LARGEST SQUARE DIVISORS OF SHIFTED PRIMES

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ABSTRACT. The author shows that there are infinitely many primes p such that for any nonzero integer a, $p-a$ is divisible by a square $d^2 > p^{\frac{1}{2}+\frac{1}{700}}$. The exponent $\frac{1}{2}+\frac{1}{700}$ improves Merikoski's $\frac{1}{2}+\frac{1}{2000}$. Many powerful devices in Harman's sieve are used for this improvement.

CONTENTS

1. INTRODUCTION

The Euler's conjecture, which states that there are infinitely many primes of the form n^2+1, is one of Landau's problems on prime numbers. There are several ways to attack this conjecture. One way is to relax the number of prime factors of $f(n)$, and the best result in this way is due to Iwaniec [4]. Building on the previous work of Richert [11], he showed that for any irreducible polynomial $f(n)=an^2+bn+c$ with $a>0$ and $c \equiv 1(\bmod 2)$, there are infinitely many x such that $f(x)$ has at most 2 prime factors.

Another possible way is to consider the square divisors of $p-1$. If we can show that there are infinitely many primes such that $p-1$ is divisible by a large square $d^2 \geqslant p^{\theta}$ with $\theta=1$, then the Euler's conjecture is solved. The first result on this direction is due to Baier and Zhao [1] [2], where they proved the above statement holds with d prime and $\theta<\frac{4}{9}$ as an application of their large sieve for sparse sets of moduli. They interpret the problem as an equidistribution problem for primes $p \equiv 1(\bmod\ d^2)$, after which the result follows from their Bombieri–Vinogradov type theorem for sparse sets of moduli [[1], Theorem 3].

In 2009, Matomäki [9] improved the above result to $\theta<\frac{1}{2}$ using Harman's sieve [3] and Type–II information obtained using the large sieve of Baier and Zhao [2]. Note that the exponent $\theta=\frac{1}{2}$ is the limit of what can be obtained under the Generalized Riemann Hypothesis (GRH). In [10], Merikoski first broke this $\frac{1}{2}$–barrier and successfully got $\theta \leqslant \frac{1}{2}+\frac{1}{2000}$ without the restriction that d is a prime. In the article [10], he mentioned that the "extra" exponent $\frac{1}{2000}$ has not been fully optimized, and one should be able to increase this to some value between $\frac{1}{500}$ and $\frac{1}{1000}$. In this paper, we increase this to $\frac{1}{700}$ by a careful decomposition on Harman's sieve.

Theorem 1.1. *Let $a \neq 0$ be an integer. There are infinitely many primes p such that $d^2 \mid (p-a)$ for some integer d with*
$$d^2 \geqslant p^{\frac{1}{2}+\frac{1}{700}}.$$

Throughout this paper, we always suppose that ε is a sufficiently small positive constant and X is sufficiently large. The letter p, with or without subscript, is reserved for prime numbers. Let $\varpi=\frac{1}{1400}$, $D=X^{\frac{1}{2}+2\varpi}$, $K=\lceil\frac{1}{\varepsilon}\rceil$ and $P=D^{\frac{1}{K}}$. Let σ be a number satifsies the condition $19\sigma+90\varpi+71\varepsilon<1$. Define
$$I_j=\left(2^{j-1}P^{\frac{1}{2}}, 2^j P^{\frac{1}{2}}\right] \text{ for } j=1,2,\ldots,K.$$

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We set

$$\mathcal{D}=\{p_1^2p_2^2\cdots p_K^2 : p_j\in I_j \text{ for } j=1,2,\ldots,K\}$$

so that $d^2\in D$ is of size $\asymp D$ and is a square of a squarefree integer.

