# Primes in short intervals

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# Ancient number theory

One of the most important topics in analytic number theory is the distribution of prime numbers. In ancient times, people knew that there were infinitely many prime numbers. Let

$$\pi(x) = \sum_{p \leqslant x} 1.$$

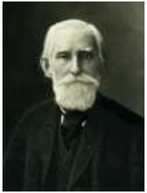
**Theorem (Euclid)**

$$\pi(x) \to \infty \text{ as } x \to \infty.$$

Euclid constructed a prime number of the form $p_1 p_2 \cdots p_n + 1$ and proved the above theorem by contradiction.

# Chebyshev's theorem

Due to the discrete distribution of individual prime numbers, mathematicians began to focus on the distribution of prime counting function $\pi(x)$.

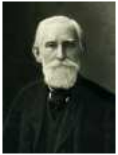
In 1845, Bertrand conjectured the following statement, which was later proved by Chebyshev in 1852.

**Bertrand's postulate / Chebyshev's theorem (1852)**

For any $x > 1$, there is at least one prime number between $x$ and $2x$. That is,

$$\pi(2x) - \pi(x) > 0.$$

# Prime Number Theorem

Chebyshev actually proved the following result.

$$0.92129\frac{x}{\log x} \leqslant \pi(x) \leqslant 1.10555\frac{x}{\log x} \text{ as } x \to \infty,$$

where Gauss and Legendre previously conjectured that

$$\pi(x) \sim \frac{x}{\log x} \text{ as } x \to \infty.$$

By Chebyshev's result, one can easily show that

$$\pi(2x) - \pi(x) \gg \frac{x}{\log x}.$$

# Riemann Hypothesis

In 1859, Riemann connected $\pi(x)$ with the zeros of complex function $\zeta(s)$ and put forward his famous hypothesis.

**Riemann Hypothesis (RH)**

All non–trivial zeros of $\zeta(s)$ lie on the straight line $\mathrm{Re}(s) = \frac{1}{2}$.

As of 2025, RH is still unsolved.

# Prime Number Theorem

Using ideas introduced by Riemann, Hadamard and de la Vallée Poussin proved the famous Prime Number Theorem independently in 1896.

**Prime Number Theorem (PNT) (Hadamard, 1896; de la Vallée Poussin, 1896)**

$$\pi(x) \sim \frac{x}{\log x} \text{ as } x \to \infty.$$

By this theorem, it is easy to prove that

$$\pi(2x) - \pi(x) \sim \frac{x}{\log x}.$$

# Primes in short intervals

**Can we find primes in intervals shorter than $x$ as $x \to \infty$?**

**Hoheisel's theorem (1930)**

There exists some $\theta < 1$ such that

$$\pi(x + x^{\theta+\varepsilon}) - \pi(x) \sim \frac{x^{\theta+\varepsilon}}{\log x}.$$

Moreover, $\theta = \frac{32999}{33000}$ is acceptable.

# Primes in short intervals

**Ingham's theorem (1936)**

If

$$\zeta\left(\frac{1}{2}+it\right) \ll t^{c},$$

then

$$\pi(x+x^{\theta+\varepsilon})-\pi(x) \sim \frac{x^{\theta+\varepsilon}}{\log x}, \qquad \theta = \frac{1+4c}{2+4c}.$$

Moreover, $c=\frac{1}{6}$ yields

$$\pi(x+x^{\frac{5}{8}+\varepsilon})-\pi(x) \sim \frac{x^{\frac{5}{8}+\varepsilon}}{\log x}.$$

# Primes in short intervals, records I

- $\frac{32999}{33000} = 0.9999$, Hoheisel, 1930;
- $\frac{249}{250} = 0.9960$, Heilbronn, 1933;
- $\frac{3}{4} = 0.7500$, Chudakov, 1936;
- $\frac{5}{8} = 0.6250$, Ingham, 1936;
- $\frac{3}{5} = 0.6000$, Montgomery, 1971;
- $\frac{7}{12} = 0.5833$, Huxley, 1972; Ivić, 1979; Heath-Brown, 1988;
- $\frac{17}{30} = 0.5667$, Guth–Maynard, 2025.

# Zero-density approach

Let

$$\Lambda(n) = \begin{cases} \log p, & n = p^k, \\ 0, & \text{otherwise.} \end{cases}$$

Because

$$\sum_{n \leqslant x} \Lambda(n) = \sum_{p \leqslant x} \log p + O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

we can study $\sum_{n\leqslant x} \Lambda(n)$ instead of $\pi(x)$. Note that we have

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

# Zero-density approach

**Perron's formula**

Let $a(n) = O(1)$. We have

$$\sum_{n \leqslant x} a(n) = \frac{1}{2\pi i} \int_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty} \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \frac{x^s}{s} ds + \text{Error}.$$

By Perron's formula we have

$$\sum_{n=1}^{\infty} \Lambda(n) = -\frac{1}{2\pi i} \int_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty} \sum_{n=1}^{\infty} \frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds + \text{Error}.$$

By moving the line of integration, we can get the Explicit Formula

$$\sum_{n \leqslant x} \Lambda(n) = x - \sum_{\substack{\rho=\beta+i\gamma \\ |\gamma|<T}} \frac{x^\rho}{\rho} + O\left(\frac{x(\log x)^2}{T}\right).$$

# Zero-density approach

Similarly, for the short interval problem we can also get the Explicit Formula

$$\sum_{x-x^\theta<n\leqslant x}\Lambda(n)=x^\theta-\sum_{\substack{\rho=\beta+i\gamma\\|\gamma|<T}}\frac{x^\rho-\left(x-x^\theta\right)^\rho}{\rho}+O\left(\frac{x(\log x)^2}{T}\right).$$

Let $T=x^{1-\theta}(\log x)^3$ and

$$N(\sigma,T)=\#\{\text{zeros of }\zeta(\beta+i\gamma):\beta\geqslant\sigma,\ 0<\gamma\leqslant T\}.$$

Let

$$E(\sigma)=\sum_{\substack{\rho=\beta+i\gamma\\|\gamma|<T\\\sigma\leqslant\beta<\sigma+(\log x)^{-1}}}\frac{x^\rho-\left(x-x^\theta\right)^\rho}{\rho}.$$

We want to show that $E(\sigma)=o\left(x^\theta(\log x)^{-1}\right)$.

# Zero-density approach

Note that

$$\frac{x^\rho - \left(x - x^\theta\right)^\rho}{\rho} = \int_{x-x^\theta}^{x} u^{\rho-1} du \ll x^\theta x^{\mathrm{Re}(\rho)-1},$$

we have

$$E(\sigma) \ll x^\theta x^{\sigma-1} N(\sigma, T).$$

Thus, by Vinogradov zero-free region and bounds of the types

$$N(\sigma, T) \ll T^{A(1-\sigma)}(\log T)^B \quad \text{or} \quad N(\sigma, T) \ll T^{A(1-\sigma)+\varepsilon},$$

we only need

$$(1-\sigma)(A(1-\theta)-1) < 0 \quad \text{or} \quad \theta > 1 - \frac{1}{A}.$$

Huxley: $A = \frac{12}{5} \implies \theta > \frac{7}{12}$. Guth–Maynard: $A = \frac{30}{13} \implies \theta > \frac{17}{30}$.

