https://teorth.github.io/expdb https://github.com/teorth/expdb https://teorth.github.io/expdb/docs

Analytic Number Theory Exponent Database

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Chapter 1

Introduction

This is the LaTeX “Blueprint” form of the *analytic number theory exponent database (ANTEDB)*, which is an ongoing project to record (both in a human-readable and computer-executable formats) the latest known bounds, conjectures, and other relationships concerning several exponents of interest in analytic number theory. It can be viewed as an expansion of the paper [**?**]. Currently, the database is recording information on the following exponents:

- Exponent pairs (k, ℓ).
- The exponential sum function $\beta(\alpha)$ dual to exponent pairs.
- The growth exponent $\mu(\sigma)$ of the zeta function $\zeta(\sigma + it)$.
- The moment exponents $M(\sigma, A)$ of the zeta function.
- Large value exponents $\mathrm{LV}(\sigma, \tau)$ for Dirichlet polynomials $\sum_{n\in[N,2N]} a_n n^{-it}$.
- Large value exponents $\mathrm{LV}_\zeta(\sigma, \tau)$ of the zeta polynomials $\sum_{n\in I} n^{-it}$.
- Large value additive energy exponents $\mathrm{LV}^*(\sigma, \tau)$, $\mathrm{LV}^*_\zeta(\sigma, \tau)$ for Dirichlet and zeta polynomials.
- Zero density exponents $\mathrm{A}(\sigma)$ for the zeta function.
- Zero density additive energy exponents $\mathrm{A}^*(\sigma)$ for the zeta function.
- The regions $\mathcal{E}$, $\mathcal{E}_\zeta$ of exponent tuples $(\sigma, \tau, \rho, \rho^*, s)$ recording possible large values, large value additive energy, and double zeta sums for Dirichlet and zeta polynomials.
- Exponents α_k for the Dirichlet divisor problem.
- The primitive Pythagorean triple exponent θ_{Pythag}.
- Exponents θ_{PNT}, $\theta_{\mathrm{PNT-AA}}$ for the prime number theorem in all or almost all short intervals.

- The maximal prime gap exponent θ_{gap}.
- The prime gap second moment exponent $\theta_{\mathrm{gap},2}$.

Possible future directions for expansion include

- Exponents for L-functions (in both q and T aspects)
- More exponents relating to prime gaps

The database aims to enumerate, as comprehensively as possible, all the various known or conjectured facts about these exponents, including "trivial" or "obsolete" such facts. Of particular interest are implications that allow new bounds on exponents to be established from existing bounds on other exponents.

Each of the facts in the database can be supported with a reference, or one or more proofs, or with executable code in Python; ideally one should have all three (and with a preference for proofs that rely as much as possible on other facts in the database). In the future we could also expand this database to support each fact with formal proofs in proof assistant languages such as Lean.

In order to facilitate the dependency tree of the python code, as well as to assist readers who wish to derive the facts in this database from first principles, the blueprint is arranged in linear order. Thus, the statement and proof of a proposition in the blueprint is only permitted to use propositions and definitions that are located earlier in the blueprint, although we do allow forward-referencing references in the remarks. As a consequence, the material relating to a single topic will not necessarily be located in a single chapter, but could be spread out over multiple chapters, depending on how much advanced material is needed to state or prove the required results. Additionally, a single proposition may occur multiple times in the blueprint, if it has multiple proofs with varying prerequisites. In the future, we plan to implement a search feature that will allow the reader to locate all propositions of relevance to a given topic (e.g., all propositions whose statement involves the concept of an exponent pair).

Chapter 2

Basic notation

We freely assume the axiom of choice in this blueprint.

Throughout this blueprint we adopt following notation. If θ is a real number, then we write

$$e(\theta) \coloneqq e^{2\pi i\theta}$$

where i is the imaginary unit. The indicator function $1_I(n)$ of a set I is defined to equal 1 when $n \in I$, and 0 otherwise.

We adopt the convention that an empty supremum is $-\infty$, and an empty infimum is $+\infty$. Thus, for instance, $\sup_{\sigma_0 \leq \sigma \leq \sigma_1} f(\sigma)$ would equal $-\infty$ if $\sigma_1 < \sigma_0$. Related to this, we also adopt the convention that $N^{-\infty} = 0$ when $N > 1$.

A sequence $a_n, n \in I$ of real or complex numbers indexed by some index set is said to be 1-*bounded* if $|a_n| \leq 1$ for all $n \in I$. Similarly, a sequence $t_1, \dots, t_R$ of real numbers is said to be 1-*separated* if $|t_r - t_{r'}| \geq 1$ for all $1 \leq r < r' \leq R$. One can define more general notions of λ-bounded or λ-separated for other $\lambda > 0$ in the obvious fashion.

2.1 Asymptotic (or "cheap nonstandard") notation

It is convenient to use a "cheap nonstandard analysis" framework for asymptotic notation, in the spirit of [**?**], as this will reduce the amount of "epsilon management" one has to do in the arguments. This framework is inspired by nonstandard analysis, but we will avoid explicitly using such nonstandard constructions as ultraproducts in the discussion below, relying instead on the more familiar notion of sequential limits.

In this framework, we assume there is some ambient index parameter i, which ranges over some ambient sequence of natural numbers going to infinity. All mathematical objects X (numbers, sequences, sets, functions, etc.), will either be *fixed* - i.e., independent of i - or *variable* - i.e., dependent on i. (These correspond to the notions of *standard* and *non-standard* objects in nonstandard analysis.) Of course, fixed objects can be considered as special cases of variable objects, in which the dependency is constant. By default, objects should be understood to be variable if not explicitly declared to be fixed. For emphasis, we shall sometimes write $X = X_\mathrm{i}$ to explicitly indicate that an object X is variable; however, to reduce clutter, we shall generally omit explicit mention of the parameter i in most of our arguments. We will often reserve the right to refine the ambient sequence to a subsequence as needed, usually in order to apply a compactness theorem such as the Bolzano–Weierstrass theorem; we refer to this process as "passing to a subsequence if necessary". When we say that a statement involving variable objects is true, we mean that it is true for all i in the ambient sequence. For instance, a variable set E of real numbers is a set $E = E_\mathrm{i}$ indexed by the ambient parameter i, and by an element of such a set, we mean a variable real number $x = x_\mathrm{i}$ such that $x_\mathrm{i} \in E_\mathrm{i}$ for all i in the ambient sequence. A variable sequence $t_1, \dots, t_R$ of real numbers of some variable length R is actually a sequence of sequences $t_{1,\mathrm{i}}, \dots, t_{R_\mathrm{i},\mathrm{i}}$ of real numbers of some i-dependent length R_i. Saying that such a sequence is 1-separated is thus asserting that

$$|t_{r,\mathrm{i}} - t_{r',\mathrm{i}}| \geq 1$$

for all i in the ambient sequence and all $1 \leq r < r' \leq R_\mathrm{i}$.

