ON THE LARGEST PRIME FACTOR OF QUADRATIC POLYNOMIALS

RUNBO LI

Abstract. Let x denote a sufficiently large integer. We show that the recent result of Grimmelt and Merikoski actually yields the largest prime factor of n^2+1 is greater than $x^{1.317}$ infinitely often. As an application, we give a new upper bound for the number of integers $n \leqslant x$ which n^2+1 has a primitive divisor.

Contents

1. Introduction

Let x, n denote sufficiently large integers, p denote a prime number, P_r denote an integer with at most r prime factors counted with multiplicity, and let f be an irreducible polynomial with degree g. It's conjectured that there are infinitely many n such that $f(n)$ is prime. The simplest case is $g=1$, which is the famous Dirichlet's theorem proved more than 100 years ago. However, for $g \geqslant 2$, this conjecture is still open.

For the second simplest case $g=2$, there are several ways to attack this conjecture. One way is to relax the number of prime factors of $f(n)$, and the best result in this way is due to Iwaniec [8]. Building on the previous work of Richert [14], he showed that for any irreducible polynomial $f(n)=an^2+bn+c$ with $a>0$ and $c \equiv 1(\bmod 2)$, there are infinitely many x such that $f(x)$ is a P_2.

Another possible way is to consider the largest prime factor of $f(n)$. Let $P^+(x)$ denote the largest prime factor of x, then we hope to show that the largest prime factor of $f(n)$ is greater than n^g for infinitely many integers n. For general polynomials, the best result is due to Tenenbaum [16], where he showed that for some $0<t<2-\log 4$, the largest prime factor of $f(n)$ is greater than $n\exp((\log n)^t)$ for infinitely many integers n. However, it's rather difficult to prove the same thing holds for $n^{1+\varepsilon}$ even for a small ε.

For the special case $f(n)=n^2+1$, the progress is far more than the general case. In 1967, Hooley [7] first proved the largest prime factor of n^2+1 is greater than $n^{1.10014}$ for infinitely many integers n by using the Weil bound for Kloosterman sums. By applying their new bounds for multilinear forms of Kloosterman sums, Deshouillers and Iwaniec [2] showed in 1982 that the largest prime factor of n^2+1 is greater than $n^{1.202468}$ infinitely often. In 2020, de la Bretèche and Drappeau [1] improved the exponent to 1.2182 by making use of the result of Kim and Sarnak [9]. In 2023, Merikoski [11] proved a new bilinear estimate and used Harman's sieve to get the exponent 1.279. This is the first attempt of using Harman's sieve on this problem. In 2024, Pascadi [13] optimized the exponent to 1.3 by inserting his new arithmetic information. Recently, using a different approach to obtain arithmetical information, Grimmelt and Merikoski [5] got 1.312, which is the value that previously obtained by Merikoski [11] under the Selberg eigenvalue conjecture. In the present paper, we shall use the exactly same sieve argument as in [5] and illustrate that this exponent can be further improved to 1.317.

2020 Mathematics Subject Classification. 11N32, 11N35, 11N36.

Key words and phrases. prime, quadratic polynomial.

Theorem 1.1. *Let $(\frac{x}{p})$ denotes the Legendre symbol. There exists some small $\varepsilon > 0$ such that the following holds for all $X > 1/\varepsilon$. Let $1 \leqslant h \leqslant X^{1+\varepsilon}$ be square–free and $1 \leqslant a \leqslant X^{\varepsilon}$ with $(a, h) = 1$. Suppose that*

$$\left| \sum_{p \leqslant Y} \frac{\log p}{p} \left(\frac{-ah}{p} \right) \right| \leqslant \varepsilon \log Y$$

for any $X^{\varepsilon} < Y \leqslant X^2$. Then there exists $n \in [X, 2X]$ such that the largest prime factor of $an^2 + h$ is greater than $n^{1.317}$. Specially, the largest prime factor of $n^2 + 1$ is greater than $n^{1.317}$ infinitely often.

