\varepsilon$	0.7079	0.5551	0.3625	—	—	—	—	—	—	—	—	—
0.09	1	1	0.7079	0.5569	0.3625	—	—	—	—	—	—	—	—
0.08	1	1	1	1	0.5633	0.3625	—	—	—	—	—	—	—
0.07	1	1	1	1	1	1	0.3625	—	—	—	—	—	—
0.06	1	1	1	1	1	1	1	0.3625	—	—	—	—	—
0.05	1	1	1	1	1	1	1	1	0.5237	—	—	—	—
0.04	1	1	1	1	1	1	1	1	0.7051	0.4419	—	—	—
0.03	1	1	1	1	1	1	1	1	1	0.6339	0.3625	—	—
0.02	1	1	1	1	1	1	1	1	1	0.7741	0.5634	0.3509	—
0.01	1	1	1	1	1	1	1	1	1	1	0.7097	0.5552	0.3449
$\theta_2 \backslash \theta_1$	**0.40**	**0.41**	**0.42**	**0.43**	**0.44**	**0.45**	**0.46**	**0.47**	**0.48**	**0.49**	**0.50**	**0.51**	**0.52**

Table 3.5: Lower Bounds for $C_0(\theta_1, \theta_2)$ (Second Method, $0.5 < \theta \leqslant 0.53$)

0.13	-0.08 **0.3477**	—	—	—	—	—	—	—	—	—	—	—	—
0.12	0.3890 **0.5487**	0.0450 **0.3625**	—	—	—	—	—	—	—	—	—	—	—
0.11	0.6857 **0.7079**	0.4072 **0.5519**	0.0450 **0.3625**	—	—	—	—	—	—	—	—	—	—
0.10	$1-\varepsilon$	0.6857 **0.7079**	0.4192 **0.5551**	0.0450 **0.3625**	—	—	—	—	—	—	—	—	—
0.09	1	1	0.6857 **0.7079**	0.4394 **0.5569**	0.0450 **0.3625**	—	—	—	—	—	—	—	—
0.08	1	1	1	1	0.4409 **0.5633**	0.0450 **0.3625**	—	—	—	—	—	—	—
0.07	1	1	1	1	1	1	0.0450 **0.3625**	—	—	—	—	—	—
0.06	1	1	1	1	1	1	1	0.0450 **0.3625**	—	—	—	—	—
0.05	1	1	1	1	1	1	1	1	0.2431 **0.5237**	—	—	—	—
0.04	1	1	1	1	1	1	1	1	0.6042 **0.7051**	0.1463 **0.4419**	—	—	—
0.03	1	1	1	1	1	1	1	1	1	0.5227 **0.6339**	0.0450 **0.3625**	—	—
0.02	1	1	1	1	1	1	1	1	1	0.7732 **0.7741**	0.4383 **0.5634**	-0.01 **0.3509**	—
0.01	1	1	1	1	1	1	1	1	1	1	0.6991 **0.7097**	0.4145 **0.5552**	-0.05 **0.3449**
$\boldsymbol{\theta_2\backslash\theta_1}$	**0.40**	**0.41**	**0.42**	**0.43**	**0.44**	**0.45**	**0.46**	**0.47**	**0.48**	**0.49**	**0.50**	**0.51**	**0.52**

Table 3.6: A Comparison of Two Methods on the Lower Bounds for $C_0(\theta_1, \theta_2)$ ($0.5 < \theta \leqslant 0.53$)

4. 2-factored Moduli, 2

In this section we focus on the second 2-factored case with bilinear weights, where the absolute values in the first case are replaced by divisor-bounded coefficients. We also call this case "the bilinear case". The initial setups on the sieves are similar to the first case. We want to get the following result with some $0 < C_0'(\theta_1, \theta_2) \leqslant 1$ and $C_1'(\theta_1, \theta_2) \geqslant 1$:

Theorem 4.1. *There exist functions ρ_0 and ρ_1 which satisfies the following properties:*

(Majorant / Minorant). $\rho_0(n)$ is a minorant for the prime indicator function $\mathbb{1}_p(n)$, and $\rho_1(n)$ is a majorant for the prime indicator function $\mathbb{1}_p(n)$. That is, we have

$$\rho_0(n) \leqslant \mathbb{1}_p(n) \leqslant \rho_1(n).$$

(Upper and Lower bounds). We have

$$\sum_{n\leqslant x}\rho_0(n)\geqslant(1+o(1))\frac{C_0'(\theta_1,\theta_2)x}{\log x}\quad\text{and}\quad\sum_{n\leqslant x}\rho_1(n)\leqslant(1+o(1))\frac{C_1'(\theta_1,\theta_2)x}{\log x}$$

for two functions $C_0'(\theta_1,\theta_2)$ and $C_1'(\theta_1,\theta_2)$ satisfy $0<C_0'(\theta_1,\theta_2)\leqslant 1$ and $C_1'(\theta_1,\theta_2)\geqslant 1$.

(Distributions in Arithmetic Progressions). Let λ_{1,q_1} and λ_{2,q_2} be divisor-bounded complex sequences. For any $a\in\mathbb{Z}\backslash\{0\}$ and any $A>0$, we have

$$\sum_{\substack{q_1\sim Q_1\\ q_2\sim Q_2\\ (q_1q_2,a)=1}}\lambda_{1,q_1}\lambda_{2,q_2}\left(\sum_{\substack{n\leqslant x\\ n\equiv a(\bmod q_1q_2)}}\rho_j(n)-\frac{1}{\varphi(q_1q_2)}\sum_{\substack{n\leqslant x\\ (n,q_1q_2)=1}}\rho_j(n)\right)\ll\frac{x}{(\log x)^A}$$

for $j=0,1$.

In order to prove Theorem 4.1 with suitable $C_0'(\theta_1,\theta_2)$ and $C_1'(\theta_1,\theta_2)$, we need results of the form

$$\sum_{\substack{q_1\sim Q_1\\ q_2\sim Q_2\\ (q_1q_2,a)=1}}\lambda_{1,q_1}\lambda_{2,q_2}\left(\sum_{\substack{n\sim x\\ n\equiv a(\bmod q_1q_2)}}f(n)-\frac{1}{\varphi(q_1q_2)}\sum_{\substack{n\sim x\\ (n,q_1q_2)=1}}f(n)\right)\ll\frac{x}{(\log x)^A}. \tag{118}$$

Again, we may want the coefficients to satisfy **Conditions A and B** mentioned in Section 2.

4.1. **Preliminary Lemmas.** Before constructing the majorant and minorant, we need estimate results of the form (118). The results from Section 2 and Section 3 are still applicable in the final decomposition, and the results here are still useful in the later Section 5.

4.1.1. *Type-II estimate.* The first lemma comes from [13]. Note that Case (1) of this lemma can be deduced easily by Lemma 3.3.

Lemma 4.2. *([13], Théorème]). Let $M_1M_2\asymp x$. Let a_{1,m_1}, a_{2,m_2}, λ_{1,q_1} and λ_{2,q_2} be divisor-bounded complex sequences. Suppose that a_{2,m_2} satisfies **Conditions A and B**. If any of the following conditions*

(1). $Q_1Q_2^2\leqslant M_2x^{1-\varepsilon},\ Q_1^8Q_2^7M_2^6\leqslant x^{4-\varepsilon}$;

(2). $Q_1Q_2^2\leqslant x^{1-\varepsilon},\ Q_1^8Q_2^7M_2^5\leqslant x^{4-\varepsilon}$;

(3). $Q_1Q_2^2\leqslant M_2x^{1-\varepsilon},\ Q_1^5Q_2^2M_2^6\leqslant x^{2-\varepsilon},\ Q_1^4Q_2^3M_2^3\leqslant x^{2-\varepsilon},\ Q_1^9Q_2^8M_2^6\leqslant x^{5-\varepsilon}$;

(4). $Q_1Q_2^2\leqslant x^{1-\varepsilon},\ Q_1^5Q_2^2M_2^5\leqslant x^{2-\varepsilon},\ Q_1^8Q_2^6M_2^5\leqslant x^{4-\varepsilon}$;

(5). $Q_1Q_2^2M_2\leqslant x^{1-\varepsilon},\ Q_1^5Q_2^2M_2^4\leqslant x^{2-\varepsilon},\ Q_1^8Q_2^6M_2^5\leqslant x^{4-\varepsilon}$

holds, then

$$\sum_{\substack{q_1\sim Q_1\\ q_2\sim Q_2\\ (q_1q_2,a)=1}}\lambda_{1,q_1}\lambda_{2,q_2}\left(\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ m_1m_2\equiv a(\bmod q_1q_2)}}a_{1,m_1}a_{2,m_2}-\frac{1}{\varphi(q_1q_2)}\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ (m_1m_2,q_1q_2)=1}}a_{1,m_1}a_{2,m_2}\right)\ll\frac{x}{(\log x)^A}.$$

The next lemma comes from [22], and it was used in [25].

Lemma 4.3. *([22], Proposition]). Let $M_1M_2\asymp x$. Let a_{1,m_1}, a_{2,m_2}, λ_{1,q_1} and λ_{2,q_2} be divisor-bounded complex sequences. Suppose that a_{2,m_2} satisfies **Conditions A and B**. If we have*

$$Q_1x^{\varepsilon}<M_2,\ Q_1Q_2^2<M_2x^{1-\varepsilon},\ Q_1^4Q_2^8M_2^6<x^{5-\varepsilon},$$

then

$$\sum_{\substack{q_1\sim Q_1\\ q_2\sim Q_2\\ (q_1q_2,a)=1}}\lambda_{1,q_1}\lambda_{2,q_2}\left(\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ m_1m_2\equiv a(\bmod q_1q_2)}}a_{1,m_1}a_{2,m_2}-\frac{1}{\varphi(q_1q_2)}\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ (m_1m_2,q_1q_2)=1}}a_{1,m_1}a_{2,m_2}\right)\ll\frac{x}{(\log x)^A}.$$

4.1.2. *Type-I estimate.* The next lemma comes from [6], and it was used in [3].

Lemma 4.4. *([6], Theorems 5 and 5*]). Let $M_1M_2\asymp x$ and $z\ll\exp\left(\log x(\log\log x)^{-2}\right)$. Let a_{1,m_1}, a_{2,m_2}, λ_{1,q_1} and λ_{2,q_2} be divisor-bounded complex sequences. Suppose that*

$$a_{1,m_1}=\mathbb{1}_{m_1\in\mathbf{M}}\qquad\text{or}\qquad a_{1,m_1}=\mathbb{1}_{\substack{m_1\in\mathbf{M}\\ (m_1,P(z))=1}}$$

for some interval $\mathbf{M}\subseteq[M_1,2M_1]$. If we have

$$Q_1M_2<x^{1-\varepsilon},\quad Q_1Q_2^4M_2<x^{2-\varepsilon},\quad Q_1Q_2^2M_2^2<x^{2-2\varepsilon},\quad Q_1^3Q_2^4M_2<x^{3-\varepsilon},$$

then

$$\sum_{\substack{q_1\sim Q_1\\ q_2\sim Q_2\\ (q_1q_2,a)=1}} \lambda_{1,q_1}\lambda_{2,q_2}\left(\sum_{\substack{m_1\in\mathbf{M}\\ m_2\sim M_2\\ m_1m_2\equiv a(\bmod q_1q_2)}} a_{1,m_1}a_{2,m_2} - \frac{1}{\varphi(q_1q_2)}\sum_{\substack{m_1\in\mathbf{M}\\ m_2\sim M_2\\ (m_1m_2,q_1q_2)=1}} a_{1,m_1}a_{2,m_2}\right) \ll \|a_2\| M_1^{\frac{1}{2}} x^{\frac{1}{2}-\varepsilon},$$

where $\|a_2\| = \left(\sum_{m_2}|a_{2,m_2}|^2\right)^{\frac{1}{2}}$ *is the* l^2 *norm.*

4.2. **Upper Bounds.** We shall construct the majorant $\rho_1(n)$ in this subsection. Before constructing, we first mention some existing results of $C_1'(\theta_1,\theta_2)$.

Theorem 4.5. *The function* $C_1'(\theta_1,\theta_2)$ *satisfies the following conditions:*
(1). $C_1'(\theta_1,\theta_2) = C_1'(\theta_2,\theta_1)$*;*
(2). $C_1'(\theta_1,\theta_2) = 1$ *for all* θ_1,θ_2 *satisfy* $\theta_1+\theta_2 < 0.5$*;*
(3). $C_1'(\theta_1,\theta_2) = 1$ *for all* θ_1,θ_2 *satisfy* $2\theta_1+\theta_2<1$, $7\theta_1+12\theta_2<4$ *and* $19\theta_1+20\theta_2<10$*;*
(4). $C_1'(\theta_1,\theta_2) = 1$ *for all* θ_1,θ_2 *satisfy* $\theta_1<\frac{1}{3}$, $\theta_2<\frac{1}{5}$ *and* $\theta_1+\theta_2<\frac{29}{56}$*;*
(5). $C_1'(\theta_1,\theta_2) = 1$ *for all* θ_1,θ_2 *satisfy* $\theta_1+3\theta_2<1$, $\theta_1+\theta_2<\frac{29}{56}$ *and* $\theta_2<\max\left(\frac{1-2\theta_1}{2},\frac{2-2\theta_1}{5}\right)$*;*
(6). $C_1'(\theta_1,\theta_2) = 1$ *for all* θ_1,θ_2 *satisfy* $\theta_1+3\theta_2<1$, $\theta_1+\theta_2<\frac{29}{56}$, $4\theta_1+\theta_2<\frac{403}{266}$ *and* $\frac{7}{4}\theta_1+\theta_2<\frac{403}{532}$*;*
(7). $C_1'(\theta_1,\theta_2) \leqslant C_1(\theta_1,\theta_2) \leqslant C_1(\theta_1+\theta_2)$ *for* $0.5\leqslant\theta_1+\theta_2\leqslant 1$*;*
(8). $C_1'(\theta_1,\theta_2) \leqslant 1+\varepsilon$ *for all* θ_1,θ_2 *satisfy* $\theta_1+\theta_2=0.5$*;*
(9). $C_1'(\theta_1,\theta_2) \leqslant 1+\varepsilon$ *for all* θ_1,θ_2 *satisfy* $\theta_1\leqslant 0.5$, $\theta_1\neq\frac{10}{21}$ *and* $\theta_2=\min\left(1-2\theta_1,\frac{4-7\theta_1}{12},\frac{10-19\theta_1}{20}\right)$*.*

Proof. The first statement is obvious. Statements (2)–(6) follow easily from the Bombieri–Vinogradov Theorem, [[28], Theorem 1.1], [[3], Theorem 3] and [[13], Page 621 and Corollaire 5]. The seventh statement holds trivially by the work done in Section 2 and Section 3. When there are no new arithmetic information inputs outside of those in previous sections, we use $C_1(\theta_1,\theta_2)$ as an upper bound for $C_1'(\theta_1,\theta_2)$. The eighth statement holds from the seven statement and statement (5) of Theorem 3.9. The ninth statement follows from statement (6) of Theorem 3.9. □

From here to the end of this section, we assume that $\theta_1\geqslant\theta_2$ to simplify the conditions. We also write $\theta=\theta_1+\theta_2$. Before performing our final decompositions, we define several regions of the pair (θ_1,θ_2) based on various arithmetic information inputs. We also use many other regions defined before, and the readers can find the definitions of them in previous sections. Moreover, the region $\boldsymbol{S}$ defined in Section 2 can be enlarged in this section (and also in Section 5) using Lemma 4.4 above.

$$\begin{aligned}
\boldsymbol{S} = {} & \{(s,t): s<1-\theta,\ s+2t<2-2\theta,\ s+4t<2-\theta \\
& \qquad \text{or } s+t<\min\left(1-\theta_1, 2-\theta_1-4\theta_2, 1-\frac{1}{2}\theta_1-\theta_2, 3-3\theta_1-4\theta_2\right)\right\}, \\
\boldsymbol{J} = {} & \left\{(\theta_1,\theta_2): (\theta_1,\theta_2)\in\boldsymbol{U};\ \theta_1+\theta_2<\frac{1}{2}\right. \\
& \qquad \text{or } 2\theta_1+\theta_2<1,\ 7\theta_1+12\theta_2<4,\ 19\theta_1+20\theta_2<10 \\
& \qquad \text{or } \theta_1<\frac{1}{3},\ \theta_2<\frac{1}{5},\ \theta_1+\theta_2<\frac{29}{56} \\
& \qquad \text{or } \theta_1+3\theta_2<1,\ \theta_1+\theta_2<\frac{29}{56},\ \theta_2<\max\left(\frac{1-2\theta_1}{2},\frac{2-2\theta_1}{5}\right) \\
& \qquad \left.\text{or } \theta_1+3\theta_2<1,\ \theta_1+\theta_2<\frac{29}{56},\ 4\theta_1+\theta_2<\frac{403}{266},\ \frac{7}{4}\theta_1+\theta_2<\frac{403}{532}\right\}, \\
\boldsymbol{E} = {} & \left\{(\theta_1,\theta_2): (\theta_1,\theta_2)\in\boldsymbol{U}\backslash\boldsymbol{J};\ \frac{1}{4}<\theta_1\leqslant\frac{2}{7},\ \frac{1}{2}(1-2\theta_1)<\theta_2\leqslant\theta_1\right. \\
& \qquad \text{or } \frac{2}{7}<\theta_1\leqslant\frac{2}{5},\ \frac{1}{2}(1-2\theta_1)<\theta_2<\frac{1}{4}(2-3\theta_1) \\
& \qquad \text{or } \frac{2}{5}<\theta_1<\frac{1}{2},\ \frac{1}{2}(1-2\theta_1)<\theta_2<\frac{1}{15}(11-20\theta_1) \\
& \qquad \left.\text{or } \frac{1}{2}\leqslant\theta_1<\frac{11}{20},\ 0<\theta_2<\frac{1}{15}(11-20\theta_1)\right\}, \\
\boldsymbol{F} = {} & \left\{(\theta_1,\theta_2): (\theta_1,\theta_2)\in\boldsymbol{U}\backslash\boldsymbol{J};\ \frac{1}{4}<\theta_1\leqslant\frac{5}{18},\ \frac{1}{2}(1-2\theta_1)<\theta_2<\theta_1\right. \\
& \qquad \text{or } \frac{5}{18}<\theta_1<\frac{1}{2},\ \frac{1}{2}(1-2\theta_1)<\theta_2<\frac{1}{10}(5-8\theta_1) \\
& \qquad \left.\text{or } \frac{1}{2}\leqslant\theta_1<\frac{11}{20},\ 0<\theta_2<\frac{1}{10}(11-20\theta_1)\right\},
\end{aligned}$$

Here, $\boldsymbol{J}$ denote the region that $C_1'(\theta_1, \theta_2) = 1$ follows by the Bombieri–Vinogradov Theorem or theorems mentioned in the proof of Statements (3)–(6) of Theorem 4.5. Region $\boldsymbol{E}$ corresponds to Lemma 4.2, and region $\boldsymbol{F}$ corresponds to Lemma 4.3. The results in Section 3 will also be applied when (θ_1, θ_2) is in $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ or $\boldsymbol{T}$. We note that the three-dimensional Harman's sieve corresponds to region $\boldsymbol{T}_2$ (see Lemma 3.7 and the discussions in Subsection 3.3) will be used a lot since we have many Type-II ranges start from 0, compare to the cases in Section 3. Again, we shall implicitly use the three-dimensional Harman's sieve in many decompositions below. Sometimes we use it even when $\kappa \leqslant \frac{1}{7}$; in this case, we only need to make some modifications to the "loss integrals": Suppose that $\frac{1}{8} < \kappa \leqslant \frac{1}{7}$, and we apply a three-dimensional Harman's sieve on a region R. Since $\alpha_1 < \frac{1}{2} = \frac{4}{8}$ and $\alpha_2 < \frac{1}{3} < \frac{3}{8}$, we have $\Omega(m_1) \leqslant 3$ and $\Omega(m_2) \leqslant 2$, and $\Omega(m_1 m_2)$ can be 3, 4 or 5. Now we have 3 cases based on different values of $\Omega(m_1 m_2)$:

(1). $\Omega(m_1) = 1,\ \Omega(m_2) = 2$ or $\Omega(m_1) = 2,\ \Omega(m_2) = 1$;
(2). $\Omega(m_1) = 2,\ \Omega(m_2) = 2$ or $\Omega(m_1) = 3,\ \Omega(m_2) = 1$;
(3). $\Omega(m_1) = 3,\ \Omega(m_2) = 2$.

These 3 cases correspond to 3 "loss integrals":

$$\frac{1}{\kappa} \int_{\substack{t_1, t_2, t_3 \geqslant \kappa \\ (t_1, t_2+t_3) \in R,\ t_2 \geqslant t_3 \\ \text{or } (t_1+t_2, t_3) \in R,\ t_1 \geqslant t_2 \\ (t_1, t_2, t_3) \notin \boldsymbol{G}_3}} \frac{\omega\left(\frac{1-t_1-t_2-t_3}{\kappa}\right)}{t_1 t_2 t_3} dt_3 dt_2 dt_1, \tag{119}$$

$$\frac{1}{\kappa} \int_{\substack{t_1, t_2, t_3, t_4 \geqslant \kappa \\ (t_1+t_2, t_3+t_4) \in R,\ t_1 \geqslant t_2,\ t_3 \geqslant t_4 \\ \text{or } (t_1+t_2+t_3, t_4) \in R,\ t_1 \geqslant t_2 \geqslant t_3 \\ (t_1, t_2, t_3, t_4) \notin \boldsymbol{G}_4}} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{\kappa}\right)}{t_1 t_2 t_3 t_4} dt_4 dt_3 dt_2 dt_1, \tag{120}$$

and

$$\frac{1}{\kappa} \int_{\substack{t_1, t_2, t_3, t_4, t_5 \geqslant \kappa \\ (t_1+t_2+t_3, t_4+t_5) \in R \\ t_1 \geqslant t_2 \geqslant t_3,\ t_4 \geqslant t_5 \\ (t_1, t_2, t_3, t_4, t_5) \notin \boldsymbol{G}_5}} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{\kappa}\right)}{t_1 t_2 t_3 t_4 t_5} dt_5 dt_4 dt_3 dt_2 dt_1. \tag{121}$$

One can compare them with the "loss integrals" in Lemma 2.20 and Lemma 2.21. When $\frac{11}{21} \leqslant \theta < \frac{17}{32}$, the three-dimensional Harman's sieve should be applied on both B and C (see Lemma 2.17) if we use it, since we can only perform straightforward decompositions on A where $\alpha_1 + \alpha_2 < \theta$. This can be proved by the discussions in [[17], Page 186, Case (vi)], and we shall prove it again for clarity. The proof is trivial when $\alpha_1 < \tau = \frac{2}{7}$. When $\frac{2}{7} \leqslant \alpha_1 \leqslant \frac{3}{7}$, we only need to show that

$$\alpha_1 < 1 - \theta, \quad \alpha_1 + 2\alpha_2 < 2 - 2\theta, \quad \alpha_1 + 4\alpha_2 < 2 - \theta.$$

Since $\theta < \frac{4}{7}$, we have

$$\alpha_1 \leqslant \frac{3}{7} < 1 - \theta, \quad \alpha_1 + 2\alpha_2 = 2\theta - \alpha_1 \leqslant 2\theta - \frac{2}{7} < 2 - 2\theta, \quad \alpha_1 + 4\alpha_2 = 4\theta - 3\alpha_1 \leqslant 4\theta - \frac{6}{7} < 2 - \theta. \tag{122}$$

Of course, we often decide to use it instead of discarding the region $\frac{2}{7} \leqslant \alpha_1 \leqslant \frac{3}{7}$ since the higher dimensional loss comes from integrals (119)–(121) is usually smaller than the one-dimensional loss.

