PRIMES IN ALMOST ALL SHORT INTERVALS III

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Abstract. A conjecture states that if the interval $[n-n^{\theta_0+\varepsilon}, n]$ contains primes for all n, then the interval $[n-n^{\theta_1+\varepsilon}, n]$ contains primes for almost all n, where $\theta_1 = 2\theta_0 - 1$. In the present paper, the author proves that $\theta_1 = \frac{1}{24}$, improving the previous exponents $\frac{1}{21.5}$ and $\frac{1}{22}$ by the author himself. If the conjecture is true, then we can get $\theta_1 = \frac{1}{25}$.

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1. Introduction

One of the famous topics in number theory is to find prime numbers in short intervals. In 1937, Cramér [6] conjectured that every interval $[n, n+f(n)(\log n)^2]$ contains prime numbers for some $f(n) \to 1$ as $n \to \infty$. The Riemann Hypothesis implies that for all large n, the interval $[n-n^\theta, n]$ contains $\sim n^\theta (\log n)^{-1}$ prime numbers for every $\frac{1}{2} < \theta \leqslant 1$. The first unconditional result of this asymptotic formula was proved by Hoheisel [17] in 1930 with $\theta = 1 - \frac{1}{33000}$. After the works of Hoheisel [17], Heilbronn [16], Chudakov [5], Ingham [19] and Montgomery [37], Huxley [18] proved in 1972 that the above asymptotic formula holds when $\theta > \frac{7}{12}$ by his zero density estimate. In 2024, Guth and Maynard [8] improved this to $\theta > \frac{17}{30}$ by a new zero density estimate.

In 1979, Iwaniec and Jutila [20] first introduced a sieve method into this problem. They established a lower bound with correct order of magnitude (instead of an asymptotic formula) with $\theta = \frac{13}{23}$. After that breakthrough, many improvements were made and the value of θ was reduced successively to

$$\frac{5}{9} \approx 0.5556,\ \frac{11}{20} = 0.5500,\ \frac{17}{31} \approx 0.5484,\ \frac{23}{42} \approx 0.5476,$$

$$\frac{1051}{1920} \approx 0.5474,\ \frac{35}{64} \approx 0.5469,\ \frac{6}{11} \approx 0.5455 \ \text{ and } \ \frac{7}{13} \approx 0.5385$$

by Iwaniec and Jutila [20], Heath-Brown and Iwaniec [15], Pintz [39] [40], Iwaniec and Pintz [21], Mozzochi [38] and Lou and Yao [32] [34] [35] [36] respectively. In 1996, Baker and Harman [1] presented an alternative approach to this problem. They used the alternative sieve developed by Harman [10] [11] to reduce θ to 0.535. Baker, Harman and Pintz [3] further developed this sieve process and combined it with Watt's theorem and showed $\theta = 0.525$. The best result now is due to the author [31], where he proved $\theta = 0.52$ is acceptable.

However, if we only consider the prime numbers in "almost all" intervals instead of "all" intervals, the intervals will be much shorter than $n^{0.52}$. In 1943, under the Riemann Hypothesis, Selberg [42] showed that Cramér's interval contains primes for almost all n if $f(n) \to \infty$ as $n \to \infty$. In the same paper, he also showed unconditionally that the interval $[n-n^{\frac{19}{77}+\varepsilon}, n]$ contains prime numbers for almost all n. In 1971, Montgomery [37] improved the exponent $\frac{19}{77}$ to $\frac{1}{5}$ with an asymptotic formula. The zero density estimate of Huxley [18] gives the exponent $\frac{1}{6}$ with an asymptotic formula, and the best asymptotic result now is also due to Guth and Maynard [8], where they proved the exponent $\frac{1}{7.5}$.

In 1982, Harman [9] used his alternative sieve method to showed that the interval $[n-n^{\frac{1}{10}+\varepsilon}, n]$ contains prime numbers for almost all n. His method can only provide a lower bound instead of an asymptotic formula. The exponent $\frac{1}{10}$ was reduced

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successively to

$$\frac{1}{12} \approx 0.0833,\ \frac{14}{159} \approx 0.0881,\ \frac{1}{13} \approx 0.0769,\ \frac{17}{227} \approx 0.0749,\ \frac{1}{13.5} \approx 0.0740,\ \frac{1}{14} \approx 0.0714,$$
$$\frac{1}{15} \approx 0.0667,\ \frac{1}{16} = 0.0625,\ \frac{1}{18} \approx 0.0556,\ \frac{1}{20} = 0.0500,\ \frac{1}{21.5} \approx 0.0465 \text{ and } \frac{1}{22} \approx 0.0455$$

by Harman [10] (and Heath-Brown [14]), Lou and Yao [33], Jia [23] [24], Lou and Yao [33], Li [27], Jia [22] (and Watt [44]), Li [28], Baker, Harman and Pintz [2], Wong [45] (and Jia [26], Harman [[13], Chapter 9]), Jia [25] and the author [29] [30] respectively.

In [30], the author proved an exponent $\frac{1}{22}$ with a lower constant $0.2 = \frac{1}{5}$, which is large enough and usually indicates further refinements in many applications of Harman's sieve. Selberg [42] conjectured that if the interval $[n - n^{\theta_0+\varepsilon}, n]$ contains primes for all n, then the interval $[n - n^{\theta_1+\varepsilon}, n]$ contains primes for almost all n, where $\theta_1 = 2\theta_0 - 1$. If his conjecture is true, then by the main result proved in [31] we know that $\theta_1 = \frac{1}{25}$ is achievable. Following this, we try to reduce the value of θ_1 to $\frac{1}{25}$ unconditionally but fail. Fortunately, we still obtain $\theta_1 = \frac{1}{24}$:

Theorem 1.1. *The interval $[n - n^{\frac{1}{24}+\varepsilon}, n]$ contains prime numbers for almost all n. Specifically, suppose that B is a sufficiently large positive constant, ε is a sufficiently small positive constant and X is sufficiently large. Then for positive integers $n \in [X, 2X]$, except for $O\left(X(\log X)^{-B}\right)$ values, the interval $[n - n^{\frac{1}{24}+\varepsilon}, n]$ contains $\gg n^{1/24+\varepsilon}(\log n)^{-1}$ prime numbers.*

Throughout this paper, we always suppose that B is a sufficiently large positive constant, ε is a sufficiently small positive constant, X is a sufficiently large integer, $x \in [X, 2X]$ and $K_0 > 0$. The appearance of K_0 in the exponent of a logarithm will always signify that the result holds for every $K_0 > 0$ with an implied constant that depends on K_0. The letter p, with or without subscript, is reserved for prime numbers. Let c_0, c_1 and c_2 denote positive constants which may have different values at different places, and we write $m \sim M$ to mean that $c_1 M < m \leqslant c_2 M$. Let $b = 1 + \frac{1}{\log X}$, $\delta = \varepsilon^{1/3}$, $\eta = (2X)^{-\frac{23}{24}+\varepsilon}$, and $\eta_1 = \exp\left(-3(\log X)^{1/3}\right)$. We use $M(s)$, $N(s)$, $H(s)$ and some other capital letters to denote divisor-bounded Dirichlet polynomials

$$M(s) = \sum_{m \sim M} a_m m^{-s}, \quad N(s) = \sum_{n \sim N} b_n n^{-s}, \quad H(s) = \sum_{h \sim H} c_h h^{-s},$$

and we use $L(s)$ to denote a "zeta-factor"

$$L(s) = \sum_{l \sim L} l^{-s}.$$

We say a Dirichlet polynomial $M(s)$ is *prime-factored* if we have

$$\left| M\left(\frac{1}{2} + it\right) \right| \ll M^{\frac{1}{2}} (\log x)^{-K_0}$$

for $\exp\left((\log x)^{1/3}\right) < |t| < x^{\frac{23}{24}}$. In fact, this holds when a_m is the characteristic function of primes or of numbers with a bounded number of prime factors restricted to certain ranges.

2. An outline of the proof

Let $\mathcal{C}$ denote a finite set of positive integers, $p_j = X^{t_j}$ in the following sections and put

$$\mathcal{A} = \{a : a \in \mathbb{Z},\ x - \eta x \leqslant a < x\}, \quad \mathcal{B}^1 = \{b : b \in \mathbb{Z},\ x < b \leqslant 2x\}, \quad \mathcal{B}^2 = \{b : b \in \mathbb{Z},\ x - \eta_1 x \leqslant b < x\},$$
$$\mathcal{C}_d = \{a : ad \in \mathcal{C}\}, \quad P(z) = \prod_{p<z} p, \quad S(\mathcal{C}, z) = \sum_{\substack{a \in \mathcal{C} \\ (a, P(z))=1}} 1.$$

Then we have

$$\pi(x) - \pi(x - \eta x) = S\left(\mathcal{A}, (2X)^{\frac{1}{2}}\right). \tag{1}$$

Buchstab's identity is the equation

$$S\left(\mathcal{C}, z\right) = S\left(\mathcal{C}, w\right) - \sum_{w \leqslant p < z} S\left(\mathcal{C}_p, p\right),$$

where $2 \leqslant w < z$.

