THE NUMBER OF PRIMES IN SHORT INTERVALS AND NUMERICAL CALCULATIONS FOR HARMAN'S SIEVE

RUNBO LI

Abstract. The author gives nontrivial upper and lower bounds for the number of primes in the interval $[x-x^{\theta}, x]$ for some $0.52 \leqslant \theta < 0.525$, showing that the interval $[x-x^{0.52}, x]$ contains prime numbers for all sufficiently large x. This refines a result of Baker, Harman and Pintz (2001) and gives an affirmative answer to Harman and Pintz's argument. Guth and Maynard's large value estimate, new arithmetical information, a delicate sieve decomposition, various techniques in Harman's sieve, and accurate estimates for integrals are used to good effect.

Contents

1. Introduction

One of the famous topics in number theory is to find prime numbers in short intervals. In 1937, Cramér [10] conjectured that every interval $[x-f(x)(\log x)^2, x]$ contains prime numbers for some $f(x) \to 1$ as $x \to \infty$. The Riemann Hypothesis implies that for all sufficiently large x, the interval $[x-x^{\theta}, x]$ contains $\sim x^{\theta}(\log x)^{-1}$ prime numbers for every $\frac{1}{2} < \theta \leqslant 1$. The first unconditional result of this asymptotic formula with some $\theta < 1$ was proved by Hoheisel [26] in 1930 with $\theta \geqslant 1 - \frac{1}{33000}$. After the works of Hoheisel [26], Heilbronn [25], Chudakov [9], Ingham [28] and Montgomery [42], Huxley [27] proved in 1972 that the above asymptotic formula holds when $\theta > \frac{7}{12}$ by his zero density estimate. In 2024, Guth and Maynard [15] improved this to $\theta > \frac{17}{30}$ by a new zero density estimate.

In 1979, Iwaniec and Jutila [30] first introduced a sieve method into this problem. They established a lower bound with correct order of magnitude (instead of an asymptotic formula) with $\theta = \frac{13}{23}$. After that breakthrough, many improvements were made and the value of θ was reduced successively to

$$\frac{5}{9} \approx 0.5556,\ \frac{11}{20} = 0.5500,\ \frac{17}{31} \approx 0.5484,\ \frac{23}{42} \approx 0.5476,$$
$$\frac{1051}{1920} \approx 0.5474,\ \frac{35}{64} \approx 0.5469,\ \frac{6}{11} \approx 0.5455 \text{ and } \frac{7}{13} \approx 0.5385$$

by Iwaniec and Jutila [30], Heath-Brown and Iwaniec [24], Pintz [45] [46], Iwaniec and Pintz [31], Mozzochi [44] and Lou and Yao [39] [40] [41] respectively.

In 1996, Baker and Harman [3] presented an alternative approach to this problem. They used the alternative sieve developed by Harman [16] [17] to reduce θ to 0.535. Finally, Baker, Harman and Pintz (BHP) [5] further developed this sieve process and combined it with Watt's mean value theorem on Dirichlet polynomials [53] and showed $\theta \geqslant 0.525$. As Friedlander and Iwaniec mentioned in their book [[13], Chapter 23], "their method uses many powerful tools and arguments, both analytic and combinatorial, and these are extremely complicated." However, they omitted almost all calculation details in [3] and [5], which makes the papers very hard to read and check. In 2014, Pintz [48] pointed out that "the Baker–Harman–Pintz result with $\theta = 0.525$ actually leads to a slightly better value" in his lecture. Harman [[20], Chapter 7.10] also mentioned that $\theta = 0.52$ might be achievable with an incredibly long and boring argument. In a personal communication, Kumchev mentioned that BHP tried and discovered that 0.52 was out of reach of the existing techniques. In 2024, Starichkova [49] provided full details

2020 Mathematics Subject Classification. **11N05**, **11N35**, **11N36**.
Key words and phrases. **Prime**, **Sieve methods**, **Short intervals**.

of [3] in her PhD thesis, so we turn our attention to [5]. In this paper, we provide the calculation details and sharpen the main theorem proved in [5].

Theorem 1. *For all sufficiently large x, the interval $[x - x^{0.52}, x]$ contains prime numbers.*

Theorem 1 is a direct corollary of the following result, which gives nontrivial upper and lower bounds with correct order of magnitude for the number of primes in intervals of length between $x^{0.52}$ and $x^{0.525}$. Here we say a trivial upper bound is the bound with upper constant $\frac{2}{\theta}$ obtained by Montgomery and Vaughan [43], and a trivial lower bound is of course zero. Note that in [29], [38], [39] and [4] nontrivial upper bounds for the number of primes in other intervals are also given.

Theorem 2. *Let $0.52 \leqslant \theta < 0.525$ and $\varepsilon > 0$. Then we have*

$$\mathbf{LB}(\theta)\frac{x^{\theta+\varepsilon}}{\log x} \leqslant \pi(x) - \pi(x - x^{\theta+\varepsilon}) \leqslant \mathbf{UB}(\theta)\frac{x^{\theta+\varepsilon}}{\log x}$$

for all sufficiently large x, where the values of functions $\mathbf{LB}(\theta)$ and $\mathbf{UB}(\theta)$ satisfy the following condition table.

$\boldsymbol{\theta}$	$\mathbf{LB}(\boldsymbol{\theta})$	$\mathbf{UB}(\boldsymbol{\theta})$
0.520	> 0.0300	< 2.7626
0.521	> 0.1006	< 2.6484
0.522	> 0.1493	< 2.5630
0.523	> 0.2221	< 2.4597
0.524	> 0.2650	< 2.3759

Obviously, our result confirms Harman and Pintz's argument and breaks the 0.52-barrier mentioned by Kumchev. Although our upper constant for $\theta = 0.52$ is a little bit weaker than Iwaniec's (see Section 6 of [29]), our result comes from Harman's sieve, which leads to much better results for intervals longer than $x^{0.522}$. The most powerful arithmetical information input in our paper is a new large value estimate due to Guth and Maynard [15] (which was used in the proof of Lemma 4.2). We also prove some new arithmetical information outside of those in [5] and [20] without using Guth–Maynard's result, see Lemmas 4.5–4.6. We also find that Lemma 18 in [5] and an improved version of Lemma 7.22 in [20] actually cover some non-overlapping three-dimensional regions when $\theta \geqslant 0.522$, so using them simultaneously yields a better result. All the numerical values of integrals in Sections 5 and 6 are calculated by Mathematica 14. We use an Intel(R) Xeon(R) Platinum 8383C CPU with 80 Wolfram kernels and a Montage Jintide(R) C6248R CPU with 48 Wolfram kernels to run the code, and it took about several months to give all numerical values.

Throughout this paper, we always suppose that K is a sufficiently large positive constant, ε is a sufficiently small positive constant and x is a sufficiently large integer. Let θ be a positive number which will be fixed later. The letter p, with or without subscript, is reserved for prime numbers. We write $m \sim M$ to mean that $M \leqslant m < 2M$. We use $M(s)$, $N(s)$, $R(s)$ and some other capital letters (with or without subscript) to denote some divisor-bounded Dirichlet polynomials

$$M(s) = \sum_{m \sim M} a_m m^{-s}, \quad N(s) = \sum_{n \sim N} b_n n^{-s}, \quad R(s) = \sum_{r \sim R} c_r r^{-s}.$$

We say a Dirichlet polynomial $M(s)$ is prime-factored if its coefficient is the characteristic function of primes or of numbers with a bounded number of prime factors restricted to certain ranges. That is, we have

$$\left| M\left(\frac{1}{2} + it\right) \right| \ll M^{\frac{1}{2}} (\log x)^{-K}$$

for $\exp\left((\log x)^{1/3}\right) < |t| < x^{1-\theta+\varepsilon}$ and the least prime factor of m is $\gg \exp\left((\log x)^{4/5}\right)$ if $M(s)$ is prime-factored. We also say a Dirichlet polynomial is decomposable if it can be written as the form

$$\sum_{p_i \sim P_i} (p_1 \ldots p_u)^{-s}.$$

(That is, we can decompose this Dirichlet polynomial.) We define the boolean function as

$$\texttt{Boole}(\mathbf{X}) = \begin{cases} 1 & \text{if } \mathbf{X} \text{ is true,} \\ 0 & \text{if } \mathbf{X} \text{ is false.} \end{cases} \tag{1}$$

2. An outline of the proof

Let $0.505 \leqslant \theta \leqslant 0.535$, $y = x^{\theta+\varepsilon}$, $y_1 = x \exp\left(-3(\log x)^{1/3}\right)$,

$$\mathcal{A} = \{a : a \in \mathbb{Z},\ x - y \leqslant a < x\}, \quad \mathcal{B} = \{b : b \in \mathbb{Z},\ x - y_1 \leqslant b < x\},$$

$$\mathcal{A}_d = \{a : ad \in \mathcal{A}\}, \quad \mathcal{B}_d = \{b : bd \in \mathcal{B}\}, \quad P(z) = \prod_{p<z} p, \quad S(\mathcal{A}, z) = \sum_{\substack{a \in \mathcal{A} \\ (a, P(z))=1}} 1, \quad S(\mathcal{B}, z) = \sum_{\substack{b \in \mathcal{B} \\ (b, P(z))=1}} 1.$$

Then we have

$$\pi(x) - \pi(x - y) = S\left(\mathcal{A}, x^{\frac{1}{2}}\right). \tag{2}$$

In order to prove Theorem 2, we only need to give upper and lower bounds for $S\left(\mathcal{A}, x^{\frac{1}{2}}\right)$. Our aim is to show that the sparser set $\mathcal{A}$ contains the expected proportion of primes compared to the bigger set $\mathcal{B}$, which requires us to decompose $S\left(\mathcal{A}, x^{\frac{1}{2}}\right)$ and prove asymptotic formulas of the form

$$S(\mathcal{A}, z)=\frac{y}{y_1}(1+o(1)) S(\mathcal{B}, z) \tag{3}$$

for some parts of it, and drop the other parts. The dropped parts must be positive for the lower bound case, and they must be negative for the upper bound case. In Section 3 we provide asymptotic formulas for terms of the form $S\left(\mathcal{A}_{p_1 \ldots p_n}, x^\nu\right)$ (which requires both Type-I and Type-II information) and in Section 4 we provide asymptotic formulas for terms of the form $S\left(\mathcal{A}_{p_1 \ldots p_n}, p_n\right)$ (which only requires Type-II information). In Sections 5 and 6 we will make further use of Buchstab's identity to decompose $S\left(\mathcal{A}, x^{\frac{1}{2}}\right)$ and prove Theorem 2 for $\theta=0.52$. We omit the proof of numerical bounds for other values of θ for the sake of simplicity.

3. Sieve asymptotic formulas I

Now we follow [5] directly to get some sieve asymptotic formulas. For a positive integer h, we define the interval

$$I_h=\left[\frac{1}{2}-2 h\left(\theta-\frac{1}{2}\right), \frac{1}{2}-(2 h-2)\left(\theta-\frac{1}{2}\right)\right), \tag{4}$$

and we define the piecewise-linear function $\nu=\nu(\alpha)$ and the function $\alpha^*=\alpha^*(\alpha)$, $0 \leqslant \alpha \leqslant \frac{1}{2}$ as follows: if $\alpha \in I_h$ then

$$\nu(\alpha)=\min \left(\frac{2(\theta-\alpha)}{2 h-1}, \gamma(\theta)\right) \text { for } h \geqslant 1, \tag{5}$$

$$\alpha^*=\max \left(\frac{2 h(1-\theta)-\alpha}{2 h-1}, \frac{2(h-1) \theta+\alpha}{2 h-1}\right), \tag{6}$$

where $\gamma(\theta)$ will be defined in the next section. Note that we have $\nu(\alpha) \geqslant 2 \theta-1$ and $1-\theta \leqslant \alpha^* \leqslant \frac{1}{2}+\varepsilon$.

Now we provide some lemmas which will be used to give asymptotic formulas for sieve functions of the form $S\left(\mathcal{A}_{p_1 \ldots p_n}, x^\nu\right)$. The next two lemmas can be deduced from [[20], Lemma 7.3], some combinatorial lemmas together with [[5], Lemma 2] which is a generalized version of Watt's theorem [53].

Lemma 3.1. *([[5], Lemma 12], [[20], Lemma 7.15]). Let $M=x^{\alpha_1}, N=x^{\alpha_2}$ where $M(s)$ and $N(s)$ are decomposable. Suppose that $\alpha_1 \leqslant \frac{1}{2}$ and*

$$\alpha_2 \leqslant \min \left(\frac{3 \theta+1-4 \alpha_1^*}{2}, \frac{3+\theta-4 \alpha_1^*}{5}\right)-2 \varepsilon .$$

Then

$$\sum_{\substack{m \sim M \\ n \sim N}} a_m b_n S\left(\mathcal{A}_{m n}, x^\nu\right)=\frac{y}{y_1}(1+o(1)) \sum_{\substack{m \sim M \\ n \sim N}} a_m b_n S\left(\mathcal{B}_{m n}, x^\nu\right)$$

holds for every $\nu \leqslant \nu(\alpha_1)$.

Lemma 3.2. *([[5], Lemma 13], [[20], Lemma 7.16]). Let $M=x^{\alpha_1}, N_1=x^{\alpha_2}, N_2=x^{\alpha_3}$ where $M(s), N_1(s)$ and $N_2(s)$ are decomposable. Suppose that $\alpha_1 \leqslant \frac{1}{2}$ and either*

$$2 \alpha_2+\alpha_3 \leqslant 1+\theta-2 \alpha_1^*-2 \varepsilon, \quad \alpha_3 \leqslant \frac{1+3 \theta}{4}-\alpha_1^*-\varepsilon, \quad 2 \alpha_2+3 \alpha_3 \leqslant \frac{3+\theta}{2}-2 \alpha_1^*-2 \varepsilon$$

or

$$\alpha_2 \leqslant \frac{1-\theta}{2}, \quad \alpha_3 \leqslant \frac{1+3 \theta-4 \alpha_1^*}{8}-\varepsilon .$$

Then

$$\sum_{\substack{m \sim M \\ n_1 \sim N_1 \\ n_2 \sim N_2}} a_m b_{n_1} c_{n_2} S\left(\mathcal{A}_{m n_1 n_2}, x^\nu\right)=\frac{y}{y_1}(1+o(1)) \sum_{\substack{m \sim M \\ n_1 \sim N_1 \\ n_2 \sim N_2}} a_m b_{n_1} c_{n_2} S\left(\mathcal{B}_{m n_1 n_2}, x^\nu\right)$$

holds for every $\nu \leqslant \nu(\alpha_1)$.

The next lemma is obtained by a two-dimensional sieve together with Lemma 3.1, and they will help us deal with the regions A_2 and $A_2^{\prime}$ in Section 6.

Lemma 3.3. *Let $M_1=x^{\alpha_1}, M_2=x^{\alpha_2}$. Suppose that*

$$\alpha_2 \leqslant \alpha_1,\ 2 \alpha_1+\alpha_2<1 \text { and } \alpha_2<\frac{7}{2} \theta-\frac{3}{2} .$$

Then

$$\sum_{\substack{p_1 \sim M_1 \\ p_2 \sim M_2}} S\left(\mathcal{A}_{p_1 p_2}, x^\nu\right)=\frac{y}{y_1}(1+o(1)) \sum_{\substack{p_1 \sim M_1 \\ p_2 \sim M_2}} S\left(\mathcal{B}_{p_1 p_2}, x^\nu\right)$$

holds for $\nu=2 \theta-1$.

Proof. The proof of Lemma 3.3 is very similar to the proofs of [[5], Lemma 16] and [[20], Lemma 7.19], so we omit it here. □

Remark 3.4. For $\theta>\frac{11}{21} \approx 0.5238$, the third condition in Lemma 3.3 can be simplified to $\alpha_2<\frac{1}{3}$.

Remark 3.5. One may use existing results on higher power moments of zeta function to get a minor improvement. For example, it is possible to use Heath-Brown's twelfth power moment [23] together with Hölder's inequality (or results in [[51], Section 2.1]) and mean value theorem to get an improvement on [[20], Lemmas 7.9 and 7.10] which are essential in proving Lemma 3.2. Here we do not consider about them for the sake of simplicity.

4. Sieve asymptotic formulas II

In this section we give asymptotic formulas for sieve functions of the form $S\left(\mathcal{A}_{p_1\ldots p_n}, p_n\right)$ or more general sums. These lemmas can be deduced from [[5], Lemma 6] together with some mean and large value theorems of Dirichlet polynomials. The first lemma in this section is cited directly from BHP's paper [5].

Lemma 4.1. *([[5], Lemma 9], [[20], Lemma 7.3]). Let $M = x^{\alpha_1}, N = x^{\alpha_2}$ with*

$$|\alpha_1 - \alpha_2| < 2\theta - 1, \quad \alpha_1 + \alpha_2 > 1 - \gamma(\theta)$$

where

$$\gamma(\theta) = \max_{g \in \mathbb{N}} \gamma_g(\theta),$$
$$\gamma_g(\theta) = \min\left(4\theta - 2,\ \frac{(8g-4)\theta - (4g-3)}{4g-1},\ \frac{24g\theta - (12g+1)}{4g-1}\right).$$

Moreover, let $R = x^{1-\alpha_1-\alpha_2}$ and suppose that $R(s)$ is prime-factored. Then we can obtain an asymptotic formula for

$$\sum_{\substack{mnr \in \mathcal{A} \\ m \sim M \\ n \sim N \\ r \sim R}} a_m b_n c_r \quad \text{or specially} \quad \sum_{\substack{p_j \sim x^{\alpha_j} \\ 1 \leqslant j \leqslant 2}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right).$$

Note that the dependencies between variables in the sieve functions (here and many others below) can be removed using a truncated Perron's formula as in [[6], Lemma 11]. Moreover, we have

$$\gamma(\theta) = \begin{cases} \gamma_6(\theta) = 4\theta - 2, & 0.52 \leqslant \theta < \frac{25}{48} \approx 0.5208, \\ \gamma_6(\theta) = \frac{44\theta - 21}{23}, & \frac{25}{48} \leqslant \theta < \frac{251}{481} \approx 0.5218, \\ \gamma_5(\theta) = \frac{120\theta - 61}{19}, & \frac{251}{481} \leqslant \theta < \frac{23}{44} \approx 0.5227, \\ \gamma_5(\theta) = 4\theta - 2, & \frac{23}{44} \leqslant \theta < 0.525. \end{cases}$$

The next lemma is the most crucial Type-II information input in this paper.

Lemma 4.2. *Let $g \in \mathbb{N}$, $MNR = x$, $M = x^{\alpha_1}, N = x^{\alpha_2}$ and $g \geqslant 2$. Suppose that we have*

$$3\alpha_1 - 2\alpha_2 < 5\theta - 2, \quad 3\alpha_2 - 2\alpha_1 < 5\theta - 2,$$
$$\alpha_1 > -\frac{5(g-1)(\theta-1)}{5g-2}, \quad \alpha_2 > -\frac{5(g-1)(\theta-1)}{5g-2},$$
$$\frac{1}{3}\alpha_1 + \left(\frac{1}{15g} - \frac{1}{6}\right)\alpha_2 < \frac{\theta}{2} - \frac{1}{6}, \quad \frac{1}{3}\alpha_2 + \left(\frac{1}{15g} - \frac{1}{6}\right)\alpha_1 < \frac{\theta}{2} - \frac{1}{6}$$

and

$$1 - \alpha_1 - \alpha_2 < \min\left(5\theta - \frac{5}{2},\ \frac{10g\theta - 5\theta + (4-5g)}{5g-1},\ \frac{30g\theta - (15g+1)}{5g-1}\right).$$

Then we can obtain an asymptotic formula for

$$\sum_{\substack{mnr \in \mathcal{A} \\ m \sim M \\ n \sim N \\ r \sim R}} a_m b_n c_r \quad \text{or specially} \quad \sum_{\substack{p_j \sim x^{\alpha_j} \\ 1 \leqslant j \leqslant 2}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right).$$

Moreover, when $\theta = 0.52$, we use this lemma with $g = 6, 7, \ldots, 12$. The union of those asymptotic regions fully covers the asymptotic region given by Lemma 4.1 when $\theta = 0.52$.

Proof. We adopt the notation used in [[20], Chapter 7.2]. It suffices to prove that for any set $\mathcal{S}$ such that

$$t_1, t_2 \in \mathcal{S} \Rightarrow \exp\left((\log x)^{1/3}\right) < t_1, t_2 < x^{1-\theta+\varepsilon} := T,\ |t_1 - t_2| > 1$$

and, for $t \in \mathcal{S}$,

$$|M(s)| \sim M^{\sigma_1 - \frac{1}{2}}, \quad |N(s)| \sim N^{\sigma_2 - \frac{1}{2}}, \quad |R(s)| \sim R^{\sigma_3 - \frac{1}{2}},$$

we have

$$V := |\mathcal{S}|\, M^{\sigma_1} N^{\sigma_2} R^{\sigma_3} \ll x(\log x)^{-A}.$$

Let $S = |\mathcal{S}| (\log x)^{-B}$ for a fixed B. The classical Mean Value Theorem for Dirichlet polynomials and the Halász–Montgomery–Huxley large values estimate together give that

$$S \ll M^{2-2\sigma_1} + TM^{\min(1-2\sigma_1, 4-6\sigma_1)}.$$

Guth–Maynard's new large values estimate, the main result in [15], gives that

$$S \ll M^{\min\left(2-2\sigma_1, \frac{18}{5}-4\sigma_1\right)} + TM^{\frac{12}{5}-4\sigma_1}.$$

Combining the above two estimates, we have

$$S \ll M^{\min\left(2-2\sigma_1, \frac{18}{5}-4\sigma_1\right)} + TM^{f(\sigma_1)},$$

where

$$f(\sigma) = \min\left(1-2\sigma, 4-6\sigma, \frac{12}{5}-4\sigma\right).$$

Similarly, we have

$$S \ll N^{\min\left(2-2\sigma_2, \frac{18}{5}-4\sigma_2\right)} + TN^{f(\sigma_2)}$$

and

$$S \ll (R^g)^{\min\left(2-2\sigma_3, \frac{18}{5}-4\sigma_3\right)} + T(R^g)^{f(\sigma_3)}.$$

We remark that we also have the simple bound

$$S \ll (R^g)^{2-2\sigma_3} + T(R^g)^{4-6\sigma_3}.$$

We also note that if we obtain three different bounds S_1, S_2, S_3 for S by the above, then we have $S \ll S_1^a S_2^b S_3^c$ for any non-negative a, b, c satisfying $a+b+c=1$.

Now we want to find a function $c(\sigma)$ such that

$$M^{c(\sigma)} \geqslant T \Rightarrow TM^{f(\sigma)} \leqslant M^{\min\left(2-2\sigma, \frac{18}{5}-4\sigma\right)}.$$

Clearly

$$c(\sigma) = \min\left(2-2\sigma, \frac{18}{5}-4\sigma\right) - \min\left(1-2\sigma, 4-6\sigma, \frac{12}{5}-4\sigma\right).$$

Note that this is equivalent to

$$c(\sigma) = \begin{cases} 1, & \sigma \leqslant \frac{7}{10}, \\ 2\sigma - \frac{2}{5}, & \sigma > \frac{7}{10}. \end{cases}$$

Next we break the proof into different cases depending on the sizes of the σ_j.

