

LARGEST SQUARE DIVISORS OF SHIFTED PRIMES

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ABSTRACT. The author shows that there are infinitely many primes p such that for any nonzero integer a , $p - a$ is divisible by a square $d^2 > p^{\frac{1}{2} + \frac{1}{700}}$. The exponent $\frac{1}{2} + \frac{1}{700}$ improves Merikoski's $\frac{1}{2} + \frac{1}{2000}$. Many powerful devices in Harman's sieve are used for this improvement.

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1. INTRODUCTION

The Euler's conjecture, which states that there are infinitely many primes of the form $n^2 + 1$, is one of Landau's problems on prime numbers. There are several ways to attack this conjecture. One way is to relax the number of prime factors of $f(n)$, and the best result in this way is due to Iwaniec [4]. Building on the previous work of Richert [11], he showed that for any irreducible polynomial $f(n) = an^2 + bn + c$ with $a > 0$ and $c \equiv 1 \pmod{2}$, there are infinitely many x such that $f(x)$ has at most 2 prime factors.

Another possible way is to consider the square divisors of $p - 1$. If we can show that there are infinitely many primes such that $p - 1$ is divisible by a large square $d^2 \geq p^\theta$ with $\theta = 1$, then the Euler's conjecture is solved. The first result on this direction is due to Baier and Zhao [1] [2], where they proved the above statement holds with d prime and $\theta < \frac{4}{9}$ as an application of their large sieve for sparse sets of moduli. They interpret the problem as an equidistribution problem for primes $p \equiv 1 \pmod{d^2}$, after which the result follows from their Bombieri–Vinogradov type theorem for sparse sets of moduli ([1], Theorem 3).

In 2009, Matomäki [9] improved the above result to $\theta < \frac{1}{2}$ using Harman's sieve [3] and Type-II information obtained using the large sieve of Baier and Zhao [2]. Note that the exponent $\theta = \frac{1}{2}$ is the limit of what can be obtained under the Generalized Riemann Hypothesis (GRH). In [10], Merikoski first broke this $\frac{1}{2}$ -barrier and successfully got $\theta \leq \frac{1}{2} + \frac{1}{2000}$ without the restriction that d is a prime. In the article [10], he mentioned that the “extra” exponent $\frac{1}{2000}$ has not been fully optimized, and one should be able to increase this to some value between $\frac{1}{500}$ and $\frac{1}{1000}$. In this paper, we increase this to $\frac{1}{700}$ by a careful decomposition on Harman's sieve.

Theorem 1.1. *Let $a \neq 0$ be an integer. There are infinitely many primes p such that $d^2 \mid (p - a)$ for some integer d with*

$$d^2 \geq p^{\frac{1}{2} + \frac{1}{700}}.$$

Throughout this paper, we always suppose that ε is a sufficiently small positive constant and X is sufficiently large. The letter p , with or without subscript, is reserved for prime numbers. Let $\varpi = \frac{1}{1400}$, $D = X^{\frac{1}{2} + 2\varpi}$, $K = \lceil \frac{1}{\varepsilon} \rceil$ and $P = D^{\frac{1}{K}}$. Let σ be a number satisfies the condition $19\sigma + 90\varpi + 71\varepsilon < 1$. Define

$$I_j = \left(2^{j-1} P^{\frac{1}{2}}, 2^j P^{\frac{1}{2}} \right] \text{ for } j = 1, 2, \dots, K.$$

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We set

$$\mathcal{D} = \{p_1^2 p_2^2 \cdots p_K^2 : p_j \in I_j \text{ for } j = 1, 2, \dots, K\}$$

so that $d^2 \in D$ is of size $\asymp D$ and is a square of a squarefree integer.

Fix an integer $a \neq 0$ and a C^∞ -smooth function $0 \leq \psi \leq 1$, supported on the interval $[1, 2]$ and satisfying $\psi(x) = 1$ for $1 + \eta \leq x \leq 2 - \eta$ for some sufficiently small positive η . For $d^2 \in D$ and $z < X$, denote

$$S(\mathcal{A}^d, z) = \sum_{\substack{n \equiv a \pmod{d^2} \\ (n, P(z))=1}} \psi\left(\frac{n}{X}\right) \quad \text{and} \quad S(\mathcal{B}^d, z) = \frac{1}{\varphi(d^2)} \sum_{\substack{(n, d^2)=1 \\ (n, P(z))=1}} \psi\left(\frac{n}{X}\right).$$