Fix an integer $a\neq 0$ and a C^∞–smooth function $0\leqslant\psi\leqslant 1$, supported on the interval $[1,2]$ and satisfying $\psi(x)=1$ for $1+\eta\leqslant x\leqslant 2-\eta$ for some sufficiently small positive η. For $d^2\in D$ and $z<X$, denote

$$S\left(\mathcal{A}^d,z\right)=\sum_{\substack{n\equiv a(\bmod d^2)\\(n,P(z))=1}}\psi\left(\frac{n}{X}\right)\quad\text{and}\quad S\left(\mathcal{B}^d,z\right)=\frac{1}{\varphi(d^2)}\sum_{\substack{(n,d^2)=1\\(n,P(z))=1}}\psi\left(\frac{n}{X}\right).$$

Then Theorem 1.1 holds if there exists $\varepsilon,\eta,c>0$ such that for all but $O\left(D^{\frac12}X^{-\eta}\right)$ of the moduli $d^2\in\mathcal{D}$, we have

$$S\left(\mathcal{A}^d,2X^{\frac12}\right)\geqslant cS\left(\mathcal{B}^d,2X^{\frac12}\right).\tag{1}$$

2. Asymptotic formulas

Lemma 2.1. *([10], Proposition 7]). Let $U\leqslant X^{\frac12+2\varpi+\varepsilon}$ and let a_u be divisor–bounded. Then for all but $O\left(D^{\frac12}X^{-\eta}\right)$ of $d^2\in\mathcal{D}$, we have*

$$\sum_{u\sim U}a_uS\left(\mathcal{A}_u^d,X^{\sigma-2\varpi}\right)=(1+o(1))\sum_{u\sim U}a_uS\left(\mathcal{B}_u^d,X^{\sigma-2\varpi}\right).$$

Lemma 2.2. *([10], Proposition 6]). Let $U\leqslant X^{\frac12-\sigma}$, $V\leqslant X^{\frac18+\frac{\sigma}{2}-\frac{5\varpi}{2}-\eta}$ and let a_u, b_v be divisor–bounded. Then for all but $O\left(D^{\frac12}X^{-\eta}\right)$ of $d^2\in\mathcal{D}$, we have*

$$\sum_{\substack{u\sim U\\v\sim V}}a_ub_vS\left(\mathcal{A}_{uv}^d,X^{\sigma-2\varpi}\right)=(1+o(1))\sum_{\substack{u\sim U\\v\sim V}}a_ub_vS\left(\mathcal{B}_{uv}^d,X^{\sigma-2\varpi}\right).$$

Lemma 2.3. *([10], Proposition 4]). Let $UV=X$, $X^{\frac12-\sigma}\leqslant U\leqslant X^{\frac12-2\varpi-\varepsilon}$ and let a_u, b_v be divisor–bounded. Then we have*

$$\sum_{d^2\in\mathcal{D}}\left|\sum_{\substack{uv\equiv a(\bmod d^2)\\u\sim U,v\sim V}}a_ub_v\psi\left(\frac{uv}{X}\right)-\frac{1}{\varphi(d^2)}\sum_{\substack{(uv,d^2)=1\\u\sim U,v\sim V}}a_ub_v\psi\left(\frac{uv}{X}\right)\right|\ll D^{-\frac12}X^{1-\eta}.$$

3. The final decomposition

Before decomposing, we define asymptotic regions I and II as

$$\begin{aligned}I(m,n)&:=\left\{m+n\leqslant\frac12+2\varpi,\ \text{or } m\leqslant\frac12-\sigma \text{ and } n<\frac18+\frac{\sigma}{2}-\frac{5\varpi}{2}\right\},\\
II(m,n)&:=\left\{\frac12-\sigma\leqslant m\leqslant\frac12-2\varpi \text{ or } \frac12-\sigma\leqslant n\leqslant\frac12-2\varpi\right.\\
&\qquad\left.\text{or } \frac12-\sigma\leqslant m+n\leqslant\frac12-2\varpi \text{ or } \frac12+2\varpi\leqslant m+n\leqslant\frac12+\sigma\right\}.\end{aligned}$$

Let $\omega(u)$ denotes the Buchstab function determined by the following differential–difference equation

$$\begin{cases}\omega(u)=\frac{1}{u}, & 1\leqslant u\leqslant 2,\\(u\omega(u))'=\omega(u-1), & u\geqslant 2.\end{cases}$$