# Primes in short intervals

One can get shorter intervals if we don't require an asymptotic formula. Using sieve methods, Iwaniec and Jutila got in 1979 that

**Theorem (Iwaniec–Jutila, 1979)**

$$\pi(x + x^{\frac{13}{23}+\varepsilon}) - \pi(x) \gg \frac{x^{\frac{13}{23}+\varepsilon}}{\log x}.$$

# Primes in short intervals, records II









- $\frac{13}{23} = 0.5652$, Iwaniec–Jutila, 1979;
- $\frac{5}{9} = 0.5556$, Iwaniec–Jutila, 1979;
- $\frac{11}{20} = 0.5500$, Heath-Brown–Iwaniec, 1979;
- $\frac{17}{31} = 0.5484$, Pintz, 1981; Iwaniec (Unpublished);
- $\frac{23}{42} = 0.5476$, Iwaniec–Pintz, 1984;
- $\frac{1051}{1920} = 0.5474$, Mozzochi, 1986;
- $\frac{35}{64} = 0.5469$, Lou–Yao (Unpublished), 1985;
- $\frac{6}{11} = 0.5455$, Lou–Yao, 1992;
- $\frac{7}{13} = 0.5385$, Lou–Yao, 1992;
- $\frac{107}{200} = 0.5350$, Baker–Harman, 1996;
- $\frac{21}{40} = 0.5250$, Baker–Harman–Pintz, 2001;
- $\frac{13}{25} = 0.5200$, L. (preprint), 2025;
- $\frac{1039}{2000} = 0.5195$, Lu–Yuan, 2025.

# Primes in short intervals: New proofs

Without using too many deep results, Motohashi and Friedlander and Iwaniec gave simplified proofs of the existence of primes in short intervals.

**Theorem (Motohashi, 1983)**

We have

$$\pi(x + x^{0.56}) - \pi(x) \gg x^{0.56}(\log x)^{-1}.$$

**Theorem (Friedlander–Iwaniec, 2010)**

We have

$$\pi(x + x^{0.58}) - \pi(x) \gg x^{0.58}(\log x)^{-1}.$$

# Primes in short intervals: New proofs

In 2019, Granville, Harper and Soundararajan gave a new proof of Hoheisel's theorem with an asymptotic formula.



**Theorem (Granville–Harper–Soundararajan, 2019)**

For some $\delta > 0$, we have

$$\pi(x + x^{1-\delta}) - \pi(x) \sim x^{1-\delta}(\log x)^{-1}.$$

# Primes in short intervals: New proofs

In 2024, Matomäki, Merikoski and Teräväinen gave a pure elementary proof of Hoheisel's theorem.

**Theorem (Matomäki–Merikoski–Teräväinen, 2024)**

For some $\delta > 0$, we have

$$\pi(x + x^{\frac{39}{40}}) - \pi(x) \gg x^{\frac{39}{40}} (\log x)^{-1}.$$

# Sieve approach

Let

$$\mathcal{A} = \{a : x - x^{\theta} < a \leqslant x\}, \quad \mathcal{A}_d = \{a : ad \in \mathcal{A}\}, \quad S(\mathcal{A}, z) = \sum_{\substack{a \in \mathcal{A} \\ (a, P(z)) = 1}} 1.$$

Then by a simple observation, we can find that

$$\pi(x) - \pi(x - x^{\theta}) = S\left(\mathcal{A}, x^{\frac{1}{2}}\right).$$

We have another useful tool:

**Buchstab's identity**

For any $w \leqslant z$, we have

$$S(\mathcal{A}, z) = S(\mathcal{A}, w) - \sum_{w \leqslant p < z} S(\mathcal{A}_p, p).$$

# Sieve approach

Iwaniec and Jutila used the following decomposition:

**Sieve decomposition (Iwaniec–Jutila $\frac{13}{23}$, Motohashi 0.56)**

For some $v \geqslant u \geqslant 2$, we have

$$S\left(\mathcal{A}, x^{\frac{1}{2}}\right) = S\left(\mathcal{A}, x^{\frac{1}{v}}\right) - \sum_{x^{\frac{1}{v}} \leqslant p < x^{\frac{1}{u}}} S\left(\mathcal{A}_p, p\right) - \sum_{x^{\frac{1}{u}} \leqslant p < x^{\frac{1}{2}}} S\left(\mathcal{A}_p, p\right).$$

They also used two important devices: weighted zero-density estimate and mean values of Dirichlet polynomials.

# Weighted zero-density estimate

Let

$$M(s) = \sum_{m \sim M} a_m m^{-s}, \quad N(s) = \sum_{n \sim N} b_n n^{-s}, \quad R(s) = \sum_{r \sim R} c_r r^{-s}, \quad K(s) = \sum_{k \sim K} k^{-s},$$

where $a_m$, $b_n$ and $c_r$ are divisor–bounded. We want to get estimates of the type

**Weighted zero-density estimate**

$$\sum_{\substack{\rho=\beta+i\gamma \\ \beta \geqslant \sigma,\ |\gamma|<T}} |M(\rho)N(\rho)| \ll x^{1-\sigma}(\log x)^c.$$

Note that by a variant of the Explicit formula above, this type of estimates lead to an asymptotic formula for sums of the form

$$\sum_{\substack{p_i \sim P_i \\ 1 \leqslant i \leqslant n}} \left( \pi\left(\frac{x}{p_1 \cdots p_n}\right) - \pi\left(\frac{x - x^{\theta}}{p_1 \cdots p_n}\right) \right).$$

# Mean values of Dirichlet polynomials

Using Iwaniec's linear sieve, one need to estimate the "error term"

$$\sum_{\substack{m\sim M\\ n\sim N}} a_m b_n \left( \left[\frac{x}{mn}\right] - \left[\frac{x-x^\theta}{mn}\right] - \frac{x^\theta}{mn} \right)$$

in order to bound sums like

$$S\left(\mathcal{A}, z\right) \quad \text{and} \quad \sum_{p\sim P} S\left(\mathcal{A}_p, z\right).$$

This can be estimated by using classical mean and large value results of Dirichlet polynomials and power moments of zeta function.

# Sieve approach

In 1979, Heath-Brown and Iwaniec used another sieve decomposition together with the above tools to obtain $\frac{11}{20}$.