Then Theorem 1.1 holds if there exists $\varepsilon, \eta, c > 0$ such that for all but $O\left(D^{\frac{1}{2}} X^{-\eta}\right)$ of the moduli $d^2 \in \mathcal{D}$, we have

$$S\left(\mathcal{A}^d, 2X^{\frac{1}{2}}\right) \geq c S\left(\mathcal{B}^d, 2X^{\frac{1}{2}}\right). \quad (1)$$

2. ASYMPTOTIC FORMULAS

Lemma 2.1. ([10], Proposition 7]. Let $U \leq X^{\frac{1}{2}+2\varpi+\varepsilon}$ and let a_u be divisor-bounded. Then for all but $O\left(D^{\frac{1}{2}} X^{-\eta}\right)$ of $d^2 \in \mathcal{D}$, we have

$$\sum_{u \sim U} a_u S(\mathcal{A}_u^d, X^{\sigma-2\varpi}) = (1 + o(1)) \sum_{u \sim U} a_u S(\mathcal{B}_u^d, X^{\sigma-2\varpi}).$$

Lemma 2.2. ([10], Proposition 6]. Let $U \leq X^{\frac{1}{2}-\sigma}$, $V \leq X^{\frac{1}{8}+\frac{\sigma}{2}-\frac{5\varpi}{2}-\eta}$ and let a_u, b_v be divisor-bounded. Then for all but $O\left(D^{\frac{1}{2}} X^{-\eta}\right)$ of $d^2 \in \mathcal{D}$, we have

$$\sum_{\substack{u \sim U \\ v \sim V}} a_u b_v S(\mathcal{A}_{uv}^d, X^{\sigma-2\varpi}) = (1 + o(1)) \sum_{\substack{u \sim U \\ v \sim V}} a_u b_v S(\mathcal{B}_{uv}^d, X^{\sigma-2\varpi}).$$

Lemma 2.3. ([10], Proposition 4]. Let $UV = X$, $X^{\frac{1}{2}-\sigma} \leq U \leq X^{\frac{1}{2}-2\varpi-\varepsilon}$ and let a_u, b_v be divisor-bounded. Then we have

$$\sum_{d^2 \in \mathcal{D}} \left| \sum_{\substack{uv \equiv a \pmod{d^2} \\ u \sim U, v \sim V}} a_u b_v \psi\left(\frac{uv}{X}\right) - \frac{1}{\varphi(d^2)} \sum_{\substack{(uv, d^2)=1 \\ u \sim U, v \sim V}} a_u b_v \psi\left(\frac{uv}{X}\right) \right| \ll D^{-\frac{1}{2}} X^{1-\eta}.$$

3. THE FINAL DECOMPOSITION

Before decomposing, we define asymptotic regions I and II as

$$\begin{aligned} I(m, n) &:= \left\{ m + n \leq \frac{1}{2} + 2\varpi, \text{ or } m \leq \frac{1}{2} - \sigma \text{ and } n < \frac{1}{8} + \frac{\sigma}{2} - \frac{5\varpi}{2} \right\}, \\ II(m, n) &:= \left\{ \frac{1}{2} - \sigma \leq m \leq \frac{1}{2} - 2\varpi \text{ or } \frac{1}{2} - \sigma \leq n \leq \frac{1}{2} - 2\varpi \right. \\ &\quad \left. \text{or } \frac{1}{2} - \sigma \leq m + n \leq \frac{1}{2} - 2\varpi \text{ or } \frac{1}{2} + 2\varpi \leq m + n \leq \frac{1}{2} + \sigma \right\}. \end{aligned}$$

Let $\omega(u)$ denotes the Buchstab function determined by the following differential-difference equation

$$\begin{cases} \omega(u) = \frac{1}{u}, & 1 \leq u \leq 2, \\ (u\omega(u))' = \omega(u-1), & u \geq 2. \end{cases}$$

Moreover, we have the upper and lower bounds for $\omega(u)$:

$$\omega(u) \geq \omega_0(u) = \begin{cases} \frac{1}{u}, & 1 \leq u < 2, \\ \frac{1+\log(u-1)}{u}, & 2 \leq u < 3, \\ \frac{1+\log(u-1)}{u} + \frac{1}{u} \int_2^{u-1} \frac{\log(t-1)}{t} dt \geq 0.5607, & 3 \leq u < 4, \\ 0.5612, & u \geq 4, \end{cases}$$

$$\omega(u) \leq \omega_1(u) = \begin{cases} \frac{1}{u}, & 1 \leq u < 2, \\ \frac{1+\log(u-1)}{u}, & 2 \leq u < 3, \\ \frac{1+\log(u-1)}{u} + \frac{1}{u} \int_2^{u-1} \frac{\log(t-1)}{t} dt \leq 0.5644, & 3 \leq u < 4, \\ 0.5617, & u \geq 4. \end{cases}$$

