A REMARK ON THE DISTRIBUTION OF $\sqrt{p}$ MODULO ONE INVOLVING PRIMES OF SPECIAL TYPE

RUNBO LI

Abstract. Let P_r denote an integer with at most r prime factors counted with multiplicity. In this paper we prove that for any $0 < \lambda < \frac{1}{4}$, the inequality $\{\sqrt{p}\} < p^{-\lambda}$ has infinitely many solutions in primes p such that $p+2 = P_r$, where $r = \lfloor \frac{8}{1-4\lambda} \rfloor$. This generalizes the previous result of Cai.

Contents

1. Introduction

Beginning with Vinogradov [8], many mathematicians have studied the inequality $\{\sqrt{p}\} < p^{-\lambda}$ with prime solutions. Now the best result is due to Harman and Lewis [5]. In [5] they proved that there are infinitely many solutions in primes p to the inequality $\{\sqrt{p}\} < p^{-\lambda}$ with $\lambda = 0.262$, which improved the previous results of Vinogradov [8], Kaufman [7], Harman [4] and Balog [1].

On the other hand, one of the famous problems in prime number theory is the twin primes problem, which states that there are infinitely many primes p such that $p+2$ is also a prime. Let P_r denote an integer with at most r prime factors counted with multiplicity. Now the best result in this aspect is due to Chen [3], who showed that there are infinitely many primes p such that $p+2 = P_2$.

In 2013, Cai [2] combined those two problems and got the following result by using a delicate sieve process and a new mean value theorem for the von Mangoldt function.

Theorem 1.1. *The inequality*

$$\{\sqrt{p}\} < p^{-\lambda} \tag{1}$$

with $\lambda = \frac{1}{15.5}$ *holds for infinitely many primes* p *such that* $p+2 = P_4$.

In this paper, we generalize Cai's result to every $0 < \lambda < \frac{1}{4}$. Actually we prove the following theorem.

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Theorem 1.2. *The inequality (1) with $0<\lambda<\frac{1}{4}$ holds for infinitely many primes p such that $p+2=P_r$, where $r=\lfloor\frac{8}{1-4\lambda}\rfloor$.*

We also have some corollaries of Theorem 1.2.

Corollary 1.3. *The inequality (1) with $\lambda=\frac{1}{15.5}$ holds for infinitely many primes p such that $p+2=P_{10}$.*

Corollary 1.4. *The inequality (1) with $0<\lambda<\frac{1}{36}$ holds for infinitely many primes p such that $p+2=P_8$.*

Clearly our Corollary 1.3 is weaker than Cai's one (in fact, the limit of our method is to prove $p+2=P_8$, see Corollary 1.4), but our goal that extending Cai's result to $0<\lambda<\frac{1}{4}$, has been accomplished. It is worth mentioning that Cai proved a new mean value theorem (see [[2], Lemma 5]) for this problem and it may be useful on improving our results. We hope someone can accomplish this work.

2. Preliminary lemmas

Let $\mathcal{A}$ denote a finite set of positive integers and $z\geqslant 2$. Suppose that $|\mathcal{A}|\sim X_{\mathcal{A}}$ and for square-free d, put

$$\mathcal{P}=\{p:(p,2)=1\},\quad \mathcal{P}(r)=\{p:p\in\mathcal{P},(p,r)=1\},$$

$$P(z)=\prod_{\substack{p\in\mathcal{P}\\ p<z}}p,\quad \mathcal{A}_d=\{a:a\in\mathcal{A},a\equiv 0(\bmod d)\},\quad S(\mathcal{A};\mathcal{P},z)=\sum_{\substack{a\in\mathcal{A}\\ (a,P(z))=1}}1.$$

Lemma 2.1. *([[6], Lemma 2]). If*

$$\sum_{z_1\leqslant p<z_2}\frac{\omega(p)}{p}=\log\frac{\log z_2}{\log z_1}+O\left(\frac{1}{\log z_1}\right),\quad z_2>z_1\geqslant 2,$$

where $\omega(d)$ is a multiplicative function, $0\leqslant\omega(p)<p, X_{\mathcal{A}}>1$ is independent of d. Then

$$S(\mathcal{A};\mathcal{P},z)\geqslant X_{\mathcal{A}}W(z)\left\{f\left(\frac{\log D}{\log z}\right)+O\left(\frac{1}{\log^{\frac{1}{3}}D}\right)\right\}-\sum_{\substack{n\leqslant D\\ n\mid P(z)}}|\eta(X_{\mathcal{A}},n)|,$$

where D is a power of z,

$$W(z)=\prod_{\substack{p<z\\ (p,2)=1}}\left(1-\frac{\omega(p)}{p}\right),\quad f(s)=\frac{2e^{\gamma}\log(s-1)}{s}\ \text{for every}\ 2\leqslant s\leqslant 4,$$

$$\eta(X_{\mathcal{A}},n)=|\mathcal{A}_n|-\frac{\omega(n)}{n}X_{\mathcal{A}}=\sum_{\substack{a\in\mathcal{A}\\ a\equiv 0(\bmod n)}}1-\frac{\omega(n)}{n}X_{\mathcal{A}}.$$