Now we assume that $(\theta_1, \theta_2) \in \boldsymbol{E}$. We divide $\boldsymbol{E}$ into 11 subregions:

$$\boldsymbol{E} = \boldsymbol{E}_{01} \cup \boldsymbol{E}_{02} \cup \boldsymbol{E}_{03} \cup \boldsymbol{E}_{04} \cup \boldsymbol{E}_{05} \cup \boldsymbol{E}_{06} \cup \boldsymbol{E}_{07} \cup \boldsymbol{E}_{08} \cup \boldsymbol{E}_{09} \cup \boldsymbol{E}_{10} \cup \boldsymbol{E}_{11},$$

where

$$\begin{aligned}
\boldsymbol{E}_{01} &= \left\{ (\theta_1, \theta_2) : \frac{1}{4} < \theta_1 \leqslant \frac{2}{7},\ \frac{1}{2}(1 - 2\theta_1) < \theta_2 \leqslant \theta_1 \right\}, \\
\boldsymbol{E}_{02} &= \left\{ (\theta_1, \theta_2) : \frac{2}{7} < \theta_1 \leqslant \frac{5}{14},\ \frac{1}{2}(1 - 2\theta_1) < \theta_2 < \frac{1}{7}(6 - 14\theta_1) \right\}, \\
\boldsymbol{E}_{03} &= \left\{ (\theta_1, \theta_2) : \frac{2}{7} < \theta_1 \leqslant \frac{5}{14},\ \frac{1}{7}(6 - 14\theta_1) < \theta_2 < \frac{1}{4}(2 - 3\theta_1) \right\}, \\
\boldsymbol{E}_{04} &= \left\{ (\theta_1, \theta_2) : \frac{5}{14} < \theta_1 \leqslant \frac{3}{8},\ \frac{1}{2}(1 - 2\theta_1) < \theta_2 < \frac{1}{3}(-1 + 4\theta_1) \right\}, \\
\boldsymbol{E}_{05} &= \left\{ (\theta_1, \theta_2) : \frac{5}{14} < \theta_1 \leqslant \frac{3}{8},\ \frac{1}{3}(-1 + 4\theta_1) < \theta_2 < \frac{1}{4}(2 - 3\theta_1) \right\}, \\
\boldsymbol{E}_{06} &= \left\{ (\theta_1, \theta_2) : \frac{3}{8} < \theta_1 \leqslant \frac{2}{5},\ \frac{1}{2}(1 - 2\theta_1) < \theta_2 < \frac{1}{3}(2 - 4\theta_1) \right\}, \\
\boldsymbol{E}_{07} &= \left\{ (\theta_1, \theta_2) : \frac{3}{8} < \theta_1 \leqslant \frac{2}{5},\ \frac{1}{3}(2 - 4\theta_1) < \theta_2 < \frac{1}{4}(2 - 3\theta_1) \right\}, \\
\boldsymbol{E}_{08} &= \left\{ (\theta_1, \theta_2) : \frac{2}{5} < \theta_1 < \frac{1}{2},\ \frac{1}{2}(1 - 2\theta_1) < \theta_2 < \frac{1}{3}(2 - 4\theta_1) \right\}, \\
\boldsymbol{E}_{09} &= \left\{ (\theta_1, \theta_2) : \frac{2}{5} < \theta_1 < \frac{1}{2},\ \frac{1}{3}(2 - 4\theta_1) < \theta_2 < 1 - 2\theta_1 \right\},
\end{aligned}$$

$$\boldsymbol{E}_{10} = \left\{(\theta_1,\theta_2) : \frac{2}{5} < \theta_1 < \frac{1}{2},\ 1-2\theta_1 < \theta_2 < \frac{1}{15}(11-20\theta_1)\right\},$$
$$\boldsymbol{E}_{11} = \left\{(\theta_1,\theta_2) : \frac{1}{2} \leqslant \theta_1 < \frac{11}{20},\ 0 < \theta_2 < \frac{1}{15}(11-20\theta_1)\right\}.$$

4.2.1. $\boldsymbol{E}_{01}$. We divide $\boldsymbol{E}_{01}$ into 7 subregions:

$$\boldsymbol{E}_{01} = \boldsymbol{E}_{0101} \cup \boldsymbol{E}_{0102} \cup \boldsymbol{E}_{0103} \cup \boldsymbol{E}_{0104} \cup \boldsymbol{E}_{0105} \cup \boldsymbol{E}_{0106} \cup \boldsymbol{E}_{0107},$$

where

$$\begin{aligned}
\boldsymbol{E}_{0101} = &\left\{(\theta_1,\theta_2) : \frac{1}{4} < \theta_1 < \frac{6}{23},\ \frac{1}{2}(1-2\theta_1) < \theta_2 \leqslant \theta_1\right.\\
&\qquad \left.\text{or } \frac{6}{23} \leqslant \theta_1 \leqslant \frac{2}{7},\ \frac{1}{2}(1-2\theta_1) < \theta_2 < \frac{1}{13}(6-10\theta_1)\right\},\\
\boldsymbol{E}_{0102} = &\left\{(\theta_1,\theta_2) : \frac{6}{23} < \theta_1 < \frac{11}{40},\ \frac{1}{13}(6-10\theta_1) < \theta_2 \leqslant \theta_1\right.\\
&\qquad \text{or } \frac{11}{40} \leqslant \theta_1 \leqslant \frac{7}{25},\ \frac{1}{13}(6-10\theta_1) < \theta_2 < \frac{1}{20}(11-20\theta_1)\\
&\qquad \left.\text{or } \frac{7}{25} < \theta_1 \leqslant \frac{2}{7},\ \frac{1}{13}(6-10\theta_1) < \theta_2 < \frac{1}{4}(5-14\theta_1)\right\},\\
\boldsymbol{E}_{0103} = &\left\{(\theta_1,\theta_2) : \frac{11}{40} \leqslant \theta_1 \leqslant \frac{5}{18},\ \frac{1}{20}(11-20\theta_1) \leqslant \theta_2 \leqslant \theta_1\right.\\
&\qquad \left.\text{or } \frac{5}{18} < \theta_1 < \frac{7}{25},\ \frac{1}{20}(11-20\theta_1) \leqslant \theta_2 < \frac{1}{4}(5-14\theta_1)\right\},\\
\boldsymbol{E}_{0104} = &\left\{(\theta_1,\theta_2) : \frac{7}{25} < \theta_1 \leqslant \frac{2}{7},\ \frac{1}{4}(5-14\theta_1) \leqslant \theta_2 < \frac{1}{20}(11-20\theta_1)\right\},\\
\boldsymbol{E}_{0105} = &\left\{(\theta_1,\theta_2) : \frac{5}{18} < \theta_1 \leqslant \frac{7}{25},\ \frac{1}{4}(5-14\theta_1) \leqslant \theta_2 < \frac{1}{8}(5-10\theta_1)\right.\\
&\qquad \left.\text{or } \frac{7}{25} < \theta_1 \leqslant \frac{2}{7},\ \frac{1}{20}(11-20\theta_1) \leqslant \theta_2 < \frac{1}{8}(5-10\theta_1)\right\},\\
\boldsymbol{E}_{0106} = &\left\{(\theta_1,\theta_2) : \frac{5}{18} < \theta_1 \leqslant \frac{2}{7},\ \frac{1}{8}(5-10\theta_1) < \theta_2 < \frac{1}{10}(5-8\theta_1)\right\},\\
\boldsymbol{E}_{0107} = &\left\{(\theta_1,\theta_2) : \frac{5}{18} < \theta_1 \leqslant \frac{2}{7},\ \frac{1}{10}(5-8\theta_1) \leqslant \theta_2 \leqslant \theta_1\right\}.
\end{aligned}$$

Note that we have $\theta < \frac{11}{20}$ for $(\theta_1,\theta_2) \in \boldsymbol{E}_{0101} \cup \boldsymbol{E}_{0102} \cup \boldsymbol{E}_{0104}$.

The Type-II range for $\boldsymbol{E}_{0101}$ is

$$\left(0,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-4\theta_1-8\theta_2)\right). \tag{123}$$

The decompositions in this case will be discussed later together with the case $\boldsymbol{E}_{0202}$.

The Type-II range for $\boldsymbol{E}_{0102}$ is

$$\left(0,\ \frac{1}{4}(2-2\theta_1-5\theta_2)\right) \cup \left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-4\theta_1-8\theta_2)\right). \tag{124}$$

The decompositions are similar to which in the case $\boldsymbol{A}_{0104}$ in Section 3. When $\theta < \frac{17}{32}$, we use the middle Type-II range $\left(2\theta-1,\ \frac{5-8\theta}{6}\right)$ to give a "starting point" $\kappa = \frac{5-8\theta}{6} - \varepsilon$, and the third Type-II range $\left(\theta_2,\ \frac{5-4\theta_1-8\theta_2}{6}\right)$ is used to subtract the contributions of those sums with products of variables lie in this range. When $\theta \geqslant \frac{17}{32}$, both the second and the third Type-II ranges are used explicitly to discard suitable sums. In this case κ is reduced to $\frac{5-8\theta}{12} - 3\varepsilon$ or $\frac{3-5\theta}{7} - 2\varepsilon$, and if $\kappa < \frac{2-2\theta_1-5\theta_2}{4}$, we can replace κ with a larger $\frac{2-2\theta_1-5\theta_2}{4}$ using the first Type-II range and Buchstab's identity (55). In many cases below (especially cases with $\theta \geqslant \frac{11}{20}$), we can also compare the value of κ with $\frac{2-2\theta_1-5\theta_2}{4}$ (or maybe other "end point values"), and we shall not state the same process again.

The Type-II range for $\boldsymbol{E}_{0103}$ is

$$\left(0,\ \frac{1}{4}(2-2\theta_1-5\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-4\theta_1-8\theta_2)\right). \tag{125}$$

The decompositions are similar to which in the case $\boldsymbol{A}_{05}$ in Section 3.

The Type-II range for $\boldsymbol{E}_{0104}$ is

$$\left(0,\ \frac{1}{4}(2-2\theta_1-5\theta_2)\right) \cup \left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right) \cup \left(\theta_1,\ \frac{1}{6}(5-4\theta_1-8\theta_2)\right). \tag{126}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0102}$.

The Type-II range for $\boldsymbol{E}_{0105}$ is

$$\left(0,\ \frac{1}{4}(2-2\theta_1-5\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right) \cup \left(\theta_1,\ \frac{1}{6}(5-4\theta_1-8\theta_2)\right). \tag{127}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0103}$.

The Type-II range for $\boldsymbol{E}_{0106}$ is

$$\left(0,\ \frac{1}{4}(2-2\theta_1-5\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right). \tag{128}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0103}$.

The Type-II range for $\boldsymbol{E}_{0107}$ is

$$\left(0,\ \frac{1}{4}(2-2\theta_1-5\theta_2)\right). \tag{129}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0103}$. When $\kappa \geqslant \frac{2-2\theta_1-5\theta_2}{4}$, the decompositions are exactly same as the work done in Section 2 since this Type-II range is not useful here.

4.2.2. $\boldsymbol{E}_{02}$. We divide $\boldsymbol{E}_{02}$ into 9 subregions:

$$\boldsymbol{E}_{02} = \boldsymbol{E}_{0201} \cup \boldsymbol{E}_{0202} \cup \boldsymbol{E}_{0203} \cup \boldsymbol{E}_{0204} \cup \boldsymbol{E}_{0205} \cup \boldsymbol{E}_{0206} \cup \boldsymbol{E}_{0207} \cup \boldsymbol{E}_{0208} \cup \boldsymbol{E}_{0209},$$

where

$$\begin{aligned}
\boldsymbol{E}_{0201} = \Bigg\{(\theta_1,\theta_2) : {} & \frac{1}{3} < \theta_1 \leqslant \frac{7}{20},\ \frac{1}{2}(1-2\theta_1) < \theta_2 < \frac{1}{14}(5-8\theta_1) \\
& \text{or } \frac{7}{20} < \theta_1 < \frac{5}{14},\ \frac{1}{2}(1-2\theta_1) < \theta_2 < \frac{1}{7}(6-14\theta_1)\Bigg\}, \\
\boldsymbol{E}_{0202} = \Bigg\{(\theta_1,\theta_2) : {} & \frac{2}{7} < \theta_1 \leqslant \frac{41}{142},\ \frac{1}{2}(1-2\theta_1) < \theta_2 < \frac{1}{13}(6-10\theta_1) \\
& \text{or } \frac{41}{142} < \theta_1 < \frac{3}{10},\ \frac{1}{2}(1-2\theta_1) < \theta_2 < \frac{1}{4}(5-14\theta_1)\Bigg\}, \\
\boldsymbol{E}_{0203} = \Bigg\{(\theta_1,\theta_2) : {} & \frac{2}{7} < \theta_1 < \frac{41}{142},\ \frac{1}{13}(6-10\theta_1) < \theta_2 \leqslant \frac{1}{4}(5-14\theta_1)\Bigg\}, \\
\boldsymbol{E}_{0204} = \Bigg\{(\theta_1,\theta_2) : {} & \frac{41}{142} < \theta_1 < \frac{3}{10},\ \frac{1}{4}(5-14\theta_1) \leqslant \theta_2 < \frac{1}{13}(6-10\theta_1) \\
& \text{or } \frac{3}{10} \leqslant \theta_1 \leqslant \frac{9}{28},\ \frac{1}{2}(1-2\theta_1) < \theta_2 < \frac{1}{13}(6-10\theta_1) \\
& \text{or } \frac{9}{28} < \theta_1 \leqslant \frac{1}{3},\ \frac{1}{2}(1-2\theta_1) < \theta_2 < \frac{1}{7}(6-14\theta_1) \\
& \text{or } \frac{1}{3} < \theta_1 < \frac{7}{20},\ \frac{1}{14}(5-8\theta_1) < \theta_2 < \frac{1}{7}(6-14\theta_1)\Bigg\}, \\
\boldsymbol{E}_{0205} = \Bigg\{(\theta_1,\theta_2) : {} & \frac{2}{7} < \theta_1 < \frac{41}{142},\ \frac{1}{4}(5-14\theta_1) \leqslant \theta_2 < \frac{1}{20}(11-20\theta_1) \\
& \text{or } \frac{41}{142} \leqslant \theta_1 \leqslant \frac{3}{10},\ \frac{1}{13}(6-10\theta_1) < \theta_2 < \frac{1}{20}(11-20\theta_1) \\
& \text{or } \frac{3}{10} < \theta_1 \leqslant \frac{13}{42},\ \frac{1}{13}(6-10\theta_1) < \theta_2 < \frac{1}{8}(5-10\theta_1) \\
& \text{or } \frac{13}{42} < \theta_1 < \frac{9}{28},\ \frac{1}{13}(6-10\theta_1) < \theta_2 < \frac{1}{7}(6-14\theta_1)\Bigg\}, \\
\boldsymbol{E}_{0206} = \Bigg\{(\theta_1,\theta_2) : {} & \frac{3}{10} < \theta_1 \leqslant \frac{43}{140},\ \frac{1}{8}(5-10\theta_1) \leqslant \theta_2 < \frac{1}{20}(11-20\theta_1) \\
& \text{or } \frac{43}{140} < \theta_1 < \frac{13}{42},\ \frac{1}{8}(5-10\theta_1) \leqslant \theta_2 < \frac{1}{7}(6-14\theta_1)\Bigg\}, \\
\boldsymbol{E}_{0207} = \Bigg\{(\theta_1,\theta_2) : {} & \frac{2}{7} < \theta_1 < \frac{3}{10},\ \frac{1}{20}(11-20\theta_1) \leqslant \theta_2 < \frac{1}{8}(5-10\theta_1)\Bigg\}, \\
\boldsymbol{E}_{0208} = \Bigg\{(\theta_1,\theta_2) : {} & \frac{2}{7} < \theta_1 \leqslant \frac{25}{84},\ \frac{1}{8}(5-10\theta_1) \leqslant \theta_2 < \frac{1}{10}(5-8\theta_1) \\
& \text{or } \frac{25}{84} < \theta_1 \leqslant \frac{3}{10},\ \frac{1}{8}(5-10\theta_1) \leqslant \theta_2 < \frac{1}{7}(6-14\theta_1) \\
& \text{or } \frac{3}{10} < \theta_1 < \frac{43}{140},\ \frac{1}{20}(11-20\theta_1) \leqslant \theta_2 < \frac{1}{7}(6-14\theta_1)\Bigg\}, \\
\boldsymbol{E}_{0209} = \Bigg\{(\theta_1,\theta_2) : {} & \frac{2}{7} < \theta_1 < \frac{25}{84},\ \frac{1}{10}(5-8\theta_1) \leqslant \theta_2 < \frac{1}{7}(6-14\theta_1)\Bigg\}.
\end{aligned}$$

Note that we have $\theta < \frac{11}{20}$ for $(\theta_1, \theta_2) \in \boldsymbol{E}_{0201} \cup \boldsymbol{E}_{0202} \cup \boldsymbol{E}_{0203} \cup \boldsymbol{E}_{0204} \cup \boldsymbol{E}_{0205} \cup \boldsymbol{E}_{0206}$.

The Type-II range for $\boldsymbol{E}_{0201}$ is

$$\left(0,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right) \cup \left(\theta_1,\ \frac{1}{6}(5-4\theta_1-8\theta_2)\right). \tag{130}$$

The decompositions in this case will be discussed later together with the case $\boldsymbol{E}_{0501}$.

The Type-II range for $\boldsymbol{E}_{0202}$ is

$$\left(0,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-4\theta_1-8\theta_2)\right). \tag{131}$$

In this case we discuss both $\boldsymbol{E}_{0101}$ and $\boldsymbol{E}_{0202}$, since the Type-II ranges for them are same:

$$\begin{aligned} \boldsymbol{\mathcal{Z}_1} = \boldsymbol{E}_{0101} \cup \boldsymbol{E}_{0202} = \Bigg\{(\theta_1,\theta_2) : \ & \frac{1}{4} < \theta_1 < \frac{6}{23},\ \frac{1}{2}(1-2\theta_1) < \theta_2 \leqslant \theta_1 \\ \text{or } & \frac{6}{23} \leqslant \theta_1 \leqslant \frac{41}{142},\ \frac{1}{2}(1-2\theta_1) < \theta_2 < \frac{1}{13}(6-10\theta_1) \\ \text{or } & \frac{41}{142} < \theta_1 < \frac{3}{10},\ \frac{1}{2}(1-2\theta_1) < \theta_2 < \frac{1}{4}(5-14\theta_1) \Bigg\}. \end{aligned}$$

First, we shall prove the following theorem, which is an equivalent form of the first case of Theorem 1.1:

Theorem 4.6. *Let $(\theta_1, \theta_2) \in \boldsymbol{\mathcal{Z}_1}$. Suppose that we have*

$$\frac{1}{2} \leqslant \theta_1 + \theta_2 < \frac{11}{21} \quad \text{and} \quad \theta_1 + 2\theta_2 < \frac{3}{4}.$$

Then (118) holds for

$$f(n) = \mathbb{1}_p(n),$$

and we have

$$C_1'(\theta_1,\theta_2) = C_0'(\theta_1,\theta_2) = 1.$$

Proof. Since we have $(\theta_1, \theta_2) \in \boldsymbol{\mathcal{Z}_1}$, $\frac{1}{2} \leqslant \theta_1 + \theta_2 < \frac{11}{21}$ and $\theta_1 + 2\theta_2 < \frac{3}{4}$, all of the following conditions hold true:

$$\kappa = \frac{5-8\theta}{6} - \varepsilon > \frac{1}{8}, \quad 11\theta_1 + 12\theta_2 < 6, \quad 8\theta_1 + 11\theta_2 < 5, \quad 3\theta_1 + 2\theta_2 < \frac{11}{7},$$
$$\theta_1 + 3\theta_2 < 1, \quad \theta_2 < \frac{1-\theta}{2} \leqslant \frac{1}{4}, \quad \text{and} \quad \frac{1}{6}(5-4\theta_1-8\theta_2) > \frac{1}{3}.$$

Now, we can decompose our $\mathbb{1}_p(n) = \psi\left(n, x^{\frac{1}{2}}\right)$ in a way similar to the decompositions in [[23], Sections 6.1–6.5]. By Buchstab's identity, we have

$$\begin{aligned} \psi\left(n, x^{\frac{1}{2}}\right) &= \psi(n, x^\kappa) - \sum_{\substack{n=p_1\beta \\ \kappa \leqslant \alpha_1 < \frac{1}{2}}} \psi(\beta, p_1) \\ &= \psi(n, x^\kappa) - \sum_{\substack{n=p_1\beta \\ \kappa \leqslant \alpha_1 < \frac{3}{7}}} \psi(\beta, p_1) - \sum_{\substack{n=p_1\beta \\ \frac{3}{7} \leqslant \alpha_1 < 1-\theta}} \psi(\beta, p_1) - \sum_{\substack{n=p_1\beta \\ 1-\theta \leqslant \alpha_1 < \frac{1}{2}}} \psi(\beta, p_1) \\ &= \psi(n, x^\kappa) - \sum_{\substack{n=p_1\beta \\ \kappa \leqslant \alpha_1 < \frac{3}{7}}} \psi(\beta, x^\kappa) + \sum_{\substack{n=p_1p_2\beta \\ \kappa \leqslant \alpha_1 < \frac{3}{7} \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in \boldsymbol{G}_2}} \psi(\beta, p_2) + \sum_{\substack{n=p_1p_2\beta \\ \kappa \leqslant \alpha_1 < \frac{3}{7} \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in A \cup B}} \psi(\beta, p_2) \\ &\quad + \sum_{\substack{n=p_1p_2\beta \\ \kappa \leqslant \alpha_1 < \frac{3}{7} \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in C}} \psi(\beta, p_2) - \sum_{\substack{n=p_1\beta \\ \frac{3}{7} \leqslant \alpha_1 < 1-\theta}} \psi(\beta, p_1) - \sum_{\substack{n=p_1\beta \\ 1-\theta \leqslant \alpha_1 < \frac{1}{2}}} \psi(\beta, p_1) \\ &= \psi(n, x^\kappa) - \sum_{\substack{n=p_1\beta \\ \kappa \leqslant \alpha_1 < \frac{3}{7}}} \psi(\beta, x^\kappa) + \sum_{\substack{n=p_1p_2\beta \\ \kappa \leqslant \alpha_1 < \frac{3}{7} \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in \boldsymbol{G}_2}} \psi(\beta, p_2) \\ &\quad + \sum_{\substack{n=p_1p_2\beta \\ \kappa \leqslant \alpha_1 < \frac{3}{7} \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in A \cup B}} \psi(\beta, x^\kappa) - \sum_{\substack{n=p_1p_2p_3\beta \\ \kappa \leqslant \alpha_1 < \frac{3}{7} \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in A \cup B \\ \kappa \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right)}} \psi(\beta, p_3) \end{aligned}$$

$$+\sum_{\substack{n=p_1p_2\beta\\ \kappa\leqslant\alpha_1<\frac{3}{7}\\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\\ \boldsymbol{\alpha}_2\in C}}\psi(\beta,p_2)-\sum_{\substack{n=p_1\beta\\ \frac{3}{7}\leqslant\alpha_1<1-\theta}}\psi(\beta,p_1)-\sum_{\substack{n=p_1\beta\\ 1-\theta\leqslant\alpha_1<\frac{1}{2}}}\psi(\beta,p_1)$$
$$=\Sigma_{42201}-\Sigma_{42202}+\Sigma_{42203}+\Sigma_{42204}-\Sigma_{42205}+\Sigma_{42206}-\Sigma_{42207}-\Sigma_{42208}. \tag{132}$$

By Lemma 2.11, (118) holds for $f(n)=\Sigma_{42201}$ and $f(n)=\Sigma_{42202}$. By Lemma 2.10 and Lemma 2.16, (118) holds for $f(n)=\Sigma_{42203}$ and $f(n)=\Sigma_{42207}$. By Lemma 2.17, (118) holds for $f(n)=\Sigma_{42204}$. For the remaining sums, Σ_{42205} only counts numbers with 4 or more prime factors.

For Σ_{42206}, since we have $3\theta_1+2\theta_2<\frac{11}{7}$, $11\theta_1+12\theta_2<6$, $\theta<\frac{11}{21}<\frac{8}{15}$ and a Type-II range $\left(0,\ \frac{5-8\theta}{6}\right)$, we can use Lemma 3.7 and a three-dimensional Harman's sieve to get a "loss term"

$$\Sigma_{42209}=\sum_{\substack{n=m_1m_2m_3\\ \kappa\leqslant\alpha_1<\frac{3}{7}\\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\\ \boldsymbol{\alpha}_2\in C\\ \Omega(m_1m_2)\geqslant 3}}\psi\left(m_1m_2m_3,x^{\kappa}\right). \tag{133}$$

Since $\Omega(m_1m_2m_3)\geqslant\Omega(m_1m_2)+1\geqslant 4$, Σ_{42209} only counts numbers with 4 or more prime factors.

For Σ_{42208}, by a two-dimensional Harman's sieve in [[2], Section 7] (see Lemma 2.18), we have

$$\begin{aligned}\Sigma_{42208}=&\sum_{\substack{n=p_1\beta\\ 1-\theta\leqslant\alpha_1<\frac{1}{2}}}\psi(\beta,p_1)\\
=&\sum_{\substack{n=m_1m_2\\ 1-\theta\leqslant\alpha_1<\frac{1}{2}}}\psi\left(m_1m_2,x^{\frac{\theta+\varepsilon}{2}}\right)\\
=&\sum_{\substack{n=m_1m_2\\ 1-\theta\leqslant\alpha_1<\frac{1}{2}}}\psi\left(m_1m_2,x^{\kappa}\right)-\sum_{\substack{n=\beta_1\beta_2p_3\\ 1-\theta\leqslant\alpha_1+\alpha_3<\theta\\ \kappa\leqslant\alpha_3<\frac{\theta+\varepsilon}{2}}}\psi(\beta_1\beta_2,p_3)\\
=&\sum_{\substack{n=m_1m_2\\ 1-\theta\leqslant\alpha_1<\frac{1}{2}}}\psi\left(m_1m_2,x^{\kappa}\right)-\sum_{\substack{n=\beta_1\beta_2p_3\\ 1-\theta\leqslant\alpha_1+\alpha_3<\theta\\ \kappa\leqslant\alpha_3<\frac{\theta+\varepsilon}{2}}}\psi(\beta_1,p_3)\,\psi(\beta_2,p_3)\\
=&\sum_{\substack{n=m_1m_2\\ 1-\theta\leqslant\alpha_1<\frac{1}{2}}}\psi\left(m_1m_2,x^{\kappa}\right)-\sum_{\substack{n=\beta_1\beta_2p_3\\ 1-\theta\leqslant\alpha_1+\alpha_3<\theta\\ \kappa\leqslant\alpha_3<\frac{\theta+\varepsilon}{2}}}\psi(\beta_1,p_3)\,\psi(\beta_2,x^{\kappa})+\sum_{\substack{n=\beta_1\beta_2p_3p_4\\ 1-\theta\leqslant\alpha_1+\alpha_3<\theta\\ \kappa\leqslant\alpha_4<\alpha_3<\frac{\theta+\varepsilon}{2}}}\psi(\beta_1,p_3)\,\psi(\beta_2,p_4)\\
=&\,\Sigma_{42210}-\Sigma_{42211}+\Sigma_{42212}.\end{aligned} \tag{134}$$

Since $\theta<\frac{11}{21}<\frac{17}{32}$, (118) holds for $f(n)=\Sigma_{42210}$ by [[2], Lemma 19]. By Lemma 2.13 and the arguments in [[2], Page 78], (118) holds for $f(n)=\Sigma_{42211}$. Clearly Σ_{42212} only counts numbers with 4 or more prime factors.

Now, the proof of Theorem 4.6 reduces to showing that (118) holds for $f(n)=$ sums that count numbers with 4 or more prime factors. Since $\kappa>\frac{1}{8}$, we have $\Omega(n)\leqslant 7$. Assume that $\Omega(n)\geqslant 4$, and we only need to consider the following 7 cases:

Case 1: $\Omega(n)=7$**.** Suppose that $n=p_1\cdots p_7$ and $\alpha_1>\alpha_2>\cdots>\alpha_7$. Now we have $\alpha_6+\alpha_7<\frac{2}{7}$. Since we have $\theta_2<\frac{1-\theta}{2}\leqslant\frac{1}{4}$ and $\frac{1}{6}(5-4\theta_1-8\theta_2)>\frac{1}{3}>\frac{2}{7}$, if $\alpha_6+\alpha_7\in\left(\theta_2,\ \frac{2}{7}\right)$, then (118) holds for $f(n)$. Otherwise we have $\alpha_6+\alpha_7\leqslant\theta_2$. But since we have $8\theta_1+11\theta_2<5$, we get

$$\alpha_6+\alpha_7\leqslant\theta_2<\frac{5-8\theta}{3}<\alpha_6+\alpha_7, \tag{135}$$

making a contradiction. Hence (118) holds for $f(n)$ in **Case 1**.

Case 2: $\Omega(n)=6$**.** Suppose that $n=p_1\cdots p_6$ and $\alpha_1>\alpha_2>\cdots>\alpha_6$. Now we have $\alpha_5+\alpha_6<\frac{1}{3}$. Since we have $\theta_2<\frac{1-\theta}{2}\leqslant\frac{1}{4}$ and $\frac{1}{6}(5-4\theta_1-8\theta_2)>\frac{1}{3}$, if $\alpha_5+\alpha_6\in\left(\theta_2,\ \frac{1}{3}\right)$, then (118) holds for $f(n)$. Otherwise we have $\alpha_5+\alpha_6\leqslant\theta_2$. But since we have $8\theta_1+11\theta_2<5$, we get

$$\alpha_5+\alpha_6\leqslant\theta_2<\frac{5-8\theta}{3}<\alpha_5+\alpha_6, \tag{136}$$

making a contradiction. Hence (118) holds for $f(n)$ in **Case 2**.

Case 3: $\Omega(n)=4$ **or** 5**, no product of variables lies in** $[1-\theta,\ \theta]$**.** Suppose that $n=p_1\cdots p_k$ and $\alpha_1>\alpha_2>\cdots>\alpha_k$. In this case we can "view" $\left(\frac{3}{7},\ \frac{4}{7}\right)$ as a "fake" Type-II range. Since we have $\theta<\frac{11}{21}<\frac{9}{17}$, the results in [[28], Section 9] can be applied here if any of the following conditions holds:

(1). $\alpha_k\geqslant\frac{1}{7}$;

(2). $\Omega(n)=5$, $\alpha_3+\alpha_4+\alpha_5\geqslant\frac{4}{7}$;

(3). $\Omega(n)=4$, $\alpha_1\leqslant\frac{3}{7}$, $\alpha_1+\alpha_4\geqslant\frac{4}{7}$.