**Sieve decomposition (Heath-Brown–Iwaniec $\frac{11}{20}$, Pintz $\frac{17}{31}$)**

For some $z^{\frac{1}{2}} \leqslant D \leqslant z^4$, we have

$$S\left(\mathcal{A}, x^{\frac{1}{2}}\right) = \left( S\left(\mathcal{A}, z\right) - \sum_{\left(\frac{D}{p_1}\right)^{\frac{1}{3}} \leqslant p_2 < p_1 < z} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \right) + \sum_{\left(\frac{D}{p_1}\right)^{\frac{1}{3}} \leqslant p_2 < p_1 < z} S\left(\mathcal{A}_{p_1 p_2}, p_2\right)$$

$$- \sum_{z \leqslant p_1 < D^{\frac{1}{2}}} S\left(\mathcal{A}_{p_1}, p_1\right) - \sum_{D^{\frac{1}{2}} \leqslant p_1 < x^{\frac{1}{2}}} S\left(\mathcal{A}_{p_1}, \left(\frac{D}{p_1}\right)^{\frac{1}{3}}\right) + \sum_{\substack{D^{\frac{1}{2}} \leqslant p_1 < x^{\frac{1}{2}} \\ \left(\frac{D}{p_1}\right)^{\frac{1}{3}} \leqslant p_2 < p_1}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right).$$

# Sieve approach

In their work ($\frac{11}{20}$), Heath-Brown and Iwaniec only used the fourth power moment of zeta function. Pintz ($\frac{17}{31}$) inserted a deep result of Deshouillers and Iwaniec:

**Deshouillers–Iwaniec's Theorem (1982)**

We have

$$\int_{T_0}^{T} \left| M\left(\frac{1}{2}+it\right)^2 K\left(\frac{1}{2}+it\right)^4 \right| \ll T^{1+\varepsilon} + M^2 T^{\frac{1}{2}+\varepsilon} + M^{\frac{5}{4}} \left( T \min\left(K, \frac{T}{K}\right)\right)^{\frac{1}{2}}.$$

This can be seen as an approximation of the sixth power moment of zeta function.

# Sieve approach

Using another delicate sieve decomposition, Iwaniec and Pintz in 1984 got $\frac{23}{42}$.

**Sieve decomposition (Iwaniec–Pintz $\frac{23}{42}$, Mozzochi $\frac{1051}{1920}$)**

For $\frac{1051}{1920} < \theta \leqslant \frac{23}{42}$, we have

$$
\begin{aligned}
S\left(\mathcal{A}, x^{\frac{1}{2}}\right) \geqslant & \left( S\left(\mathcal{A}, x^{7-12\theta}\right) - \sum_{\substack{p_2 < p_1 < x^{7-12\theta} \\ p_1 p_2^3 \geqslant x^{\frac{12\theta-2}{5}}}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \right) \\
& + \sum_{\substack{p_2 < p_1 \\ x^{\frac{8-8\theta}{5}} < p_1 p_2^2 < x^{\frac{13\theta-3}{5}}}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) - \sum_{x^{7-12\theta} \leqslant p_1 < x^{\frac{6\theta-1}{5}}} S\left(\mathcal{A}_{p_1}, p_1\right)
\end{aligned}
$$

# Sieve approach

**Sieve decomposition (Iwaniec–Pintz $\frac{23}{42}$, Mozzochi $\frac{1051}{1920}$)**

$$-\sum_{x^{\frac{6\theta-1}{5}}\leqslant p_1<x^{\frac{8\theta-1}{7}}} S\left(\mathcal{A}_{p_1}, \min\left(\frac{x^{\frac{4\theta+1}{5}}}{p_1},\ x^{\frac{20\theta-9}{11}}\right)\right)-\sum_{x^{\frac{8\theta-1}{7}}\leqslant p_1<x^{\frac{1}{2}}} S\left(\mathcal{A}_{p_1}, \left(\frac{x^{\frac{12\theta-2}{5}}}{p_1}\right)^{\frac{1}{3}}\right)$$

$$+\sum_{\substack{x^{\frac{6\theta-1}{5}}\leqslant p_1<x^{\frac{1}{2}} \\ p_1p_2\leqslant x^{\frac{3\theta+2}{5}} \\ p_1p_2^{-1}\leqslant x^{\frac{8\theta-3}{5}}}} S\left(\mathcal{A}_{p_1p_2}, p_2\right).$$

# Vaughan's identity

While working on the Bombieri–Vinogradov theorem, Vaughan introduced a finite approximation to $-\frac{\zeta'(s)}{\zeta(s)}$. Note that

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)} = F(s) - \zeta(s)F(s)G(s) - \zeta'(s)G(s)$$
$$+ \left(-\frac{\zeta'(s)}{\zeta(s)} - F(s)\right)(1 - \zeta(s)G(s)),$$

$$F(s) = \sum_{m \leqslant U} \Lambda(n)n^{-s}, \quad G(s) = \sum_{d \leqslant V} \mu(d)d^{-s}$$

and all functions of the form $n^{-s}$ are linearly independent, we have the following

# Vaughan's identity

## Vaughan's identity

$$\Lambda(n) = a_1(n) + a_2(n) + a_3(n) + a_4(n),$$

where

$$a_1(n) = \begin{cases} \Lambda(n), & n \leqslant U, \\ 0, & n > U, \end{cases} \quad a_2(n) = -\sum_{\substack{mdr=n \\ m\leqslant U,\ d\leqslant V}} \Lambda(n)\mu(d),$$

$$a_3(n) = \sum_{\substack{dh\leqslant n \\ d\leqslant V}} \mu(d)\log h, \quad a_4(n) = \sum_{\substack{mk=n \\ m>U,\ k>1}} \Lambda(m) \sum_{\substack{d|k \\ d\leqslant V}} \mu(d).$$

# Vaughan's identity

This identity helps us break $\sum_{n\sim N}\Lambda(n)f(n)$ into sums (taking $U = V = x^{\beta}$ for some $0<\beta<\frac{1}{2}$)

$$\sum_{\substack{m\leqslant M\\ mn\leqslant x}} a_m f(mn), \quad M\leqslant \max(x^{1-\beta}, x^{2\beta})$$

and

$$\sum_{\substack{m\sim K\\ mn\leqslant x}} a_m b_n f(mn), \quad x^{\beta}\leqslant K\leqslant x^{1-\beta}.$$

# Heath-Brown's identity

In 1982, Heath-Brown produced what he called a generalized Vaughan identity by using the following formula, which is valid for all $k \in \mathbb{N}$ and any function $M(s)$:

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{1 \leqslant j \leqslant k} (-1)^{j-1} \binom{k}{j} \zeta(s)^{j-1} \zeta'(s) M(s)^j + \zeta(s)^{-1} (1 - \zeta(s) M(s))^k \zeta'(s).$$

Heath-Brown used this to give another proof of Huxley's $\frac{7}{12}$ with an asymptotic formula. Let

$$M(s) = \sum_{m \leqslant M} \mu(m) m^{-s},$$

this implies an identity

# Heath-Brown's identity

**Heath-Brown's identity**

$$\Lambda(n) = \sum_{1 \leqslant j \leqslant k} (-1)^{j-1} \binom{k}{j} a_j(n),$$

where

$$a_j(n) = \sum_{\substack{n = r_1 \cdots r_{2j} \\ i > j \Rightarrow r_i \leqslant x^{\frac{1}{k}}}} (\log r_1) \mu(r_{j+1}) \cdots \mu(r_{2j}).$$

One can use Heath-Brown's identity to construct several identities that do not follow from Vaughan's identity.