Lemma 2.2. *([2], Lemma 4]). For any given constant $A>0$ and $0<\lambda<\frac{1}{4}, 0<\theta<\frac{1}{4}-\lambda$ we have*

$$\sum_{d\leqslant x^{\theta}}\max_{(l,d)=1}\left|\sum_{\substack{x<p\leqslant 2x\\ \{\sqrt{p}\}<p^{-\lambda}\\ p\equiv l(\bmod d)}}1-\frac{(2x)^{1-\lambda}-x^{1-\lambda}}{\varphi(d)(1-\lambda)\log x}\right|\ll\frac{x^{1-\lambda}}{\log^{A}x}.$$

3. Proof of Theorem 1.2

In this section, we define the function ω as $\omega(p)=0$ for $p=2$ and $\omega(p)=\frac{p}{p-1}$ for other primes. Put

$$D=x^{\frac{1}{4}-\lambda-\varepsilon},\quad \mathcal{A}=\left\{p+2\mid x<p\leqslant 2x,\{\sqrt{p}\}<p^{-\lambda}\right\}.$$

By the definition of $S(\mathcal{A};\mathcal{P},z)$, any element in $S\left(\mathcal{A};\mathcal{P},x^{\frac{1}{k}}\right)$ has at most $k-1$ prime factors. Let S denote the number of prime solutions to the inequality (1) such that $p+2=P_r$, then we have

$$S\geqslant S\left(\mathcal{A};\mathcal{P},x^{\frac{1}{r+1}}\right)+O\left(x^{1-\frac{1}{r+1}}\right). \tag{2}$$

By similar arguments as in [2] we can take

$$X_{\mathcal{A}}=\frac{(2x)^{1-\lambda}-x^{1-\lambda}}{(1-\lambda)\log x}. \tag{3}$$

And by the similar arguments as in [6] we know that

$$W\left(x^{\frac{1}{r+1}}\right)=\frac{2(r+1)e^{-\gamma}C_2(1+o(1))}{\log x}, \tag{4}$$

where

$$C_2:=\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right). \tag{5}$$

To deal with the error terms, by Lemma 2.2 we can easily show that

$$\sum_{\substack{n\leqslant D\\ n\mid P\left(x^{\frac{1}{r+1}}\right)}}|\eta(X_{\mathcal{A}},n)|\ll\sum_{n\leqslant D}\mu^2(n)\,|\eta(X_{\mathcal{A}},n)|\ll x^{1-\lambda}(\log x)^{-5}. \tag{6}$$

Then by Lemma 2.1 we have

$$\begin{aligned}S\left(\mathcal{A};\mathcal{P},x^{\frac{1}{r+1}}\right)\geqslant& X_{\mathcal{A}}W\left(x^{\frac{1}{r+1}}\right)\left\{f\left(\frac{\log D}{\log x^{\frac{1}{r+1}}}\right)+O\left(\frac{1}{\log^{\frac{1}{3}}D}\right)\right\}-\sum_{\substack{n\leqslant D\\ n\mid P\left(x^{\frac{1}{r+1}}\right)}}|\eta(X_{\mathcal{A}},n)|\\ \geqslant&\frac{(2x)^{1-\lambda}-x^{1-\lambda}}{(1-\lambda)\log x}\frac{2(r+1)e^{-\gamma}C_2(1+o(1))}{\log x}f\left(\frac{\frac{1}{4}-\lambda-\varepsilon}{\frac{1}{r+1}}\right),\end{aligned} \tag{7}$$

so we only need $\frac{\frac{1}{4}-\lambda-\varepsilon}{\frac{1}{r+1}}\geqslant 2$ to provide a positive lower bound for S. Clearly this is equivalent to $r>\frac{8}{1-4\lambda}-1$. Now because $\lfloor\frac{8}{1-4\lambda}\rfloor>\frac{8}{1-4\lambda}-1$, Theorem 1.2 is proved.

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International Curriculum Center, The High School Affiliated to Renmin University of China, Beijing, China

Email address: runbo.li.carey@gmail.com