ON CHEN'S THEOREM, GOLDBACH'S CONJECTURE AND ALMOST PRIME TWINS III

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Abstract. Let N denote a sufficiently large even integer and x denote a sufficiently large integer, we define $D_{1,r}(N,c,\theta)$ as the number of primes p such that $N-p$ has at most r prime factors and $p \in \left[cN, cN+N^{\theta}\right]$. In this paper, we mention an important 1979 result of Lou and Yao, and use this result to give positive lower bounds for $D_{1,3}\left(N,\frac{1}{2},0.872\right)$ and $D_{1,3}(N,0,0.817)$, improving on the author's previous results. Similar methods can be used to give a simplified proof for the positive lower bounds for $D_{1,2}\left(N,\frac{1}{2},0.97\right)$ and $D_{1,2}(N,0,0.9409)$.

Contents

1. Introduction

One of the most famous open problem in number theory is the Goldbach's conjecture, which states that any even integers can be written as the sum of two primes. Since the original conjecture is so hard, mathematicians try to consider the problem of writing a large even integer as a sum of a prime and a number with few prime factors. Let N denote a sufficiently large even integer, p denote a prime, and P_r denote an integer with at most r prime factors counted with multiplicity. We define

$$D_{1,r}(N,c,\theta) = \left|\left\{p : p \leqslant N,\ N-p = P_r,\ p \in \left[cN, cN+N^{\theta}\right]\right\}\right|. \tag{1}$$

In 1973 Chen [10] established his remarkable Chen's theorem:

$$D_{1,2}(N,0,1) \geqslant 0.67 \frac{C(N)N}{(\log N)^2}, \tag{2}$$

where

$$C(N) = \prod_{\substack{p \mid N \\ p>2}} \frac{p-1}{p-2} \prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right). \tag{3}$$

Chen's constant 0.67 was improved successively to

$$0.689,\ 0.7544,\ 0.81,\ 0.8285,\ 0.836,\ 0.867,\ 0.899$$

by Halberstam and Richert [17] [16], Chen [12] [11], Cai and Lu [9], Wu [45], Cai [4] and Wu [46] respectively. Chen [13] announced a better constant 0.9, but this work has not been published. In the author's first two preprints in this series, [27] and [28], this constant has been improved to 1.733 and 1.9728 respectively.

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On the other hand, the distribution of primes in short intervals $\left[N, N+N^{\theta}\right]$ has been studied by many mathematicians. The first unconditional result of the asymptotic formula

$$\pi\left(N+N^{\theta}\right)-\pi(N) \sim \frac{N^{\theta}}{\log N} \tag{4}$$

with some $\theta<1$ was proved by Hoheisel [20] in 1930 with $\theta \geqslant 1-\frac{1}{33000}$. After the works of Hoheisel [20], Heilbronn [19], Chudakov [14], Ingham [22] and Montgomery [36], Huxley [21] proved in 1972 that the above asymptotic formula holds when $\theta>\frac{7}{12}$ by his zero density estimate. In 2025, Guth and Maynard [15] improved this to $\theta>\frac{17}{30}$ by a new zero density estimate.

In 1979, Iwaniec and Jutila [23] first introduced a sieve method into this problem. They established a lower bound with correct order of magnitude (instead of an asymptotic formula) with $\theta=\frac{13}{23}$. After that breakthrough, many improvements were made and the value of θ was reduced successively to

$$\frac{5}{9} \approx 0.5556,\ \frac{11}{20}=0.5500,\ \frac{17}{31} \approx 0.5484,\ \frac{23}{42} \approx 0.5476,\ \frac{1051}{1920} \approx 0.5474,\ \frac{35}{64} \approx 0.5469,$$

$$\frac{6}{11} \approx 0.5455,\ \frac{7}{13} \approx 0.5385,\ \frac{107}{200}=0.5350,\ \frac{21}{40}=0.5250 \text{ and } \frac{13}{25}=0.5200$$

by Iwaniec and Jutila [23], Heath-Brown and Iwaniec [18], Pintz [38] [39], Iwaniec and Pintz [24], Mozzochi [37], Lou and Yao [32] [33] [34] [35], Baker and Harman [1], Baker, Harman and Pintz [2] and the author [30] respectively.