In order to prove Theorem 1.1, we only need to show that $S\left(\mathcal{A}, (2X)^{\frac{1}{2}}\right) > 0$. By Buchstab's identity, we have

$$\begin{aligned} S\left(\mathcal{A}, (2X)^{\frac{1}{2}}\right) &= S\left(\mathcal{A}, X^{\frac{11}{126}}\right) - \sum_{\frac{11}{126} \leqslant t_1 < \frac{1}{2}} S\left(\mathcal{A}_{p_1}, X^{\frac{11}{126}}\right) \\ &\quad + \sum_{\substack{\frac{11}{126} \leqslant t_1 < \frac{1}{2} \\ \frac{11}{126} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\ &= S_1 - S_2 + S_3. \end{aligned} \tag{2}$$

Our aim is to show that the sparser set $\mathcal{A}$ contains the expected proportion of primes compared to the larger sets $\mathcal{B}^i$, which requires us to decompose $S\left(\mathcal{A},(2X)^{\frac{1}{2}}\right)$ using the above Buchstab's identity, prove asymptotic formulas (for almost $x \in [X,2X]$ except for $O\left(X(\log X)^{-B}\right)$ values) of the forms

$$S(\mathcal{A},z)=\eta(1+o(1))S\left(\mathcal{B}^1,z\right) \quad \text{or} \quad S(\mathcal{A},z)=\frac{\eta}{\eta_1}(1+o(1))S\left(\mathcal{B}^2,z\right) \tag{3}$$

for some parts of it, and drop the remaining positive parts. We say a term $S(\mathcal{A},z)$ *has an asymptotic formula* if any of the two forms of (3) holds for this term (or all of its decomposed parts).

In Sections 3 and 4 we provide some arithmetic information by proving mean value bounds for Dirichlet polynomials. In Section 5 we provide additional arithmetic information used in Harman's monograph [[13], Chapter 9]. We shall use them to prove asymptotic formulas for terms of the form $S\left(\mathcal{A}_{p_1\ldots p_n},X^{\delta}\right)$ and $S\left(\mathcal{A}_{p_1\ldots p_n},p_n\right)$ in Section 6. In Section 7 we make further use of Buchstab's identity to decompose $S\left(\mathcal{A},(2X)^{\frac{1}{2}}\right)$ and prove Theorem 1.1.

3. Arithmetic Information I

In the following three sections we provide some arithmetic information which will help us prove the asymptotic formulas for sieve functions. In this section we only use the classical mean value estimate and Halász method.

Lemma 3.1. *Suppose that $MH=X$, $b_h=\Lambda(h)$ and $X^{\delta} \ll H \ll X^{\frac{11}{126}}$. Then for $(\log X)^{B/\varepsilon} \leqslant T \leqslant X$, we have*

$$\left(\min\left(\eta,\frac{1}{T}\right)\right)^2 \int_T^{2T} |M(b+it)H(b+it)|^2 dt \ll \eta^2(\log x)^{-10B}.$$

Proof. The proof is similar to that of [[30], Lemma 3.1]. □

Lemma 3.2. *Suppose that $MHK=X$, and M and H satisfy one of the following 7 conditions:*

(1) $MH \ll X^{\frac{187}{348}},\ X^{\frac{23}{132}} \ll H,\ M^{29}H^{-1} \ll X^{\frac{59}{6}},\ X^{\frac{23}{76}} \ll M,\ M^{-1}H^{29} \ll X^{6},\ X^{\frac{23}{4}} \ll M^{12}H^{11}$;

(2) $MH \ll X^{\frac{149}{264}},\ X^{\frac{23}{120}} \ll H,\ M^{6}H \ll X^{\frac{13}{6}},\ X^{\frac{23}{84}} \ll M \ll X^{\frac{1}{3}},\ MH^{8} \ll X^{\frac{9}{4}},\ X^{\frac{23}{8}} \ll M^{6}H^{5}$;

(3) $MH \ll X^{\frac{53}{76}},\ X^{\frac{23}{132}} \ll H,\ M^{12}H \ll X^{\frac{25}{4}},\ X^{\frac{161}{348}} \ll M,\ M^{-1}H^{19} \ll X^{\frac{11}{3}},\ X^{\frac{23}{3}} \ll M^{12}H^{11}$;

(4) $X^{\frac{2}{3}} \ll MH \ll X^{\frac{61}{84}},\ X^{\frac{23}{120}} \ll H,\ M^{6}H \ll X^{\frac{25}{8}},\ X^{\frac{115}{264}} \ll M,\ M^{-1}H^{7} \ll X^{\frac{5}{4}},\ X^{\frac{23}{6}} \ll M^{6}H^{5}$;

(5) $X^{\frac{1}{2}} \ll MH \ll X^{\frac{149}{264}},\ M^{35}H^{23} \ll X^{\frac{71}{4}},\ X^{\frac{115}{132}} \ll M^{2}H,\ M^{2}H^{13} \ll X^{\frac{35}{12}},\ X^{\frac{69}{2}} \ll M^{70}H^{59}$;

(6) $MH \ll X^{\frac{29}{52}},\ M^{41}H^{27} \ll X^{\frac{167}{8}},\ X^{\frac{23}{26}} \ll M^{2}H,\ M^{2}H^{15} \ll X^{\frac{37}{12}},\ X^{\frac{161}{4}} \ll M^{82}H^{69}$;

(7) $M^{2}H \ll X^{\frac{149}{132}},\ M^{70}H^{11} \ll X^{\frac{71}{2}},\ X^{\frac{115}{264}} \ll M \ll X^{\frac{1}{2}},\ MH^{6} \ll X^{\frac{35}{24}},\ X^{\frac{69}{4}} \ll M^{35}H^{12}$.

Assume that for $(\log X)^{B/\varepsilon} \leqslant |t| \leqslant 2X, M(b+it) \ll (\log x)^{-B/\varepsilon}$ and $H(b+it) \ll (\log x)^{-B/\varepsilon}$. Then for $(\log X)^{B/\varepsilon} \leqslant T \leqslant X$, we have

$$\left(\min\left(\eta,\frac{1}{T}\right)\right)^2 \int_T^{2T} |M(b+it)H(b+it)K(b+it)|^2 dt \ll \eta^2(\log x)^{-10B}.$$

Proof. The proof is similar to that of [[30], Lemma 3.2, Cases 1,3,4,6,7,8,9]. □

Lemma 3.3. *Suppose that $NQRK=X$, and N, Q and R satisfy one of the following 6 conditions:*

(1) $X^{\frac{139}{336}} \ll NQ \ll X^{\frac{605}{1456}},\ (NQ)^{-\frac{82}{69}}X^{\frac{7}{12}} \ll R \ll Q \ll (NQ)^{-1}X^{\frac{29}{52}}$;

(2) $X^{\frac{605}{1456}} \ll NQ \ll X^{\frac{2875}{6888}},\ (NQ)^{-\frac{82}{69}}X^{\frac{7}{12}} \ll R \ll Q \ll (NQ)^{-\frac{41}{27}}X^{\frac{167}{216}}$;

(3) $X^{\frac{2875}{6888}} \ll NQ \ll X^{\frac{17}{40}},\ X^{\frac{11}{126}} \ll R \ll Q \ll (NQ)^{-\frac{41}{27}}X^{\frac{167}{216}}$;

(4) $X^{\frac{461}{1008}} \ll NQ \ll X^{\frac{11}{24}},\ X^{\frac{1}{6}} \ll R \ll Q \ll (NQ)^{-\frac{1}{6}}X^{\frac{35}{144}}$;

(5) $X^{\frac{265}{552}} \ll NQ \ll X^{\frac{4727}{9816}},\ X^{\frac{11}{126}} \ll R \ll Q \ll (NQ)^{-\frac{1}{6}}X^{\frac{35}{144}}$;

(6) $X^{\frac{4727}{9816}} \ll NQ \ll X^{\frac{151}{312}},\ X^{\frac{11}{126}} \ll R \ll Q \ll (NQ)^{-\frac{70}{11}}X^{\frac{71}{22}}$.