Case 1. $M^{c(\sigma_1)} \geqslant T$, $N^{c(\sigma_2)} \geqslant T$. Then

$$\begin{aligned} V &\ll \left(M^{\min\left(2-2\sigma_1, \frac{18}{5}-4\sigma_1\right)} N^{\min\left(2-2\sigma_2, \frac{18}{5}-4\sigma_2\right)}\right)^{\frac{1}{2}} M^{\sigma_1} N^{\sigma_2} R^{\sigma_3} \\ &\ll \left(M^{2-2\sigma_1} N^{2-2\sigma_2}\right)^{\frac{1}{2}} M^{\sigma_1} N^{\sigma_2} R (\log x)^{-A} \\ &\ll x(\log x)^{-A}. \end{aligned}$$

Case 2.1. $M^{c(\sigma_1)} \leqslant T$, $N^{c(\sigma_2)} \leqslant T$, $\sigma_3 \leqslant \frac{7}{10}$. Then

$$\begin{aligned} V &\ll \left(TM^{f(\sigma_1)} TN^{f(\sigma_2)}\right)^{\frac{1}{2}} M^{\sigma_1} N^{\sigma_2} R^{\sigma_3} \\ &\ll \left(TM^{1-2\sigma_1} TN^{1-2\sigma_2}\right)^{\frac{1}{2}} M^{\sigma_1} N^{\sigma_2} R^{\frac{7}{10}} \\ &= Tx^{\frac{1}{2}} R^{\frac{1}{5}}. \end{aligned}$$

In order to get $Tx^{\frac{1}{2}} R^{\frac{1}{5}} \ll x(\log x)^{-A}$, we only need

$$(1-\theta) + \frac{1}{2} + \frac{1}{5}(1-\alpha_1-\alpha_2) < 1,$$

which is equivalent to

$$1-\alpha_1-\alpha_2 < 5\theta - \frac{5}{2}.$$

Case 2.2. $M^{c(\sigma_1)} \leqslant T$, $N^{c(\sigma_2)} \leqslant T$, $\sigma_3 > \frac{7}{10}$. Now we have

$$S \ll TM^{f(\sigma_1)}, \quad S \ll TN^{f(\sigma_2)}, \quad S \ll (R^g)^{2-2\sigma_3} + T(R^g)^{4-6\sigma_3},$$

hence, for $a_1+b_1+c_1=1, a_2+b_2+c_2=1$ we have

$$V \ll \left(TM^{f(\sigma_1)}\right)^{a_1} \left(TN^{f(\sigma_2)}\right)^{b_1} \left(R^{g(2-2\sigma_3)}\right)^{c_1} M^{\sigma_1} N^{\sigma_2} R^{\sigma_3} + \left(TM^{f(\sigma_1)}\right)^{a_2} \left(TN^{f(\sigma_2)}\right)^{b_2} \left(TR^{g(4-6\sigma_3)}\right)^{c_2} M^{\sigma_1} N^{\sigma_2} R^{\sigma_3}.$$

In order to remove the dependency on σ_3, we put $c_1 = \frac{1}{2g}$ and $c_2 = \frac{1}{6g}$. We also put $a_1 = b_1 = \frac{1}{2} - \frac{1}{4g}$ and $a_2 = b_2 = \frac{1}{2} - \frac{1}{12g}$. Note that $g \geqslant 1$ implies $\frac{1}{4} \leqslant a_1, b_1 < \frac{1}{2}$ and $\frac{5}{12} \leqslant a_2, b_2 < \frac{1}{2}$. Now, the above bound for V becomes

$$\begin{aligned} V &\ll T^{2a_1} R \left(M^{a_1 f(\sigma_1)+\sigma_1} N^{a_1 f(\sigma_2)+\sigma_2}\right) + TR^{\frac{2}{3}} \left(M^{a_2 f(\sigma_1)+\sigma_1} N^{a_2 f(\sigma_2)+\sigma_2}\right) \\ &= V_1 + V_2. \end{aligned}$$

By a simple calculation, we find that for any $\frac{1}{4} \leqslant a \leqslant \frac{1}{2}$, the inequality

$$af(\sigma) + \sigma \leqslant af\left(\frac{7}{10}\right) + \frac{7}{10} = \frac{7}{10} - \frac{4}{10}a$$

holds. Using this upper bound, we have

$$\begin{aligned} V_1 &\ll T^{2a_1} R (MN)^{\frac{7}{10}-\frac{4}{10}a_1} \\ &= T^{2\left(\frac{1}{2}-\frac{1}{4g}\right)} x^{\frac{7}{10}-\frac{4}{10}\left(\frac{1}{2}-\frac{1}{4g}\right)} R^{\frac{4}{10}\left(\frac{1}{2}-\frac{1}{4g}\right)+\frac{3}{10}} \end{aligned}$$

and

$$\begin{aligned} V_2 &\ll T R^{\frac{2}{3}} (MN)^{\frac{7}{10}-\frac{4}{10}a_2} \\ &= T x^{\frac{7}{10}-\frac{4}{10}\left(\frac{1}{2}-\frac{1}{12g}\right)} R^{\frac{2}{3}-\frac{7}{10}+\frac{4}{10}\left(\frac{1}{2}-\frac{1}{12g}\right)}. \end{aligned}$$

In order to make $V_1 \ll x(\log x)^{-A}$ and $V_2 \ll x(\log x)^{-A}$, we only need the conditions

$$\left(2(1-\theta)\left(\frac{1}{2}-\frac{1}{4g}\right)\right) + \left(\frac{7}{10}-\frac{4}{10}\left(\frac{1}{2}-\frac{1}{4g}\right)\right) + (1-\alpha_1-\alpha_2)\left(\frac{4}{10}\left(\frac{1}{2}-\frac{1}{4g}\right)+\frac{3}{10}\right) < 1$$

and

$$(1-\theta) + \left(\frac{7}{10}-\frac{4}{10}\left(\frac{1}{2}-\frac{1}{12g}\right)\right) + (1-\alpha_1-\alpha_2)\left(\frac{2}{3}-\frac{7}{10}+\frac{4}{10}\left(\frac{1}{2}-\frac{1}{12g}\right)\right) < 1,$$

which are equivalent to

$$1-\alpha_1-\alpha_2 < \frac{10g\theta - 5\theta + (4-5g)}{5g-1}$$

and

$$1-\alpha_1-\alpha_2 < \frac{30g\theta - (15g+1)}{5g-1}.$$

Case 3.1. $M^{c(\sigma_1)} \geqslant T > N^{c(\sigma_2)}$, $\sigma_3 \leqslant \frac{7}{10}$. Now we have

$$\begin{aligned} V &\ll \left(M^{2-2\sigma_1}\right)^{\frac{1}{2}} \left(TN^{f(\sigma_2)}\right)^{\frac{1}{2}} M^{\sigma_1} N^{\sigma_2} R^{\frac{7}{10}} \\ &= M T^{\frac{1}{2}} R^{\frac{7}{10}} N^{\frac{1}{2}f(\sigma_2)+\sigma_2} \\ &\ll M T^{\frac{1}{2}} R^{\frac{7}{10}} N^{\frac{7}{10}-\frac{4}{10}\cdot\frac{1}{2}} \\ &\ll M (NT)^{\frac{1}{2}} R^{\frac{7}{10}} \\ &\ll x(\log x)^{-A} \end{aligned}$$

provided that

$$\alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{2}(1-\theta) + \frac{7}{10}(1-\alpha_1-\alpha_2) < 1$$

or

$$3\alpha_1 - 2\alpha_2 < 5\theta - 2.$$

Case 3.2. $M^{c(\sigma_1)} \geqslant T > N^{c(\sigma_2)}$, $\sigma_3 > \frac{7}{10}$. Again, we have

$$S \ll M^{\min\left(2-2\sigma_1, \frac{18}{5}-4\sigma_1\right)} \ll M^{2-2\sigma_1}, \quad S \ll TN^{f(\sigma_2)}, \quad S \ll (R^g)^{2-2\sigma_3} + T(R^g)^{4-6\sigma_3},$$

hence, for $a_3+b_3+c_3=1, a_4+b_4+c_4=1$ we have

$$V \ll \left(M^{2-2\sigma_1}\right)^{a_3} \left(TN^{f(\sigma_2)}\right)^{b_3} \left(R^{g(2-2\sigma_3)}\right)^{c_3} M^{\sigma_1} N^{\sigma_2} R^{\sigma_3} + \left(M^{2-2\sigma_1}\right)^{a_4} \left(TN^{f(\sigma_2)}\right)^{b_4} \left(TR^{g(4-6\sigma_3)}\right)^{c_4} M^{\sigma_1} N^{\sigma_2} R^{\sigma_3}.$$

In order to remove the dependency on σ_3, we put $c_3 = \frac{1}{2g}$ and $c_4 = \frac{1}{6g}$. We also put $a_3 = a_4 = \frac{1}{2}$, $b_3 = \frac{1}{2} - \frac{1}{2g}$ and $b_4 = \frac{1}{2} - \frac{1}{6g}$. Note that $g \geqslant 2$ implies $\frac{1}{4} \leqslant b_3 < \frac{1}{2}$ and $\frac{5}{12} \leqslant b_4 < \frac{1}{2}$. Now, the above bound for V becomes

$$\begin{aligned} V &\ll MRT^{b_3} N^{b_3 f(\sigma_2)+\sigma_2} + MR^{\frac{2}{3}} T^{\frac{1}{2}} N^{b_4 f(\sigma_2)+\sigma_2} \\ &\ll MRT^{\frac{1}{2}-\frac{1}{2g}} N^{\frac{7}{10}-\frac{4}{10}\left(\frac{1}{2}-\frac{1}{2g}\right)} + MR^{\frac{2}{3}} T^{\frac{1}{2}} N^{\frac{7}{10}-\frac{4}{10}\left(\frac{1}{2}-\frac{1}{6g}\right)} \\ &= xT^{\frac{1}{2}-\frac{1}{2g}} N^{\frac{1}{5g}-\frac{1}{2}} + MR^{\frac{2}{3}} T^{\frac{1}{2}} N^{\frac{1}{2}+\frac{1}{15g}} \\ &= V_3 + V_4. \end{aligned}$$

In order to make $V_3 \ll x(\log x)^{-A}$, we need

$$1 + (1-\theta)\left(\frac{1}{2}-\frac{1}{2g}\right) + \left(\frac{1}{5g}-\frac{1}{2}\right)\alpha_2 < 1,$$

which holds under the condition

$$\alpha_2 > -\frac{5(g-1)(\theta-1)}{5g-2}.$$

In order to make $V_4 \ll x(\log x)^{-A}$, we need

$$\alpha_1 + \frac{2}{3}(1-\alpha_1-\alpha_2) + \left(\frac{1}{2}+\frac{1}{15g}\right)\alpha_2 + \frac{1}{2}(1-\theta) < 1,$$

which holds under the condition

$$\frac{1}{3}\alpha_1+\left(\frac{1}{15g}-\frac{1}{6}\right)\alpha_2<\frac{\theta}{2}-\frac{1}{6}.$$

Case 4. $N^{c(\sigma_2)}\geqslant T>M^{c(\sigma_1)}$. In this case only the roles of M and N in *Case 3* reversed. Consequently, the corresponding conditions are just

$$3\alpha_2-2\alpha_1<5\theta-2,$$
$$\alpha_1>-\frac{5(g-1)(\theta-1)}{5g-2}$$

and

$$\frac{1}{3}\alpha_2+\left(\frac{1}{15g}-\frac{1}{6}\right)\alpha_1<\frac{\theta}{2}-\frac{1}{6}.$$

Finally, by combining all above cases, the proof of Lemma 4.2 is completed. □

The next two lemmas are cited from [5] and [20] with a small modification.

Lemma 4.3. *Let $L_1L_2L_3L_4=x, L_j=x^{\alpha_j}, \alpha_j\geqslant\varepsilon$ and suppose that $L_j(s)$ is prime-factored for $j\geqslant 2$. If any of the following conditions hold:*

$$\alpha_1\geqslant 1-\theta,\ \alpha_2\geqslant\frac{(1-\theta)}{2},\ \alpha_3\geqslant\frac{(1-\theta)}{4},\ 1-\alpha_1-\alpha_2-\alpha_3\geqslant\frac{2(1-\theta)}{7};$$
$$\alpha_1\geqslant 1-\theta,\ \alpha_2\geqslant\frac{(1-\theta)}{2},\ \alpha_3\geqslant\frac{(1-\theta)}{3},\ 1-\alpha_1-\alpha_2-\alpha_3\geqslant\frac{2(1-\theta)}{11};$$
$$\alpha_1\geqslant 1-\theta,\ \alpha_2\geqslant\frac{(1-\theta)}{3},\ 1-\alpha_1-\alpha_2+\alpha_3\geqslant 1-\theta,\ 1-\alpha_1-\alpha_2-\alpha_3\geqslant\frac{2(1-\theta)}{5};$$
$$\alpha_1\geqslant 1-\theta,\ \alpha_2\leqslant\frac{(1-\theta)}{3},\ \alpha_3\leqslant\frac{(1-\theta)}{3},\ \alpha_2+\alpha_3\geqslant\frac{4(1-\theta)}{7},\ 1-\alpha_1\geqslant\frac{14(1-\theta)}{13}.$$

Then we can obtain an asymptotic formula for

$$\sum_{\substack{l_1l_2l_3l_4\in\mathcal{A}\\ l_1\sim L_1\\ l_2\sim L_2\\ l_3\sim L_3\\ l_4\sim L_4}}a_{l_1}b_{l_2}c_{l_3}d_{l_4}\quad\text{or specially}\quad\sum_{\substack{p_j\sim x^{\alpha_j}\\ 1\leqslant j\leqslant 3}}S\left(\mathcal{A}_{p_1p_2p_3},p_3\right).$$

Proof. The proof is completely same as the proof of [[20], Lemma 7.22] except for the second case. For the second case, we need to make a small modification by choosing $h=5$ instead of $h=3$ in [[20], Lemma 7.21]. Note that this modification also occurred in [22] with $\theta=0.53$. □

Lemma 4.4. *Let L_j satisfies the conditions in Lemma 4.3. Then we can obtain an asymptotic formula for*

$$\sum_{\substack{l_1l_2l_3l_4\in\mathcal{A}\\ l_1\sim L_1\\ l_2\sim L_2\\ l_3\sim L_3\\ l_4\sim L_4}}a_{l_1}b_{l_2}c_{l_3}d_{l_4}\quad\text{or specially}\quad\sum_{\substack{p_j\sim x^{\alpha_j}\\ 1\leqslant j\leqslant 3}}S\left(\mathcal{A}_{p_1p_2p_3},p_3\right)$$

if the following conditions hold:

$$\begin{aligned}
&1-\alpha_1-\alpha_2-\alpha_3\geqslant\frac{1}{g_4}(1-\theta),\\
&\alpha_2\left(\frac{1}{4}h+\frac{1}{2}g_2b_1\right)+\alpha_3\left(-\frac{1}{2}g_3c_1+\frac{1}{4}h-\frac{hk_1}{4g_1}+\frac{1}{2}k_2b_1\right)>b_1(1-\theta),\\
&\alpha_1\left(\frac{1}{4}h+\frac{1}{2}g_1a_2\right)+\alpha_3\left(-\frac{1}{2}g_3c_2+\frac{1}{4}h-\frac{hk_2}{4g_2}+\frac{1}{2}k_1a_2\right)>a_2(1-\theta),\\
&\alpha_2\left(\frac{1}{4}h+\frac{1}{2}g_2b_3\right)+\alpha_3\left(\frac{1}{2}g_3c_3+\frac{1}{4}h-\frac{hk_1}{4g_1}+\frac{1}{2}k_2b_3\right)>\left(u-\frac{h}{2g_1}\right)(1-\theta),\\
&\alpha_1\left(\frac{1}{4}h+\frac{1}{2}g_1a_4\right)+\alpha_3\left(\frac{1}{2}g_3c_4+\frac{1}{4}h-\frac{hk_2}{4g_2}+\frac{1}{2}k_1a_4\right)>\left(u-\frac{h}{2g_2}\right)(1-\theta),\\
&\alpha_1\left(\frac{1}{4}h+\frac{1}{2}g_1a_5\right)+\alpha_2\left(\frac{1}{4}h+\frac{1}{2}g_2b_5\right)+\alpha_3\left(-\frac{1}{2}g_3c_5+\frac{1}{4}h+\frac{1}{2}k_1a_5+\frac{1}{2}k_2b_5\right)>\left(a_5+b_5\right)(1-\theta),\\
&\alpha_1\left(\frac{1}{4}h+\frac{1}{2}g_1a_6\right)+\alpha_2\left(\frac{1}{4}h+\frac{1}{2}g_2b_6\right)+\alpha_3\left(\frac{1}{2}g_3c_6+\frac{1}{4}h+\frac{1}{2}k_1a_6+\frac{1}{2}k_2b_6\right)>u(1-\theta),\\
&\alpha_3\left(\frac{1}{2}g_3\left(u-\frac{h}{2g_1}-\frac{h}{2g_2}\right)+\frac{1}{4}hv\right)>\left(u-\frac{h}{2g_1}-\frac{h}{2g_2}\right)(1-\theta),
\end{aligned}$$

where $h=1,\ g_1=1,\ g_2=2,\ g_3=3,\ g_4=d,\ d=4$ or $5,\ k_1=k_2=0,\ u=1-\frac{1}{2d},\ v=1,$

$$\left(b_1,c_1\right)=\left(\frac{1}{3}-\frac{1}{2d},\frac{1}{6}\right),\quad\left(a_2,c_2\right)=\left(\frac{7}{12}-\frac{1}{2d},\frac{1}{6}\right),$$

$$
\begin{aligned}
(b_3,c_3) = &\ \textit{either } \left(\frac{1}{3}-\frac{1}{2d},\frac{1}{6}\right) \textit{ or } \left(\frac{1}{4},\frac{1}{4}-\frac{1}{2d}\right),\\
(a_4,c_4) = &\ \textit{either } \left(\frac{1}{2},\frac{1}{4}-\frac{1}{2d}\right) \textit{ or } \left(\frac{7}{12}-\frac{1}{2d},\frac{1}{6}\right),\\
(a_5,b_5,c_5) = &\ \textit{either } \left(\frac{1}{2},\frac{1}{3}-\frac{1}{2d},\frac{1}{6}\right) \textit{ or } \left(\frac{7}{12}-\frac{1}{2d},\frac{1}{4},\frac{1}{6}\right),\\
(a_6,b_6,c_6) = &\ \textit{either } \left(\frac{1}{2},\frac{1}{4},\frac{1}{4}-\frac{1}{2d}\right) \textit{ or } \left(\frac{7}{12}-\frac{1}{2d},\frac{1}{4},\frac{1}{6}\right) \textit{ or } \left(\frac{1}{2},\frac{1}{3}-\frac{1}{2d},\frac{1}{6}\right).
\end{aligned}
$$

Proof. This is a special case of [[5], Lemma 18]. $\square$

The next two lemmas can be seen as generalizations of [[20], Lemma 7.22]. These can be used to estimate "Type-II$_5$" and "Type-II$_6$" sums mentioned in [18]. The proof is similar to the proof of first and second case of Lemma 4.3. One can generalize these to "Type-II$_n$" sums with $n \geqslant 7$, but the corresponding results will be very complicated and not very numerically significant since the contribution of those high-dimensional sums is already quite small.

Lemma 4.5. *Let $L_1L_2L_3L_4L_5 = x, L_j = x^{\alpha_j}, \alpha_j \geqslant \varepsilon$ and suppose that $L_j(s)$ is prime-factored for $j \geqslant 2$. If any of the following conditions hold:*

$$
\begin{aligned}
&\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{7},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4 \geqslant \frac{2(1-\theta)}{83};\\
&\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{8},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4 \geqslant \frac{2(1-\theta)}{47};\\
&\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{9},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4 \geqslant \frac{2(1-\theta)}{35};\\
&\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{10},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4 \geqslant \frac{2(1-\theta)}{29};\\
&\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{12},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4 \geqslant \frac{2(1-\theta)}{23};\\
&\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{5},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4 \geqslant \frac{2(1-\theta)}{39};\\
&\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{6},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4 \geqslant \frac{2(1-\theta)}{23};\\
&\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{8},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4 \geqslant \frac{2(1-\theta)}{15};\\
&\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{5},\ \alpha_4 \geqslant \frac{(1-\theta)}{5},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4 \geqslant \frac{2(1-\theta)}{19}.
\end{aligned}
$$

Then we can obtain an asymptotic formula for

$$
\sum_{\substack{l_1l_2l_3l_4l_5 \in \mathcal{A}\\ l_j \sim L_j\\ 1\leqslant j\leqslant 5}} a_{l_1}b_{l_2}c_{l_3}d_{l_4}e_{l_5} \quad \textit{or specially} \quad \sum_{\substack{p_j \sim x^{\alpha_j}\\ 1\leqslant j\leqslant 4}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right).
$$

Lemma 4.6. *Let $L_1L_2L_3L_4L_5L_6 = x, L_j = x^{\alpha_j}, \alpha_j \geqslant \varepsilon$ and suppose that $L_j(s)$ is prime-factored for $j \geqslant 2$. If any of the following conditions hold:*

$$
\begin{aligned}
&\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{7},\ \alpha_5 \geqslant \frac{(1-\theta)}{43},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{3611};\\
&\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{7},\ \alpha_5 \geqslant \frac{(1-\theta)}{44},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{1847};\\
&\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{7},\ \alpha_5 \geqslant \frac{(1-\theta)}{45},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{1259};\\
&\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{7},\ \alpha_5 \geqslant \frac{(1-\theta)}{46},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{965};\\
&\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{7},\ \alpha_5 \geqslant \frac{(1-\theta)}{48},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{671};\\
&\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{7},\ \alpha_5 \geqslant \frac{(1-\theta)}{49},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{587};\\
&\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{7},\ \alpha_5 \geqslant \frac{(1-\theta)}{51},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{475};\\
&\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{7},\ \alpha_5 \geqslant \frac{(1-\theta)}{54},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{377};\\
&\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{7},\ \alpha_5 \geqslant \frac{(1-\theta)}{56},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{335};
\end{aligned}
$$

$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{7},\ \alpha_5 \geqslant \frac{(1-\theta)}{60},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{279};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{7},\ \alpha_5 \geqslant \frac{(1-\theta)}{63},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{251};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{7},\ \alpha_5 \geqslant \frac{(1-\theta)}{70},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{209};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{7},\ \alpha_5 \geqslant \frac{(1-\theta)}{78},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{181};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{7},\ \alpha_5 \geqslant \frac{(1-\theta)}{84},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{167};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{8},\ \alpha_5 \geqslant \frac{(1-\theta)}{25},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{1199};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{8},\ \alpha_5 \geqslant \frac{(1-\theta)}{26},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{623};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{8},\ \alpha_5 \geqslant \frac{(1-\theta)}{27},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{431};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{8},\ \alpha_5 \geqslant \frac{(1-\theta)}{28},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{335};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{8},\ \alpha_5 \geqslant \frac{(1-\theta)}{30},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{239};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{8},\ \alpha_5 \geqslant \frac{(1-\theta)}{32},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{191};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{8},\ \alpha_5 \geqslant \frac{(1-\theta)}{33},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{175};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{8},\ \alpha_5 \geqslant \frac{(1-\theta)}{36},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{143};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{8},\ \alpha_5 \geqslant \frac{(1-\theta)}{40},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{119};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{8},\ \alpha_5 \geqslant \frac{(1-\theta)}{42},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{111};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{8},\ \alpha_5 \geqslant \frac{(1-\theta)}{48},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{95};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{9},\ \alpha_5 \geqslant \frac{(1-\theta)}{19},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{683};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{9},\ \alpha_5 \geqslant \frac{(1-\theta)}{20},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{359};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{9},\ \alpha_5 \geqslant \frac{(1-\theta)}{21},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{251};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{9},\ \alpha_5 \geqslant \frac{(1-\theta)}{22},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{197};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{9},\ \alpha_5 \geqslant \frac{(1-\theta)}{24},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{143};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{9},\ \alpha_5 \geqslant \frac{(1-\theta)}{27},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{107};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{9},\ \alpha_5 \geqslant \frac{(1-\theta)}{30},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{89};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{9},\ \alpha_5 \geqslant \frac{(1-\theta)}{36},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{71};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{10},\ \alpha_5 \geqslant \frac{(1-\theta)}{16},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{479};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{10},\ \alpha_5 \geqslant \frac{(1-\theta)}{18},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{179};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{10},\ \alpha_5 \geqslant \frac{(1-\theta)}{20},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{119};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{10},\ \alpha_5 \geqslant \frac{(1-\theta)}{24},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{79};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{10},\ \alpha_5 \geqslant \frac{(1-\theta)}{30},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{59};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{11},\ \alpha_5 \geqslant \frac{(1-\theta)}{14},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{461};$$

$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{11},\ \alpha_5 \geqslant \frac{(1-\theta)}{15},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{219};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{11},\ \alpha_5 \geqslant \frac{(1-\theta)}{22},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{65};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{12},\ \alpha_5 \geqslant \frac{(1-\theta)}{13},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{311};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{12},\ \alpha_5 \geqslant \frac{(1-\theta)}{14},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{167};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{12},\ \alpha_5 \geqslant \frac{(1-\theta)}{15},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{119};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{12},\ \alpha_5 \geqslant \frac{(1-\theta)}{16},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{95};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{12},\ \alpha_5 \geqslant \frac{(1-\theta)}{18},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{71};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{12},\ \alpha_5 \geqslant \frac{(1-\theta)}{20},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{59};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{12},\ \alpha_5 \geqslant \frac{(1-\theta)}{21},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{55};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{13},\ \alpha_5 \geqslant \frac{(1-\theta)}{13},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{155};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{14},\ \alpha_5 \geqslant \frac{(1-\theta)}{15},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{69};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{14},\ \alpha_5 \geqslant \frac{(1-\theta)}{21},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{41};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{15},\ \alpha_5 \geqslant \frac{(1-\theta)}{15},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{59};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{3},\ \alpha_4 \geqslant \frac{(1-\theta)}{15},\ \alpha_5 \geqslant \frac{(1-\theta)}{20},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{39};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{5},\ \alpha_5 \geqslant \frac{(1-\theta)}{21},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{839};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{5},\ \alpha_5 \geqslant \frac{(1-\theta)}{22},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{439};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{5},\ \alpha_5 \geqslant \frac{(1-\theta)}{24},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{239};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{5},\ \alpha_5 \geqslant \frac{(1-\theta)}{25},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{199};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{5},\ \alpha_5 \geqslant \frac{(1-\theta)}{28},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{139};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{5},\ \alpha_5 \geqslant \frac{(1-\theta)}{30},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{119};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{5},\ \alpha_5 \geqslant \frac{(1-\theta)}{36},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{89};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{5},\ \alpha_5 \geqslant \frac{(1-\theta)}{40},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{79};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{6},\ \alpha_5 \geqslant \frac{(1-\theta)}{13},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{311};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{6},\ \alpha_5 \geqslant \frac{(1-\theta)}{14},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{167};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{6},\ \alpha_5 \geqslant \frac{(1-\theta)}{15},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{119};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{6},\ \alpha_5 \geqslant \frac{(1-\theta)}{16},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{95};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{6},\ \alpha_5 \geqslant \frac{(1-\theta)}{18},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{71};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{6},\ \alpha_5 \geqslant \frac{(1-\theta)}{20},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{59};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{6},\ \alpha_5 \geqslant \frac{(1-\theta)}{21},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{55};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{6},\ \alpha_5 \geqslant \frac{(1-\theta)}{24},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{47};$$

$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{7},\ \alpha_5 \geqslant \frac{(1-\theta)}{10},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{279};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{7},\ \alpha_5 \geqslant \frac{(1-\theta)}{14},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{55};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{8},\ \alpha_5 \geqslant \frac{(1-\theta)}{9},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{143};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{8},\ \alpha_5 \geqslant \frac{(1-\theta)}{10},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{79};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{8},\ \alpha_5 \geqslant \frac{(1-\theta)}{16},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{31};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{4},\ \alpha_4 \geqslant \frac{(1-\theta)}{9},\ \alpha_5 \geqslant \frac{(1-\theta)}{9},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{71};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{5},\ \alpha_4 \geqslant \frac{(1-\theta)}{5},\ \alpha_5 \geqslant \frac{(1-\theta)}{11},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{219};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{5},\ \alpha_4 \geqslant \frac{(1-\theta)}{5},\ \alpha_5 \geqslant \frac{(1-\theta)}{12},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{119};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{5},\ \alpha_4 \geqslant \frac{(1-\theta)}{5},\ \alpha_5 \geqslant \frac{(1-\theta)}{14},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{69};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{5},\ \alpha_4 \geqslant \frac{(1-\theta)}{5},\ \alpha_5 \geqslant \frac{(1-\theta)}{15},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{59};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{5},\ \alpha_4 \geqslant \frac{(1-\theta)}{6},\ \alpha_5 \geqslant \frac{(1-\theta)}{8},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{239};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{5},\ \alpha_4 \geqslant \frac{(1-\theta)}{6},\ \alpha_5 \geqslant \frac{(1-\theta)}{9},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{89};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{5},\ \alpha_4 \geqslant \frac{(1-\theta)}{6},\ \alpha_5 \geqslant \frac{(1-\theta)}{10},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{59};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{5},\ \alpha_4 \geqslant \frac{(1-\theta)}{6},\ \alpha_5 \geqslant \frac{(1-\theta)}{15},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{29};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{5},\ \alpha_4 \geqslant \frac{(1-\theta)}{7},\ \alpha_5 \geqslant \frac{(1-\theta)}{7},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{139};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{6},\ \alpha_4 \geqslant \frac{(1-\theta)}{7},\ \alpha_5 \geqslant \frac{(1-\theta)}{7},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{41};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{6},\ \alpha_4 \geqslant \frac{(1-\theta)}{9},\ \alpha_5 \geqslant \frac{(1-\theta)}{9},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{17};$$
$$\alpha_1 \geqslant 1-\theta,\ \alpha_2 \geqslant \frac{(1-\theta)}{2},\ \alpha_3 \geqslant \frac{(1-\theta)}{7},\ \alpha_4 \geqslant \frac{(1-\theta)}{7},\ \alpha_5 \geqslant \frac{(1-\theta)}{7},\ 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5 \geqslant \frac{2(1-\theta)}{27}.$$