Since we have a lot of useful results on each side, it is natural to study the "mixed" problem. In 1976, Ross [40] first obtained nontrivial lower bounds for $D_{1,2}(N, c, \theta)$ for some $\theta<1$. He showed that for $c=\frac{1}{2}, \theta=0.98$ and $c=0, \theta=0.959$, we have

$$D_{1,2}(N, c, \theta) \gg \frac{C(N) N^{\theta}}{(\log N)^{2}}. \tag{5}$$

His results had been improved by many authors. For the case $c=\frac{1}{2}$, his $\theta=0.98$ was improved successively to

$$0.974,\ 0.973,\ 0.9729,\ 0.972,\ 0.971,\ 0.97$$

by Wu [43] [44], Salerno and Vitolo [41], Cai and Lu [8], Wu [45] and Cai [5] respectively. For the case $c=0$, his $\theta=0.959$ was improved successively to

$$0.95,\ 0.945,\ 0.941,\ 0.9409$$

by Cai [3], Cai and Li [7], Cai [6] and the author [29] respectively. In 2024, the author [26] considered the lower bound for $D_{1,3}(N, c, \theta)$ with smaller θ, and proved that for $c=\frac{1}{2}, \theta=0.919$ and $c=0, \theta=0.838$, we have

$$D_{1,3}(N, c, \theta) \gg \frac{C(N) N^{\theta}}{(\log N)^{2}}. \tag{6}$$

While working on [26], the author found an interesting phenomenon that one can get 0.919 and 0.838 without using Chen's switching principle and Wu's distribution result [43], but one cannot get either 0.919 or 0.838 if using any of them. In fact, Wu's result put a restriction on the size of elements in some "sieved sets", which will affect some terms in the construction of some weights.

Fortunately, the author found a 1979 paper by Lou and Yao [31], where they gave a stronger distribution result with less restriction. They proved this result by proving a generalized version of the Barban–Davenport–Halberstam Theorem. Using their results together with modern sieve weights and Chen's switching principle, we slightly improve the two exponents 0.919 and 0.838 in [26].

Theorem 1.1. *For $c=\frac{1}{2}, \theta=0.87125$ and $c=0, \theta=0.817$, we have*

$$D_{1,3}(N, c, \theta) \gg \frac{C(N) N^{\theta}}{(\log N)^{2}}.$$

Using the results in [31], we can also give a simplified proof of the main theorems in [6] and [29] without using [[6], Lemma 8], for example.

2. Weighted sieve method

Let $\mathcal{A}_i$ denote finite sets of positive integers, $\mathcal{P}$ denote an infinite set of primes and $z \geqslant 2$. Put

$$\mathcal{A}_1=\left\{N-p: p \in\left[\frac{1}{2} N,\ \frac{1}{2} N+N^{0.87125}\right]\right\}, \quad \mathcal{A}_2=\left\{N-p: p \in\left[0,\ N^{0.817}\right]\right\},$$

$$\mathcal{P}=\{p:(p, N)=1\}, \quad \mathcal{P}(q)=\{p: p \in \mathcal{P},(p, q)=1\},$$

$$P(z)=\prod_{\substack{p \in \mathcal{P} \\ p<z}} p, \quad \mathcal{A}_d=\{a: a \in \mathcal{A}, a \equiv 0(\bmod d)\}, \quad S(\mathcal{A} ; \mathcal{P}, z)=\sum_{\substack{a \in \mathcal{A} \\ (a, P(z))=1}} 1.$$