Assume further that for $(\log X)^{B/\varepsilon} \leqslant |t| \leqslant 2X, N(b+it)Q(b+it) \ll (\log x)^{-B/\varepsilon}$ and $R(b+it) \ll (\log x)^{-B/\varepsilon}$. Then for $(\log X)^{B/\varepsilon} \leqslant T \leqslant X$, we have

$$\left(\min\left(\eta,\frac{1}{T}\right)\right)^2 \int_T^{2T} |N(b+it)Q(b+it)R(b+it)K(b+it)|^2 dt \ll \eta^2(\log x)^{-10B}.$$

Proof. Let $M(s)=N(s)Q(s)$ and $H(s)=R(s)$. Then by Lemma 3.2, Lemma 3.3 is proved. □

Lemma 3.4. *Suppose that $NQRK=X$, and N, Q and R satisfy $X^{\frac{11}{126}} \ll R \ll Q$ and one of the following 3 conditions:*

(1) $N(QR) \ll X^{\frac{187}{348}},\ X^{\frac{23}{132}} \ll (QR),\ N^{29}(QR)^{-1} \ll X^{\frac{59}{6}},\ X^{\frac{23}{76}} \ll N,\ N^{-1}(QR)^{29} \ll X^{6},\ X^{\frac{23}{4}} \ll N^{12}(QR)^{11}$;

(2) $N(QR) \ll X^{\frac{149}{264}},\ X^{\frac{23}{120}} \ll (QR),\ N^{6}(QR) \ll X^{\frac{13}{6}},\ X^{\frac{23}{84}} \ll N \ll X^{\frac{1}{3}},\ N(QR)^{8} \ll X^{\frac{9}{4}},\ X^{\frac{23}{8}} \ll N^{6}(QR)^{5}$;

(3) $N(QR) \ll X^{\frac{53}{76}},\ X^{\frac{23}{132}} \ll (QR),\ N^{12}(QR) \ll X^{\frac{25}{4}},\ X^{\frac{161}{348}} \ll N,\ N^{-1}(QR)^{19} \ll X^{\frac{11}{3}},\ X^{\frac{23}{3}} \ll N^{12}(QR)^{11}$.

Assume further that for $(\log X)^{B/\varepsilon} \leqslant |t| \leqslant 2X, Q(b+it)R(b+it) \ll (\log x)^{-B/\varepsilon}$ and $N(b+it) \ll (\log x)^{-B/\varepsilon}$. Then for $(\log X)^{B/\varepsilon} \leqslant T \leqslant X$, we have

$$\left(\min\left(\eta,\frac{1}{T}\right)\right)^2 \int_T^{2T} |N(b+it)Q(b+it)R(b+it)K(b+it)|^2 dt \ll \eta^2(\log x)^{-10B}.$$

Proof. Let $M(s) = N(s)$ and $H(s) = Q(s)R(s)$ and by Lemma 3.2 (Cases 1,2,3), Lemma 3.4 is proved. □

4. Arithmetic Information II

In this section we are looking for more Type-I information. In [25], Jia used a mean value bound of Deshouillers and Iwaniec [7], which makes an approximation to the sixth-power moment of the Riemann zeta-function. Now we shall use another mean value bound of Watt [43], which is stronger than that of Deshouillers and Iwaniec when the length of interval is $n^{\frac{1}{24}+\varepsilon}$. Note that in [29] and [30] we also use this bound.

Lemma 4.1. *([30], Lemma 4.1]). Let $T \geqslant 1$, then*

$$\int_T^{2T} \left| L\left(\frac{1}{2}+it\right)\right|^4 \left| N\left(\frac{1}{2}+it\right)\right|^2 dt \ll \left(T + N^2 T^{\frac{1}{2}} + NL^2 T^{-2}\right) T^{\varepsilon^2}.$$

Lemma 4.2. *Suppose that $MHL = X$, and M and H satisfy one of the following 2 conditions:*
(1) $M^2 H \ll X^{\frac{25}{24}}$, $M^4 H^6 \ll X^{\frac{73}{24}}$, $H^4 \ll X^{\frac{9}{8}}$;
(2) $M \ll X^{\frac{25}{48}}$, $H \ll X^{\frac{23}{192}}$.
Then for $\sqrt{L} \leqslant T \leqslant X$, we have

$$\left(\min\left(\eta, \frac{1}{T}\right)\right)^2 \int_T^{2T} |M(b+it)H(b+it)L(b+it)|^2 dt \ll \eta^2 x^{-2\varepsilon^2}.$$

Proof. We can prove this by using Lemma 4.1 and the methods in [[25], Lemmas 3,4]. For condition (1) we apply Lemma 4.1 to L and N, while for condition (2) we apply Lemma 4.1 to L and N^2. One can see [25] for an explanation. □

Lemma 4.3. *Suppose that $NQRL = X$, and N, Q and R satisfy the following condition:*
$X^{\frac{115}{264}} \ll NQ \ll X^{\frac{11}{24}}$, $X^{\frac{11}{126}} \ll R \ll Q \ll (NQ)^{-\frac{1}{6}} X^{\frac{35}{144}}$.
Assume further that for $\sqrt{L} \leqslant |t| \leqslant 2X, N(b+it)Q(b+it) \ll (\log x)^{-B/\varepsilon}$ and $R(b+it) \ll (\log x)^{-B/\varepsilon}$. Then for $\sqrt{L} \leqslant T \leqslant X$, we have

$$\left(\min\left(\eta, \frac{1}{T}\right)\right)^2 \int_T^{2T} |N(b+it)Q(b+it)R(b+it)L(b+it)|^2 dt \ll \eta^2 (\log x)^{-10B}.$$

Proof. Let $M(s) = N(s)Q(s)$ and $H(s) = R(s)$. If $R \gg \max\left((NQ)^{-2} X^{\frac{25}{24}}, X^{\frac{23}{192}}\right)$, we apply Lemma 3.2 (Case 7). Otherwise we apply Lemma 4.2. Then Lemma 4.3 is proved. □

5. Arithmetic Information III

In this section we provide the "extra" arithmetic information proved by Harman [[13], Chapter 9]. Some parts of it bring new asymptotic regions. In the next section we shall show that it is useful when combining with the variable role-reversal.

Lemma 5.1. *([13], Lemma 9.3]). Let $T_0 = \exp\left((\log x)^{1/3}\right)$ and $T = X^{\varepsilon}/\eta$. Suppose that*

$$\int_{T_0}^{T} \left| M\left(\frac{1}{2}+it\right)\right|^2 dt \ll X(\log x)^{-K_0},$$

then

$$\sum_{k \in \mathcal{A}} a_m = \frac{\eta}{\delta}(1+o(1)) \sum_{k \in \mathcal{B}^2} a_m$$

for all $x \in [X, 2X]$ except for $O\left(X(\log X)^{-K_0}\right)$ values.

Lemma 5.2. *([13], Lemma 9.4]). Let $T_0 = \exp\left((\log x)^{1/3}\right)$ and $T = X^{\varepsilon}/\eta$. Suppose that $MHK = X$, $M = X^{\alpha_1}$, $H = X^{\alpha_2}$ and $0 < \alpha_2 \leqslant \alpha_1 < \frac{1}{2}$. Suppose that M, H and K satisfy one of the following 3 conditions:*
(1) $\alpha_2 > \frac{1}{12}$, $2\alpha_2 - \frac{1}{12} < \alpha_1 \leqslant \frac{1}{6}$, $M(s)$ prime-factored;
(2) $|4\alpha_1 + 3\alpha_2 - 2| < \frac{1}{12}$, $\frac{1}{12} \leqslant \alpha_2 < \frac{1}{6}$, $H(s)$ prime-factored;
(3) $|4\alpha_1 + \alpha_2 - 2| < \frac{1}{12}$, $\frac{1}{12} \leqslant \alpha_2 < \frac{1}{6}$, $H(s)$ prime-factored.
Then we have

$$\int_{T_0}^{T} \left| M\left(\frac{1}{2}+it\right) |H\left(\frac{1}{2}+it\right)| K\left(\frac{1}{2}+it\right)\right|^2 dt \ll X(\log x)^{-K_0}.$$

Lemma 5.3. *Let $T_0 = \exp\left((\log x)^{1/3}\right)$ and $T = X^{\varepsilon}/\eta$. Suppose that $NQRK = X$, $R(s)$ is prime-factored, and N, Q and R satisfy one of the following 5 conditions:*
(1) $X^{\frac{139}{336}} \ll NQ \ll X^{\frac{2875}{6888}}$, $X^{\frac{11}{126}} \ll R \ll Q \ll (NQ)^{-\frac{82}{69}} X^{\frac{7}{12}}$;
(2) $X^{\frac{17}{40}} \ll NQ \ll X^{\frac{51}{112}}$, $X^{\frac{11}{126}} \ll R \ll Q \ll (NQ)^{-\frac{4}{3}} X^{\frac{25}{36}}$;
(3) $X^{\frac{461}{1008}} \ll NQ \ll X^{\frac{23}{48}}$, $X^{\frac{11}{126}} \ll R \ll Q \ll X^{\frac{1}{6}}$;
(4) $X^{\frac{23}{48}} \ll NQ \ll X^{\frac{265}{552}}$, $X^{\frac{11}{126}} \ll R \ll Q \ll (NQ)^{-4} X^{\frac{25}{12}}$;
(5) $X^{\frac{151}{312}} \ll NQ \ll X^{\frac{503}{1008}}$, $X^{\frac{11}{126}} \ll R \ll Q \ll (NQ)^{-4} X^{\frac{25}{12}}$.