We isolate some special types of variable numerical quantities $X = X_\mathrm{i}$ (which could be a natural number, real number, or complex number):

- X is *bounded* if there exists a fixed C such that $|X| \leq C$. In this case we also write $X = O(1)$.
- X is *unbounded* if $|X_\mathrm{i}| \to \infty$ as $\mathrm{i} \to \infty$; equivalently, for every fixed C, one has $|X| \geq C$ for i sufficiently large.
- X is *infinitesimal* if $|X_\mathrm{i}| \to 0$ as $\mathrm{i} \to \infty$; equivalently, for every fixed $\varepsilon > 0$, one has $|X| \leq \varepsilon$ for i sufficiently large. In this case we also write $X = o(1)$.

Note that any quantity X will be either bounded or unbounded, after passing to a subsequence if necessary; similarly, by the Bolzano–Weierstrass theorem, any bounded (variable) quantity X will be of the form $X_0 + o(1)$ for some fixed X_0, after passing to a subsequence if necessary. Thus, for instance, if $T, N > 1$ are (variable) quantities with $N = T^{O(1)}$ (or equivalently, $T^{-C} \leq N \leq T^C$ for some fixed C), then, after passing to a subsequence if necessary, we may write $N = T^{\alpha + o(1)}$ for some fixed real number α. Note that any further passage to subsequences do not these concepts; quantities that are bounded, unbounded, or infinitesimal remain so under any additional restriction to subsequences.

We observe the *underspill principle*: if X, Y are (variable) real numbers, then the relation

$$X \leq Y + o(1)$$

is equivalent to the relation

$$X \leq Y + \varepsilon + o(1)$$

holding for all fixed $\varepsilon > 0$.

We can develop other standard asymptotic notation in the natural fashion: given two (variable) quantities X, Y, we write $X = O(Y)$, $X \ll Y$, or $Y \gg X$ if $|X| \leq CY$ for some fixed C, and $X = o(Y)$ if $|X| \leq cY$ for some infinitesimal c. We also write $X \asymp Y$ for $X \ll Y \ll X$.

A convenient property of this asymptotic formalism, analogous to the property of *ω-saturation* in nonstandard analysis, is that certain asymptotic bounds are automatically uniform in variable parameters.

Proposition 2.1 (Automatic uniformity)**.** *Let $E = E_{\mathrm{i}}$ be a non-empty variable set, and let $f = f_{\mathrm{i}} : E \to \mathbf{C}$ be a variable function.*

(i) Suppose that $f(x) = O(1)$ for all (variable) $x \in E$. Then after passing to a subsequence if necessary, the bound is uniform, that is to say, there exists a fixed C such that $|f(x)| \leq C$ for all $x \in E$.

(ii) Suppose that $f(x) = o(1)$ for all (variable) $x \in E$. Then after passing to a subsequence if necessary, the bound is uniform, that is to say, there exists an infitesimal c such that $|f(x)| \leq c$ for all $x \in E$.

Proof. We begin with (i). Suppoe that there is no uniform bound. Then for any fixed natural number n, one can find arbitrarily large i_n in the sequence and $x_{\mathrm{i}_n} \in E_{\mathrm{i}_n}$ such that $|f_{\mathrm{i}_n}(x_{\mathrm{i}_n})| \geq n$. Clearly one can arrange matters so that the sequence i_n is increasing. If one then restricts to this sequence and sets x to be the variable element x_{i_n} of E, then $f(x)$ is unbounded, a contradiction.

Now we prove (ii). We can assume for each fixed n that there exists i_n in the ambient sequence such that $|f_{\mathrm{i}}(x_{\mathrm{i}})| \leq 1/n$ for all $\mathrm{i} \geq \mathrm{i}_n$ and $x_{\mathrm{i}} \in E_{\mathrm{i}}$, since if this were not the case one can construct an $x = x_{\mathrm{i}} \in E$ such that $|f_{\mathrm{i}}(x_{\mathrm{i}})| \geq 1/n$ for i sufficiently large, contradicting the hypothesis. Again, we may take the i_n to be increasing. Restricting to this sequence, and writing $c_{\mathrm{i}_n} := 1/n$, we see that $c = o(1)$ and $|f(x)| \leq c$ for all $x \in E$, as required. □

Remark 2.2. It is important in Proposition 2.1 that the hypotheses in (i), (ii) are assumed for all *variable* $x \in E$, rather than merely the *fixed* $x \in E$. For instance, let $E = \mathbf{R}$ and consider the variable function $f_{\mathrm{i}}(x) := x/\mathrm{i}$. Then $f(x) = o(1)$ for any fixed $x \in E$, but the decay rate is not uniform, and we do not have $f(x) = o(1)$ for all variable $x \in E$ (e.g., $x_{\mathrm{i}} := \mathrm{i}$ is a counterexample).

Remark 2.3. There are two caveats to keep in mind when using this asymptotic formalism. Firstly, the law of the excluded middle is only valid after passing to subsequences. For instance, it is possible for a nonstandard natural number to neither be even or odd, since it could be even for some i and odd for others. However, one can pass to a subsequence in which it becomes either even or odd. Secondly, one cannot combine the "external" concepts of asymptotic notation with the "internal" framework of (variable) set theory. For instance, one cannot view the collection of all bounded (variable) real numbers as a variable set, since the notion of boundedness is not "pointwise" to each index i, but instead describes the "global" behavior of this index set. Thus, for instance, set builder notation such as $\{x : x = O(1)\}$ should be avoided.