Lemma 2.1. *Let $100>k_1>k_2>4$. For $\mathcal{A}=\mathcal{A}_1$ or $\mathcal{A}_2$, we have*

$$\begin{aligned}
3 D_{1,3}(N, c, \theta) \geqslant\ & 3 S\left(\mathcal{A} ; \mathcal{P}, N^{\frac{1}{k_1}}\right)-\sum_{\substack{N^{\frac{1}{k_1}} \leqslant p<N^{\frac{1}{k_2}} \\ (p, N)=1}} S\left(\mathcal{A}_p ; \mathcal{P}, N^{\frac{1}{k_1}}\right) \\
& -\sum_{\substack{N^{\frac{1}{k_1}} \leqslant p_1<p_2<N^{\frac{1}{k_2}} \leqslant p_3<\left(\frac{N}{p_1 p_2}\right)^{\frac{1}{2}} \\ (p_1 p_2 p_3, N)=1}} S\left(\mathcal{A}_{p_1 p_2 p_3} ; \mathcal{P}(p_1 p_2), p_3\right) \\
& -2 \sum_{\substack{N^{\frac{1}{k_1}} \leqslant p_1<N^{\frac{1}{k_2}} \leqslant p_2<p_3<\left(\frac{N}{p_1 p_2}\right)^{\frac{1}{2}} \\ (p_1 p_2 p_3, N)=1}} S\left(\mathcal{A}_{p_1 p_2 p_3} ; \mathcal{P}(p_1 p_2), p_3\right) \\
& -3 \sum_{\substack{N^{\frac{1}{k_2}} \leqslant p_1<p_2<p_3<\left(\frac{N}{p_1 p_2}\right)^{\frac{1}{2}} \\ (p_1 p_2 p_3, N)=1}} S\left(\mathcal{A}_{p_1 p_2 p_3} ; \mathcal{P}(p_1 p_2), p_3\right)+O\left(N^{0.99}\right) \\
=\ & 3 S_1-S_2-S_3-2 S_4-3 S_5+O\left(N^{0.99}\right).
\end{aligned}$$

Proof. The proof is very similar to those in [[42], Lemma 3] and [[25], Proposition 4.1] so we omit it here. □

3. A result of Lou and Yao

In this section we shall state the important result proved by Lou and Yao.

Lemma 3.1. *([31], Lemma 2]). Let*

$$\psi(x ; a, d, l)=\sum_{\substack{a n \leqslant x \\ a n \equiv l(\bmod d)}} \Lambda(n),$$

and let $g(a)$ denote a real function such that $|g(a)| \leqslant 1$. Then, for any given constant $A>0$, there exists a constant $B=B(A)>0$ such that

$$\sum_{d \leqslant D} \sum_{\substack{1 \leqslant l \leqslant d \\ (l, d)=1}} \sum_{\substack{A_1<a \leqslant A_2 \\ (a, d)=1}} g(a)\left(\psi\left(y+y^\theta ; a, d, l\right)-\psi(y ; a, d, l)-\frac{y^\theta}{a \varphi(d)}\right)^2 \ll \frac{y^{2 \theta}}{(\log y)^A},$$

where

$$D \leqslant \min \left(y^{2 \theta-1}(\log y)^{-B},\ y^{\frac{24 \theta-14}{5}}\right), \quad 1<A_1<A_2 \leqslant y^{\theta-\varepsilon}, \quad \frac{7}{12}<\theta \leqslant 1, \quad B(A)=A+70.$$

Using the same method as in the proof of Lemma 3.1, we can prove the following result, which is the most crucial result that we shall use later.