Moreover, we have the upper and lower bounds for $\omega(u)$:

$$\omega(u)\geqslant\omega_0(u)=\begin{cases}\frac{1}{u}, & 1\leqslant u<2,\\ \frac{1+\log(u-1)}{u}, & 2\leqslant u<3,\\ \frac{1+\log(u-1)}{u}+\frac{1}{u}\int_2^{u-1}\frac{\log(t-1)}{t}dt\geqslant 0.5607, & 3\leqslant u<4,\\ 0.5612, & u\geqslant 4,\end{cases}$$

$$\omega(u) \leqslant \omega_1(u) = \begin{cases} \frac{1}{u}, & 1 \leqslant u < 2, \\ \frac{1+\log(u-1)}{u}, & 2 \leqslant u < 3, \\ \frac{1+\log(u-1)}{u} + \frac{1}{u}\int_2^{u-1} \frac{\log(t-1)}{t} dt \leqslant 0.5644, & 3 \leqslant u < 4, \\ 0.5617, & u \geqslant 4. \end{cases}$$

We shall use $\omega_0(u)$ and $\omega_1(u)$ to give numerical bounds for some sieve functions discussed below. We shall also use the simple upper bound $\omega(u) \leqslant \max(\frac{1}{u}, 0.5672)$ (see Lemma 8(iii) of [5]) to estimate high–dimensional integrals. Fix $\sigma = \frac{1}{20.31}$ and let $p_j = X^{\alpha_j}$. By Buchstab's identity, we have

$$\begin{aligned} S\left(\mathcal{A}^d, 2X^{\frac{1}{2}}\right) &= S\left(\mathcal{A}^d, X^{\sigma-2\varpi}\right) - \sum_{\sigma-2\varpi \leqslant \alpha_1 < \frac{1}{2}} S\left(\mathcal{A}^d_{p_1}, X^{\sigma-2\varpi}\right) + \sum_{\substack{\sigma-2\varpi \leqslant \alpha_1 < \frac{1}{2} \\ \sigma-2\varpi \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1-\alpha_1}{2}\right)}} S\left(\mathcal{A}^d_{p_1p_2}, p_2\right) \\ &= S\left(\mathcal{A}^d, X^{\sigma-2\varpi}\right) - \sum_{\sigma-2\varpi \leqslant \alpha_1 < \frac{1}{2}} S\left(\mathcal{A}^d_{p_1}, X^{\sigma-2\varpi}\right) + \sum_{(\alpha_1,\alpha_2)\in II} S\left(\mathcal{A}^d_{p_1p_2}, p_2\right) \\ &\quad + \sum_{(\alpha_1,\alpha_2)\in A} S\left(\mathcal{A}^d_{p_1p_2}, p_2\right) + \sum_{(\alpha_1,\alpha_2)\in B} S\left(\mathcal{A}^d_{p_1p_2}, p_2\right) + \sum_{(\alpha_1,\alpha_2)\in C} S\left(\mathcal{A}^d_{p_1p_2}, p_2\right) \\ &= S_1 - S_2 + S_{II} + S_A + S_B + S_C, \end{aligned} \tag{2}$$

where

$$\begin{aligned} A(\alpha_1,\alpha_2) = &\left\{\sigma - 2\varpi \leqslant \alpha_1 < \frac{1}{2},\ \sigma - 2\varpi \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1-\alpha_1}{2}\right),\ (\alpha_1,\alpha_2) \notin II,\right. \\ &\quad \left.(\alpha_1,\alpha_2,\alpha_2) \text{ can be partitioned into } (m,n) \in I\right\}, \\ B(\alpha_1,\alpha_2) = &\left\{\sigma - 2\varpi \leqslant \alpha_1 < \frac{1}{2},\ \sigma - 2\varpi \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1-\alpha_1}{2}\right),\ (\alpha_1,\alpha_2) \notin II,\right. \\ &\quad \left.(\alpha_1,\alpha_2,\alpha_2) \text{ cannot be partitioned into } (m,n) \in I,\ (\alpha_1,\alpha_2) \in I,\ (1-\alpha_1-\alpha_2,\alpha_2) \in I\right\}, \\ C(\alpha_1,\alpha_2) = &\left\{\sigma - 2\varpi \leqslant \alpha_1 < \frac{1}{2},\ \sigma - 2\varpi \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1-\alpha_1}{2}\right),\ (\alpha_1,\alpha_2) \notin II \cup A \cup B\right\}. \end{aligned}$$