Then we have

$$\int_{T_0}^{T}\left|N\left(\frac{1}{2}+it\right)Q\left(\frac{1}{2}+it\right)R\left(\frac{1}{2}+it\right)K\left(\frac{1}{2}+it\right)\right|^2 dt \ll X(\log x)^{-K_0}.$$

Proof. Let $M(s)=N(s)Q(s)$ and $H(s)=R(s)$. Then by Lemma 5.2, Lemma 5.3 is proved. □

6. Sieve asymptotic formulas

In this section we transform the arithmetic given in Sections 3–5 into asymptotic formulas of the form (3). The first two lemmas deal with the case $z=X^{\delta}$.

Lemma 6.1. *Suppose that $MHL=X$, and M and H satisfy one of the 2 conditions in Lemma 4.2. Then for real numbers $x\in[X,2X]$, except for $O\left(X(\log X)^{-B}\right)$ values, we have*

$$\sum_{\substack{m\sim M\\ h\sim H}} a_m b_h \left(\sum_{x-\eta x\leqslant mhl<x} 1-\frac{\eta x}{mh}\right) \ll \eta x(\log x)^{-B}.$$

Proof. The proof is similar to that of [[30], Lemma 6.1] where Lemma 4.2 is used. □

Lemma 6.2. *Suppose that $NQRL=X$, and N, Q and R satisfy the following condition:*
$X^{\frac{115}{264}} \ll NQ \ll X^{\frac{11}{24}},\ X^{\frac{11}{126}} \ll R \ll Q \ll (NQ)^{-\frac{1}{6}}X^{\frac{35}{144}}$.
Assume further that for $\sqrt{L}\leqslant|t|\leqslant 2X$, $N(b+it)Q(b+it)\ll(\log x)^{-B/\varepsilon}$ and $R(b+it)\ll(\log x)^{-B/\varepsilon}$. Then for real numbers $x\in[X,2X]$, except for $O\left(X(\log X)^{-B}\right)$ values, we have Then for real numbers $x\in[X,2X]$, except for $O\left(X(\log X)^{-B}\right)$ values, we have

$$\sum_{\substack{n\sim N\\ q\sim Q\\ r\sim R}} a_n b_q c_r \left(\sum_{x-\eta x\leqslant nqrl<x} 1-\frac{\eta x}{nqr}\right) \ll \eta x(\log x)^{-B}.$$

Proof. The proof is similar to that of Lemma 6.1 where Lemma 4.3 is used. □

The next three lemmas focus on the case $z=p$, or the general case.

Lemma 6.3. *Suppose that $X^{\frac{115}{126}} \ll M \ll X^{1-\delta}$, $a_m\geqslant 0$, and $a_m=0$ if m has a prime factor $<X^{\delta}$. Then for real numbers $x\in[X,2X]$, except for $O\left(X(\log X)^{-B}\right)$ values, we have*

$$\sum_{\substack{x-\eta x\leqslant mp<x\\ m\sim M}} a_m = \eta\left(1+O\left(\frac{1}{\log x}\right)\right)\sum_{\substack{x<mp\leqslant 2x\\ m\sim M}} a_m + O\left(\eta x(\log x)^{-B}\right).$$

Proof. The proof is similar to that of [[30], Lemma 6.3] where Lemma 3.1 is used. □

Lemma 6.4. *Suppose that $MHK=X$, $c_k=\Lambda(k)$, $X^{\frac{11}{126}} \ll MH \ll X^{\frac{115}{126}}$, and $a_m, b_h\geqslant 0$. Suppose that $a_m=0$ if m has a prime factor $<X^{\delta}$, and $b_h=0$ if h has a prime factor $<X^{\delta}$. If we have*

$$\left(\min\left(\eta,\frac{1}{T}\right)\right)^2\int_{T}^{2T}|M(b+it)H(b+it)K(b+it)|^2 dt \ll \eta^2(\log x)^{-10B}$$

for $(\log X)^{B/\varepsilon}\leqslant T\leqslant X$, then for real numbers $x\in[X,2X]$, except for $O\left(X(\log X)^{-B}\right)$ values, we have

$$\sum_{\substack{x-\eta x\leqslant mhp<x\\ m\sim M\\ h\sim H}} a_m b_h = \eta\left(1+O\left(\frac{1}{\log x}\right)\right)\sum_{\substack{x<mhp\leqslant 2x\\ m\sim M\\ h\sim H}} a_m b_h + O\left(\eta x(\log x)^{-B}\right).$$

Proof. The proof is similar to that of Lemma 6.3. □

Lemma 6.5. *Suppose that $MNHK=X$, $d_k=\Lambda(k)$, $X^{\frac{11}{126}} \ll MNH \ll X^{\frac{115}{126}}$, and $a_m, b_n, c_h\geqslant 0$. Suppose that $a_m=0$ if m has a prime factor $<X^{\delta}$, $b_n=0$ if n has a prime factor $<X^{\delta}$, and $c_h=0$ if h has a prime factor $<X^{\delta}$. If we have*

$$\left(\min\left(\eta,\frac{1}{T}\right)\right)^2\int_{T}^{2T}|M(b+it)N(b+it)H(b+it)K(b+it)|^2 dt \ll \eta^2(\log x)^{-10B}$$

for $(\log X)^{B/\varepsilon}\leqslant T\leqslant X$, then for real numbers $x\in[X,2X]$, except for $O\left(X(\log X)^{-B}\right)$ values, we have

$$\sum_{\substack{x-\eta x\leqslant mnhp<x\\ m\sim M\\ n\sim N\\ h\sim H}} a_m b_n c_h = \eta\left(1+O\left(\frac{1}{\log x}\right)\right)\sum_{\substack{x<mhp\leqslant 2x\\ m\sim M\\ n\sim N\\ h\sim H}} a_m b_n c_h + O\left(\eta x(\log x)^{-B}\right).$$

Proof. The proof is similar to that of Lemma 6.3. □

The next two lemmas give asymptotic formulas for specific sieve functions.

Lemma 6.6. *Suppose that $P_0P_1P_2\cdots P_n = X$, $P_n < \cdots < P_2 < P_1 < X^{\frac{1}{2}}$ and $p_i \sim P_i$ for $i \geqslant 1$. Suppose that $P_1, P_2, \ldots, P_n, X^{10^{-1000}}$ can be partitioned into M, H satisfy Lemma 6.1 or N, Q, R satisfy Lemma 6.2, Then for real numbers $x \in [X, 2X]$, except for $O\left(X(\log X)^{-B}\right)$ values, we have*

$$\sum_{(t_1,\ldots,t_n)} S\left(\mathcal{A}_{p_1p_2\cdots p_n}, X^{\frac{11}{126}}\right) = \eta(1+o(1)) \sum_{(t_1,\ldots,t_n)} S\left(\mathcal{B}^1_{p_1p_2\cdots p_n}, X^{\frac{11}{126}}\right).$$

Proof. The proof is similar to that of [[30], Lemma 6.6] where Lemmas 6.1, 6.2 and 6.3 are used. □

Lemma 6.7. *Suppose that $P_0P_1P_2P_3 = X$, $P_3 < P_2 < P_1 < X^{\frac{1}{2}}$ and $p_i \sim P_i$ for $i \geqslant 1$. Suppose that P_1, P_2, P_3 satisfy one of the conditions:*