As an application of our Theorem 1.1, we consider the polynomial $n^2 + 1$ with a primitive divisor.

Definition 1.2. Let (A_n) denote a sequence with integer terms. We say an integer $d > 1$ is a primitive divisor of A_n if $d \mid A_n$ and $(d, A_m) = 1$ for all non-zero terms A_m with $m < n$.

Proposition 1.3. *For all $n > 1$, the term $n^2 + 1$ has a primitive divisor if and only if $P^+(n^2 + 1) > 2n$. For all $n > 1$, if $n^2 + 1$ has a primitive divisor then that primitive divisor is a prime and it is unique.*

Contrary to the previous works on the lower bounds for the largest prime factor, a result due to Schinzel [15] showed that for any $\varepsilon > 0$, the largest prime factor of $n^2 + 1$ is less than n^{ε} infinitely often. In fact, from his result we can easily get the following.

Theorem 1.4. *([3], Theorem 1.2]). The polynomial $n^2 + 1$ does not have a primitive divisor for infinitely many terms.*

We are interested in finding good upper and lower bound for the number of terms $n^2 + 1$ with a primitive divisor. We define

$$\rho(x) = \left| \left\{ n \leqslant x : n^2 + 1 \text{ has a primitive divisor} \right\} \right| .$$

Then we have the following simple upper bound

$$\rho(x) < x - \frac{Cx}{\log x} \tag{1}$$

for some constant $C > 0$. In [4] the following stronger result is mentioned.

$$\rho(x) < x - \frac{x \log \log x}{\log x}. \tag{2}$$

In [3], Everest and Harman first proved a lower bound with positive density and a better upper bound for $\rho(x)$. More precisely, they got the following bounds:

Theorem 1.5. *([3], Theorem 1.4]). We have*

$$0.5324x < \rho(x) < 0.905x.$$

They also conjectured the asymptotic $\rho(x) \sim (\log 2)x$ in their paper. In 2024, Harman [6] used Merikoski's work on the largest prime factor of $n^2 + 1$ and sharpened the upper and lower bounds for $\rho(x)$.

Theorem 1.6. *([6], Theorem 5.5]). We have*

$$0.5377x < \rho(x) < 0.86x.$$

In the same year, Li [10] further improved the upper bound for $\rho(x)$ by using Pascadi's work.

Theorem 1.7. *([10], Theorem 1.6]). We have*

$$\rho(x) < 0.847x.$$

Mine [12] got a better lower bound for $\rho(x)$.

Theorem 1.8. *([12], Theorem 1.3]). We have*

$$\rho(x) > 0.543x.$$

In the present paper, we use the same sieve argument as in [10] and a recent result of Grimmelt and Merikoski to give a better upper bound for $\rho(x)$.

Theorem 1.9. *We have*

$$\rho(x) < 0.838x.$$

2. Merikoski's sieve decompositions

Let ε denote a sufficient small positive number and P_x denote the largest prime factor of $\prod_{x \leqslant n \leqslant 2x} (n^2+1)$. In this section we briefly introduce Grimmelt and Merikoski's work on finding a lower bound for P_x. Let $b(x)$ denote a non-nagative C^∞–smooth function supported on $[x, 2x]$ and its derivatives satisfy $b^{(j)}(x) \ll x^{-j}$ for all $j \geqslant 0$. We define

$$|\mathcal{A}_d| := \sum_{n^2+1 \equiv 0 (\bmod d)} b(n) \quad \text{and} \quad X := \int b(x) dx.$$

Then by the method of Chebyshev–Hooley and the discussion in [11], we only need to find an upper bound for

$$S(x) := \sum_{x < p \leqslant P_x} |\mathcal{A}_p| \log p = X \log x + O(x) \tag{3}$$

with a constant less than 1. By a smooth dyadic partition we have

$$S(x) = \sum_{\substack{x \leqslant P \leqslant P_x \\ P = 2^j x}} S(x, P) + O(x), \tag{4}$$

where

$$S(x, P) = \sum_{P \leqslant p \leqslant 4P} \psi_P(p) \, |\mathcal{A}_p| \log p \tag{5}$$

for some C^∞–smooth functions ψ_P supported on $[P, 4P]$ satisfying $\psi_P^{(l)}(x) \ll P^{-l}$ for all $l \geqslant 0$.