Chapter 3

Basic L^2 estimates

Lemma 3.1 (L^2 integral estimate)**.** *Let $\xi_1, \ldots, \xi_R$ be real numbers that are $1/N$-separated. Then for any interval I of length T, and any sequence $a_1, \ldots, a_R$ of complex numbers one has*

$$\int_I |\sum_{r=1}^R a_r e(\xi_r t)|^2 \, dt = (T + O(N)) \sum_{r=1}^R |a_r|^2.$$

Proof. We adapt the proof of [**?**, Theorem 9.1]. Without loss of generality we may normalize $\sum_{r=1}^R |a_r|^2 = 1$. From the Plancherel identity we have

$$\int_{\mathbf{R}} |\sum_{r=1}^R a_r e(\xi_r t)|^2 |\hat{\psi}((t-t_0)/N)|^2 \, dt = N \tag{3.1}$$

whenever $t_0 \in \mathbf{R}$ and ψ is a smooth function supported on $[-1/4, 1/4]$ of L^2 norm 1. By suitable choice of ψ, this implies that

$$\int_J |\sum_{r=1}^R a_r e(\xi_r t)|^2 \, dt \ll N \tag{3.2}$$

whenever J is an interval of length N. If one integrates (3.1) for all $t_0 \in I$, we see that

$$\int_I |\sum_{r=1}^R a_r e(\xi_r t)|^2 \, dt = T - \int_{\mathbf{R}} |\sum_{r=1}^R a_r e(\xi_r t)|^2 (\int_I |\hat{\psi}((t-t_0)/N)|^2 \, dt_0 - 1_I(t)) \, dt.$$

Since $\hat{\psi}$ is rapidly decreasing and has L^2 norm 1, one can compute

$$\int_I |\hat{\psi}((t-t_0)/N)|^2 \, dt_0 - 1_I(t) \ll (1 + \operatorname{dist}(t, \partial I)/N)^{-10}$$

and hence by (3.2) and the triangle inequality

$$\int_{\mathbf{R}} |\sum_{r=1}^R a_r e(\xi_r t)|^2 (\int_I |\hat{\psi}((t-t_0)/N)|^2 \, dt_0 - 1_I(t)) \, dt \ll N$$

giving the claim. □

Chapter 4

Exponential sum growth exponents

4.1 Phase functions

Definition 4.1 (Phase function)**.** A *phase function* is a (variable) smooth function $F\colon [1,2] \to \mathbf{R}$. A phase function F will be called a *model phase function* if there exists a fixed exponent $\sigma > 0$ with the property that

$$F^{(p+1)}(u) - \frac{d^p}{du^p} u^{-\sigma} = o(1) \tag{4.1}$$

for all (variable) $u \in [1,2]$ and all fixed $p \geq 0$, where $F^{(p+1)}$ denotes the $(p+1)^{\text{st}}$ derivative of F.

For instance, $u \mapsto \log u$ is a model phase function (with $\sigma = 1$), and for any fixed $\sigma \neq 1$, $u \mapsto u^{1-\sigma}/(1-\sigma)$ is a model phase function. Informally, a model phase function is a function which asymptotically behaves like $u \mapsto \log u$ (for $\sigma = 1$) or $u \mapsto u^{1-\sigma}/(1-\sigma)$ (for $\sigma \neq 1$), up to constants.

Note from Proposition 2.1 that the $o(1)$ decay rate in (4.1) can be made uniform, after passing to a subsequence if necessary.

4.2 Exponential sum exponent

The main purpose of this chapter is to introduce and establish the basic properties of the following exponent function.

Definition 4.2 (Exponent sum growth exponent)**.** For any fixed $\alpha \geq 0$, let $\beta(\alpha) \in \mathbf{R}$ denote the least possible (fixed) exponent for which the following claim holds: whenever $N, T \geq 1$ are (variable) quantities with T unbounded and $N = T^{\alpha+o(1)}$, F is a model phase function, and $I \subset [N, 2N]$ is an interval, then

$$\sum_{n \in I} e(TF(n/N)) \ll T^{\beta(\alpha)+o(1)}.$$

Implemented at `|bound`$_b$*eta*|*.py* as:
`|Bound`$_b eta|$

`It is easy to see that the set of possible candidates for` $\beta(\alpha)$ `is closed (thanks to underspill), non-empty, and bounded from below, so` β `is well-defined as a (fixed) function from` $[0, +\infty)$ `to` $\mathbf{R}$`. Specializing to the logarithmic phase` $F(u) = \log u$`, and performing a complex conjugation, we see in particular that`

$$\sum_{n \in I} n^{-iT} \ll T^{\beta(\alpha)+o(1)} \tag{4.2}$$

`whenever` T `is unbounded,` $N = T^{\alpha+o(1)}$`, and` I `is an interval in` $[N, 2N]$`. Thus it is clear that knowledge of` β `is of relevance to understanding the Riemann zeta function.`

`The quantity` $\beta(\alpha)$ `can also be formulated without asymptotic notation, but at the cost of introducing some ``epsilon and delta'' parameters:`

Lemma 4.3 (Non-asymptotic definition of β)**.** *Let* $\alpha \geq 0$ *and* $\overline{\beta} \in \mathbf{R}$ *be fixed. Then the following are equivalent:*