We have asymptotic formulas for S_1 and S_2 by Lemma 2.1. We can also give an asymptotic formula for S_{II} by Lemma 2.3. For S_A, we can apply Buchstab's identity to get

$$\begin{aligned} S_A = &\sum_{(\alpha_1,\alpha_2)\in A} S\left(\mathcal{A}^d_{p_1p_2}, p_2\right) \\ = &\sum_{(\alpha_1,\alpha_2)\in A} S\left(\mathcal{A}^d_{p_1p_2}, X^{\sigma-2\varpi}\right) - \sum_{\substack{(\alpha_1,\alpha_2)\in A \\ \sigma-2\varpi \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1-\alpha_1-\alpha_2}{2}\right) \\ (\alpha_1,\alpha_2,\alpha_3) \text{ can be partitioned into } (m,n)\in II}} S\left(\mathcal{A}^d_{p_1p_2p_3}, p_3\right) \\ &- \sum_{\substack{(\alpha_1,\alpha_2)\in A \\ \sigma-2\varpi \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1-\alpha_1-\alpha_2}{2}\right) \\ (\alpha_1,\alpha_2,\alpha_3) \text{ cannot be partitioned into } (m,n)\in II}} S\left(\mathcal{A}^d_{p_1p_2p_3}, X^{\sigma-2\varpi}\right) \\ &+ \sum_{\substack{(\alpha_1,\alpha_2)\in A \\ \sigma-2\varpi \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1-\alpha_1-\alpha_2}{2}\right) \\ (\alpha_1,\alpha_2,\alpha_3) \text{ cannot be partitioned into } (m,n)\in II \\ \sigma-2\varpi \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1-\alpha_1-\alpha_2-\alpha_3}{2}\right) \\ (\alpha_1,\alpha_2,\alpha_3,\alpha_4) \text{ can be partitioned into } (m,n)\in II}} S\left(\mathcal{A}^d_{p_1p_2p_3p_4}, p_4\right) \\ &+ \sum_{\substack{(\alpha_1,\alpha_2)\in A \\ \sigma-2\varpi \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1-\alpha_1-\alpha_2}{2}\right) \\ (\alpha_1,\alpha_2,\alpha_3) \text{ cannot be partitioned into } (m,n)\in II \\ \sigma-2\varpi \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1-\alpha_1-\alpha_2-\alpha_3}{2}\right) \\ (\alpha_1,\alpha_2,\alpha_3,\alpha_4) \text{ cannot be partitioned into } (m,n)\in II}} S\left(\mathcal{A}^d_{p_1p_2p_3p_4}, p_4\right) \end{aligned}$$

$$= S_{A1} - S_{A2} - S_{A3} + S_{A4} + S_{A5}. \tag{3}$$

By Lemmas 2.1–2.2 we have asymptotic formulas for S_{A1} and S_{A3}. For S_{A2} and S_{A4} we can also give asymptotic formulas by Lemma 2.3. We discard part of S_{A5} if we can neither give an asymptotic formula nor decompose it further. For the remaining part, we can perform Buchstab's identity twice more to reach a six–dimensional sum if we can group $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_4)$ into $(m, n) \in I$, and we can use reversed Buchstab's identity to make some almost–primes visible. Working as in [6] and [8], the total loss from S_A can be bounded by

