On Chen's theorem, Goldbach's conjecture and applications of sieve methods

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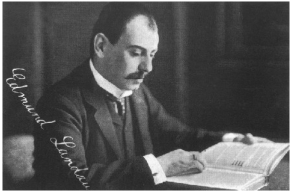
Overview

1. Historical Background

Landau's problems



At the 1912 ICM, Landau listed four basic problems about prime numbers.

- **Goldbach's conjecture**: Can every even integer greater than 2 be written as the sum of two primes?
- **Twin prime conjecture**: Are there infinitely many primes p such that $p + 2$ is prime?
- **Legendre's conjecture**: Does there always exist a prime between consecutive perfect squares?
- **Euler's conjecture**: Are there infinitely many primes of the form $n^2 + 1$?

As of 2024, all four problems are unresolved.

Goldbach's conjecture

Goldbach (1690-1764) and Euler (1707-1783)

Goldbach's conjecture

Every even integer greater than 2 can be written as the sum of two primes.

Goldbach's conjecture for large integers

Every sufficiently large even integer can be written as the sum of two primes.

Sieve approach on Goldbach's conjecture

Writing an even number as the sum of two primes (p, P_1)

$\downarrow$

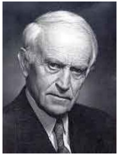
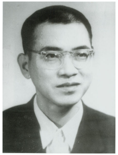
Writing an even number as the sum of two numbers with at most r prime factors (r-almost primes, P_r)

Proposition ($a + b$)

Every sufficiently large even integer can be written as the sum of an a-almost prime and a b-almost prime.

Especially, the Goldbach's conjecture is equivalent to Proposition $(1 + 1)$.

Historical records

- $9+9$, Brun, 1920; First make a and b explicit
- $1+c$, Rényi, 1948; (c is very large) First get $a=1$
- $1+9$, Barban, 1961; First make b explicit under $a=1$
- $1+5$, Pan, 1962;
- $1+4$, Pan-Wang, 1962;
- $1+3$, Barban, Bombieri, Vinogradov, 1965;
- $1+2$, Chen, 1973. Best result until now

Chen's Theorem

Theorem (Chen, 1973)

Every sufficiently large even integer can be written as the sum of a prime and a P_2. Moreover, let N denote a sufficiently large even integer and define

$$D_{1,2}(N) := \left|\{p : p \leqslant N, N - p = P_2\}\right|,$$

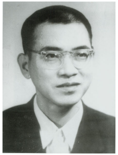
we have

$$D_{1,2}(N) \geqslant 0.67 \frac{C(N)N}{(\log N)^2},$$

where

$$C(N) := \prod_{\substack{p|N \\ p>2}} \frac{p-1}{p-2} \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right).$$

Numerical refinements

- 0.67, Chen, 1973;
- 0.689, Halberstam-Richert, 1975;
- 0.7544, Chen, 1978;
- 0.81, Chen, 1978;
- 0.8285, Cai-Lu, 2002;
- 0.836, Wu, 2004;
- 0.867, Cai, 2008;
- 0.899, Wu, 2008. Best published result until now

Two results claimed by Chen

In 1980, Chen announced two unpublished results of himself:

1. 0.9;
2. 9 $\Rightarrow$ Goldbach's conjecture.

$$C_x = \prod_{\substack{p \mid x \\ p>2}} \frac{p-1}{p-2} \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) .$$

Let $P_x(1,2)$ be the number of primes p satisfying the following conditions: either $x-p = p_1$ or $x-p = p_2p_3$, where p_1, p_2, p_3 are primes. In 1966, I proved that

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$$P_x(1,2) \ge \frac{BC_x x}{(\log x)^2} \qquad (1)$$

where $B \ge 0.098$. This was successively improved by myself in 1973 to $B \ge 0.67$, by Halberstam and Richert in 1975 to $B \ge 0.689$, and by myself in 1978 to $B \ge 0.81$. Recently, in an unpublished manuscript, I have shown that $B \ge 0.9$. The improvement of the constant B in the inequality (1) is important because it implies an increase in the number of solutions of our problem. Moreover, if B is larger than 9, then we can solve (1,1).