# Heath-Brown's identity

## Heath-Brown's identity

Suppose that $u \leqslant N^{\frac{1}{10}}$, then

$$\sum_{n \sim N} \Lambda(n) f(n)$$

can be written as $\ll (\log x)^5$ sums of the forms

$$\sum_{\substack{m \leqslant M \\ n \sim N}} a_m f(mn), \quad M \ll Nu$$

and

$$\sum_{\substack{m \sim M \\ n \sim N}} a_m b_n f(mn), \quad u^2 \leqslant M \ll N^{\frac{1}{3}}.$$

# Heath-Brown's identity

Heath-Brown's identity has the advantage that more flexible sums are produced. However, the disadvantage persists that if one makes a problem harder, the method collapses. There is no "grey area" between an asymptotic formula and no result at all. Heath-Brown produced another identity that can be applied to remove this disadvantage.

**Heath-Brown–Linnik identity**

For $z > x^{\frac{1}{k}}$, we have

$$S\left(\mathcal{A}, x^{\frac{1}{2}}\right) = \sum_{1 \leqslant j \leqslant k} \frac{(-1)^{j-1}}{j} S\left(\mathcal{A}^k, z\right) + O\left(x^{\frac{1}{2}}\right),$$

where $\mathcal{A}^k = \{n_1 \cdots n_k \in \mathcal{A}\}$.

In 1988, he used this identity with $k = 7$ to prove $\frac{7}{12} - \varepsilon$ with an asymptotic formula.

# Heath-Brown's identity

Heath-Brown (unpublished) used this identity with $k = 7$ to prove

$$0.99\frac{x^{\frac{11}{20}+\varepsilon}}{\log x} \leqslant \pi(x) - \pi(x - x^{\frac{11}{20}+\varepsilon}) \leqslant 1.01\frac{x^{\frac{11}{20}+\varepsilon}}{\log x}.$$

Lou and Yao (1992) used this identity with $k = 7$ to prove

$$0.969\frac{x^{\frac{6}{11}+\varepsilon}}{\log x} \leqslant \pi(x) - \pi(x - x^{\frac{6}{11}+\varepsilon}) \leqslant 1.031\frac{x^{\frac{6}{11}+\varepsilon}}{\log x}$$

and

$$\pi(x) - \pi(x - x^{\frac{7}{13}+\varepsilon}) \gg \frac{x^{\frac{7}{13}+\varepsilon}}{\log x}.$$

# Weighted zero-density estimate

In 1996, Baker and Harman used a stronger version of the weighted zero-density estimate:

**Weighted zero-density estimate, stronger version**

$$\sum_{\substack{\rho=\beta+i\gamma \\ |\gamma|<T}} x^{\beta-1} \left|M(\rho)N(\rho)\right| \ll (\log x)^{-A}.$$

Using this estimate and a truncated Perron's formula, they got

$$\sum_{\substack{mnr\in\mathcal{A} \\ m\sim M \\ n\sim N}} a_m b_n \Lambda(r) - x^{\theta} \sum_{\substack{mnr\in\mathcal{A} \\ m\sim M \\ n\sim N}} \frac{a_m b_n}{mn} \leqslant x^{\theta} \sum_{\substack{\rho=\beta+i\gamma \\ 0\leqslant\beta\leqslant1 \\ |\gamma|<T}} x^{\beta-1} \left|M(\rho)N(\rho)\right| + O\left(x^{\theta-\varepsilon}\right) \ll x^{\theta}(\log x)^{-A}.$$

# Sieve approach

By using a truncated Perron's formula to remove the dependencies between variables, they obtained an asymptotic formula for sums of the form

$$\sum_{\substack{p_i \sim P_i \\ 1 \leqslant i \leqslant n}} S\left(\mathcal{A}_{p_1 \cdots p_n}, p_n\right).$$

# Sieve approach

The most important observation of Baker and Harman is that we can use Buchstab's identity in this way:

$$\sum_{\substack{p_i \sim P_i \\ 1 \leqslant i \leqslant n}} S\left(\mathcal{A}_{p_1 \cdots p_n}, z\right) = \sum_{\substack{p_i \sim P_i \\ 1 \leqslant i \leqslant n}} S\left(\mathcal{A}_{p_1 \cdots p_n}, x^{\varepsilon}\right) - \sum_{\substack{p_i \sim P_i \\ 1 \leqslant i \leqslant n \\ x^{\varepsilon} \leqslant p_{i+1} < z}} S\left(\mathcal{A}_{p_1 \cdots p_{n+1}}, p_{n+1}\right).$$

The estimate of the first sum on the right-hand side using Iwaniec's linear sieve is asymptotic. This means that if we can find $z = x^{\delta}$ with $\delta > 0$ as large as possible such that the second sum on the right-hand side has an asymptotic formula, then we can obtain an asymptotic formula for the sum on the left-hand side. This estimate is better than the bounds we get using only Iwaniec's linear sieve.

# Sieve approach

Suppose that we want to give a lower bound for $\sum_{\substack{p_i \sim P_i \\ 1 \leqslant i \leqslant n}} S\left(\mathcal{A}_{p_1 \cdots p_n}, z\right)$.

Using Iwaniec's linear sieve directly, we only have

$$\sum_{\substack{p_i \sim P_i \\ 1 \leqslant i \leqslant n}} S\left(\mathcal{A}_{p_1 \cdots p_n}, z\right) \geqslant (1 + o(1)) \frac{x^\theta}{\log x} e^{-\gamma} f(u).$$

Using the above procedure, we can get

$$\sum_{\substack{p_i \sim P_i \\ 1 \leqslant i \leqslant n}} S\left(\mathcal{A}_{p_1 \cdots p_n}, z\right) = (1 + o(1)) \frac{x^\theta}{\log x} \omega(u).$$

Note that

$$\omega(u) = \frac{e^{-\gamma}(f(u) + F(u))}{2}.$$

# Sieve approach

In 2001, Baker, Harman and Pintz (BHP) developed a new method of estimating $\sum_{\substack{p_i \sim P_i \\ 1 \leqslant i \leqslant n}} S\left(\mathcal{A}_{p_1 \cdots p_n}, p_n\right)$. They used mean value results of Dirichlet polynomials instead of weighted zero-density estimates. Specifically, they proved that

**Theorem (BHP, 2001)**

If

$$\int_{T_0}^{T} \left| M\left(\frac{1}{2} + it\right) \right| \ll x^{\frac{1}{2}} (\log x)^{-A},$$

then

$$\sum_{m \in \mathcal{A}} a_m = (1 + o(1)) \frac{x^{\theta}}{X} \sum_{x - X < m \leqslant x} a_m,$$

where $X = x \exp\left(-3 \log x\right)^{\frac{1}{3}}$.