We shall use $\omega_0(u)$ and $\omega_1(u)$ to give numerical bounds for some sieve functions discussed below. We shall also use the simple upper bound $\omega(u) \leq \max(\frac{1}{u}, 0.5672)$ (see Lemma 8(iii) of [5]) to estimate high-dimensional integrals. Fix $\sigma = \frac{1}{20.31}$ and let $p_j = X^{\alpha_j}$. By Buchstab's identity, we have

$$\begin{aligned} S(\mathcal{A}^d, 2X^{\frac{1}{2}}) &= S(\mathcal{A}^d, X^{\sigma-2\varpi}) - \sum_{\sigma-2\varpi \leq \alpha_1 < \frac{1}{2}} S(\mathcal{A}_{p_1}^d, X^{\sigma-2\varpi}) + \sum_{\substack{\sigma-2\varpi \leq \alpha_1 < \frac{1}{2} \\ \sigma-2\varpi \leq \alpha_2 < \min(\alpha_1, \frac{1-\alpha_1}{2})}} S(\mathcal{A}_{p_1 p_2}^d, p_2) \\ &= S(\mathcal{A}^d, X^{\sigma-2\varpi}) - \sum_{\sigma-2\varpi \leq \alpha_1 < \frac{1}{2}} S(\mathcal{A}_{p_1}^d, X^{\sigma-2\varpi}) + \sum_{(\alpha_1, \alpha_2) \in II} S(\mathcal{A}_{p_1 p_2}^d, p_2) \\ &\quad + \sum_{(\alpha_1, \alpha_2) \in A} S(\mathcal{A}_{p_1 p_2}^d, p_2) + \sum_{(\alpha_1, \alpha_2) \in B} S(\mathcal{A}_{p_1 p_2}^d, p_2) + \sum_{(\alpha_1, \alpha_2) \in C} S(\mathcal{A}_{p_1 p_2}^d, p_2) \\ &= S_1 - S_2 + S_{II} + S_A + S_B + S_C, \end{aligned} \tag{2}$$

where

$$\begin{aligned} A(\alpha_1, \alpha_2) &= \left\{ \sigma - 2\varpi \leq \alpha_1 < \frac{1}{2}, \sigma - 2\varpi \leq \alpha_2 < \min\left(\alpha_1, \frac{1-\alpha_1}{2}\right), (\alpha_1, \alpha_2) \notin II, \right. \\ &\quad \left. (\alpha_1, \alpha_2, \alpha_2) \text{ can be partitioned into } (m, n) \in I \right\}, \\ B(\alpha_1, \alpha_2) &= \left\{ \sigma - 2\varpi \leq \alpha_1 < \frac{1}{2}, \sigma - 2\varpi \leq \alpha_2 < \min\left(\alpha_1, \frac{1-\alpha_1}{2}\right), (\alpha_1, \alpha_2) \notin II, \right. \\ &\quad \left. (\alpha_1, \alpha_2, \alpha_2) \text{ cannot be partitioned into } (m, n) \in I, (\alpha_1, \alpha_2) \in I, (1-\alpha_1-\alpha_2, \alpha_2) \in I \right\}, \\ C(\alpha_1, \alpha_2) &= \left\{ \sigma - 2\varpi \leq \alpha_1 < \frac{1}{2}, \sigma - 2\varpi \leq \alpha_2 < \min\left(\alpha_1, \frac{1-\alpha_1}{2}\right), (\alpha_1, \alpha_2) \notin II \cup A \cup B \right\}. \end{aligned}$$