In [5], Grimmelt and Merikoski proved the following upper bound for $S(x)$ with $P_x = x^{1.312}$ by using Harman's sieve method together with their new arithmetic information.

Lemma 2.1. *(See [11]). We have*

$$\begin{aligned} \sum_{\substack{x \leqslant P \leqslant x^{1.312} \\ P = 2^j x}} S(x, P) \leqslant & \left(G_0 + G_1 + G_2 + G_3 + G_4 + G_5 - G_6 + G_7\right) X \log x \\ < & \ 0.998 X \log x, \end{aligned}$$

where

$$G_0 = \int_1^{\frac{7}{6}} 1 d\alpha = \frac{1}{6},$$

$$G_1 = \int_1^{\frac{17}{16}} \int_{\sigma(\alpha)}^{\alpha - 2\sigma(\alpha)} \alpha \frac{\omega\left(\frac{\alpha-\beta}{\beta}\right)}{\beta^2} d\beta d\alpha + \int_1^{\frac{17}{16}} \int_{\xi(\alpha)}^{\frac{\alpha}{2}} \alpha \frac{\omega\left(\frac{\alpha-\beta}{\beta}\right)}{\beta^2} d\beta d\alpha < 0.0287,$$

$$G_2 = \int_{\frac{17}{16}}^{\frac{8}{7}} \int_{\sigma(\alpha)}^{\frac{\alpha}{2}} \alpha \frac{\omega\left(\frac{\alpha-\beta}{\beta}\right)}{\beta^2} d\beta d\alpha < 0.08622,$$

$$G_3 = \int_{\frac{8}{7}}^{\frac{7}{6}} \int_{\sigma(\alpha)}^{\frac{\alpha}{2}} \alpha \frac{\omega\left(\frac{\alpha-\beta}{\beta}\right)}{\beta^2} d\beta d\alpha < 0.03107,$$

$$G_4 = \int_{\frac{8}{7}}^{\frac{7}{6}} \int_{\sigma(\alpha)-\alpha+1}^{\alpha-1} \int_{\sigma(\alpha)-\alpha+1}^{\beta_1} \int_{\sigma(\alpha)-\alpha+1}^{\beta_2} f_4\left(\alpha, \beta_1, \beta_2, \beta_3\right) \alpha \frac{\omega\left(\frac{\alpha-\beta_1-\beta_2-\beta_3}{\beta_3}\right)}{\beta_1 \beta_2 \beta_3^2} d\beta_3 d\beta_2 d\beta_1 d\alpha < 0.00011,$$

$$G_5 = 4 \int_{\frac{7}{6}}^{\frac{5}{4}} \alpha d\alpha = \frac{29}{72},$$

$$G_6 = \int_{\frac{7}{6}}^{\frac{5}{4}} \int_{\alpha-1}^{\sigma(\alpha)} \alpha \frac{\omega\left(\frac{\alpha-\beta}{\beta}\right)}{\beta^2} d\beta d\alpha > 0.035631,$$

$$G_7 = 4 \int_{\frac{5}{4}}^{1.312} \alpha d\alpha < 0.31769,$$

where

$$\sigma(\alpha) := \frac{2-\alpha}{3}, \qquad \xi(\alpha) = \frac{3}{2} - \alpha, \tag{6}$$

f_4 denotes the characteristic function of the set

$$\{\beta_1+\beta_2, \beta_1+\beta_3, \beta_2+\beta_3, \beta_1+\beta_2+\beta_3 \notin [\alpha-1, \sigma(\alpha)]\},$$

and $\omega(u)$ denotes the Buchstab function determined by the following differential-difference equation

$$\begin{cases} \omega(u) = \frac{1}{u}, & 1 \leqslant u \leqslant 2, \\ (u\omega(u))' = \omega(u-1), & u \geqslant 2. \end{cases}$$

However, their bounds for those integrals are not very accurate. Using Mathematica 14, we can get the following better bounds. We remark that for G_6 the new lower bound gives a 67% improvement over the bound mentioned in [11].