# Sieve approach

Using the above Theorem, one can easily show the two relations between mean value results and asymptotic formulas:

$$\int_{T_0}^{T} |MNR| \ll x^{\frac{1}{2}} (\log x)^{-A} \Longrightarrow \sum_{\substack{mnr \in \mathcal{A} \\ m \sim M \\ n \sim N}} a_m b_n c_r \tag{A}$$

and

$$\int_{T_0}^{T} |MNK| \ll x^{\frac{1}{2}} (\log x)^{-A} \Longrightarrow \sum_{\substack{m \sim M \\ n \sim N}} a_m b_n S\left(\mathcal{A}_{mn}, \exp\left(\frac{\log x}{\log\log x}\right)\right). \tag{B}$$

Thus, one only need to find longer ranges of $M$ and $N$ such that (A) or (B) holds.

# Sieve approach

BHP used Watt's Theorem together with Hölder's inequality to get more type (B) estimates.

**Watt's Theorem (1995)**

We have

$$\int_{T_0}^{T} \left| M\left(\frac{1}{2}+it\right)^2 K\left(\frac{1}{2}+it\right)^4 \right| \ll T^{1+\varepsilon} + M^2 T^{\frac{1}{2}+\varepsilon}.$$

Watt's Theorem improves Deshouillers–Iwaniec's Theorem.

# Sieve approach

$$\sum_{\substack{p_i \sim P_i \\ 1 \leqslant i \leqslant n}} S\left(\mathcal{A}_{p_1 \cdots p_n}, z\right) = \sum_{\substack{p_i \sim P_i \\ 1 \leqslant i \leqslant n}} S\left(\mathcal{A}_{p_1 \cdots p_n}, \exp\left(\frac{\log x}{\log \log x}\right)\right) - \sum_{\substack{p_i \sim P_i \\ 1 \leqslant i \leqslant n \\ p_{i+1} < z}} S\left(\mathcal{A}_{p_1 \cdots p_{n+1}}, p_{n+1}\right),$$

$$\begin{aligned}
S\left(\mathcal{A}, x^{\frac{1}{2}}\right) &= S\left(\mathcal{A}, z\right) - \sum_{z \leqslant p_1 < x^{\frac{1}{2}}} S\left(\mathcal{A}_{p_1}, z\right) + \sum_{z \leqslant p_2 < p_1 < x^{\frac{1}{2}}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
&= S\left(\mathcal{A}, z\right) - \sum_{z \leqslant p_1 < x^{\frac{1}{2}}} S\left(\mathcal{A}_{p_1}, z\right) + \sum_{z \leqslant p_2 < p_1 < x^{\frac{1}{2}}} S\left(\mathcal{A}_{p_1 p_2}, z\right) \\
&\quad - \sum_{z \leqslant p_3 < p_2 < p_1 < x^{\frac{1}{2}}} S\left(\mathcal{A}_{p_1 p_2 p_3}, z\right) + \sum_{z \leqslant p_4 < p_3 < p_2 < p_1 < x^{\frac{1}{2}}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right) \\
&= \cdots
\end{aligned}$$

# Sieve approach

- BHP (1996);

$$0.9953\frac{x^{\frac{11}{20}+\varepsilon}}{\log x} \leqslant \pi(x) - \pi(x - x^{\frac{11}{20}+\varepsilon}) \leqslant 1.0001\frac{x^{\frac{11}{20}+\varepsilon}}{\log x}.$$

- BHP (2001), 0.525;
- L. (preprint, 2025), 0.52;
  1. More type (A) estimates with 5 or more variables (the lower bound of the sum of all variables decreases as the number of variables increases);
  2. An optimized sieve argument (one can get 0.523 with BHP's original argument).
- Lu–Yuan (2025), 0.5195.
  1. More type (B) estimates obtained by Hölder's inequality and higher power means of zeta function

$$\int_{T_0}^{T} \left|K^A\right| \ll T^{1+\frac{A-4}{8}+\varepsilon} \text{ for } 4 \leqslant A \leqslant 12.$$

# Legendre's conjecture

**Legendre's conjecture**

For any $x > 1$, there is at least one prime number between $x^2$ and $(x+1)^2$.

**Legendre's conjecture ($x \to \infty$)**

We have

$$\pi(x + x^{\frac{1}{2}}) - \pi(x) > 0.$$

# Cramér's conjecture

**Cramér's conjecture (1937)**

The interval

$$[x, x + f(x) \log^2 x]$$

contains primes for some $f(x) \to 1$ as $x \to \infty$.

# Lindelöf Hypothesis

**Lindelöf Hypothesis (LH)**

For any $\varepsilon > 0$, we have

$$\zeta\left(\frac{1}{2} + it\right) \ll t^{\varepsilon}.$$

Clearly, we have RH $\Rightarrow$ LH.

# Primes in short intervals

## Under RH

We have

$$\pi(x + x^{\frac{1}{2}} \log x) - \pi(x) \gg x^{\frac{1}{2}}.$$

## Under LH

We have

$$\pi(x + x^{\frac{1}{2}+\varepsilon}) - \pi(x) \sim x^{\frac{1}{2}+\varepsilon}(\log x)^{-1}.$$

## Unconditional

We have

$$\pi(x + x^{\frac{17}{30}+\varepsilon}) - \pi(x) \sim x^{\frac{17}{30}+\varepsilon}(\log x)^{-1},$$

$$\pi(x + x^{0.5195}) - \pi(x) \gg x^{0.5195}(\log x)^{-1}.$$

# Primes in almost all short intervals

In 1943, Selberg obtained the following two results.

**Theorem (Selberg, 1943)**

1. Under RH, Cramér's interval contains primes for almost all $x$ if $f(x) \to \infty$ as $x \to \infty$.
2. The interval

$$[x, x + x^{\frac{19}{77}+\varepsilon}]$$

contains $\sim \frac{x^{\frac{19}{77}+\varepsilon}}{\log x}$ primes for almost all $x$.