By Condition (1), we can assume that $\frac{1}{8} < \alpha_k < \frac{1}{7}$. We first consider the subcase $\Omega(n) = 5$. By Condition (2) and the "fake" Type-II range $\left(\frac{3}{7},\ \frac{4}{7}\right)$, we can also assume that $\alpha_3 + \alpha_4 + \alpha_5 \leqslant \frac{3}{7}$. This means that $\alpha_1 + \alpha_2 \geqslant \frac{4}{7}$, and thus $\alpha_1 \geqslant \frac{2}{7}$. Since $\theta_2 < \frac{1}{4} < \frac{2}{7} < \frac{1}{3} < \frac{1}{6}(5 - 4\theta_1 - 8\theta_2)$, we can assume that $\alpha_1 \geqslant \frac{1}{3}$. Now we have $\frac{3}{7} < \frac{11}{24} = \frac{1}{3} + \frac{1}{8} < \alpha_1 + \alpha_5$, and we can assume that $\alpha_1 + \alpha_5 \geqslant \frac{4}{7}$. Since $\alpha_5 < \frac{1}{7}$, we have $\alpha_1 > \frac{3}{7}$. Now we can assume that $\alpha_1 \geqslant \frac{4}{7}$. But now we have $1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 > \frac{4}{7} + 4 \cdot \frac{1}{8} > 1$, making a contradiction. Hence (118) holds for $f(n)$ in **Case 3** if $\Omega(n) = 5$.

Next, we consider the second subcase $\Omega(n) = 4$. Suppose that $\alpha_1 \leqslant \frac{3}{7}$. By Condition (3) and the "fake" Type-II range $\left(\frac{3}{7},\ \frac{4}{7}\right)$, we can assume that $\alpha_1 + \alpha_4 \leqslant \frac{3}{7}$. Since $\alpha_4 > \frac{1}{8}$, we have $\alpha_1 < \frac{17}{56} = \frac{3}{7} - \frac{1}{8}$. Since $\theta_2 < \frac{1}{4} < \frac{17}{56} < \frac{1}{3} < \frac{1}{6}(5 - 4\theta_1 - 8\theta_2)$, we can assume that $\alpha_1 \leqslant \frac{1}{4}$. But now we have $1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < 4 \cdot \frac{1}{4} = 1$, making a contradiction.

Now, by the "fake" Type-II range $\left(\frac{3}{7},\ \frac{4}{7}\right)$, we only need to consider the subcase $\Omega(n) = 4$ with $\alpha_1 \geqslant \frac{4}{7}$. By a simple observation, one can easily find that this type of n will not be counted in Σ_{42209} (note that Σ_{42209} only counts numbers with all prime factors smaller than $\max(m_1, m_2, m_3) \leqslant x^{\frac{3}{7}}$). Also, this type of n will not be counted in Σ_{42212}, since Σ_{42212} only counts numbers $n = \beta_1\beta_2$ with $\Omega(\beta_1), \Omega(\beta_2) \geqslant 2$ and $x^{1-\theta} \leqslant \beta_1, \beta_2 \leqslant x^{\theta}$. Hence, this type of n will only be counted in one part of Σ_{42205}:

$$\sum_{\substack{n = p_1 p_2 p_3 \beta \\ \alpha_1 + \alpha_2 + \alpha_3 \leqslant \frac{3}{7}}} \psi\left(\beta, p_3\right), \tag{137}$$

where β in (137) is a large prime. Using Buchstab's identity, we have

$$\sum_{\substack{n = p_1 p_2 p_3 \beta \\ \alpha_1 + \alpha_2 + \alpha_3 \leqslant \frac{3}{7}}} \psi\left(\beta, p_3\right) = \sum_{\substack{n = p_1 p_2 p_3 \beta \\ \alpha_1 + \alpha_2 + \alpha_3 \leqslant \frac{3}{7}}} \psi\left(\beta, x^{\kappa}\right) - \sum_{\substack{n = p_1 p_2 p_3 p_4 \beta \\ \alpha_1 + \alpha_2 + \alpha_3 \leqslant \frac{3}{7} \\ \kappa \leqslant \alpha_4 < \alpha_3}} \psi\left(\beta, p_4\right). \tag{138}$$

By Lemma 2.17, (118) holds for the first sum in (138) since $(\alpha_1 + \alpha_2, \alpha_3) \in A$, and we only need to deal with the second sum in (138) that counts numbers with 5 or more prime factors. Now we reduce this subcase to other cases with $\Omega(n) \geqslant 5$.

Case 4: $\Omega(n) = 5$, one variable lies in $[1-\theta,\ \theta]$. Suppose that $n = p_1 \cdots p_5$ and $\alpha_1 > \alpha_2 > \cdots > \alpha_5$. Now we have $\alpha_1 \geqslant 1 - \theta$ and $\alpha_2 > \cdots > \alpha_5 \geqslant \frac{5-8\theta}{6}$. But since $\theta < \frac{11}{21} < \frac{10}{19}$, we have

$$1 - \theta + 4 \cdot \frac{5 - 8\theta}{6} > 1, \tag{139}$$

making a contradiction. Hence (118) holds for (empty) $f(n)$ in **Case 4**.

Case 5: $\Omega(n) = 5$, a product of two variables lies in $[1-\theta,\ \theta]$. Suppose that $n = p_1 \cdots p_5$ and $\alpha_1 + \alpha_2 \in [1-\theta,\ \theta]$. Since $\alpha_i + \alpha_j > \frac{5-8\theta}{3} > \theta_2$ (see **Case 1**), we can assume that $\alpha_i + \alpha_j \geqslant \frac{1}{6}(5 - 4\theta_1 - 8\theta_2) > \frac{1}{3}$ for all $i, j \in \{1, 2, 3, 4, 5\}$ (otherwise $\alpha_i + \alpha_j \in \left(\theta_2,\ \frac{1}{6}(5 - 4\theta_1 - 8\theta_2)\right)$, and (118) holds for $f(n)$). Since $\alpha_1 + \alpha_2 \geqslant 1 - \theta$, we have $\max(\alpha_1, \alpha_2) \geqslant \frac{1-\theta}{2} > \theta_2$. If $\max(\alpha_1, \alpha_2) \in \left(\theta_2,\ \frac{1}{6}(5 - 4\theta_1 - 8\theta_2)\right)$, then (118) holds for $f(n)$. Otherwise we have $\max(\alpha_1, \alpha_2) \geqslant \frac{1}{6}(5 - 4\theta_1 - 8\theta_2) > \frac{1}{3}$, and thus

$$\alpha_1 + \cdots + \alpha_5 = \max(\alpha_1, \alpha_2) + (\min(\alpha_1, \alpha_2) + \alpha_3) + (\alpha_4 + \alpha_5) > \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1. \tag{140}$$

But since we have $\alpha_1 + \cdots + \alpha_5 = 1$, this makes a contradiction. Hence (118) holds for $f(n)$ in **Case 5**.

Case 6: $\Omega(n) = 4$, a product of two variables lies in $[1-\theta,\ \theta]$. Suppose that $n = p_1 p_2 p_3 p_4$ and $\alpha_1 + \alpha_2 \in [1-\theta,\ \theta]$ (of course, $\alpha_3 + \alpha_4 \in [1-\theta,\ \theta]$ too). Without loss of generality, we further assume that $\alpha_1 > \alpha_2$ and $\alpha_3 > \alpha_4$. Now we have $\alpha_2, \alpha_4 < \frac{\theta}{2}$. Since $\frac{1}{2} \leqslant \theta < \frac{11}{21}$, we have $\theta_2 < \frac{1-\theta}{2} < \frac{\theta}{2} < \frac{11}{42} < \frac{1}{3} < \frac{1}{6}(5 - 4\theta_1 - 8\theta_2)$. Now we can assume that $\alpha_2, \alpha_4 \leqslant \theta_2$ (otherwise α_2 or $\alpha_4 \in \left(\theta_2,\ \frac{1}{6}(5 - 4\theta_1 - 8\theta_2)\right)$, and (118) holds for $f(n)$). Since we can assume that $\alpha_2 + \alpha_4 > \frac{1}{3}$ (see **Case 5**), if $\alpha_1, \alpha_3 \geqslant \frac{1}{3}$, then $1 = \alpha_1 + \alpha_3 + (\alpha_2 + \alpha_4) > \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$, making a contradiction. Now we assume that $\min(\alpha_1, \alpha_3) < \frac{1}{3} < \frac{1}{6}(5 - 4\theta_1 - 8\theta_2)$. Without loss of generality, we further assume that $\alpha_1 > \alpha_3$. If $\alpha_3 \in \left(\theta_2,\ \frac{1}{6}(5 - 4\theta_1 - 8\theta_2)\right)$, then (118) holds for $f(n)$. Otherwise we have $\alpha_2, \alpha_3, \alpha_4 \leqslant \theta_2$. But now we have $\alpha_3 + \alpha_4 \leqslant 2\theta_2 < 1 - \theta$ (since $\theta_2 < \frac{1-\theta}{2}$), contradicting the assumption $\alpha_3 + \alpha_4 \in [1-\theta,\ \theta]$. Hence (118) holds for $f(n)$ in **Case 6**.

Case 7: $\Omega(n) = 4$, one variable lies in $[1-\theta,\ \theta]$. Suppose that $n = p_0 p_1 p_2 p_3$ and $\alpha_0 > \alpha_1 > \alpha_2 > \alpha_3$. Now we have $\alpha_0 \in [1-\theta,\ \theta]$ and $\alpha_1 + \alpha_2 + \alpha_3 \in [1-\theta,\ \theta]$. We also have $\alpha_1 + \alpha_3 < \frac{1}{2}$ and $\alpha_2 < \frac{1}{3}$. By a simple observation similar to that in **Case 3**, one can easily find that this type of n will only be counted in one part of Σ_{42205}:

$$\sum_{\substack{n = p_1 p_2 p_3 \beta \\ \alpha_1 + \alpha_2 + \alpha_3 \leqslant \theta}} \psi\left(\beta, p_3\right), \tag{141}$$

where β in (141) is a large prime. Using Buchstab's identity, we have

$$\sum_{\substack{n = p_1 p_2 p_3 \beta \\ \alpha_1 + \alpha_2 + \alpha_3 \leqslant \theta}} \psi\left(\beta, p_3\right) = \sum_{\substack{n = p_1 p_2 p_3 \beta \\ \alpha_1 + \alpha_2 + \alpha_3 \leqslant \theta}} \psi\left(\beta, x^{\kappa}\right) - \sum_{\substack{n = p_1 p_2 p_3 p_4 \beta \\ \alpha_1 + \alpha_2 + \alpha_3 \leqslant \theta \\ \kappa \leqslant \alpha_4 < \alpha_3}} \psi\left(\beta, p_4\right). \tag{142}$$

By Lemma 2.17, (118) holds for the first sum in (142) since $(\alpha_1 + \alpha_2, \alpha_3) \in A$, and we only need to deal with the second sum in (142) that counts numbers with 5 or more prime factors. Now we reduce this case to other cases with $\Omega(n) \geqslant 5$.

Combining all the 7 cases above, the proof of Theorem 4.6 is completed. ∎

Remark. Throughout the proof of Theorem 4.6, the **Case 6** is the most important case since it gives an asymptotic formula for the sum Σ_{42212}. The sum Σ_{42208} is very easy to handle when we have a Type-II range $(1-\theta,\ \theta)$. Without this range, we need to carefully discuss every subcases on the sizes of different variables. The proof of the other two cases of Theorem 1.1 is much more easier than the proof of Theorem 4.6, and we shall provide it in the cases $\boldsymbol{E}_{05}$ and $\boldsymbol{E}_{09}$.

When (θ_1, θ_2) lies in the boundary of this region:

$$\frac{1}{4} \leqslant \theta_1 < \frac{3}{10}, \quad \theta_2 = \min\left(\frac{3-4\theta_1}{8}, \frac{5-14\theta_1}{4}\right),$$

we can use the exactly same decomposing process to prove that $C_1'(\theta_1, \theta_2) = 1+\varepsilon$. For example, we have $C_1'(0.29, 0.23) = 1+\varepsilon$. However, we cannot use the same method to show $C_0'(\theta_1, \theta_2) = 1-\varepsilon$ since the sums Σ_{409} and Σ_{412} are negative and cannot be discarded in the lower bound case.

For the remaining parts of $\boldsymbol{E}_{0101}$ and $\boldsymbol{E}_{0202}$, the decompositions are similar to which in the case $\boldsymbol{A}_{0101}$ in Section 3. We also use the Type-II range $\left(\theta_2,\ \frac{5-4\theta_1-8\theta_2}{6}\right)$ to discard sums with products of variables lie in this range.

The Type-II range for $\boldsymbol{E}_{0203}$ is

$$\left(0,\ \frac{1}{4}(2-2\theta_1-5\theta_2)\right) \cup \left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-4\theta_1-8\theta_2)\right). \tag{143}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0102}$.

The Type-II range for $\boldsymbol{E}_{0204}$ is

$$\left(0,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right) \cup \left(\theta_1,\ \frac{1}{6}(5-4\theta_1-8\theta_2)\right). \tag{144}$$

The decompositions are similar to which in the non-asymptotic parts in the case $\boldsymbol{E}_{0202}$.

The Type-II range for $\boldsymbol{E}_{0205}$ is

$$\left(0,\ \frac{1}{4}(2-2\theta_1-5\theta_2)\right) \cup \left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right) \cup \left(\theta_1,\ \frac{1}{6}(5-4\theta_1-8\theta_2)\right). \tag{145}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0104}$.

The Type-II range for $\boldsymbol{E}_{0206}$ is

$$\left(0,\ \frac{1}{4}(2-2\theta_1-5\theta_2)\right) \cup \left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right). \tag{146}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0104}$.

The Type-II range for $\boldsymbol{E}_{0207}$ is

$$\left(0,\ \frac{1}{4}(2-2\theta_1-5\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right) \cup \left(\theta_1,\ \frac{1}{6}(5-4\theta_1-8\theta_2)\right). \tag{147}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0105}$.

The Type-II range for $\boldsymbol{E}_{0208}$ is

$$\left(0,\ \frac{1}{4}(2-2\theta_1-5\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right). \tag{148}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0106}$.

The Type-II range for $\boldsymbol{E}_{0209}$ is

$$\left(0,\ \frac{1}{4}(2-2\theta_1-5\theta_2)\right). \tag{149}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0107}$.

4.2.3. $\boldsymbol{E}_{03}$. We divide $\boldsymbol{E}_{03}$ into 7 subregions:

$$\boldsymbol{E}_{03} = \boldsymbol{E}_{0301} \cup \boldsymbol{E}_{0302} \cup \boldsymbol{E}_{0303} \cup \boldsymbol{E}_{0304} \cup \boldsymbol{E}_{0305} \cup \boldsymbol{E}_{0306} \cup \boldsymbol{E}_{0307},$$

where

$$\begin{aligned}
\boldsymbol{E}_{0301} &= \left\{(\theta_1,\theta_2) : \frac{7}{20} < \theta_1 \leqslant \frac{5}{14},\ \frac{1}{7}(6-14\theta_1) \leqslant \theta_2 < \frac{1}{14}(5-8\theta_1)\right\}, \\
\boldsymbol{E}_{0302} &= \left\{(\theta_1,\theta_2) : \frac{9}{28} < \theta_1 \leqslant \frac{9}{26},\ \frac{1}{7}(6-14\theta_1) \leqslant \theta_2 < \frac{1}{18}(9-16\theta_1)\right. \\
&\qquad \text{or } \frac{9}{26} < \theta_1 < \frac{7}{20},\ \frac{1}{7}(6-14\theta_1) \leqslant \theta_2 < \frac{1}{8}(5-10\theta_1) \\
&\qquad \left.\text{or } \frac{7}{20} \leqslant \theta_1 \leqslant \frac{5}{14},\ \frac{1}{14}(5-8\theta_1) < \theta_2 < \frac{1}{8}(5-10\theta_1)\right\}, \\
\boldsymbol{E}_{0303} &= \left\{(\theta_1,\theta_2) : \frac{9}{26} < \theta_1 \leqslant \frac{5}{14},\ \frac{1}{8}(5-10\theta_1) \leqslant \theta_2 < \frac{1}{18}(9-16\theta_1)\right\}, \\
\boldsymbol{E}_{0304} &= \left\{(\theta_1,\theta_2) : \frac{13}{42} < \theta_1 < \frac{9}{28},\ \frac{1}{7}(6-14\theta_1) \leqslant \theta_2 < \frac{1}{8}(5-10\theta_1)\right.
\end{aligned}$$

$$
\text{or } \frac{9}{28} \leqslant \theta_1 < \frac{9}{26},\ \frac{1}{18}(9-16\theta_1) < \theta_2 < \frac{1}{8}(5-10\theta_1) \Big\},
$$

$$
\begin{aligned}
\boldsymbol{E}_{0305} = \Big\{(\theta_1,\theta_2): &\frac{43}{140} < \theta_1 \leqslant \frac{13}{24},\ \frac{1}{7}(6-14\theta_1) \leqslant \theta_2 < \frac{1}{20}(11-20\theta_1) \\
\text{or } &\frac{13}{42} < \theta_1 < \frac{9}{26},\ \frac{1}{8}(5-10\theta_1) \leqslant \theta_2 < \frac{1}{20}(11-20\theta_1) \\
\text{or } &\frac{9}{26} \leqslant \theta_1 \leqslant \frac{5}{14},\ \frac{1}{18}(9-16\theta_1) < \theta_2 < \frac{1}{20}(11-20\theta_1) \Big\},
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{E}_{0306} = \Big\{(\theta_1,\theta_2): &\frac{25}{84} < \theta_1 \leqslant \frac{43}{140},\ \frac{1}{7}(6-14\theta_1) \leqslant \theta_2 < \frac{1}{10}(5-8\theta_1) \\
\text{or } &\frac{43}{140} < \theta_1 \leqslant \frac{5}{14},\ \frac{1}{20}(11-20\theta_1) \leqslant \theta_2 < \frac{1}{10}(5-8\theta_1) \Big\},
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{E}_{0307} = \Big\{(\theta_1,\theta_2): &\frac{2}{7} < \theta_1 \leqslant \frac{25}{84},\ \frac{1}{7}(6-14\theta_1) \leqslant \theta_2 < \frac{1}{4}(2-3\theta_1) \\
\text{or } &\frac{25}{84} < \theta_1 \leqslant \frac{5}{14},\ \frac{1}{10}(5-8\theta_1) \leqslant \theta_2 < \frac{1}{4}(2-3\theta_1) \Big\}.
\end{aligned}
$$

Note that we have $\theta < \frac{11}{20}$ for $(\theta_1,\theta_2) \in \boldsymbol{E}_{0301} \cup \boldsymbol{E}_{0302} \cup \boldsymbol{E}_{0303} \cup \boldsymbol{E}_{0304} \cup \boldsymbol{E}_{0305}$.

The Type-II range for $\boldsymbol{E}_{0301}$ is

$$
\left(0,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right) \cup \left(\theta_1,\ \frac{1}{6}(5-4\theta_1-8\theta_2)\right). \tag{150}
$$

The decompositions in this case will be discussed later together with the case $\boldsymbol{E}_{0501}$.

The Type-II range for $\boldsymbol{E}_{0302}$ is

$$
\left(0,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right) \cup \left(\theta_1,\ \frac{1}{6}(5-4\theta_1-8\theta_2)\right). \tag{151}
$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0204}$.

The Type-II range for $\boldsymbol{E}_{0303}$ is

$$
\left(0,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right). \tag{152}
$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0204}$.

The Type-II range for $\boldsymbol{E}_{0304}$ is

$$
\left(0,\ \frac{1}{5}(4-6\theta_1-8\theta_2)\right) \cup \left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right) \cup \left(\theta_1,\ \frac{1}{6}(5-4\theta_1-8\theta_2)\right). \tag{153}
$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0104}$.

The Type-II range for $\boldsymbol{E}_{0305}$ is

$$
\left(0,\ \frac{1}{5}(4-6\theta_1-8\theta_2)\right) \cup \left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right). \tag{154}
$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0206}$.

The Type-II range for $\boldsymbol{E}_{0306}$ is

$$
\left(0,\ \frac{1}{5}(4-6\theta_1-8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right). \tag{155}
$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0106}$.

The Type-II range for $\boldsymbol{E}_{0307}$ is

$$
\left(0,\ \frac{1}{5}(4-6\theta_1-8\theta_2)\right). \tag{156}
$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0107}$.

4.2.4. $\boldsymbol{E}_{04}$. We divide $\boldsymbol{E}_{04}$ into 3 subregions:

$$
\boldsymbol{E}_{04} = \boldsymbol{E}_{0401} \cup \boldsymbol{E}_{0402} \cup \boldsymbol{E}_{0403},
$$

where

$$
\begin{aligned}
\boldsymbol{E}_{0401} = \Big\{(\theta_1,\theta_2): &\frac{5}{14} < \theta_1 \leqslant \frac{29}{80},\ \frac{1}{2}(1-2\theta_1) \leqslant \theta_2 < \frac{1}{3}(-1+4\theta_1) \\
\text{or } &\frac{29}{80} < \theta_1 \leqslant \frac{3}{8},\ \frac{1}{2}(1-2\theta_1) \leqslant \theta_2 < \frac{1}{14}(5-8\theta_1) \Big\},
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{E}_{0402} = \Big\{(\theta_1,\theta_2): &\frac{29}{80} < \theta_1 \leqslant \frac{23}{62},\ \frac{1}{14}(5-8\theta_1) \leqslant \theta_2 < \frac{1}{3}(-1+4\theta_1) \\
\text{or } &\frac{23}{62} < \theta_1 \leqslant \frac{3}{8},\ \frac{1}{14}(5-8\theta_1) \leqslant \theta_2 < \frac{1}{8}(5-10\theta_1) \Big\},
\end{aligned}
$$

$$\boldsymbol{E}_{0403} = \left\{ (\theta_1, \theta_2) : \frac{23}{62} < \theta_1 \leqslant \frac{3}{8},\ \frac{1}{8}(5 - 10\theta_1) \leqslant \theta_2 < \frac{1}{3}(-1 + 4\theta_1) \right\}.$$

Note that we have $\theta < \frac{11}{20}$ for $(\theta_1, \theta_2) \in \boldsymbol{E}_{04}$.

The Type-II range for $\boldsymbol{E}_{0401}$ is

$$\left(0,\ \frac{1}{6}(5 - 8\theta_1 - 4\theta_2)\right) \cup \left(\theta_1,\ \frac{1}{6}(5 - 4\theta_1 - 8\theta_2)\right). \tag{157}$$

The decompositions in this case will be discussed later together with the case $\boldsymbol{E}_{0501}$.

The Type-II range for $\boldsymbol{E}_{0402}$ is

$$\left(0,\ \frac{1}{6}(5 - 8\theta_1 - 8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5 - 8\theta_1 - 4\theta_2)\right) \cup \left(\theta_1,\ \frac{1}{6}(5 - 4\theta_1 - 8\theta_2)\right). \tag{158}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0204}$.

The Type-II range for $\boldsymbol{E}_{0403}$ is

$$\left(0,\ \frac{1}{6}(5 - 8\theta_1 - 8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5 - 8\theta_1 - 4\theta_2)\right). \tag{159}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0204}$.

4.2.5. $\boldsymbol{E}_{05}$. We divide $\boldsymbol{E}_{05}$ into 6 subregions:

$$\boldsymbol{E}_{05} = \boldsymbol{E}_{0501} \cup \boldsymbol{E}_{0502} \cup \boldsymbol{E}_{0503} \cup \boldsymbol{E}_{0504} \cup \boldsymbol{E}_{0505} \cup \boldsymbol{E}_{0506},$$

where

$$\begin{aligned}
\boldsymbol{E}_{0501} = &\left\{ (\theta_1, \theta_2) : \frac{5}{14} < \theta_1 < \frac{29}{80},\ \frac{1}{3}(-1 + 4\theta_1) \leqslant \theta_2 < \frac{1}{14}(5 - 8\theta_1) \right\}, \\
\boldsymbol{E}_{0502} = &\left\{ (\theta_1, \theta_2) : \frac{5}{14} < \theta_1 \leqslant \frac{29}{80},\ \frac{1}{14}(5 - 8\theta_1) \leqslant \theta_2 < \frac{1}{8}(5 - 10\theta_1) \right. \\
&\qquad \left. \text{or } \frac{29}{80} < \theta_1 < \frac{23}{62},\ \frac{1}{3}(-1 + 4\theta_1) \leqslant \theta_2 < \frac{1}{8}(5 - 10\theta_1) \right\}, \\
\boldsymbol{E}_{0503} = &\left\{ (\theta_1, \theta_2) : \frac{5}{14} < \theta_1 \leqslant \frac{23}{62},\ \frac{1}{8}(5 - 10\theta_1) \leqslant \theta_2 < \frac{1}{18}(9 - 16\theta_1) \right. \\
&\qquad \left. \text{or } \frac{23}{62} < \theta_1 < \frac{3}{8},\ \frac{1}{3}(-1 + 4\theta_1) \leqslant \theta_2 < \frac{1}{18}(9 - 16\theta_1) \right\}, \\
\boldsymbol{E}_{0504} = &\left\{ (\theta_1, \theta_2) : \frac{5}{14} < \theta_1 \leqslant \frac{3}{8},\ \frac{1}{18}(9 - 16\theta_1) \leqslant \theta_2 < \frac{1}{20}(11 - 20\theta_1) \right\}, \\
\boldsymbol{E}_{0505} = &\left\{ (\theta_1, \theta_2) : \frac{5}{14} < \theta_1 \leqslant \frac{3}{8},\ \frac{1}{20}(11 - 20\theta_1) \leqslant \theta_2 < \frac{1}{10}(5 - 8\theta_1) \right\}, \\
\boldsymbol{E}_{0506} = &\left\{ (\theta_1, \theta_2) : \frac{5}{14} < \theta_1 \leqslant \frac{3}{8},\ \frac{1}{10}(5 - 8\theta_1) \leqslant \theta_2 < \frac{1}{4}(2 - 3\theta_1) \right\}.
\end{aligned}$$

Note that we have $\theta < \frac{11}{20}$ for $(\theta_1, \theta_2) \in \boldsymbol{E}_{0501} \cup \boldsymbol{E}_{0502} \cup \boldsymbol{E}_{0503} \cup \boldsymbol{E}_{0504}$.