(1) $X^{\frac{139}{336}} \ll NQ \ll X^{\frac{605}{1456}}$, $X^{\frac{11}{126}} \ll R \ll Q \ll (NQ)^{-1}X^{\frac{29}{52}}$;
(2) $X^{\frac{605}{1456}} \ll NQ \ll X^{\frac{17}{40}}$, $X^{\frac{11}{126}} \ll R \ll Q \ll (NQ)^{-\frac{41}{27}}X^{\frac{167}{216}}$;
(3) $X^{\frac{17}{40}} \ll NQ \ll X^{\frac{51}{112}}$, $X^{\frac{11}{126}} \ll R \ll Q \ll (NQ)^{-\frac{4}{3}}X^{\frac{25}{36}}$;
(4) $X^{\frac{461}{1008}} \ll NQ \ll X^{\frac{11}{24}}$, $X^{\frac{11}{126}} \ll R \ll Q \ll (NQ)^{-\frac{1}{6}}X^{\frac{35}{144}}$;
(5) $X^{\frac{11}{24}} \ll NQ \ll X^{\frac{23}{48}}$, $X^{\frac{11}{126}} \ll R \ll Q \ll X^{\frac{1}{6}}$;
(6) $X^{\frac{23}{48}} \ll NQ \ll X^{\frac{265}{552}}$, $X^{\frac{11}{126}} \ll R \ll Q \ll (NQ)^{-4}X^{\frac{25}{12}}$;
(7) $X^{\frac{265}{552}} \ll NQ \ll X^{\frac{4727}{9816}}$, $X^{\frac{11}{126}} \ll R \ll Q \ll (NQ)^{-\frac{1}{6}}X^{\frac{35}{144}}$;
(8) $X^{\frac{4727}{9816}} \ll NQ \ll X^{\frac{151}{312}}$, $X^{\frac{11}{126}} \ll R \ll Q \ll (NQ)^{-\frac{70}{11}}X^{\frac{71}{22}}$;
(9) $X^{\frac{151}{312}} \ll NQ \ll X^{\frac{503}{1008}}$, $X^{\frac{11}{126}} \ll R \ll Q \ll (NQ)^{-4}X^{\frac{25}{12}}$.

Then

$$\sum_{(t_1,t_2,t_3)} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right)$$

has an asymptotic formula of the form (3).

Proof. Let $N(s) = P_1(s)$, $Q(s) = P_2(s)$ and $R(s) = P_3(s)$. Combining Lemmas 3.3 and 5.3 and use Lemmas 6.5 and 5.1, Lemma 6.7 is proved. □

7. The final decomposition

In this section, sets $\mathcal{A}$, $\mathcal{B}^1$ and $\mathcal{B}^2$ are defined respectively. Let $\omega(u)$ denote the Buchstab function determined by the following differential-difference equation

$$\begin{cases} \omega(u) = \frac{1}{u}, & 1 \leqslant u \leqslant 2, \\ (u\omega(u))' = \omega(u-1), & u \geqslant 2. \end{cases}$$

Moreover, we have the upper and lower bounds for $\omega(u)$:

$$\omega(u) \geqslant \omega_0(u) = \begin{cases} \frac{1}{u}, & 1 \leqslant u < 2, \\ \frac{1+\log(u-1)}{u}, & 2 \leqslant u < 3, \\ \frac{1+\log(u-1)}{u} + \frac{1}{u}\int_2^{u-1} \frac{\log(t-1)}{t} dt \geqslant 0.5607, & 3 \leqslant u < 4, \\ 0.5612, & u \geqslant 4, \end{cases}$$

$$\omega(u) \leqslant \omega_1(u) = \begin{cases} \frac{1}{u}, & 1 \leqslant u < 2, \\ \frac{1+\log(u-1)}{u}, & 2 \leqslant u < 3, \\ \frac{1+\log(u-1)}{u} + \frac{1}{u}\int_2^{u-1} \frac{\log(t-1)}{t} dt \leqslant 0.5644, & 3 \leqslant u < 4, \\ 0.5617, & u \geqslant 4. \end{cases}$$

We shall use $\omega_0(u)$ and $\omega_1(u)$ to give numerical bounds for some sieve functions discussed below.

By Prime Number Theorem with Vinogradov's error term and the inductive arguments in [[13], Chapter A.2], we know that, for sufficiently large z,

$$S\left(\mathcal{B}^1, z\right) = \sum_{\substack{a \in \mathcal{B}^1 \\ (a, P(z))=1}} 1 = (1+o(1)) \frac{x}{\log z} \omega\left(\frac{\log x}{\log z}\right), \tag{4}$$

$$S\left(\mathcal{B}^2, z\right) = \sum_{\substack{a \in \mathcal{B}^2 \\ (a, P(z))=1}} 1 = (1+o(1)) \frac{\eta_1 x}{\log z} \omega\left(\frac{\log x}{\log z}\right), \tag{5}$$

and we expect that the similar relation also holds for $S(\mathcal{A}, z)$:

$$S(\mathcal{A}, z) = \sum_{\substack{a \in \mathcal{A} \\ (a, P(z))=1}} 1 = (1+o(1)) \frac{\eta x}{\log z} \omega\left(\frac{\log x}{\log z}\right). \tag{6}$$

If (3) holds for $S(\mathcal{A}, z)$, then we can deduce (6) easily from (4) or (5). Otherwise we must drop this $S(\mathcal{A}, z)$. We define the *loss* from this term by the size of corresponding $S\left(\mathcal{B}^i, z\right)$:

$$S\left(\mathcal{B}^1, z\right) = (loss + o(1))\frac{x}{\log x} \quad \text{or} \quad S\left(\mathcal{B}^2, z\right) = (loss + o(1))\frac{\eta_1 x}{\log x}. \tag{7}$$

We note that we can only drop positive parts and the total loss of the dropped parts must be less than 1.

Beginning with (2), we can easily give asymptotic formulas for S_1 and S_2 by Lemma 6.6. Before estimating S_3, we first define regions U_{01}–U_{04} and V as

$$\begin{aligned}
U_{01}(t_1,t_2) := &\ \Big\{(t_1,t_2) \in U_{0101} \cup U_{0102} \cup \cdots \cup U_{0107},\\
&\ \frac{11}{126} \leqslant t_1 < \frac{1}{2},\ \frac{11}{126} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Big\},\\
V(t_1,t_2) := &\ \Big\{(t_1,t_2) \in V_0 \cup V_{11} \cup V_{12},\\
&\ \frac{11}{126} \leqslant t_1 < \frac{1}{2},\ \frac{11}{126} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Big\},\\
U_{02}(t_1,t_2) := &\ \Big\{(t_1,t_2) \in U_{0201} \cup U_{0202} \cup \cdots \cup U_{0212},\ (t_1,t_2) \notin U_{01} \cup V,\\
&\ \frac{11}{126} \leqslant t_1 < \frac{1}{2},\ \frac{11}{126} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Big\},\\
U_{03}(t_1,t_2) := &\ \Big\{(t_1,t_2) \in U_{0301} \cup U_{0302} \cup U_{0303},\ (t_1,t_2) \notin U_{01} \cup V \cup U_{02},\\
&\ \frac{11}{126} \leqslant t_1 < \frac{1}{2},\ \frac{11}{126} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Big\},\\
U_{04}(t_1,t_2) := &\ \Big\{(t_1,t_2) \in U_{0401} \cup U_{0402},\ (t_1,t_2) \notin U_{01} \cup V \cup U_{02} \cup U_{03},\\
&\ \frac{11}{126} \leqslant t_1 < \frac{1}{2},\ \frac{11}{126} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Big\},
\end{aligned}$$