2. Main Result

Main result

Theorem 1.1 (L. 2024)

We have

$$D_{1,2}(N) \geq 1.733 \frac{C(N)N}{(\log N)^2}.$$

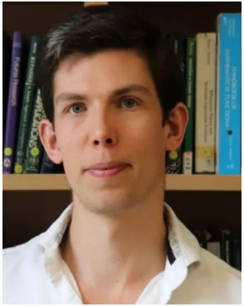
One important significance of our Theorem 1.1 is to make us truly achieve and exceed the constant 0.9 claimed by Chen.

Our constant 1.733 gives a 93% refinement of Wu's prior record 0.899. This is the greatest refinement on this problem since Chen from 1973.

Main tools

In order to prove our Theorem 1.1, we mainly utilize the following tools:

1. **Weighted sieve inequalities**;
2. **Lichtman's new distribution levels**;
3. **Chen's double sieve**;
4. **Optimization of various bounds**.

James Maynard and Jared Duker Lichtman

3. Tools for Proving

Weighted sieve inequalities

Let

$$\mathcal{A} = \{N - p : p \leqslant N\}, \quad \mathcal{P} = \{p : (p, 2) = 1\},$$
$$\mathcal{P}(q) = \{p : p \in \mathcal{P}, (p, q) = 1\}, \quad P(z) = \prod_{\substack{p \in \mathcal{P} \\ p < z}} p,$$
$$\mathcal{A}_d = \{a : a \in \mathcal{A}, a \equiv 0 (\bmod d)\}, \quad S(\mathcal{A}; \mathcal{P}, z) = \sum_{\substack{a \in \mathcal{A} \\ (a, P(z)) = 1}} 1.$$

Then we have

Lemma (Wu, 2008)

$$\begin{aligned} 4D_{1,2}(N) \geqslant\ & S_0 - 2S_1 - 2S_2 - 2S_3 - S_4 - S_5 + S_6 + S_7 \\ & + S_8 + S_9 - 2S_{10} - S_{11} - S_{12} - S_{13} - S_{14} + O\left(N^{\frac{12.27}{13.27}}\right), \end{aligned}$$

Weighted sieve inequalities

where

$$S_0 = 3S\left(\mathcal{A};\mathcal{P}(N),N^{\frac{1}{13.27}}\right)+S\left(\mathcal{A};\mathcal{P}(N),N^{\frac{1}{8.24}}\right),$$

$$S_1 = \sum_{\substack{N^{\frac{1}{13.27}}\leqslant p<N^{\frac{25}{128}}\\(p,N)=1}} S\left(\mathcal{A}_p;\mathcal{P}(N),N^{\frac{1}{13.27}}\right),$$

$$S_2 = \sum_{\substack{N^{\frac{25}{128}}\leqslant p<N^{\frac{1}{4}}\\(p,N)=1}} S\left(\mathcal{A}_p;\mathcal{P}(N),N^{\frac{1}{13.27}}\right),$$

$$S_3 = \sum_{\substack{N^{\frac{1}{4}}\leqslant p<N^{\frac{57}{224}}\\(p,N)=1}} S\left(\mathcal{A}_p;\mathcal{P}(N),N^{\frac{1}{13.27}}\right),$$

Weighted sieve inequalities

$$S_4 = \sum_{\substack{N^{\frac{57}{224}} \leqslant p < N^{\frac{1}{3}} \\ (p,N)=1}} S\left(\mathcal{A}_p; \mathcal{P}(N), N^{\frac{1}{13.27}}\right),$$

$$S_5 = \sum_{\substack{N^{\frac{57}{224}} \leqslant p < N^{\frac{1}{2}-\frac{3}{13.27}} \\ (p,N)=1}} S\left(\mathcal{A}_p; \mathcal{P}(N), N^{\frac{1}{13.27}}\right),$$

$$S_6 = \sum_{\substack{N^{\frac{1}{13.27}} \leqslant p_2 < p_1 < N^{\frac{1}{8.24}} \\ (p_1 p_2,N)=1}} S\left(\mathcal{A}_{p_1 p_2}; \mathcal{P}(N), N^{\frac{1}{13.27}}\right),$$

$$S_7 = \sum_{\substack{N^{\frac{1}{13.27}} \leqslant p_2 < N^{\frac{1}{8.24}} \leqslant p_1 < N^{\frac{25}{128}} \\ (p_1 p_2,N)=1}} S\left(\mathcal{A}_{p_1 p_2}; \mathcal{P}(N), N^{\frac{1}{13.27}}\right),$$