(i) $\beta(\alpha) \leq \overline{\beta}$*.*

(ii) For every (fixed) $\varepsilon > 0$ *and* $\sigma > 0$ *there exists (fixed)* $\delta > 0$*,* $P \geq 1$*,* $C \geq 1$ *with the following property: if* $T \geq C$*,* $T^{\alpha-\delta} \leq N \leq T^{\alpha+\delta}$ *are (fixed) real numbers,* $I \subset [N, 2N]$ *is a (fixed) interval, and* F *is a (fixed) phase function such that*

$$|F^{(p+1)}(u) - \frac{d^p}{du^p} u^{-\sigma}| \leq \delta \tag{4.3}$$

for all (fixed) $0 \leq p \leq P$ *and* $u \in [1, 2]$*, then*

$$|\sum_{n \in I} e(TF(n/N))| \leq CT^{\overline{\beta}+\varepsilon}.$$

Proof. It is easy to see that (ii) implies (i) by expanding out all the definitions (and using Proposition 2.1 to resolve any uniformity issues). Conversely, suppose that (ii) fails. Carefully negating all the quantifiers, we conclude that there exists a fixed $\varepsilon, \sigma > 0$ such that for any (fixed) natural number i, one can find real numbers $T = T_{\mathrm{i}} \geq \mathrm{i}$, $T^{\alpha-1/\mathrm{i}} \leq N = N_{\mathrm{i}} \leq T^{\alpha+1/\mathrm{i}}$, an interval $I = I_{\mathrm{i}} \subset [N_{\mathrm{i}}, 2N_{\mathrm{i}}]$, and a phase function $F = F_{\mathrm{i}}$ such that

$$|F_{\mathrm{i}}^{(p+1)}(u) - \frac{d^p}{du^p} u^{-\sigma}| \leq 1/\mathrm{i}$$

for all (fixed) $0 \leq p \leq \mathrm{i}$ and $u \in [1, 2]$, but that

$$|\sum_{n \in I} e(TF(n/N))| \geq \mathrm{i}T^{\overline{\beta}+\varepsilon}.$$

But then $F = F_{\mathrm{i}}$ is a model phase function which gives a counterexample to the claim $\beta(\alpha) \leq \overline{\beta}$. $\square$

`We will however work with the asymptotic formulation of` β `throughout this database, as it makes the proofs somewhat cleaner.`

`We record the trivial bounds on` β`:`

Lemma 4.4 (Trivial bounds on β)**.** *beta-def For any fixed $\alpha > 1$, we have*

$$\beta(\alpha) = \alpha - 1.$$

For fixed $0 \leq \alpha \leq 1$, we have

$$\frac{\alpha}{2} \leq \beta(\alpha) \leq \alpha.$$

In particular

$$\beta(0) = 0. \tag{4.4}$$

`Implemented at |bound`$_b eta|.py$ `as:`
`|trivial`$_b eta_b ound_1||trivial_b eta_b ound_2|$

Proof. Let $T > 1$ be unbounded, $N = T^{\alpha+o(1)}$, $I \subset [N, 2N]$ an interval, and F a model phase function.

For $\alpha > 1$, the Euler–Maclaurin formula (see e.g. [**?**, (2.1.2)]) gives

$$\left| \sum_{N \leq n \leq 2N} n^{iT} \right| = \left| \frac{2^{1+iT} - 1}{1 + iT} N^{1+iT} + O(1) \right| \asymp \frac{N}{T} \tag{4.5}$$

which gives the lower bound $\beta(\alpha) \geq \alpha - 1$; applying the Euler–Maclaurin formula for model phase functions F then gives the matching upper bound.

The triangle inequality bound

$$\sum_{n \in I} e(TF(n/N)) \ll N$$

gives the upper bound $\beta(\alpha) \leq \alpha$. Next, if $0 \leq \alpha \leq 1$, then from Lemma 3.1 (and the $\gg 1/N$-separated nature of the $F(n/N)$ for model phase functions F, after passing to a subsequence if necessary) that

$$\int_{T^{2T}} \left| \sum_{n \in [N, 2N]} e(tF(n/N)) \right|^2 \, dt \asymp TN$$

for $N = cT^\alpha$ for c a fixed small enough constant, which by the pigeonhole principle implies that

$$\left| \sum_{n \in [N, 2N]} e(tF(n/N)) \right|^2 \, dt \gg N^{1/2} = T^{\alpha/2}$$

for at least one $t \asymp T$, giving the claim. $\square$

As we shall see, the exponent pair conjecture is equivalent to the lower bound here being sharp, thus it is conjectured that

$$\beta(\alpha) = \begin{cases} \alpha/2, & 0 \leq \alpha \leq 1 \\ \alpha - 1, & \alpha > 1 \end{cases}.$$

Note the discontinuity at 1. Despite this, we have:

Lemma 4.5 (Upper semicontinuity)**.** *beta-triv β is an upper semicontinuous function.*

Proof. Routine from the definition. □

We record the classical bounds on β:

Proposition 4.6 (Van der Corput inequality)**.** *beta-def For any natural number $k \geq 2$ and any $\alpha > 0$, one has*

$$\beta(\alpha) \leq \max\left(\alpha + \frac{1 - k\alpha}{2^k - 2}, (1 - 2^{2-k})\alpha - \frac{1 - \alpha}{2^k - 2}\right).$$

Thus for instance when $k = 2$ we have

$$\beta(\alpha) \leq \max\left(\frac{1}{2}, \frac{2\alpha - 1}{2}\right),$$

so in particular

$$\beta(1) = \frac{1}{2}, \tag{4.6}$$

by Lemma 4.4, when $k = 3$ one has

$$\beta(\alpha) \leq \max\left(\frac{1 + 3\alpha}{6}, \frac{6\alpha - 1}{3}\right),$$

and when $k = 4$ one has

$$\beta(\alpha) \leq \max\left(\frac{10\alpha + 1}{14}, \frac{29\alpha - 2}{28}\right).$$

This form of upper bound of $\beta(\alpha)$ - as the maximum of a finite number of linear functions of α - is extremely common in the literature.