where

$$\begin{aligned}
U_{0101}(t_1,t_2) &:= \left\{t_1+t_2 \leqslant \frac{187}{348},\ \frac{23}{132} \leqslant t_2,\ 29t_1 - t_2 \leqslant \frac{59}{6},\ \frac{23}{76} \leqslant t_1,\ 29t_2 - t_1 \leqslant 6,\ \frac{23}{4} \leqslant 12t_1 + 11t_2\right\},\\
U_{0102}(t_1,t_2) &:= \left\{t_1+t_2 \leqslant \frac{149}{264},\ \frac{23}{120} \leqslant t_2,\ 6t_1 + t_2 \leqslant \frac{13}{6},\ \frac{23}{84} \leqslant t_1 \leqslant \frac{1}{3},\ t_1 + 8t_2 \leqslant \frac{9}{4},\ \frac{23}{8} \leqslant 6t_1 + 5t_2\right\},\\
U_{0103}(t_1,t_2) &:= \left\{t_1+t_2 \leqslant \frac{53}{76},\ \frac{23}{132} \leqslant t_2,\ 12t_1 + t_2 \leqslant \frac{25}{4},\ \frac{161}{348} \leqslant t_1,\ 19t_2 - t_1 \leqslant \frac{11}{3},\ \frac{23}{3} \leqslant 12t_1 + 11t_2\right\},\\
U_{0104}(t_1,t_2) &:= \left\{\frac{2}{3} \leqslant t_1+t_2 \leqslant \frac{61}{84},\ \frac{23}{120} \leqslant t_2,\ 6t_1 + t_2 \leqslant \frac{25}{8},\ \frac{115}{264} \leqslant t_1,\ 7t_2 - t_1 \leqslant \frac{5}{4},\ \frac{23}{6} \leqslant 6t_1 + 5t_2\right\},\\
U_{0105}(t_1,t_2) &:= \left\{\frac{1}{2} \leqslant t_1+t_2 \leqslant \frac{149}{264},\ 35t_1 + 23t_2 \leqslant \frac{71}{4},\ \frac{115}{132} \leqslant 2t_1 + t_2,\ 2t_1 + 13t_2 \leqslant \frac{35}{12},\ \frac{69}{2} \leqslant 70t_1 + 59t_2\right\},\\
U_{0106}(t_1,t_2) &:= \left\{t_1+t_2 \leqslant \frac{29}{52},\ 41t_1 + 27t_2 \leqslant \frac{167}{8},\ \frac{23}{26} \leqslant 2t_1 + t_2,\ 2t_1 + 15t_2 \leqslant \frac{37}{12},\ \frac{161}{4} \leqslant 82t_1 + 69t_2\right\},\\
U_{0107}(t_1,t_2) &:= \left\{2t_1+t_2 \leqslant \frac{149}{132},\ 70t_1 + 11t_2 \leqslant \frac{71}{2},\ \frac{115}{264} \leqslant t_1 \leqslant \frac{1}{2},\ t_1 + 6t_2 \leqslant \frac{35}{24},\ \frac{69}{4} \leqslant 35t_1 + 12t_2\right\},\\
V_0(t_1,t_2) &:= \left\{t_2 > \frac{1}{12},\ 2t_2 - \frac{1}{12} < t_1 < \frac{1}{16}\right\},\\
V_{11}(t_1,t_2) &:= \left\{|4t_1 + 3t_2 - 2| < \frac{1}{12},\ \frac{1}{12} < t_2 < \frac{1}{6}\right\},\\
V_{11}(t_1,t_2) &:= \left\{|4t_1 + t_2 - 2| < \frac{1}{12},\ \frac{1}{12} < t_2 < \frac{1}{6}\right\},\\
U_{0201}(t_1,t_2) &:= \left\{\frac{139}{336} \leqslant t_1+t_2 \leqslant \frac{605}{1456},\ \frac{11}{126} \leqslant t_2 \leqslant -(t_1+t_2) + \frac{29}{52}\right\},\\
U_{0202}(t_1,t_2) &:= \left\{\frac{605}{1456} \leqslant t_1+t_2 \leqslant \frac{17}{40},\ \frac{11}{126} \leqslant t_2 \leqslant -\frac{41}{27}(t_1+t_2) + \frac{167}{216}\right\},\\
U_{0203}(t_1,t_2) &:= \left\{\frac{17}{40} \leqslant t_1+t_2 \leqslant \frac{51}{112},\ \frac{11}{126} \leqslant t_2 \leqslant -\frac{4}{3}(t_1+t_2) + \frac{25}{36}\right\},\\
U_{0204}(t_1,t_2) &:= \left\{\frac{461}{1008} \leqslant t_1+t_2 \leqslant \frac{11}{24},\ \frac{11}{126} \leqslant t_2 \leqslant -\frac{1}{6}(t_1+t_2) + \frac{35}{144}\right\},\\
U_{0205}(t_1,t_2) &:= \left\{\frac{11}{24} \leqslant t_1+t_2 \leqslant \frac{23}{48},\ \frac{11}{126} \leqslant t_2 \leqslant \frac{1}{6}\right\},\\
U_{0206}(t_1,t_2) &:= \left\{\frac{23}{48} \leqslant t_1+t_2 \leqslant \frac{265}{552},\ \frac{11}{126} \leqslant t_2 \leqslant -4(t_1+t_2) + \frac{25}{12}\right\},
\end{aligned}$$

$$U_{0207}(t_1,t_2) := \left\{\frac{265}{552} \leqslant t_1+t_2 \leqslant \frac{4727}{9816},\ \frac{11}{126} \leqslant t_2 \leqslant -\frac{1}{6}(t_1+t_2)+\frac{35}{144}\right\},$$
$$U_{0208}(t_1,t_2) := \left\{\frac{4727}{9816} \leqslant t_1+t_2 \leqslant \frac{151}{312},\ \frac{11}{126} \leqslant t_2 \leqslant -\frac{70}{11}(t_1+t_2)+\frac{71}{22}\right\},$$
$$U_{0209}(t_1,t_2) := \left\{\frac{151}{312} \leqslant t_1+t_2 \leqslant \frac{503}{1008},\ \frac{11}{126} \leqslant t_2 \leqslant -4(t_1+t_2)+\frac{25}{12}\right\},$$
$$U_{0210}(t_1,t_2) := \left\{t_1+(t_2+t_2) \leqslant \frac{187}{348},\ \frac{23}{132} \leqslant \left(t_2+\frac{11}{126}\right),\ 29t_1-\left(t_2+\frac{11}{126}\right) \leqslant \frac{59}{6},\ \frac{23}{76} \leqslant t_1,\right.$$
$$\left. 29(t_2+t_2)-t_1 \leqslant 6,\ \frac{23}{4} \leqslant 12t_1+11\left(t_2+\frac{11}{126}\right)\right\},$$
$$U_{0211}(t_1,t_2) := \left\{t_1+(t_2+t_2) \leqslant \frac{149}{264},\ \frac{23}{120} \leqslant \left(t_2+\frac{11}{126}\right),\ 6t_1+(t_2+t_2) \leqslant \frac{13}{6},\ \frac{23}{84} \leqslant t_1 \leqslant \frac{1}{3},\right.$$
$$\left. t_1+8(t_2+t_2) \leqslant \frac{9}{4},\ \frac{23}{8} \leqslant 6t_1+5\left(t_2+\frac{11}{126}\right)\right\},$$
$$U_{0212}(t_1,t_2) := \left\{t_1+(t_2+t_2) \leqslant \frac{53}{76},\ \frac{23}{132} \leqslant \left(t_2+\frac{11}{126}\right),\ 12t_1+(t_2+t_2) \leqslant \frac{25}{4},\ \frac{161}{348} \leqslant t_1,\right.$$
$$\left. 19(t_2+t_2)-t_1 \leqslant \frac{11}{3},\ \frac{23}{3} \leqslant 12t_1+11\left(t_2+\frac{11}{126}\right)\right\},$$
$$U_{0301}(t_1,t_2) := \left\{2(t_1+t_2)+t_2 \leqslant \frac{25}{24},\ 4(t_1+t_2)+6t_2 \leqslant \frac{73}{24},\ 4t_2 \leqslant \frac{9}{8}\right\},$$
$$U_{0302}(t_1,t_2) := \left\{2(t_2+t_2)+t_1 \leqslant \frac{25}{24},\ 4(t_2+t_2)+6t_1 \leqslant \frac{73}{24},\ 4t_1 \leqslant \frac{9}{8}\right\},$$
$$U_{0303}(t_1,t_2) := \left\{\frac{115}{264} \leqslant (t_1+t_2) \leqslant \frac{11}{24},\ t_2 \leqslant -\frac{1}{6}(t_1+t_2)+\frac{35}{144}\right\},$$
$$U_{0401}(t_1,t_2) := \left\{2t_1+t_2 \leqslant \frac{25}{24},\ 4t_1+6t_2 \leqslant \frac{73}{24},\ 4t_2 \leqslant \frac{9}{8},\right.$$
$$\left. 2(1-t_1-t_2)+t_2 \leqslant \frac{25}{24},\ 4(1-t_1-t_2)+6t_2 \leqslant \frac{73}{24}\right\},$$
$$U_{0402}(t_1,t_2) := \left\{t_1 \leqslant \frac{25}{48},\ t_2 \leqslant \frac{23}{192},\ (1-t_1-t_2) \leqslant \frac{25}{48}\right\},$$

and let $U_0 = U_{01} \cup V \cup U_{02} \cup U_{03} \cup U_{04}$. Clearly U_{01} corresponds to Lemma 3.2, V corresponds to Lemma 5.2, U_{02} corresponds to Lemmas 3.3–3.4 and U_{03} corresponds to Lemmas 4.2–4.3. Then we have

$$\begin{aligned} S_3 = & \sum_{(t_1,t_2)\in U_{01}\cup V} S\left(\mathcal{A}_{p_1p_2}, p_2\right) + \sum_{(t_1,t_2)\in U_{02}} S\left(\mathcal{A}_{p_1p_2}, p_2\right) \\ & + \sum_{(t_1,t_2)\in U_{03}} S\left(\mathcal{A}_{p_1p_2}, p_2\right) + \sum_{(t_1,t_2)\in U_{04}} S\left(\mathcal{A}_{p_1p_2}, p_2\right) \\ & + \sum_{(t_1,t_2)\notin U_0} S\left(\mathcal{A}_{p_1p_2}, p_2\right) \\ = & S_{301}+S_{302}+S_{303}+S_{304}+S_{305}. \end{aligned} \tag{8}$$

For S_{301}, by Lemmas 3.2, 6.4, 5.1 and 5.2, we can give an asymptotic formula.