The Type-II range for $\boldsymbol{E}_{0501}$ is

$$\left(0,\ \frac{1}{6}(5 - 8\theta_1 - 4\theta_2)\right) \cup \left(\theta_1,\ \frac{1}{6}(5 - 4\theta_1 - 8\theta_2)\right). \tag{160}$$

In this case we discuss $\boldsymbol{E}_{0201}$, $\boldsymbol{E}_{0301}$, $\boldsymbol{E}_{0401}$ and $\boldsymbol{E}_{0501}$, since the Type-II ranges for them are same:

$$\boldsymbol{\mathcal{Z}_2} = \boldsymbol{E}_{0201} \cup \boldsymbol{E}_{0301} \cup \boldsymbol{E}_{0401} \cup \boldsymbol{E}_{0501} = \left\{ (\theta_1, \theta_2) : \frac{1}{3} < \theta_1 \leqslant \frac{3}{8},\ \frac{1}{2}(1 - 2\theta_1) \leqslant \theta_2 < \frac{1}{14}(5 - 8\theta_1) \right\}.$$

First, we shall prove the following theorem, which is an equivalent form of the second case of Theorem 1.1:

Theorem 4.7. *Let $(\theta_1, \theta_2) \in \boldsymbol{\mathcal{Z}_2}$. Suppose that we have*

$$16\theta_1 + 8\theta_2 < 7.$$

Then (118) holds for

$$f(n) = \mathbb{1}_p(n),$$

and we have

$$C_1'(\theta_1, \theta_2) = C_0'(\theta_1, \theta_2) = 1.$$

Proof. Since we have $(\theta_1, \theta_2) \in \boldsymbol{\mathcal{Z}_2}$ and $16\theta_1 + 8\theta_2 < 7$, all of the following conditions hold true:

$$\kappa = \frac{5 - 8\theta_1 - 4\theta_2}{6} - \varepsilon > \frac{1}{4}, \quad \frac{1}{2} \leqslant \theta < \frac{11}{21}.$$

Now, we can decompose our $\mathbb{1}_p(n) = \psi\left(n, x^{\frac{1}{2}}\right)$ in a way similar to the decompositions in the proof of Theorem 4.6. By the discussions in Theorem 4.6, we need to show that (118) holds for $f(n) =$ sums that count numbers with 4 or more prime factors. Since $\kappa > \frac{1}{4}$, any $n \sim x$ with 4 or more prime factors must has at least one factor smaller than x^κ, and we can use our Type-II range $\left(0,\ \frac{5-8\theta_1-4\theta_2}{6}\right)$ to give an asymptotic formula. The proof of Theorem 4.6 is now completed. □

For the remaining parts, we use $\kappa = \frac{5-8\theta_1-4\theta_2}{6} - \varepsilon$ as the "starting point" and use the second Type-II range $\left(\theta_1,\ \frac{1}{6}(5-4\theta_1-8\theta_2)\right)$ to discard sums explicitly. The decompositions are similar to which in the case E_{0204}. Note that in those parts of $\boldsymbol{\mathcal{Z}_2}$ we still have $\kappa > \frac{1}{5}$, hence the only "loss contribution" comes from numbers with 4 prime factors that we cannot give asymptotic formulas, and that is why $\kappa > \frac{1}{4}$ is crucial in the proof of Theorem 4.7.

The Type-II range for $\boldsymbol{E}_{0502}$ is

$$\left(0,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right) \cup \left(\theta_1,\ \frac{1}{6}(5-4\theta_1-8\theta_2)\right). \tag{161}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0204}$.

The Type-II range for $\boldsymbol{E}_{0503}$ is

$$\left(0,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right). \tag{162}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0204}$.

The Type-II range for $\boldsymbol{E}_{0504}$ is

$$\left(0,\ \frac{1}{5}(4-6\theta_1-8\theta_2)\right) \cup \left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right). \tag{163}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0206}$.

The Type-II range for $\boldsymbol{E}_{0505}$ is

$$\left(0,\ \frac{1}{5}(4-6\theta_1-8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right). \tag{164}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0106}$.

The Type-II range for $\boldsymbol{E}_{0506}$ is

$$\left(0,\ \frac{1}{5}(4-6\theta_1-8\theta_2)\right). \tag{165}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0107}$.

4.2.6. $\boldsymbol{E}_{06}$. We divide $\boldsymbol{E}_{06}$ into 4 subregions:

$$\boldsymbol{E}_{06} = \boldsymbol{E}_{0601} \cup \boldsymbol{E}_{0602} \cup \boldsymbol{E}_{0603} \cup \boldsymbol{E}_{0604},$$

where

$$\begin{aligned}
\boldsymbol{E}_{0601} = &\left\{(\theta_1,\theta_2) : \frac{3}{8} < \theta_1 \leqslant \frac{15}{38},\ \frac{1}{2}(1-2\theta_1) \leqslant \theta_2 < \frac{1}{14}(5-8\theta_1)\right.\\
&\left.\qquad \text{or } \frac{15}{38} < \theta_1 \leqslant \frac{2}{5},\ \frac{1}{2}(1-2\theta_1) \leqslant \theta_2 < \frac{1}{8}(5-10\theta_1)\right\},\\
\boldsymbol{E}_{0602} = &\left\{(\theta_1,\theta_2) : \frac{15}{38} < \theta_1 \leqslant \frac{2}{5},\ \frac{1}{8}(5-10\theta_1) \leqslant \theta_2 < \frac{1}{14}(5-8\theta_1)\right\},\\
\boldsymbol{E}_{0603} = &\left\{(\theta_1,\theta_2) : \frac{3}{8} < \theta_1 < \frac{15}{38},\ \frac{1}{14}(5-8\theta_1) \leqslant \theta_2 < \frac{1}{8}(5-10\theta_1)\right\},\\
\boldsymbol{E}_{0604} = &\left\{(\theta_1,\theta_2) : \frac{3}{8} < \theta_1 < \frac{15}{38},\ \frac{1}{8}(5-10\theta_1) \leqslant \theta_2 < \frac{1}{3}(2-4\theta_1)\right.\\
&\left.\qquad \text{or } \frac{15}{38} \leqslant \theta_1 \leqslant \frac{2}{5},\ \frac{1}{14}(5-8\theta_1) \leqslant \theta_2 < \frac{1}{3}(2-4\theta_1)\right\}.
\end{aligned}$$

Note that we have $\theta < \frac{11}{20}$ for $(\theta_1,\theta_2) \in \boldsymbol{E}_{06}$.

The Type-II range for $\boldsymbol{E}_{0601}$ is

$$\left(0,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right) \cup \left(\theta_1,\ \frac{1}{6}(5-4\theta_1-8\theta_2)\right). \tag{166}$$

The decompositions are similar to which in the non-asymptotic parts in the case $\boldsymbol{E}_{0501}$.

The Type-II range for $\boldsymbol{E}_{0602}$ is

$$\left(0,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right). \tag{167}$$

The decompositions are similar to which in the non-asymptotic parts in the case $\boldsymbol{E}_{0501}$.

The Type-II range for $\boldsymbol{E}_{0603}$ is

$$\left(0,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right) \cup \left(\theta_1,\ \frac{1}{6}(5-4\theta_1-8\theta_2)\right). \tag{168}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0204}$.

The Type-II range for $\boldsymbol{E}_{0604}$ is

$$\left(0,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right)\cup\left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right). \tag{169}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0204}$.

4.2.7. $\boldsymbol{E}_{07}$. We divide $\boldsymbol{E}_{07}$ into 4 subregions:

$$\boldsymbol{E}_{07}=\boldsymbol{E}_{0701}\cup\boldsymbol{E}_{0702}\cup\boldsymbol{E}_{0703}\cup\boldsymbol{E}_{0704},$$

where

$$\begin{aligned}
\boldsymbol{E}_{0701}&=\left\{(\theta_1,\theta_2):\frac{3}{8}<\theta_1\leqslant\frac{2}{5},\ \frac{1}{3}(2-4\theta_1)\leqslant\theta_2<\frac{1}{18}(9-16\theta_1)\right\},\\
\boldsymbol{E}_{0702}&=\left\{(\theta_1,\theta_2):\frac{3}{8}<\theta_1\leqslant\frac{2}{5},\ \frac{1}{18}(9-16\theta_1)\leqslant\theta_2<\frac{1}{20}(11-20\theta_1)\right\},\\
\boldsymbol{E}_{0703}&=\left\{(\theta_1,\theta_2):\frac{3}{8}<\theta_1\leqslant\frac{2}{5},\ \frac{1}{20}(11-20\theta_1)\leqslant\theta_2<\frac{1}{10}(5-8\theta_1)\right\},\\
\boldsymbol{E}_{0704}&=\left\{(\theta_1,\theta_2):\frac{3}{8}<\theta_1\leqslant\frac{2}{5},\ \frac{1}{10}(5-8\theta_1)\leqslant\theta_2<\frac{1}{4}(2-3\theta_1)\right\}.
\end{aligned}$$

Note that we have $\theta<\frac{11}{20}$ for $(\theta_1,\theta_2)\in\boldsymbol{E}_{0701}\cup\boldsymbol{E}_{0702}$.

The Type-II range for $\boldsymbol{E}_{0701}$ is

$$\left(0,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right)\cup\left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right). \tag{170}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0204}$.

The Type-II range for $\boldsymbol{E}_{0702}$ is

$$\left(0,\ \frac{1}{5}(4-6\theta_1-8\theta_2)\right)\cup\left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right)\cup\left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right). \tag{171}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0206}$.

The Type-II range for $\boldsymbol{E}_{0703}$ is

$$\left(0,\ \frac{1}{5}(4-6\theta_1-8\theta_2)\right)\cup\left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right). \tag{172}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0106}$.

The Type-II range for $\boldsymbol{E}_{0704}$ is

$$\left(0,\ \frac{1}{5}(4-6\theta_1-8\theta_2)\right). \tag{173}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0107}$.

4.2.8. $\boldsymbol{E}_{08}$. By statement (3) of Theorem 4.5, we know that $C_1'(\theta_1,\theta_2)=C_0'(\theta_1,\theta_2)=1$ for $(\theta_1,\theta_2)\in\boldsymbol{E}_{08}$ if $7\theta_1+12\theta_2<4$. We divide the ramaining parts of $\boldsymbol{E}_{08}$ into 3 subregions:

$$\begin{aligned}
\boldsymbol{E}_{0801}&=\left\{(\theta_1,\theta_2):\frac{2}{5}<\theta_1<\frac{7}{16},\ \frac{1}{12}(4-7\theta_1)\leqslant\theta_2<\frac{1}{8}(5-10\theta_1)\right\},\\
\boldsymbol{E}_{0802}&=\left\{(\theta_1,\theta_2):\frac{2}{5}<\theta_1\leqslant\frac{13}{32},\ \frac{1}{8}(5-10\theta_1)\leqslant\theta_2<\frac{1}{14}(5-8\theta_1)\right.\\
&\qquad\text{or }\frac{13}{32}<\theta_1\leqslant\frac{7}{16},\ \frac{1}{8}(5-10\theta_1)\leqslant\theta_2<\frac{1}{3}(2-4\theta_1)\\
&\qquad\left.\text{or }\frac{7}{16}<\theta_1<\frac{4}{9},\ \frac{1}{12}(4-7\theta_1)\leqslant\theta_2<\frac{1}{3}(2-4\theta_1)\right\},\\
\boldsymbol{E}_{0803}&=\left\{(\theta_1,\theta_2):\frac{2}{5}<\theta_1<\frac{13}{32},\ \frac{1}{14}(5-8\theta_1)\leqslant\theta_2<\frac{1}{3}(2-4\theta_1)\right\}.
\end{aligned}$$

Note that we have $\theta<\frac{11}{20}$ for $(\theta_1,\theta_2)\in\boldsymbol{E}_{0801}\cup\boldsymbol{E}_{0802}\cup\boldsymbol{E}_{0803}$.

The Type-II range for $\boldsymbol{E}_{0801}$ is

$$\left(0,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right)\cup\left(\theta_1,\ \frac{1}{6}(5-4\theta_1-8\theta_2)\right). \tag{174}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0601}$.

The Type-II range for $\boldsymbol{E}_{0802}$ is

$$\left(0,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right). \tag{175}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0602}$.

The Type-II range for $\boldsymbol{E}_{0803}$ is

$$\left(0,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right)\cup\left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right). \tag{176}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0204}$.

4.2.9. $\boldsymbol{E}_{09}$. By statement (3) of Theorem 4.5, we know that $C_1'(\theta_1,\theta_2)=C_0'(\theta_1,\theta_2)=1$ for $(\theta_1,\theta_2)\in\boldsymbol{E}_{09}$ if $\theta_2<\min\left(\frac{4-7\theta_1}{12},\frac{10-19\theta_1}{20}\right)$. We divide the ramaining parts of $\boldsymbol{E}_{09}$ into 5 subregions:

$$\begin{aligned}
\boldsymbol{E}_{0901}=\ &\Big\{(\theta_1,\theta_2):\frac{13}{32}<\theta_1\leqslant\frac{4}{9},\ \frac{1}{3}(2-4\theta_1)\leqslant\theta_2<\frac{1}{14}(5-8\theta_1)\\
&\quad\text{or } \frac{4}{9}<\theta_1\leqslant\frac{9}{20},\ \frac{1}{12}(4-7\theta_1)\leqslant\theta_2<\frac{1}{14}(5-8\theta_1)\\
&\quad\text{or } \frac{9}{20}<\theta_1\leqslant\frac{5}{11},\ \frac{1}{12}(4-7\theta_1)\leqslant\theta_2<1-2\theta_1\\
&\quad\text{or } \frac{5}{11}<\theta_1<\frac{10}{21},\ \frac{1}{20}(10-19\theta_1)\leqslant\theta_2<1-2\theta_1\Big\},\\
\boldsymbol{E}_{0902}=\ &\Big\{(\theta_1,\theta_2):\frac{2}{5}<\theta_1<\frac{13}{32},\ \frac{1}{3}(2-4\theta_1)\leqslant\theta_2<\frac{1}{18}(9-16\theta_1)\\
&\quad\text{or } \frac{13}{32}\leqslant\theta_1<\frac{9}{20},\ \frac{1}{14}(5-8\theta_1)<\theta_2<\frac{1}{18}(9-16\theta_1)\Big\},\\
\boldsymbol{E}_{0903}=\ &\Big\{(\theta_1,\theta_2):\frac{2}{5}<\theta_1<\frac{9}{20},\ \frac{1}{18}(9-16\theta_1)\leqslant\theta_2<\frac{1}{20}(11-20\theta_1)\Big\},\\
\boldsymbol{E}_{0904}=\ &\Big\{(\theta_1,\theta_2):\frac{2}{5}<\theta_1\leqslant\frac{5}{12},\ \frac{1}{20}(11-20\theta_1)\leqslant\theta_2<\frac{1}{10}(5-8\theta_1)\\
&\quad\text{or } \frac{5}{12}<\theta_1<\frac{9}{20},\ \frac{1}{20}(11-20\theta_1)\leqslant\theta_2<1-2\theta_1\Big\},\\
\boldsymbol{E}_{0905}=\ &\Big\{(\theta_1,\theta_2):\frac{2}{5}<\theta_1<\frac{5}{12},\ \frac{1}{10}(5-8\theta_1)\leqslant\theta_2<1-2\theta_1\Big\}.
\end{aligned}$$

Note that we have $\theta<\frac{11}{20}$ for $(\theta_1,\theta_2)\in\boldsymbol{E}_{0901}\cup\boldsymbol{E}_{0902}\cup\boldsymbol{E}_{0903}$.

The Type-II range for $\boldsymbol{E}_{0901}$ is

$$\left(0,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right). \tag{177}$$

First, we shall prove the following theorem, which, combined with [[28], Theorem 1.1], implies the third case of Theorem 1.1:

Theorem 4.8. *Define*

$$\begin{aligned}
\boldsymbol{\mathcal{Z}_3}=\ &\Big\{(\theta_1,\theta_2):\frac{5}{11}<\theta_1\leqslant\frac{8}{17},\ \frac{1}{20}(10-19\theta_1)\leqslant\theta_2<\frac{1}{12}(4-7\theta_1)\\
&\quad\text{or } \frac{8}{17}<\theta_1<\frac{10}{21},\ \frac{1}{20}(10-19\theta_1)\leqslant\theta_2<1-2\theta_1\Big\}.
\end{aligned}$$

Let $(\theta_1,\theta_2)\in\boldsymbol{\mathcal{Z}_3}$. Then (118) holds for

$$f(n)=\mathbb{1}_p(n),$$

and we have

$$C_1'(\theta_1,\theta_2)=C_0'(\theta_1,\theta_2)=1.$$

Proof. Since we have $(\theta_1,\theta_2)\in\boldsymbol{\mathcal{Z}_3}$, all of the following conditions hold true:

$$\kappa=\frac{5-8\theta_1-4\theta_2}{6}-\varepsilon>\frac{1}{6},\quad \frac{1}{2}\leqslant\theta<\frac{9}{17},\quad 11\theta_1+12\theta_2<6,\quad 3\theta_1+2\theta_2<\frac{11}{7}.$$

Now, we can decompose our $\mathbb{1}_p(n)=\psi\left(n,x^{\frac{1}{2}}\right)$ in a way similar to the decompositions in the proof of Theorem 4.6. By Buchstab's identity, we have

$$\begin{aligned}
\psi\left(n,x^{\frac{1}{2}}\right)=\ &\psi\left(n,x^{\kappa}\right)-\sum_{\substack{n=p_1\beta\\ \kappa\leqslant\alpha_1<\frac{3}{7}}}\psi\left(\beta,x^{\kappa}\right)+\sum_{\substack{n=p_1p_2\beta\\ \kappa\leqslant\alpha_1<\frac{3}{7}\\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\\ \boldsymbol{\alpha}_2\in\boldsymbol{G}_2}}\psi\left(\beta,p_2\right)\\
&+\sum_{\substack{n=p_1p_2\beta\\ \kappa\leqslant\alpha_1<\frac{3}{7}\\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\\ \boldsymbol{\alpha}_2\in A}}\psi\left(\beta,x^{\kappa}\right)-\sum_{\substack{n=p_1p_2p_3\beta\\ \kappa\leqslant\alpha_1<\frac{3}{7}\\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\\ \boldsymbol{\alpha}_2\in A\\ \kappa\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)}}\psi\left(\beta,p_3\right)\\
&+\sum_{\substack{n=p_1p_2\beta\\ \kappa\leqslant\alpha_1<\frac{3}{7}\\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\\ \boldsymbol{\alpha}_2\in B\cup C}}\psi\left(\beta,p_2\right)-\sum_{\substack{n=p_1\beta\\ \frac{3}{7}\leqslant\alpha_1<1-\theta}}\psi\left(\beta,p_1\right)-\sum_{\substack{n=p_1\beta\\ 1-\theta\leqslant\alpha_1<\frac{1}{2}}}\psi\left(\beta,p_1\right)
\end{aligned}$$

$$= \Sigma_{42901} - \Sigma_{42902} + \Sigma_{42903} + \Sigma_{42904} - \Sigma_{42905} + \Sigma_{42906} - \Sigma_{42907} - \Sigma_{42908}. \tag{178}$$

By Lemma 2.11 and a Type-II range $\left(0,\ \frac{5-8\theta_1-4\theta_2}{6}\right)$, (118) holds for $f(n) = \Sigma_{42901}$ and $f(n) = \Sigma_{42902}$. By Lemma 2.10 and Lemma 2.16, (118) holds for $f(n) = \Sigma_{42903}$ and $f(n) = \Sigma_{42907}$. By the Type-II range $\left(0,\ \frac{5-8\theta_1-4\theta_2}{6}\right)$ and the discussions in the end of the three-dimensional sieves (122) in this section, (118) holds for $f(n) = \Sigma_{42904}$. For the remaining sums, Σ_{42905} only counts numbers with 4 or more prime factors.

For Σ_{42906}, since we have $3\theta_1 + 2\theta_2 < \frac{11}{7}$, $11\theta_1 + 12\theta_2 < 6$, $\theta < \frac{9}{17} < \frac{8}{15}$ and a Type-II range $\left(0,\ \frac{5-8\theta_1-4\theta_2}{6}\right)$, we can use Lemma 3.7 and a three-dimensional Harman's sieve to get a "loss term"

$$\Sigma_{42909} = \sum_{\substack{n=m_1m_2m_3 \\ \kappa \leqslant \alpha_1 < \frac{3}{7} \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in B \cup C \\ \Omega(m_1m_2) \geqslant 3}} \psi\left(m_1m_2m_3, x^{\kappa}\right). \tag{179}$$

Since $\Omega(m_1m_2m_3) \geqslant \Omega(m_1m_2) + 1 \geqslant 4$, Σ_{42909} only counts numbers with 4 or more prime factors.

For Σ_{42908}, by a two-dimensional Harman's sieve in [[2], Section 7] (see Lemma 2.18), we have

$$\begin{aligned}
\Sigma_{42908} &= \sum_{\substack{n=p_1\beta \\ 1-\theta \leqslant \alpha_1 < \frac{1}{2}}} \psi\left(\beta, p_1\right) \\
&= \sum_{\substack{n=m_1m_2 \\ 1-\theta \leqslant \alpha_1 < \frac{1}{2}}} \psi\left(m_1m_2, x^{\frac{\theta+\varepsilon}{2}}\right) \\
&= \sum_{\substack{n=m_1m_2 \\ 1-\theta \leqslant \alpha_1 < \frac{1}{2}}} \psi\left(m_1m_2, x^{\kappa}\right) - \sum_{\substack{n=\beta_1\beta_2p_3 \\ 1-\theta \leqslant \alpha_1+\alpha_3 < \theta \\ \kappa \leqslant \alpha_3 < \frac{\theta+\varepsilon}{2}}} \psi\left(\beta_1\beta_2, p_3\right) \\
&= \sum_{\substack{n=m_1m_2 \\ 1-\theta \leqslant \alpha_1 < \frac{1}{2}}} \psi\left(m_1m_2, x^{\kappa}\right) - \sum_{\substack{n=\beta_1\beta_2p_3 \\ 1-\theta \leqslant \alpha_1+\alpha_3 < \theta \\ \kappa \leqslant \alpha_3 < \frac{\theta+\varepsilon}{2}}} \psi\left(\beta_1, p_3\right) \psi\left(\beta_2, p_3\right) \\
&= \sum_{\substack{n=m_1m_2 \\ 1-\theta \leqslant \alpha_1 < \frac{1}{2}}} \psi\left(m_1m_2, x^{\kappa}\right) - \sum_{\substack{n=\beta_1\beta_2p_3 \\ 1-\theta \leqslant \alpha_1+\alpha_3 < \theta \\ \kappa \leqslant \alpha_3 < \frac{\theta+\varepsilon}{2}}} \psi\left(\beta_1, p_3\right) \psi\left(\beta_2, x^{\kappa}\right) + \sum_{\substack{n=\beta_1\beta_2p_3p_4 \\ 1-\theta \leqslant \alpha_1+\alpha_3 < \theta \\ \kappa \leqslant \alpha_4 < \alpha_3 < \frac{\theta+\varepsilon}{2}}} \psi\left(\beta_1, p_3\right) \psi\left(\beta_2, p_4\right) \\
&= \Sigma_{42910} - \Sigma_{42911} + \Sigma_{42912}.
\end{aligned} \tag{180}$$

Since $\theta < \frac{9}{17} < \frac{17}{32}$, (118) holds for $f(n) = \Sigma_{42910}$ by [[2], Lemma 19] and a Type-II range $\left(0,\ \frac{5-8\theta_1-4\theta_2}{6}\right)$. By Lemma 2.13, the Type-II range $\left(0,\ \frac{5-8\theta_1-4\theta_2}{6}\right)$ and the arguments in [[2], Page 78], (118) holds for $f(n) = \Sigma_{42911}$. Clearly Σ_{42912} only counts numbers with 4 or more prime factors.

Now, the proof of Theorem 4.8 reduces to showing that (118) holds for $f(n) =$ sums that count numbers with 4 or more prime factors. The proof of Theorem 4.8 is thus completed by applying Lemma 3.8 on those sums, since we have $\kappa > \frac{1}{6} > \frac{1}{7}$. ☐

For the remaining parts, the decompositions are similar to which in the case E_{0602}.

The Type-II range for $\boldsymbol{E}_{0902}$ is

$$\left(0,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right). \tag{181}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0204}$.

The Type-II range for $\boldsymbol{E}_{0903}$ is

$$\left(0,\ \frac{1}{5}(4-6\theta_1-8\theta_2)\right) \cup \left(2\theta_1+2\theta_2-1,\ \frac{1}{6}(5-8\theta_1-8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right). \tag{182}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0206}$.

The Type-II range for $\boldsymbol{E}_{0904}$ is

$$\left(0,\ \frac{1}{5}(4-6\theta_1-8\theta_2)\right) \cup \left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right). \tag{183}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0106}$.

The Type-II range for $\boldsymbol{E}_{0905}$ is

$$\left(0,\ \frac{1}{5}(4-6\theta_1-8\theta_2)\right). \tag{184}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{0107}$.

4.2.10. $\boldsymbol{E}_{10}$. We divide $\boldsymbol{E}_{10}$ into 3 subregions:

$$\boldsymbol{E}_{10}=\boldsymbol{E}_{1001}\cup\boldsymbol{E}_{1002}\cup\boldsymbol{E}_{1003},$$

where

$$\begin{aligned}
\boldsymbol{E}_{1001}=&\left\{(\theta_1,\theta_2):\frac{5}{11}<\theta_1<\frac{1}{2},\ 1-2\theta_1\leqslant\theta_2<\frac{1}{15}(5-8\theta_1)\right\},\\
\boldsymbol{E}_{1002}=&\left\{(\theta_1,\theta_2):\frac{5}{12}<\theta_1\leqslant\frac{7}{16},\ 1-2\theta_1\leqslant\theta_2<\frac{1}{10}(5-8\theta_1)\right.\\
&\quad\text{or }\frac{7}{16}<\theta_1<\frac{5}{11},\ 1-2\theta_1\leqslant\theta_2<\frac{1}{15}(11-20\theta_1)\\
&\quad\left.\text{or }\frac{5}{11}\leqslant\theta_1<\frac{1}{2},\ \frac{1}{15}(5-8\theta_1)\leqslant\theta_2<\frac{1}{15}(11-20\theta_1)\right\},\\
\boldsymbol{E}_{1003}=&\left\{(\theta_1,\theta_2):\frac{2}{5}<\theta_1\leqslant\frac{5}{12},\ 1-2\theta_1\leqslant\theta_2<\frac{1}{15}(11-20\theta_1)\right.\\
&\quad\left.\text{or }\frac{5}{12}<\theta_1<\frac{7}{16},\ \frac{1}{10}(5-8\theta_1)\leqslant\theta_2<\frac{1}{15}(11-20\theta_1)\right\}.
\end{aligned}$$

Note that we have $\theta>\frac{17}{32}$ for $(\theta_1,\theta_2)\in\boldsymbol{E}_{1002}\cup\boldsymbol{E}_{1003}$.