Then we can obtain an asymptotic formula for

$$\sum_{\substack{l_1 l_2 l_3 l_4 l_5 l_6 \in \mathcal{A} \\ l_j \sim L_j \\ 1 \leqslant j \leqslant 6}} a_{l_1} b_{l_2} c_{l_3} d_{l_4} e_{l_5} f_{l_6} \quad \textit{or specially} \quad \sum_{\substack{p_j \sim x^{\alpha_j} \\ 1 \leqslant j \leqslant 5}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5\right).$$

5. The final decomposition I: Lower Bounds

In this section, we ignore the presence of ε for clarity. Let $\omega(u)$ denotes the Buchstab function determined by the following differential-difference equation

$$\begin{cases} \omega(u)=\frac{1}{u}, & 1 \leqslant u \leqslant 2, \\ (u\omega(u))'=\omega(u-1), & u \geqslant 2. \end{cases}$$

Moreover, we have the upper and lower bounds for $\omega(u)$:

$$\omega(u) \geqslant \omega_0(u) = \begin{cases} \frac{1}{u}, & 1 \leqslant u < 2, \\ \frac{1+\log(u-1)}{u}, & 2 \leqslant u < 3, \\ \frac{1+\log(u-1)}{u} + \frac{1}{u}\int_2^{u-1} \frac{\log(t-1)}{t} dt, & 3 \leqslant u < 4, \\ 0.5612, & u \geqslant 4, \end{cases}$$

$$\omega(u) \leqslant \omega_1(u) = \begin{cases} \frac{1}{u}, & 1 \leqslant u < 2, \\ \frac{1+\log(u-1)}{u}, & 2 \leqslant u < 3, \\ \frac{1+\log(u-1)}{u} + \frac{1}{u}\int_2^{u-1} \frac{\log(t-1)}{t} dt, & 3 \leqslant u < 4, \\ 0.5617, & u \geqslant 4. \end{cases}$$

We shall use $\omega_0(u)$ and $\omega_1(u)$ to give numerical bounds for some sieve functions discussed below. We shall also use the simple upper bound $\omega(u) \leqslant \max(\frac{1}{u}, 0.5672)$ (see Lemma 8(iii) of [32]) to estimate high-dimensional integrals. Fix $\theta = 0.52$, $\nu_0 = \nu_{\min} = 2\theta - 1 = 0.04$ and let $p_j = x^{\alpha_j}$. By Buchstab's identity, we have

$$
\begin{aligned}
S\left(\mathcal{A}, x^{\frac{1}{2}}\right) &= S\left(\mathcal{A}, x^{\nu(0)}\right) - \sum_{\nu(0) \leqslant \alpha_1 < \frac{1}{2}} S\left(\mathcal{A}_{p_1}, x^{\nu(\alpha_1)}\right) + \sum_{\substack{\nu(0) \leqslant \alpha_1 < \frac{1}{2} \\ \nu(\alpha_1) \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
&= \sum_1 - \sum_2 + \sum_3 .
\end{aligned} \tag{7}
$$

We can give asymptotic formulas for $\sum_1$ and $\sum_2$. For $\sum_3$, We begin with some notation needed to describe the further decompositions. Following [20] directly, we use the bold capital letters $\boldsymbol{G}$ and $\boldsymbol{D}$ to represent sets that have asymptotic formulas and that can perform further decompositions directly. We write $\boldsymbol{\alpha}_n$ to denote $(\alpha_1, \ldots, \alpha_n)$. Let $\boldsymbol{G}_n$ denotes the set of $\boldsymbol{\alpha}_n$ such that an asymptotic formula can be obtained for

$$
\sum_{\alpha_1, \ldots, \alpha_n} S\left(\mathcal{A}_{p_1 \ldots p_n}, p_n\right),
$$

so we can define sets $\boldsymbol{G}_2$ and $\boldsymbol{G}_3$ by using Lemmas 4.1–4.4. We also need to define $\boldsymbol{G}_i$ with $i \geqslant 4$ in order to perform our calculation. For this, we can define them by checking whether a region $\boldsymbol{\alpha}_i$ can be partitioned into $(m, n) \in \boldsymbol{G}_2$ or $(m, n, h) \in \boldsymbol{G}_3$ in any order as long as the conditions are satisfied. By Lemmas 3.1–3.2, we put

$$
\begin{aligned}
\boldsymbol{D}_0 &= \left\{\boldsymbol{\alpha}_2 : 0 \leqslant \alpha_1 \leqslant \frac{1}{2},\ 0 \leqslant \alpha_2 \leqslant \min\left(\frac{3\theta + 1 - 4\alpha_1^*}{2}, \frac{3 + \theta - 4\alpha_1^*}{5}\right)\right\}, \\
\boldsymbol{D}_1 &= \left\{\boldsymbol{\alpha}_3 : 0 \leqslant \alpha_1 \leqslant \frac{1}{2},\ \alpha_3 \leqslant \frac{1 + 3\theta}{4} - \alpha_1^*,\ 2\alpha_2 + \alpha_3 \leqslant 1 + \theta - 2\alpha_1^*,\ 2\alpha_2 + 3\alpha_3 \leqslant \frac{3 + \theta}{2} - 2\alpha_1^*\right\}, \\
\boldsymbol{D}_2 &= \left\{\boldsymbol{\alpha}_3 : 0 \leqslant \alpha_1 \leqslant \frac{1}{2},\ \alpha_2 \leqslant \frac{1 - \theta}{2},\ \alpha_3 \leqslant \frac{1 + 3\theta - 4\alpha_1^*}{8}\right\}, \\
\boldsymbol{D}_0' &= \left\{\boldsymbol{\alpha}_2 : 0 \leqslant \alpha_1 \leqslant \frac{1}{2},\ 0 \leqslant \alpha_2 \leqslant \min\left(\frac{3\theta - 1}{2}, \frac{1 + \theta}{5}\right)\right\}, \\
\boldsymbol{D}_1' &= \left\{\boldsymbol{\alpha}_3 : 0 \leqslant \alpha_1 \leqslant \frac{1}{2},\ \alpha_3 \leqslant \frac{3\theta - 1}{4},\ 2\alpha_2 + \alpha_3 \leqslant \theta,\ 2\alpha_2 + 3\alpha_3 \leqslant \frac{1 + \theta}{2}\right\}, \\
\boldsymbol{D}_2' &= \left\{\boldsymbol{\alpha}_3 : 0 \leqslant \alpha_1 \leqslant \frac{1}{2},\ \alpha_2 \leqslant \frac{1 - \theta}{2},\ \alpha_3 \leqslant \frac{3\theta - 1}{8}\right\}, \\
\boldsymbol{D}^* &= \left\{\boldsymbol{\alpha}_4 : (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_4) \text{ can be partitioned into } (m, n) \in \boldsymbol{D}_0' \text{ or } (m, n, h) \in \boldsymbol{D}_1' \cup \boldsymbol{D}_2'\right\}, \\
\boldsymbol{D}^{**} &= \left\{\boldsymbol{\alpha}_6 : (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_6) \text{ can be partitioned into } (m, n) \in \boldsymbol{D}_0' \text{ or } (m, n, h) \in \boldsymbol{D}_1' \cup \boldsymbol{D}_2'\right\}, \\
\boldsymbol{D}^{\dagger} &= \left\{\boldsymbol{\alpha}_4 : \boldsymbol{\alpha}_4 \text{ can be partitioned into } (m, n) \in \boldsymbol{D}_0 \text{ or } (m, n, h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2\right\}, \\
\boldsymbol{D}^{\ddagger} &= \left\{\boldsymbol{\alpha}_4 : \boldsymbol{\alpha}_4 \in \boldsymbol{D}^{\dagger},\ (1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4, \alpha_2, \alpha_3, \alpha_4) \in \boldsymbol{D}^{\dagger}\right\},
\end{aligned}
$$

where the sets $\boldsymbol{D}_0$, $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ correspond to conditions on variables that allow a further decomposition (that is, we apply Buchstab's identity twice), $\boldsymbol{D}_i'$ is a simplified version of $\boldsymbol{D}_i$ for $i \in \{0, 1, 2\}$, $\boldsymbol{D}^*$ and $\boldsymbol{D}^{**}$ allow two and three further decompositions respectively, and $\boldsymbol{D}^{\ddagger}$ allows two further decompositions with a role-reversal. For example, if we have $\boldsymbol{\alpha}_4 \in \boldsymbol{D}^*$ after applying Buchstab's identity twice, we can apply Buchstab's identity twice more because we can obtain an asymptotic formula for $S\left(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, x^{\nu}\right)$ for all $\nu \leqslant \alpha_5 < \alpha_4$. In regions corresponding to neither $\boldsymbol{G}$ nor $\boldsymbol{D}$, we sometimes need role-reversals to perform further decompositions.

We remark that if $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_4)$ can be partitioned into $(m, n) \in \boldsymbol{D}_0$ with $m < n$ or one element α_4 is partitioned into n, then we still have this $\boldsymbol{\alpha}_4 \in \boldsymbol{D}^*$ even if $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_4)$ cannot be partitioned into $(m, n) \in \boldsymbol{D}_0'$. This is because sometimes we can group $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_4)$ into $(m, n) \in \boldsymbol{D}_0$ but cannot group $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ into $(m, n) \in \boldsymbol{D}_0$ for some $\nu \leqslant \alpha_5 < \alpha_4$ due to the involvement of $\alpha^*(m)$ in the upper bound of n. That is, if we group $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_4)$ into (m, n) which lies in the areas above the line $n = \frac{3\theta - 1}{2}$ (see the protrusions at the top of the region $\boldsymbol{D}_0$ in Figure 1 in Appendix 1) and all of two elements α_4 are partitioned into m (note that n is a constant in this partition), then we cannot group all $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ with $\nu \leqslant \alpha_5 < \alpha_4$ because some values of α_5 may leads to some smaller m where the new (m, n) lies in the concave areas on the left of the original (m, n). Otherwise we have at least one α_4 is in n, and we can let this α_4 to be the variable α_5 runs over values less than α_4. In $\boldsymbol{D}_0'$ the function α^* is replaced by an upper bound $\frac{1}{2}$, and the shape of $\boldsymbol{D}_0'$ is a rectangle with bounds $0 \leqslant m \leqslant \frac{1}{2}$ and $0 \leqslant n \leqslant \frac{3\theta - 1}{2}$. Thus, we can group $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ into $(m, n) \in \boldsymbol{D}_0'$ for any $\nu \leqslant \alpha_5 < \alpha_4$ if we can group $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_4)$ into $(m, n) \in \boldsymbol{D}_0'$. However, if $m < n$, we can change the role of m and n so that at least one element α_4 is in n, hence we can group $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ into $(m, n) \in \boldsymbol{D}_0$ for any $\nu \leqslant \alpha_5 < \alpha_4$. The similar phenomenon holds for $\boldsymbol{D}^{**}$, and it was implicitly used in the estimation of S_{310} in [35].

Now we split the region defined by $\sum_3$ into three subregions A, B, C corresponding to the different techniques that should be applied. The plot of these regions can be found in Appendix 1.

$$
A = \left\{\boldsymbol{\alpha}_2 : \frac{1}{4} \leqslant \alpha_1 \leqslant \frac{2}{5},\ \frac{1}{3}\left(1 - \alpha_1\right) \leqslant \alpha_2 \leqslant \min\left(\alpha_1, 1 - 2\alpha_1\right)\right\};
$$

$$B=\left\{\boldsymbol{\alpha}_2: \frac{1}{3}\leqslant \alpha_1\leqslant \frac{1}{2},\ \max\left(\frac{1}{2}\alpha_1, 1-2\alpha_1\right)\leqslant \alpha_2\leqslant \frac{1}{2}\left(1-\alpha_1\right)\right\};$$
$$C=\left\{\boldsymbol{\alpha}_2: \nu(0)\leqslant \alpha_1\leqslant \frac{1}{2},\ \nu\left(\alpha_1\right)\leqslant \alpha_2\leqslant \min\left(\alpha_1, \frac{1}{2}\left(1-\alpha_1\right)\right),\ \boldsymbol{\alpha}_2\notin A\cup B\right\}.$$

We note that $(\alpha_1,\alpha_2)\in A \Leftrightarrow (1-\alpha_1-\alpha_2,\alpha_2)\in B$. Since in $A\cup B$ only products of three primes are counted, we have

$$\sum_{\boldsymbol{\alpha}_2\in A} S\left(\mathcal{A}_{p_1p_2}, p_2\right)=\sum_{\boldsymbol{\alpha}_2\in B} S\left(\mathcal{A}_{p_1p_2}, p_2\right), \tag{8}$$

hence

$$\begin{aligned}\sum_3 &= 2\sum_{\boldsymbol{\alpha}_2\in A} S\left(\mathcal{A}_{p_1p_2}, p_2\right)+\sum_{\boldsymbol{\alpha}_2\in C} S\left(\mathcal{A}_{p_1p_2}, p_2\right)\\ &= 2\sum_A+\sum_C.\end{aligned} \tag{9}$$

We first consider $\sum_A$. Discarding the whole of $\sum_A$ leads to a loss of

$$\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)} \texttt{Boole}[(t_1,t_2)\in A]\frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1t_2^2}dt_2dt_1<0.240227. \tag{10}$$

We shall make a relatively small saving over this sum whenever $\alpha_1+3\alpha_2<1.005$ and $\alpha_1\geqslant 0.38$. Let A' denotes this part of region A and by Buchstab's identity, we have

$$\sum_{\boldsymbol{\alpha}_2\in A'} S\left(\mathcal{A}_{p_1p_2}, p_2\right)=\sum_{\boldsymbol{\alpha}_2\in A'} S\left(\mathcal{A}_{p_1p_2}, x^{\nu_0}\right)-\sum_{\substack{\boldsymbol{\alpha}_2\in A'\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right). \tag{11}$$

We can give an asymptotic formula for the first sum on the right-hand side. For the second sum, we can perform a straightforward decomposition by applying Buchstab's identity twice more if we can group $\boldsymbol{\alpha}_3$ into $(m,n)\in\boldsymbol{D}_0$ or $(m,n,h)\in\boldsymbol{D}_1\cup\boldsymbol{D}_2$. For the remaining part of the second sum, we note that $\alpha_1+\alpha_2\geqslant\frac{1}{2}$ and $\alpha_3<\alpha_2\leqslant\frac{3\theta-1}{2}\leqslant\frac{3\theta+1-4\alpha_1^*}{2}$. Let $p_1p_2p_3\beta$ denote the numbers counted by $S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right)$, we have $\beta\sim x^{1-\alpha_1-\alpha_2-\alpha_3}$. Thus, we can perform a role-reversal (see the estimation of $\sum_6$ in [35] for an explanation of role-reversals) on the ramaining part because we have $((1-\alpha_1-\alpha_2-\alpha_3)+\alpha_3,\alpha_2)\in\boldsymbol{D}_0$ in this case. Altogether, we have the following expression after the first decomposition procedure:

$$\begin{aligned}\sum_{\boldsymbol{\alpha}_2\in A'} S\left(\mathcal{A}_{p_1p_2}, p_2\right)=&\sum_{\boldsymbol{\alpha}_2\in A'} S\left(\mathcal{A}_{p_1p_2}, x^{\nu_0}\right)-\sum_{\substack{\boldsymbol{\alpha}_2\in A'\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right)\\
=&\sum_{\boldsymbol{\alpha}_2\in A'} S\left(\mathcal{A}_{p_1p_2}, x^{\nu_0}\right)-\sum_{\substack{\boldsymbol{\alpha}_2\in A'\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\in\boldsymbol{G}_3}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right)\\
&-\sum_{\substack{\boldsymbol{\alpha}_2\in A'\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3\\ \boldsymbol{\alpha}_3\text{ can be partitioned into }(m,n)\in\boldsymbol{D}_0\text{ or }(m,n,h)\in\boldsymbol{D}_1\cup\boldsymbol{D}_2}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right)\\
&-\sum_{\substack{\boldsymbol{\alpha}_2\in A'\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3\\ \boldsymbol{\alpha}_3\text{ cannot be partitioned into }(m,n)\in\boldsymbol{D}_0\text{ or }(m,n,h)\in\boldsymbol{D}_1\cup\boldsymbol{D}_2}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right)\\
=&\sum_{\boldsymbol{\alpha}_2\in A'} S\left(\mathcal{A}_{p_1p_2}, x^{\nu_0}\right)-\sum_{\substack{\boldsymbol{\alpha}_2\in A'\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\in\boldsymbol{G}_3}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right)\\
&-\sum_{\substack{\boldsymbol{\alpha}_2\in A'\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3\\ \boldsymbol{\alpha}_3\text{ can be partitioned into }(m,n)\in\boldsymbol{D}_0\text{ or }(m,n,h)\in\boldsymbol{D}_1\cup\boldsymbol{D}_2}} S\left(\mathcal{A}_{p_1p_2p_3}, x^{\nu_0}\right)\\
&+\sum_{\substack{\boldsymbol{\alpha}_2\in A'\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3\\ \boldsymbol{\alpha}_3\text{ can be partitioned into }(m,n)\in\boldsymbol{D}_0\text{ or }(m,n,h)\in\boldsymbol{D}_1\cup\boldsymbol{D}_2\\ \nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right)\\ \boldsymbol{\alpha}_4\in\boldsymbol{G}_4}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right)\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\substack{\boldsymbol{\alpha}_2 \in A' \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right) \\
&- \sum_{\substack{\boldsymbol{\alpha}_2 \in A' \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ cannot be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2}} S\left(\mathcal{A}_{\beta p_2 p_3}, x^{\nu_0}\right) \\
&+ \sum_{\substack{\boldsymbol{\alpha}_2 \in A' \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ cannot be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \frac{1}{2}\alpha_1 \\ \boldsymbol{\alpha}_4' \in \boldsymbol{G}_4}} S\left(\mathcal{A}_{\beta p_2 p_3 p_4}, p_4\right) \\
&+ \sum_{\substack{\boldsymbol{\alpha}_2 \in A' \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ cannot be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \frac{1}{2}\alpha_1 \\ \boldsymbol{\alpha}_4' \notin \boldsymbol{G}_4}} S\left(\mathcal{A}_{\beta p_2 p_3 p_4}, p_4\right) \\
&= S_{01} - S_{02} - S_{03} + S_{04} + T_{01} - S_{05} + S_{06} + T_{02}, \qquad (12)
\end{aligned}
$$

where $(\beta, P(p_3)) = 1$ and $\boldsymbol{\alpha}_4' = (1-\alpha_1-\alpha_2-\alpha_3,\ \alpha_2,\ \alpha_3,\ \alpha_4)$.

We can give asymptotic formulas for S_{01}–S_{06}. For T_{01} we can perform Buchstab's identity twice more to reach a six-dimensional sum if $\boldsymbol{\alpha}_4 \in \boldsymbol{D}^*$, and we can use Buchstab's identity in reverse (see the estimation of S_{312} and S_{313} in [35] for an explanation) to make some savings on the remaining parts. After the second decomposition procedure on T_{01}, we have

$$
\begin{aligned}
T_{01} &= \sum_{\substack{\boldsymbol{\alpha}_2 \in A' \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right) \\
&= \sum_{\substack{\boldsymbol{\alpha}_2 \in A' \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^*}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right) \\
&\quad + \sum_{\substack{\boldsymbol{\alpha}_2 \in A' \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^*}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, x^{\nu_0}\right) \\
&= \sum_{\substack{\boldsymbol{\alpha}_2 \in A' \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^*}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right) \\
&\quad + \sum_{\substack{\boldsymbol{\alpha}_2 \in A' \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^*}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, x^{\nu_0}\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{\substack{\boldsymbol{\alpha}_2 \in A' \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^* \\ \nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5 \in \boldsymbol{G}_5}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5\right) \\
&- \sum_{\substack{\boldsymbol{\alpha}_2 \in A' \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^* \\ \nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5 \notin \boldsymbol{G}_5}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, x^{\nu_0}\right) \\
&+ \sum_{\substack{\boldsymbol{\alpha}_2 \in A' \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^* \\ \nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5 \notin \boldsymbol{G}_5 \\ \nu_0 \leqslant \alpha_6 < \min\left(\alpha_5, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_5-\alpha_6)\right) \\ \boldsymbol{\alpha}_6 \in \boldsymbol{G}_6}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5 p_6}, p_6\right) \\
&+ \sum_{\substack{\boldsymbol{\alpha}_2 \in A' \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^* \\ \nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5 \notin \boldsymbol{G}_5 \\ \nu_0 \leqslant \alpha_6 < \min\left(\alpha_5, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_5-\alpha_6)\right) \\ \boldsymbol{\alpha}_6 \notin \boldsymbol{G}_6}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5 p_6}, p_6\right) \\
&= T_{011} + S_{07} - S_{08} - S_{09} + S_{10} + T_{012},
\end{aligned}
\tag{13}
$$

where the sum T_{011} can be further decomposed to

$$
\begin{aligned}
T_{011} =& \sum_{\substack{\boldsymbol{\alpha}_2 \in A' \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^*}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right) \\
=& \sum_{\substack{\boldsymbol{\alpha}_2 \in A' \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^*,\ \alpha_4 \geqslant \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right) \\
&+ \sum_{\substack{\boldsymbol{\alpha}_2 \in A' \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^*,\ \alpha_4 < \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right) \\
=& \sum_{\substack{\boldsymbol{\alpha}_2 \in A' \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^*,\ \alpha_4 \geqslant \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right)
\end{aligned}
$$

$$+\sum_{\substack{\boldsymbol{\alpha}_2\in A'\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3\\ \boldsymbol{\alpha}_3\text{ can be partitioned into }(m,n)\in\boldsymbol{D}_0\text{ or }(m,n,h)\in\boldsymbol{D}_1\cup\boldsymbol{D}_2\\ \nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right)\\ \boldsymbol{\alpha}_4\notin\boldsymbol{G}_4,\ \boldsymbol{\alpha}_4\notin\boldsymbol{D}^*,\ \alpha_4<\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)}} S\left(\mathcal{A}_{p_1p_2p_3p_4},\left(\frac{x}{p_1p_2p_3p_4}\right)^{\frac{1}{2}}\right)$$

$$+\sum_{\substack{\boldsymbol{\alpha}_2\in A'\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3\\ \boldsymbol{\alpha}_3\text{ can be partitioned into }(m,n)\in\boldsymbol{D}_0\text{ or }(m,n,h)\in\boldsymbol{D}_1\cup\boldsymbol{D}_2\\ \nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right)\\ \boldsymbol{\alpha}_4\notin\boldsymbol{G}_4,\ \boldsymbol{\alpha}_4\notin\boldsymbol{D}^*\\ \alpha_4<\alpha_5<\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\\ \boldsymbol{\alpha}_5\notin\boldsymbol{G}_5}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5},p_5\right)$$

$$+\sum_{\substack{\boldsymbol{\alpha}_2\in A'\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3\\ \boldsymbol{\alpha}_3\text{ can be partitioned into }(m,n)\in\boldsymbol{D}_0\text{ or }(m,n,h)\in\boldsymbol{D}_1\cup\boldsymbol{D}_2\\ \nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right)\\ \boldsymbol{\alpha}_4\notin\boldsymbol{G}_4,\ \boldsymbol{\alpha}_4\notin\boldsymbol{D}^*\\ \alpha_4<\alpha_5<\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\\ \boldsymbol{\alpha}_5\in\boldsymbol{G}_5}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5},p_5\right). \tag{14}$$

We can give asymptotic formulas for S_{07}–S_{10}. We can also give an asymptotic formula for the last sum on the right-hand side of (14), hence we can subtract its contribution from the loss from T_{011}. The same process can also be used to deal with T_{02}, but we choose to discard all of it for the sake of simplicity. There are also many possible decompositions in some subregions, but we don't consider them here. In fact, we shall consider some of them when decomposing $\sum_C$.