We have asymptotic formulas for S_1 and S_2 by Lemma 2.1. We can also give an asymptotic formula for S_{II} by Lemma 2.3. For S_A , we can apply Buchstab's identity to get

$$\begin{aligned} S_A &= \sum_{(\alpha_1, \alpha_2) \in A} S(\mathcal{A}_{p_1 p_2}^d, p_2) \\ &= \sum_{(\alpha_1, \alpha_2) \in A} S(\mathcal{A}_{p_1 p_2}^d, X^{\sigma-2\varpi}) - \sum_{\substack{(\alpha_1, \alpha_2) \in A \\ \sigma-2\varpi \leq \alpha_3 < \min(\alpha_2, \frac{1-\alpha_1-\alpha_2}{2}) \\ (\alpha_1, \alpha_2, \alpha_3) \text{ can be partitioned into } (m, n) \in II}} S(\mathcal{A}_{p_1 p_2 p_3}^d, p_3) \\ &\quad - \sum_{\substack{(\alpha_1, \alpha_2) \in A \\ \sigma-2\varpi \leq \alpha_3 < \min(\alpha_2, \frac{1-\alpha_1-\alpha_2}{2}) \\ (\alpha_1, \alpha_2, \alpha_3) \text{ cannot be partitioned into } (m, n) \in II}} S(\mathcal{A}_{p_1 p_2 p_3}^d, X^{\sigma-2\varpi}) \\ &\quad + \sum_{\substack{(\alpha_1, \alpha_2) \in A \\ \sigma-2\varpi \leq \alpha_3 < \min(\alpha_2, \frac{1-\alpha_1-\alpha_2}{2}) \\ (\alpha_1, \alpha_2, \alpha_3) \text{ cannot be partitioned into } (m, n) \in II \\ \sigma-2\varpi \leq \alpha_4 < \min(\alpha_3, \frac{1-\alpha_1-\alpha_2-\alpha_3}{2}) \\ (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \text{ can be partitioned into } (m, n) \in II}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}^d, p_4) \\ &\quad + \sum_{\substack{(\alpha_1, \alpha_2) \in A \\ \sigma-2\varpi \leq \alpha_3 < \min(\alpha_2, \frac{1-\alpha_1-\alpha_2}{2}) \\ (\alpha_1, \alpha_2, \alpha_3) \text{ cannot be partitioned into } (m, n) \in II \\ \sigma-2\varpi \leq \alpha_4 < \min(\alpha_3, \frac{1-\alpha_1-\alpha_2-\alpha_3}{2}) \\ (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \text{ cannot be partitioned into } (m, n) \in II}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}^d, p_4) \end{aligned}$$

$$= S_{A1} - S_{A2} - S_{A3} + S_{A4} + S_{A5}. \quad (3)$$

By Lemmas 2.1–2.2 we have asymptotic formulas for S_{A1} and S_{A3} . For S_{A2} and S_{A4} we can also give asymptotic formulas by Lemma 2.3. We discard part of S_{A5} if we can neither give an asymptotic formula nor decompose it further. For the remaining part, we can perform Buchstab's identity twice more to reach a six-dimensional sum if we can group $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ into $(m, n) \in I$, and we can use reversed Buchstab's identity to make some almost-primes visible. Working as in [6] and [8], the total loss from S_A can be bounded by

$$\begin{aligned} & \left(\int_{(t_1, t_2, t_3, t_4) \in S_{A51}} \frac{\omega_1 \left(\frac{1-t_1-t_2-t_3-t_4}{t_4} \right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1 \right) \\ & - \left(\int_{(t_1, t_2, t_3, t_4, t_5) \in S_{A52}} \frac{\omega_0 \left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5} \right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\ & + \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6) \in S_{A53}} \frac{\omega_1 \left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{t_6} \right)}{t_1 t_2 t_3 t_4 t_5 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\ & + \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \in S_{A54}} \frac{\max \left(\frac{t_8}{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7-t_8}, 0.5672 \right)}{t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\ & < 0.002515, \end{aligned} \quad (4)$$

where

$$S_{A51}(t_1, t_2, t_3, t_4) := \{(t_1, t_2) \in S_A,$$

$$\sigma - 2\varpi \leq t_3 < \min \left(t_2, \frac{1}{2}(1 - t_1 - t_2) \right),$$

$$(t_1, t_2, t_3) \text{ cannot be partitioned into } (m, n) \in II,$$

$$\sigma - 2\varpi \leq t_4 < \min \left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3) \right),$$

$$(t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in II,$$

$$(t_1, t_2, t_3, t_4, t_5) \text{ cannot be partitioned into } (m, n) \in I,$$

$$\sigma - 2\varpi \leq t_1 < \frac{1}{2}, \quad \sigma - 2\varpi \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \Big\},$$

$$S_{A52}(t_1, t_2, t_3, t_4, t_5) := \{(t_1, t_2) \in S_A,$$

$$\sigma - 2\varpi \leq t_3 < \min \left(t_2, \frac{1}{2}(1 - t_1 - t_2) \right),$$