# Primes in almost all short intervals, records I





- $\frac{19}{77} = 0.2468$, Selberg, 1943; ($\theta_1 = 2\theta_0 - 1$)
- $\frac{1}{5} = 0.2000$, Montgomery, 1971; ($\frac{3}{5} \Leftrightarrow \frac{1}{5}$)
- $\frac{1}{6} = 0.1667$, Huxley, 1972; ($\frac{7}{12} \Leftrightarrow \frac{1}{6}$)
- $\frac{1}{7.5} = 0.1333$, Guth–Maynard, 2025. ($\frac{17}{30} \Leftrightarrow \frac{1}{7.5}$)

# Primes in almost all short intervals

Using sieve methods, Harman got in 1982 that

**Theorem (Harman, 1982)**

The interval

$$[x, x + x^{\frac{1}{10}+\varepsilon}]$$

contains $\gg \frac{x^{\frac{1}{10}+\varepsilon}}{\log x}$ primes for almost all $x$.

# Primes in almost all short intervals, records II

- $\frac{1}{10} = 0.1000$, Harman, 1982;
- $\frac{14}{159} = 0.0881$, Lou–Yao (Unpublished), 1985;
- $\frac{1}{12} = 0.0833$, Harman, 1983; Heath-Brown, 1984;
- $\frac{1}{13} = 0.0769$, Jia, 1995;
- $\frac{17}{227} = 0.0749$, Lou–Yao (Unpublished), 1985;
- $\frac{1}{13.5} = 0.0740$, H. Li, 1995;
- $\frac{1}{14} = 0.0714$, Jia, 1995; Watt, 1995;
- $\frac{1}{15} = 0.0667$, H. Li, 1997;
- $\frac{1}{16} = 0.0625$, Baker–Harman–Pintz, 1997;
- $\frac{1}{18} = 0.0556$, Wong, 1996; Jia, 1996; Harman, 2007;
- $\frac{1}{20} = 0.0500$, Jia, 1996;
- $\frac{1}{21.5} = 0.0476$, L. (preprint), 2024.

# A weak Legendre's conjecture

**Legendre's conjecture**

The interval

$$[x, x + x^{\frac{1}{2}}]$$

contains an integer with a prime factor larger than $x^{1}$.

**Conjecture $LPF(\theta)$**

The interval

$$[x, x + x^{\frac{1}{2}}]$$

contains an integer with a prime factor larger than $x^{\theta}$ for some $\theta > 0$.

# Largest prime factors of integers in short intervals

In 1969, Ramachandra got the first result in this direction.

**Theorem (Ramachandra, 1969)**

The interval

$$[x, x + x^{\frac{1}{2}}]$$

contains an integer with a prime factor larger than $x^{0.576}$.

# Largest prime factors of integers in short intervals






- 0.576, Ramachandra, 1969;
- 0.625, Ramachandra, 1970;
- 0.662, Graham, 1981;
- 0.675225, Zhu, 1987;
- 0.692, Jia, 1986;
- 0.7, Baker, 1986;
- 0.71, Jia, 1989;
- 0.723, Jia, 1993; H.-Q. Liu, 1993;
- 0.728, Jia, 1996;
- 0.732, Baker–Harman, 1995;
- 0.738, H.-Q. Liu–Wu, 1999;
- 0.74, Harman, 2007;
- 0.7428, Baker–Harman, 2009;
- 0.7437, L. (preprint), 2025.

# Largest prime factors of integers in short intervals

If we increase the interval length to $x^{\frac{1}{2}+\varepsilon}$, then we can get better results since many powerful analytic tools, such as the estimation of Dirichlet polynomials, can be used. In 1973, Jutila obtained

**Theorem (Jutila, 1973)**

The interval

$$[x, x + x^{\frac{1}{2}+\varepsilon}]$$

contains an integer with a prime factor larger than $x^{\frac{2}{3}-\varepsilon}$.

# Largest prime factors of integers in short intervals








- $\frac{2}{3} = 0.6666$, Jutila, 1973;
- $\frac{73}{100} = 0.7300$, Balog, 1980;
- $\frac{193}{250} = 0.7720$, Balog, 1984;
- $\frac{41}{50} = 0.8200$, Balog–Harman–Pintz, 1983;
- $\frac{11}{12} = 0.9166$, Heath-Brown, 1996;
- $\frac{17}{18} = 0.9444$, Heath-Brown–Jia, 1998;
- $\frac{19}{20} = 0.9500$, Harman, 2007;
- $\frac{24}{25} = 0.9600$, Haugland, 1998;
- $\frac{25}{26} = 0.9615$, Jia–M.-C. Liu, 2000;
- $\frac{51}{53} = 0.9622$, L. (preprint), 2024.

# Largest prime factors of integers in short intervals

In 1983, Balog, Harman and Pintz proved a result with "medium" interval lengths.



**Theorem (Balog–Harman–Pintz, 1983)**

The interval

$$[x, x + x^{\frac{1}{2}}(\log x)^A]$$

contains an integer with a prime factor larger than $x^{0.712-\varepsilon}$.

# Largest prime factors of integers in short intervals

- 0.7120, Balog–Harman–Pintz, 1983;
- $\frac{5}{6} = 0.8333$, Lou, 1984;
- $\frac{18}{19} = 0.9473$, Merikoski, 2021; ($A < 1.39$)
- $\frac{37}{39} = 0.9487$, L. (Unpublished); ($A < 1.39$)

# Almost–primes in short intervals

Instead of considering the size of prime factors, one can also consider the number of prime factors. We define the "Almost–primes" $P_r$ and $E_r$ as

**Definition (Almost–primes)**

An integer $n$ is a $P_r$ if $n$ has at most $r$ prime factors counted with multiplicity.
An integer $n$ is an $E_r$ if $n$ has exactly $r$ prime factors counted with multiplicity.

Of course, short–interval results for $P_r$ are easier to obtain than corresponding results for $E_r$.

# Almost–primes in short intervals

**Theorem (Brun, 1920)**

The interval $[x, x + x^{\frac{1}{2}}]$ contains a $P_{11}$. $LPF(\frac{1}{11})$ is true.

**Theorem (Wang, 1957)**

The interval $[x, x + x^{\frac{1}{2}}]$ contains a $P_3$. $LPF(\frac{1}{3})$ is true.

**Theorem (Wang, 1959)**

The interval $[x, x + x^{\frac{10}{17}}]$ contains a $P_2$.
The interval $[x, x + x^{\frac{20}{49}}]$ contains a $P_3$.

# Almost–primes in short intervals

- $\frac{10}{17} = 0.5882$, Wang, 1959;
- $\frac{14}{25} = 0.5600$, Jurkat–Richert, 1965;
- $\frac{6}{11} = 0.5454$, Richert, 1969;
- $\frac{1}{2} = 0.5000$ ($LPF(\frac{1}{2})$ is true), Chen, 1975;
- 0.4856, Laborde, 1978;
- 0.4770, Chen, 1979;
- 0.4550, Halberstam–Heath-Brown–Richert, 1981;
- 0.4500, Iwaniec–Laborde, 1981;
- 0.4476, Halberstam–Richert, 1985;
- $\frac{63}{142} = 0.4436$, Fouvry, 1990;
- 0.4400, Wu, 1992;
- 0.4382, H. Li, 1994;
- 0.4378, Cao, 1995;
- 0.4360, H.-Q. Liu, 1996;
- 0.43596, Sargos–Wu, 2000;
- $\frac{101}{232} = 0.43535$, Wu, 2010.