Combining all the sums above with remaining parts of A we get a loss from $\sum_A$ of

$$\begin{aligned}
&\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\texttt{Boole}[(t_1,t_2)\in U_{A1}]\frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1t_2^2}dt_2dt_1\right)\\
&+\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\right.\\
&\qquad\left.\texttt{Boole}[(t_1,t_2,t_3,t_4)\in U_{A2}]\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1t_2t_3t_4^2}dt_4dt_3dt_2dt_1\right)\\
&-\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\int_{t_4}^{\frac{1-t_1-t_2-t_3-t_4}{2}}\right.\\
&\qquad\left.\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5)\in U_{A3}]\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1t_2t_3t_4t_5^2}dt_5dt_4dt_3dt_2dt_1\right)\\
&+\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\right.\\
&\qquad\int_{\nu_0}^{\min\left(t_4,\frac{1-t_1-t_2-t_3-t_4}{2}\right)}\int_{\nu_0}^{\min\left(t_5,\frac{1-t_1-t_2-t_3-t_4-t_5}{2}\right)}\\
&\qquad\left.\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6)\in U_{A4}]\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{t_6}\right)}{t_1t_2t_3t_4t_5t_6^2}dt_6dt_5dt_4dt_3dt_2dt_1\right)\\
&+\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\frac{1}{2}t_1}\right.\\
&\qquad\left.\texttt{Boole}[(t_1,t_2,t_3,t_4)\in U_{A5}]\frac{\omega\left(\frac{t_1-t_4}{t_4}\right)\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2t_3^2t_4^2}dt_4dt_3dt_2dt_1\right)\\
&\leqslant\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\texttt{Boole}[(t_1,t_2)\in U_{A1}]\frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1t_2^2}dt_2dt_1\right)
\end{aligned}$$

$$
\begin{aligned}
&+\Bigg(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)} \\
&\qquad\qquad \texttt{Boole}[(t_1,t_2,t_3,t_4)\in U_{A2}]\frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1t_2t_3t_4^2}dt_4dt_3dt_2dt_1\Bigg) \\
&-\Bigg(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\int_{t_4}^{\frac{1-t_1-t_2-t_3-t_4}{2}} \\
&\qquad\qquad \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5)\in U_{A3}]\frac{\omega_0\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1t_2t_3t_4t_5^2}dt_5dt_4dt_3dt_2dt_1\Bigg) \\
&+\Bigg(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)} \\
&\qquad \int_{\nu_0}^{\min\left(t_4,\frac{1-t_1-t_2-t_3-t_4}{2}\right)}\int_{\nu_0}^{\min\left(t_5,\frac{1-t_1-t_2-t_3-t_4-t_5}{2}\right)} \\
&\qquad\qquad \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6)\in U_{A4}]\frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{t_6}\right)}{t_1t_2t_3t_4t_5t_6^2}dt_6dt_5dt_4dt_3dt_2dt_1\Bigg) \\
&+\Bigg(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\frac{1}{2}t_1} \\
&\qquad\qquad \texttt{Boole}[(t_1,t_2,t_3,t_4)\in U_{A5}]\frac{\omega_1\left(\frac{t_1-t_4}{t_4}\right)\omega_1\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2t_3^2t_4^2}dt_4dt_3dt_2dt_1\Bigg) \\
&\leqslant (0.239221+0-0+0+0)=0.239221,
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
U_{A1}(\alpha_1,\alpha_2) :=\ & \left\{\boldsymbol{\alpha}_2\in A\backslash A',\ \nu_0\leqslant\alpha_1<\frac{1}{2},\ \nu_0\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\right\}, \\
U_{A2}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) :=\ & \Bigg\{\boldsymbol{\alpha}_2\in A',\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right),\ \boldsymbol{\alpha}_3\notin \boldsymbol{G}_3, \\
& \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n)\in\boldsymbol{D}_0 \text{ or } (m,n,h)\in\boldsymbol{D}_1\cup\boldsymbol{D}_2, \\
& \nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right),\ \boldsymbol{\alpha}_4\notin\boldsymbol{G}_4,\ \boldsymbol{\alpha}_4\notin\boldsymbol{D}^*, \\
& \nu_0\leqslant\alpha_1<\frac{1}{2},\ \nu_0\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\Bigg\}, \\
U_{A3}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) :=\ & \Bigg\{\boldsymbol{\alpha}_2\in A',\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right),\ \boldsymbol{\alpha}_3\notin \boldsymbol{G}_3, \\
& \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n)\in\boldsymbol{D}_0 \text{ or } (m,n,h)\in\boldsymbol{D}_1\cup\boldsymbol{D}_2, \\
& \nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right),\ \boldsymbol{\alpha}_4\notin\boldsymbol{G}_4,\ \boldsymbol{\alpha}_4\notin\boldsymbol{D}^*, \\
& \alpha_4<\alpha_5<\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4),\ \boldsymbol{\alpha}_5\in\boldsymbol{G}_5, \\
& \nu_0\leqslant\alpha_1<\frac{1}{2},\ \nu_0\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\Bigg\}, \\
U_{A4}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6) :=\ & \Bigg\{\boldsymbol{\alpha}_2\in A',\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right),\ \boldsymbol{\alpha}_3\notin \boldsymbol{G}_3, \\
& \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n)\in\boldsymbol{D}_0 \text{ or } (m,n,h)\in\boldsymbol{D}_1\cup\boldsymbol{D}_2, \\
& \nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right),\ \boldsymbol{\alpha}_4\notin\boldsymbol{G}_4,\ \boldsymbol{\alpha}_4\in\boldsymbol{D}^*, \\
& \nu_0\leqslant\alpha_5<\min\left(\alpha_4,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right),\ \boldsymbol{\alpha}_5\notin\boldsymbol{G}_5, \\
& \nu_0\leqslant\alpha_6<\min\left(\alpha_5,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5)\right),\ \boldsymbol{\alpha}_6\notin\boldsymbol{G}_6, \\
& \nu_0\leqslant\alpha_1<\frac{1}{2},\ \nu_0\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
U_{A5}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) := \Bigg\{ & \boldsymbol{\alpha}_2 \in A',\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right),\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3, \\
& \boldsymbol{\alpha}_3 \text{ cannot be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2, \\
& \nu_0 \leqslant \alpha_4 < \frac{1}{2}\alpha_1,\ \boldsymbol{\alpha}_4' \notin \boldsymbol{G}_4, \\
& \nu_0 \leqslant \alpha_1 < \frac{1}{2},\ \nu_0 \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)\Bigg\}.
\end{aligned}
$$

Note that the above integrals arise from sums T_{011}, T_{012}, T_{02} and the two-dimensional sum over region $A \backslash A'$, and one can compare our integrals to those in [5] and [20]. For example, one can see the integrals corresponding to U_{A2} and U_{A3} as an simple explicit expression of the function $w(\boldsymbol{\alpha}_4)$ defined in [[20], Chapter 7.9]. In [5] and [20] reversed Buchstab's identity has been used many many times, but we don't consider using it repeatedly since the savings over high-dimensional sums produced by this technique are very small. We choose an extremely small region A' so that the loss from this region is zero. If A' is large, then we may get a high-dimensional loss which exceeds the original two-dimensional loss from A'.

Remember that we have $\alpha_3 < \alpha_2 \leqslant \frac{3\theta-1}{2} \leqslant \frac{3\theta+1-4\alpha_1^*}{2}$ and at least one of $\alpha_1+\alpha_2$ and $1-\alpha_1-\alpha_2$ is $\leqslant \frac{1}{2}$ when $(\alpha_1,\alpha_2) \in C$, further decompositions in region C are possible. For $\sum_C$ we can redo the above decomposition procedure (12) on the entire region C to reach two four-dimensional sums

$$
T_{03} := \sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right) \tag{16}
$$

and

$$
T_{04} := \sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ cannot be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \frac{1}{2}\alpha_1 \\ \boldsymbol{\alpha}_4' \notin \boldsymbol{G}_4}} S\left(\mathcal{A}_{\beta p_2p_3p_4}, p_4\right). \tag{17}
$$

For T_{03}, we can perform Buchstab's identity twice more if $\boldsymbol{\alpha}_4 \in \boldsymbol{D}^*$, and we can use Buchstab's identity twice with a role-reversal if $\boldsymbol{\alpha}_4 \in \boldsymbol{D}^{\ddagger}$. Again, we can apply Buchstab's identity in reverse to gain some savings by making almost-primes visible. Similar to the decomposition procedure (13), we have the following expression after the decomposition of T_{03}:

$$
\begin{aligned}
T_{03} = & \sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right) \\
= & \sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^*,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^{\ddagger}}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right) \\
& + \sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^*}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right) \\
& + \sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^*,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^{\ddagger}}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right)
\end{aligned}
$$

$$
= \sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^*,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^{\ddagger}}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right)
$$

$$
+ \sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^*}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, x^{\nu_0}\right)
$$

$$
- \sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^* \\ \nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5 \in \boldsymbol{G}_5}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5\right)
$$

$$
- \sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^* \\ \nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5 \notin \boldsymbol{G}_5}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, x^{\nu_0}\right)
$$

$$
+ \sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^* \\ \nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5 \notin \boldsymbol{G}_5 \\ \nu_0 \leqslant \alpha_6 < \min\left(\alpha_5, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5)\right) \\ \boldsymbol{\alpha}_6 \in \boldsymbol{G}_6}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5 p_6}, p_6\right)
$$

$$
+ \sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^* \\ \nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5 \notin \boldsymbol{G}_5 \\ \nu_0 \leqslant \alpha_6 < \min\left(\alpha_5, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5)\right) \\ \boldsymbol{\alpha}_6 \notin \boldsymbol{G}_6}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5 p_6}, p_6\right)
$$

$$
+ \sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^*,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^{\ddagger}}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, x^{\nu_0}\right)
$$

$$
- \sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^*,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^{\ddagger} \\ \nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5 \in \boldsymbol{G}_5}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5\right)
$$

$$
\begin{aligned}
&- \sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^*,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^{\ddagger} \\ \nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5 \notin \boldsymbol{G}_5}} S\left(\mathcal{A}_{\gamma p_2 p_3 p_4 p_5}, x^{\nu_0}\right) \\
&+ \sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^*,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^{\ddagger} \\ \nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5 \notin \boldsymbol{G}_5 \\ \nu_0 \leqslant \alpha_6 < \frac{1}{2}\alpha_1 \\ \boldsymbol{\alpha}_6^{\ddagger} \in \boldsymbol{G}_6}} S\left(\mathcal{A}_{\gamma p_2 p_3 p_4 p_5 p_6}, p_6\right) \\
&+ \sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^*,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^{\ddagger} \\ \nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5 \notin \boldsymbol{G}_5 \\ \nu_0 \leqslant \alpha_6 < \frac{1}{2}\alpha_1 \\ \boldsymbol{\alpha}_6^{\ddagger} \notin \boldsymbol{G}_6}} S\left(\mathcal{A}_{\gamma p_2 p_3 p_4 p_5 p_6}, p_6\right) \\
&= T_{031} + S_{11} - S_{12} - S_{13} + S_{14} + T_{032} + S_{15} - S_{16} - S_{17} + S_{18} + T_{033}, \qquad (18)
\end{aligned}
$$

where $\gamma \sim x^{1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5}$, $(\gamma, P(p_5)) = 1$ and $\boldsymbol{\alpha}_6^{\ddagger} = (1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6)$. We can give asymptotic formulas for S_{11}–S_{18}. For T_{031} we can use Buchstab's identity in reverse to subtract the sum

$$
\sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^*,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^{\ddagger} \\ \alpha_4 < \alpha_5 < \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4) \\ \boldsymbol{\alpha}_5 \in \boldsymbol{G}_5}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5\right) \qquad (19)
$$

from the loss, and for T_{032} we can perform a straightforward decomposition if $\boldsymbol{\alpha}_6 \in \boldsymbol{D}^{**}$, leading to an eight-dimensional sum

$$
\sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^* \\ \nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5 \notin \boldsymbol{G}_5 \\ \nu_0 \leqslant \alpha_6 < \min\left(\alpha_5, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5)\right) \\ \boldsymbol{\alpha}_6 \notin \boldsymbol{G}_6,\ \boldsymbol{\alpha}_6 \in \boldsymbol{D}^{**} \\ \nu_0 \leqslant \alpha_7 < \min\left(\alpha_6, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6)\right) \\ \boldsymbol{\alpha}_7 \notin \boldsymbol{G}_7 \\ \nu_0 \leqslant \alpha_8 < \min\left(\alpha_7, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7)\right) \\ \boldsymbol{\alpha}_8 \notin \boldsymbol{G}_8}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5 p_6 p_7 p_8}, p_8\right) \qquad (20)
$$

Note that in T_{033} we counts numbers of the form $\gamma p_2 p_3 p_4 p_5 p_6 \gamma_1$ with two almost-prime variables γ and $\gamma_1 \sim x^{\alpha_1-\alpha_6}$, hence we can perform a straightforward decomposition if either $\boldsymbol{\alpha}_6^{\ddagger} \in \boldsymbol{D}^{**}$ or $\boldsymbol{\alpha}_6^{\ddagger\prime} \in \boldsymbol{D}^{**}$, where $\boldsymbol{\alpha}_6^{\ddagger\prime} \in (\alpha_1-\alpha_6,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_6,\ \alpha_5)$.

That is, we write

$$
\begin{aligned}
T_{033} = & \sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^*,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^{\ddagger} \\ \nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5 \notin \boldsymbol{G}_5 \\ \nu_0 \leqslant \alpha_6 < \frac{1}{2}\alpha_1 \\ \boldsymbol{\alpha}_6^{\ddagger} \notin \boldsymbol{G}_6}} S\left(\mathcal{A}_{\gamma p_2 p_3 p_4 p_5 p_6}, p_6\right) \\
= & \sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^*,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^{\ddagger} \\ \nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5 \notin \boldsymbol{G}_5 \\ \nu_0 \leqslant \alpha_6 < \frac{1}{2}\alpha_1 \\ \boldsymbol{\alpha}_6^{\ddagger} \notin \boldsymbol{G}_6,\ \boldsymbol{\alpha}_6^{\ddagger} \notin \boldsymbol{D}^{**},\ \boldsymbol{\alpha}_6^{\ddagger\prime} \notin \boldsymbol{D}^{**}}} S\left(\mathcal{A}_{\gamma p_2 p_3 p_4 p_5 p_6}, p_6\right) \\
& + \sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^*,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^{\ddagger} \\ \nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5 \notin \boldsymbol{G}_5 \\ \nu_0 \leqslant \alpha_6 < \frac{1}{2}\alpha_1 \\ \boldsymbol{\alpha}_6^{\ddagger} \notin \boldsymbol{G}_6,\ \boldsymbol{\alpha}_6^{\ddagger} \in \boldsymbol{D}^{**}}} S\left(\mathcal{A}_{\gamma p_2 p_3 p_4 p_5 p_6}, p_6\right) \\
& + \sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^*,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^{\ddagger} \\ \nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5 \notin \boldsymbol{G}_5 \\ \nu_0 \leqslant \alpha_6 < \frac{1}{2}\alpha_1 \\ \boldsymbol{\alpha}_6^{\ddagger} \notin \boldsymbol{G}_6,\ \boldsymbol{\alpha}_6^{\ddagger} \notin \boldsymbol{D}^{**},\ \boldsymbol{\alpha}_6^{\ddagger\prime} \in \boldsymbol{D}^{**}}} S\left(\mathcal{A}_{\gamma p_2 p_3 p_4 p_5 p_6}, p_6\right) \\
= & T_{0331} + T_{0332} + T_{0333}.
\end{aligned}
\tag{21}
$$

We discard the whole of T_{0331}. For T_{0332} we perform a straightforward decomposition to reach an eight-dimensional sum

$$
\sum_{\substack{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2 \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^*,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^{\ddagger} \\ \nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5 \notin \boldsymbol{G}_5 \\ \nu_0 \leqslant \alpha_6 < \frac{1}{2}\alpha_1 \\ \boldsymbol{\alpha}_6^{\ddagger} \notin \boldsymbol{G}_6,\ \boldsymbol{\alpha}_6^{\ddagger} \in \boldsymbol{D}^{**} \\ \nu_0 \leqslant \alpha_7 < \min\left(\alpha_6, \frac{1}{2}(\alpha_1-\alpha_6)\right) \\ \boldsymbol{\alpha}_7^{\ddagger} \notin \boldsymbol{G}_7 \\ \nu_0 \leqslant \alpha_8 < \min\left(\alpha_7, \frac{1}{2}(\alpha_1-\alpha_6-\alpha_7)\right) \\ \boldsymbol{\alpha}_8^{\ddagger} \notin \boldsymbol{G}_8}} S\left(\mathcal{A}_{\gamma p_2 p_3 p_4 p_5 p_6 p_7 p_8}, p_8\right), \tag{22}
$$

where $\boldsymbol{\alpha}_7^{\ddagger} = (1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6,\ \alpha_7)$ and $\boldsymbol{\alpha}_8^{\ddagger} = (1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6,\ \alpha_7,\ \alpha_8)$. For T_{0333} we reverse the roles of γ and γ_1 to get

$$T_{0333} = \sum_{\substack{\boldsymbol{\alpha}_2\in C,\ \boldsymbol{\alpha}_2\notin \boldsymbol{G}_2 \\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n)\in\boldsymbol{D}_0 \text{ or } (m,n,h)\in\boldsymbol{D}_1\cup\boldsymbol{D}_2 \\ \nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4\notin\boldsymbol{G}_4,\ \boldsymbol{\alpha}_4\notin\boldsymbol{D}^*,\ \boldsymbol{\alpha}_4\in\boldsymbol{D}^{\ddagger} \\ \nu_0\leqslant\alpha_5<\min\left(\alpha_4,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5\notin\boldsymbol{G}_5 \\ \nu_0\leqslant\alpha_6<\frac{1}{2}\alpha_1 \\ \boldsymbol{\alpha}_6^{\ddagger}\notin\boldsymbol{G}_6,\ \boldsymbol{\alpha}_6^{\ddagger}\notin\boldsymbol{D}^{**},\ \boldsymbol{\alpha}_6^{\ddagger\prime}\in\boldsymbol{D}^{**}}} S\left(\mathcal{A}_{\gamma p_2 p_3 p_4 p_5 p_6}, p_6\right)$$

$$= \sum_{\substack{\boldsymbol{\alpha}_2\in C,\ \boldsymbol{\alpha}_2\notin \boldsymbol{G}_2 \\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n)\in\boldsymbol{D}_0 \text{ or } (m,n,h)\in\boldsymbol{D}_1\cup\boldsymbol{D}_2 \\ \nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4\notin\boldsymbol{G}_4,\ \boldsymbol{\alpha}_4\notin\boldsymbol{D}^*,\ \boldsymbol{\alpha}_4\in\boldsymbol{D}^{\ddagger} \\ \nu_0\leqslant\alpha_5<\min\left(\alpha_4,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5\notin\boldsymbol{G}_5 \\ \nu_0\leqslant\alpha_6<\frac{1}{2}\alpha_1 \\ \boldsymbol{\alpha}_6^{\ddagger}\notin\boldsymbol{G}_6,\ \boldsymbol{\alpha}_6^{\ddagger}\notin\boldsymbol{D}^{**},\ \boldsymbol{\alpha}_6^{\ddagger\prime}\in\boldsymbol{D}^{**}}} S\left(\mathcal{A}_{\gamma_1 p_2 p_3 p_4 p_5 p_6}, p_5\right), \tag{23}$$

where we can perform a straightforward decomposition on the sum on the right hand side to reach an eight-dimensional sum

$$\sum_{\substack{\boldsymbol{\alpha}_2\in C,\ \boldsymbol{\alpha}_2\notin \boldsymbol{G}_2 \\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n)\in\boldsymbol{D}_0 \text{ or } (m,n,h)\in\boldsymbol{D}_1\cup\boldsymbol{D}_2 \\ \nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4\notin\boldsymbol{G}_4,\ \boldsymbol{\alpha}_4\notin\boldsymbol{D}^*,\ \boldsymbol{\alpha}_4\in\boldsymbol{D}^{\ddagger} \\ \nu_0\leqslant\alpha_5<\min\left(\alpha_4,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5\notin\boldsymbol{G}_5 \\ \nu_0\leqslant\alpha_6<\frac{1}{2}\alpha_1 \\ \boldsymbol{\alpha}_6^{\ddagger}\notin\boldsymbol{G}_6,\ \boldsymbol{\alpha}_6^{\ddagger}\notin\boldsymbol{D}^{**},\ \boldsymbol{\alpha}_6^{\ddagger\prime}\in\boldsymbol{D}^{**} \\ \nu_0\leqslant\alpha_7<\min\left(\alpha_5,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5)\right) \\ \boldsymbol{\alpha}_7^{\ddagger\prime}\notin\boldsymbol{G}_7 \\ \nu_0\leqslant\alpha_8<\min\left(\alpha_7,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_7)\right) \\ \boldsymbol{\alpha}_8^{\ddagger\prime}\notin\boldsymbol{G}_8}} S\left(\mathcal{A}_{\gamma_1 p_2 p_3 p_4 p_5 p_6 p_7 p_8}, p_8\right), \tag{24}$$

where $\boldsymbol{\alpha}_7^{\ddagger\prime} = (\alpha_1-\alpha_6,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6,\ \alpha_7)$ and $\boldsymbol{\alpha}_8^{\ddagger\prime} = (\alpha_1-\alpha_6,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6,\ \alpha_7,\ \alpha_8)$. We can also use Buchstab's identity in reverse on those sums, but the corresponding savings are quite small.

For T_{04} we can also use the devices mentioned earlier to take into account the savings over this sum. Note that there are two almost-prime variables counted by this sum, so the use of straightforward decompositions is just like the case in T_{033}. The sum T_{04} counts numbers of the form $\beta p_2 p_3 p_4 \beta_1$, where $\beta \sim x^{1-\alpha_1-\alpha_2-\alpha_3}$, $(\beta, P(p_3)) = 1$, $\beta_1 \sim x^{\alpha_1-\alpha_4}$ and $(\beta_1, P(p_4)) = 1$. Here we can decompose either β or β_1, leading to the six-dimensional sums

$$T_{041} := \sum_{\substack{\boldsymbol{\alpha}_2\in C,\ \boldsymbol{\alpha}_2\notin \boldsymbol{G}_2 \\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ cannot be partitioned into } (m,n)\in\boldsymbol{D}_0 \text{ or } (m,n,h)\in\boldsymbol{D}_1\cup\boldsymbol{D}_2 \\ \nu_0\leqslant\alpha_4<\frac{1}{2}\alpha_1 \\ \boldsymbol{\alpha}_4'\notin\boldsymbol{G}_4,\ \boldsymbol{\alpha}_4'\in\boldsymbol{D}^* \\ \nu_0\leqslant\alpha_5<\min\left(\alpha_4,\frac{1}{2}(\alpha_1-\alpha_4)\right) \\ \boldsymbol{\alpha}_5'\notin\boldsymbol{G}_5 \\ \nu_0\leqslant\alpha_6<\min\left(\alpha_5,\frac{1}{2}(\alpha_1-\alpha_4-\alpha_5)\right) \\ \boldsymbol{\alpha}_6'\notin\boldsymbol{G}_6}} S\left(\mathcal{A}_{\beta p_2 p_3 p_4 p_5 p_6}, p_6\right) \tag{25}$$

and

$$T_{042} := \sum_{\substack{\boldsymbol{\alpha}_2\in C,\ \boldsymbol{\alpha}_2\notin \boldsymbol{G}_2 \\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3 \\ \boldsymbol{\alpha}_3 \text{ cannot be partitioned into } (m,n)\in\boldsymbol{D}_0 \text{ or } (m,n,h)\in\boldsymbol{D}_1\cup\boldsymbol{D}_2 \\ \nu_0\leqslant\alpha_4<\frac{1}{2}\alpha_1 \\ \boldsymbol{\alpha}_4'\notin\boldsymbol{G}_4,\ \boldsymbol{\alpha}_4'\notin\boldsymbol{D}^*,\ \boldsymbol{\alpha}_4''\in\boldsymbol{D}^* \\ \nu_0\leqslant\alpha_5<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_5''\notin\boldsymbol{G}_5 \\ \nu_0\leqslant\alpha_6<\min\left(\alpha_5,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_5)\right) \\ \boldsymbol{\alpha}_6''\notin\boldsymbol{G}_6}} S\left(\mathcal{A}_{\beta_1 p_2 p_3 p_4 p_5 p_6}, p_6\right), \tag{26}$$

where $\boldsymbol{\alpha}_5' = (1-\alpha_1-\alpha_2-\alpha_3,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5)$, $\boldsymbol{\alpha}_6' = (1-\alpha_1-\alpha_2-\alpha_3,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6)$, $\boldsymbol{\alpha}_4'' = (\alpha_1-\alpha_4,\ \alpha_2,\ \alpha_4,\ \alpha_3)$, $\boldsymbol{\alpha}_5'' = (\alpha_1-\alpha_4,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5)$ and $\boldsymbol{\alpha}_6'' = (\alpha_1-\alpha_4,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6)$. On the remaining of T_{04} (with $\boldsymbol{\alpha}_4' \notin \boldsymbol{D}^*$ and $\boldsymbol{\alpha}_4'' \notin \boldsymbol{D}^*$) we can use Buchstab's identity in reverse to make savings, and the use of reversed Buchstab's identity over this sum can be seen as the following: the remaining sum counts numbers of the form $\beta p_2 p_3 p_4 \beta_1$, hence we can decompose either β or β_1, leading to the savings of numbers of the forms

$$\beta p_2 p_3 p_4 (p_5 \beta_2) \quad \text{where } \beta_2 \sim x^{\alpha_1-\alpha_4-\alpha_5},\ (\beta_2, P(p_5)) = 1,\ \boldsymbol{\alpha}_5' \in \boldsymbol{G}_5$$

and

$$(\beta_3 p_5) p_2 p_3 p_4 \beta_1 \quad \text{where } \beta_3 \sim x^{1-\alpha_1-\alpha_2-\alpha_3-\alpha_5},\ (\beta_3, P(p_5)) = 1,\ \boldsymbol{\alpha}_5'' \in \boldsymbol{G}_5.$$

Here, the numbers of the form

$$(\beta_3 p_6) p_2 p_3 p_4 (p_5 \beta_2) \quad \text{where } \boldsymbol{\alpha}_5' \in \boldsymbol{G}_5 \text{ and } \boldsymbol{\alpha}_5''' := (\alpha_1-\alpha_4, \alpha_2, \alpha_3, \alpha_4, \alpha_6) \in \boldsymbol{G}_5$$

are counted twice, hence we need to subtract them from the savings. For simplicity we omit the sieve iteration process of this sum. One can see the second part of the estimation of Φ_7 in [33] to understand this decomposing procedure.