$$(t_1, t_2, t_3) \text{ cannot be partitioned into } (m, n) \in II,$$

$$\sigma - 2\varpi \leq t_4 < \min \left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3) \right),$$

$$(t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in II,$$

$$(t_1, t_2, t_3, t_4, t_5) \text{ cannot be partitioned into } (m, n) \in I,$$

$$t_4 < t_5 < \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4),$$

$$(t_1, t_2, t_3, t_4, t_5) \text{ can be partitioned into } (m, n) \in II,$$

$$\sigma - 2\varpi \leq t_1 < \frac{1}{2}, \quad \sigma - 2\varpi \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \Big\},$$

$$S_{A53}(t_1, t_2, t_3, t_4, t_5, t_6) := \{(t_1, t_2) \in S_A,$$

$$\begin{aligned}
& \sigma - 2\varpi \leq t_3 < \min \left(t_2, \frac{1}{2}(1 - t_1 - t_2) \right), \\
& (t_1, t_2, t_3) \text{ cannot be partitioned into } (m, n) \in II, \\
& \sigma - 2\varpi \leq t_4 < \min \left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3) \right), \\
& (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in II, \\
& (t_1, t_2, t_3, t_4, t_5) \text{ can be partitioned into } (m, n) \in I, \\
& \sigma - 2\varpi \leq t_5 < \min \left(t_4, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4) \right), \\
& (t_1, t_2, t_3, t_4, t_5) \text{ cannot be partitioned into } (m, n) \in II, \\
& \sigma - 2\varpi \leq t_6 < \min \left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5) \right), \\
& (t_1, t_2, t_3, t_4, t_5, t_6) \text{ cannot be partitioned into } (m, n) \in II, \\
& (t_1, t_2, t_3, t_4, t_5, t_6, t_7) \text{ cannot be partitioned into } (m, n) \in I, \\
& \left. \sigma - 2\varpi \leq t_1 < \frac{1}{2}, \sigma - 2\varpi \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \right\}, \\
S_{A54}(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) := & \{(t_1, t_2) \in S_A, \\
& \sigma - 2\varpi \leq t_3 < \min \left(t_2, \frac{1}{2}(1 - t_1 - t_2) \right), \\
& (t_1, t_2, t_3) \text{ cannot be partitioned into } (m, n) \in II, \\
& \sigma - 2\varpi \leq t_4 < \min \left(t_3, \frac{1}{2}(1 - t_1 - t_2 - t_3) \right), \\
& (t_1, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in II, \\
& (t_1, t_2, t_3, t_4, t_5) \text{ can be partitioned into } (m, n) \in I, \\
& \sigma - 2\varpi \leq t_5 < \min \left(t_4, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4) \right), \\
& (t_1, t_2, t_3, t_4, t_5) \text{ cannot be partitioned into } (m, n) \in II, \\
& \sigma - 2\varpi \leq t_6 < \min \left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5) \right), \\
& (t_1, t_2, t_3, t_4, t_5, t_6) \text{ cannot be partitioned into } (m, n) \in II, \\
& (t_1, t_2, t_3, t_4, t_5, t_6, t_7) \text{ can be partitioned into } (m, n) \in I, \\
& \sigma - 2\varpi \leq t_7 < \min \left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5 - t_6) \right), \\
& (t_1, t_2, t_3, t_4, t_5, t_6, t_7) \text{ cannot be partitioned into } (m, n) \in II, \\
& \sigma - 2\varpi \leq t_8 < \min \left(t_5, \frac{1}{2}(1 - t_1 - t_2 - t_3 - t_4 - t_5 - t_6 - t_7) \right), \\
& (t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \text{ cannot be partitioned into } (m, n) \in II, \\
& \left. \sigma - 2\varpi \leq t_1 < \frac{1}{2}, \sigma - 2\varpi \leq t_2 < \min \left(t_1, \frac{1}{2}(1 - t_1) \right) \right\}.
\end{aligned}$$

For S_B we use Buchstab's identity to get

$$S_B = \sum_{(\alpha_1, \alpha_2) \in B} S(\mathcal{A}_{p_1 p_2}^d, X^{\sigma-2\varpi}) - \sum_{\substack{(\alpha_1, \alpha_2) \in B \\ \sigma-2\varpi \leq \alpha_3 < \min(\alpha_2, \frac{1-\alpha_1-\alpha_2}{2})}} S(\mathcal{A}_{p_1 p_2 p_3}^d, p_3). \quad (5)$$