Proof. Follows from [**?**, Theorem 8.20]. □

Corollary 4.7 (Optimizing the van der Corput inequality)**.** *beta-def For any $\alpha > 0$ one has*

$$\beta(\alpha) \leq \inf_{k \in \mathbf{N}: k \geq 2} \alpha + \frac{1 - k\alpha}{2^k - 2}.$$

Thus for instance

$$\beta(\alpha) \leq \min\left(\frac{1}{2}, \frac{1 + 3\alpha}{6}, \frac{10\alpha + 1}{14}\right).$$

Proof. beta-vdc Let $\beta_k(\alpha) = \alpha + (1 - k\alpha)/(2^k - 2)$ and

$$\alpha_k = \frac{2^k}{(k-1)2^k + 2}.$$

Via a routine computation, $\beta_{k+1}(\alpha) \leq \beta_k(\alpha)$ for $\alpha \geq \alpha_k$ and any $k \geq 2$. Thus, to verify that $\beta(\alpha) \leq \beta_k(\alpha)$ for $0 \leq \alpha \leq 1/2$, it suffices to just show that the same result holds for $0 \leq \alpha \leq \alpha_k$. However, for $0 \leq \alpha \leq \alpha_k$ and $k \geq 2$, we have

$$0 \leq \alpha \leq \alpha_k \leq \frac{2^{k+1}}{(k-3)2^k + 8}$$

which rearranges to give

$$\alpha + \frac{1 - k\alpha}{2^k - 2} \geq (1 - 2^{2-k})\alpha - \frac{1 - \alpha}{2^k - 2}, \qquad (0 \leq \alpha \leq \alpha_k, k \geq 2)$$

which completes the proof in view of Proposition 4.6. See Figure 4.3. □

`We can remove the role of` I `in the definition of` β`:`

Lemma 4.8. *In Definition 4.2, one can take the interval I to be $[N, 2N]$.*

Proof. Suppose that $\alpha, \overline{\beta}$ are fixed quantities such that the bounds in Definition 4.2 hold just for $I = [N, 2N]$, thus whenever $T > 1$ is unbounded, $N = T^{\alpha + o(1)}$, and F is a model phase function one has

$$\sum_{N \leq n \leq 2N} e(TF(n/N)) \ll T^{\overline{\beta} + o(1)}. \tag{4.7}$$

Our task is then to show that

$$\sum_{n \in I} e(TfF(n/N)) \ll T^{\overline{\beta} + o(1)}$$

under the same hypotheses. Similarly with $\alpha = 1$ we can use the proof of Lemma 4.6 to obtain $\overline{\beta} \geq 1/2$, and we are again done. Thus we may assume that $\alpha < 1$.

For $n \in [N, 2N]$, the constraint $n \in I$ is equivalent to restricting $F(n/N)$ to an interval J of length $O(1)$, which we can also smooth out by $O(1/N)$ without affecting the sum. Applying a Fourier expansion and the triangle inequality, we can thus bound the left-hand side by

$$\ll T^{o(1)} + \int_{-N^{1+o(1)}}^{N^{1+o(1)}} \left| \sum_{n \in [N, 2N]} e(TF(n/N) - tF(n/N)) \right| \frac{dt}{1 + |t|}.$$

Since $\alpha > 1$, we have $|t - T| \leq T/2$ for all t in the integral if T is large enough. From hypothesis (4.7) (with T replaced by $T - t$)we have

$$\left| \sum_{n \in [N, 2N]} e(TF(n/N) - tF(n/N)) \right| \ll T^{\overline{\beta} + o(1)}$$

for all such t, and the claim follows. See also Sargos [**?**, p 310]. □

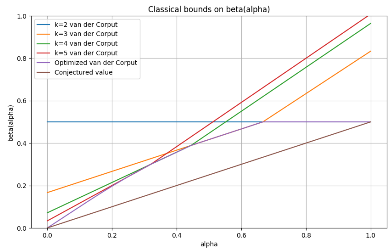

Figure 4.1: The bounds in Proposition 4.6 for $k = 2, 3, 4, 5$, compared against the optimized bound in Corollary 4.7.

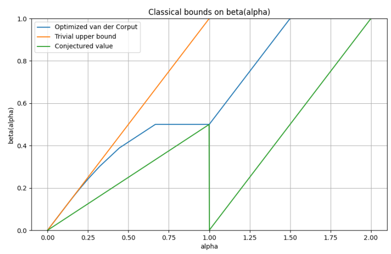

Figure 4.2: The bound in Corollary 4.7, compared against the trivial upper and lower bounds in Lemma 4.4.

Lemma 4.9 (Reflection). *For any* $0 < \alpha < 1$, *we have* $\beta(\alpha) - \frac{\alpha}{2} = \beta(1-\alpha) - \frac{1-\alpha}{2}$, *or equivalently* $\beta(1-\alpha) = \frac{1}{2} - \alpha + \beta(\alpha)$.

TODO: implement this in python

Proof. This is the van der Corput B-process. See e.g., [**?**, p 370]. ☐

4.3 Known bounds on β

`We remark that this corollary also follows from Proposition 5.10.`

Theorem 4.10 (Watt bound). *beta-def For any* $3/7 \leq \alpha \leq 1/2$, *one has*

$$\beta(\alpha) \leq \frac{89}{560} + \frac{1}{2}\alpha.$$

`Recorded in literature.py as:`
|add$_{b}eta_{b}ound_{w}att_{1}989()$|

Proof. See [**?**, Theorem 5]. ☐

Theorem 4.11 (1991 Huxley–Kolesnik bound)**.** *beta-def For any* $2/5 \le \alpha \le 1/2$ *one has*

$$\beta(\alpha) \le \max\left(\frac{1+8\alpha}{22}, \frac{11+112\alpha}{158}, \frac{1+17\alpha}{22}\right).$$

Recorded in literature.py as:
|add$_beta_bound_huxley_kolesnik_1991()$|

Proof. See [**?**, Theorem 3]. Note that the paper contains an error, however this result was reinstated with the corrections given in [**?**]. □

Theorem 4.12 (1993 Huxley bound)**.** *beta-def For any* $0 \le \alpha \le 49/114$*, one has*

$$\beta(\alpha) \le \max\left(\frac{13}{60} + \frac{7}{20}\alpha, \frac{11}{120} + \frac{13}{20}\alpha\right).$$