Altogether we get a loss from $\sum_C$ of

$$\begin{aligned}
&\Bigg(\int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\nu_0}^{\min\left(t_3, \frac{1-t_1-t_2-t_3}{2}\right)} \\
&\qquad \texttt{Boole}[(t_1,t_2,t_3,t_4) \in U_{C01}] \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1\Bigg) \\
-&\Bigg(\int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\nu_0}^{\min\left(t_3, \frac{1-t_1-t_2-t_3}{2}\right)} \int_{t_4}^{\frac{1-t_1-t_2-t_3-t_4}{2}} \\
&\qquad \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5) \in U_{C02}] \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1\Bigg) \\
+&\Bigg(\int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\nu_0}^{\min\left(t_3, \frac{1-t_1-t_2-t_3}{2}\right)} \\
&\qquad \int_{\nu_0}^{\min\left(t_4, \frac{1-t_1-t_2-t_3-t_4}{2}\right)} \int_{\nu_0}^{\min\left(t_5, \frac{1-t_1-t_2-t_3-t_4-t_5}{2}\right)} \\
&\qquad \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6) \in U_{C03}] \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{t_6}\right)}{t_1 t_2 t_3 t_4 t_5 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1\Bigg) \\
+&\Bigg(\int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\nu_0}^{\min\left(t_3, \frac{1-t_1-t_2-t_3}{2}\right)} \int_{\nu_0}^{\min\left(t_4, \frac{1-t_1-t_2-t_3-t_4}{2}\right)} \int_{\nu_0}^{\frac{1}{2}t_1} \\
&\qquad \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6) \in U_{C04}] \frac{\omega\left(\frac{t_1-t_6}{t_6}\right) \omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_2 t_3 t_4 t_5^2 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1\Bigg) \\
+&\Bigg(\int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\nu_0}^{\min\left(t_3, \frac{1-t_1-t_2-t_3}{2}\right)} \\
&\qquad \int_{\nu_0}^{\min\left(t_4, \frac{1-t_1-t_2-t_3-t_4}{2}\right)} \int_{\nu_0}^{\min\left(t_5, \frac{1-t_1-t_2-t_3-t_4-t_5}{2}\right)} \\
&\qquad \int_{\nu_0}^{\min\left(t_6, \frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{2}\right)} \int_{\nu_0}^{\min\left(t_7, \frac{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7}{2}\right)} \\
&\qquad \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) \in U_{C05}] \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7-t_8}{t_8}\right)}{t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1\Bigg) \\
+&\Bigg(\int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\nu_0}^{\min\left(t_3, \frac{1-t_1-t_2-t_3}{2}\right)} \int_{\nu_0}^{\min\left(t_4, \frac{1-t_1-t_2-t_3-t_4}{2}\right)} \\
&\qquad \int_{\nu_0}^{\frac{1}{2}t_1} \int_{\nu_0}^{\min\left(t_6, \frac{t_1-t_6}{2}\right)} \int_{\nu_0}^{\min\left(t_7, \frac{t_1-t_6-t_7}{2}\right)} \\
&\qquad \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) \in U_{C06}] \frac{\omega\left(\frac{t_1-t_6-t_7-t_8}{t_8}\right) \omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_2 t_3 t_4 t_5^2 t_6 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1\Bigg)
\end{aligned}$$

$$+\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\int_{\nu_0}^{\min\left(t_4,\frac{1-t_1-t_2-t_3-t_4}{2}\right)}\right.$$

$$\int_{\nu_0}^{\frac{1}{2}t_1}\int_{\nu_0}^{\min\left(t_5,\frac{1-t_1-t_2-t_3-t_4-t_5}{2}\right)}\int_{\nu_0}^{\min\left(t_7,\frac{1-t_1-t_2-t_3-t_4-t_5-t_7}{2}\right)}$$

$$\left.\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8)\in U_{C07}]\frac{\omega\left(\frac{t_1-t_6}{t_6}\right)\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_7-t_8}{t_8}\right)}{t_2t_3t_4t_5t_6^2t_7t_8^2}dt_8dt_7dt_6dt_5dt_4dt_3dt_2dt_1\right)$$

$$+\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\frac{1}{2}t_1}\right.$$

$$\left.\texttt{Boole}[(t_1,t_2,t_3,t_4)\in U_{C08}]\frac{\omega\left(\frac{t_1-t_4}{t_4}\right)\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2t_3^2t_4^2}dt_4dt_3dt_2dt_1\right)$$

$$-\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\frac{1}{2}t_1}\int_{t_4}^{\frac{t_1-t_4}{2}}\right.$$

$$\left.\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5)\in U_{C09}]\frac{\omega\left(\frac{t_1-t_4-t_5}{t_5}\right)\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2t_3^2t_4t_5^2}dt_5dt_4dt_3dt_2dt_1\right)$$

$$-\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\frac{1}{2}t_1}\int_{t_3}^{\frac{1-t_1-t_2-t_3}{2}}\right.$$

$$\left.\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5)\in U_{C10}]\frac{\omega\left(\frac{t_1-t_4}{t_4}\right)\omega\left(\frac{1-t_1-t_2-t_3-t_5}{t_5}\right)}{t_2t_3t_4^2t_5^2}dt_5dt_4dt_3dt_2dt_1\right)$$

$$+\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\frac{1}{2}t_1}\int_{t_4}^{\frac{t_1-t_4}{2}}\int_{t_3}^{\frac{1-t_1-t_2-t_3}{2}}\right.$$

$$\left.\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6)\in U_{C11}]\frac{\omega\left(\frac{t_1-t_4-t_5}{t_5}\right)\omega\left(\frac{1-t_1-t_2-t_3-t_6}{t_6}\right)}{t_2t_3t_4t_5^2t_6^2}dt_6dt_5dt_4dt_3dt_2dt_1\right)$$

$$+\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\frac{1}{2}t_1}\int_{\nu_0}^{\min\left(t_4,\frac{t_1-t_4}{2}\right)}\int_{\nu_0}^{\min\left(t_5,\frac{t_1-t_4-t_5}{2}\right)}\right.$$

$$\left.\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6)\in U_{C12}]\frac{\omega\left(\frac{t_1-t_4-t_5-t_6}{t_6}\right)\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2t_3^2t_4t_5t_6^2}dt_6dt_5dt_4dt_3dt_2dt_1\right)$$

$$+\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\frac{1}{2}t_1}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\int_{\nu_0}^{\min\left(t_5,\frac{1-t_1-t_2-t_3-t_5}{2}\right)}\right.$$

$$\left.\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6)\in U_{C13}]\frac{\omega\left(\frac{t_1-t_4}{t_4}\right)\omega\left(\frac{1-t_1-t_2-t_3-t_5-t_6}{t_6}\right)}{t_2t_3t_4^2t_5t_6^2}dt_6dt_5dt_4dt_3dt_2dt_1\right)$$

$$\leqslant\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\right.$$

$$\left.\texttt{Boole}[(t_1,t_2,t_3,t_4)\in U_{C01}]\frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1t_2t_3t_4^2}dt_4dt_3dt_2dt_1\right)$$

$$-\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\int_{t_4}^{\frac{1-t_1-t_2-t_3-t_4}{2}}\right.$$

$$\left.\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5)\in U_{C02}]\frac{\omega_0\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1t_2t_3t_4t_5^2}dt_5dt_4dt_3dt_2dt_1\right)$$

$$+\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\right.$$

$$\int_{\nu_0}^{\min\left(t_4,\frac{1-t_1-t_2-t_3-t_4}{2}\right)}\int_{\nu_0}^{\min\left(t_5,\frac{1-t_1-t_2-t_3-t_4-t_5}{2}\right)}$$

$$\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6)\in U_{C03}]\frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{t_6}\right)}{t_1t_2t_3t_4t_5t_6^2}dt_6dt_5dt_4dt_3dt_2dt_1\Bigg)$$

$$+\Bigg(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\int_{\nu_0}^{\min\left(t_4,\frac{1-t_1-t_2-t_3-t_4}{2}\right)}\int_{\nu_0}^{\frac{1}{2}t_1}$$

$$\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6)\in U_{C04}]\frac{\omega_1\left(\frac{t_1-t_6}{t_6}\right)\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_2t_3t_4t_5^2t_6^2}dt_6dt_5dt_4dt_3dt_2dt_1\Bigg)$$

$$+\Bigg(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}$$

$$\int_{\nu_0}^{\min\left(t_4,\frac{1-t_1-t_2-t_3-t_4}{2}\right)}\int_{\nu_0}^{\min\left(t_5,\frac{1-t_1-t_2-t_3-t_4-t_5}{2}\right)}$$

$$\int_{\nu_0}^{\min\left(t_6,\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{2}\right)}\int_{\nu_0}^{\min\left(t_7,\frac{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7}{2}\right)}$$

$$\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8)\in U_{C05}]\frac{\max\left(\frac{t_8}{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7-t_8},0.5672\right)}{t_1t_2t_3t_4t_5t_6t_7t_8^2}dt_8dt_7dt_6dt_5dt_4dt_3dt_2dt_1\Bigg)$$

$$+\Bigg(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\int_{\nu_0}^{\min\left(t_4,\frac{1-t_1-t_2-t_3-t_4}{2}\right)}$$

$$\int_{\nu_0}^{\frac{1}{2}t_1}\int_{\nu_0}^{\min\left(t_6,\frac{t_1-t_6}{2}\right)}\int_{\nu_0}^{\min\left(t_7,\frac{t_1-t_6-t_7}{2}\right)}\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8)\in U_{C06}]\times$$

$$\frac{\max\left(\frac{t_8}{t_1-t_6-t_7-t_8},0.5672\right)\max\left(\frac{t_5}{1-t_1-t_2-t_3-t_4-t_5},0.5672\right)}{t_2t_3t_4t_5^2t_6t_7t_8^2}dt_8dt_7dt_6dt_5dt_4dt_3dt_2dt_1\Bigg)$$

$$+\Bigg(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\int_{\nu_0}^{\min\left(t_4,\frac{1-t_1-t_2-t_3-t_4}{2}\right)}$$

$$\int_{\nu_0}^{\frac{1}{2}t_1}\int_{\nu_0}^{\min\left(t_5,\frac{1-t_1-t_2-t_3-t_4-t_5}{2}\right)}\int_{\nu_0}^{\min\left(t_7,\frac{1-t_1-t_2-t_3-t_4-t_5-t_7}{2}\right)}\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8)\in U_{C07}]\times$$

$$\frac{\max\left(\frac{t_6}{t_1-t_6},0.5672\right)\max\left(\frac{t_8}{1-t_1-t_2-t_3-t_4-t_5-t_7-t_8},0.5672\right)}{t_2t_3t_4t_5t_6^2t_7t_8^2}dt_8dt_7dt_6dt_5dt_4dt_3dt_2dt_1\Bigg)$$

$$+\Bigg(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\frac{1}{2}t_1}$$

$$\texttt{Boole}[(t_1,t_2,t_3,t_4)\in U_{C08}]\frac{\omega_1\left(\frac{t_1-t_4}{t_4}\right)\omega_1\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2t_3^2t_4^2}dt_4dt_3dt_2dt_1\Bigg)$$

$$-\Bigg(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\frac{1}{2}t_1}\int_{t_4}^{\frac{t_1-t_4}{2}}$$

$$\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5)\in U_{C09}]\frac{\omega_0\left(\frac{t_1-t_4-t_5}{t_5}\right)\omega_0\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2t_3^2t_4t_5^2}dt_5dt_4dt_3dt_2dt_1\Bigg)$$

$$-\Bigg(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\frac{1}{2}t_1}\int_{t_3}^{\frac{1-t_1-t_2-t_3}{2}}$$

$$\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5)\in U_{C10}]\frac{\omega_0\left(\frac{t_1-t_4}{t_4}\right)\omega_0\left(\frac{1-t_1-t_2-t_3-t_5}{t_5}\right)}{t_2t_3t_4^2t_5^2}dt_5dt_4dt_3dt_2dt_1\Bigg)$$

$$+\Bigg(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\frac{1}{2}t_1}\int_{t_4}^{\frac{t_1-t_4}{2}}\int_{t_3}^{\frac{1-t_1-t_2-t_3}{2}}$$

$$\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6)\in U_{C11}]\frac{\omega_1\left(\frac{t_1-t_4-t_5}{t_5}\right)\omega_1\left(\frac{1-t_1-t_2-t_3-t_6}{t_6}\right)}{t_2t_3t_4t_5^2t_6^2}dt_6dt_5dt_4dt_3dt_2dt_1\Bigg)$$

$$+\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\frac{1}{2}t_1}\int_{\nu_0}^{\min\left(t_4,\frac{t_1-t_4}{2}\right)}\int_{\nu_0}^{\min\left(t_5,\frac{t_1-t_4-t_5}{2}\right)}\right.$$

$$\left.\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6)\in U_{C12}]\frac{\omega_1\left(\frac{t_1-t_4-t_5-t_6}{t_6}\right)\omega_1\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2t_3^2t_4t_5t_6^2}dt_6dt_5dt_4dt_3dt_2dt_1\right)$$

$$+\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\frac{1}{2}t_1}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\int_{\nu_0}^{\min\left(t_5,\frac{1-t_1-t_2-t_3-t_5}{2}\right)}\right.$$

$$\left.\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6)\in U_{C13}]\frac{\omega_1\left(\frac{t_1-t_4}{t_4}\right)\omega_1\left(\frac{1-t_1-t_2-t_3-t_5-t_6}{t_6}\right)}{t_2t_3t_4^2t_5t_6^2}dt_6dt_5dt_4dt_3dt_2dt_1\right)$$

$$\leqslant (0.197907-0.001607+0.020936+0.065033+0.000101+0.000131+0$$
$$+0.20109-0.000693-0.000222+0.000048+0.008809+0)$$
$$=0.491533, \tag{27}$$

where

$$\begin{aligned}
U_{C01}(\alpha_1,\alpha_2,\alpha_3,\alpha_4):=\Bigg\{&\boldsymbol{\alpha}_2\in C,\ \boldsymbol{\alpha}_2\notin\boldsymbol{G}_2,\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right),\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\\
&\boldsymbol{\alpha}_3\text{ can be partitioned into }(m,n)\in\boldsymbol{D}_0\text{ or }(m,n,h)\in\boldsymbol{D}_1\cup\boldsymbol{D}_2,\\
&\nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right),\ \boldsymbol{\alpha}_4\notin\boldsymbol{G}_4,\ \boldsymbol{\alpha}_4\notin\boldsymbol{D}^*,\ \boldsymbol{\alpha}_4\notin\boldsymbol{D}^\ddagger,\\
&\nu_0\leqslant\alpha_1<\frac{1}{2},\ \nu_0\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\Bigg\},\\
U_{C02}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5):=\Bigg\{&\boldsymbol{\alpha}_2\in C,\ \boldsymbol{\alpha}_2\notin\boldsymbol{G}_2,\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right),\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\\
&\boldsymbol{\alpha}_3\text{ can be partitioned into }(m,n)\in\boldsymbol{D}_0\text{ or }(m,n,h)\in\boldsymbol{D}_1\cup\boldsymbol{D}_2,\\
&\nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right),\ \boldsymbol{\alpha}_4\notin\boldsymbol{G}_4,\ \boldsymbol{\alpha}_4\notin\boldsymbol{D}^*,\ \boldsymbol{\alpha}_4\notin\boldsymbol{D}^\ddagger,\\
&\alpha_4<\alpha_5<\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4),\ \boldsymbol{\alpha}_5\in\boldsymbol{G}_5,\\
&\nu_0\leqslant\alpha_1<\frac{1}{2},\ \nu_0\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\Bigg\},\\
U_{C03}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6):=\Bigg\{&\boldsymbol{\alpha}_2\in C,\ \boldsymbol{\alpha}_2\notin\boldsymbol{G}_2,\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right),\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\\
&\boldsymbol{\alpha}_3\text{ can be partitioned into }(m,n)\in\boldsymbol{D}_0\text{ or }(m,n,h)\in\boldsymbol{D}_1\cup\boldsymbol{D}_2,\\
&\nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right),\ \boldsymbol{\alpha}_4\notin\boldsymbol{G}_4,\ \boldsymbol{\alpha}_4\in\boldsymbol{D}^*,\\
&\nu_0\leqslant\alpha_5<\min\left(\alpha_4,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right),\ \boldsymbol{\alpha}_5\notin\boldsymbol{G}_5,\\
&\nu_0\leqslant\alpha_6<\min\left(\alpha_5,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5)\right),\ \boldsymbol{\alpha}_6\notin\boldsymbol{G}_6,\ \boldsymbol{\alpha}_6\notin\boldsymbol{D}^{**},\\
&\nu_0\leqslant\alpha_1<\frac{1}{2},\ \nu_0\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\Bigg\},\\
U_{C04}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6):=\Bigg\{&\boldsymbol{\alpha}_2\in C,\ \boldsymbol{\alpha}_2\notin\boldsymbol{G}_2,\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right),\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\\
&\boldsymbol{\alpha}_3\text{ can be partitioned into }(m,n)\in\boldsymbol{D}_0\text{ or }(m,n,h)\in\boldsymbol{D}_1\cup\boldsymbol{D}_2,\\
&\nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right),\ \boldsymbol{\alpha}_4\notin\boldsymbol{G}_4,\ \boldsymbol{\alpha}_4\notin\boldsymbol{D}^*,\ \boldsymbol{\alpha}_4\in\boldsymbol{D}^\ddagger,\\
&\nu_0\leqslant\alpha_5<\min\left(\alpha_4,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right),\ \boldsymbol{\alpha}_5\notin\boldsymbol{G}_5,\\
&\nu_0\leqslant\alpha_6<\frac{1}{2}\alpha_1,\ \boldsymbol{\alpha}_6^\ddagger\notin\boldsymbol{G}_6,\ \boldsymbol{\alpha}_6^\ddagger\notin\boldsymbol{D}^{**},\ \boldsymbol{\alpha}_6^{\ddagger\prime}\notin\boldsymbol{D}^{**},\\
&\nu_0\leqslant\alpha_1<\frac{1}{2},\ \nu_0\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\Bigg\},\\
U_{C05}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8):=\Bigg\{&\boldsymbol{\alpha}_2\in C,\ \boldsymbol{\alpha}_2\notin\boldsymbol{G}_2,\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right),\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\\
&\boldsymbol{\alpha}_3\text{ can be partitioned into }(m,n)\in\boldsymbol{D}_0\text{ or }(m,n,h)\in\boldsymbol{D}_1\cup\boldsymbol{D}_2,
\end{aligned}$$

$$
\begin{aligned}
&\nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right),\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^*,\\
&\nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right),\ \boldsymbol{\alpha}_5 \notin \boldsymbol{G}_5,\\
&\nu_0 \leqslant \alpha_6 < \min\left(\alpha_5, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5)\right),\ \boldsymbol{\alpha}_6 \notin \boldsymbol{G}_6,\ \boldsymbol{\alpha}_6 \in \boldsymbol{D}^{**},\\
&\nu_0 \leqslant \alpha_7 < \min\left(\alpha_6, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6)\right),\ \boldsymbol{\alpha}_7 \notin \boldsymbol{G}_7,\\
&\nu_0 \leqslant \alpha_8 < \min\left(\alpha_7, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7)\right),\ \boldsymbol{\alpha}_8 \notin \boldsymbol{G}_8,\\
&\left.\nu_0 \leqslant \alpha_1 < \frac{1}{2},\ \nu_0 \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
U_{C06}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8) := &\left\{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2,\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right),\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\right.\\
&\boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2,\\
&\nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right),\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^*,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^{\ddagger},\\
&\nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right),\ \boldsymbol{\alpha}_5 \notin \boldsymbol{G}_5,\\
&\nu_0 \leqslant \alpha_6 < \frac{1}{2}\alpha_1,\ \boldsymbol{\alpha}_6^{\ddagger} \notin \boldsymbol{G}_6,\ \boldsymbol{\alpha}_6^{\ddagger} \in \boldsymbol{D}^{**},\\
&\nu_0 \leqslant \alpha_7 < \min\left(\alpha_6, \frac{1}{2}(\alpha_1-\alpha_6)\right),\ \boldsymbol{\alpha}_7^{\ddagger} \notin \boldsymbol{G}_7,\\
&\nu_0 \leqslant \alpha_8 < \min\left(\alpha_7, \frac{1}{2}(\alpha_1-\alpha_6-\alpha_7)\right),\ \boldsymbol{\alpha}_8^{\ddagger} \notin \boldsymbol{G}_8,\\
&\left.\nu_0 \leqslant \alpha_1 < \frac{1}{2},\ \nu_0 \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
U_{C07}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8) := &\left\{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2,\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right),\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\right.\\
&\boldsymbol{\alpha}_3 \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2,\\
&\nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right),\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4 \notin \boldsymbol{D}^*,\ \boldsymbol{\alpha}_4 \in \boldsymbol{D}^{\ddagger},\\
&\nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right),\ \boldsymbol{\alpha}_5 \notin \boldsymbol{G}_5,\\
&\nu_0 \leqslant \alpha_6 < \frac{1}{2}\alpha_1,\ \boldsymbol{\alpha}_6^{\ddagger} \notin \boldsymbol{G}_6,\ \boldsymbol{\alpha}_6^{\ddagger} \notin \boldsymbol{D}^{**},\ \boldsymbol{\alpha}_6^{\ddagger\prime} \in \boldsymbol{D}^{**},\\
&\nu_0 \leqslant \alpha_7 < \min\left(\alpha_5, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5)\right),\ \boldsymbol{\alpha}_7^{\ddagger\prime} \notin \boldsymbol{G}_7,\\
&\nu_0 \leqslant \alpha_8 < \min\left(\alpha_7, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_7)\right),\ \boldsymbol{\alpha}_8^{\ddagger\prime} \notin \boldsymbol{G}_8,\\
&\left.\nu_0 \leqslant \alpha_1 < \frac{1}{2},\ \nu_0 \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
U_{C08}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) := &\left\{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2,\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right),\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\right.\\
&\boldsymbol{\alpha}_3 \text{ cannot be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2,\\
&\nu_0 \leqslant \alpha_4 < \frac{1}{2}\alpha_1,\ \boldsymbol{\alpha}_4' \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4' \notin \boldsymbol{D}^*,\ \boldsymbol{\alpha}_4'' \notin \boldsymbol{D}^*,\\
&\left.\nu_0 \leqslant \alpha_1 < \frac{1}{2},\ \nu_0 \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
U_{C09}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) := &\left\{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2,\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right),\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\right.\\
&\boldsymbol{\alpha}_3 \text{ cannot be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2,\\
&\nu_0 \leqslant \alpha_4 < \frac{1}{2}\alpha_1,\ \boldsymbol{\alpha}_4' \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4' \notin \boldsymbol{D}^*,\ \boldsymbol{\alpha}_4'' \notin \boldsymbol{D}^*,\\
&\alpha_4 < \alpha_5 < \frac{1}{2}(\alpha_1-\alpha_4),\ \boldsymbol{\alpha}_5' \in \boldsymbol{G}_5,
\end{aligned}
$$

$$\nu_0 \leqslant \alpha_1 < \frac{1}{2},\ \nu_0 \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)\Big\},$$

$$U_{C10}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) := \Big\{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2,\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right),\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,$$
$$\boldsymbol{\alpha}_3 \text{ cannot be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2,$$
$$\nu_0 \leqslant \alpha_4 < \frac{1}{2}\alpha_1,\ \boldsymbol{\alpha}_4' \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4' \notin \boldsymbol{D}^*,\ \boldsymbol{\alpha}_4'' \notin \boldsymbol{D}^*,$$
$$\alpha_3 < \alpha_5 < \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3),\ \boldsymbol{\alpha}_5'' \in \boldsymbol{G}_5,$$
$$\nu_0 \leqslant \alpha_1 < \frac{1}{2},\ \nu_0 \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)\Big\},$$

$$U_{C11}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6) := \Big\{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2,\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right),\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,$$
$$\boldsymbol{\alpha}_3 \text{ cannot be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2,$$
$$\nu_0 \leqslant \alpha_4 < \frac{1}{2}\alpha_1,\ \boldsymbol{\alpha}_4' \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4' \notin \boldsymbol{D}^*,\ \boldsymbol{\alpha}_4'' \notin \boldsymbol{D}^*,$$
$$\alpha_4 < \alpha_5 < \frac{1}{2}(\alpha_1-\alpha_4),\ \boldsymbol{\alpha}_5' \in \boldsymbol{G}_5,$$
$$\alpha_3 < \alpha_6 < \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3),\ \boldsymbol{\alpha}_5''' \in \boldsymbol{G}_5,$$
$$\nu_0 \leqslant \alpha_1 < \frac{1}{2},\ \nu_0 \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)\Big\},$$

$$U_{C12}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6) := \Big\{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2,\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right),\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,$$
$$\boldsymbol{\alpha}_3 \text{ cannot be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2,$$
$$\nu_0 \leqslant \alpha_4 < \frac{1}{2}\alpha_1,\ \boldsymbol{\alpha}_4' \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4' \in \boldsymbol{D}^*,$$
$$\nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(\alpha_1-\alpha_4)\right),\ \boldsymbol{\alpha}_5' \notin \boldsymbol{G}_5,$$
$$\nu_0 \leqslant \alpha_6 < \min\left(\alpha_5, \frac{1}{2}(\alpha_1-\alpha_4-\alpha_5)\right),\ \boldsymbol{\alpha}_6' \notin \boldsymbol{G}_6,$$
$$\nu_0 \leqslant \alpha_1 < \frac{1}{2},\ \nu_0 \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)\Big\},$$

$$U_{C13}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6) := \Big\{\boldsymbol{\alpha}_2 \in C,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2,\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right),\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,$$
$$\boldsymbol{\alpha}_3 \text{ cannot be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2,$$
$$\nu_0 \leqslant \alpha_4 < \frac{1}{2}\alpha_1,\ \boldsymbol{\alpha}_4' \notin \boldsymbol{G}_4,\ \boldsymbol{\alpha}_4' \in \boldsymbol{D}^*,$$
$$\nu_0 \leqslant \alpha_5 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right),\ \boldsymbol{\alpha}_5'' \notin \boldsymbol{G}_5,$$
$$\nu_0 \leqslant \alpha_6 < \min\left(\alpha_5, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_5)\right),\ \boldsymbol{\alpha}_6'' \notin \boldsymbol{G}_6,$$
$$\nu_0 \leqslant \alpha_1 < \frac{1}{2},\ \nu_0 \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)\Big\}.$$

One can also see the integrals corresponding to U_{C08}–U_{C11} as an simple explicit expression of the function $w^*(\boldsymbol{\alpha}_4)$ defined in [[20], Chapter 7.9]. We remark that a small part of C is actually covered by $\boldsymbol{G}_2$. If we discard the whole of $\sum_C$, we would have a loss larger than 1 which leads to a trivial lower bound.

Finally, by (9), (15) and (27), the total loss from $\sum_3$ is less than

$$2 \times 0.239221 + 0.491533 < 0.97$$

and we conclude that

$$\pi(x) - \pi(x - x^{0.52}) = S\left(\mathcal{A}, x^{\frac{1}{2}}\right) \geqslant 0.03 \frac{x^{0.52}}{\log x}.$$

The lower constant 0.03 can be slightly improved by more careful decompositions. The lower bounds for other values of θ between 0.52 and 0.525 can be proved in the same way, so we omit the calculation details. One can check our Mathematica code for them to verify the numerical calculations.

6. The final decomposition II: Upper Bounds

In this section, we ignore the presence of ε for clarity. Let $\omega(u)$, $\omega_0(u)$ and $\omega_1(u)$ denote the same functions as in Section 5. Fix $\theta = 0.52$, $\nu_0 = \nu_{\min} = 2\theta - 1 = 0.04$ and let $p_j = x^{\alpha_j}$. By Buchstab's identity, we have

$$\begin{aligned} S\left(\mathcal{A}, x^{\frac{1}{2}}\right) &= S\left(\mathcal{A}, x^{\nu(0)}\right) - \sum_{\nu(0) \leqslant \alpha_1 < \frac{1}{2}} S\left(\mathcal{A}_{p_1}, p_1\right) \\ &= \sum_1 - \sum_2{}' . \end{aligned} \tag{28}$$

We can give asymptotic formulas for $\sum_1$. For $\sum_2'$, We need to split the whole summation range over p_1 into different ranges and consider further decompositions in each range because we can only drop negative parts on the upper bound problem. The sets $\boldsymbol{G}$ and $\boldsymbol{D}$ with same superscripts and subscripts as in Section 5 represent the same asymptotic regions. We shall define some new sets and subsets using Lemma 3.3 and partition technique. Put

$$\begin{aligned} \boldsymbol{D}_3 &= \left\{\boldsymbol{\alpha}_2 : \alpha_2 \leqslant \alpha_1,\ 2\alpha_1 + \alpha_2 < 1,\ \alpha_2 < \frac{7}{2}\theta - \frac{3}{2}\right\}, \\ \boldsymbol{D}^{+} &= \left\{\boldsymbol{\alpha}_3 : (\alpha_1, \alpha_2, \alpha_3, \alpha_3) \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0' \text{ or } (m,n,h) \in \boldsymbol{D}_1' \cup \boldsymbol{D}_2'\right\}, \\ \boldsymbol{D}^{++} &= \left\{\boldsymbol{\alpha}_5 : (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_5) \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0' \text{ or } (m,n,h) \in \boldsymbol{D}_1' \cup \boldsymbol{D}_2'\right\}, \\ \boldsymbol{D}^{\#} &= \left\{\boldsymbol{\alpha}_3 : \text{both } \boldsymbol{\alpha}_3 \text{ and } (1-\alpha_1-\alpha_2-\alpha_3, \alpha_2, \alpha_3) \text{ can be partitioned into } (m,n) \in \boldsymbol{D}_0 \text{ or } (m,n,h) \in \boldsymbol{D}_1 \cup \boldsymbol{D}_2\right\}, \\ H &= \left\{\boldsymbol{\alpha}_1 : \frac{7}{2}\theta - \frac{3}{2} \leqslant \alpha_1 \leqslant 4 - 7\theta\right\}, \end{aligned}$$

where $\boldsymbol{D}_3$ correspond to conditions on variables that allow a further decomposition, $\boldsymbol{D}^{+}$ and $\boldsymbol{D}^{++}$ allow two and three further decompositions respectively, and $\boldsymbol{D}^{\#}$ allows two further decompositions with a role-reversal. In the sum corresponding to region H, we need to discard the whole of it because we cannot use Lemmas 3.1–3.3 to give an asymptotic formula for the two-dimensional sum with $\alpha_2 \geqslant \frac{7}{2}\theta - \frac{3}{2}$ after a Buchstab iteration. We remark that H is empty when $\theta > \frac{11}{21} \approx 0.5238$.