Lemma 2.2. *For G_i $(0 \leqslant i \leqslant 6)$ defined in Lemma 2.1, we have*

$$G_0 = \frac{1}{6}, \quad G_1 < 0.028611(0.0287), \quad G_2 < 0.086062(0.08622), \quad G_3 < 0.030992(0.03107),$$

$$G_4 < 0.0001(00.00011), \quad G_5 = \frac{29}{72}, \quad G_6 > 0.059841(0.035631).$$

Moreover, with these new bounds for G_i $(0 \leqslant i \leqslant 6)$ we have

$$G_0 + G_1 + G_2 + G_3 + G_4 + G_5 - G_6 + 4\int_{\frac{5}{4}}^{1.317} \alpha d\alpha < 0.9993.$$

By Lemma 2.2 and the same arguments as in [5], we complete the proof of Theorem 1.1.

3. Proof of Theorem 1.9

Let $V(u)$ denote an infinitely differentiable non-negative function such that

$$V(u) \begin{cases} < 2, & 1 < u < 2, \\ = 0, & u \leqslant 1 \text{ or } u \geqslant 2, \end{cases}$$

with

$$\frac{d^r V(u)}{du^r} \ll 1 \quad \text{and} \quad \int_{\mathbb{R}} V(u) du = 1.$$

By the discussion in [3] and [6], we wish to get an upper bound for sum of $\sum_{p|k^2+1} V(k/x)$ of the form

$$\sum_{1 \leqslant px^{-\alpha} \leqslant e} \sum_{p|k^2+1} V\left(\frac{k}{x}\right) \leqslant K(\alpha)(1+o(1))\frac{X}{\log x}$$

where $K(\alpha)$ is the sum of sieve theoretical functions related to the sieve decomposition on the problem of the largest prime factor of n^2+1. This requires us to prove that for some τ, we have

$$\int_1^{\tau} \alpha K(\alpha) d\alpha < 1.$$

By Lemma 2.2 we can take $\tau = 1.317$, and $K(\alpha)$ is defined as the piecewise function in Section 2. Combining this with the bound proved in [3], we have

$$\begin{aligned} \rho(x) &\leqslant (1+o(1))x \int_1^{1.317} K(\alpha) d\alpha \\ &\leqslant \left(G_0' + G_1' + G_2' + G_3' + G_4' + G_5' - G_6' + G_7'\right) x, \end{aligned} \tag{7}$$

where

$$G_0' = \int_1^{\frac{7}{6}} \frac{1}{\alpha} d\alpha = \log \frac{7}{6} < 0.154151,$$

$$G_1' = \int_1^{\frac{17}{16}} \int_{\sigma(\alpha)}^{\alpha-2\sigma(\alpha)} \frac{\omega\left(\frac{\alpha-\beta}{\beta}\right)}{\beta^2} d\beta d\alpha + \int_1^{\frac{17}{16}} \int_{\xi(\alpha)}^{\frac{\alpha}{2}} \frac{\omega\left(\frac{\alpha-\beta}{\beta}\right)}{\beta^2} d\beta d\alpha < 0.027475,$$

$$G_2' = \int_{\frac{17}{16}}^{\frac{8}{7}} \int_{\sigma(\alpha)}^{\frac{\alpha}{2}} \frac{\omega\left(\frac{\alpha-\beta}{\beta}\right)}{\beta^2} d\beta d\alpha < 0.077933,$$

$$G_3' = \int_{\frac{8}{7}}^{\frac{7}{6}} \int_{\sigma(\alpha)}^{\frac{\alpha}{2}} \frac{\omega\left(\frac{\alpha-\beta}{\beta}\right)}{\beta^2} d\beta d\alpha < 0.026835,$$