For S_{302}, we can apply Buchstab's identity again to get

$$\begin{aligned} S_{302} = & \sum_{(t_1,t_2)\in U_{02}} S\left(\mathcal{A}_{p_1p_2}, p_2\right) \\ = & \sum_{(t_1,t_2)\in U_{02}} S\left(\mathcal{A}_{p_1p_2}, X^{\frac{11}{126}}\right) - \sum_{\substack{(t_1,t_2)\in U_{02} \\ \frac{11}{126}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right). \end{aligned} \tag{9}$$

By Lemma 6.6 we can give an asymptotic formula for

$$\sum_{(t_1,t_2)\in U_{02}} S\left(\mathcal{A}_{p_1p_2}, X^{\frac{11}{126}}\right). \tag{10}$$

By Lemma 6.7 we can give an asymptotic formula for

$$\sum_{\substack{(t_1,t_2)\in U_{02} \\ \frac{11}{126}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right). \tag{11}$$

Combining (30)–(32), we can give an asymptotic formula for S_{302}.

Before decomposing S_{303}, we first make a definition: we say a set of variables $(t_1, \ldots, t_n)$ is *good* if the variables can be partitioned into $(t_i, h) \in U_{01}$, $(h, t_i) \in U_{01}$ or $(m, h) \in V$. Now we can apply Buchstab's identity twice more to get

$$\begin{aligned}
S_{303} &= \sum_{(t_1,t_2)\in U_{03}} S\left(\mathcal{A}_{p_1p_2}, p_2\right) \\
&= \sum_{(t_1,t_2)\in U_{03}} S\left(\mathcal{A}_{p_1p_2}, X^{\frac{11}{126}}\right) - \sum_{\substack{(t_1,t_2)\in U_{03} \\ \frac{11}{126}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ is good}}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right) \\
&\quad - \sum_{\substack{(t_1,t_2)\in U_{03} \\ \frac{11}{126}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ is not good}}} S\left(\mathcal{A}_{p_1p_2p_3}, X^{\frac{11}{126}}\right) + \sum_{\substack{(t_1,t_2)\in U_{03} \\ \frac{11}{126}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ is not good} \\ \frac{11}{126}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ is good}}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right) \\
&\quad + \sum_{\substack{(t_1,t_2)\in U_{03} \\ \frac{11}{126}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ is not good} \\ \frac{11}{126}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ is not good}}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right) \\
&= S_{3031} - S_{3032} - S_{3033} + S_{3034} + S_{3035}.
\end{aligned} \tag{12}$$

By Lemma 6.6 we can give asymptotic formulas for

$$S_{3031} = \sum_{(t_1,t_2)\in U_{03}} S\left(\mathcal{A}_{p_1p_2}, X^{\frac{11}{126}}\right) \quad \text{and} \quad S_{3033} = \sum_{\substack{(t_1,t_2)\in U_{03} \\ \frac{11}{126}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ is not good}}} S\left(\mathcal{A}_{p_1p_2p_3}, X^{\frac{11}{126}}\right). \tag{13}$$

By Lemmas 3.2, 6.5, 5.1 and 5.2, we can give asymptotic formulas for

$$S_{3032} = \sum_{\substack{(t_1,t_2)\in U_{03} \\ \frac{11}{126}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ is good}}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right) \quad \text{and} \quad S_{3034} = \sum_{\substack{(t_1,t_2)\in U_{03} \\ \frac{11}{126}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ is not good} \\ \frac{11}{126}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ is good}}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right). \tag{14}$$

For the remaining S_{3035}, we can simply discard the whole sum, leading to a loss of size

$$\int_{(t_1,t_2,t_3,t_4)\in R_1} \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1t_2t_3t_4^2} dt_4 dt_3 dt_2 dt_1 < 0.02, \tag{15}$$

where

$$\begin{aligned}
R_1(t_1,t_2,t_3,t_4) := \Bigg\{ &(t_1,t_2)\in U_{03},\ \frac{11}{126}\leqslant t_3<\min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \\
&(t_1,t_2,t_3) \text{ is not good}, \\
&\frac{11}{126}\leqslant t_4<\min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\
&(t_1,t_2,t_3,t_4) \text{ is not good}, \\
&\frac{11}{126}\leqslant t_1<\frac{1}{2},\ \frac{11}{126}\leqslant t_2<\min\left(t_1, \frac{1}{2}(1-t_1)\right)\Bigg\}.
\end{aligned}$$

We note that further decompositions on S_{3035} are possible, but we do not consider them here.

We shall use the *variable role-reversal* to deal with S_{304}. For the definition of a variable role-reversal, see [30]. By Buchstab's identity, we have

$$\begin{aligned}
S_{304} &= \sum_{(t_1,t_2)\in U_{04}} S\left(\mathcal{A}_{p_1p_2}, p_2\right) \\
&= \sum_{(t_1,t_2)\in U_{04}} S\left(\mathcal{A}_{p_1p_2}, X^{\frac{11}{126}}\right) - \sum_{\substack{(t_1,t_2)\in U_{04} \\ \frac{11}{126}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ is good}}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right) \\
&\quad - \sum_{\substack{(t_1,t_2)\in U_{04} \\ \frac{11}{126}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ is not good}}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right) \\
&= S_{3041} - S_{3042} - S_{3043}.
\end{aligned} \tag{16}$$

By Lemma 6.6 we can give an asymptotic formula for

$$S_{3041} = \sum_{(t_1,t_2)\in U_{04}} S\left(\mathcal{A}_{p_1p_2}, X^{\frac{11}{126}}\right). \tag{17}$$

By Lemmas 3.2, 6.5, 5.1 and 5.2, we can give asymptotic formulas for

$$S_{3042} = \sum_{\substack{(t_1,t_2)\in U_{04} \\ \frac{11}{126}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ is good}}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right). \tag{18}$$

Note that S_{3043} counts numbers of the form $p_1p_2p_3\beta$, where the implicit variable β satisfies $\beta \sim X^{1-t_1-t_2-t_3}$ and $(\beta, P(p_3)) = 1$. Now, we reverse the roles of variables β and p_1, making p_1 the implicit one. In this way, S_{3043} can be rewritten as

$$S_{3043} = \sum_{\substack{(t_1,t_2)\in U_{04} \\ \frac{11}{126}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ is not good}}} S\left(\mathcal{A}_{\beta p_2p_3}, \left(\frac{2X}{\beta p_2p_3}\right)^{\frac{1}{2}}\right). \tag{19}$$

Using Buchstab's identity, we have

$$\begin{aligned}
S_{3043} &= \sum_{\substack{(t_1,t_2)\in U_{04} \\ \frac{11}{126}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ is not good}}} S\left(\mathcal{A}_{\beta p_2p_3}, \left(\frac{2X}{\beta p_2p_3}\right)^{\frac{1}{2}}\right) \\
&= \sum_{\substack{(t_1,t_2)\in U_{04} \\ \frac{11}{126}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ is not good}}} S\left(\mathcal{A}_{\beta p_2p_3}, X^{\frac{11}{126}}\right) \\
&\quad - \sum_{\substack{(t_1,t_2)\in U_{04} \\ \frac{11}{126}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ is not good} \\ \frac{11}{126}\leqslant t_4<\frac{1}{2}t_1 \\ (1-t_1-t_2-t_3,t_2,t_3,t_4) \text{ is good}}} S\left(\mathcal{A}_{\beta p_2p_3p_4}, p_4\right) \\
&\quad - \sum_{\substack{(t_1,t_2)\in U_{04} \\ \frac{11}{126}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ is not good} \\ \frac{11}{126}\leqslant t_4<\frac{1}{2}t_1 \\ (1-t_1-t_2-t_3,t_2,t_3,t_4) \text{ is not good}}} S\left(\mathcal{A}_{\beta p_2p_3p_4}, p_4\right) \\
&= S_{30431} - S_{30432} - S_{30433}.
\end{aligned} \tag{20}$$