The Type-II range for $\boldsymbol{E}_{1001}$ is

$$\left(2\theta_1+\theta_2-1,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right). \tag{185}$$

We first assume that $\theta<\frac{17}{32}$. Similar to the case $(\theta_1,\theta_2)\in\boldsymbol{A}_{1101}$, we want to replace $\boldsymbol{U}_j$ with $\boldsymbol{U}_j''$ in the decompositions. Let $\boldsymbol{\alpha}_k\in\boldsymbol{A}_k$. By a decomposing process similar to (98), we only need to prove that if

$$\alpha_{j+1}+\cdots+\alpha_{k-1}<2\theta_1+\theta_2-1+\varepsilon\leqslant\alpha_{j+1}+\cdots+\alpha_k$$

and

$$\alpha_{j+1}<\frac{5-8\theta_1-4\theta_2}{6}-\varepsilon$$

hold for some j $(0\leqslant j\leqslant k-1)$, then $\boldsymbol{\alpha}_k\in\boldsymbol{G}_k$.

When $\alpha_k<\frac{11-20\theta_1-10\theta_2}{6}-2\varepsilon$, then

$$2\theta_1+\theta_2-1+\varepsilon\leqslant\alpha_{j+1}+\cdots+\alpha_k<(2\theta_1+\theta_2-1+\varepsilon)+\frac{11-20\theta_1-10\theta_2}{6}-2\varepsilon=\frac{5-8\theta_1-4\theta_2}{6}-\varepsilon$$

and $\boldsymbol{\alpha}_k\in\boldsymbol{G}_k$.

Suppose that $\alpha_k\geqslant\frac{11-20\theta_1-10\theta_2}{6}-2\varepsilon$. Since $\boldsymbol{\alpha}_k\in\boldsymbol{A}_k$, we have $\alpha_k<\alpha_{j+1}<\frac{5-8\theta_1-4\theta_2}{6}-\varepsilon$. Now we only need to prove that

$$\frac{11-20\theta_1-10\theta_2}{6}-2\varepsilon\geqslant2\theta_1+\theta_2-1+\varepsilon,$$

or

$$32\theta_1+16\theta_2\leqslant17-18\varepsilon$$

when $\theta<\frac{17}{32}$ and $(\theta_1,\theta_2)\in\boldsymbol{E}_{1001}$. A simple verification then completes the proof. The remaining decompositions are similar to which in the case $\boldsymbol{A}_{1101}$ in Section 3.

Now we assume that $\theta\geqslant\frac{17}{32}$. In this case we use the Type-II range $\left(2\theta_1+\theta_2-1,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right)$ to remove sums explicitly.

The Type-II range for $\boldsymbol{E}_{1002}$ is

$$\left(2\theta_1+\theta_2-1,\ \frac{1}{6}(5-8\theta_1-9\theta_2)\right)\cup\left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right). \tag{186}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{1001}$, where the Type-II range $\left(\theta_2,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right)$ is used to remove sums explicitly.

The Type-II range for $\boldsymbol{E}_{1003}$ is

$$\left(2\theta_1+\theta_2-1,\ \frac{1}{6}(5-8\theta_1-9\theta_2)\right). \tag{187}$$

Since we have $\theta>0.57$ in this region, we do not perform any decompositions here.

4.2.11. $\boldsymbol{E}_{11}$. Since we have $\frac{1}{15}(11-20\theta_1)<\frac{1}{10}(11-20\theta_1)$ when $\theta_1<\frac{11}{20}$, by Lemma 4.3 and the definition of the region $\boldsymbol{F}$, we can extend the region $\boldsymbol{E}_{11}$ to a larger one:

$$\boldsymbol{E}_{11}'=\left\{(\theta_1,\theta_2):\frac{1}{2}\leqslant\theta_1<\frac{11}{20},\ 0<\theta_2<\frac{1}{10}(11-20\theta_1)\right\}.$$

The Type-II range for $\boldsymbol{E}_{11}'$ is

$$\left(2\theta_1+\theta_2-1,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right). \tag{188}$$

The decompositions are similar to which in the case $\boldsymbol{E}_{1001}$. When $\theta<\frac{17}{32}$, we can replace $\boldsymbol{U}_j$ with $\boldsymbol{U}_j''$ in the decompositions. When $\theta\geqslant\frac{17}{32}$, we use the Type-II range $\left(2\theta_1+\theta_2-1,\ \frac{1}{6}(5-8\theta_1-4\theta_2)\right)$ to remove sums explicitly.

The decompositions in other parts of $\boldsymbol{U}\backslash\boldsymbol{J}$ stay the same as in [23] and in Section 2. Working on each case above, we can get the following upper bounds for $C_1'(\theta_1,\theta_2)$ $(0.5<\theta\leqslant 0.56)$:

0.28	1.8027	2.2243	2.5345	—	—	—	—	—	—	—	—	—	—	—	—
0.27	1.6342	1.8782	2.2243	2.5345	—	—	—	—	—	—	—	—	—	—	—
0.26	1.0336	1.6342	1.8027	2.2091	2.5345	—	—	—	—	—	—	—	—	—	—
0.25	1.0022	1.0201	1.5567	1.7861	2.2255	2.5048	—	—	—	—	—	—	—	—	—
0.24	1	1	1.0013	1.5197	1.8116	2.2230	2.5197	—	—	—	—	—	—	—	—
0.23	1	1	1	$1+\varepsilon$	1.5624	1.8324	2.2613	2.5022	—	—	—	—	—	—	—
0.22	1	1	1	1	1.0100	1.6019	1.8812	2.2844	2.5041	—	—	—	—	—	—
0.21	1	1	1	1	1	1.0100	1.6271	1.8812	2.3149	2.4790	—	—	—	—	—
0.20	1	1	1	1	1	1	1.0066	1.6442	1.8988	2.3137	2.4699	—	—	—	—
0.19	1	1	1	1	1	1	1	1.0027	1.6526	1.8835	2.2852	2.4592	—	—	—
0.18	1	1	1	1	1	1	1	1	1.0055	1.6779	1.8846	2.2861	2.4446	—	—
0.17	1	1	1	1	1	1	1	1	1.0005	1.0085	1.7048	1.8606	2.2883	2.4274	—
0.16	1	1	1	1	1	1	1	1	1	1.0007	1.0097	1.0957	1.8643	2.2729	2.4124
0.15	1	1	1	1	1	1	1	1	1	1	1	1.0069	1.1167	1.8644	2.2680
0.14	1	1	1	1	1	1	1	1	1	1	1	1.0001	1.0028	1.0704	1.8592
0.13	1	1	1	1	1	1	1	1	1	1	1	1	1.0006	1.0033	1.0274
0.12	1	1	1	1	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.0006	1.0031
0.11	1	1	1	1	1	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.0003
$\boldsymbol{\theta_2\backslash\theta_1}$	**0.26**	**0.27**	**0.28**	**0.29**	**0.30**	**0.31**	**0.32**	**0.33**	**0.34**	**0.35**	**0.36**	**0.37**	**0.38**	**0.39**	**0.40**

Table 4.1: Upper Bounds for $C_1'(\theta_1,\theta_2)$ $(0.5<\theta\leqslant 0.56)$ 1/2

0.15	2.4075	—	—	—	—	—	—	—	—	—	—	—	—	—	—
0.14	2.2416	2.4033	—	—	—	—	—	—	—	—	—	—	—	—	—
0.13	1.8541	2.2451	2.3900	—	—	—	—	—	—	—	—	—	—	—	—
0.12	1.0138	1.8484	2.2388	2.3930	—	—	—	—	—	—	—	—	—	—	—
0.11	1.0030	1.0174	1.8455	2.2375	2.3891	—	—	—	—	—	—	—	—	—	—
0.10	1.0005	1.0041	1.0265	1.8464	2.2577	2.3885	—	—	—	—	—	—	—	—	—
0.09	1	1.0003	1.0072	1.0397	1.8463	2.2082	2.3888	—	—	—	—	—	—	—	—
0.08	1	1	1	1.0125	1.0545	1.8476	2.2383	2.3907	—	—	—	—	—	—	—
0.07	1	1	1	1	1	1.0641	1.8498	2.2535	2.4108	—	—	—	—	—	—
0.06	1	1	1	1	1	1	1.7218	1.8561	2.2675	2.4313	—	—	—	—	—
0.05	1	1	1	1	1	1	1	1.6848	1.8642	2.2987	2.4608	—	—	—	—
0.04	1	1	1	1	1	1	1	$1+\varepsilon$	1.7104	1.8806	2.3027	2.4918	—	—	—
0.03	1	1	1	1	1	1	1	1	1.0121	1.7659	1.8963	2.3271	2.5324	—	—
0.02	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.0579	1.7923	1.9316	2.3483	2.5345	—
0.01	1	1	1	1	1	1	1	1	1	1.0110	1.6301	1.8169	1.9658	2.3633	2.5345
$\boldsymbol{\theta_2\backslash\theta_1}$	**0.41**	**0.42**	**0.43**	**0.44**	**0.45**	**0.46**	**0.47**	**0.48**	**0.49**	**0.50**	**0.51**	**0.52**	**0.53**	**0.54**	**0.55**

Table 4.2: Upper Bounds for $C_1'(\theta_1,\theta_2)$ $(0.5<\theta\leqslant 0.56)$ 2/2

4.3. **Lower Bounds.** We shall construct the minorant $\rho_0(n)$ in this subsection. Before constructing, we first mention some existing results of $C_0'(\theta_1,\theta_2)$.

Theorem 4.9. *The function $C_0(\theta_1,\theta_2)$ satisfies the following conditions:*

(1). $C_0'(\theta_1,\theta_2)=C_0'(\theta_2,\theta_1)$;
(2). $C_0'(\theta_1,\theta_2)=1$ for all θ_1,θ_2 satisfy $\theta_1+\theta_2<0.5$;
(3). $C_0'(\theta_1,\theta_2)=1$ for all θ_1,θ_2 satisfy $2\theta_1+\theta_2<1$, $7\theta_1+12\theta_2<4$ and $19\theta_1+20\theta_2<10$;
(4). $C_0'(\theta_1,\theta_2)=1$ for all θ_1,θ_2 satisfy $\theta_1<\frac{1}{3}$, $\theta_2<\frac{1}{5}$ and $\theta_1+\theta_2<\frac{29}{56}$;
(5). $C_0'(\theta_1,\theta_2)=1$ for all θ_1,θ_2 satisfy $\theta_1+3\theta_2<1$, $\theta_1+\theta_2<\frac{29}{56}$ and $\theta_2<\max\left(\frac{1-2\theta_1}{2},\frac{2-2\theta_1}{5}\right)$;
(6). $C_0'(\theta_1,\theta_2)=1$ for all θ_1,θ_2 satisfy $\theta_1+3\theta_2<1$, $\theta_1+\theta_2<\frac{29}{56}$, $4\theta_1+\theta_2<\frac{403}{266}$ and $\frac{7}{4}\theta_1+\theta_2<\frac{403}{532}$;
(7). $C_0'(\theta_1,\theta_2)\geqslant C_0(\theta_1,\theta_2)\geqslant C_0(\theta_1+\theta_2)$ for $0.5\leqslant\theta_1+\theta_2\leqslant 1$;
(8). $C_0'(\theta_1,\theta_2)\geqslant 1-\varepsilon$ for all θ_1,θ_2 satisfy $\theta_1+\theta_2=0.5$.

Proof. The first statement is obvious. Statements (2)–(6) follow easily from the Bombieri–Vinogradov Theorem, [[28], Theorem 1.1], [[3], Theorem 3] and [[13], Page 621 and Corollaire 5]. The seventh statement holds trivially by the work done in Section 2 and Section 3. When there are no new arithmetic information inputs outside of those in previous sections, we use $C_0(\theta_1,\theta_2)$ as a lower bound for $C_0'(\theta_1,\theta_2)$. The eighth statement holds from the seven statement and statement (5) of Theorem 3.10. □

Again, we use two different methods to construct $\rho_0(n)$: The first is Harman's sieve, and the second is due to Mikawa [32].

4.3.1. *First Method.* The first method is to use Harman's sieve to construct $\rho_0(n)$. Again, we can only discard positive terms that do not have asymptotic formulas in this case.

Suppose first that $\frac{1}{2} \leqslant \theta < \frac{17}{32}$. The main steps remain the same as in Subsubsection 2.5.1, but now we can use the new Type-II information corresponding to different (θ_1, θ_2) given in Subsection 3.3 and Subsection 4.2. Modifications need to do in the lower bound case are similar to those in the upper bound case; However, we do not need to consider the validity of three-dimensional sieves since they only give upper bounds for positive terms.

Now suppose that $\frac{17}{32} \leqslant \theta < \frac{7}{13}$. Using Buchstab's identity, we get

$$\begin{aligned}
\psi\left(n, x^{\frac{1}{2}}\right) &= \psi\left(n, x^{\kappa}\right) - \sum_{\substack{n=p_1\beta \\ \kappa \leqslant \alpha_1 < \frac{1}{2}}} \psi\left(\beta, p_1\right) \\
&= \psi\left(n, x^{\kappa}\right) - \sum_{\substack{n=p_1\beta \\ \kappa \leqslant \alpha_1 < \frac{3}{7}}} \psi\left(\beta, p_1\right) - \sum_{\substack{n=p_1\beta \\ \frac{3}{7} \leqslant \alpha_1 < 1-\theta}} \psi\left(\beta, p_1\right) - \sum_{\substack{n=p_1\beta \\ 1-\theta \leqslant \alpha_1 < \frac{1}{2}}} \psi\left(\beta, p_1\right) \\
&= \psi\left(n, x^{\kappa}\right) - \sum_{\substack{n=p_1\beta \\ \kappa \leqslant \alpha_1 < \frac{3}{7}}} \psi\left(\beta, x^{\kappa}\right) + \sum_{\substack{n=p_1p_2\beta \\ \kappa \leqslant \alpha_1 < \frac{3}{7} \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)}} \psi\left(\beta, p_2\right) \\
&\quad - \sum_{\substack{n=p_1\beta \\ \frac{3}{7} \leqslant \alpha_1 < 1-\theta}} \psi\left(\beta, p_1\right) - \sum_{\substack{n=p_1\beta \\ 1-\theta \leqslant \alpha_1 < \frac{1}{2}}} \psi\left(\beta, p_1\right) \\
&= \Sigma_{431101} - \Sigma_{431102} + \Sigma_{431103} - \Sigma_{431104} - \Sigma_{431105}. \qquad (189)
\end{aligned}$$

By Lemma 2.13, (118) holds for $f(n) = \Sigma_{431101}$. By Lemma 2.16, (118) holds for $f(n) = \Sigma_{431104}$. We can ignore the positive sum Σ_{431103} since it can either be decomposed further if $(\alpha_1, \alpha_2, \alpha_2) \in \boldsymbol{U}_3$ (or $\boldsymbol{U}_3'$ and $\boldsymbol{S}_3$ in some cases) or simply be discarded. For Σ_{431102}, by Lemma 2.13, we only need to show that $(\alpha_1, 2\theta - 1) \in \boldsymbol{S}_2$. Since $\alpha_1 < \frac{3}{7}$ and $\theta < \frac{7}{13} < \frac{4}{7} < \frac{25}{42} < \frac{13}{21}$, we have

$$\alpha_1 < \frac{3}{7} < 1 - \theta, \quad \alpha_1 + 2\alpha_2 < \frac{3}{7} + 2(2\theta - 1) < 2 - 2\theta, \quad \alpha_1 + 4\alpha_2 < \frac{3}{7} + 4(2\theta - 1) < 2 - \theta.$$

For Σ_{431105}, we can use a two-dimensional Harman's sieve as in the decompositions on Σ_{42208} in Theorem 4.6:

$$\begin{aligned}
\Sigma_{431107} &= \sum_{\substack{n=p_1\beta \\ 1-\theta \leqslant \alpha_1 < \frac{1}{2}}} \psi\left(\beta, p_1\right) \\
&= \sum_{\substack{n=m_1m_2 \\ 1-\theta \leqslant \alpha_1 < \frac{1}{2}}} \psi\left(m_1m_2, x^{\frac{\theta+\varepsilon}{2}}\right) \\
&= \sum_{\substack{n=m_1m_2 \\ 1-\theta \leqslant \alpha_1 < \frac{1}{2}}} \psi\left(m_1m_2, x^{\kappa'}\right) - \sum_{\substack{n=\beta_1\beta_2p_3 \\ 1-\theta \leqslant \alpha_1+\alpha_3 < \theta \\ \kappa' \leqslant \alpha_3 < \frac{\theta+\varepsilon}{2}}} \psi\left(\beta_1\beta_2, p_3\right) \\
&= \sum_{\substack{n=m_1m_2 \\ 1-\theta \leqslant \alpha_1 < \frac{1}{2}}} \psi\left(m_1m_2, x^{\kappa'}\right) - \sum_{\substack{n=\beta_1\beta_2p_3 \\ 1-\theta \leqslant \alpha_1+\alpha_3 < \theta \\ \kappa' \leqslant \alpha_3 < \frac{\theta+\varepsilon}{2}}} \psi\left(\beta_1, p_3\right)\psi\left(\beta_2, p_3\right) \\
&= \Sigma_{431106} - \Sigma_{431107}. \qquad (190)
\end{aligned}$$

Since $\theta < \frac{7}{13} < \frac{11}{20}$, (118) holds for $f(n) = \Sigma_{431106}$ by [[2], Remarks on Lemma 19]. We discard the parts of Σ_{431107} that do not satisfy (118), leading to a loss similar to the last sum in Lemma 2.19. Hence, the total loss in this case includes the loss from Σ_{431103} after possible further decompositions and the loss from Σ_{431105} after a two-dimensional Harman's sieve.

Finally, suppose that $\frac{7}{13} \leqslant \theta < \frac{6}{11}$. Using Buchstab's identity, we get

$$\begin{aligned}
\psi\left(n, x^{\frac{1}{2}}\right) &= \psi\left(n, x^{\kappa}\right) - \sum_{\substack{n=p_1\beta \\ \kappa \leqslant \alpha_1 < \frac{1}{2}}} \psi\left(\beta, p_1\right) \\
&= \psi\left(n, x^{\kappa}\right) - \sum_{\substack{n=p_1\beta \\ \kappa \leqslant \alpha_1 < \tau'}} \psi\left(\beta, p_1\right) - \sum_{\substack{n=p_1\beta \\ \tau' \leqslant \alpha_1 < \frac{3}{7}}} \psi\left(\beta, p_1\right) - \sum_{\substack{n=p_1\beta \\ \frac{3}{7} \leqslant \alpha_1 < 1-\theta}} \psi\left(\beta, p_1\right) - \sum_{\substack{n=p_1\beta \\ 1-\theta \leqslant \alpha_1 < \frac{1}{2}}} \psi\left(\beta, p_1\right) \\
&= \psi\left(n, x^{\kappa}\right) - \sum_{\substack{n=p_1\beta \\ \kappa \leqslant \alpha_1 < \tau'}} \psi\left(\beta, x^{\kappa}\right) - \sum_{\substack{n=p_1\beta \\ \tau' \leqslant \alpha_1 < \frac{3}{7}}} \psi\left(\beta, x^{\kappa'}\right) \\
&\quad + \sum_{\substack{n=p_1p_2\beta \\ \kappa \leqslant \alpha_1 < \tau' \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)}} \psi\left(\beta, p_2\right) + \sum_{\substack{n=p_1p_2\beta \\ \tau' \leqslant \alpha_1 < \frac{3}{7} \\ \kappa' \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)}} \psi\left(\beta, p_2\right)
\end{aligned}$$

$$- \sum_{\substack{n=p_1\beta \\ \frac{3}{7}\leqslant\alpha_1<1-\theta}} \psi(\beta, p_1) - \sum_{\substack{n=p_1\beta \\ 1-\theta\leqslant\alpha_1<\frac{1}{2}}} \psi(\beta, p_1)$$

$$= \Sigma_{431201} - \Sigma_{431202} - \Sigma_{431203} + \Sigma_{431204} + \Sigma_{431205} - \Sigma_{431206} - \Sigma_{431207}. \tag{191}$$

By Lemma 2.13, (118) holds for $f(n) = \Sigma_{431201}$. By Lemma 2.16, (118) holds for $f(n) = \Sigma_{431206}$. By the arguments in [[23], Subsection 6.9], (118) holds for $f(n) = \Sigma_{431202}$. By Lemma 2.13 and the same discussions above, we have $(\alpha_1, 2\theta - 1) \in \boldsymbol{S}_2$ and (118) holds for $f(n) = \Sigma_{431203}$. We can ignore the sums Σ_{431204} and Σ_{431205} since they are positive, and they can either be decomposed further if $(\alpha_1, \alpha_2, \alpha_2) \in \boldsymbol{U}_3$ (or $\boldsymbol{U}_3'$ and $\boldsymbol{S}_3$ in some cases) or simply be discarded. For the remaining Σ_{431207}, we can use the above two-dimensional Harman's sieve technique (190) to get a loss since $\theta < \frac{6}{11} < \frac{11}{20}$. Hence, the total loss in this case includes the loss from Σ_{431204} and Σ_{431205} after possible further decompositions and the loss from Σ_{431207} after a two-dimensional Harman's sieve.

Since the two-dimensional Harman's sieve (190) is valid only when $\theta < \frac{11}{20}$, we cannot extend the range of θ to $\geqslant \frac{11}{20}$ using the first method. Working on each region and subregion carefully, we can obtain the following lower bounds for $C_0'(\theta_1, \theta_2)$ $(0.5 < \theta \leqslant 0.54)$:

0.27	0.2905	−10.3	—	—	—	—	—	—	—	—	—	—	—	—
0.26	0.7039	0.2905	−8.84	—	—	—	—	—	—	—	—	—	—	—
0.25	0.9494	0.7721	0.4105	−5.29	—	—	—	—	—	—	—	—	—	—
0.24	1	1	0.8233	0.4735	−3.34	—	—	—	—	—	—	—	—	—
0.23	1	1	1	0.8493	0.4471	−2.15	—	—	—	—	—	—	—	—
0.22	1	1	1	1	0.8334	0.4225	−1.46	—	—	—	—	—	—	—
0.21	1	1	1	1	1	0.8117	0.4245	−1.13	—	—	—	—	—	—
0.20	1	1	1	1	1	1	0.8165	0.4380	−0.86	—	—	—	—	—
0.19	1	1	1	1	1	1	1	0.8085	0.4516	−0.60	—	—	—	—
0.18	1	1	1	1	1	1	1	1	0.8104	0.4458	−0.38	—	—	—
0.17	1	1	1	1	1	1	1	1	0.9134	0.8060	0.4450	−0.16	—	—
0.16	1	1	1	1	1	1	1	1	1	0.9260	0.8003	0.4419	−0.11	—
0.15	1	1	1	1	1	1	1	1	1	1	1	0.7967	0.4671	−0.05
0.14	1	1	1	1	1	1	1	1	1	1	1	0.9188	0.8287	0.5356
0.13	1	1	1	1	1	1	1	1	1	1	1	1	0.8985	0.7999
0.12	1	1	1	1	1	1	1	1	1	1	1	1	$1-\varepsilon$	0.8776
$\boldsymbol{\theta_2\backslash\theta_1}$	**0.26**	**0.27**	**0.28**	**0.29**	**0.30**	**0.31**	**0.32**	**0.33**	**0.34**	**0.35**	**0.36**	**0.37**	**0.38**	**0.39**

Table 4.3: Lower Bounds for $C_0'(\theta_1, \theta_2)$ (First Method, $0.5 < \theta \leqslant 0.54$) 1/2

0.14	0.0570	—	—	—	—	—	—	—	—	—	—	—	—	—
0.13	0.6092	0.1628	—	—	—	—	—	—	—	—	—	—	—	—
0.12	0.7702	0.6209	0.2919	—	—	—	—	—	—	—	—	—	—	—
0.11	0.8540	0.7420	0.5812	0.3862	—	—	—	—	—	—	—	—	—	—
0.10	$1-\varepsilon$	0.8337	0.7175	0.5486	0.3307	—	—	—	—	—	—	—	—	—
0.09	1	1	0.8152	0.6973	0.5095	0.2705	—	—	—	—	—	—	—	—
0.08	1	1	1	1	0.6711	0.4696	0.0012	—	—	—	—	—	—	—
0.07	1	1	1	1	1	1	0.4270	−0.08	—	—	—	—	—	—
0.06	1	1	1	1	1	1	1	0.3789	−0.16	—	—	—	—	—
0.05	1	1	1	1	1	1	1	1	0.4585	−0.25	—	—	—	—
0.04	1	1	1	1	1	1	1	1	0.6800	0.3509	−4.03	—	—	—
0.03	1	1	1	1	1	1	1	1	1	0.5868	0.1963	−7.06	—	—
0.02	1	1	1	1	1	1	1	1	1	0.7823	0.4839	0.1176	−8.71	—
0.01	1	1	1	1	1	1	1	1	1	1	0.7078	0.4362	0.0196	−9.80
$\boldsymbol{\theta_2\backslash\theta_1}$	**0.40**	**0.41**	**0.42**	**0.43**	**0.44**	**0.45**	**0.46**	**0.47**	**0.48**	**0.49**	**0.50**	**0.51**	**0.52**	**0.53**

Table 4.4: Lower Bounds for $C_0'(\theta_1, \theta_2)$ (First Method, $0.5 < \theta \leqslant 0.54$) 2/2

4.3.2. *Second Method.* The second method is to use Mikawa's sieve [32] and is discussed in Section 2 and Section 3. By the discussions in the end of Section 3, we need to use new Type-I information to extend the "Mikawa applicable range" of θ to $\geqslant \frac{17}{32}$. The only new Type-I information input in this section is Lemma 4.4. Thus, we can assume that

$$\theta_1 + \theta_2 \geqslant \frac{17}{32}, \quad \min\left(1 - \theta_1, 2 - \theta_1 - 4\theta_2, 1 - \frac{1}{2}\theta_1 - \theta_2, 3 - 3\theta_1 - 4\theta_2\right) > \frac{1}{2} + \frac{3}{2}\kappa > \frac{1}{2} + \frac{5 - 8\theta}{4}, \tag{192}$$

since we need $\kappa > \frac{5-8\theta}{6}$. However, careful verifications show that every region with a Type-II range $\left(0,\ \frac{1}{6}(5 - 8\theta_1 - 4\theta_2)\right)$ or $\left(2\theta_1 + \theta_2 - 1,\ \frac{1}{6}(5 - 8\theta_1 - 4\theta_2)\right)$ does not satisfy (192), which means that the "outermost" part of the rectangular region $(M \ll IG^{-1}, N \ll H^2$, defined in [[32], Page 148]) cannot be covered by either Lemma 2.8 or Lemma 4.4.