$$G_4' = \int_{\frac{8}{7}}^{\frac{7}{6}} \int_{\sigma(\alpha)-\alpha+1}^{\alpha-1} \int_{\sigma(\alpha)-\alpha+1}^{\beta_1} \int_{\sigma(\alpha)-\alpha+1}^{\beta_2} f_4\left(\alpha, \beta_1, \beta_2, \beta_3\right) \frac{\omega\left(\frac{\alpha-\beta_1-\beta_2-\beta_3}{\beta_3}\right)}{\beta_1\beta_2\beta_3^2} d\beta_3 d\beta_2 d\beta_1 d\alpha < 0.00009,$$

$$G_5' = 4\int_{\frac{7}{6}}^{\frac{5}{4}} 1 d\alpha = \frac{1}{3},$$

$$G_6' = \int_{\frac{7}{6}}^{\frac{5}{4}} \int_{\alpha-1}^{\sigma(\alpha)} \frac{\omega\left(\frac{\alpha-\beta}{\beta}\right)}{\beta^2} d\beta d\alpha > 0.05016,$$

$$G_7' = 4\int_{\frac{5}{4}}^{1.317} 1 d\alpha = 0.268.$$

By a simple calculation, the value of the right hand side of (7) is less than $0.838x$, and the proof of Theorem 1.9 is completed.

References

[1] R. de la Bretèche and S. Drappeau. Niveau de répartition des polynômes quadratiques et crible majorant pour les entiers friables. *J. Eur. Math. Soc.*, 22:1577–1624, 2020.

[2] J.-M. Deshouillers and H. Iwaniec. On the greatest prime factor of n^2+1. *Ann. Inst. Fourier (Grenoble)*, 32(4):1–11, 1982.

[3] G. R. Everest and G. Harman. On primitive divisors of n^2+b. In *Number Theory and Polynomials*, volume 352 of *London Math. Soc. Lecture Note Ser.*, pages 142–154. Cambridge Univ. Press, 2008.

[4] G. R. Everest, S. Stevens, D. Tamsett, and T. Ward. Primes generated by recurrence sequences. *Amer. Math. Monthly*, 114:417–431, 2007.

[5] L. Grimmelt and J. Merikoski. On the greatest prime factor and uniform equidistribution of quadratic polynomials. *arXiv e-prints*, page arXiv:2505.00493v1, 2025.

[6] G. Harman. Two problems on the greatest prime factor of n^2+1. *Acta Arith.*, 213(3):273–287, 2024.

[7] C. Hooley. On the greatest prime factor of a quadratic polynomial. *Acta Math.*, 117:281–299, 1967.

[8] H. Iwaniec. Almost–primes represented by quadratic polynomials. *Invent. Math.*, 47:171–188, 1978.

[9] H. H. Kim. Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2, with appendix 1 by D. Ramakrishnan and appendix 2 by H. H. Kim and P. Sarnak. *J. Amer. Math. Soc.*, 16:139–183, 2003.

[10] R. Li. On the primitive divisors of quadratic polynomials. *arXiv e-prints*, page arXiv:2406.07575v1, 2024.

[11] J. Merikoski. On the greatest prime factor of n^2+1. *J. Eur. Math. Soc.*, 25:1253–1284, 2023.

[12] M. Mine. An upper bound for the number of smooth values of a polynomial and its applications. *arXiv e-prints*, page arXiv:2410.09558v1, 2024.

[13] A. Pascadi. Large sieve inequalities for exceptional Maass forms and applications. *arXiv e-prints*, page arXiv:2404.04239v1, 2024.

[14] H.-E. Richert. Selberg's sieve with weights. *Mathmatika*, 16:1–22, 1969.

[15] A. Schinzel. On two theorems of Gelfond and some of their applications. *Acta Arith.*, 13:177–236, 1967.

[16] G. Tenenbaum. *Introduction to analytic and probabilistic number theory*, volume 163 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, third edition, 2015.

International Curriculum Center, The High School Affiliated to Renmin University of China, Beijing, China
Email address: runbo.li.carey@gmail.com