Lemma 3.2. *([31], Lemma 3]). Let $\psi(x;a,d,l)$ and $g(a)$ denote the same things as in Lemma 3.1. Then, for any given constant $A>0$, there exists a constant $B=B(A)>0$ such that*

$$\sum_{d\leqslant D}\max_{y\leqslant x}\max_{(l,d)=1}\left|\sum_{\substack{A_1<a\leqslant A_2\\(a,d)=1}} g(a)\left(\psi\left(y+y^{\theta};a,d,l\right)-\psi\left(y;a,d,l\right)-\frac{y^{\theta}}{a\varphi(d)}\right)\right|\ll\frac{x^{\theta}}{(\log x)^{A}},$$

where

$$D\leqslant\min\left(x^{\theta-\frac{1}{2}}(\log x)^{-B},\ x^{\frac{12\theta-7}{5}}\right),\quad 1<A_1<A_2\leqslant y^{\theta-\varepsilon},\quad \frac{7}{12}<\theta\leqslant 1,\quad B(A)=2A+77.$$

Moreover, we have

$$\sum_{d\leqslant D}\max_{y\leqslant x}\max_{(l,d)=1}\left|\sum_{\substack{A_1<a\leqslant A_2\\(a,d)=1}} g(a)\left(\pi\left(y+y^{\theta};a,d,l\right)-\pi\left(y;a,d,l\right)-\frac{1}{\varphi(d)}\int_{\frac{y}{a}}^{\frac{y+y^{\theta}}{a}}\frac{dt}{\log t}\right)\right|\ll\frac{x^{\theta}}{(\log x)^{A}},$$

and we can replace the boundaries $\frac{y}{a}$ and $\frac{y+y^{\theta}}{a}$ in the prime-counting functions by any positive functions $r_1(a,y)$ and $r_2(a,y)$ such that

$$\frac{y}{a}\leqslant r_1(a,y)\leqslant r_2(a,y)\leqslant\frac{y+y^{\theta}}{a}$$

and prove the corresponding version of Lemma 3.2. (See [[8], Remark] and [[6], Lemma 7], for example.)

Remark 3.3. In Wu's 1993 result on the generalized Bombieri–Vinogradov Theorem in short intervals [[43], Théorème 1 and 2], the upper bound for a in the inner summation (or the upper bound for the size of elements in some "sieved sets" after using Chen's switching principle) is $y^{\frac{5\theta-3}{2}}$, which is smaller than y^{θ} when $\theta<1$. Note that $x^{\theta-\frac{1}{2}}<x^{\frac{12\theta-7}{5}}$ for $\theta>\frac{9}{14}$, the sizes of the moduli in [[43], Théorème 1 and 2] and in Lemma 3.2 are same. In fact, the reason we cannot get better results by inserting weights into [26] is that Wu's result is not sufficient to deal with some error terms when θ decreases. With the help of Lou and Yao's results, we can deal with those error terms now.

4. Proof of Theorem 1.1

Let γ denote the Euler's constant, $F(s)$ and $f(s)$ are determined by the following differential-difference equation

$$\begin{cases}F(s)=\frac{2e^{\gamma}}{s},\quad f(s)=0, & 0<s\leqslant 2,\\ (sF(s))'=f(s-1),\quad (sf(s))'=F(s-1), & s\geqslant 2,\end{cases}$$

and let $\omega(u)$ denote the Buchstab function determined by the following differential-difference equation

$$\begin{cases}\omega(u)=\frac{1}{u}, & 1\leqslant u\leqslant 2,\\ (u\omega(u))'=\omega(u-1), & u\geqslant 2.\end{cases}$$

We first show that

$$D_{1,3}\left(N,\frac{1}{2},0.87125\right)\gg\frac{C(N)N^{0.87125}}{(\log N)^2}. \tag{7}$$