Again, we fail to extend the "Mikawa applicable range" of θ to $\geqslant \frac{17}{32}$ in the bilinear case. Hence, we can only make improvements over the results in Section 3 by discarding new Type-II sums come from Lemma 4.2 and Lemma 4.3. Working on each region and subregion carefully, we can obtain the following lower bounds for $C_0(\theta_1, \theta_2)$ $(0.5 < \theta \leqslant 0.53)$:

0.26	0.7685	0.5563	—	—	—	—	—	—	—	—	—	—	—	—
0.25	0.9217	0.7989	0.6027	—	—	—	—	—	—	—	—	—	—	—
0.24	1	1	0.8213	0.6373	—	—	—	—	—	—	—	—	—	—
0.23	1	1	1	0.8439	0.6201	—	—	—	—	—	—	—	—	—
0.22	1	1	1	1	0.8377	0.6203	—	—	—	—	—	—	—	—
0.21	1	1	1	1	1	0.8331	0.6115	—	—	—	—	—	—	—
0.20	1	1	1	1	1	1	0.8191	0.6113	—	—	—	—	—	—
0.19	1	1	1	1	1	1	1	0.8089	0.6007	—	—	—	—	—
0.18	1	1	1	1	1	1	1	1	0.8063	0.5745	—	—	—	—
0.17	1	1	1	1	1	1	1	1	0.8887	0.7965	0.5521	—	—	—
0.16	1	1	1	1	1	1	1	1	1	0.8855	0.7757	0.5323	—	—
0.15	1	1	1	1	1	1	1	1	1	1	1	0.7605	0.5051	—
0.14	1	1	1	1	1	1	1	1	1	1	1	0.8741	0.7469	0.5113
0.13	1	1	1	1	1	1	1	1	1	1	1	1	0.8431	0.7115
0.12	1	1	1	1	1	1	1	1	1	1	1	1	$1-\varepsilon$	0.8101
$\boldsymbol{\theta_2 \backslash \theta_1}$	**0.26**	**0.27**	**0.28**	**0.29**	**0.30**	**0.31**	**0.32**	**0.33**	**0.34**	**0.35**	**0.36**	**0.37**	**0.38**	**0.39**

Table 4.5: Lower Bounds for $C_0'(\theta_1, \theta_2)$ (Second Method, $0.5 < \theta \leqslant 0.53$) 1/2

0.13	0.5137	—	—	—	—	—	—	—	—	—	—	—	—
0.12	0.6783	0.5105	—	—	—	—	—	—	—	—	—	—	—
0.11	0.7859	0.6549	0.4981	—	—	—	—	—	—	—	—	—	—
0.10	$1-\varepsilon$	0.7643	0.6459	0.4841	—	—	—	—	—	—	—	—	—
0.09	1	1	0.7511	0.6365	0.4689	—	—	—	—	—	—	—	—
0.08	1	1	1	1	0.6279	0.4547	—	—	—	—	—	—	—
0.07	1	1	1	1	1	1	0.4417	—	—	—	—	—	—
0.06	1	1	1	1	1	1	1	0.4277	—	—	—	—	—
0.05	1	1	1	1	1	1	1	1	0.5405	—	—	—	—
0.04	1	1	1	1	1	1	1	1	0.7051	0.4669	—	—	—
0.03	1	1	1	1	1	1	1	1	1	0.6425	0.3849	—	—
0.02	1	1	1	1	1	1	1	1	1	0.7741	0.5693	0.3683	—
0.01	1	1	1	1	1	1	1	1	1	1	0.7141	0.5633	0.3509
$\boldsymbol{\theta_2 \backslash \theta_1}$	**0.40**	**0.41**	**0.42**	**0.43**	**0.44**	**0.45**	**0.46**	**0.47**	**0.48**	**0.49**	**0.50**	**0.51**	**0.52**

Table 4.6: Lower Bounds for $C_0'(\theta_1, \theta_2)$ (Second Method, $0.5 < \theta \leqslant 0.53$) 2/2

0.27	0.2905 **0.5563**	—	—	—	—	—	—	—	—	—	—	—	—	—
0.26	0.7039 **0.7685**	0.2905 **0.5563**	—	—	—	—	—	—	—	—	—	—	—	—
0.25	**0.9494** 0.9217	0.7721 **0.7989**	0.4105 **0.6027**	—	—	—	—	—	—	—	—	—	—	—
0.24	1	1	**0.8233** 0.8213	0.4735 **0.6373**	—	—	—	—	—	—	—	—	—	—
0.23	1	1	1	**0.8493** 0.8439	0.4471 **0.6201**	—	—	—	—	—	—	—	—	—
0.22	1	1	1	1	0.8334 **0.8377**	0.4225 **0.6203**	—	—	—	—	—	—	—	—
0.21	1	1	1	1	1	0.8117 **0.8331**	0.4245 **0.6115**	—	—	—	—	—	—	—
0.20	1	1	1	1	1	1	0.8165 **0.8191**	0.4380 **0.6113**	—	—	—	—	—	—
0.19	1	1	1	1	1	1	1	0.8085 **0.8089**	0.4516 **0.6007**	—	—	—	—	—
0.18	1	1	1	1	1	1	1	1	**0.8104** 0.8063	0.4458 **0.5745**	—	—	—	—
0.17	1	1	1	1	1	1	1	1	**0.9134** 0.8887	**0.8060** 0.7965	0.4450 **0.5521**	—	—	—
0.16	1	1	1	1	1	1	1	1	1	**0.9260** 0.8855	**0.8003** 0.7757	0.4419 **0.5323**	—	—
0.15	1	1	1	1	1	1	1	1	1	1	1	**0.7967** 0.7605	0.4671 **0.5051**	—
0.14	1	1	1	1	1	1	1	1	1	1	1	**0.9188** 0.8741	**0.8287** 0.7469	**0.5356** 0.5113
0.13	1	1	1	1	1	1	1	1	1	1	1	1	**0.8985** 0.8431	**0.7999** 0.7115
0.12	1	1	1	1	1	1	1	1	1	1	1	1	$1-\varepsilon$	**0.8776** 0.8101
$\theta_2\backslash\theta_1$	**0.26**	**0.27**	**0.28**	**0.29**	**0.30**	**0.31**	**0.32**	**0.33**	**0.34**	**0.35**	**0.36**	**0.37**	**0.38**	**0.39**

Table 4.7: A Comparison of Two Methods on the Lower Bounds for $C_0'(\theta_1, \theta_2)$ $(0.5 < \theta \leqslant 0.54)$ 1/2

$\theta_2 \backslash \theta_1$	0.40	0.41	0.42	0.43	0.44	0.45	0.46	0.47	0.48	0.49	0.50	0.51	0.52	0.53
0.14	**0.0570** —	—	—	—	—	—	—	—	—	—	—	—	—	—
0.13	**0.6092** 0.5137	**0.1628** —	—	—	—	—	—	—	—	—	—	—	—	—
0.12	**0.7702** 0.6783	**0.6209** 0.5105	**0.2919** —	—	—	—	—	—	—	—	—	—	—	—
0.11	**0.8540** 0.7859	**0.7420** 0.6549	**0.5812** 0.4981	**0.3862** —	—	—	—	—	—	—	—	—	—	—
0.10	$1-\varepsilon$	**0.8337** 0.7643	**0.7175** 0.6459	**0.5486** 0.4841	**0.3307** —	—	—	—	—	—	—	—	—	—
0.09	1	1	**0.8152** 0.7511	**0.6973** 0.6365	**0.5095** 0.4689	**0.2705** —	—	—	—	—	—	—	—	—
0.08	1	1	1	1	**0.6711** 0.6279	**0.4696** 0.4547	**0.0012** —	—	—	—	—	—	—	—
0.07	1	1	1	1	1	1	0.4270 **0.4417**	—	—	—	—	—	—	—
0.06	1	1	1	1	1	1	1	0.3789 **0.4277**	—	—	—	—	—	—
0.05	1	1	1	1	1	1	1	1	0.4585 **0.5405**	—	—	—	—	—
0.04	1	1	1	1	1	1	1	1	0.6800 **0.7051**	0.3509 **0.4669**	—	—	—	—
0.03	1	1	1	1	1	1	1	1	1	0.5868 **0.6425**	0.1963 **0.3849**	—	—	—
0.02	1	1	1	1	1	1	1	1	1	**0.7823** 0.7741	0.4839 **0.5693**	0.1176 **0.3683**	—	—
0.01	1	1	1	1	1	1	1	1	1	1	0.7078 **0.7141**	0.4362 **0.5633**	0.0196 **0.3509**	—

Table 4.8: A Comparison of Two Methods on the Lower Bounds for $C_0'(\theta_1, \theta_2)$ ($0.5 < \theta \leqslant 0.54$) 2/2

5. 3-factored Moduli

In this section we focus on the 3-factored case. Since we almost do not have any new arithmetic information inputs, the 3-factored case with absolute values is almost same as the first 2-factored case, with only one additional Type-II information input [[30], Proposition 5.2] that is only valid for $Q_1 Q_2 Q_3 < x^{0.501}$. Hence we only discuss the trilinear case, or 3-factored case with divisor-bounded coefficient weights. The initial setups on the sieves are similar to the bilinear case. We want to get the following result with some $0 < C_0'(\theta_1, \theta_2, \theta_3) \leqslant 1$ and $C_1'(\theta_1, \theta_2, \theta_3) \geqslant 1$:

Theorem 5.1. *There exist functions ρ_0 and ρ_1 which satisfies the following properties:*

(Majorant / Minorant). $\rho_0(n)$ is a minorant for the prime indicator function $\mathbb{1}_p(n)$, and $\rho_1(n)$ is a majorant for the prime indicator function $\mathbb{1}_p(n)$. That is, we have

$$\rho_0(n) \leqslant \mathbb{1}_p(n) \leqslant \rho_1(n).$$

(Upper and Lower bounds). We have

$$\sum_{n \leqslant x} \rho_0(n) \geqslant (1+o(1)) \frac{C_0'(\theta_1,\theta_2,\theta_3)x}{\log x} \quad \text{and} \quad \sum_{n \leqslant x} \rho_1(n) \leqslant (1+o(1)) \frac{C_1'(\theta_1,\theta_2,\theta_3)x}{\log x}$$

for two functions $C_0'(\theta_1,\theta_2,\theta_3)$ and $C_1'(\theta_1,\theta_2,\theta_3)$ satisfy $0 < C_0'(\theta_1,\theta_2,\theta_3) \leqslant 1$ and $C_1'(\theta_1,\theta_2,\theta_3) \geqslant 1$.

(Distributions in Arithmetic Progressions). Let λ_{j,q_j} $(j=1,2,3)$ be divisor-bounded complex sequences. For any $a \in \mathbb{Z}\backslash\{0\}$ and any $A>0$, we have

$$\sum_{\substack{q_1 \sim Q_1 \\ q_2 \sim Q_2 \\ q_3 \sim Q_3 \\ (q_1q_2q_3,a)=1}} \lambda_{1,q_1}\lambda_{2,q_2}\lambda_{3,q_3} \left(\sum_{\substack{n \leqslant x \\ n \equiv a (\bmod q_1q_2q_3)}} \rho_j(n) - \frac{1}{\varphi(q_1q_2q_3)} \sum_{\substack{n \leqslant x \\ (n,q_1q_2q_3)=1}} \rho_j(n) \right) \ll \frac{x}{(\log x)^A}$$

for $j=0,1$.

In order to prove Theorem 5.1 with suitable $C_0'(\theta_1,\theta_2,\theta_3)$ and $C_1'(\theta_1,\theta_2,\theta_3)$, we need results of the form

$$\sum_{\substack{q_1 \sim Q_1 \\ q_2 \sim Q_2 \\ q_3 \sim Q_3 \\ (q_1q_2q_3,a)=1}} \lambda_{1,q_1}\lambda_{2,q_2}\lambda_{3,q_3} \left(\sum_{\substack{n \sim x \\ n \equiv a (\bmod q_1q_2q_3)}} f(n) - \frac{1}{\varphi(q_1q_2q_3)} \sum_{\substack{n \sim x \\ (n,q_1q_2q_3)=1}} f(n) \right) \ll \frac{x}{(\log x)^A}. \tag{193}$$

Again, we may want the coefficients to satisfy **Conditions A and B** mentioned in Section 2.

5.1. **Preliminary Lemmas.** Before constructing the majorant and minorant, we need estimate results of the form (193). The results from Sections 2–4 are applicable in the final decompositions here.

5.1.1. *Type-II estimate.* The next lemma comes from [25], and the readers can compare it with Lemma 3.4 to see a difference. It is the only new arithmetic information input in the trilinear case.

Lemma 5.2. *([25], Proposition 8.3]). Let $M_1M_2 \asymp x$. Let a_{1,m_1}, a_{2,m_2}, λ_{1,q_1}, λ_{2,q_2} and λ_{3,q_3} be divisor-bounded complex sequences. Suppose that a_{2,m_2} satisfies **Conditions A and B**. If we have*

$$Q_1^7 Q_2^{12} Q_3^{10} < x^{4-20\varepsilon},\ Q_2 < Q_1 Q_3^3,\ Q_1 x^{\varepsilon} < M_2 < Q_1^{-1} Q_3^{-2} x^{1-6\varepsilon},$$

then

$$\sum_{\substack{q_1 \sim Q_1 \\ q_2 \sim Q_2 \\ q_3 \sim Q_3 \\ (q_1q_2q_3,a)=1}} \lambda_{1,q_1}\lambda_{2,q_2}\lambda_{3,q_3} \left(\sum_{\substack{m_1 \sim M_1 \\ m_2 \sim M_2 \\ m_1m_2 \equiv a (\bmod q_1q_2q_3)}} a_{1,m_1}a_{2,m_2} - \frac{1}{\varphi(q_1q_2q_3)} \sum_{\substack{m_1 \sim M_1 \\ m_2 \sim M_2 \\ (m_1m_2,q_1q_2q_3)=1}} a_{1,m_1}a_{2,m_2} \right) \ll \frac{x}{(\log x)^A}.$$

5.2. **Sieve Asymptotic Formulas.** Using Lemma 5.2 together with Lemma 2.6 and Lemma 2.16, we can get the following result for the trilinear case.

Lemma 5.3. *([25], Proposition 5.3]). Let $j \geqslant 4$, $P_1P_2\cdots P_j \asymp x$ and $P_1 \geqslant P_2 \geqslant \cdots \geqslant P_j \geqslant x^{\frac{1}{7}+10\varepsilon}$. Suppose that*

$$\theta_1+\theta_2 < \frac{1}{2}+\varepsilon, \quad \theta_1+\theta_3 < \frac{1}{2}-2\varepsilon \quad \text{and} \quad \theta_3 < \theta_2 < \frac{1}{32}-\varepsilon.$$

Then (193) holds for

$$f(n) = \sum_{\substack{n=p_1\cdots p_j \\ p_i \sim P_i,\ 1 \leqslant i \leqslant j}} 1.$$

Again, many asymptotic formulas used in the decompositions in this section will be adopted from previous sections.

5.3. **Upper and Lower Bounds.** We shall construct the majorant $\rho_1(n)$ and the minorant $\rho_0(n)$ in this subsection. Before constructing, we first mention some existing results of $C_1'(\theta_1,\theta_2,\theta_3)$ and $C_0'(\theta_1,\theta_2,\theta_3)$.

Theorem 5.4. *The functions $C_1'(\theta_1,\theta_2,\theta_3)$ and $C_0'(\theta_1,\theta_2,\theta_3)$ satisfy the following conditions:*

(1.1). $C_1'(\theta_1,\theta_2,\theta_3) = C_1'(\theta_1,\theta_3,\theta_2) = C_1'(\theta_2,\theta_1,\theta_3) = C_1'(\theta_2,\theta_3,\theta_1) = C_1'(\theta_3,\theta_1,\theta_2) = C_1'(\theta_3,\theta_2,\theta_1)$;

(1.2). $C_0'(\theta_1,\theta_2,\theta_3) = C_0'(\theta_1,\theta_3,\theta_2) = C_0'(\theta_2,\theta_1,\theta_3) = C_0'(\theta_2,\theta_3,\theta_1) = C_0'(\theta_3,\theta_1,\theta_2) = C_0'(\theta_3,\theta_2,\theta_1)$;

(2.1). $C_1'(\theta_1,\theta_2,\theta_3) = C_0'(\theta_1,\theta_2,\theta_3) = 1$ for all $\theta_1,\theta_2,\theta_3$ satisfy $\theta_1+\theta_2+\theta_3 < 0.5$;

(2.2). $C_1'(\theta_1,\theta_2,\theta_3) = C_0'(\theta_1,\theta_2,\theta_3) = 1$ for all $\theta_1,\theta_2,\theta_3$ satisfy $\theta_1+\theta_2+\theta_3 = 0.5+r$, $0<r<0.001$, $40r < \theta_2 < \frac{1}{20}-7r$ and $\frac{1}{10}-\theta_2+12r < \theta_3 < \frac{1}{10}-\frac{3}{5}\theta_2-4r$;

(3.1). $C_1'(\theta_1,\theta_2,\theta_3) = C_0'(\theta_1,\theta_2,\theta_3) = 1$ for all $\theta_1,\theta_2,\theta_3$ satisfy $\theta_1 < \frac{1}{3}$, $\theta_2+\theta_3 < \frac{1}{5}$ and $\theta_1+\theta_2+\theta_3 < \frac{29}{56}$;

(3.2). $C_1'(\theta_1,\theta_2,\theta_3)=C_0'(\theta_1,\theta_2,\theta_3)=1$ *for all* $\theta_1,\theta_2,\theta_3$ *satisfy* $\theta_1+\theta_3<\frac{1}{3}$, $\theta_2<\frac{1}{5}$ *and* $\theta_1+\theta_2+\theta_3<\frac{29}{56}$*;*

(4.1). $C_1'(\theta_1,\theta_2,\theta_3)=C_0'(\theta_1,\theta_2,\theta_3)=1$ *for all* $\theta_1,\theta_2,\theta_3$ *satisfy* $\theta_1+3\theta_2+3\theta_3<1$, $\theta_1+\theta_2+\theta_3<\frac{29}{56}$ *and* $\theta_2+\theta_3<\max\left(\frac{1-2\theta_1}{2},\frac{2-2\theta_1}{5}\right)$*;*

(4.2). $C_1'(\theta_1,\theta_2,\theta_3)=C_0'(\theta_1,\theta_2,\theta_3)=1$ *for all* $\theta_1,\theta_2,\theta_3$ *satisfy* $\theta_1+3\theta_2+\theta_3<1$, $\theta_1+\theta_2+\theta_3<\frac{29}{56}$ *and* $\theta_2<\max\left(\frac{1-2\theta_1-2\theta_3}{2},\frac{2-2\theta_1-2\theta_3}{5}\right)$*;*

(5.1). $C_1'(\theta_1,\theta_2,\theta_3)=C_0'(\theta_1,\theta_2,\theta_3)=1$ *for all* $\theta_1,\theta_2,\theta_3$ *satisfy* $\theta_1+3\theta_2+3\theta_3<1$, $\theta_1+\theta_2+\theta_3<\frac{29}{56}$, $4\theta_1+\theta_2+\theta_3<\frac{403}{266}$ *and* $\frac{7}{4}\theta_1+\theta_2+\theta_3<\frac{403}{532}$*;*

(5.2). $C_1'(\theta_1,\theta_2,\theta_3)=C_0'(\theta_1,\theta_2,\theta_3)=1$ *for all* $\theta_1,\theta_2,\theta_3$ *satisfy* $\theta_1+3\theta_2+\theta_3<1$, $\theta_1+\theta_2+\theta_3<\frac{29}{56}$, $4\theta_1+\theta_2+4\theta_3<\frac{403}{266}$ *and* $\frac{7}{4}\theta_1+\theta_2+\frac{7}{4}\theta_3<\frac{403}{532}$*;*

(6.1). $C_1'(\theta_1,\theta_2,\theta_3)=C_0'(\theta_1,\theta_2,\theta_3)=1$ *for all* $\theta_1,\theta_2,\theta_3$ *satisfy* $\frac{1}{4}\leqslant\theta_1<\frac{3}{10}$ *and* $\frac{1-2\theta_1}{2}\leqslant\theta_2+\theta_3<\min\left(\frac{3-4\theta_1}{8},\frac{5-14\theta_1}{4}\right)$*;*

(6.2). $C_1'(\theta_1,\theta_2,\theta_3)=C_0'(\theta_1,\theta_2,\theta_3)=1$ *for all* $\theta_1,\theta_2,\theta_3$ *satisfy* $\frac{1}{4}\leqslant\theta_1+\theta_3<\frac{3}{10}$ *and* $\frac{1-2\theta_1-2\theta_3}{2}\leqslant\theta_2<\min\left(\frac{3-4\theta_1-4\theta_3}{8},\frac{5-14\theta_1-14\theta_3}{4}\right)$*;*

(7.1). $C_1'(\theta_1,\theta_2,\theta_3)=C_0'(\theta_1,\theta_2,\theta_3)=1$ *for all* $\theta_1,\theta_2,\theta_3$ *satisfy* $\frac{1}{3}<\theta_1<\frac{3}{8}$ *and* $\frac{1-2\theta_1}{2}\leqslant\theta_2+\theta_3<\min\left(\frac{5-8\theta_1}{14},\frac{7-16\theta_1}{8}\right)$*;*

(7.2). $C_1'(\theta_1,\theta_2,\theta_3)=C_0'(\theta_1,\theta_2,\theta_3)=1$ *for all* $\theta_1,\theta_2,\theta_3$ *satisfy* $\frac{1}{3}<\theta_1+\theta_3<\frac{3}{8}$ *and* $\frac{1-2\theta_1-2\theta_3}{2}\leqslant\theta_2<\min\left(\frac{5-8\theta_1-8\theta_3}{14},\frac{7-16\theta_1-16\theta_3}{8}\right)$*;*

(8.1). $C_1'(\theta_1,\theta_2,\theta_3)=C_0'(\theta_1,\theta_2,\theta_3)=1$ *for all* $\theta_1,\theta_2,\theta_3$ *satisfy* $2\theta_1+\theta_2+\theta_3<1$ *and* $7\theta_1+12\theta_2+12\theta_3<4$*;*

(8.2). $C_1'(\theta_1,\theta_2,\theta_3)=C_0'(\theta_1,\theta_2,\theta_3)=1$ *for all* $\theta_1,\theta_2,\theta_3$ *satisfy* $2\theta_1+\theta_2+2\theta_3<1$ *and* $7\theta_1+12\theta_2+7\theta_3<4$*;*

(9.1). $C_1'(\theta_1,\theta_2,\theta_3)\leqslant\min\left(C_1'(\theta_1,\theta_2+\theta_3),C_1'(\theta_2,\theta_1+\theta_3),C_1'(\theta_3,\theta_1+\theta_2),C_1'(\theta_1+\theta_2,\theta_3),C_1'(\theta_1+\theta_3,\theta_2),C_1'(\theta_2+\theta_3,\theta_1)\right)$ *for* $0.5\leqslant\theta_1+\theta_2\leqslant 1$*;*

(9.2). $C_0'(\theta_1,\theta_2,\theta_3)\geqslant\max\left(C_0'(\theta_1,\theta_2+\theta_3),C_0'(\theta_2,\theta_1+\theta_3),C_0'(\theta_3,\theta_1+\theta_2),C_0'(\theta_1+\theta_2,\theta_3),C_0'(\theta_1+\theta_3,\theta_2),C_0'(\theta_2+\theta_3,\theta_1)\right)$ *for* $0.5\leqslant\theta_1+\theta_2\leqslant 1$*;*

(10.1) $C_1'(\theta_1,\theta_2,\theta_3)\leqslant 1+\varepsilon$ *for all* $\theta_1,\theta_2,\theta_3$ *satisfy* $\theta_1+\theta_2+\theta_3=0.5$*;*

(10.2) $C_0'(\theta_1,\theta_2,\theta_3)\geqslant 1-\varepsilon$ *for all* $\theta_1,\theta_2,\theta_3$ *satisfy* $\theta_1+\theta_2+\theta_3=0.5$*;*

(11.1). $C_1'(\theta_1,\theta_2,\theta_3)\leqslant 1+\varepsilon$ *for all* $\theta_1,\theta_2,\theta_3$ *satisfy* $\frac{1}{4}\leqslant\theta_1\leqslant\frac{3}{10}$ *and* $\theta_2+\theta_3=\min\left(\frac{3-4\theta_1}{8},\frac{5-14\theta_1}{4}\right)$*;*

(11.2). $C_1'(\theta_1,\theta_2,\theta_3)\leqslant 1+\varepsilon$ *for all* $\theta_1,\theta_2,\theta_3$ *satisfy* $\frac{1}{4}\leqslant\theta_1+\theta_3\leqslant\frac{3}{10}$ *and* $\theta_2=\min\left(\frac{3-4\theta_1-4\theta_3}{8},\frac{5-14\theta_1-14\theta_3}{4}\right)$*;*

(12.1). $C_1'(\theta_1,\theta_2,\theta_3)\leqslant 1+\varepsilon$ *for all* $\theta_1,\theta_2,\theta_3$ *satisfy* $\frac{1}{3}\leqslant\theta_1\leqslant\frac{3}{8}$ *and* $\theta_2+\theta_3=\min\left(\frac{5-8\theta_1}{14},\frac{7-16\theta_1}{8}\right)$*;*

(12.2). $C_1'(\theta_1,\theta_2,\theta_3)\leqslant 1+\varepsilon$ *for all* $\theta_1,\theta_2,\theta_3$ *satisfy* $\frac{1}{3}\leqslant\theta_1+\theta_3\leqslant\frac{3}{8}$ *and* $\theta_2=\min\left(\frac{5-8\theta_1-8\theta_3}{14},\frac{7-16\theta_1-16\theta_3}{8}\right)$*;*

(13.1). $C_1'(\theta_1,\theta_2,\theta_3)\leqslant 1+\varepsilon$ *for all* $\theta_1,\theta_2,\theta_3$ *satisfy* $\theta_1\leqslant 0.5$ *and* $\theta_2+\theta_3=\min\left(1-2\theta_1,\frac{4-7\theta_1}{12}\right)$*;*

(13.2). $C_1'(\theta_1,\theta_2,\theta_3)\leqslant 1+\varepsilon$ *for all* $\theta_1,\theta_2,\theta_3$ *satisfy* $\theta_1+\theta_3\leqslant 0.5$ *and* $\theta_2=\min\left(1-2\theta_1-2\theta_3,\frac{4-7\theta_1-7\theta_3}{12}\right)$.

Proof. This theorem follows from Theorems 1.1, 2.22, 2.23, 3.9, 3.10, 4.5, 4.9 and [[30], Theorem 1.2]. ☐

Next, We shall prove the following theorem.