Next, we shall define some subregions of A and B defined in Section 5. The plot of these regions can also be found in Appendix 1.

$$\begin{aligned} A_1 &= \left\{\boldsymbol{\alpha}_2 : \boldsymbol{\alpha}_2 \in A,\ \alpha_2 < \min\left(\frac{3\theta-1}{2}, \frac{1+\theta}{5}\right)\right\}; \\ A_2 &= \left\{\boldsymbol{\alpha}_2 : \boldsymbol{\alpha}_2 \in A,\ \alpha_2 \geqslant \min\left(\frac{3\theta-1}{2}, \frac{1+\theta}{5}\right)\right\}; \\ B_1 &= \left\{\boldsymbol{\alpha}_2 : \boldsymbol{\alpha}_2 \in B,\ \alpha_2 < \min\left(\frac{3\theta-1}{2}, \frac{1+\theta}{5}\right)\right\}; \\ B_2 &= \left\{\boldsymbol{\alpha}_2 : \boldsymbol{\alpha}_2 \in B,\ \alpha_2 \geqslant \min\left(\frac{3\theta-1}{2}, \frac{1+\theta}{5}\right)\right\}; \\ A_1' &= \left\{\boldsymbol{\alpha}_2 : (1-\alpha_1-\alpha_2, \alpha_2) \in B_1,\ (1-\alpha_1-\alpha_2) \notin H\right\}; \\ A_2' &= \left\{\boldsymbol{\alpha}_2 : (1-\alpha_1-\alpha_2, \alpha_2) \in B_2,\ (1-\alpha_1-\alpha_2) \notin H\right\}. \end{aligned}$$

Hence, by Buchstab's identity, we have

$$\begin{aligned} -\sum_2{}' = -\sum_{\nu(0) \leqslant \alpha_1 < \frac{1}{2}} S\left(\mathcal{A}_{p_1}, p_1\right) &= -\sum_{\substack{\nu(0) \leqslant \alpha_1 < \frac{1}{2} \\ \boldsymbol{\alpha}_1 \in H}} S\left(\mathcal{A}_{p_1}, p_1\right) - \sum_{\substack{\nu(0) \leqslant \alpha_1 < \frac{1}{2} \\ \boldsymbol{\alpha}_1 \notin H}} S\left(\mathcal{A}_{p_1}, p_1\right) \\ &= -\sum_{\substack{\nu(0) \leqslant \alpha_1 < \frac{1}{2} \\ \boldsymbol{\alpha}_1 \in H}} S\left(\mathcal{A}_{p_1}, p_1\right) - \sum_{\substack{\nu(0) \leqslant \alpha_1 < \frac{1}{2} \\ \boldsymbol{\alpha}_1 \notin H}} S\left(\mathcal{A}_{p_1}, x^{\nu(\alpha_1)}\right) \\ &\quad + \sum_{\substack{\nu(0) \leqslant \alpha_1 < \frac{1}{2} \\ \boldsymbol{\alpha}_1 \notin H \\ \nu(\alpha_1) \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\ &= -\sum_{\substack{\nu(0) \leqslant \alpha_1 < \frac{1}{2} \\ \boldsymbol{\alpha}_1 \in H}} S\left(\mathcal{A}_{p_1}, p_1\right) - \sum_{\substack{\nu(0) \leqslant \alpha_1 < \frac{1}{2} \\ \boldsymbol{\alpha}_1 \notin H}} S\left(\mathcal{A}_{p_1}, x^{\nu(\alpha_1)}\right) \\ &\quad + \sum_{\substack{\boldsymbol{\alpha}_1 \notin H \\ \boldsymbol{\alpha}_2 \in A_1}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) + \sum_{\substack{\boldsymbol{\alpha}_1 \notin H \\ \boldsymbol{\alpha}_2 \in A_2}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\ &\quad + \sum_{\substack{\boldsymbol{\alpha}_1 \notin H \\ \boldsymbol{\alpha}_2 \in B_1}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) + \sum_{\substack{\boldsymbol{\alpha}_1 \notin H \\ \boldsymbol{\alpha}_2 \in B_2}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) + \sum_{\substack{\boldsymbol{\alpha}_1 \notin H \\ \boldsymbol{\alpha}_2 \in C}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \end{aligned}$$

$$= -\sum_{H}' - \sum_{3}' + \sum_{A_1}' + \sum_{A_2}' + \sum_{B_1}' + \sum_{B_2}' + \sum_{C}' . \tag{29}$$

By a similar discussion as in Section 5, we know that

$$\sum_{B_1}' = \sum_{A_1'}' := \sum_{\boldsymbol{\alpha}_2 \in A_1'} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \tag{30}$$

and

$$\sum_{B_2}' = \sum_{A_2'}' := \sum_{\boldsymbol{\alpha}_2 \in A_2'} S\left(\mathcal{A}_{p_1 p_2}, p_2\right), \tag{31}$$

hence

$$-\sum_{2}' = -\sum_{H}' - \sum_{3}' + \sum_{A_1}' + \sum_{A_2}' + \sum_{A_1'}' + \sum_{A_2'}' + \sum_{C}' . \tag{32}$$

We have an asymptotic formula for $\sum_3'$. For $\sum_H'$ which cannot be decomposed anymore, we discard the whole of the sum leading to a loss of

$$\int_{\frac{7}{2}\theta-\frac{3}{2}}^{4-7\theta} \texttt{Boole}[(t_1) \in H] \frac{\omega\left(\frac{1-t_1}{t_1}\right)}{t_1^2} dt_1 < 0.182012. \tag{33}$$

For $\sum_{A_1}'$ we can use Buchstab's identity to reach

$$\begin{aligned} \sum_{A_1}' &= \sum_{\substack{\boldsymbol{\alpha}_1 \notin H \\ \boldsymbol{\alpha}_2 \in A_1}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\ &= \sum_{\substack{\boldsymbol{\alpha}_1 \notin H \\ \boldsymbol{\alpha}_2 \in A_1}} S\left(\mathcal{A}_{p_1 p_2}, x^{\nu_0}\right) - \sum_{\substack{\boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in A_1 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right)}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right). \end{aligned} \tag{34}$$

By Lemma 3.1, we can give an asymptotic formula for the first sum on the right-hand side. For the second sum, we can perform a straightforward decomposition if we have $\boldsymbol{\alpha}_3 \in \boldsymbol{D}^{+}$, and we can perform a role-reversal if we have $\boldsymbol{\alpha}_3 \in \boldsymbol{D}^{\#}$. We can also use Buchstab's identity in reverse to gain some four-dimensional savings. Altogether, we have the following expression after this decomposition procedure:

$$\begin{aligned} & - \sum_{\substack{\boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in A_1 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right)}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right) \\ = & - \sum_{\substack{\boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in A_1 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \in \boldsymbol{G}_3}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right) - \sum_{\substack{\boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in A_1 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\ \boldsymbol{\alpha}_3 \notin \boldsymbol{D}^{+}}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right) \\ & - \sum_{\substack{\boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in A_1 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\ \boldsymbol{\alpha}_3 \in \boldsymbol{D}^{+}}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right) \\ = & - \sum_{\substack{\boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in A_1 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \in \boldsymbol{G}_3}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right) - \sum_{\substack{\boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in A_1 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\ \boldsymbol{\alpha}_3 \notin \boldsymbol{D}^{+}}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right) \\ & - \sum_{\substack{\boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in A_1 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\ \boldsymbol{\alpha}_3 \in \boldsymbol{D}^{+}}} S\left(\mathcal{A}_{p_1 p_2 p_3}, x^{\nu_0}\right) + \sum_{\substack{\boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in A_1 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\ \boldsymbol{\alpha}_3 \in \boldsymbol{D}^{+} \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \in \boldsymbol{G}_4}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right) \\ & + \sum_{\substack{\boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in A_1 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\ \boldsymbol{\alpha}_3 \in \boldsymbol{D}^{+} \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, x^{\nu_0}\right) - \sum_{\substack{\boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in A_1 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\ \boldsymbol{\alpha}_3 \in \boldsymbol{D}^{+} \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4 \\ \nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5 \in \boldsymbol{G}_5}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5\right) \end{aligned}$$

$$
\begin{aligned}
&- \sum_{\substack{\boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in A_1 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\ \boldsymbol{\alpha}_3 \in \boldsymbol{D}^+ \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4 \\ \nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5 \notin \boldsymbol{G}_5}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5}, p_5\right) \\
&= -S_{01}' - T_{01}' - S_{02}' + S_{03}' + S_{04}' - S_{05}' - T_{02}'. \qquad (35)
\end{aligned}
$$

We can give asymptotic formulas for S_{01}'–S_{05}', and we can subtract the contribution of the sum

$$
\sum_{\substack{\boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in A_1 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\ \boldsymbol{\alpha}_3 \notin \boldsymbol{D}^+ \\ \alpha_3 < \alpha_4 < \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3) \\ \boldsymbol{\alpha}_4 \in \boldsymbol{G}_4}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right) \qquad (36)
$$

from the loss from T_{01}' by using Buchstab's identity in reverse.

To sum up, the loss from $\sum_{A_1}'$ can be bounded by

$$
\begin{aligned}
&\left(\int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \texttt{Boole}[(t_1,t_2,t_3) \in V_{A1}] \frac{\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 dt_2 dt_1\right) \\
&- \left(\int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{t_3}^{\frac{1-t_1-t_2-t_3}{2}} \right. \\
&\qquad \left. \texttt{Boole}[(t_1,t_2,t_3,t_4) \in V_{A2}] \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1\right) \\
&+ \left(\int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\nu_0}^{\min\left(t_3, \frac{1-t_1-t_2-t_3}{2}\right)} \int_{\nu_0}^{\min\left(t_4, \frac{1-t_1-t_2-t_3-t_4}{2}\right)} \right. \\
&\qquad \left. \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5) \in V_{A3}] \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1\right) \\
&\leqslant \left(\int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \texttt{Boole}[(t_1,t_2,t_3) \in V_{A1}] \frac{\omega_1\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 dt_2 dt_1\right) \\
&- \left(\int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{t_3}^{\frac{1-t_1-t_2-t_3}{2}} \right. \\
&\qquad \left. \texttt{Boole}[(t_1,t_2,t_3,t_4) \in V_{A2}] \frac{\omega_0\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1\right) \\
&+ \left(\int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\nu_0}^{\min\left(t_3, \frac{1-t_1-t_2-t_3}{2}\right)} \int_{\nu_0}^{\min\left(t_4, \frac{1-t_1-t_2-t_3-t_4}{2}\right)} \right. \\
&\qquad \left. \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5) \in V_{A3}] \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1\right) \\
&\leqslant (0.179773 - 0.004874 + 0.043475) < 0.218374, \qquad (37)
\end{aligned}
$$

where

$$
\begin{aligned}
V_{A1}(\alpha_1,\alpha_2,\alpha_3) := &\left\{\boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in A_1,\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right),\right. \\
&\boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\ \boldsymbol{\alpha}_3 \notin \boldsymbol{D}^+, \\
&\left.\nu_0 \leqslant \alpha_1 < \frac{1}{2},\ \nu_0 \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)\right\}, \\
V_{A2}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) := &\left\{\boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in A_1,\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right),\right. \\
&\boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\ \boldsymbol{\alpha}_3 \notin \boldsymbol{D}^+, \\
&\alpha_3 < \alpha_4 < \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3),\ \boldsymbol{\alpha}_4 \in \boldsymbol{G}_4,
\end{aligned}
$$

$$\nu_0 \leqslant \alpha_1 < \frac{1}{2},\ \nu_0 \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)\Bigg\},$$

$$V_{A3}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) := \Bigg\{\boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in A_1,\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right),$$
$$\boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\ \boldsymbol{\alpha}_3 \in \boldsymbol{D}^+,$$
$$\nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right),\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,$$
$$\nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right),\ \boldsymbol{\alpha}_5 \notin \boldsymbol{G}_5,$$
$$\nu_0 \leqslant \alpha_1 < \frac{1}{2},\ \nu_0 \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)\Bigg\}.$$

By the essentially identical decomposing process, the loss from $\sum'_{A'_1}$ is less than

$$\left(\int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \texttt{Boole}[(t_1,t_2,t_3) \in V_{A4}] \frac{\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 dt_2 dt_1\right)$$
$$-\left(\int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{t_3}^{\frac{1-t_1-t_2-t_3}{2}} \texttt{Boole}[(t_1,t_2,t_3,t_4) \in V_{A5}] \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1\right)$$
$$+\left(\int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\nu_0}^{\min\left(t_3, \frac{1-t_1-t_2-t_3}{2}\right)} \int_{\nu_0}^{\min\left(t_4, \frac{1-t_1-t_2-t_3-t_4}{2}\right)} \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5) \in V_{A6}] \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1\right)$$
$$\leqslant \left(\int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \texttt{Boole}[(t_1,t_2,t_3) \in V_{A4}] \frac{\omega_1\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 dt_2 dt_1\right)$$
$$-\left(\int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{t_3}^{\frac{1-t_1-t_2-t_3}{2}} \texttt{Boole}[(t_1,t_2,t_3,t_4) \in V_{A5}] \frac{\omega_0\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1\right)$$
$$+\left(\int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\nu_0}^{\min\left(t_3, \frac{1-t_1-t_2-t_3}{2}\right)} \int_{\nu_0}^{\min\left(t_4, \frac{1-t_1-t_2-t_3-t_4}{2}\right)} \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5) \in V_{A6}] \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1\right)$$
$$\leqslant (0.310609 - 0.008299 + 0.051108) < 0.353418, \tag{38}$$

where

$$V_{A4}(\alpha_1,\alpha_2,\alpha_3) := \Bigg\{\boldsymbol{\alpha}_2 \in A'_1,\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right),$$
$$\boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\ \boldsymbol{\alpha}_3 \notin \boldsymbol{D}^+,$$
$$\nu_0 \leqslant \alpha_1 < \frac{1}{2},\ \nu_0 \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)\Bigg\},$$

$$V_{A5}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) := \Bigg\{\boldsymbol{\alpha}_2 \in A'_1,\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right),$$
$$\boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\ \boldsymbol{\alpha}_3 \notin \boldsymbol{D}^+,$$
$$\alpha_3 < \alpha_4 < \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3),\ \boldsymbol{\alpha}_4 \in \boldsymbol{G}_4,$$
$$\nu_0 \leqslant \alpha_1 < \frac{1}{2},\ \nu_0 \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)\Bigg\},$$

$$
\begin{aligned}
V_{A6}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) := \Bigg\{ & \boldsymbol{\alpha}_2 \in A_1',\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right), \\
& \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\ \boldsymbol{\alpha}_3 \in \boldsymbol{D}^+, \\
& \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right),\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4, \\
& \nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right),\ \boldsymbol{\alpha}_5 \notin \boldsymbol{G}_5, \\
& \nu_0 \leqslant \alpha_1 < \frac{1}{2},\ \nu_0 \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \Bigg\}.
\end{aligned}
$$

For $\sum_{A_2}'$, we apply Buchstab's identity as in (29) to get

$$
\begin{aligned}
\sum_{A_2}{}' &= \sum_{\substack{\boldsymbol{\alpha}_1 \notin H \\ \boldsymbol{\alpha}_2 \in A_2}} S\left(\mathcal{A}_{p_1p_2}, p_2\right) \\
&= \sum_{\substack{\boldsymbol{\alpha}_1 \notin H \\ \boldsymbol{\alpha}_2 \in A_2}} S\left(\mathcal{A}_{p_1p_2}, x^{\nu_0}\right) - \sum_{\substack{\boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in A_2 \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right)}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right). \qquad (39)
\end{aligned}
$$

Although Lemma 3.1 is not applicable in this case, we can use Lemma 3.3 to give an asymptotic formula for the first sum on the right-hand side. For the second sum, we cannot perform any further decompositions because we cannot give asymptotic formula for the four-dimensional sum after applying Buchstab's identity twice. Thus, the loss from $\sum_{A_2}'$ is just

$$
\begin{aligned}
& \int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \texttt{Boole}[(t_1,t_2,t_3) \in V_{A7}] \frac{\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 dt_2 dt_1 \\
\leqslant & \int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \texttt{Boole}[(t_1,t_2,t_3) \in V_{A7}] \frac{\omega_1\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 dt_2 dt_1 < 0.102865, \qquad (40)
\end{aligned}
$$

where

$$
\begin{aligned}
V_{A7}(\alpha_1,\alpha_2,\alpha_3) := \Bigg\{ & \boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in A_2,\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right),\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3, \\
& \nu_0 \leqslant \alpha_1 < \frac{1}{2},\ \nu_0 \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \Bigg\}.
\end{aligned}
$$

Similarly, the loss from $\sum_{A_2'}'$ is

$$
\begin{aligned}
& \int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \texttt{Boole}[(t_1,t_2,t_3) \in V_{A8}] \frac{\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 dt_2 dt_1 \\
\leqslant & \int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \texttt{Boole}[(t_1,t_2,t_3) \in V_{A8}] \frac{\omega_1\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 dt_2 dt_1 < 0.201264, \qquad (41)
\end{aligned}
$$

where

$$
\begin{aligned}
V_{A8}(\alpha_1,\alpha_2,\alpha_3) := \Bigg\{ & \boldsymbol{\alpha}_2 \in A_2',\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right),\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3, \\
& \nu_0 \leqslant \alpha_1 < \frac{1}{2},\ \nu_0 \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \Bigg\}.
\end{aligned}
$$

For the remaining $\sum_{C}'$, we have

$$
\begin{aligned}
\sum_{C}{}' &= \sum_{\substack{\boldsymbol{\alpha}_1 \notin H \\ \boldsymbol{\alpha}_2 \in C}} S\left(\mathcal{A}_{p_1p_2}, p_2\right) \\
&= \sum_{\substack{\boldsymbol{\alpha}_1 \notin H \\ \boldsymbol{\alpha}_2 \in C}} S\left(\mathcal{A}_{p_1p_2}, x^{\nu_0}\right) - \sum_{\substack{\boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in C \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right)}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right). \qquad (42)
\end{aligned}
$$

We can give an asymptotic formula for the first sum on the right-hand side. For the second sum, we need to consider role-reversals because we may have a large α_1 in this case. We can perform a straightforward decomposition if we have $\boldsymbol{\alpha}_3 \in \boldsymbol{D}^+$, and we can perform a role-reversal if we have $\boldsymbol{\alpha}_3 \in \boldsymbol{D}^{\#}$. Using Buchstab's identity, we write

$$
- \sum_{\substack{\boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in C \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right)}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right)
$$

$$
\begin{aligned}
= &-\sum_{\substack{\boldsymbol{\alpha}_1\notin H,\ \boldsymbol{\alpha}_2\in C\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\in\boldsymbol{G}_3}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right)-\sum_{\substack{\boldsymbol{\alpha}_1\notin H,\ \boldsymbol{\alpha}_2\in C\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\ \boldsymbol{\alpha}_3\notin\boldsymbol{D}^{+},\ \boldsymbol{\alpha}_3\notin\boldsymbol{D}^{\#}}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right)\\
&-\sum_{\substack{\boldsymbol{\alpha}_1\notin H,\ \boldsymbol{\alpha}_2\in C\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\ \boldsymbol{\alpha}_3\in\boldsymbol{D}^{+}}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right)-\sum_{\substack{\boldsymbol{\alpha}_1\notin H,\ \boldsymbol{\alpha}_2\in C\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\ \boldsymbol{\alpha}_3\notin\boldsymbol{D}^{+},\ \boldsymbol{\alpha}_3\in\boldsymbol{D}^{\#}}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right)\\
= &-\sum_{\substack{\boldsymbol{\alpha}_1\notin H,\ \boldsymbol{\alpha}_2\in C\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\in\boldsymbol{G}_3}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right)-\sum_{\substack{\boldsymbol{\alpha}_1\notin H,\ \boldsymbol{\alpha}_2\in C\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\ \boldsymbol{\alpha}_3\notin\boldsymbol{D}^{+},\ \boldsymbol{\alpha}_3\notin\boldsymbol{D}^{\#}}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right)\\
&-\sum_{\substack{\boldsymbol{\alpha}_1\notin H,\ \boldsymbol{\alpha}_2\in C\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\ \boldsymbol{\alpha}_3\in\boldsymbol{D}^{+}}} S\left(\mathcal{A}_{p_1p_2p_3},x^{\nu_0}\right)+\sum_{\substack{\boldsymbol{\alpha}_1\notin H,\ \boldsymbol{\alpha}_2\in C\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\ \boldsymbol{\alpha}_3\in\boldsymbol{D}^{+}\\ \nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right)\\ \boldsymbol{\alpha}_4\in\boldsymbol{G}_4}} S\left(\mathcal{A}_{p_1p_2p_3p_4},p_4\right)\\
&+\sum_{\substack{\boldsymbol{\alpha}_1\notin H,\ \boldsymbol{\alpha}_2\in C\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\ \boldsymbol{\alpha}_3\in\boldsymbol{D}^{+}\\ \nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right)\\ \boldsymbol{\alpha}_4\notin\boldsymbol{G}_4}} S\left(\mathcal{A}_{p_1p_2p_3p_4},x^{\nu_0}\right)-\sum_{\substack{\boldsymbol{\alpha}_1\notin H,\ \boldsymbol{\alpha}_2\in C\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\ \boldsymbol{\alpha}_3\in\boldsymbol{D}^{+}\\ \nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right)\\ \boldsymbol{\alpha}_4\notin\boldsymbol{G}_4\\ \nu_0\leqslant\alpha_5<\min\left(\alpha_4,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right)\\ \boldsymbol{\alpha}_5\in\boldsymbol{G}_5}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5},p_5\right)\\
&-\sum_{\substack{\boldsymbol{\alpha}_1\notin H,\ \boldsymbol{\alpha}_2\in C\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\ \boldsymbol{\alpha}_3\in\boldsymbol{D}^{+}\\ \nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right)\\ \boldsymbol{\alpha}_4\notin\boldsymbol{G}_4\\ \nu_0\leqslant\alpha_5<\min\left(\alpha_4,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right)\\ \boldsymbol{\alpha}_5\notin\boldsymbol{G}_5}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5},p_5\right)-\sum_{\substack{\boldsymbol{\alpha}_1\notin H,\ \boldsymbol{\alpha}_2\in C\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\ \boldsymbol{\alpha}_3\notin\boldsymbol{D}^{+},\ \boldsymbol{\alpha}_3\in\boldsymbol{D}^{\#}}} S\left(\mathcal{A}_{p_1p_2p_3},x^{\nu_0}\right)\\
&+\sum_{\substack{\boldsymbol{\alpha}_1\notin H,\ \boldsymbol{\alpha}_2\in C\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\ \boldsymbol{\alpha}_3\notin\boldsymbol{D}^{+},\ \boldsymbol{\alpha}_3\in\boldsymbol{D}^{\#}\\ \nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right)\\ \boldsymbol{\alpha}_4\in\boldsymbol{G}_4}} S\left(\mathcal{A}_{p_1p_2p_3p_4},p_4\right)+\sum_{\substack{\boldsymbol{\alpha}_1\notin H,\ \boldsymbol{\alpha}_2\in C\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\ \boldsymbol{\alpha}_3\notin\boldsymbol{D}^{+},\ \boldsymbol{\alpha}_3\in\boldsymbol{D}^{\#}\\ \nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right)\\ \boldsymbol{\alpha}_4\notin\boldsymbol{G}_4}} S\left(\mathcal{A}_{\eta p_2p_3p_4},x^{\nu_0}\right)\\
&-\sum_{\substack{\boldsymbol{\alpha}_1\notin H,\ \boldsymbol{\alpha}_2\in C\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\ \boldsymbol{\alpha}_3\notin\boldsymbol{D}^{+},\ \boldsymbol{\alpha}_3\in\boldsymbol{D}^{\#}\\ \nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right)\\ \boldsymbol{\alpha}_4\notin\boldsymbol{G}_4\\ \nu_0\leqslant\alpha_5<\frac{1}{2}\alpha_1\\ \boldsymbol{\alpha}_5^{\#}\in\boldsymbol{G}_5}} S\left(\mathcal{A}_{\eta p_2p_3p_4p_5},p_5\right)-\sum_{\substack{\boldsymbol{\alpha}_1\notin H,\ \boldsymbol{\alpha}_2\in C\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\ \boldsymbol{\alpha}_3\notin\boldsymbol{D}^{+},\ \boldsymbol{\alpha}_3\in\boldsymbol{D}^{\#}\\ \nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right)\\ \boldsymbol{\alpha}_4\notin\boldsymbol{G}_4\\ \nu_0\leqslant\alpha_5<\frac{1}{2}\alpha_1\\ \boldsymbol{\alpha}_5^{\#}\notin\boldsymbol{G}_5}} S\left(\mathcal{A}_{\eta p_2p_3p_4p_5},p_5\right)\\
= &-S_{06}'-T_{03}'-S_{07}'+S_{08}'+S_{09}'-S_{10}'-T_{04}'-S_{11}'+S_{12}'+S_{13}'-S_{14}'-T_{05}', \qquad (43)
\end{aligned}
$$

where $\eta\sim x^{1-\alpha_1-\alpha_2-\alpha_3-\alpha_4}$, $(\eta,P(p_4))=1$ and $\boldsymbol{\alpha}_5^{\#}=(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5)$. We can give asymptotic formulas for S_{06}'–S_{14}'. For T_{03}' we can use Buchstab's identity in reverse to subtract the contribution of the sum

$$
\sum_{\substack{\boldsymbol{\alpha}_1\notin H,\ \boldsymbol{\alpha}_2\in C\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\ \boldsymbol{\alpha}_3\notin\boldsymbol{D}^{+}\\ \alpha_3<\alpha_4<\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\\ \boldsymbol{\alpha}_4\in\boldsymbol{G}_4}} S\left(\mathcal{A}_{p_1p_2p_3p_4},p_4\right) \qquad (44)
$$

from the loss. For the remaining

$$
T_{04}'=\sum_{\substack{\boldsymbol{\alpha}_1\notin H,\ \boldsymbol{\alpha}_2\in C\\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)\\ \boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\ \boldsymbol{\alpha}_3\in\boldsymbol{D}^{+}\\ \nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right)\\ \boldsymbol{\alpha}_4\notin\boldsymbol{G}_4\\ \nu_0\leqslant\alpha_5<\min\left(\alpha_4,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right)\\ \boldsymbol{\alpha}_5\notin\boldsymbol{G}_5}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5},p_5\right) \qquad (45)
$$