Let $k_1=7.726$ and $k_2=4.24$ in Lemma 2.1. Then we have

$$\begin{aligned}3D_{1,3}\left(N,\frac{1}{2},0.87125\right)\geqslant{}& 3S\left(\mathcal{A};\mathcal{P},N^{\frac{1}{7.726}}\right)-\sum_{\substack{N^{\frac{1}{7.726}}\leqslant p<N^{\frac{1}{4.24}}\\(p,N)=1}}S\left(\mathcal{A}_p;\mathcal{P},N^{\frac{1}{7.726}}\right)\\&-\sum_{\substack{N^{\frac{1}{7.726}}\leqslant p_1<p_2<N^{\frac{1}{4.24}}\leqslant p_3<\left(\frac{N}{p_1p_2}\right)^{\frac{1}{2}}\\(p_1p_2p_3,N)=1}}S\left(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(p_1p_2),p_3\right)\end{aligned}$$

$$
\begin{aligned}
&-2 \sum_{\substack{N^{\frac{1}{7.726}} \leqslant p_1 < N^{\frac{1}{4.24}} \leqslant p_2 < p_3 < \left(\frac{N}{p_1 p_2}\right)^{\frac{1}{2}} \\ (p_1 p_2 p_3, N)=1}} S\left(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(p_1 p_2), p_3\right) \\
&-3 \sum_{\substack{N^{\frac{1}{4.24}} \leqslant p_1 < p_2 < p_3 < \left(\frac{N}{p_1 p_2}\right)^{\frac{1}{2}} \\ (p_1 p_2 p_3, N)=1}} S\left(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(p_1 p_2), p_3\right) + O\left(N^{0.99}\right) \\
&= 3T_1 - T_2 - T_3 - 2T_4 - 3T_5 + O\left(N^{0.99}\right) \qquad (8)
\end{aligned}
$$

where $\mathcal{A} = \mathcal{A}_1$. By our Lemma 3.2, the linear sieve and similar arguments as in [8], we have

$$
T_1 \geqslant (1+o(1)) \frac{2}{e^\gamma} \left(\frac{f\left(7.726\left(0.87125 - \frac{1}{2}\right)\right)}{\frac{1}{7.726}} \right) \frac{C(N) N^{0.87125}}{(\log N)^2} \geqslant 6.81949 \frac{C(N) N^{0.87125}}{(\log N)^2}, \qquad (9)
$$

$$
\begin{aligned}
T_2 &\leqslant (1+o(1)) \frac{2}{e^\gamma} \left(\int_{\frac{1}{7.726}}^{\frac{1}{4.24}} \frac{7.726 F\left(7.726\left(0.87125 - \frac{1}{2} - t_1\right)\right)}{t_1} \right) \frac{C(N) N^{0.87125}}{(\log N)^2} \\
&\leqslant 12.79877 \frac{C(N) N^{0.87125}}{(\log N)^2}. \qquad (10)
\end{aligned}
$$

By Chen's switching principle, our Lemma 3.2, the linear sieve and similar arguments as in [8], we have

$$
\begin{aligned}
T_3 &\leqslant (1+o(1)) \frac{8}{0.87125 - \frac{1}{2}} \left(\int_{\frac{1}{7.726}}^{\frac{1}{4.24}} \int_{t_1}^{\frac{1}{4.24}} \int_{\frac{1}{4.24}}^{\frac{1-t_1-t_2}{2}} \frac{\omega\left(\frac{0.87125 - t_1 - t_2 - t_3}{t_3}\right)}{t_1 t_2 t_3^2} \right) \frac{C(N) N^{0.87125}}{(\log N)^2} \\
&\leqslant 5.12555 \frac{C(N) N^{0.87125}}{(\log N)^2}, \qquad (11) \\
T_4 &\leqslant (1+o(1)) \frac{8}{0.87125 - \frac{1}{2}} \left(\int_{\frac{1}{7.726}}^{\frac{1}{4.24}} \int_{\frac{1}{4.24}}^{\frac{1-t_1}{3}} \int_{t_2}^{\frac{1-t_1-t_2}{2}} \frac{\omega\left(\frac{0.87125 - t_1 - t_2 - t_3}{t_3}\right)}{t_1 t_2 t_3^2} \right) \frac{C(N) N^{0.87125}}{(\log N)^2} \\
&\leqslant 1.24518 \frac{C(N) N^{0.87125}}{(\log N)^2}, \qquad (12) \\
T_5 &\leqslant (1+o(1)) \frac{8}{0.87125 - \frac{1}{2}} \left(\int_{\frac{1}{4.24}}^{\frac{1}{4}} \int_{t_1}^{\frac{1-t_1}{3}} \int_{t_2}^{\frac{1-t_1-t_2}{2}} \frac{\omega\left(\frac{0.87125 - t_1 - t_2 - t_3}{t_3}\right)}{t_1 t_2 t_3^2} \right) \frac{C(N) N^{0.87125}}{(\log N)^2} \\
&\leqslant 0.01357 \frac{C(N) N^{0.87125}}{(\log N)^2}. \qquad (13)
\end{aligned}
$$