$$\begin{aligned}
&\left(\int_{(t_1,t_2,t_3,t_4)\in S_{A51}} \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1t_2t_3t_4^2} dt_4dt_3dt_2dt_1\right)\\
&-\left(\int_{(t_1,t_2,t_3,t_4,t_5)\in S_{A52}} \frac{\omega_0\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1t_2t_3t_4t_5^2} dt_5dt_4dt_3dt_2dt_1\right)\\
&+\left(\int_{(t_1,t_2,t_3,t_4,t_5,t_6)\in S_{A53}} \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{t_6}\right)}{t_1t_2t_3t_4t_5t_6^2} dt_6dt_5dt_4dt_3dt_2dt_1\right)\\
&+\left(\int_{(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8)\in S_{A54}} \frac{\max\left(\frac{t_8}{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7-t_8}, 0.5672\right)}{t_1t_2t_3t_4t_5t_6t_7t_8^2} dt_8dt_7dt_6dt_5dt_4dt_3dt_2dt_1\right)\\
&< 0.002515,
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
S_{A51}(t_1,t_2,t_3,t_4) := \{ &(t_1,t_2)\in S_A,\\
&\sigma - 2\varpi \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\\
&(t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in II,\\
&\sigma - 2\varpi \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in II,\\
&(t_1,t_2,t_3,t_4,t_4) \text{ cannot be partitioned into } (m,n)\in I,\\
&\sigma - 2\varpi \leqslant t_1 < \frac{1}{2},\ \sigma - 2\varpi \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Bigg\},\\
S_{A52}(t_1,t_2,t_3,t_4,t_5) := \{ &(t_1,t_2)\in S_A,\\
&\sigma - 2\varpi \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\\
&(t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in II,\\
&\sigma - 2\varpi \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in II,\\
&(t_1,t_2,t_3,t_4,t_4) \text{ cannot be partitioned into } (m,n)\in I,\\
&t_4 < t_5 < \frac{1}{2}(1-t_1-t_2-t_3-t_4),\\
&(t_1,t_2,t_3,t_4,t_5) \text{ can be partitioned into } (m,n)\in II,\\
&\sigma - 2\varpi \leqslant t_1 < \frac{1}{2},\ \sigma - 2\varpi \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Bigg\},
\end{aligned}$$

$$S_{A53}(t_1,t_2,t_3,t_4,t_5,t_6) := \{ (t_1,t_2)\in S_A,$$

$$
\begin{aligned}
&\sigma - 2\varpi \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \\
&(t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n) \in II, \\
&\sigma - 2\varpi \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in II, \\
&(t_1,t_2,t_3,t_4,t_4) \text{ can be partitioned into } (m,n) \in I, \\
&\sigma - 2\varpi \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right), \\
&(t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (m,n) \in II, \\
&\sigma - 2\varpi \leqslant t_6 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right), \\
&(t_1,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n) \in II, \\
&(t_1,t_2,t_3,t_4,t_5,t_6,t_6) \text{ cannot be partitioned into } (m,n) \in I, \\
&\left.\sigma - 2\varpi \leqslant t_1 < \frac{1}{2},\ \sigma - 2\varpi \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
S_{A54}(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) := \ &\{(t_1,t_2) \in S_A, \\
&\sigma - 2\varpi \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \\
&(t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n) \in II, \\
&\sigma - 2\varpi \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in II, \\
&(t_1,t_2,t_3,t_4,t_4) \text{ can be partitioned into } (m,n) \in I, \\
&\sigma - 2\varpi \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right), \\
&(t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (m,n) \in II, \\
&\sigma - 2\varpi \leqslant t_6 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right), \\
&(t_1,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n) \in II, \\
&(t_1,t_2,t_3,t_4,t_5,t_6,t_6) \text{ can be partitioned into } (m,n) \in I, \\
&\sigma - 2\varpi \leqslant t_7 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6)\right), \\
&(t_1,t_2,t_3,t_4,t_5,t_6,t_7) \text{ cannot be partitioned into } (m,n) \in II, \\
&\sigma - 2\varpi \leqslant t_8 < \min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6-t_7)\right), \\
&(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) \text{ cannot be partitioned into } (m,n) \in II, \\
&\left.\sigma - 2\varpi \leqslant t_1 < \frac{1}{2},\ \sigma - 2\varpi \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\}.
\end{aligned}
$$

For S_B we use Buchstab's identity to get

$$
S_B = \sum_{(\alpha_1,\alpha_2)\in B} S\left(\mathcal{A}^d_{p_1p_2}, X^{\sigma-2\varpi}\right) - \sum_{\substack{(\alpha_1,\alpha_2)\in B \\ \sigma-2\varpi\leqslant\alpha_3<\min\left(\alpha_2, \frac{1-\alpha_1-\alpha_2}{2}\right)}} S\left(\mathcal{A}^d_{p_1p_2p_3}, p_3\right). \tag{5}
$$