Here we cannot decompose it directly using Buchstab's identity once more, since we cannot give an asymptotic formula for part of the negative sum

$$\sum_{\substack{(\alpha_1, \alpha_2) \in B \\ \sigma - 2\varpi \leq \alpha_3 < \min\left(\alpha_2, \frac{1-\alpha_1-\alpha_2}{2}\right) \\ (\alpha_1, \alpha_2, \alpha_3) \text{ cannot be partitioned into } (m, n) \in II}} S(\mathcal{A}_{p_1 p_2 p_3}^d, X^{\sigma-2\varpi}).$$

However, we can use a role-reversal to transfer the last sum in (5) into a form that have an asymptotic formula. Note that in [10] role-reversals were not used. We refer the readers to [7] and [8] for more applications of role-reversals. By a standard process, we have (where $\beta \sim X^{1-\alpha_1-\alpha_2-\alpha_3}$ and $(\beta, P(p_3)) = 1$)

$$\begin{aligned} S_B &= \sum_{(\alpha_1, \alpha_2) \in B} S(\mathcal{A}_{p_1 p_2}^d, p_2) \\ &= \sum_{(\alpha_1, \alpha_2) \in B} S(\mathcal{A}_{p_1 p_2}^d, X^{\sigma-2\varpi}) - \sum_{\substack{(\alpha_1, \alpha_2) \in B \\ \sigma - 2\varpi \leq \alpha_3 < \min\left(\alpha_2, \frac{1-\alpha_1-\alpha_2}{2}\right) \\ (\alpha_1, \alpha_2, \alpha_3) \text{ can be partitioned into } (m, n) \in II}} S(\mathcal{A}_{p_1 p_2 p_3}^d, p_3) \\ &\quad - \sum_{\substack{(\alpha_1, \alpha_2) \in B \\ \sigma - 2\varpi \leq \alpha_3 < \min\left(\alpha_2, \frac{1-\alpha_1-\alpha_2}{2}\right) \\ (\alpha_1, \alpha_2, \alpha_3) \text{ cannot be partitioned into } (m, n) \in II}} S(\mathcal{A}_{\beta p_2 p_3}^d, X^{\sigma-2\varpi}) \\ &\quad + \sum_{\substack{(\alpha_1, \alpha_2) \in B \\ \sigma - 2\varpi \leq \alpha_3 < \min\left(\alpha_2, \frac{1-\alpha_1-\alpha_2}{2}\right) \\ (\alpha_1, \alpha_2, \alpha_3) \text{ cannot be partitioned into } (m, n) \in II \\ \sigma - 2\varpi \leq \alpha_4 < \frac{1}{2}\alpha_1 \\ (1-\alpha_1-\alpha_2-\alpha_3, \alpha_2, \alpha_3, \alpha_4) \text{ can be partitioned into } (m, n) \in II}} S(\mathcal{A}_{\beta p_2 p_3 p_4}^d, p_4) \\ &\quad + \sum_{\substack{(\alpha_1, \alpha_2) \in B \\ \sigma - 2\varpi \leq \alpha_3 < \min\left(\alpha_2, \frac{1-\alpha_1-\alpha_2}{2}\right) \\ (\alpha_1, \alpha_2, \alpha_3) \text{ cannot be partitioned into } (m, n) \in II \\ \sigma - 2\varpi \leq \alpha_4 < \frac{1}{2}\alpha_1 \\ (1-\alpha_1-\alpha_2-\alpha_3, \alpha_2, \alpha_3, \alpha_4) \text{ cannot be partitioned into } (m, n) \in II}} S(\mathcal{A}_{\beta p_2 p_3 p_4}^d, p_4) \\ &= S_{B1} - S_{B2} - S_{B3} + S_{B4} + S_{B5}. \end{aligned} \tag{6}$$