and

$$T'_{05} = \sum_{\substack{\boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in C \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\ \boldsymbol{\alpha}_3 \notin \boldsymbol{D}^+,\ \boldsymbol{\alpha}_3 \in \boldsymbol{D}^{\#} \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4 \\ \nu_0 \leqslant \alpha_5 < \frac{1}{2}\alpha_1 \\ \boldsymbol{\alpha}_5^{\#} \notin \boldsymbol{G}_5}} S\left(\mathcal{A}_{\eta p_2 p_3 p_4 p_5}, p_5\right), \tag{46}$$

we can perform a further straightforward decomposition on T'_{04} if $\boldsymbol{\alpha}_5 \in \boldsymbol{D}^{++}$, and on T'_{05} if either $\boldsymbol{\alpha}_5^{\#} \in \boldsymbol{D}^{++}$ or $\boldsymbol{\alpha}_5^{\#\prime} \in \boldsymbol{D}^{++}$. Note that for T_{032} and T_{033} in Section 5 we use similar discussion. This leads to the loss of three seven-dimensional sums

$$\sum_{\substack{\boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in C \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\ \boldsymbol{\alpha}_3 \in \boldsymbol{D}^+ \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4 \\ \nu_0 \leqslant \alpha_5 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_5 \notin \boldsymbol{G}_5,\ \boldsymbol{\alpha}_5 \in \boldsymbol{D}^{++} \\ \nu_0 \leqslant \alpha_6 < \min\left(\alpha_5, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5)\right) \\ \boldsymbol{\alpha}_6 \notin \boldsymbol{G}_6 \\ \nu_0 \leqslant \alpha_7 < \min\left(\alpha_6, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6)\right) \\ \boldsymbol{\alpha}_7 \notin \boldsymbol{G}_7}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4 p_5 p_6 p_7}, p_7\right), \tag{47}$$

$$\sum_{\substack{\boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in C \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\ \boldsymbol{\alpha}_3 \notin \boldsymbol{D}^+,\ \boldsymbol{\alpha}_3 \in \boldsymbol{D}^{\#} \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4 \\ \nu_0 \leqslant \alpha_5 < \frac{1}{2}\alpha_1 \\ \boldsymbol{\alpha}_5^{\#} \notin \boldsymbol{G}_5,\ \boldsymbol{\alpha}_5^{\#} \in \boldsymbol{D}^{++} \\ \nu_0 \leqslant \alpha_6 < \min\left(\alpha_5, \frac{1}{2}(\alpha_1-\alpha_5)\right) \\ \boldsymbol{\alpha}_6^{\#} \notin \boldsymbol{G}_6 \\ \nu_0 \leqslant \alpha_7 < \min\left(\alpha_6, \frac{1}{2}(\alpha_1-\alpha_5-\alpha_6)\right) \\ \boldsymbol{\alpha}_7^{\#} \notin \boldsymbol{G}_7}} S\left(\mathcal{A}_{\eta p_2 p_3 p_4 p_5 p_6 p_7}, p_7\right), \tag{48}$$

and

$$\sum_{\substack{\boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in C \\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\ \boldsymbol{\alpha}_3 \notin \boldsymbol{D}^+,\ \boldsymbol{\alpha}_3 \in \boldsymbol{D}^{\#} \\ \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right) \\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4 \\ \nu_0 \leqslant \alpha_5 < \frac{1}{2}\alpha_1 \\ \boldsymbol{\alpha}_5^{\#} \notin \boldsymbol{G}_5,\ \boldsymbol{\alpha}_5^{\#} \notin \boldsymbol{D}^{++},\ \boldsymbol{\alpha}_5^{\#\prime} \in \boldsymbol{D}^{++} \\ \nu_0 \leqslant \alpha_6 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \boldsymbol{\alpha}_6^{\#\prime} \notin \boldsymbol{G}_6 \\ \nu_0 \leqslant \alpha_7 < \min\left(\alpha_6, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_6)\right) \\ \boldsymbol{\alpha}_7^{\#\prime} \notin \boldsymbol{G}_7}} S\left(\mathcal{A}_{\eta_1 p_2 p_3 p_4 p_5 p_6 p_7}, p_7\right), \tag{49}$$

where $\eta_1 \sim x^{\alpha_1-\alpha_5}$, $(\eta_1, P(p_5)) = 1$, $\boldsymbol{\alpha}_6^{\#} = (1-\alpha_1-\alpha_2-\alpha_3-\alpha_4,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6)$, $\boldsymbol{\alpha}_7^{\#} = (1-\alpha_1-\alpha_2-\alpha_3-\alpha_4,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6,\ \alpha_7)$, $\boldsymbol{\alpha}_5^{\#\prime} = (\alpha_1-\alpha_5,\ \alpha_2,\ \alpha_3,\ \alpha_5,\ \alpha_4)$, $\boldsymbol{\alpha}_6^{\#\prime} = (\alpha_1-\alpha_5,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6)$ and $\boldsymbol{\alpha}_7^{\#\prime} = (\alpha_1-\alpha_5,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6,\ \alpha_7)$. Again, the loss from $\sum'_C$ is no more than

$$\begin{aligned}
&\left(\int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \texttt{Boole}[(t_1,t_2,t_3) \in V_{C1}] \frac{\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 dt_2 dt_1\right) \\
&- \left(\int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{t_3}^{\frac{1-t_1-t_2-t_3}{2}} \right. \\
&\qquad \left. \texttt{Boole}[(t_1,t_2,t_3,t_4) \in V_{C2}] \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1\right) \\
&+ \left(\int_{\nu_0}^{\frac{1}{2}} \int_{\nu_0}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\nu_0}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\nu_0}^{\min\left(t_3, \frac{1-t_1-t_2-t_3}{2}\right)} \int_{\nu_0}^{\min\left(t_4, \frac{1-t_1-t_2-t_3-t_4}{2}\right)} \right. \\
&\qquad \left. \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5) \in V_{C3}] \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1\right)
\end{aligned}$$

$$+\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\int_{\nu_0}^{\frac{1}{2}t_1}\right.$$
$$\left.\mathtt{Boole}[(t_1,t_2,t_3,t_4,t_5)\in V_{C4}]\frac{\omega\left(\frac{t_1-t_5}{t_5}\right)\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_2t_3t_4^2t_5^2}dt_5dt_4dt_3dt_2dt_1\right)$$
$$+\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\right.$$
$$\int_{\nu_0}^{\min\left(t_4,\frac{1-t_1-t_2-t_3-t_4}{2}\right)}\int_{\nu_0}^{\min\left(t_5,\frac{1-t_1-t_2-t_3-t_4-t_5}{2}\right)}\int_{\nu_0}^{\min\left(t_6,\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{2}\right)}$$
$$\left.\mathtt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6,t_7)\in V_{C5}]\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7}{t_7}\right)}{t_1t_2t_3t_4t_5t_6t_7^2}dt_7dt_6dt_5dt_4dt_3dt_2dt_1\right)$$
$$+\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\int_{\nu_0}^{\frac{1}{2}t_1}\int_{\nu_0}^{\min\left(t_5,\frac{t_1-t_5}{2}\right)}\int_{\nu_0}^{\min\left(t_6,\frac{t_1-t_5-t_6}{2}\right)}\right.$$
$$\left.\mathtt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6,t_7)\in V_{C6}]\frac{\omega\left(\frac{t_1-t_5-t_6-t_7}{t_7}\right)\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_2t_3t_4^2t_5t_6t_7^2}dt_7dt_6dt_5dt_4dt_3dt_2dt_1\right)$$
$$+\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\right.$$
$$\int_{\nu_0}^{\frac{1}{2}t_1}\int_{\nu_0}^{\min\left(t_4,\frac{1-t_1-t_2-t_3-t_4}{2}\right)}\int_{\nu_0}^{\min\left(t_6,\frac{1-t_1-t_2-t_3-t_4-t_6}{2}\right)}$$
$$\left.\mathtt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6,t_7)\in V_{C7}]\frac{\omega\left(\frac{t_1-t_5}{t_5}\right)\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_6-t_7}{t_7}\right)}{t_2t_3t_4t_5^2t_6t_7^2}dt_7dt_6dt_5dt_4dt_3dt_2dt_1\right)$$
$$\leqslant\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\mathtt{Boole}[(t_1,t_2,t_3)\in V_{C1}]\frac{\omega_1\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_1t_2t_3^2}dt_3dt_2dt_1\right)$$
$$-\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{t_3}^{\frac{1-t_1-t_2-t_3}{2}}\right.$$
$$\left.\mathtt{Boole}[(t_1,t_2,t_3,t_4)\in V_{C2}]\frac{\omega_0\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1t_2t_3t_4^2}dt_4dt_3dt_2dt_1\right)$$
$$+\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\int_{\nu_0}^{\min\left(t_4,\frac{1-t_1-t_2-t_3-t_4}{2}\right)}\right.$$
$$\left.\mathtt{Boole}[(t_1,t_2,t_3,t_4,t_5)\in V_{C3}]\frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1t_2t_3t_4t_5^2}dt_5dt_4dt_3dt_2dt_1\right)$$
$$+\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\int_{\nu_0}^{\frac{1}{2}t_1}\right.$$
$$\left.\mathtt{Boole}[(t_1,t_2,t_3,t_4,t_5)\in V_{C4}]\frac{\omega_1\left(\frac{t_1-t_5}{t_5}\right)\omega_1\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_2t_3t_4^2t_5^2}dt_5dt_4dt_3dt_2dt_1\right)$$
$$+\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\right.$$
$$\int_{\nu_0}^{\min\left(t_4,\frac{1-t_1-t_2-t_3-t_4}{2}\right)}\int_{\nu_0}^{\min\left(t_5,\frac{1-t_1-t_2-t_3-t_4-t_5}{2}\right)}\int_{\nu_0}^{\min\left(t_6,\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{2}\right)}$$
$$\left.\mathtt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6,t_7)\in V_{C5}]\frac{\max\left(\frac{t_7}{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7},0.5672\right)}{t_1t_2t_3t_4t_5t_6t_7^2}dt_7dt_6dt_5dt_4dt_3dt_2dt_1\right)$$
$$+\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\int_{\nu_0}^{\frac{1}{2}t_1}\right.$$

$$\int_{\nu_0}^{\min\left(t_5,\frac{t_1-t_5}{2}\right)}\int_{\nu_0}^{\min\left(t_6,\frac{t_1-t_5-t_6}{2}\right)}\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6,t_7)\in V_{C6}]\times$$
$$\left.\frac{\max\left(\frac{t_7}{t_1-t_5-t_6-t_7},0.5672\right)\max\left(\frac{t_4}{1-t_1-t_2-t_3-t_4},0.5672\right)}{t_2t_3t_4^2t_5t_6t_7^2}dt_7dt_6dt_5dt_4dt_3dt_2dt_1\right)$$
$$+\left(\int_{\nu_0}^{\frac{1}{2}}\int_{\nu_0}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{\nu_0}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)}\int_{\nu_0}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)}\int_{\nu_0}^{\frac{1}{2}t_1}\int_{\nu_0}^{\min\left(t_4,\frac{1-t_1-t_2-t_3-t_4}{2}\right)}\right.$$
$$\int_{\nu_0}^{\min\left(t_6,\frac{1-t_1-t_2-t_3-t_4-t_6}{2}\right)}\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6,t_7)\in V_{C7}]\times$$
$$\left.\frac{\max\left(\frac{t_5}{t_1-t_5},0.5672\right)\max\left(\frac{t_7}{1-t_1-t_2-t_3-t_4-t_6-t_7},0.5672\right)}{t_2t_3t_4t_5^2t_6t_7^2}dt_7dt_6dt_5dt_4dt_3dt_2dt_1\right)$$
$$\leqslant(0.261034-0.000575+0.128160+0.307367+0.004722+0.003889+0.000013)<0.70461, \tag{50}$$

where

$$\begin{aligned}
V_{C1}(\alpha_1,\alpha_2,\alpha_3):=\ &\Big\{\boldsymbol{\alpha}_1\notin H,\ \boldsymbol{\alpha}_2\in C,\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right),\\
&\boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\ \boldsymbol{\alpha}_3\notin\boldsymbol{D}^+,\ \boldsymbol{\alpha}_3\notin\boldsymbol{D}^\#,\\
&\nu_0\leqslant\alpha_1<\frac{1}{2},\ \nu_0\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\Big\},\\
V_{C2}(\alpha_1,\alpha_2,\alpha_3,\alpha_4):=\ &\Big\{\boldsymbol{\alpha}_1\notin H,\ \boldsymbol{\alpha}_2\in C,\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right),\\
&\boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\ \boldsymbol{\alpha}_3\notin\boldsymbol{D}^+,\ \boldsymbol{\alpha}_3\notin\boldsymbol{D}^\#,\\
&\alpha_3<\alpha_4<\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3),\ \boldsymbol{\alpha}_4\in\boldsymbol{G}_4,\\
&\nu_0\leqslant\alpha_1<\frac{1}{2},\ \nu_0\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\Big\},\\
V_{C3}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5):=\ &\Big\{\boldsymbol{\alpha}_1\notin H,\ \boldsymbol{\alpha}_2\in C,\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right),\\
&\boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\ \boldsymbol{\alpha}_3\in\boldsymbol{D}^+,\\
&\nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right),\ \boldsymbol{\alpha}_4\notin\boldsymbol{G}_4,\\
&\nu_0\leqslant\alpha_5<\min\left(\alpha_4,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right),\ \boldsymbol{\alpha}_5\notin\boldsymbol{G}_5,\ \boldsymbol{\alpha}_5\notin\boldsymbol{D}^{++},\\
&\nu_0\leqslant\alpha_1<\frac{1}{2},\ \nu_0\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\Big\},\\
V_{C4}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5):=\ &\Big\{\boldsymbol{\alpha}_1\notin H,\ \boldsymbol{\alpha}_2\in C,\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right),\\
&\boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\ \boldsymbol{\alpha}_3\notin\boldsymbol{D}^+,\ \boldsymbol{\alpha}_3\in\boldsymbol{D}^\#,\\
&\nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right),\ \boldsymbol{\alpha}_4\notin\boldsymbol{G}_4,\\
&\nu_0\leqslant\alpha_5<\frac{1}{2}\alpha_1,\ \boldsymbol{\alpha}_5^\#\notin\boldsymbol{G}_5,\ \boldsymbol{\alpha}_5^\#\notin\boldsymbol{D}^{++},\ \boldsymbol{\alpha}_5^{\#\prime}\notin\boldsymbol{D}^{++}\\
&\nu_0\leqslant\alpha_1<\frac{1}{2},\ \nu_0\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\Big\},\\
V_{C5}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7):=\ &\Big\{\boldsymbol{\alpha}_1\notin H,\ \boldsymbol{\alpha}_2\in C,\ \nu_0\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right),\\
&\boldsymbol{\alpha}_3\notin\boldsymbol{G}_3,\ \boldsymbol{\alpha}_3\in\boldsymbol{D}^+,\\
&\nu_0\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right),\ \boldsymbol{\alpha}_4\notin\boldsymbol{G}_4,\\
&\nu_0\leqslant\alpha_5<\min\left(\alpha_4,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right),\ \boldsymbol{\alpha}_5\notin\boldsymbol{G}_5,\ \boldsymbol{\alpha}_5\in\boldsymbol{D}^{++},\\
&\nu_0\leqslant\alpha_6<\min\left(\alpha_5,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5)\right),\ \boldsymbol{\alpha}_6\notin\boldsymbol{G}_6,
\end{aligned}$$

$$\nu_0 \leqslant \alpha_7 < \min\left(\alpha_6, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6)\right),\ \boldsymbol{\alpha}_7 \notin \boldsymbol{G}_7,$$

$$\nu_0 \leqslant \alpha_1 < \frac{1}{2},\ \nu_0 \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)\Bigg\},$$

$$\begin{aligned} V_{C6}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7) := \Bigg\{ & \boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in C,\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right), \\ & \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\ \boldsymbol{\alpha}_3 \notin \boldsymbol{D}^+,\ \boldsymbol{\alpha}_3 \in \boldsymbol{D}^{\#}, \\ & \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right),\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4, \\ & \nu_0 \leqslant \alpha_5 < \frac{1}{2}\alpha_1,\ \boldsymbol{\alpha}_5^{\#} \notin \boldsymbol{G}_5,\ \boldsymbol{\alpha}_5^{\#} \in \boldsymbol{D}^{++} \\ & \nu_0 \leqslant \alpha_6 < \min\left(\alpha_5, \frac{1}{2}(\alpha_1-\alpha_5)\right),\ \boldsymbol{\alpha}_6^{\#} \notin \boldsymbol{G}_6, \\ & \nu_0 \leqslant \alpha_7 < \min\left(\alpha_6, \frac{1}{2}(\alpha_1-\alpha_5-\alpha_6)\right),\ \boldsymbol{\alpha}_7^{\#} \notin \boldsymbol{G}_7, \\ & \nu_0 \leqslant \alpha_1 < \frac{1}{2},\ \nu_0 \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)\Bigg\}, \end{aligned}$$

$$\begin{aligned} V_{C7}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7) := \Bigg\{ & \boldsymbol{\alpha}_1 \notin H,\ \boldsymbol{\alpha}_2 \in C,\ \nu_0 \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right), \\ & \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\ \boldsymbol{\alpha}_3 \notin \boldsymbol{D}^+,\ \boldsymbol{\alpha}_3 \in \boldsymbol{D}^{\#}, \\ & \nu_0 \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right),\ \boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4, \\ & \nu_0 \leqslant \alpha_5 < \frac{1}{2}\alpha_1,\ \boldsymbol{\alpha}_5^{\#} \notin \boldsymbol{G}_5,\ \boldsymbol{\alpha}_5^{\#} \notin \boldsymbol{D}^{++},\ \boldsymbol{\alpha}_5^{\#\prime} \in \boldsymbol{D}^{++} \\ & \nu_0 \leqslant \alpha_6 < \min\left(\alpha_4, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\right),\ \boldsymbol{\alpha}_6^{\#\prime} \notin \boldsymbol{G}_6, \\ & \nu_0 \leqslant \alpha_7 < \min\left(\alpha_6, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_6)\right),\ \boldsymbol{\alpha}_7^{\#\prime} \notin \boldsymbol{G}_7, \\ & \nu_0 \leqslant \alpha_1 < \frac{1}{2},\ \nu_0 \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)\Bigg\}. \end{aligned}$$

Finally, by (32), (33), (37), (38), (40), (41) and (50), the total loss from $\sum_2'$ is less than

$$0.182012 + 0.218374 + 0.353418 + 0.102865 + 0.201264 + 0.70461 < 1.7626$$

and we conclude that

$$\pi(x) - \pi(x - x^{0.52}) = S\left(\mathcal{A}, x^{\frac{1}{2}}\right) \leqslant 2.7626 \frac{x^{0.52}}{\log x}.$$

The upper constant 2.7626 can be slightly improved by more careful decompositions. We remark that we can also use Lemma 3.3 together with a variant of [[5], Lemma 17] on the lower bound problem, but the new four-dimensional loss after using them on $\sum_A$ exceeds the original two-dimensional loss when $\theta = 0.52$. Our upper bound result is weaker than Iwaniec's upper constant $\frac{4}{1+\theta} \approx 2.6316$ when $\theta = 0.52$, but our sieve approach gives better results for slightly longer intervals (of length $x^{0.522}$, $x^{0.523}$ and so on, see the values in following table). In fact, the upper constant rises rapidly as θ increases. The upper bounds for other values of θ between 0.52 and 0.525 can be proved in the same way, so we omit the calculation details. One can check our Mathematica code for them to verify the numerical calculations.

$\boldsymbol{\theta}$	**New UB($\boldsymbol{\theta}$)**	**Iwaniec's UB($\boldsymbol{\theta}$)**
0.520	< 2.7626	$\frac{4}{1+0.52} < 2.6316$
0.521	< 2.6484	$\frac{4}{1+0.521} < 2.6299$
0.522	$\boldsymbol{< 2.5630}$	$\frac{4}{1+0.522} < 2.6282$
0.523	$\boldsymbol{< 2.4597}$	$\frac{4}{1+0.523} < 2.6264$
0.524	$\boldsymbol{< 2.3759}$	$\frac{4}{1+0.524} < 2.6247$

7. Applications

Clearly our Theorem 1 has many interesting applications (just like the previous BHP's result), and we state some of them in this section. Note that we ignore the presence of ε because we can use the same method to prove Theorem 1 with a slightly smaller θ, such as $0.52 - 10^{-100}$. The first application is about primes in arithmetic progressions in short intervals, which improves upon the result of Harman [[20], Theorem 10.8].

Theorem 3. *For all $q \leqslant (\log x)^K$ and any a coprime to q, we have*

$$\pi(x; q, a) - \pi(x - x^{0.52}; q, a) \geqslant 0.03 \frac{x^{0.52}}{\varphi(q) \log x}.$$

Another application is about bounded gaps between primes in short intervals, which improves the result of Alweiss and Luo [[1], Corollary 1.2].

Theorem 4. *There exist positive integers k, d such that the interval $[x-x^{0.52}, x]$ contains $\gg x^{0.52}(\log x)^{-k}$ pairs of consecutive primes differing by at most d.*

The third application is about primes with prime subscripts (or prime-primes) in short intervals. By using the numerical bound in Theorem 2, we can derive the following theorem.

Theorem 5. *We have*

$$\pi(\pi(x)) - \pi(\pi(x-x^{0.52})) \geqslant 0.03\frac{x^{0.52}}{(\log x)^2}.$$

The bound for the number of prime-primes in interval of length $x^{0.525}$ was obtained by Broughan and Barnett [8], where they also proved the analogs of Prime Number Theorem and weak Dirichlet's Theorem for prime-primes.

The next two applications focus on Goldbach numbers (sum of two primes) in short intervals. By replacing [[20], Theorem 10.8] by our Theorem 3 in the proof of the main theorem in [14], we can easily deduce the following result.

Theorem 6. *Almost all even numbers in the interval $[x, x+x^{\frac{13}{225}}]$ are Goldbach numbers.*

By combining our Theorem 1 with the main theorem proved in [35], we can easily show the following result.

Theorem 7. *The interval $[x, x+x^{\frac{26}{1075}}]$ contains Goldbach numbers.*

Note that $\frac{13}{225} \approx 0.0578$ and $\frac{26}{1075} \approx 0.0242$. Previous exponents $\frac{7}{120} \approx 0.0583$ [[14], Theorem 1.1] and $\frac{21}{860} \approx 0.0244$ [[35], Theorem 8.1] come from BHP's 0.525. We remark that if we focus on Maillet numbers (difference of two primes) instead of Goldbach numbers in short intervals, Pintz [47] improved the exponent in Theorem 7 to any $\varepsilon > 0$.

The next four applications of Theorem 1 are not direct corollaries of Theorem 1 and Theorem 2. However, since the arithmetical information inputs are quite similar, we can easily get these results with a slight modification of our calculation. We remark that we can only get an exponent 0.524 in these applications since the corresponding arithmetical information is weaker than that in Section 3 and 4. The first one is about the distribution of prime ideals of imaginary quadratic fields.

Theorem 8. *Let $d < 0$ be the discriminant of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$, and let $Q(x, y) \in \mathbb{Z}[x, y]$ be a positive definite quadratic form with discriminant d. Then, for every pair $(s, t) \in \mathbb{R}^2$, there is another pair $(m, n) \in \mathbb{Z}^2$ for which $Q(m, n)$ is prime and*

$$Q(s-m, t-n) \ll Q(s, t)^{0.524} + 1.$$

Specially, For every $z \in \mathbb{C}$, one can find a Gaussian prime $\mathfrak{p} \neq z$ satisfying

$$|z - \mathfrak{p}| \ll |z|^{0.524} + 1.$$

Theorem 8 improves previous results of Lewis [34] and Harman, Kumchev and Lewis [22], who got exponents 0.528 and 0.53 respectively.

The second one focuses on primes in arithmetic progressions valid except for a small set of exceptional moduli.

Theorem 9. *There exists a $\delta > 0$ such that if q is large, all prime factors of q is less than q^{δ}, and we have*

$$L(s, \chi) \neq 0 \text{ for } \operatorname{Re} s > 1 - \frac{1}{(\log q)^{\frac{3}{4}}}, \qquad |t| \leqslant \exp\left(\varepsilon(\log q)^{\frac{3}{4}}\right)$$

for every $d \mid q$ with χ a primitive character mod d, then for any a coprime to q, we have

$$\pi(x; q, a) \gg \frac{x}{\varphi(q)\log x}$$

whenever $q < x^{0.476}$.

The third one is a corollary of Theorem 9, which concerns the number of Carmichael numbers less than x. By combining our exponent 0.524 with [[36], Theorem 1.1], we know that

Theorem 10. *Let $Carm(x)$ denotes the number of Carmichael numbers less than x. Then we have*

$$Carm(x) > x^{(1-0.2844)(1-0.524)} > x^{0.3406}.$$

Theorem 10 improves previous results of Lichtman [36] and Harman [21] [19], who got exponents 0.3389, $\frac{1}{3}$ and 0.33 respectively.

The fourth one is another corollary of Theorem 9, which focuses on Linnik's constant for prime power moduli with a fixed prime and improves the result of Banks and Shparlinski [7].

Theorem 11. *For any $q = p^R$ with a large integer R and any a coprime to p, we have*

$$\sum_{\substack{n \leqslant x \\ n \equiv a (\bmod q)}} \Lambda(n) \gg \frac{x}{\varphi(q)}$$

for any $x > q^{\frac{1}{0.476}}$. Specially, we can bound Linnik's constant by $\frac{1}{0.476} < 2.1009$ if q is a power of a fixed prime.

The tenth application of Theorem 1 concerns the work of Erdős and Rényi [12] on Turán's problem 10. By applying the methods in [2] together with our Theorem 1, we can obtain the following upper bound of the power sum of complex z:

Theorem 12. *We have*

$$\inf_{|z_k|\geqslant 1}\max_{v=1,\ldots,n^2}\left|\sum_{1\leqslant k\leqslant n} z_k^v\right| = \sqrt{n} + O\left(n^{0.26}\right).$$

Theorem 12 improves upon the result of Andersson [2], which has an error of $O\left(n^{0.2625}\right)$.

Next application of Theorem 1 gives a better lower bound for the pairs of "symmetric primes", which was first considered by Tang and Wu [50]. By applying our Theorem 1 directly, we can get the following bound.

Theorem 13. *We have*

$$\sum_{\substack{p\leqslant x\\ \exists p' \text{ such that } [x/p']=p}} 1 \gg \frac{x^{\frac{12}{37}}}{\log x}.$$

Note that $\frac{12}{37} = \frac{1-0.52}{2-0.52}$. Theorem 13 improves upon the result of Tang and Wu [50], which has a lower bound $x^{\frac{19}{59}}(\log x)^{-1}$ comes from BHP's 0.525.

The twelfth application of Theorem 1 focuses on the size of a Sidon set and the sum of elements in it. By applying Theorem 1 together with the methods in [11], we can get the following result.