Here we cannot decompose it directly using Buchstab's identity once more, since we cannot give an asymptotic formula for part of the negative sum

$$\sum_{\substack{(\alpha_1,\alpha_2)\in B\\ \sigma-2\varpi\leqslant\alpha_3<\min\left(\alpha_2,\frac{1-\alpha_1-\alpha_2}{2}\right)\\ (\alpha_1,\alpha_2,\alpha_3)\text{ cannot be partitioned into }(m,n)\in II}} S\left(\mathcal{A}^d_{p_1p_2p_3}, X^{\sigma-2\varpi}\right).$$

However, we can use a role–reversal to transfer the last sum in (5) into a form that have an asymptotic formula. Note that in [10] role–reversals were not used. We refer the readers to [7] and [8] for more applications of role–reversals. By a standard process, we have (where $\beta\sim X^{1-\alpha_1-\alpha_2-\alpha_3}$ and $(\beta,P(p_3))=1$)

$$\begin{aligned}
S_B =& \sum_{(\alpha_1,\alpha_2)\in B} S\left(\mathcal{A}^d_{p_1p_2}, p_2\right)\\
=& \sum_{(\alpha_1,\alpha_2)\in B} S\left(\mathcal{A}^d_{p_1p_2}, X^{\sigma-2\varpi}\right) - \sum_{\substack{(\alpha_1,\alpha_2)\in B\\ \sigma-2\varpi\leqslant\alpha_3<\min\left(\alpha_2,\frac{1-\alpha_1-\alpha_2}{2}\right)\\ (\alpha_1,\alpha_2,\alpha_3)\text{ can be partitioned into }(m,n)\in II}} S\left(\mathcal{A}^d_{p_1p_2p_3}, p_3\right)\\
&- \sum_{\substack{(\alpha_1,\alpha_2)\in B\\ \sigma-2\varpi\leqslant\alpha_3<\min\left(\alpha_2,\frac{1-\alpha_1-\alpha_2}{2}\right)\\ (\alpha_1,\alpha_2,\alpha_3)\text{ cannot be partitioned into }(m,n)\in II}} S\left(\mathcal{A}^d_{\beta p_2p_3}, X^{\sigma-2\varpi}\right)\\
&+ \sum_{\substack{(\alpha_1,\alpha_2)\in B\\ \sigma-2\varpi\leqslant\alpha_3<\min\left(\alpha_2,\frac{1-\alpha_1-\alpha_2}{2}\right)\\ (\alpha_1,\alpha_2,\alpha_3)\text{ cannot be partitioned into }(m,n)\in II\\ \sigma-2\varpi\leqslant\alpha_4<\frac{1}{2}\alpha_1\\ (1-\alpha_1-\alpha_2-\alpha_3,\alpha_2,\alpha_3,\alpha_4)\text{ can be partitioned into }(m,n)\in II}} S\left(\mathcal{A}^d_{\beta p_2p_3p_4}, p_4\right)\\
&+ \sum_{\substack{(\alpha_1,\alpha_2)\in B\\ \sigma-2\varpi\leqslant\alpha_3<\min\left(\alpha_2,\frac{1-\alpha_1-\alpha_2}{2}\right)\\ (\alpha_1,\alpha_2,\alpha_3)\text{ cannot be partitioned into }(m,n)\in II\\ \sigma-2\varpi\leqslant\alpha_4<\frac{1}{2}\alpha_1\\ (1-\alpha_1-\alpha_2-\alpha_3,\alpha_2,\alpha_3,\alpha_4)\text{ cannot be partitioned into }(m,n)\in II}} S\left(\mathcal{A}^d_{\beta p_2p_3p_4}, p_4\right)\\
=& S_{B1}-S_{B2}-S_{B3}+S_{B4}+S_{B5}.
\end{aligned}\tag{6}$$