Furthermore, for any $49/114 \le \alpha \le 1/2$*, one has*

$$\beta(\alpha) \le \frac{89}{570} + \frac{1}{2}\alpha.$$

Recorded in literature.py as:
|add$_beta_bound_huxley_1993()$|

Proof. See [**?**, Theorem 1]. □

Theorem 4.13 (Second 1993 Huxley bound)**.** *beta-def If* $0 \le \alpha \le 1$*, then* $\beta(\alpha)$ *is bounded by*

$$\begin{aligned}
\frac{1}{146}(13+94\alpha) &\textit{ for } \alpha \le \frac{87}{275}\\
\frac{1}{244}(11+191\alpha) &\textit{ for } \frac{87}{275} \le \alpha \le \frac{423}{1295}\\
\frac{1}{1282}(89+908\alpha) &\textit{ for } \frac{423}{1295} \le \alpha \le \frac{227}{601}\\
\frac{1}{280}(29+173\alpha) &\textit{ for } \frac{227}{601} \le \alpha \le \frac{12}{31}\\
\frac{1}{128}(4+103\alpha) &\textit{ for } \frac{12}{31} \le \alpha \le 1.
\end{aligned}$$

Recorded in literature.py as:
|add$_beta_bound_huxley_1993_3()$|

Proof. See [**?**, Theorem 3]. □

Theorem 4.14 (1996 Huxley table). *One can bound $\beta(\alpha)$ by*

$$\begin{aligned}
&\frac{4+39\alpha}{60} \text{ for } \frac{7}{12} \le \alpha \le \frac{517}{873}\\
&\frac{29+42\alpha}{120} \text{ for } \frac{65}{114} \le \alpha \le \frac{7}{12}\\
&\frac{89+285\alpha}{570} \text{ for } \frac{49}{114} \le \alpha \le \frac{65}{114}\\
&\frac{11+78\alpha}{120} \text{ for } \frac{5}{12} \le \alpha \le \frac{49}{114}\\
&\frac{13+21\alpha}{60} \text{ for } \frac{356}{873} \le \alpha \le \frac{5}{12}\\
&\frac{4+103\alpha}{128} \text{ for } \frac{12}{31} \le \alpha \le \frac{356}{873}\\
&\frac{29+173\alpha}{280} \text{ for } \frac{227}{601} \le \alpha \le \frac{12}{31}\\
&\frac{89+908\alpha}{1282} \text{ for } \frac{423}{1295} \le \alpha \le \frac{227}{601}\\
&\frac{11+191\alpha}{244} \text{ for } \frac{87}{275} \le \alpha \le \frac{423}{1295}\\
&\frac{13+94\alpha}{146} \text{ for } \frac{1424}{4747} \le \alpha \le \frac{87}{275}\\
&\frac{4+235\alpha}{264} \text{ for } \frac{120}{419} \le \alpha \le \frac{1424}{4747}\\
&\frac{49+1351\alpha}{1614} \text{ for } \frac{967}{3428} \le \alpha \le \frac{120}{419}\\
&\frac{29+464\alpha}{600} \text{ for } \frac{199}{716} \le \alpha \le \frac{967}{3428}\\
&\frac{89+2243\alpha}{2706} \text{ for } \frac{19}{74} \le \alpha \le \frac{199}{716}\\
&\frac{11+428\alpha}{492} \text{ for } \frac{161}{646} \le \alpha \le \frac{19}{74}\\
&\frac{13+253\alpha}{318} \text{ for } \frac{2848}{12173} \le \alpha \le \frac{161}{646}.
\end{aligned}$$

and also

$$\begin{aligned}
\frac{89+285\alpha}{570} &\quad \textit{for}\ \frac{106822}{246639} \le \alpha \le \frac{139817}{246639}\\
\frac{2387+17972\alpha}{27290} &\quad \textit{for}\ \frac{1033325}{2642746} \le \alpha \le \frac{106822}{246639}\\
\frac{2819+19177\alpha}{29855} &\quad \textit{for}\ \frac{699371}{1647930} \le \alpha \le \frac{1033325}{2642746}\\
\frac{11897+88442\alpha}{134680} &\quad \textit{for}\ \frac{156527}{370694} \le \alpha \le \frac{699371}{1647930}\\
\frac{113+897\alpha}{1345} &\quad \textit{for}\ \frac{263}{638} \le \alpha \le \frac{156527}{370694}\\
\frac{491+3624\alpha}{5530} &\quad \textit{for}\ \frac{143}{349} \le \alpha \le \frac{263}{638}\\
\frac{569+1053\alpha}{2800} &\quad \textit{for}\ \frac{307}{761} \le \alpha \le \frac{143}{349}\\
\frac{1273+2484\alpha}{6410} &\quad \textit{for}\ \frac{68682}{171139} \le \alpha \le \frac{307}{761}\\
\frac{4+103\alpha}{128} &\quad \textit{for}\ \frac{12}{31} \le \alpha \le \frac{68682}{171139}\\
\frac{29+173\alpha}{280} &\quad \textit{for}\ \frac{227}{601} \le \alpha \le \frac{12}{31}.
\end{aligned}$$

Recorded in literature.py as:
|add$_b eta_b ound_h uxley_1 996()$|
|$add_b eta_b ound_h uxley_1 996_2()$|

Proof. See [?, Table 17.1, Table 19.2]. ☐

Theorem 4.15 (2001 Huxley–Kolesnik bound)**.** *beta-def For any* $2/5 \le \alpha \le 1/2$ *one has*

$$\beta(\alpha) \le \max\left(\frac{7}{80} + \frac{79}{120}\alpha, \frac{3}{32} + \frac{103}{160}\alpha, \frac{9}{40} + \frac{13}{40}\alpha\right).$$

Recorded in literature.py as:
|add$_b eta_b ound_h uxley_k olesnik_2 001()$|

Proof. See [?, Theorem 1]. ☐

Theorem 4.16 (Robert–Sargos bound)**.** *beta-def For any* $\alpha > 0$ *one has*

$$\beta(\alpha) \le \max\left(\alpha + \frac{1-4\alpha}{13}, -\frac{7(1-4\alpha)}{13}\right).$$