Theorem 14. *Let S be a Sidon set in $\{1,2,\ldots,n\}$ with $|S| = S_n$, then we have*

$$S_n = n^{\frac{1}{2}} + O\left(n^{\frac{13}{50}}\right)$$

for positive integers n, and

$$\sum_{a\in S} a = \frac{1}{2}n^{\frac{3}{2}} + O\left(n^{\frac{69}{50}}\right).$$

The second part of Theorem 14 improves upon the result of Ding [[11], Corollary 1.3], where he proved an error of $O\left(n^{\frac{221}{160}}\right)$. The first part of Theorem 14 is a direct corollary of Theorem 1. One can see [[11], Lemma 2.3] for a proof with $\theta = 0.525$.

The last application of Theorem 1 is Waring–Goldbach problem in short intervals, which improves the previous result of Wang [[52], Corollary 2]. Using our new Theorem 3 together with [[52], Theorem 1], we can get the following result.

Theorem 15. *Let $v = v(p,k)$ denotes the integer such that $p^v \mid k$ but $p^{v+1} \nmid k$. Define*

$$y = y(p,k) = \begin{cases} v+2, & \textit{if } p=2 \textit{ and } v>0,\\ v+1, & \textit{otherwise}\end{cases}$$

and

$$R_k = \prod_{(p-1)\mid k} p^y.$$

Then, when $k\geqslant 2$, $\theta > 0.52$ and $s > \max\left(\frac{5000}{39}, k(k+1)\right)$, for all sufficiently large $n \equiv s(\bmod R_k)$, there are primes

$$p_1,\ldots,p_s \in \left[\left(\frac{n}{s}\right)^{\frac{1}{k}} - n^{\frac{\theta}{k}}, \left(\frac{n}{s}\right)^{\frac{1}{k}} + n^{\frac{\theta}{k}}\right]$$

such that

$$n = p_1^k + \cdots + p_s^k.$$

Note that $\frac{5000}{39} = \frac{2}{0.03\times 0.52} \approx 128.2$.

Appendix 1: Plot of Regions

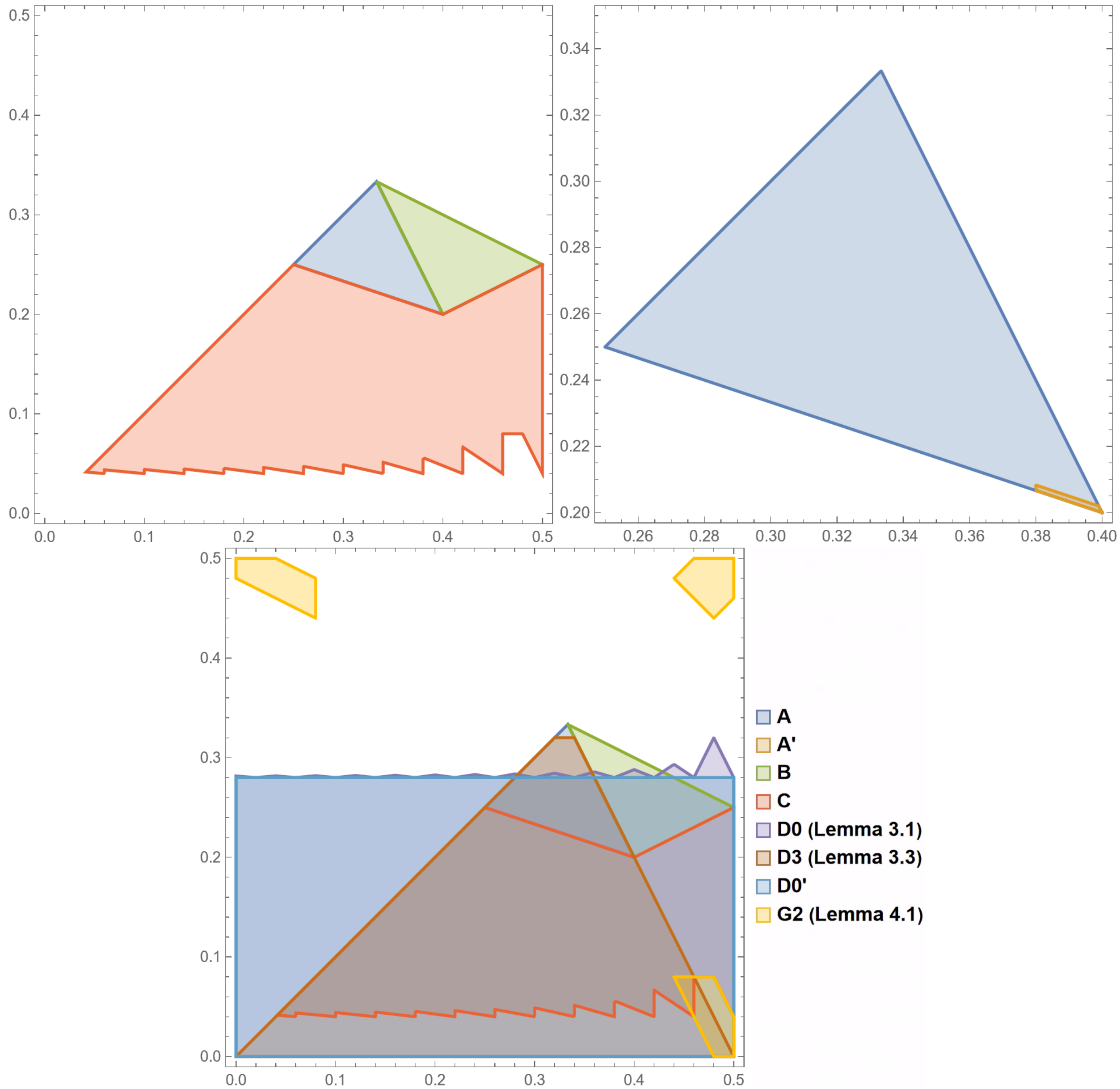

Figure 1: Plot for the Lower Bound ($\theta = 0.52$)

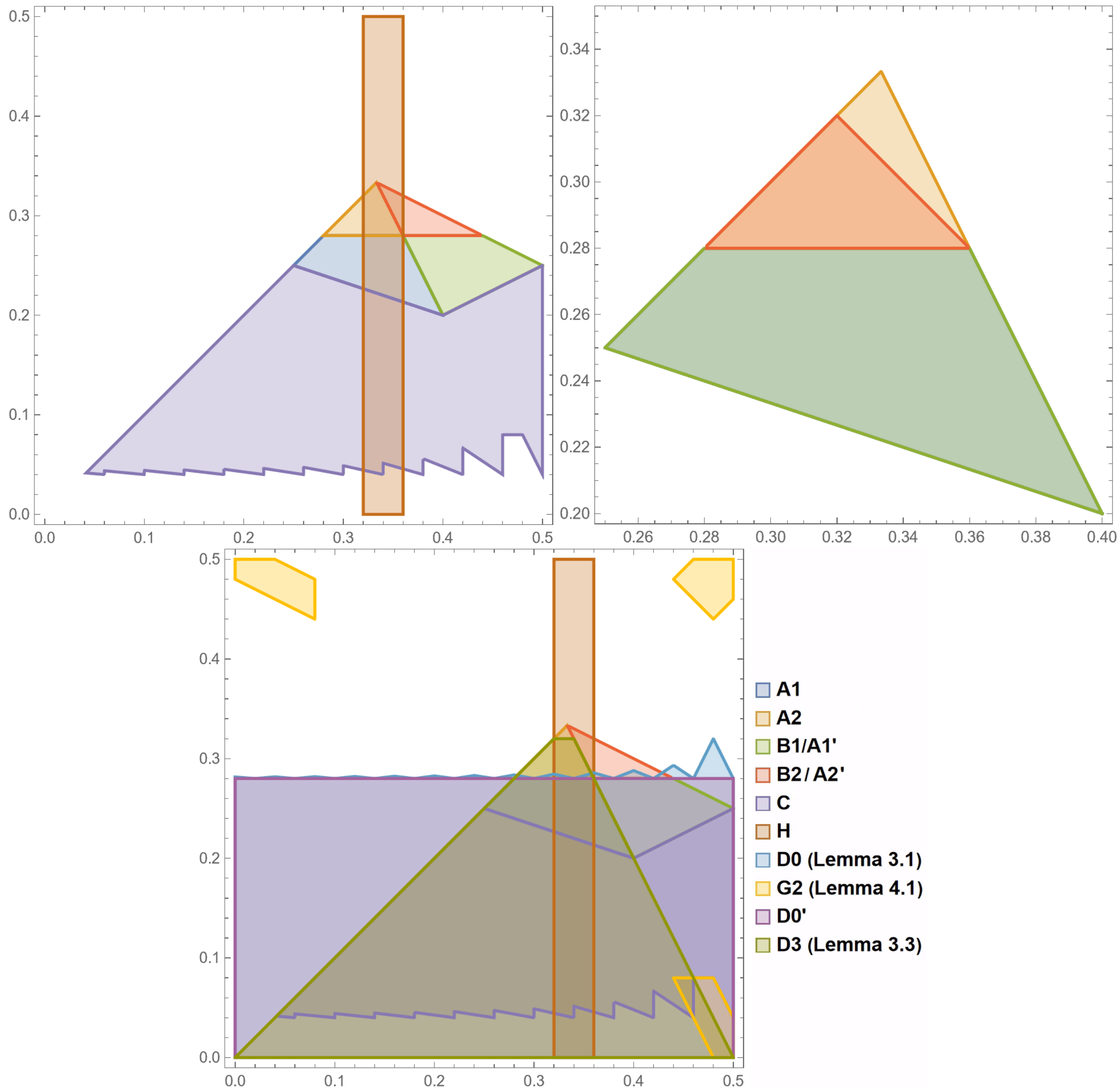

Figure 2: Plot for the Upper Bound ($\theta = 0.52$)

Appendix 2: Values of Integrals

$\boldsymbol{\theta}$	**0.52**	**0.521**	**0.522**	**0.523**	**0.524**
U_{A1}	0.239221	0.239221	0.239221	0.239221	0.239221
U_{A2}	0	0	0	0	0
U_{A3}	0	0	0	0	0
U_{A4}	0	0	0	0	0
U_{A5}	0	0	0	0	0
Loss from $\boldsymbol{A}$	0.239221	0.239221	0.239221	0.239221	0.239221
U_{C01}	0.197907	0.178493	0.158194	0.136616	0.119466
U_{C02}	0.001607	0.001789	0.001688	0.001790	0.001787
U_{C03}	0.020936	0.014611	0.010405	0.005868	0.003499
U_{C04}	0.065033	0.043988	0.037108	0.014831	0.007705
U_{C05}	0.000101	0.000043	0.000018	0.000006	0.000002
U_{C06}	0.000131	0.000106	0.000073	0	0
U_{C07}	0	0	0	0	0
U_{C08}	0.201090	0.181571	0.165022	0.143845	0.128240
U_{C09}	0.000693	0.001054	0.001193	0.001259	0.001338
U_{C10}	0.000222	0.000251	0.000286	0.000295	0.000293
U_{C11}	0.000048	0.000040	0.000029	0.000028	0.000265
U_{C12}	0.008809	0.005143	0.004523	0.001541	0.000727
U_{C13}	0	0	0	0	0
Loss from $\boldsymbol{C}$	0.491533	0.420901	0.372205	0.299391	0.256486
Total Loss	0.969975	0.899343	0.850647	0.777833	0.734928
Lower Bound	0.030025	0.100657	0.149353	0.222167	0.265072

Table 1: Values for the Lower Bounds

$\boldsymbol{\theta}$	**0.52**	**0.521**	**0.522**	**0.523**	**0.524**
Loss from $\boldsymbol{H}$	0.182012	0.133815	0.085930	0.038334	0
V_{A1}	0.179773	0.217159	0.254821	0.292355	0.323686
V_{A2}	0.004874	0.007200	0.010359	0.017561	0.023389
V_{A3}	0.043475	0.035114	0.027426	0.020820	0.015243
Loss from $\boldsymbol{A_1}$	0.218374	0.245073	0.271888	0.295614	0.315540
V_{A4}	0.310609	0.313652	0.316896	0.320119	0.323686
V_{A5}	0.008299	0.010006	0.012635	0.019583	0.023389
V_{A6}	0.051108	0.038581	0.028772	0.021186	0.015243
Loss from $\boldsymbol{A_1'}$	0.353418	0.342227	0.333033	0.321722	0.315540
Loss from $\boldsymbol{A_2}$	0.102865	0.109021	0.122256	0.140969	0.155383
Loss from $\boldsymbol{A_2'}$	0.201264	0.195899	0.187831	0.173941	0.155383
V_{C1}	0.261034	0.260555	0.257913	0.254700	0.249854
V_{C2}	0.000575	0.000787	0.000850	0.000815	0.000795
V_{C3}	0.128160	0.107541	0.092325	0.070907	0.055342
V_{C4}	0.307367	0.249849	0.210236	0.163109	0.128740
V_{C5}	0.004722	0.002606	0.001446	0.000670	0.000322
V_{C6}	0.003889	0.002529	0.000965	0.000461	0.000512
V_{C7}	0.000013	0	0	0	0
Loss from $\boldsymbol{C}$	0.704610	0.622293	0.562035	0.489032	0.433975
Total Loss	1.762543	1.648328	1.562973	1.459612	1.375821
Upper Bound	2.762543	2.648328	2.562973	2.459612	2.375821

Table 2: Values for the Upper Bounds

Appendix 3: Mathematica Code for Numerical Calculations

$\boldsymbol{\theta}$	**LB($\boldsymbol{\theta}$)**	**Code**
0.520	> 0.0300	`https://notebookarchive.org/2025-04-3pjf0hd`
0.521	> 0.1006	`https://notebookarchive.org/2025-04-3pju4ml`
0.522	> 0.1493	`https://notebookarchive.org/2025-04-3pk3cy5`
0.523	> 0.2221	`https://notebookarchive.org/2025-04-3pmvmzo`
0.524	> 0.2650	`https://notebookarchive.org/2025-04-48d2pum`
$\boldsymbol{\theta}$	**UB($\boldsymbol{\theta}$)**	**Code**
0.520	< 2.7626	`https://notebookarchive.org/2025-04-3pjlqxi`
0.521	< 2.6484	`https://notebookarchive.org/2025-04-3pk0dpg`
0.522	< 2.5630	`https://notebookarchive.org/2025-04-3pmmkfv`
0.523	< 2.4597	`https://notebookarchive.org/2025-04-48cy9rw`
0.524	< 2.3759	`https://notebookarchive.org/2025-04-48d5k5h`

Table 3: Code websites

It's important to notice that the above code contains a mass of error warning messages such as `NIntegrate::slwcon` and `NIntegrate::eincr`. However, this is not a big problem, since we can eliminate the impact of error messages by the following ways:

1. the warning message `NIntegrate::slwcon` only means that numerical integration converging too slowly, and it will not affect the numerical results at all. In Lichtman's code (see the ancillary files of [37]) this message was switched off by a line of `Off[NIntegrate::slwcon]`.

2. the warning message `NIntegrate::eincr` provides a *result* with an *error estimation*, which is helpful since we can use *result* $\pm$ *error estimation* to give upper and lower bounds for our integrals. For many of such integrals involving Buchstab's function, this method has been proved to be safe. Another useful trick is that when *result* $<$ *error estimation*, we can use 0 instead of *result*$\pm$*error estimation* to give a lower bound since all sieve functions are non-negative. We have done the numerical calculations for all integrals with different values of `MaxErrorIncreases`, including 100, 1000, 10000, 30000 and 100000. We have even done some calculations on other sieve integrals with `MaxErrorIncreases` = 10000000. The numerical results reveal that for all integrals above, the value of *error estimation* will decrease since `MaxErrorIncreases` increases, and the values of *result* $\pm$ *error estimation* will become closer to the real value. Again, this message can be switched off by adding a line of `Off[NIntegrate::eincr]`, but for the sake of integrity, we didn't do that.

3. If we reduce the accuracy of bounds to 2 or 3 significant digits (just like the things done in BHP), then we can eliminate all messages like `NIntegrate::eincr` without adding `Off[NIntegrate::eincr]`. This is because the calculation complexity decreases since our accuracy goal reduces. This way, Mathematica still gave good results: the new bounds are so close to the high-accuracy bounds (before taking account of *error estimation*), and many of them are even numerically better than the safer high-accuracy bounds with the consideration of *error estimation*. For this, one can see the table below as a theoretical reference. Of course, both of them can produce nontrivial lower and upper bounds.

4. If one carefully check our code, he or she can find out that our code doesn't use the full power of our method. This can be seen as 2 aspects:

4.1. In Lemmas 3.1–3.2, the asymptotic formulas hold for every $\nu \leqslant \nu(\alpha_1)$. For high-dimensional sums, this upper bound becomes more complicated since it involves a maximum of things like $\nu(\alpha_1)$, $\nu(\alpha_2)$, $\nu(\alpha_1+\alpha_2)$ depending on optimal partition of the variables to make $\nu\left(\frac{\log M}{\log x}\right)$ as large as possible. However, our calculation only use $\nu \leqslant \nu_0 = \nu_{\min} = 2\theta - 1$ for variables except for p_1 and p_2 because of the running time and internal storage restrictions. This leads to an overestimation of the loss since $\nu_0 \leqslant \nu\left(\frac{\log M}{\log x}\right)$ and we actually have asymptotic formulas for those sums with a variable lies in $\left[\nu_0, \nu\left(\frac{\log M}{\log x}\right)\right]$.

4.2. Many asymptotic formulas for Type-II sums are not used in the calculations. We don't use our new Lemmas 4.5–4.6 in all calculations; we don't use the powerful Lemma 4.4 to give asymptotic formulas for "Type-II$_n$" sums with $n \geqslant 6$; we don't consider regions $\boldsymbol{G}_n$ with all components made up by at least two variables, like $(\alpha_1+\alpha_2+\alpha_6, \alpha_3+\alpha_4) \in \boldsymbol{G}_2 \implies \boldsymbol{\alpha}_6 \in \boldsymbol{G}_6$; we don't even use Lemma 4.3 for "Type-II$_n$" sums with $n \geqslant 8$. This is also due to running time and internal storage restrictions. We hope that these restrictions can be removed using more powerful supercomputers.

LB($\boldsymbol{\theta}$)	**Code**
0.520	`https://notebookarchive.org/2025-04-9p1on6y`
0.521	`https://notebookarchive.org/2025-04-9p1t3hj`
0.522	`https://notebookarchive.org/2025-04-9p1xmq7`
0.523	`https://notebookarchive.org/2025-04-9p4hloi`
0.524	`https://notebookarchive.org/2025-04-9p4m3bt`
UB($\boldsymbol{\theta}$)	**Code**
0.520	`https://notebookarchive.org/2025-04-9p1qy87`
0.521	`https://notebookarchive.org/2025-04-9p1va6j`
0.522	`https://notebookarchive.org/2025-04-9p4dl87`
0.523	`https://notebookarchive.org/2025-04-9p4jsrs`
0.524	`https://notebookarchive.org/2025-04-9p4o9kp`

Table 4: Code websites (without error messages)

References

[1] R. Alweiss and S. Luo. Bounded gaps between primes in short intervals. *Res. Number Theory*, 4(2):No. 15, 2018.
[2] J. Andersson. Turán's problem 10 revisited. *arXiv Mathematics e-prints*, page math/0609271v3, 2007.
[3] R. C. Baker and G. Harman. The difference between consecutive primes. *Proc. London Math. Soc.*, 72(3):261–280, 1996.
[4] R. C. Baker, G. Harman, and J. Pintz. The exceptional set for Goldbach's problem in short intervals. In *Sieve methods, exponential sums and their applications in number theory*, ed. G. R. H. Greaves, G. Harman and M. N. Huxley, Cambridge University Press, pages 1–54. Cambridge University Press, Cambridge, 1997.
[5] R. C. Baker, G. Harman, and J. Pintz. The difference between consecutive primes, II. *Proc. London Math. Soc.*, 83(3):532–562, 2001.
[6] R. C. Baker, G. Harman, and J. Rivat. Primes of the form $[n^c]$. *J. Number Theory*, 50:261–277, 1995.
[7] W. D. Banks and I. E. Shparlinski. Bounds on short character sums and L–functions with characters to a powerful modulus. *Journal d Analyse Mathématique*, 139:239–263, 2019.
[8] K. A. Broughan and A. R. Barnett. On the subsequence of primes having prime subscripts. *J. Int. Seq.*, 12(2):Article 09.2.3, 2009.
[9] N. G. Chudakov. On the difference between two neighboring prime numbers. *Mat. Sb.*, 43(1):799–813, 1936.
[10] H. Cramér. On the order of magnitude of the difference between consecutive prime numbers. *Acta Arith.*, 2:23–46, 1937.
[11] Y. Ding. Sum of elements in finite Sidon sets. II. *Publ. Math. Debrecen*, 103:243–256, 2023.
[12] P. Erdős and A. Rényi. A probabilistic approach to problems of Diophantine approximation. *Illinois J. Math.*, 1(3):303–315, 1957.
[13] J. Friedlander and H. Iwaniec. *Opera de cribro*, volume 57 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2010.
[14] L. Grimmelt. Goldbach numbers in short intervals. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, 23(3):1395–1416, 2022.
[15] L. Guth and J. Maynard. New large value estimates for Dirichlet polynomials. *arXiv e-prints*, page arXiv:2405.20552v1, 2024.
[16] G. Harman. On the distribution of αp modulo one. *J. London Math. Soc.*, 27(2):9–18, 1983.
[17] G. Harman. On the distribution of αp modulo one II. *Proc. London Math. Soc.*, 72(3):241–260, 1996.
[18] G. Harman. Eratosthenes, Legendre, Vinogradov and beyond: The hidden power of the simplest sieve. In *Sieve methods, exponential sums and their applications in number theory*, ed. G. R. H. Greaves, G. Harman and M. N. Huxley, Cambridge University Press, pages 161–173. Cambridge University Press, Cambridge, 1997.
[19] G. Harman. On the number of Carmichael numbers up to x. *Bull. London Math. Soc.*, 37:641–650, 2005.
[20] G. Harman. *Prime–detecting Sieves*, volume 33 of *London Mathematical Society Monographs (New Series)*. Princeton University Press, Princeton, NJ, 2007.
[21] G. Harman. Watt's mean value theorem and Carmichael numbers. *Int. J. Number Theory*, 4(2):241–248, 2008.
[22] G. Harman, A. Kumchev, and P. A. Lewis. The distribution of prime ideals of imaginary quadratic fields. *Trans. Am. Math. Soc.*, 356(2):599–620, 2004.
[23] D. R. Heath-Brown. The twelfth power moment of the Riemann–function. *Q. J. Math.*, 29(4):443–462, 1978.
[24] D. R. Heath-Brown and H. Iwaniec. On the difference between consecutive primes. *Invent. Math.*, 55:49–69, 1979.
[25] H. Heilbronn. Über den Primzahlsatz von Herrn Hoheisel. *Math. Z.*, 36:394–423, 1933.
[26] G. Hoheisel. Primzahlprobleme in der analysis. *Sitz. Preuss. Akad. Wiss.*, 2:1–13, 1930.
[27] M. N. Huxley. On the difference between consecutive primes. *Invent. Math.*, 15:164–170, 1972.
[28] A. E. Ingham. On the difference between consecutive primes. *Q. J. Math.*, 8:255–266, 1936.
[29] H. Iwaniec. On the Brun–Titchmarsh theorem. *J. Math. Soc. Japan*, 34(1):95–123, 1982.
[30] H. Iwaniec and M. Jutila. Primes in short intervals. *Ark. Mat.*, 17:167–176, 1979.
[31] H. Iwaniec and J. Pintz. Primes in short intervals. *Monatsh. Math.*, 98:115–143, 1984.
[32] C. Jia. On the Piatetski–Shapiro–Vinogradov Theorem. *Acta Arith.*, 73(1):1–28, 1995.
[33] C. Jia. On the distribution of αp modulo one (II). *Sci. China Ser. A*, 43:703–721, 2000.
[34] P. A. Lewis. Finding Gaussian primes by analytic number theory sieve methods. *Ph.D. Thesis*, Cardiff University, 2002.
[35] R. Li. Primes in almost all short intervals. *arXiv e-prints*, page arXiv:2407.05651v5, 2024.
[36] J. D. Lichtman. Primes in arithmetic progressions to large moduli, and shifted primes without large prime factors. *Mathematische Annalen, to appear. arXiv e-prints*, page arXiv:2211.09641v1, 2022.
[37] J. D. Lichtman. Primes in arithmetic progressions to large moduli, and Goldbach beyond the square–root barrier. *arXiv e-prints*, page arXiv:2309.08522v1, 2023.
[38] S. Lou and Q. Yao. An upper bound for primes in an interval. *Chinese Ann. Math. Ser. A*, 10(3):255–262, 1989.
[39] S. Lou and Q. Yao. A Chebyshev's type of prime number theorem in a short interval–II. *Hardy-Ramanujan J.*, 15:1–33, 1992.
[40] S. Lou and Q. Yao. The number of primes in a short interval. *Hardy-Ramanujan J.*, 16:21–43, 1993.
[41] S. Lou and Q. Yao. Estimate of sums of Dirichlet series. *Hardy-Ramanujan J.*, 17:1–31, 1994.
[42] H. L. Montgomery. *Topics in Multiplicative Number Theory*. Lecture Notes in Math. 227. Springer, Berlin, 1971.
[43] H. L. Montgomery and R. C. Vaughan. The large sieve. *Mathmatika*, 20:119–134, 1973.
[44] C. J. Mozzochi. On the difference between consecutive primes. *J. Number Theory*, 24:181–187, 1986.
[45] J. Pintz. On primes in short intervals I. *Studia Sci. Math. Hungar.*, 16:395–414, 1981.
[46] J. Pintz. On primes in short intervals II. *Studia Sci. Math. Hungar.*, 19:89–96, 1984.
[47] J. Pintz. On the difference of primes. *arXiv e-prints*, page arXiv:1206.0149v1, 2012.
[48] J. Pintz. *A Goldbach–sejtésről*. Székfoglaló előadások a Magyar Tudományos Akadémián. Magyar Tudományos Akadémia, Budapest, 2014.
[49] V. Starichkova. The distribution of prime numbers in short intervals. *Ph.D. Thesis*, UNSW Canberra, 2024.
[50] H. Tang and J. Wu. Mean values of arithmetic functions on a sparse set and applications. *HAL open science*, pages hal–04557186, 2024.
[51] T. S. Trudgian and A. Yang. Toward optimal exponent pairs. *arXiv e-prints*, page arXiv:2306.05599v3, 2024.
[52] M. Wang. Waring–Goldbach problem in short intervals. *Isr. J. Math.*, 261:637–669, 2024.
[53] N. Watt. Kloosterman sums and a mean value for Dirichlet polynomials. *J. Number Theory*, 53:179–210, 1995.

International Curriculum Center, The High School Affiliated to Renmin University of China, Beijing, China
Email address: runbo.li.carey@gmail.com