By Lemma 6.6 and the definition of U_{04}, we can give an asymptotic formula for

$$S_{30431} = \sum_{\substack{(t_1,t_2)\in U_{04} \\ \frac{11}{126}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ is not good}}} S\left(\mathcal{A}_{\beta p_2p_3}, X^{\frac{11}{126}}\right). \tag{21}$$

By Lemmas 3.2, 6.5, 5.1 and 5.2, we can give asymptotic formulas for

$$S_{30432} = \sum_{\substack{(t_1,t_2)\in U_{04} \\ \frac{11}{126}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right) \\ (t_1,t_2,t_3) \text{ is not good} \\ \frac{11}{126}\leqslant t_4<\frac{1}{2}t_1 \\ (1-t_1-t_2-t_3,t_2,t_3,t_4) \text{ is good}}} S\left(\mathcal{A}_{\beta p_2p_3p_4}, p_4\right). \tag{22}$$

We discard the whole sum S_{30433}, leading to a loss of size

$$\int_{(t_1,t_2,t_3,t_4)\in R_2} \frac{\omega_1\left(\frac{t_1-t_4}{t_4}\right)\omega_1\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2t_3^2t_4^2} dt_4dt_3dt_2dt_1 < 0.03, \tag{23}$$

where

$$\begin{aligned}
R_2(t_1,t_2,t_3,t_4) := \Bigg\{ & (t_1,t_2)\in U_{04},\ \frac{11}{126}\leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \\
& (t_1,t_2,t_3) \text{ is not good}, \\
& \frac{11}{126}\leqslant t_4 < \frac{1}{2}t_1, \\
& (1-t_1-t_2-t_3,t_2,t_3,t_4) \text{ is not good},
\end{aligned}$$

$$\frac{11}{126} \leqslant t_1 < \frac{1}{2},\ \frac{11}{126} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\}.$$

For the remaining S_{305}, discarding the whole sum lead to a two-dimensional loss of

$$\int_{\frac{11}{126}}^{\frac{1}{2}} \int_{\frac{11}{126}}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \mathbb{1}_{(t_1,t_2)\notin U_0} \frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1 t_2^2} dt_2 dt_1 < 0.97, \tag{24}$$

However, we can use Buchstab's identity in reverse to get asymptotic formulas for some almost-primes counted. When $t_2 < \frac{1}{2}(1-t_1-t_2)$, we have

$$\sum_{\substack{(t_1,t_2)\notin U_0 \\ t_2<\frac{1}{2}(1-t_1-t_2)}} S\left(\mathcal{A}_{p_1p_2}, p_2\right) = \sum_{\substack{(t_1,t_2)\notin U_0 \\ t_2<\frac{1}{2}(1-t_1-t_2)}} S\left(\mathcal{A}_{p_1p_2}, \left(\frac{2X}{p_1p_2}\right)^{\frac{1}{2}}\right) + \sum_{\substack{(t_1,t_2)\notin U_0 \\ t_2<t_3<\frac{1}{2}(1-t_1-t_2)}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right). \tag{25}$$

This can be seen as a "reversed" application of Buchstab's identity. The first term on the right-hand side counts primes, while the second term counts almost-primes. We cannot give an asymptotic formula for the first term on the right-hand side, but we can give asymptotic formulas for part of the second term if (t_1, t_2, t_3) is good. We can then subtract the size of this part from the loss of S_{305}. The savings from this part are larger than

$$\int_{(t_1,t_2,t_3)\in R_3} \frac{\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 dt_2 dt_1 > 0.04, \tag{26}$$

where

$$R_3(t_1,t_2,t_3) := \left\{(t_1,t_2) \notin U_0,\ t_2 < t_3 < \frac{1}{2}(1-t_1-t_2),\ (t_1,t_2,t_3) \text{ is good},\right.$$
$$\left.\frac{11}{126} \leqslant t_1 < \frac{1}{2},\ \frac{11}{126} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\}.$$

Finally, by (1)–(26) we have

$$\pi(x+\eta x) - \pi(x) = S\left(\mathcal{A}, (2X)^{\frac{1}{2}}\right) \geqslant (1 - 0.02 - 0.03 - 0.97 + 0.04)\frac{\eta x}{\log x} = 0.02\frac{\eta x}{\log x},$$

and the proof of Theorem 1.1 is completed.

8. Applications of Theorem 1.1

Clearly our Theorem 1.1 has many interesting applications. The following application of our Theorem 1.1 is about Goldbach numbers (sum of two primes) in short intervals. By combining our Theorem 1.1 with the main theorem proved in [31], we can easily deduce the following theorem.

Theorem 8.1. *The interval $[X, X+X^{\frac{13}{600}}]$ contains Goldbach numbers. That is,*

$$g_{n+1} - g_n \ll g_n^{\frac{13}{600}},$$

where g_n denote the n-th Goldbach number.

Previous exponent $\frac{13}{550}$ [[30], Theorem 8.1] comes from the exponent $\frac{1}{22}$ [30]. Note that $\frac{13}{550} \approx 0.0236$ and $\frac{13}{600} \approx 0.0217$. If we focus on Maillet numbers (difference of two primes) instead of Goldbach numbers in short intervals, Pintz [41] improved this exponent to any $\varepsilon > 0$.

Another application of our Theorem 1.1 is about prime values of the integer parts of real sequences, improving the previous result of the author [30] by adding two more term on both of the sequences $[p^k\alpha]$ and $[(p\alpha)^k]$.

Theorem 8.2. *For almost all $\alpha > 0$, both of the following statements are true:*
(1) Infinitely often $p, [p\alpha], [p^2\alpha], \ldots, [p^{24}\alpha]$ are all prime.
(2) Infinitely often $p, [p\alpha], [(p\alpha)^2], \ldots, [(p\alpha)^{24}]$ are all prime.

The proof of Theorem 8.2 can be done by replacing [[12], Lemma 4] by a variant of our Theorem 1.1. In this way, the ratio $\frac{20}{19}$ in [[12], Theorem 4] can be improved to $\frac{24}{23}$.

The last application of our Theorem 1.1 is about the Three Primes Theorem with small prime solutions, improving the previous result of the author [30] by reducing the exponent $\frac{1}{40}$ to $\frac{11}{480}$.

Theorem 8.3. *Let Y denote a sufficiently large odd integer. The equation*

$$Y = p_1 + p_2 + p_3, \quad p_1 \leqslant Y^{\frac{11}{480}}$$

is solvable.

We define u_0, u_1, v_0, v_1 as in [4]. Note that we have $u_0 > 0.02$ by Theorem 1.1, and we can show that $u_1 < 3.26$ by similar arguments as in [4]. Using the vector sieve together with the same bounds for v_0 and v_1 as in [4], the proof of our Theorem 8.3 is essentially the same as the proof of [[4], Theorem].

Appendix: Plot of Regions

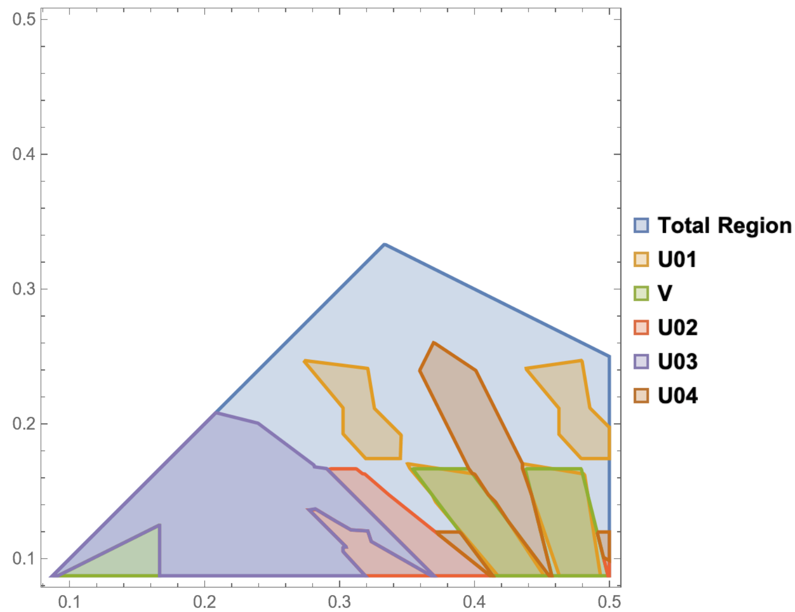

Figure 1: Plot of Regions

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