We can give asymptotic formulas for S_{B1}–S_{B4} by Lemmas 2.1–2.3. We can also decompose part of S_{B5} if the variables (with α_1 replaced by $1-\alpha_1-\alpha_2-\alpha_3$) satisfy the same conditions as in the decomposable part of S_{B5}. Again, the total loss from S_B can be bounded by

$$\begin{aligned}
&\left(\int_{(t_1,t_2,t_3,t_4)\in S_{B51}} \frac{\omega_1\left(\frac{t_1-t_4}{t_4}\right)\omega_1\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2t_3^2t_4^2}dt_4dt_3dt_2dt_1\right)\\
&-\left(\int_{(t_1,t_2,t_3,t_4,t_5)\in S_{B52}} \frac{\omega_0\left(\frac{t_1-t_4-t_5}{t_5}\right)\omega_0\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2t_3^2t_4t_5^2}dt_5dt_4dt_3dt_2dt_1\right)\\
&+\left(\int_{(t_1,t_2,t_3,t_4,t_5,t_6)\in S_{B53}} \frac{\omega_1\left(\frac{t_1-t_4-t_5-t_6}{t_6}\right)\omega_1\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2t_3^2t_4t_5t_6^2}dt_6dt_5dt_4dt_3dt_2dt_1\right)\\
&<0.006249,
\end{aligned}\tag{7}$$

where

$$\begin{aligned}
S_{B51}(t_1,t_2,t_3,t_4) :=\ & \{(t_1,t_2)\in S_B,\\
&\sigma-2\varpi\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right),
\end{aligned}$$

$$
\begin{aligned}
& (t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in II,\\
& \sigma-2\varpi \leqslant t_4 < \frac{1}{2}t_1,\\
& (1-t_1-t_2-t_3,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in II,\\
& (1-t_1-t_2-t_3,t_2,t_3,t_4,t_4) \text{ cannot be partitioned into } (m,n)\in I,\\
& \left. \sigma-2\varpi \leqslant t_1 < \frac{1}{2},\ \sigma-2\varpi \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\},\\
S_{B52}(t_1,t_2,t_3,t_4,t_5) := \ & \{(t_1,t_2)\in S_B,\\
& \sigma-2\varpi \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\\
& (t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in II,\\
& \sigma-2\varpi \leqslant t_4 < \frac{1}{2}t_1,\\
& (1-t_1-t_2-t_3,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in II,\\
& (1-t_1-t_2-t_3,t_2,t_3,t_4,t_4) \text{ cannot be partitioned into } (m,n)\in I,\\
& t_4 < t_5 < \frac{1}{2}(t_1-t_4),\\
& (1-t_1-t_2-t_3,t_2,t_3,t_4,t_5) \text{ can be partitioned into } (m,n)\in II,\\
& \left. \sigma-2\varpi \leqslant t_1 < \frac{1}{2},\ \sigma-2\varpi \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\},\\
S_{B53}(t_1,t_2,t_3,t_4,t_5,t_6) := \ & \{(t_1,t_2)\in S_B,\\
& \sigma-2\varpi \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\\
& (t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in II,\\
& \sigma-2\varpi \leqslant t_4 < \frac{1}{2}t_1,\\
& (1-t_1-t_2-t_3,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in II,\\
& (1-t_1-t_2-t_3,t_2,t_3,t_4,t_4) \text{ can be partitioned into } (m,n)\in I,\\
& \sigma-2\varpi \leqslant t_5 < \min\left(t_4, \frac{1}{2}(t_1-t_4)\right),\\
& (1-t_1-t_2-t_3,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (m,n)\in II,\\
& \sigma-2\varpi \leqslant t_6 < \min\left(t_5, \frac{1}{2}(t_1-t_4-t_5)\right),\\
& (1-t_1-t_2-t_3,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n)\in II,\\
& \left. \sigma-2\varpi \leqslant t_1 < \frac{1}{2},\ \sigma-2\varpi \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\}.
\end{aligned}
$$

For S_C we cannot use Buchstab's identity in a straightforward manner, but we can use Buchstab's identity in reverse to make some almost–primes visible. The details of using Buchstab's identity in reverse are similar to those in [7] and [8]. By using Buchstab's identity in reverse twice, we have