Weighted sieve inequalities

$$S_8 = \sum_{\substack{N^{\frac{1}{13.27}} \leqslant p_2 < N^{\frac{1}{8.24}} < N^{\frac{25}{128}} \leqslant p_1 < N^{\frac{57}{224}} \\ (p_1 p_2, N)=1}} S\left(\mathcal{A}_{p_1 p_2}; \mathcal{P}(N), N^{\frac{1}{13.27}}\right),$$

$$S_9 = \sum_{\substack{N^{\frac{1}{13.27}} \leqslant p_2 < N^{\frac{1}{8.24}} < N^{\frac{57}{224}} \leqslant p_1 < N^{\frac{1}{2} - \frac{3}{13.27}} \\ (p_1 p_2, N)=1}} S\left(\mathcal{A}_{p_1 p_2}; \mathcal{P}(N), N^{\frac{1}{13.27}}\right),$$

$$S_{10} = \sum_{\substack{N^{\frac{1}{2} - \frac{3}{13.27}} \leqslant p_1 < p_2 < \left(\frac{N}{p_1}\right)^{\frac{1}{2}} \\ (p_1 p_2, N)=1}} S\left(\mathcal{A}_{p_1 p_2}; \mathcal{P}(Np_1), p_2\right),$$

$$S_{11} = \sum_{\substack{N^{\frac{1}{13.27}} \leqslant p_1 < N^{\frac{1}{3}} \leqslant p_2 < \left(\frac{N}{p_1}\right)^{\frac{1}{2}} \\ (p_1 p_2, N)=1}} S\left(\mathcal{A}_{p_1 p_2}; \mathcal{P}(Np_1), p_2\right),$$

Weighted sieve inequalities

$$S_{12} = \sum_{\substack{N^{\frac{1}{8.24}} \leqslant p_1 < N^{\frac{1}{2} - \frac{3}{13.27}} \leqslant p_2 < \left(\frac{N}{p_1}\right)^{\frac{1}{2}} \\ (p_1 p_2, N) = 1}} S\left(\mathcal{A}_{p_1 p_2}; \mathcal{P}(Np_1), \left(\frac{N}{p_1 p_2}\right)^{\frac{1}{2}}\right),$$

$$S_{13} = \sum_{\substack{N^{\frac{1}{13.27}} \leqslant p_1 < p_2 < p_3 < p_4 < N^{\frac{1}{8.24}} \\ (p_1 p_2 p_3 p_4, N) = 1}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}; \mathcal{P}(N), p_2\right),$$

$$S_{14} = \sum_{\substack{N^{\frac{1}{13.27}} \leqslant p_1 < p_2 < p_3 < N^{\frac{1}{8.24}} \leqslant p_4 < N^{\frac{1}{2} - \frac{2}{13.27}} p_3^{-1} \\ (p_1 p_2 p_3 p_4, N) = 1}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}; \mathcal{P}(N), p_2\right) + O\left(N^{\frac{12.27}{13.27}}\right).$$

New distribution levels

In order to handle the error terms occurred in the estimation of sieve functions, we need Bombieri-Vinogradov type results.
The classical Bombieri-Vinogradov theorem is of the form

$$\sum_{d \leqslant x^{\frac{1}{2}-\varepsilon}} \max_{(l,d)=1} \left| \pi(x; d, l) - \frac{\pi(x)}{\varphi(d)} \right| \ll \frac{x}{(\log x)^A}.$$

The exponent $\frac{1}{2}$ is called the distribution level θ of primes in arithmetic progressions.