# Almost–primes in short intervals


**Theorem (Matomäki–Teräväinen, 2023)**

The interval $[x, x + x^{\frac{1}{2}}(\log x)^{1.55}]$ contains an $E_3$.

# Almost–primes in almost all short intervals




| Author | Form | Length | Year |
|---|---|---|---|
| Wolke | $E_2$ | $(\log x)^{5000000}$ | 1979 |
| Harman | $E_2$ | $(\log x)^{7+\varepsilon}$ | 1979 |
| Bourgain | $E_2$ | $(\log x)^{6.86}$ | 2000 |
| Teräväinen | $E_2$ | $(\log x)^{3.51+\varepsilon}$ | 2016 |
| Matomäki–Teräväinen | $E_2$ | $(\log x)^{2.1+\varepsilon}$ | 2023 |

# Almost–primes in almost all short intervals

| Author | Form | Length | Year |
|---|---|---|---|
| Heath-Brown | $P_2$ | $x^{\frac{1}{11}}$ | 1978 |
| Heath-Brown | $P_3$ | $(\log x)^{35+\varepsilon}$ | 1978 |
| Friedlander | $P_4$ | $(\log x)^5$ | 1982 |
| Motohashi | $P_2$ | $x^{\varepsilon}$ | Unpublished |
| Mikawa | $P_2$ | $h(x)(\log x)^5$ | 1989 |
| Matomäki | $P_2$ | $h(x)\log x$ | 2022 |
| Teräväinen | $E_3$ | $(\log x)(\log\log x)^{6+\varepsilon}$ | 2016 |
| Teräväinen | $E_k$ | $(\log x)(\log_{k-1} x)^{C_k+\varepsilon}$ | 2016 |

# Mean square gap between primes

In 1943, Selberg proved the following result under RH.

**Theorem (Selberg, 1943)**

Under RH, we have

$$\sum_{p_n \leqslant x} (p_{n+1} - p_n)^2 \ll x(\log x)^3.$$

# Mean square gap between primes

In 1978, Heath-Brown obtained a weaker bound of Selberg's mean square gap unconditionally.

**Theorem (Heath-Brown, 1978)**

We have

$$\sum_{p_n \leqslant x} (p_{n+1} - p_n)^2 \ll x^{\frac{4}{3}+\varepsilon}.$$

# Mean square gap between primes





- 1 (on RH), Selberg, 1943;
- $\frac{4}{3} = 1.3333$, Heath-Brown, 1978;
- $\frac{1413}{1067} = 1.3242$, Heath-Brown, 1979;
- $\frac{7}{6} = 1.1666$ (on LH), Heath-Brown, 1979;
- $\frac{23}{18} = 1.2777$, Heath-Brown, 1979;
- 1 (on LH), Yu, 1996;
- $\frac{5}{4} = 1.25$, Peck, 1996; Maynard, 2012;
- $\frac{123}{100} = 1.23$, Stadlmann, 2022.

# Large differences between primes

In the same paper, Selberg also considered a variant of the mean square gap.

**Theorem (Selberg, 1943)**

Under RH, we have

$$\sum_{\substack{p_n \leqslant x \\ p_{n+1}-p_n \geqslant x^{\frac{1}{2}+\varepsilon}}} (p_{n+1} - p_n) \ll x^{\frac{1}{2}+\varepsilon}.$$

We call this $LD(\frac{1}{2}+\varepsilon, \frac{1}{2})$.

# Large differences between primes

- $LD(\frac{1}{2}+\varepsilon, \frac{1}{2})$ (on RH), Selberg, 1943;
- $LD(\frac{1}{2}, \frac{29}{30} = 0.9666)$, Wolke, 1975;
- $LD(\frac{1}{2}+\varepsilon, \frac{85}{98} = 0.8673)$, Cook, 1979;
- $LD(\frac{1}{2}+\varepsilon, \frac{1759}{2134} = 0.8242)$, Huxley, 1980;
- $LD(\frac{1}{2}+\varepsilon, \frac{3}{4})$ (on LH), Huxley, 1980;
- $LD(\frac{1}{2}, \frac{215}{266} = 0.8082)$, Ivić, 1979;
- $LD(\frac{1}{2}, \frac{3}{4})$, Heath-Brown, 1979;
- $LD(\frac{1}{2}+\varepsilon, \frac{5}{8})$, Heath-Brown, 1979;
- $LD(\frac{1}{2}, \frac{25}{36} = 0.6944)$, Peck, 1998;
- $LD(\frac{1}{2}, \frac{2}{3})$, Matomäki, 2007;
- $LD(\frac{1}{2}-\delta, \frac{2}{3}+5\delta)$, Islam, 2015 $(0 \leqslant \delta \leqslant \frac{1}{6}\sqrt{327}-3 = 0.01385)$;
- $LD(\frac{1}{2}, \frac{3}{5})$, Heath-Brown, 2021;
- $LD(\frac{1}{2}, 0.57)$, Järviniemi, 2022;
- $LD(0.45, 0.63)$, Järviniemi, 2022;

# Primes in short intervals: Explicit results

**Primes in short intervals: Explicit version 1**

For all $x \geqslant N_0$, we have
$$\pi(x + x^{\theta}) - \pi(x) > 0.$$





| **Author** | $\boldsymbol{\theta}$ | $\boldsymbol{N_0}$ | **Year** |
|---|---|---|---|
| Caldwell–Cheng | $\frac{2}{3}$ | 1 (on RH) | 2005 |
| Dudek | $\frac{2}{3}$ | $\exp(\exp(33.217))$ | 2014 |
| Mattner | $\frac{2}{3}$ | $\exp(\exp(33.1981))$ | 2017 |
| Cully-Hugill | $\frac{2}{3}$ | $\exp(\exp(32.892))$ | 2021 |
| Mossinghoff–Trudgian–Yang | $\frac{2}{3}$ | $\exp(\exp(32.76))$ | 2024 |
| Cully-Hugill | $\frac{2}{3}$ | $\exp(\exp(32.537))$ | 2023 |

# Primes in short intervals: Explicit results

| Author | $\theta$ | $N_0$ | Year |
|---|---|---|---|
| Caldwell–Cheng | $\frac{2}{3}$ | 1 (on RH) | 2005 |
| Dudek | $1 - \frac{1}{5\cdot 10^9}$ | 1 | 2014 |
| Mattner | $1 - \frac{1}{1.5\cdot 10^6}$ | 1 | 2017 |
| Cully-Hugill | $1 - \frac{1}{296}$ | 1 | 2021 |
| Cully-Hugill | $1 - \frac{1}{180}$ | 1 | 2021 |
| Cully-Hugill | $1 - \frac{1}{155}$ | 1 | 2023 |
| Dudek–Johnston | $\frac{1}{2}$ $(P_4)$ | 1 | 2025 |