Recorded in literature.py as:
|add$_b eta_b ound_r obert_s argos_2 002()$|

Proof. See [?, Theorem 1]. ☐

Theorem 4.17 (Sargos bound)**.** *For any $\alpha > 0$ one has*

$$\beta(\alpha) \leq \max\left(\alpha + \frac{1-8\alpha}{204}, -\frac{95(1-8\alpha)}{204}\right)$$

and

$$\beta(\alpha) \leq \max\left(\alpha + \frac{7(1-9\alpha)}{2640}, -\frac{1001(1-9\alpha)}{2640}\right).$$

Recorded in literature.py as:
|add$_beta_bound_sargos_2003()$|

Proof. See [**?**, Theorems 3, 4]. □

Theorem 4.18 (Huxley bound)**.** *beta-def For any $1/3 \leq \alpha \leq 1/2$, one has*

$$\beta(\alpha) \leq \max\left(\frac{37+59\alpha}{170}, \frac{63+449\alpha}{690}\right).$$

Recorded in literature.py as:
|add$_beta_bound_huxley_2005()$|

Proof. See [**?**, Proposition 1, Theorem 1]. □

Theorem 4.19 (Robert bound)**.** *beta-def For any $0 < \alpha \leq 3/7$ one has*

$$\beta(\alpha) \leq \max\left(\alpha + \frac{1-4\alpha}{12}, \frac{11}{12}\alpha\right).$$

Recorded in literature.py as:
|add$_beta_bound_robert_2016()$|

Proof. See [**?**, Theorem 1]. □

Theorem 4.20 (Second Robert bound)**.** *If $k \geq 4$ and $\alpha \geq -(1-k\alpha)\frac{k-1}{2k-3}$ then*

$$\beta(\alpha) \leq \alpha + \max(\frac{1-k\alpha}{2(k-1)(k-2)}, -\frac{1}{2(k-1)(k-2)}).$$

Recorded in literature.py as:
|add$_beta_bound_robert_2016_2(Constants.BETA_TRUNCATION)$|

Proof. See [**?**, Theorem 10]. □

Theorem 4.21 (Heath-Brown bound)**.** *beta-def For any $\alpha > 0$ and any natural number $k \geq 3$ one has*

$$\beta(\alpha) \leq \alpha + \max\left(\frac{1-k\alpha}{k(k-1)}, -\frac{\alpha}{k(k-1)}, -\frac{2\alpha}{k(k-1)} - \frac{2(1-k\alpha)}{k^2(k-1)}\right).$$

Recorded in literature.py as:
|add$_beta_bound_heath_brown_2017(Constants.BETA_TRUNCATION)$|

Proof. See [**?**, Theorem 1]. ☐

Theorem 4.22 (Bourgain bound)**.** *beta-def One has*

$$\beta(\alpha) \leq \begin{cases} \dfrac{2}{9} + \dfrac{1}{3}\alpha, & \dfrac{1}{3} < \alpha \leq \dfrac{5}{12}, \\[2ex] \dfrac{1}{12} + \dfrac{2}{3}\alpha, & \dfrac{5}{12} < \alpha \leq \dfrac{3}{7}, \\[2ex] \dfrac{13}{84} + \dfrac{1}{2}\alpha, & \dfrac{3}{7} < \alpha \leq \dfrac{1}{2}. \end{cases}$$

`Recorded in literature.py as:`
|`add`$_beta_bound_bourgain_2017()|$

Proof. See [**?**, Equation (3.18)]. ☐

TODO: provide a graphic of the best bounds available of beta, compared against the classical van der Corput bounds

Chapter 5

Exponent pairs

Definition 5.1 (Exponent pair)**.** An exponent pair is a (fixed) element (k, ℓ) of the triangle

$$\{(k, \ell) \in \mathbf{R}^2 : 0 \leq k \leq 1/2 \leq \ell \leq 1, k + \ell \leq 1\} \tag{5.1}$$

with the following property: for all model phase functions F, all $T \geq N \leq 1$, and all intervals $I \subset [N, 2N]$, one has

$$\sum_{n \in I} e(TF(n/N)) \ll (T/N)^{k+o(1)} N^{\ell+o(1)} \tag{5.2}$$

whenever $T \geq N \geq 1$, I is an interval in $[N, 2N]$, and $F \in \mathcal{U}$.

`Implemented at |exponent`$_p air|.py$ `as:`
`|Exp`$_p air|$
`One can formulate the notion of an exponent pair without recourse to asymptotic notation:`

Lemma 5.2 (Non-asymptotic definition of exponent pair)**.** *Let (k, ℓ) be a fixed element of (5.1). Then the following are equivalent:*

(i) (k, ℓ) is an exponent pair.

(ii) For every (fixed) $\varepsilon > 0$ there exist (fixed) $C, P > 0$ such that, whenever $T \geq N \geq 1$, $I \subset [N, 2N]$, and F is a phase function obeying (4.3) for for all (fixed) $0 \leq p \leq P$ and $u \in [1, 2]$, then

$$|\sum_{n \in I} e(TF(n/N))| \leq C(T/N)^{k+\varepsilon} N^{\ell+\varepsilon}.$$

`The proof of this lemma is similar to that of Lemma 4.3 and is omitted.`
`Exponent pairs are closely related to the function` β `from the previous chapter:`

Lemma 5.3 (Duality between exponent pairs and β)**.** *Let* (k, ℓ) *be in the triangle* (5.1)*. Then the following are equivalent:*

(i) (k, ℓ) *is an exponent pair.*

(ii) $\beta(\alpha) \leq k + (\ell - k)\alpha$ *for all* $0 \leq \alpha \leq 1$.

Implemented at |exponent$_p air|.py$ as:
|exponent$_p airs_t o_b eta_b ounds()||beta_b ounds_t o_e xponent_p airs()|$

Thus exponent pairs are dual to the convex hull of the graph of β. But β is not known to be convex, so one could have bounds on β that do not directly correspond to exponent pairs.