New distribution levels

In 2023, Lichtman proved some new Bombieri-Vinogradov type results on well-factorable sequences. Here is one of his new results:

Lemma (Lichtman, 2023)

We have

$$\sum_{\substack{q \leqslant N^{\frac{19101}{32000}} \\ q \mid P\left(N^{1/500}\right) \\ (q,N)=1}} \widetilde{\lambda}^{\pm}(q) \left(\pi(N; q, N) - \frac{\pi(N)}{\varphi(q)} \right) \ll \frac{N}{(\log N)^A},$$

where $\widetilde{\lambda}^{\pm}(q)$ is Iwaniec's modified linear sieve weight (well-factorable).

Clearly we can take $\theta = \frac{19101}{32000}$, which is larger than $\frac{1}{2}$.

Chen's double sieve

In classical linear sieve, we can estimate the upper bound for S_1 as

Estimation of S_1, I

$$S_1 \leqslant (1+o(1))\frac{2}{e^\gamma}\left(\int_{\frac{1}{13.27}}^{\frac{25}{128}} \frac{13.27F(13.27(\frac{1}{2}-t_1))}{t_1}\right)\frac{C(N)N}{(\log N)^2},$$

where the functions $F(s)$ (and $f(s)$ below) are defined by a differential-difference equation.

Chen's double sieve

In 1978, Chen first introduced his double sieve method, which was later refined by Wu, to give a better estimation of some sieve functions.

Estimation of S_1, II

$$S_1 \leqslant (1+o(1))\frac{2}{e^\gamma}\int_{\frac{1}{13.27}}^{\frac{25}{128}}\left(\frac{13.27F(13.27(\frac{1}{2}-t_1))}{t_1}-\frac{26.54e^\gamma H(13.27(\frac{1}{2}-t_1))}{(13.27(\frac{1}{2}-t_1))t_1}\right)\frac{C(N)N}{(\log N)^2},$$

where the functions $H(s)$ (and $h(s)$ below) are extremely complicated. The main work of Chen and Wu on this topic is to show $H(s) \geqslant 0$ and $h(s) \geqslant 0$.

Numerical values of $H(s)$ and $h(s)$

s	$H(s)$	$h(s)$
$2.0 < s \leqslant 2.1$	0.0223939	0.0211041
$2.1 < s \leqslant 2.2$	0.0223939	0.0191556
$2.2 < s \leqslant 2.3$	0.0217196	0.0173631
$2.3 < s \leqslant 2.4$	0.0202876	0.0157035
$2.4 < s \leqslant 2.5$	0.0181433	0.0141585
$2.5 < s \leqslant 2.6$	0.0158644	0.0127132
$2.6 < s \leqslant 2.7$	0.0129923	0.0113556
$2.7 < s \leqslant 2.8$	0.0100686	0.0100756
$2.8 < s \leqslant 2.9$	0.0078162	0.0088648
$2.9 < s \leqslant 3.0$	0.0072943	0.0077612

Table: Lower bounds for $H(s)$ and $h(s)$, $2 < s \leqslant 3$

Optimization of various bounds

In some cases, we can either use Lichtman's distribution level or Chen's double sieve. Hence we need to optimize the two options.

Estimation of S_1, III

$$
\begin{aligned}
S_1 \leqslant (1+o(1))\frac{2}{e^\gamma}\Bigg(\int_{\frac{1}{13.27}}^{\frac{25}{128}} \min\Bigg(& 13.27\frac{F(13.27(\vartheta_1(t_1,\frac{1}{13.27},\frac{1}{13.27})-t_1))}{t_1} \\
&- \frac{26.54e^\gamma H(13.27(\frac{1}{2}-t_1))}{(13.27(\frac{1}{2}-t_1))t_1}, \min_{13.27\leqslant k\leqslant 500}\Bigg(k\frac{F(k(\vartheta_1(t_1,\frac{1}{k},\frac{1}{k})-t_1))}{t_1} \\
&- \frac{2ke^\gamma H(k(\frac{1}{2}-t_1))}{(k(\frac{1}{2}-t_1))t_1} - k\int_{\frac{1}{k}}^{\frac{1}{13.27}} \frac{f(k(\vartheta_1(t_1,t_2,\frac{1}{k})-t_1-t_2))}{t_1 t_2}dt_2
\end{aligned}
$$