Theorem 5.5. *Let* $Q_1=x^{\theta_1}$, $Q_2=x^{\theta_2}$ *and* $Q_3=x^{\theta_3}$*. Suppose that* θ_1, θ_2 *and* θ_3 *satisfy the following conditions:*

$$\theta_1+\theta_2<\frac{1}{2}+\varepsilon,\quad \theta_1+\theta_3<\frac{1}{2}-2\varepsilon,\quad \theta_3<\theta_2<\frac{1}{32}-\varepsilon$$

Let λ_{1,q_1}, λ_{2,q_2} *and* λ_{3,q_3} *be divisor-bounded complex sequences. Then, for any fixed* $a\in\mathbb{Z}\backslash\{0\}$ *and any* $A>0$*, we have*

$$\sum_{\substack{q_1\sim Q_1\\ q_2\sim Q_2\\ q_3\sim Q_3\\ (q_1q_2q_3,a)=1}}\lambda_{1,q_1}\lambda_{2,q_2}\lambda_{3,q_3}\left(\pi(x;q_1q_2q_3,a)-\frac{\pi(x)}{\varphi(q_1q_2q_3)}\right)\ll\frac{x}{(\log x)^A}.$$

Proof. Define

$$\boldsymbol{\mathcal{W}_4}=\left\{(\theta_1,\theta_2,\theta_3):\theta_1+\theta_2<\frac{1}{2}+\varepsilon,\ \theta_1+\theta_3<\frac{1}{2}-2\varepsilon,\ \theta_3<\theta_2<\frac{1}{32}-\varepsilon\right\}.$$

By Case (3) of Theorem 1.1 and Statement (8.1) of Theorem 5.4, we only need to prove that for $(\theta_1,\theta_2,\theta_3)\in\boldsymbol{\mathcal{W}_4}$, (193) holds for

$$f(n)=\mathbb{1}_p(n).$$

Write $\theta'=\theta_2+\theta_3$ and $\theta=\theta_1+\theta_2+\theta_3$. We can define a two-variable region $\boldsymbol{\mathcal{Z}_4}$ that is analogous to $\boldsymbol{\mathcal{W}_4}$:

$$\boldsymbol{\mathcal{Z}_4}=\left\{(\theta_1,\theta_2):2\theta_1+\theta_2<1,\ \theta_2<\frac{1}{16}\right\}.$$

Now, $(\theta_1,\theta_2,\theta_3)\in\boldsymbol{\mathcal{W}_4}$ implies $(\theta_1,\theta')\in\boldsymbol{\mathcal{Z}_4}$. Since $(\theta_1,\theta')\in\boldsymbol{\mathcal{Z}_4}$ implies $(\theta_1,\theta')\in\boldsymbol{J}\cup\boldsymbol{E}_{0901}$, we have a Type-II range $\left(0,\ \frac{1}{6}(5-8\theta_1-4\theta')\right)$ and $\kappa=\frac{5-8\theta_1-4\theta'}{6}-\varepsilon$.

We use the same decomposing process as (178) to decompose $\mathbb{1}_p(n)=\psi\left(n,x^{\frac{1}{2}}\right)$ and reduce the proof of Theorem 5.5 to showing that (193) holds for $f(n)=$ sums that count numbers with 4 or more prime factors. The proof of Theorem 5.5 is thus completed by applying Lemma 5.3 on those sums, since we have $\kappa>\frac{1}{6}>\frac{1}{7}$. ☐

Remark. If we choose, for example,

$$\theta_1 = \frac{15}{32} + \varepsilon, \quad \theta_2 = \frac{1}{32} - 2\varepsilon, \quad \theta_3 = \frac{1}{32} - 4\varepsilon,$$

then Theorem 5.5 extends the range of θ in Theorem 1.1 to $\frac{17}{32}$ under a trilinear form of moduli. Theorem 5.5 also generalizes the main result of Lichtman [25].

6. Smooth Moduli

In this section we focus on the smooth case, where the moduli $q \leqslant Q = x^\theta$ and $P^+(q) < x^\delta$. We also suppose that q is square-free, hence $q \mid P(x^\delta)$. Similarly, we want to get the following result with some $0 < C_0^\delta(\theta) \leqslant 1$ and $C_1^\delta(\theta) \geqslant 1$:

Theorem 6.1. *There exist functions ρ_0 and ρ_1 which satisfies the following properties:*

(Majorant / Minorant). $\rho_0(n)$ is a minorant for the prime indicator function $\mathbb{1}_p(n)$, and $\rho_1(n)$ is a majorant for the prime indicator function $\mathbb{1}_p(n)$. That is, we have

$$\rho_0(n) \leqslant \mathbb{1}_p(n) \leqslant \rho_1(n).$$

(Upper and Lower bounds). We have

$$\sum_{n \leqslant x} \rho_0(n) \geqslant (1 + o(1)) \frac{C_0^\delta(\theta) x}{\log x} \quad \text{and} \quad \sum_{n \leqslant x} \rho_1(n) \leqslant (1 + o(1)) \frac{C_1^\delta(\theta) x}{\log x}$$

for two functions $C_0^\delta(\theta)$ and $C_1^\delta(\theta)$ satisfy $0 < C_0^\delta(\theta) \leqslant 1$ and $C_1^\delta(\theta) \geqslant 1$.

(Distributions in Arithmetic Progressions). For any $a \in \mathbb{Z}\backslash\{0\}$ and any $A > 0$, we have

$$\sum_{\substack{q \leqslant Q \\ q \mid P(x^\delta) \\ (q,a)=1}} \left| \sum_{\substack{n \leqslant x \\ n \equiv a (\bmod q)}} \rho_j(n) - \frac{1}{\varphi(q)} \sum_{\substack{n \leqslant x \\ (n,q)=1}} \rho_j(n) \right| \ll \frac{x}{(\log x)^A}$$

for $j = 0, 1$.

In order to prove Theorem 6.1 with suitable $C_0^\delta(\theta)$ and $C_1^\delta(\theta)$, we need results of the form

$$\sum_{\substack{q \leqslant Q \\ q \mid P(x^\delta) \\ (q,a)=1}} \left| \sum_{\substack{n \asymp x \\ n \equiv a (\bmod q)}} f(n) - \frac{1}{\varphi(q)} \sum_{\substack{n \asymp x \\ (n,q)=1}} f(n) \right| \ll \frac{x}{(\log x)^A}. \tag{194}$$

As in previous sections, we sometimes want the coefficients to satisfy the Siegel-Wafisz condition (or **Condition A**). In our Type-I estimate, we also want the coefficients to satisfy an extra condition. Again, we use λ_l as an example.

(**Condition C(L)**: Smooth at scale L) λ_l has the form of $\eta(\frac{l}{L})$ for some smooth function $\eta : \mathbb{R} \to \mathbb{C}$ supported on $[c_1, c_2]$ for fixed $0 < c_1 < c_2$. The function η also satisfies the bound

$$\left| \eta^{(j)}(x) \right| \ll (\log x)^A$$

for all fixed $j \geqslant 0$, where $\eta^{(j)}$ denote the j-th derivative of η.

6.1. Preliminary Lemmas.

6.1.1. *Type-II estimate.* The first estimate comes from [38], and it is nontrivial when $\theta < \frac{19}{36} \approx 0.5278$. In the proof of [[38], Theorem 2], Stadlmann used this lemma as the Type-II information input to construct a minorant for $\theta = 0.5253$.

Lemma 6.2. *([38], Proposition 1]). Let $M_1 M_2 \asymp x$. Let a_{1,m_1} and a_{2,m_2} be divisor-bounded complex sequences. Suppose that a_{2,m_2} satisfies **Condition A**. If we have*

$$x^{\frac{1}{2}-\sigma} \leqslant M_2 \leqslant x^{\frac{1}{2}+\sigma},$$

where σ and θ satisfy

$$\sigma > 0,\ \theta > \frac{1}{2},\ 36\theta + 24\delta < 19,\ 24\theta + 4\sigma + 16\delta < 13,\ 32\theta + 2\sigma + 20\delta < 17,$$

then

$$\sum_{\substack{q \leqslant Q \\ q \mid P(x^\delta) \\ (q,a)=1}} \left| \sum_{\substack{m_1 \asymp M_1 \\ m_2 \asymp M_2 \\ m_1 m_2 \equiv a (\bmod q)}} a_{1,m_1} a_{2,m_2} - \frac{1}{\varphi(q)} \sum_{\substack{m_1 \asymp M_1 \\ m_2 \asymp M_2 \\ (m_1 m_2, q)=1}} a_{1,m_1} a_{2,m_2} \right| \ll \frac{x}{(\log x)^A}.$$

The first estimate comes from [36], and it is nontrivial when $\theta < \frac{9}{17} \approx 0.5294$. In the proof of [[4], Theorem 1.1], Baker and Irving used this lemma as the Type-II information input to construct a minorant for $\theta \approx 0.5242$. Of course, we have $C_0^\delta(0.5242) = C_1^\delta(0.5242) = 1$ now by Stadlmann's result [38].

Lemma 6.3. *([36], Theorem 2.8(iii)]). Let $M_1M_2 \asymp x$. Let a_{1,m_1} and a_{2,m_2} be divisor-bounded complex sequences. Suppose that a_{2,m_2} satisfies **Condition A**. If we have*

$$x^{\frac{1}{2}-\sigma} \leqslant M_2 \leqslant x^{\frac{1}{2}+\sigma},$$

where σ and θ satisfy

$$\sigma > 0,\ \theta > \frac{1}{2},\ 17\theta + 7\delta < 9,\ \frac{80}{3}\theta + \frac{34}{9}\sigma + 16\delta < \frac{43}{3},\ 32\theta + 2\sigma + 18\delta < 17,$$

then

$$\sum_{\substack{q \leqslant Q \\ q \mid P(x^\delta) \\ (q,a)=1}} \left| \sum_{\substack{m_1 \asymp M_1 \\ m_2 \asymp M_2 \\ m_1 m_2 \equiv a (\bmod q)}} a_{1,m_1} a_{2,m_2} - \frac{1}{\varphi(q)} \sum_{\substack{m_1 \asymp M_1 \\ m_2 \asymp M_2 \\ (m_1 m_2, q)=1}} a_{1,m_1} a_{2,m_2} \right| \ll \frac{x}{(\log x)^A}.$$

When $\theta < \frac{19}{36}$, Lemma 6.2 gives more Type-II information than Lemma 6.3. However, Lemma 6.2 is not applicable when $\theta \geqslant \frac{19}{36}$. Lemma 6.3 is the only Type-II information input when $\frac{19}{36} \leqslant \theta < \frac{9}{17}$.

6.1.2. *Type-I estimate.* The first Type-I estimate was proved by Baker and Irving [4] by combining 3 cases depending on the size of M_2, and it plays an important role in both [4] and [38]. In [24] the author used this as Type-I information input to construct a minorant for $\theta = \frac{10}{19} - \varepsilon$.

Lemma 6.4. *([4], Lemma 5]). Let $M_1M_2 \asymp x$. Let a_{1,m_1} and a_{2,m_2} be divisor-bounded complex sequences. Suppose that a_{1,m_1} satisfies **Condition A** and a_{2,m_2} satisfies **Condition C**(M_2). If we have*

$$\frac{1}{2} < \theta < \frac{10}{19},\ M_1 \leqslant x^{\frac{1}{2}(5-7\theta)-3\delta},$$

then

$$\sum_{\substack{q \leqslant Q \\ q \mid P(x^\delta) \\ (q,a)=1}} \left| \sum_{\substack{m_1 \asymp M_1 \\ m_2 \asymp M_2 \\ m_1 m_2 \equiv a (\bmod q)}} a_{1,m_1} a_{2,m_2} - \frac{1}{\varphi(q)} \sum_{\substack{m_1 \asymp M_1 \\ m_2 \asymp M_2 \\ (m_1 m_2, q)=1}} a_{1,m_1} a_{2,m_2} \right| \ll \frac{x}{(\log x)^A}.$$

For $\theta \geqslant \frac{10}{19}$, we cannot apply Lemma 6.4. Fortunately, we still have two valid Type-I information ranges. The first one can be proved by the method used in the discussion of the "Polymath Type-0 sums" in [36]. Readers can see the end of [[36], Section 3] for more details.

Lemma 6.5. *Let $M_1M_2 \asymp x$. Let a_{1,m_1} and a_{2,m_2} be divisor-bounded complex sequences. Suppose that a_{1,m_1} satisfies **Condition A** and a_{2,m_2} satisfies **Condition C**(M_2). If we have*

$$\theta > \frac{1}{2},\ \sigma > \theta - \frac{1}{2},\ M_1 \leqslant x^{\frac{1}{2}-\sigma},$$

then

$$\sum_{\substack{q \leqslant Q \\ q \mid P(x^\delta) \\ (q,a)=1}} \left| \sum_{\substack{m_1 \asymp M_1 \\ m_2 \asymp M_2 \\ m_1 m_2 \equiv a (\bmod q)}} a_{1,m_1} a_{2,m_2} - \frac{1}{\varphi(q)} \sum_{\substack{m_1 \asymp M_1 \\ m_2 \asymp M_2 \\ (m_1 m_2, q)=1}} a_{1,m_1} a_{2,m_2} \right| \ll \frac{x}{(\log x)^A}.$$

The second one is [[4], Lemma 3].

Lemma 6.6. *([4], Lemma 3]). Let $M_1M_2 \asymp x$. Let a_{1,m_1} and a_{2,m_2} be divisor-bounded complex sequences. Suppose that a_{1,m_1} satisfies **Condition A** and a_{2,m_2} satisfies **Condition C**(M_2). If we have*

$$\sigma > 0,\ \theta > \frac{1}{2},\ 14\theta + 4\sigma + 8\delta < 8,\ x^{\frac{1}{2}} \leqslant M_1 \leqslant x^{\frac{1}{2}+\sigma},$$

then

$$\sum_{\substack{q \leqslant Q \\ q \mid P(x^\delta) \\ (q,a)=1}} \left| \sum_{\substack{m_1 \asymp M_1 \\ m_2 \asymp M_2 \\ m_1 m_2 \equiv a (\bmod q)}} a_{1,m_1} a_{2,m_2} - \frac{1}{\varphi(q)} \sum_{\substack{m_1 \asymp M_1 \\ m_2 \asymp M_2 \\ (m_1 m_2, q)=1}} a_{1,m_1} a_{2,m_2} \right| \ll \frac{x}{(\log x)^A}.$$

Lemmas 6.5, 6.6 and 6.3 help us to obtain an analog of Lemma 6.4 when $\theta \geqslant \frac{10}{19}$.

Lemma 6.7. *Let $M_1M_2 \asymp x$. Let a_{1,m_1} and a_{2,m_2} be divisor-bounded complex sequences. Suppose that a_{1,m_1} satisfies* ***Condition A*** *and a_{2,m_2} satisfies* ***Condition C***(M_2)*. If we have*

$$\frac{1}{2} < \theta < \frac{9}{17} - \delta,\ M_1 \leqslant x^{\frac{1}{2}(5-7\theta)-3\delta},$$

then

$$\sum_{\substack{q \leqslant Q \\ q \mid P(x^\delta) \\ (q,a)=1}} \left| \sum_{\substack{m_1 \asymp M_1 \\ m_2 \asymp M_2 \\ m_1m_2 \equiv a(\bmod q)}} a_{1,m_1}a_{2,m_2} - \frac{1}{\varphi(q)} \sum_{\substack{m_1 \asymp M_1 \\ m_2 \asymp M_2 \\ (m_1m_2,q)=1}} a_{1,m_1}a_{2,m_2} \right| \ll \frac{x}{(\log x)^A}.$$

Proof. When $\frac{1}{2} < \theta < \frac{10}{19}$, this is Lemma 6.4.

When $\frac{10}{19} \leqslant \theta < \frac{9}{17} - \delta$, we follow the proof of Lemma 6.4 by splitting the range of M_1 into 3 subranges:

(1). $M_1 \leqslant x^{1-\theta-\delta}$: We take $\sigma = \theta - \frac{1}{2} + \delta$ and apply Lemma 6.5;

(2). $x^{1-\theta-\delta} < M_1 \leqslant x^{\frac{1}{2}}$: We take $\sigma = \theta - \frac{1}{2} + \delta$ and apply Lemma 6.3;

(3). $x^{\frac{1}{2}} < M_1 \leqslant x^{\frac{1}{2}(5-7\theta)}$: We take $\sigma = 2 - \frac{7}{2}\theta - 3\delta$ and apply Lemma 6.6.

Combining the above 3 cases, Lemma 6.7 is proved. □

6.1.3. *Type-I$_3$ estimate.* The next lemma is used together with Heath-Brown's identity to handle sums that count products of three large primes, and it can only give asymptotic formulas for such sums when $\theta < \frac{1}{2} + \frac{2}{79} \approx 0.525314$. Because of the lack of the Type-II information with a small variable, we cannot construct a three-dimensional Harman's sieve as previous sections based on this lemma.

Lemma 6.8. *([36], Theorem 2.8(v)). Let $M_0M_1M_2M_3 \asymp x$. Let a_{0,m_0}, a_{1,m_1}, a_{2,m_2} and a_{3,m_3} be divisor-bounded complex sequences. Suppose that a_{i,m_i} satisfies* ***Condition C***(M_i) *for $1 \leqslant i \leqslant 3$. If we have*

$$\theta < \frac{2}{3},\ \min\left(M_1M_2, M_1M_3, M_2M_3\right) > x^{\frac{14\theta-2}{9}},\ x^{\frac{28\theta-13}{9}} < M_1, M_2, M_3 < x^{\frac{11-14\theta}{9}},$$

then

$$\sum_{\substack{q \leqslant Q \\ q \mid P(x^\delta) \\ (q,a)=1}} \left| \sum_{\substack{m_0 \asymp M_0 \\ m_1 \asymp M_1 \\ m_2 \asymp M_2 \\ m_3 \asymp M_3 \\ m_0m_1m_2m_3 \equiv a(\bmod q)}} a_{0,m_0}a_{1,m_1}a_{2,m_2}a_{3,m_3} - \frac{1}{\varphi(q)} \sum_{\substack{m_0 \asymp M_0 \\ m_1 \asymp M_1 \\ m_2 \asymp M_2 \\ m_3 \asymp M_3 \\ (m_0m_1m_2m_3,q)=1}} a_{0,m_0}a_{1,m_1}a_{2,m_2}a_{3,m_3} \right| \ll \frac{x}{(\log x)^A}.$$

6.2. **Sieve Asymptotic Formulas.** In this subsection we prove results of the form (194) for some functions $f(n)$. We write

$$\nu = \nu(\theta) = 1 - 2\max\left(6\theta - \frac{11}{4}, 16\theta - 8\right)$$

and

$$\nu' = \nu'(\theta) = 1 - 2\max\left(\frac{120}{17}\theta - \frac{56}{17}, 16\theta - 8\right).$$

Lemma 6.9. *Let $\frac{1}{2} < \theta < \frac{19}{36}$ and $\alpha_1, \alpha_2, \ldots, \alpha_k > \varepsilon$. Suppose that we can partition $\{1, \ldots, k\}$ into I and J such that*

$$\frac{1-\nu}{2} < \sum_{i \in I} \alpha_i < \frac{1+\nu}{2},$$

then (194) holds for

$$f(n) = \sum_{n = p_1p_2\cdots p_k} 1.$$

Let $\frac{1}{2} < \theta < \frac{9}{17}$ and $\alpha_1, \alpha_2, \ldots, \alpha_k > \varepsilon$. Suppose that we can partition $\{1, \ldots, k\}$ into I and J such that

$$\frac{1-\nu'}{2} < \sum_{i \in I} \alpha_i < 1 - \frac{1-\nu'}{2},$$

then (194) holds for

$$f(n) = \sum_{n = p_1p_2\cdots p_k} 1.$$

Proof. This lemma follows easily from Lemma 6.2 and Lemma 6.3, taking $\sigma = \frac{\nu}{2} - 11\delta$ and $\sigma = \frac{\nu'}{2} - 11\delta$ respectively. □

Lemma 6.10. *Let $\frac{1}{2} < \theta < \frac{19}{36}$ and $\alpha_1, \alpha_2, \ldots, \alpha_k > \varepsilon$. Suppose that we can partition $\{1, \ldots, k\}$ into I and J such that*

$$\sum_{i \in I} \alpha_i < \frac{1-\nu}{2}, \ \sum_{j \in J} \alpha_j < \frac{4-7\theta+\nu}{2},$$

then (194) holds for

$$f(n) = \sum_{n=p_1 p_2 \cdots p_k \beta} \psi\left(\beta, x^{\nu}\right) \quad \text{and} \quad f(n) = \psi\left(n, x^{\nu}\right).$$

Let $\frac{1}{2} < \theta < \frac{9}{17}$ and $\alpha_1, \alpha_2, \ldots, \alpha_k > \varepsilon$. Suppose that we can partition $\{1, \ldots, k\}$ into I and J such that

$$\sum_{i \in I} \alpha_i < \frac{1-\nu'}{2}, \ \sum_{j \in J} \alpha_j < \frac{4-7\theta+\nu'}{2},$$

then (194) holds for

$$f(n) = \sum_{n=p_1 p_2 \cdots p_k \beta} \psi\left(\beta, x^{\nu'}\right) \quad \text{and} \quad f(n) = \psi\left(n, x^{\nu'}\right).$$

Proof. This lemma can be proved by applying [[1], Lemma 14], with Lemma 6.7 as Type-I information and Lemmas 6.2–6.3 as Type-II information. ☐

Lemma 6.11. *Let $\frac{1}{2} < \theta < \frac{1}{2} + \frac{1}{79}$. We have (194) holds for*

$$f(n) = \sum_{\substack{n=p_1 p_2 p_3 \\ \nu \leqslant \alpha_2 < \alpha_1 < \frac{1-\nu}{2} \\ \alpha_1 + \alpha_2 \geqslant \frac{1+\nu}{2} \\ \alpha_3 > \alpha_2 \geqslant \frac{4-7\theta+\nu}{2}}} 1.$$

Proof. See [[38], Section 4.5]. Lemma 6.8 was used in the proof. ☐

6.3. **Lower Bounds.** We shall construct the minorant $\rho_0(n)$ in this subsection. Before constructing, we first mention existing results of $C_0^{\delta}(\theta)$ proved by Stadlmann [38].

Theorem 6.12. *([[38], Theorem 1]). The function $C_1^{\delta}(\theta)$ satisfies the following conditions:*
(1). $C_0^{\delta}(\theta) = 1$ for all $\theta < 0.525$.
(2). $C_0^{\delta}(\theta) \geqslant 0.9999$ for all $0.525 \leqslant \theta < 0.5253$.

Recalling that our aim is to decompose $\psi\left(n, x^{\frac{1}{2}}\right)$ using Buchstab's identity and show that (194) holds for most of the sums after the decomposition. For the remaining sums that we cannot ensure (194) holds, we must make them positive so that we can drop them in order to get a lower bound. Now we split the range $\theta \in \left[0.525, \frac{9}{17}\right)$ to 2 subranges.

6.3.1. *Case 1.* $0.525 \leqslant \theta < \frac{19}{36}$. Using Buchstab's identity twice, we have

$$\begin{aligned}
\psi\left(n, x^{\frac{1}{2}}\right) &= \psi\left(n, x^{\nu}\right) - \sum_{\substack{n=p_1 \beta \\ \nu \leqslant \alpha_1 < \frac{1}{2}}} \psi\left(\beta, p_1\right) \\
&= \psi\left(n, x^{\nu}\right) - \sum_{\substack{n=p_1 \beta \\ \nu \leqslant \alpha_1 < \frac{1-\nu}{2}}} \psi\left(\beta, p_1\right) - \sum_{\substack{n=p_1 \beta \\ \frac{1-\nu}{2} \leqslant \alpha_1 < \frac{1}{2}}} \psi\left(\beta, p_1\right) \\
&= \psi\left(n, x^{\nu}\right) - \sum_{\substack{n=p_1 \beta \\ \nu \leqslant \alpha_1 < \frac{1-\nu}{2}}} \psi\left(\beta, x^{\nu}\right) - \sum_{\substack{n=p_1 \beta \\ \frac{1-\nu}{2} \leqslant \alpha_1 < \frac{1}{2}}} \psi\left(\beta, p_1\right) + \sum_{\substack{n=p_1 p_2 \beta \\ \nu \leqslant \alpha_1 < \frac{1-\nu}{2} \\ \nu \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)}} \psi\left(\beta, p_2\right) \\
&= \psi\left(n, x^{\nu}\right) - \sum_{\substack{n=p_1 \beta \\ \nu \leqslant \alpha_1 < \frac{1-\nu}{2}}} \psi\left(\beta, x^{\nu}\right) - \sum_{\substack{n=p_1 \beta \\ \frac{1-\nu}{2} \leqslant \alpha_1 < \frac{1}{2}}} \psi\left(\beta, p_1\right) \\
&\quad + \sum_{\substack{n=p_1 p_2 \beta \\ \nu \leqslant \alpha_1 < \frac{1-\nu}{2} \\ \nu \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \alpha_1+\alpha_2 \in \left(\frac{1-\nu}{2}, \frac{1+\nu}{2}\right)}} \psi\left(\beta, p_2\right) + \sum_{\substack{n=p_1 p_2 \beta \\ \nu \leqslant \alpha_1 < \frac{1-\nu}{2} \\ \nu \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \alpha_1+\alpha_2 \notin \left(\frac{1-\nu}{2}, \frac{1+\nu}{2}\right)}} \psi\left(\beta, p_2\right) \\
&= S_{61} - S_{62} - S_{63} - S_{64} - S_{65}.
\end{aligned} \tag{195}$$

By Lemma 6.10 we know that (194) holds for $f(n) = S_{61}$ and $f(n) = S_{62}$, and by Lemma 6.9 we know that (194) holds for $f(n) = S_{63}$ and $f(n) = S_{64}$.

We divide the sum S_{65} into 4 subsums that are summing over 4 regions A, B, C and D (only defined in this subsection) respectively. These regions are defined as

$$A = A(\theta) = \left\{\boldsymbol{\alpha}_2 : \nu \leqslant \alpha_1 < \frac{1-\nu}{2},\ \nu \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right),\ \alpha_1+\alpha_2 \leqslant \frac{1-\nu}{2},\ \alpha_2 < \frac{4-7\theta+\nu}{2}\right\},$$
$$B = B(\theta) = \left\{\boldsymbol{\alpha}_2 : \nu \leqslant \alpha_1 < \frac{1-\nu}{2},\ \nu \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right),\ \alpha_1+\alpha_2 \geqslant \frac{1+\nu}{2},\ \alpha_2 < \frac{4-7\theta+\nu}{2}\right\},$$
$$C = C(\theta) = \left\{\boldsymbol{\alpha}_2 : \nu \leqslant \alpha_1 < \frac{1-\nu}{2},\ \nu \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right),\ \alpha_1+\alpha_2 \geqslant \frac{1+\nu}{2},\ \alpha_2 \geqslant \frac{4-7\theta+\nu}{2}\right\},$$
$$D = D(\theta) = \left\{\boldsymbol{\alpha}_2 : \nu \leqslant \alpha_1 < \frac{1-\nu}{2},\ \nu \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right),\ \alpha_1+\alpha_2 \leqslant \frac{1-\nu}{2},\ \alpha_2 \geqslant \frac{4-7\theta+\nu}{2}\right\}.$$

Now, we have

$$\begin{aligned}
S_{65} &= \sum_{\substack{n=p_1p_2\beta \\ \nu\leqslant\alpha_1<\frac{1-\nu}{2} \\ \nu\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right) \\ \alpha_1+\alpha_2\notin\left(\frac{1-\nu}{2},\frac{1+\nu}{2}\right)}} \psi(\beta,p_2) \\
&= \sum_{\substack{n=p_1p_2\beta \\ \boldsymbol{\alpha}_2\in A}} \psi(\beta,p_2) + \sum_{\substack{n=p_1p_2\beta \\ \boldsymbol{\alpha}_2\in B}} \psi(\beta,p_2) + \sum_{\substack{n=p_1p_2\beta \\ \boldsymbol{\alpha}_2\in C}} \psi(\beta,p_2) + \sum_{\substack{n=p_1p_2\beta \\ \boldsymbol{\alpha}_2\in D}} \psi(\beta,p_2) \\
&= S_{65A} + S_{65B} + S_{65C} + S_{65D}.
\end{aligned}$$

For S_{65A}, since $\boldsymbol{\alpha}_3$ satisfies Lemma 6.10, we can use Buchstab's identity twice again to reach a four-dimensional sum

$$\sum_{\substack{n=p_1p_2p_3p_4\beta \\ \boldsymbol{\alpha}_2\in A_1 \\ \nu\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \text{ does not satisfy Lemma 6.9} \\ \nu\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \text{ does not satisfy Lemma 6.9}}} \psi(\beta,p_4). \tag{196}$$

In the above sum, we can still perform Buchstab's identity twice when $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_4)$ satisfies Lemma 6.10. This process leads to a six-dimensional loss

$$\sum_{\substack{n=p_1p_2p_3p_4p_5p_6\beta \\ \boldsymbol{\alpha}_2\in A_1 \\ \nu\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \text{ does not satisfy Lemma 6.9} \\ \nu\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \text{ does not satisfy Lemma 6.9} \\ (\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_4) \text{ satisfies Lemma 6.10} \\ \nu\leqslant\alpha_5<\min\left(\alpha_4,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5 \text{ does not satisfy Lemma 6.9} \\ \nu\leqslant\alpha_6<\min\left(\alpha_5,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5)\right) \\ \boldsymbol{\alpha}_6 \text{ does not satisfy Lemma 6.9}}} \psi(\beta,p_6). \tag{197}$$

Note that we have $\nu > \frac{1}{9}$ when $\theta < \frac{19}{36}$, only $\Omega(n) \leqslant 8$ will be counted in the sums, and further straightforward decompositions are not applicable since $n = p_1 \cdots p_8 m$ implies $\Omega(n) \geqslant 9$. For (196) and (197), we can also use reversed Buchstab's identity to gain possible savings.