$$
\begin{aligned}
S_C = & \sum_{(\alpha_1,\alpha_2)\in C} S\left(\mathcal{A}^d_{p_1p_2}, p_2\right)\\
= & \sum_{(\alpha_1,\alpha_2)\in C} S\left(\mathcal{A}^d_{p_1p_2}, 2\left(\frac{X}{p_1p_2}\right)^{\frac{1}{2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{\substack{(\alpha_1,\alpha_2)\in C\\ \alpha_2<\alpha_3<\frac{1-\alpha_1-\alpha_2}{2}\\ (\alpha_1,\alpha_2,\alpha_3) \text{ can be partitioned into } (m,n)\in II}} S\left(\mathcal{A}^d_{p_1p_2p_3}, p_3\right)\\
&+ \sum_{\substack{(\alpha_1,\alpha_2)\in C\\ \alpha_2<\alpha_3<\frac{1-\alpha_1-\alpha_2}{2}\\ (\alpha_1,\alpha_2,\alpha_3) \text{ cannot be partitioned into } (m,n)\in II}} S\left(\mathcal{A}^d_{p_1p_2p_3}, 2\left(\frac{X}{p_1p_2p_3}\right)^{\frac{1}{2}}\right)\\
&+ \sum_{\substack{(\alpha_1,\alpha_2)\in C\\ \alpha_2<\alpha_3<\frac{1-\alpha_1-\alpha_2}{2}\\ (\alpha_1,\alpha_2,\alpha_3) \text{ cannot be partitioned into } (m,n)\in II\\ \alpha_3<\alpha_4<\frac{1-\alpha_1-\alpha_2-\alpha_3}{2}\\ (\alpha_1,\alpha_2,\alpha_3,\alpha_4) \text{ can be partitioned into } (m,n)\in II}} S\left(\mathcal{A}^d_{p_1p_2p_3p_4}, p_4\right)\\
&+ \sum_{\substack{(\alpha_1,\alpha_2)\in C\\ \alpha_2<\alpha_3<\frac{1-\alpha_1-\alpha_2}{2}\\ (\alpha_1,\alpha_2,\alpha_3) \text{ cannot be partitioned into } (m,n)\in II\\ \alpha_3<\alpha_4<\frac{1-\alpha_1-\alpha_2-\alpha_3}{2}\\ (\alpha_1,\alpha_2,\alpha_3,\alpha_4) \text{ cannot be partitioned into } (m,n)\in II}} S\left(\mathcal{A}^d_{p_1p_2p_3p_4}, p_4\right)\\
=\;& S_{C1}+S_{C2}+S_{C3}+S_{C4}+S_{C5}. \qquad (8)
\end{aligned}
$$

We can give asymptotic formulas for S_{C2} and S_{C4} by Lemma 2.3, hence we can subtract them from the loss. In this way we obtain a loss from S_C of

$$
\begin{aligned}
&\left(\int_{(t_1,t_2)\in S_C} \frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1t_2^2} dt_2dt_1\right)\\
&-\left(\int_{(t_1,t_2,t_3)\in S_{C2}} \frac{\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_1t_2t_3^2} dt_3dt_2dt_1\right)\\
&-\left(\int_{(t_1,t_2,t_3,t_4)\in S_{C4}} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1t_2t_3t_4^2} dt_4dt_3dt_2dt_1\right)\\
&< 0.990258, \qquad (9)
\end{aligned}
$$

where

$$
\begin{aligned}
S_{C2}(t_1,t_2,t_3) := \Bigg\{ & (t_1,t_2)\in S_C,\ t_2<t_3<\frac{1}{2}(1-t_1-t_2),\\
& (t_1,t_2,t_3) \text{ can be partitioned into } (m,n)\in II,\\
& \sigma-2\varpi \leqslant t_1 < \frac{1}{2},\ \sigma-2\varpi \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Bigg\},\\
S_{C4}(t_1,t_2,t_3,t_4) := \Bigg\{ & (t_1,t_2)\in S_C,\ t_2<t_3<\frac{1}{2}(1-t_1-t_2),\\
& (t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in II,\\
& t_3<t_4<\frac{1}{2}(1-t_1-t_2-t_3),\\
& (t_1,t_2,t_3,t_4) \text{ can be partitioned into } (m,n)\in II,\\
& \sigma-2\varpi \leqslant t_1 < \frac{1}{2},\ \sigma-2\varpi \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Bigg\}.
\end{aligned}
$$

Finally, by (2)–(9), the total loss is less than

$$0.990258 + 0.002515 + 0.006249 < 0.9991 < 1$$

and the proof of Theorem 1.1 is completed.

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