# Primes in short intervals: Explicit results

**Legendre's conjecture**

We have
$$\pi(x + x^{\frac{1}{2}}) - \pi(x) > 0.$$

**Primes in short intervals: Under RH / LH**

We have
$$\pi(x + x^{\frac{1}{2}+\varepsilon}) - \pi(x) \sim x^{\frac{1}{2}+\varepsilon}(\log x)^{-1}.$$

**Primes in short intervals: Explicit version 2 (Under RH)**

For all $x \geqslant N_0$, we have
$$\pi(x + c x^{\frac{1}{2}} \log x) - \pi(x) > 0.$$

# Primes in short intervals: Explicit results

| **Author** | $c$ | $N_0$ | **Year** |
|---|---|---|---|
| von Koch | $c_0 \log x$ | $< \infty$ (on RH) | 1901 |
| Schoenfeld | $\frac{1}{4\pi} \log x$ | 599 (on RH) | 1976 |
| Cramér | $< \infty$ | sufficiently large (on RH) | 1920 |
| Goldston | 5 | sufficiently large (on RH) | 1983 |
| Ramaré–Saouter | $\frac{8}{5} = 1.6$ | 2 (on RH) | 2003 |
| Dudek | $\frac{4}{\pi} = 1.2732$ | 2 (on RH) | 2015 |
| Dudek–Grenié–Molteni | 1.2204 | 2 (on RH) | 2016 |
| Dudek–Grenié–Molteni | $1 + \frac{4}{\log x}$ | 2 (on RH) | 2016 |
| Carneiro–Milinovich–Soundararajan | $\frac{22}{25} = 0.88$ | 4 (on RH) | 2019 |

# Exceptional characters and primes in short intervals

In 2001, Friedlander and Iwaniec first proved an asymptotic formula for the number of primes in intervals shorter than $x^{\frac{1}{2}}$ under the existence of exceptional characters.

**Theorem (Friedlander–Iwaniec, 2001)**

We have

$$\pi(x + x^{\frac{39}{79}}) - \pi(x) \gg x^{\frac{39}{79}} (\log x)^{-1} \left(1 + O\left(L(1,\chi)(\log x)^A\right)\right).$$

In 2024, L. (preprint) improved the exponent $\frac{39}{79} = 0.4937$ to $0.4923$.

# Upper bounds

In 1973, Montgomery and Vaughan considered the upper bounds for the number of primes in short intervals.

**Theorem (Montgomery–Vaughan, 1973)**

For any $0 < \theta < 1$, we have

$$\pi(x + x^\theta) - \pi(x) \leqslant \frac{2}{\theta} \frac{x^\theta}{\log x}.$$

# Upper bounds

- $\frac{2}{\theta}$ $(0 < \theta < 1)$, Montgomery–Vaughan, 1973;
- $\frac{18}{15\theta - 2}$ $(\frac{1}{3} < \theta < 1)$, Iwaniec, 1982;
- $\frac{4}{\theta + 1}$ $(\frac{1}{2} < \theta < 1)$, Iwaniec, 1982;
- $\frac{22}{100\theta - 45}$ $(\frac{6}{11} < \theta < \frac{11}{20})$, Lou–Yao, 1989;
- $1.031$ $(\frac{6}{11} < \theta < 1)$, Lou–Yao, 1992;
- $1.0001$ $(\frac{11}{20} < \theta < 1)$, Baker–Harman–Pintz, 1997;
- $1$ $(\frac{17}{30} < \theta < 1)$, Guth–Maynard, 2025.

# Upper bounds

**Theorem (L. (preprint), 2025)**

For any $0.52 < \theta \leqslant 0.535$, we have

$$\pi(x + x^{\theta}) - \pi(x) \leqslant C(\theta)\frac{x^{\theta}}{\log x},$$

where

$$C(\theta) \leqslant \begin{cases} 2.7626, & 0.52 < \theta \leqslant 0.521, \\ 2.6484, & 0.521 < \theta \leqslant 0.522, \\ 2.5630, & 0.522 < \theta \leqslant 0.523, \\ 2.4597, & 0.523 < \theta \leqslant 0.524, \\ 2.3759, & 0.524 < \theta \leqslant 0.535. \end{cases}$$

# Exceptional sets in PNT in short intervals

We also want to know how frequently an asymptotic formula in PNT in short intervals "does not hold".

**Definition ($E(\theta)$)**

For any $0 < \theta < 1$, let $E(\theta)$ denote the least exponent such that

$$\pi(x + x^\theta) - \pi(x) \sim x^\theta (\log x)^{-1}$$

holds for all $x \in [X, 2X]$ except for a set of measure $O(X^{E(\theta)+\varepsilon})$.

Note that we have the following simple relations:

$$E(\theta) = -\infty,\ \theta > \frac{17}{30}; \quad E(\theta) \geqslant 0,\ \theta \leqslant \frac{17}{30}; \quad E(\theta) < 1,\ \theta \geqslant \frac{1}{21.5}.$$

# Exceptional sets in PNT in short intervals

- $E(\theta) \leqslant 1 - \theta$ for $0 < \theta \leqslant \frac{1}{2}$ (on RH), Bazzanella–Perelli, 2000;
- $E(\theta) \leqslant \frac{3(1-\theta)}{2}$ for $\frac{1}{2} < \theta \leqslant \frac{11}{21}$, Bazzanella, 2000;
- $E(\theta) \leqslant \frac{47-42\theta}{35}$ for $\frac{11}{21} < \theta \leqslant \frac{23}{42}$, Bazzanella, 2000;
- $E(\theta) \leqslant \frac{36\theta^2-96\theta+55}{39-36\theta}$ for $\frac{23}{42} < \theta \leqslant \frac{17}{30}$, Bazzanella, 2000;
- $E(\frac{1}{2}) \leqslant \frac{3}{5}$, Heath-Brown, 2021;
- various bounds for $E(\theta)$, Gafni–Tao, 2025.

# Bounded gaps between primes

Let
$$H_m = \liminf_{n\to\infty}(p_{n+m} - p_n).$$
Then we have the following bounds:

- $H_1 \leqslant 246$, Polymath8b, 2014;
- $H_2 \leqslant 396504$, Stadlmann, 2025;
- $H_3 \leqslant 24407016$, Stadlmann, 2025;
- $H_4 \leqslant 1391051532$, Stadlmann, 2025;
- $H_5 \leqslant 77510685234$, Stadlmann, 2025;
- $H_m \ll e^{3.8075m}$, Stadlmann, 2025.

**Thank you!**