Combining (9)–(13), we have

$$
3T_1 - T_2 - T_3 - 2T_4 - 3T_5 \geqslant 0.003 \frac{C(N) N^{0.87125}}{(\log N)^2}. \qquad (14)
$$

Now by (8) and (14), (7) is proved.

We then show that

$$
D_{1,3}(N, 0, 0.817) \gg \frac{C(N) N^{0.817}}{(\log N)^2}. \qquad (15)
$$

Let $k_1 = 5.4644$ and $k_2 = 4.24$ in Lemma 2.1. Then we have

$$
\begin{aligned}
3D_{1,3}(N, 0, 0.817) \geqslant\ & 3S\left(\mathcal{A}; \mathcal{P}, N^{\frac{1}{5.4644}}\right) - \sum_{\substack{N^{\frac{1}{5.4644}} \leqslant p < N^{\frac{1}{4.24}} \\ (p, N)=1}} S\left(\mathcal{A}_p; \mathcal{P}, N^{\frac{1}{5.4644}}\right) \\
& - \sum_{\substack{N^{\frac{1}{5.4644}} \leqslant p_1 < p_2 < N^{\frac{1}{4.24}} \leqslant p_3 < \left(\frac{N}{p_1 p_2}\right)^{\frac{1}{2}} \\ (p_1 p_2 p_3, N)=1}} S\left(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(p_1 p_2), p_3\right)
\end{aligned}
$$

$$
\begin{aligned}
&-2\sum_{\substack{N^{\frac{1}{5.4644}}\leqslant p_1<N^{\frac{1}{4.24}}\leqslant p_2<p_3<\left(\frac{N}{p_1p_2}\right)^{\frac{1}{2}}\\(p_1p_2p_3,N)=1}} S\left(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(p_1p_2),p_3\right)\\
&-3\sum_{\substack{N^{\frac{1}{4.24}}\leqslant p_1<p_2<p_3<\left(\frac{N}{p_1p_2}\right)^{\frac{1}{2}}\\(p_1p_2p_3,N)=1}} S\left(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(p_1p_2),p_3\right)+O\left(N^{0.99}\right)\\
&=3T_1'-T_2'-T_3'-2T_4'-3T_5'+O\left(N^{0.99}\right) \qquad (16)
\end{aligned}
$$

where $\mathcal{A}=\mathcal{A}_2$. By the Bombieri–Vinogradov Theorem, the linear sieve and similar arguments as in [3], we have

$$
T_1'\geqslant(1+o(1))\frac{2}{0.817e^{\gamma}}\left(\frac{f\left(5.4644\left(\frac{0.817}{2}\right)\right)}{\frac{1}{5.4644}}\right)\frac{C(N)N^{0.817}}{(\log N)^2}\geqslant 2.50259\frac{C(N)N^{0.817}}{(\log N)^2}, \qquad (17)
$$

$$
\begin{aligned}
T_2'&\leqslant(1+o(1))\frac{2}{0.817e^{\gamma}}\left(\int_{\frac{1}{5.4644}}^{\frac{1}{4.24}}\frac{5.4644F\left(5.4644\left(\frac{0.817}{2}-t_1\right)\right)}{t_1}\right)\frac{C(N)N^{0.817}}{(\log N)^2}\\
&\leqslant 6.24103\frac{C(N)N^{0.817}}{(\log N)^2}. \qquad (18)
\end{aligned}
$$