We can give asymptotic formulas for $S_{B1}-S_{B4}$ by Lemmas 2.1–2.3. We can also decompose part of S_{B5} if the variables (with α_1 replaced by $1-\alpha_1-\alpha_2-\alpha_3$) satisfy the same conditions as in the decomposable part of S_{B5} . Again, the total loss from S_B can be bounded by

$$\begin{aligned} &\left(\int_{(t_1, t_2, t_3, t_4) \in S_{B51}} \frac{\omega_1\left(\frac{t_1-t_4}{t_4}\right) \omega_1\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2 t_3^2 t_4^2} dt_4 dt_3 dt_2 dt_1 \right) \\ &- \left(\int_{(t_1, t_2, t_3, t_4, t_5) \in S_{B52}} \frac{\omega_0\left(\frac{t_1-t_4-t_5}{t_5}\right) \omega_0\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2 t_3^2 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\ &+ \left(\int_{(t_1, t_2, t_3, t_4, t_5, t_6) \in S_{B53}} \frac{\omega_1\left(\frac{t_1-t_4-t_5-t_6}{t_6}\right) \omega_1\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2 t_3^2 t_4 t_5 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\ &< 0.006249, \end{aligned} \tag{7}$$

where

$$\begin{aligned} S_{B51}(t_1, t_2, t_3, t_4) &:= \{(t_1, t_2) \in S_B, \\ &\quad \sigma - 2\varpi \leq t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \end{aligned}$$

$$\begin{aligned}
& (t_1, t_2, t_3) \text{ cannot be partitioned into } (m, n) \in II, \\
& \sigma - 2\varpi \leq t_4 < \frac{1}{2}t_1, \\
& (1 - t_1 - t_2 - t_3, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in II, \\
& (1 - t_1 - t_2 - t_3, t_2, t_3, t_4, t_4) \text{ cannot be partitioned into } (m, n) \in I, \\
& \left. \sigma - 2\varpi \leq t_1 < \frac{1}{2}, \sigma - 2\varpi \leq t_2 < \min\left(t_1, \frac{1}{2}(1 - t_1)\right) \right\}, \\
S_{B52}(t_1, t_2, t_3, t_4, t_5) &:= \{(t_1, t_2) \in S_B, \\
& \sigma - 2\varpi \leq t_3 < \min\left(t_2, \frac{1}{2}(1 - t_1 - t_2)\right), \\
& (t_1, t_2, t_3) \text{ cannot be partitioned into } (m, n) \in II, \\
& \sigma - 2\varpi \leq t_4 < \frac{1}{2}t_1, \\
& (1 - t_1 - t_2 - t_3, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in II, \\
& (1 - t_1 - t_2 - t_3, t_2, t_3, t_4, t_4) \text{ cannot be partitioned into } (m, n) \in I, \\
& t_4 < t_5 < \frac{1}{2}(t_1 - t_4), \\
& (1 - t_1 - t_2 - t_3, t_2, t_3, t_4, t_5) \text{ can be partitioned into } (m, n) \in II, \\
& \left. \sigma - 2\varpi \leq t_1 < \frac{1}{2}, \sigma - 2\varpi \leq t_2 < \min\left(t_1, \frac{1}{2}(1 - t_1)\right) \right\}, \\
S_{B53}(t_1, t_2, t_3, t_4, t_5, t_6) &:= \{(t_1, t_2) \in S_B, \\
& \sigma - 2\varpi \leq t_3 < \min\left(t_2, \frac{1}{2}(1 - t_1 - t_2)\right), \\
& (t_1, t_2, t_3) \text{ cannot be partitioned into } (m, n) \in II, \\
& \sigma - 2\varpi \leq t_4 < \frac{1}{2}t_1, \\
& (1 - t_1 - t_2 - t_3, t_2, t_3, t_4) \text{ cannot be partitioned into } (m, n) \in II, \\
& (1 - t_1 - t_2 - t_3, t_2, t_3, t_4, t_4) \text{ can be partitioned into } (m, n) \in I, \\
& \sigma - 2\varpi \leq t_5 < \min\left(t_4, \frac{1}{2}(t_1 - t_4)\right), \\
& (1 - t_1 - t_2 - t_3, t_2, t_3, t_4, t_5) \text{ cannot be partitioned into } (m, n) \in II, \\
& \sigma - 2\varpi \leq t_6 < \min\left(t_5, \frac{1}{2}(t_1 - t_4 - t_5)\right), \\
& (1 - t_1 - t_2 - t_3, t_2, t_3, t_4, t_5, t_6) \text{ cannot be partitioned into } (m, n) \in II, \\
& \left. \sigma - 2\varpi \leq t_1 < \frac{1}{2}, \sigma - 2\varpi \leq t_2 < \min\left(t_1, \frac{1}{2}(1 - t_1)\right) \right\}.
\end{aligned}$$