For S_{65B} we cannot perform a straightforward decomposition. However, since $\alpha_1+\alpha_2 \geqslant \frac{1+\nu}{2}$ implies $1-\alpha_1-\alpha_2 \leqslant \frac{1-\nu}{2}$, a role-reversal can be applied. We first use Buchstab's identity once to get

$$\begin{aligned}
S_{65B} &= \sum_{\substack{n=p_1p_2\beta \\ \boldsymbol{\alpha}_2\in B_1}} \psi(\beta,x^{\nu}) - \sum_{\substack{n=p_1p_2p_3\beta \\ \boldsymbol{\alpha}_2\in B_1 \\ \nu\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \text{ satisfies Lemma 6.9}}} \psi(\beta,p_3) - \sum_{\substack{n=p_1p_2p_3\beta \\ \boldsymbol{\alpha}_2\in B_1 \\ \nu\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \text{ does not satisfy Lemma 6.9}}} \psi(\beta,p_3) \\
&= S_{65B1} - S_{65B2} - S_{65B3}.
\end{aligned} \tag{198}$$

We know that (194) holds for S_{65B1} and S_{65B2} by Lemma 6.10 (since we have $\alpha_1 < \frac{1-\nu}{2}$ and $\alpha_2 < \frac{4-7\theta+\nu}{2}$ in B) and Lemma 6.9 respectively. For S_{65B3}, we change the roles of p_1 and m in S_{65B}, and use Buchstab's identity on p_1 to get

$$S_{65B3} = \sum_{\substack{n=p_1p_2p_3\beta \\ \boldsymbol{\alpha}_2\in B_1 \\ \nu\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \text{ does not satisfy Lemma 6.9}}} \psi(\beta,p_3)$$

$$
\begin{aligned}
&= \sum_{\substack{n=\beta_1 p_2 p_3 \beta \\ \boldsymbol{\alpha}_2 \in B_1 \\ \nu \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \text{ does not satisfy Lemma 6.9}}} \psi(\beta, p_3)\, \psi\left(\beta_1, \left(\frac{2x}{p_2 p_3 m}\right)^{\frac{1}{2}}\right) \\
&= \sum_{\substack{n=\beta_1 p_2 p_3 \beta \\ \boldsymbol{\alpha}_2 \in B_1 \\ \nu \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \text{ does not satisfy Lemma 6.9}}} \psi(\beta, p_3)\, \psi(\beta_1, x^{\nu}) \\
&\quad - \sum_{\substack{n=\beta_1 p_2 p_3 p_4 \beta \\ \boldsymbol{\alpha}_2 \in B_1 \\ \nu \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \text{ does not satisfy Lemma 6.9} \\ \nu \leqslant \alpha_4 < \frac{1}{2}\alpha_1 \\ (1-\alpha_1-\alpha_2-\alpha_3, \alpha_2, \alpha_3, \alpha_4) \text{ satisfies Lemma 6.9}}} \psi(\beta, p_3)\, \psi(\beta_1, p_4) \\
&\quad - \sum_{\substack{n=\beta_1 p_2 p_3 p_4 \beta \\ \boldsymbol{\alpha}_2 \in B_1 \\ \nu \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \text{ does not satisfy Lemma 6.9} \\ \nu \leqslant \alpha_4 < \frac{1}{2}\alpha_1 \\ (1-\alpha_1-\alpha_2-\alpha_3, \alpha_2, \alpha_3, \alpha_4) \text{ does not satisfy Lemma 6.9}}} \psi(\beta, p_3)\, \psi(\beta_1, p_4) \\
&= S_{65B31} - S_{65B32} - S_{65B33}. \qquad (199)
\end{aligned}
$$

In the above 3 sums, we have $\beta \asymp x^{1-\alpha_1-\alpha_2-\alpha_3}$. We have $\beta_1 \asymp x^{\alpha_1}$ in S_{65B31} and $\beta_1 \asymp x^{\alpha_1-\alpha_4}$ in S_{65B32} and S_{65B33}. Here, (194) holds for S_{65B31} and S_{65B32} by Lemma 6.10 (since we have $(1-\alpha_1-\alpha_2-\alpha_3)+\alpha_2 = 1-\alpha_1-\alpha_2 < \frac{1-\nu}{2}$ and $\alpha_2 < \frac{4-7\theta+\nu}{2}$ in B) and Lemma 6.9 respectively. We discard S_{65B33} which gives a four-dimensional loss. Note that further decompositions and the reversed Buchstab's identity can still be performed to gain possible savings.

For S_{65C} and For S_{65D} we cannot perform either straightforward decompositions or decompositions with role-reversals, and we can only discard the whole of them. However, Lemma 6.11 is applicable to show that (194) holds for S_{65C} when $\theta \leqslant 0.5253$. Since $\alpha_2 \geqslant \frac{4-7\cdot 0.5253+\nu(0.5253)}{2} > 0.256 > \frac{1}{4}$ when $0.525 \leqslant \theta \leqslant 0.5253$, we know that S_{65C} only counts products of 3 primes, and we do not need to discard it in this range of θ by an application of Lemma 6.11.

Combining the loss from all 4 subsums, we can get the total loss and the lower bounds for $C_0^{\delta}(\theta)$.

$\boldsymbol{\theta}$	$\boldsymbol{C_0^{\delta}(\theta)}$	$\boldsymbol{\theta}$	$\boldsymbol{C_0^{\delta}(\theta)}$
0.5250	0.9999	0.5264	0.7407
0.5251	0.9999	0.5265	0.7259
0.5252	0.9999	0.5266	0.7141
0.5253	0.9999	0.5267	0.6947
0.5254	0.8416	0.5268	0.6807
0.5255	0.8337	0.5269	0.6662
0.5256	0.8251	0.5270	0.6509
0.5257	0.8166	0.5271	0.6312
0.5258	0.8075	0.5272	0.6107
0.5259	0.7979	0.5273	0.5903
0.5260	0.7868	0.5274	0.5654
0.5261	0.7759	0.5275	0.5374
0.5262	0.7641	0.5276	0.5135
0.5263	0.7527	0.5277	0.4760

Table 6.1: Lower Bounds for $\boldsymbol{C_0^{\delta}(\theta)}$ ($\boldsymbol{0.525 \leqslant \theta < \frac{19}{36}}$)

6.3.2. *Case 2.* $\frac{19}{36} \leqslant \theta < \frac{9}{17}$. The decompositions in this case are very similar to the first case; one can just replace the parameter ν occurred above with ν' and calculate the total loss. Note that Lemma 6.11 is not applicable in this case. Working like the above case we get the following lower bounds for $C_0^{\delta}(\theta)$.

$\boldsymbol{\theta}$	$\boldsymbol{C_0^{\delta}(\theta)}$
0.5278	0.4291
0.5279	0.3856
0.5280	0.3354
0.5281	0.2833
0.5282	0.2189
0.5283	0.1489
0.5284	0.0648
0.5285	-0.023

Table 6.2: Lower Bounds for $\boldsymbol{C_0^{\delta}(\theta)}$ ($\boldsymbol{\frac{19}{36} \leqslant \theta < \frac{9}{17}}$)

Note that the lower bound becomes trivial when $\theta \geqslant 0.5285$.

6.4. **Upper Bounds.** We shall construct the majorant $\rho_1(n)$ in this subsection. Before constructing, we first mention an existing result of $C_1^{\delta}(\theta)$ proved by Stadlmann [38].

Theorem 6.13. *([38], Theorem 1]). The function $C_1^{\delta}(\theta)$ satisfies the following condition:*
$C_1^{\delta}(\theta) = 1$ for all $\theta < 0.525$.

Recalling that our aim is to decompose $\psi\left(n, x^{\frac{1}{2}}\right)$ using Buchstab's identity and show that (194) holds for most of the sums after the decomposition. For the remaining sums that we cannot ensure (194) holds, we must make them negative so that we can drop them in order to get an upper bound. Now we split the range $\theta \in \left[0.525, \frac{9}{17}\right)$ to 2 subranges.

6.4.1. *Case 1.* $0.525 \leqslant \theta < \frac{19}{36}$. Using Buchstab's identity, we get

$$\begin{aligned}
\psi\left(n, x^{\frac{1}{2}}\right) =& \psi\left(n, x^{\nu}\right) - \sum_{\substack{n=p_1\beta \\ \nu \leqslant \alpha_1 < \frac{1}{2}}} \psi\left(\beta, p_1\right) \\
=& \psi\left(n, x^{\nu}\right) - \sum_{\substack{n=p_1\beta \\ \nu \leqslant \alpha_1 < \frac{1+\nu}{4}}} \psi\left(\beta, p_1\right) - \sum_{\substack{n=p_1\beta \\ \frac{1+\nu}{4} \leqslant \alpha_1 < \frac{1-\nu}{2}}} \psi\left(\beta, p_1\right) - \sum_{\substack{n=p_1\beta \\ \frac{1-\nu}{2} \leqslant \alpha_1 < \frac{1}{2}}} \psi\left(\beta, p_1\right) \\
=& \psi\left(n, x^{\nu}\right) - \sum_{\substack{n=p_1\beta \\ \frac{1+\nu}{4} \leqslant \alpha_1 < \frac{1-\nu}{2}}} \psi\left(\beta, p_1\right) - \sum_{\substack{n=p_1\beta \\ \frac{1-\nu}{2} \leqslant \alpha_1 < \frac{1}{2}}} \psi\left(\beta, p_1\right) \\
& - \sum_{\substack{n=p_1\beta \\ \nu \leqslant \alpha_1 < \frac{1+\nu}{4}}} \psi\left(\beta, x^{\nu}\right) + \sum_{\substack{n=p_1 p_2\beta \\ \nu \leqslant \alpha_1 < \frac{1+\nu}{4} \\ \nu \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)}} \psi\left(\beta, p_2\right) \\
=& \psi\left(n, x^{\nu}\right) - \sum_{\substack{n=p_1\beta \\ \frac{1+\nu}{4} \leqslant \alpha_1 < \frac{1-\nu}{2}}} \psi\left(\beta, p_1\right) - \sum_{\substack{n=p_1\beta \\ \frac{1-\nu}{2} \leqslant \alpha_1 < \frac{1}{2}}} \psi\left(\beta, p_1\right) \\
& - \sum_{\substack{n=p_1\beta \\ \nu \leqslant \alpha_1 < \frac{1+\nu}{4}}} \psi\left(\beta, x^{\nu}\right) + \sum_{\substack{n=p_1 p_2\beta \\ \nu \leqslant \alpha_1 < \frac{1+\nu}{4} \\ \nu \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \alpha_1+\alpha_2 < \frac{1-\nu}{2}}} \psi\left(\beta, p_2\right) + \sum_{\substack{n=p_1 p_2\beta \\ \nu \leqslant \alpha_1 < \frac{1+\nu}{4} \\ \nu \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \frac{1-\nu}{2} \leqslant \alpha_1+\alpha_2 < \frac{1+\nu}{2}}} \psi\left(\beta, p_2\right) \\
=& \psi\left(n, x^{\nu}\right) - \sum_{\substack{n=p_1\beta \\ \frac{1+\nu}{4} \leqslant \alpha_1 < \frac{1-\nu}{2}}} \psi\left(\beta, p_1\right) - \sum_{\substack{n=p_1\beta \\ \frac{1-\nu}{2} \leqslant \alpha_1 < \frac{1}{2}}} \psi\left(\beta, p_1\right) \\
& - \sum_{\substack{n=p_1\beta \\ \nu \leqslant \alpha_1 < \frac{1+\nu}{4}}} \psi\left(\beta, x^{\nu}\right) + \sum_{\substack{n=p_1 p_2\beta \\ \nu \leqslant \alpha_1 < \frac{1+\nu}{4} \\ \nu \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \frac{1-\nu}{2} \leqslant \alpha_1+\alpha_2 < \frac{1+\nu}{2}}} \psi\left(\beta, p_2\right) \\
& + \sum_{\substack{n=p_1 p_2\beta \\ \nu \leqslant \alpha_1 < \frac{1+\nu}{4} \\ \nu \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \alpha_1+\alpha_2 < \frac{1-\nu}{2}}} \psi\left(\beta, x^{\nu}\right) - \sum_{\substack{n=p_1 p_2 p_3\beta \\ \nu \leqslant \alpha_1 < \frac{1+\nu}{4} \\ \nu \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \alpha_1+\alpha_2 < \frac{1-\nu}{2} \\ \nu \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \text{ satisfies Lemma 6.9}}} \psi\left(\beta, p_3\right) \\
& - \sum_{\substack{n=p_1 p_2 p_3\beta \\ \nu \leqslant \alpha_1 < \frac{1+\nu}{4} \\ \nu \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \alpha_1+\alpha_2 < \frac{1-\nu}{2} \\ \nu \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \text{ does not satisfy Lemma 6.9}}} \psi\left(\beta, p_3\right) \\
=& S_{661} - S_{662} - S_{663} - S_{664} + S_{665} + S_{666} - S_{667} - S_{668}. \qquad (200)
\end{aligned}$$

We know that (194) holds for S_{661}, S_{664}, S_{666} (by Lemma 6.10) and S_{663}, S_{665}, S_{667} (by Lemma 6.9). We can perform a further decomposition on S_{668} if $(\alpha_1, \alpha_2, \alpha_3, \alpha_3)$ satisfies Lemma 6.10, leading to a five-dimensional sum similar to (197). We discard this sum and remaining parts of S_{668} where $(\alpha_1, \alpha_2, \alpha_3, \alpha_3)$ does not satisfy Lemma 6.10. Again, reversed Buchstab's identity and role-reversals can be applied.

When $\theta \geqslant \frac{1}{2}+\frac{1}{79}$, we discard the whole of S_{662}. When $\theta < \frac{1}{2}+\frac{1}{79}$, we perform a further decomposition on S_{662} to get

$$
\begin{aligned}
S_{662} = & \sum_{\substack{n=p_1\beta \\ \frac{1+\nu}{4}\leqslant\alpha_1<\frac{1-\nu}{2}}} \psi(\beta, p_1) \\
= & \sum_{\substack{n=p_1\beta \\ \frac{1+\nu}{4}\leqslant\alpha_1<\frac{1-\nu}{2}}} \psi(\beta, x^\nu) - \sum_{\substack{n=p_1p_2\beta \\ \frac{1+\nu}{4}\leqslant\alpha_1<\frac{1-\nu}{2} \\ \nu\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)}} \psi(\beta, p_2) \\
= & \sum_{\substack{n=p_1\beta \\ \frac{1+\nu}{4}\leqslant\alpha_1<\frac{1-\nu}{2}}} \psi(\beta, x^\nu) - \sum_{\substack{n=p_1p_2\beta \\ \frac{1+\nu}{4}\leqslant\alpha_1<\frac{1-\nu}{2} \\ \nu\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right) \\ \alpha_1+\alpha_2\in\left(\frac{1-\nu}{2},\frac{1+\nu}{2}\right)}} \psi(\beta, p_2) - \sum_{\substack{n=p_1p_2\beta \\ \frac{1+\nu}{4}\leqslant\alpha_1<\frac{1-\nu}{2} \\ \nu\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right) \\ \alpha_1+\alpha_2\notin\left(\frac{1-\nu}{2},\frac{1+\nu}{2}\right) \\ \alpha_2\geqslant\frac{4-7\theta+\nu}{2}}} \psi(\beta, p_2) \\
& - \sum_{\substack{n=p_1p_2\beta \\ \frac{1+\nu}{4}\leqslant\alpha_1<\frac{1-\nu}{2} \\ \nu\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right) \\ \alpha_1+\alpha_2\notin\left(\frac{1-\nu}{2},\frac{1+\nu}{2}\right) \\ \alpha_2<\frac{4-7\theta+\nu}{2}}} \psi(\beta, p_2) \\
= & \sum_{\substack{n=p_1\beta \\ \frac{1+\nu}{4}\leqslant\alpha_1<\frac{1-\nu}{2}}} \psi(\beta, x^\nu) - \sum_{\substack{n=p_1p_2\beta \\ \frac{1+\nu}{4}\leqslant\alpha_1<\frac{1-\nu}{2} \\ \nu\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right) \\ \alpha_1+\alpha_2\in\left(\frac{1-\nu}{2},\frac{1+\nu}{2}\right)}} \psi(\beta, p_2) - \sum_{\substack{n=p_1p_2\beta \\ \frac{1+\nu}{4}\leqslant\alpha_1<\frac{1-\nu}{2} \\ \nu\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right) \\ \alpha_1+\alpha_2\notin\left(\frac{1-\nu}{2},\frac{1+\nu}{2}\right) \\ \alpha_2\geqslant\frac{4-7\theta+\nu}{2}}} \psi(\beta, p_2) \\
& - \sum_{\substack{n=p_1p_2\beta \\ \frac{1+\nu}{4}\leqslant\alpha_1<\frac{1-\nu}{2} \\ \nu\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right) \\ \alpha_1+\alpha_2\notin\left(\frac{1-\nu}{2},\frac{1+\nu}{2}\right) \\ \alpha_2<\frac{4-7\theta+\nu}{2}}} \psi(\beta, x^\nu) + \sum_{\substack{n=p_1p_2p_3\beta \\ \frac{1+\nu}{4}\leqslant\alpha_1<\frac{1-\nu}{2} \\ \nu\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right) \\ \alpha_1+\alpha_2\notin\left(\frac{1-\nu}{2},\frac{1+\nu}{2}\right) \\ \alpha_2<\frac{4-7\theta+\nu}{2} \\ \nu\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \text{ satisfies Lemma 6.9}}} \psi(\beta, p_3) + \sum_{\substack{n=p_1p_2p_3\beta \\ \frac{1+\nu}{4}\leqslant\alpha_1<\frac{1-\nu}{2} \\ \nu\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right) \\ \alpha_1+\alpha_2\notin\left(\frac{1-\nu}{2},\frac{1+\nu}{2}\right) \\ \alpha_2<\frac{4-7\theta+\nu}{2} \\ \nu\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \text{ does not satisfy Lemma 6.9}}} \psi(\beta, p_3) \\
= & S_{6621} - S_{6622} - S_{6623} - S_{6624} + S_{6625} + S_{6626}. \qquad (201)
\end{aligned}
$$

We have (194) holds for S_{6621}, S_{6624} (by Lemma 6.10) and S_{6622}, S_{6625} (by Lemma 6.9). For S_{6623} we note that $\alpha_1 > \alpha_2 \geqslant \frac{4-7\theta+\nu}{2} > \frac{1-\nu}{4}$ when $0.525 \leqslant \theta \leqslant 0.5253$, hence the condition $\alpha_1+\alpha_2 \notin \left(\frac{1-\nu}{2}, \frac{1+\nu}{2}\right)$ is equivalent to $\alpha_1+\alpha_2 \geqslant \frac{1+\nu}{2}$ in this sum. Then we have

$$
S_{6623} = \sum_{\substack{n=p_1p_2\beta \\ \frac{1+\nu}{4}\leqslant\alpha_1<\frac{1-\nu}{2} \\ \nu\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right) \\ \alpha_1+\alpha_2\notin\left(\frac{1-\nu}{2},\frac{1+\nu}{2}\right) \\ \alpha_2\geqslant\frac{4-7\theta+\nu}{2}}} \psi(\beta, p_2) = \sum_{\substack{n=p_1p_2\beta \\ \frac{1+\nu}{4}\leqslant\alpha_1<\frac{1-\nu}{2} \\ \nu\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right) \\ \alpha_1+\alpha_2\geqslant\frac{1+\nu}{2} \\ \alpha_2\geqslant\frac{4-7\theta+\nu}{2}}} \psi(\beta, p_2) = S_{65C} \qquad (202)
$$

where S_{65C} is defined in Subsection 6.3. By Lemma 6.11, we know that (194) holds for this sum. We discard the remaining sum S_{6626}. Note that this process replaces a "larger" one-dimensional loss from S_{662} with a "smaller" three-dimensional loss S_{6626}.

Numerical calculations show the following upper bounds for $C_1^\delta(\theta)$.

$\boldsymbol{\theta}$	$\boldsymbol{C_1^{\delta}(\theta)}$	$\boldsymbol{\theta}$	$\boldsymbol{C_1^{\delta}(\theta)}$
0.5250	1.0001	0.5264	1.6367
0.5251	1.0004	0.5265	1.6515
0.5252	1.0011	0.5266	1.6660
0.5253	1.0028	0.5267	1.6811
0.5254	1.5112	0.5268	1.6949
0.5255	1.5231	0.5269	1.7110
0.5256	1.5349	0.5270	1.7269
0.5257	1.5471	0.5271	1.7452
0.5258	1.5594	0.5272	1.7584
0.5259	1.5719	0.5273	1.7788
0.5260	1.5843	0.5274	1.7961
0.5261	1.5972	0.5275	1.8139
0.5262	1.6103	0.5276	1.8332
0.5263	1.6238	0.5277	1.8542

Table 6.3: Upper Bounds for $\boldsymbol{C_1^{\delta}(\theta)}$ ($\boldsymbol{0.525 \leqslant \theta < \frac{19}{36}}$)

6.4.2. *Case 2.* $\frac{19}{36} \leqslant \theta < \frac{9}{17}$. Again, one can just replace the parameter ν occurred above with ν' and calculate the total loss. Working like the first case we get the following upper bounds for $C_1^{\delta}(\theta)$.

$\boldsymbol{\theta}$	$\boldsymbol{C_1^{\delta}(\theta)}$
0.5278	1.8722
0.5279	1.8885
0.5280	1.9126
0.5281	1.9315
0.5282	1.9563
0.5283	1.9751
0.5284	1.9971
0.5285	2.0156
0.5286	2.0401
0.5287	2.0649
0.5288	2.0859
0.5289	2.1117
0.5290	2.1353
0.5291	2.1628
0.5292	2.1954
0.5293	2.2446
0.5294	2.2963

Table 6.4: Upper Bounds for $\boldsymbol{C_1^{\delta}(\theta)}$ ($\boldsymbol{\frac{19}{36} \leqslant \theta < \frac{9}{17}}$)

7. Lower Bounds: A General Case

In this section, we focus on a general form of Mikawa's modified sieve [32]. The sets $\mathcal{A}$ and $\mathcal{B}$ in this section can be other "comparison" sets, and their "proportion" may not be $\frac{1}{\varphi(q)}$ as in previous sections. Assume that variants of (27) and (31), with different proportion, hold true. Assume further that there are lots of Type-II information inputs so that the corresponding loss integrals (41) and (42) are zero. We want to find the minimum value of κ such that the sum of other loss integrals that cannot be reduced using Type-II information is less than 1. Let $\kappa \leqslant \frac{1}{8}$. For simplicity, we ignore the new integrals corresponding to sums that count numbers with 7 or more prime factors, and only consider the 3 loss integrals corresponding to (36)–(38). Define

$$\begin{aligned} L_7(\kappa) = {} & 2\int_{(t_1,t_2,t_3)\in U_{71}} \frac{1}{t_1t_2t_3(1-t_1-t_2-t_3)} dt_3dt_2dt_1 \\ & + 2\int_{(t_1,t_2,t_3,t_4)\in U_{72}} \frac{1}{t_1t_2t_3t_4(1-t_1-t_2-t_3-t_4)} dt_4dt_3dt_2dt_1 \\ & + 20\int_{(t_1,t_2,t_3,t_4,t_5)\in U_{73}} \frac{1}{t_1t_2t_3t_4t_5(1-t_1-t_2-t_3-t_4-t_5)} dt_5dt_4dt_3dt_2dt_1, \end{aligned} \tag{203}$$

where

$$\begin{aligned} U_{71}(\boldsymbol{\alpha}_3) &:= \left\{\kappa < \alpha_3 < \alpha_2 < \alpha_1,\ \alpha_1+\alpha_2+\alpha_3 > \frac{1}{2},\ 2\alpha_1+2\alpha_2+\alpha_3 < 1\right\}, \\ U_{72}(\boldsymbol{\alpha}_4) &:= \{\kappa < \alpha_4 < \alpha_3 < \alpha_2 < \alpha_1,\ 2\alpha_1+2\alpha_2+\alpha_3 < 1\}, \\ U_{73}(\boldsymbol{\alpha}_5) &:= \{\kappa < \alpha_5 < \alpha_4 < \alpha_3 < \alpha_2 < \alpha_1,\ 2\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5 < 1\}. \end{aligned}$$

Numerical calculations show that $L_7(\frac{1}{11}) < 0.84$ and $L_7(\frac{1}{12}) > 1.2$. This means that we cannot get a nontrivial lower bound using Mikawa's sieve if we only have a Type-II range $\left(0, \frac{1}{12}\right)$ or even "weaker" ranges.

Another interesting question about Mikawa's sieve is: Can we apply this sieve on other sieve problems? In another paper, Mikawa [31] used his method to study the distribution of Goldbach numbers in arithmetic progressions. However, we cannot apply Mikawa's sieve on many other classical sieve problems, such as primes in all short intervals and primes in almost all short intervals. The most important reason of that is the arithmetic information inputs in those problems often have extra restrictions, such as **Condition B** on one coefficient and the "prime-factored" condition (see [[17], Chapter 7]). We cannot use those "restricted" arithmetic information inputs to prove that variants of (27) and (31) hold true.

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