By Chen's switching principle, our Lemma 3.2, the linear sieve and similar arguments as in [3], we have

$$
\begin{aligned}
T_3'&\leqslant(1+o(1))\frac{4}{0.817-\frac{1}{2}}\left(\int_{\frac{1}{5.4644}}^{\frac{1}{4.24}}\int_{t_1}^{\frac{1}{4.24}}\int_{\frac{1}{4.24}}^{\frac{1-t_1-t_2}{2}}\frac{\omega\left(\frac{0.817-t_1-t_2-t_3}{t_3}\right)}{t_1t_2t_3^2}\right)\frac{C(N)N^{0.817}}{(\log N)^2}\\
&\leqslant 0.62937\frac{C(N)N^{0.817}}{(\log N)^2}, \qquad (19)\\
T_4'&\leqslant(1+o(1))\frac{4}{0.817-\frac{1}{2}}\left(\int_{\frac{1}{5.4644}}^{\frac{1}{4.24}}\int_{\frac{1}{4.24}}^{\frac{1-t_1}{3}}\int_{t_2}^{\frac{1-t_1-t_2}{2}}\frac{\omega\left(\frac{0.817-t_1-t_2-t_3}{t_3}\right)}{t_1t_2t_3^2}\right)\frac{C(N)N^{0.817}}{(\log N)^2}\\
&\leqslant 0.2886\frac{C(N)N^{0.817}}{(\log N)^2}, \qquad (20)\\
T_5'&\leqslant(1+o(1))\frac{4}{0.817-\frac{1}{2}}\left(\int_{\frac{1}{4.24}}^{\frac{1}{4}}\int_{t_1}^{\frac{1-t_1}{3}}\int_{t_2}^{\frac{1-t_1-t_2}{2}}\frac{\omega\left(\frac{0.817-t_1-t_2-t_3}{t_3}\right)}{t_1t_2t_3^2}\right)\frac{C(N)N^{0.817}}{(\log N)^2}\\
&\leqslant 0.01323\frac{C(N)N^{0.817}}{(\log N)^2}. \qquad (21)
\end{aligned}
$$

Combining (17)–(21), we have

$$
3T_1'-T_2'-T_3'-2T_4'-3T_5'\geqslant 0.02\frac{C(N)N^{0.817}}{(\log N)^2}. \qquad (22)
$$

Now by (16) and (22), (15) is proved and the proof of Theorem 1.1 is completed. One can replace [[6], Lemmas 7 and 8] by our Lemma 3.2 and give another proof of

$$
D_{1,2}\left(N,0,0.9409\right)\gg\frac{C(N)N^{0.9409}}{(\log N)^2} \qquad (23)
$$

using the methods in [6] and parameters in [29].

Appendix: Lou and Yao's calculations

In [31], Lou and Yao mentioned that they can give a nontrivial lower bound for $D_{1,2}\left(N,\frac{1}{2},\theta\right)$ with $\theta=0.92$, improving the result of Ross (which has some errors in the calculations for $\theta=0.94$ and was corrected to $\theta = 0.98$ in [40]). However, the calculations in Lou and Yao's paper also contain serious errors, leading to an invalid proof of $\theta = 0.92$. The exact θ in their proof should be a value between 0.97 and 0.98.

The main error that occurred in their calculations is their use of the "distribution level" D. For $\theta > 0.9$, we know that $D = x^{\theta - \frac{1}{2} - \varepsilon}$ in Lemma 3.2. However, in [31] they used

$$D = x^{\frac{\theta}{2} - \varepsilon} \quad \text{or even} \quad D = x^{\theta - \varepsilon}$$

to calculate many sieve main terms and values of linear sieve functions $F(s)$ and $f(s)$, leading to an inexact result 0.92.

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