For S_C we cannot use Buchstab's identity in a straightforward manner, but we can use Buchstab's identity in reverse to make some almost-primes visible. The details of using Buchstab's identity in reverse are similar to those in [7] and [8]. By using Buchstab's identity in reverse twice, we have

$$\begin{aligned}
S_C &= \sum_{(\alpha_1, \alpha_2) \in C} S(\mathcal{A}_{p_1 p_2}^d, p_2) \\
&= \sum_{(\alpha_1, \alpha_2) \in C} S\left(\mathcal{A}_{p_1 p_2}^d, 2\left(\frac{X}{p_1 p_2}\right)^{\frac{1}{2}}\right)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{\substack{(\alpha_1, \alpha_2) \in C \\ \alpha_2 < \alpha_3 < \frac{1-\alpha_1-\alpha_2}{2} \\ (\alpha_1, \alpha_2, \alpha_3) \text{ can be partitioned into } (m, n) \in II}} S(\mathcal{A}_{p_1 p_2 p_3}^d, p_3) \\
& + \sum_{\substack{(\alpha_1, \alpha_2) \in C \\ \alpha_2 < \alpha_3 < \frac{1-\alpha_1-\alpha_2}{2} \\ (\alpha_1, \alpha_2, \alpha_3) \text{ cannot be partitioned into } (m, n) \in II}} S\left(\mathcal{A}_{p_1 p_2 p_3}^d, 2\left(\frac{X}{p_1 p_2 p_3}\right)^{\frac{1}{2}}\right) \\
& + \sum_{\substack{(\alpha_1, \alpha_2) \in C \\ \alpha_2 < \alpha_3 < \frac{1-\alpha_1-\alpha_2}{2} \\ (\alpha_1, \alpha_2, \alpha_3) \text{ cannot be partitioned into } (m, n) \in II \\ \alpha_3 < \alpha_4 < \frac{1-\alpha_1-\alpha_2-\alpha_3}{2} \\ (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \text{ can be partitioned into } (m, n) \in II}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}^d, p_4) \\
& + \sum_{\substack{(\alpha_1, \alpha_2) \in C \\ \alpha_2 < \alpha_3 < \frac{1-\alpha_1-\alpha_2}{2} \\ (\alpha_1, \alpha_2, \alpha_3) \text{ cannot be partitioned into } (m, n) \in II \\ \alpha_3 < \alpha_4 < \frac{1-\alpha_1-\alpha_2-\alpha_3}{2} \\ (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \text{ cannot be partitioned into } (m, n) \in II}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}^d, p_4) \\
& = S_{C1} + S_{C2} + S_{C3} + S_{C4} + S_{C5}.
\end{aligned} \tag{8}$$

We can give asymptotic formulas for S_{C2} and S_{C4} by Lemma 2.3, hence we can subtract them from the loss. In this way we obtain a loss from S_C of

$$\begin{aligned}
& \left(\int_{(t_1, t_2) \in S_C} \frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1 t_2^2} dt_2 dt_1 \right) \\
& - \left(\int_{(t_1, t_2, t_3) \in S_{C2}} \frac{\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 dt_2 dt_1 \right) \\
& - \left(\int_{(t_1, t_2, t_3, t_4) \in S_{C4}} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1 \right) \\
& < 0.990258,
\end{aligned} \tag{9}$$

where

$$\begin{aligned}
S_{C2}(t_1, t_2, t_3) &:= \left\{ (t_1, t_2) \in S_C, \ t_2 < t_3 < \frac{1}{2}(1-t_1-t_2), \right. \\
& \quad (t_1, t_2, t_3) \text{ can be partitioned into } (m, n) \in II, \\
& \quad \left. \sigma - 2\varpi \leq t_1 < \frac{1}{2}, \ \sigma - 2\varpi \leq t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \right\}, \\
S_{C4}(t_1, t_2, t_3, t_4) &:= \left\{ (t_1, t_2) \in S_C, \ t_2 < t_3 < \frac{1}{2}(1-t_1-t_2), \right. \\
& \quad (t_1, t_2, t_3) \text{ cannot be partitioned into } (m, n) \in II, \\
& \quad t_3 < t_4 < \frac{1}{2}(1-t_1-t_2-t_3), \\
& \quad (t_1, t_2, t_3, t_4) \text{ can be partitioned into } (m, n) \in II, \\
& \quad \left. \sigma - 2\varpi \leq t_1 < \frac{1}{2}, \ \sigma - 2\varpi \leq t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \right\}.
\end{aligned}$$

Finally, by (2)–(9), the total loss is less than

$$0.990258 + 0.002515 + 0.006249 < 0.9991 < 1$$

and the proof of Theorem 1.1 is completed.

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