Optimization of various bounds

Estimation of S_1, III

$$
\begin{aligned}
&- 2ke^{\gamma}\int_{\frac{1}{k}}^{\frac{1}{13.27}} \frac{h(k(\frac{1}{2}-t_1-t_2))}{(k(\frac{1}{2}-t_1-t_2))t_1t_2}dt_2 \\
&+ \int_{\frac{1}{k}}^{\frac{1}{13.27}}\int_{\frac{1}{k}}^{t_2} \frac{F\left(\frac{(\vartheta_1(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1t_2t_3^2}dt_3dt_2\Bigg)\Bigg)\,dt_1\Bigg)\frac{C(N)N}{(\log N)^2},
\end{aligned}
$$

where ϑ_1 is Lichtman's function of distribution level.

4. Applications

Application: Chen's theorem with coefficients

In 1976, Ross considered Chen's theorem with two fixed positive integer coefficients a and b. Let M denote a sufficiently large integer and fix $a, b < M^{\varepsilon}$, and we define

$$R_{a,b}(M) := |\{p : ap \leqslant M, M - ap = bP_2\}| .$$

Then Ross proved in 1976 that

Theorem (Ross, 1976)

We have

$$R_{a,b}(M) \geqslant 0.608 \frac{C(abM)M}{(\log M)^2}.$$

0.68, H. Li, 2023; 0.8671, L., 2023;

Application: Chen's theorem with coefficients

A direct consequence of Theorem 1.1 is:

Theorem 1.2 (L., 2024)

We have

$$R_{a,b}(M) \geqslant 1.733 \frac{C(abM)M}{(\log M)^2}.$$

Application: Chen's theorem on almost prime twins

We can also generalize Chen's theorem to the twin prime problem.

Theorem (Chen, 1973)

Let x denote a sufficiently large integer and define

$$\pi_{1,2}(x) := \left|\{p : p \leqslant x, p + 2 = P_2\}\right|,$$

we have

$$\pi_{1,2}(x) \geqslant 0.335 \frac{C_2 N}{(\log N)^2},$$

where

$$C_2 = 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right).$$

Note that $0.335 = \frac{1}{2} 0.67$.

Application: Chen's theorem on almost prime twins

- 0.335, Chen, 1973;
- 0.3445, Halberstam-Richert, 1975;
- 0.3772, Chen, 1978;
- 0.405, Chen, 1978;
- 0.71, Fouvry-Grupp, 1986;
- 1.015, Liu, 1990;
- 1.05, Wu, 1990;
- 1.0971, Cai, 2002;
- 1.104, Wu, 2004;
- 1.123, Cai, 2008;
- 1.13, Cai, 2008;
- 1.145, Wu, 2008. Best result until now

Application: Chen's theorem on almost prime twins

By the similar arguments (the four tools above), we can prove the following.

Theorem 1.3 (L., 2024)

We have

$$\pi_{1,2}(x) \geqslant 1.238 \frac{C_2 N}{(\log N)^2}.$$

Application: Chen's theorem with small primes

We want to restrict the prime p in Chen's theorem to be smaller than N^{θ} for some $\theta < 1$. Define

$$D_{1,2}^{\theta}(N) := \left|\left\{p : p \leqslant N^{\theta}, N - p = P_2\right\}\right|,$$

then Ross proved in 1976 that $D_{1,2}^{\theta}(N) \gg \frac{C(N)N^{\theta}}{(\log N)^2}$ for any $0.959 \leqslant \theta \leqslant 1$.

$0.95 \leqslant \theta \leqslant 1$, Cai, 2002;

$0.945 \leqslant \theta \leqslant 1$, Li-Cai, 2011;

$0.941 \leqslant \theta \leqslant 1$, Cai, 2015.

Application: Chen's theorem with small primes

A simple calculation yields

Theorem 1.4 (L., 2024)

We have

$$D_{1,2}^{\theta}(N) \gg \frac{C(N)N^{\theta}}{(\log N)^2}$$

for any $0.9409 \leqslant \theta \leqslant 1$.

Thank you!