Proof. If (i) holds, then for any $0 < \alpha < 1$, any unbounded $T \geq 1$, any $N = T^{\alpha+o(1)}$, interval $I \subset [N, 2N]$, and model phase function F, we have from (i) that

$$\sum_{n \in I} e(TF(n/N)) \ll (T/N)^{k+o(1)} N^{\ell+o(1)} = T^{k+(\ell-k)\alpha+o(1)}.$$

From Definition 4.2 we conclude that $\beta(\alpha) \leq k + (\ell - k)\alpha$. Also since (k, ℓ) lies in (5.1), we see from (4.4), (4.6) that we also have $\beta(\alpha) \leq k + (\ell - k)\alpha$ for $\alpha = 0, 1$.

Now suppose that (ii) holds. Let F, T, N, I be as in Definition 5.1. By underspill it suffices to show that

$$\sum_{n \in I} e(TF(n/N)) \ll (T/N)^{k+\varepsilon+o(1)} N^{\ell+\varepsilon+o(1)}$$

for any fixed $\varepsilon > 0$. We may assume that $T \leq N^{1/\varepsilon+1}$, since the claim follows from the trivial bound $\sum_{n \in I} e(TF(n/N)) \ll N$ otherwise. We may also assume that N is unbounded, since the claim is clear for N bounded; this forces T to be unbounded as well.

By passing to a subsequence we may assume that $N = T^{\alpha+o(1)}$ for some fixed $0 \leq \alpha \leq 1$. By Definition 4.2 we then have

$$\sum_{n \in I} e(TF(n/N)) \ll T^{\beta(\alpha)+o(1)}$$

and hence by (ii)

$$\sum_{n \in I} e(TF(n/N)) \ll (T/N)^{k+o(1)} N^{\ell+o(1)}$$

giving the claim. $\square$

Corollary 5.4 (Exponent pairs closed and convex)**.** *exp-pair-def The set of exponent pairs is closed and convex.*

Proof. Immediate from Lemma 5.3. $\square$

Proposition 5.5 (Trivial exponent pairs)**.** *exp-pair-def* $(0,1)$ *and* $(1/2,1/2)$ *are exponent pairs.*

Proof. Immediate from Lemma 5.3 and Lemma 4.4. ☐

Conjecture 5.6 (Exponent pairs conjecture)**.** $(0,1/2)$ *is an exponent pair. (Equivalently, by Corollary 5.4, every point in the triangle* (5.1) *is an exponent pair.)*

Implemented at |exponent$_p air|.py$ as:
|exponent$_p air_c onjecture|$

Lemma 5.7. *The exponent pair conjecture is equivalent to* $\beta(\alpha) = \alpha/2$ *holding true for all* $0 \le \alpha \le 1$*.*

Proof. Clear from Lemma 5.3 and Lemma 4.4. ☐

Proposition 5.8 (Van der Corput A-process)**.** *If* (k,ℓ) *is an exponent pair, then so is*

$$A(k,\ell) := \left(\frac{k}{2k+2}, \frac{\ell}{2k+2} + \frac{1}{2}\right).$$

Recorded in literature.py as:
|A$_t ransform()|$

Proof. See [**?**, Lemma 2.8]. ☐

TODO: determine the analogous A-process for $\beta()$ and state it as a lemma.

Proposition 5.9 (Van der Corput B-process)**.** *If* (k,ℓ) *is an exponent pair, then so is*

$$B(k,\ell) := \left(\ell - \frac{1}{2}, k + \frac{1}{2}\right).$$

Recorded in literature.py as:
|B$_t ransform()|$

Proof. See [**?**, Lemma 2.9]. Alternatively, this can be derived from Lemma 4.9 and Lemma 5.3. ☐

5.1 Known exponent pairs

Proposition 5.10 (Classical van der Corput exponent pairs)**.** *For any natural number* $k \ge 2$*,*

$$A^{k-2}B(0,1) = \left(\frac{1}{2^k - 2}, 1 - \frac{k-1}{2^k-2}\right)$$

is an exponent pair. In particular,

$$\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{6}, \frac{2}{3}\right), \left(\frac{1}{14}, \frac{11}{14}\right)$$

are exponent pairs.

Proof. Follows by induction from Proposition 5.8 and Proposition 5.5; alternatively, follows from (and is equivalent to) Corollary 4.7 and Lemma 5.3. □

Derived in derived.py as:
|van$_d er_c orput_p air()$|

Theorem 5.11 (Exponent pairs on the line of symmetry). $(k, k+1/2)$ *is an exponent pair for*

(i) $k = 9/56$ *[?, Theorem 1];*

(ii) $k = 89/560$ *[?, Theorem 6];*

(iii) $k = 17/108$ *[?, p. 467];*

(iv) $k = 89/570$ *[?, p. 40];*

(v) $k = 32/205$ *[?, Theorem 1];*

(vi) $k = 13/84$ *[?, p. 206].*

Recorded in literature.py as:
|add$_l iterature_e xponent_p airs()$|

Theorem 5.12 (Exponent pairs from the Bombieri–Iwaniec method). *The following pairs are exponent pairs:*

(i) $(\frac{2}{13}, \frac{35}{52})$ *[?];*

(ii) $(\frac{6299}{43860}, \frac{29507}{43860})$ *[?, Table 17.3];*

(iii) $(\frac{771}{8116}, \frac{1499}{2029})$ *[?, p. 285];*

(iv) $(\frac{21}{232}, \frac{173}{232})$ *[?, p. 286];*

(v) $(\frac{1959}{21656}, \frac{16135}{21656})$ *[?, p. 286];*

(vi) $(\frac{516247}{6629696}, \frac{5080955}{6629696})$ *[?], [?, Table 19.2], [?].*

Recorded in literature.py as:
|add$_l iterature_e xponent_p airs()$|

Theorem 5.13 (Exponent pairs from derivative tests). $(k, 1-mk)$ *is an exponent pair when*

(i) $k = \frac{1}{13}$ *and* $m = 3$ *[?, Theorem 1];*

(ii) $k = \frac{1}{204}$ *and* $m = 7$ *[?, p. 231];*

(iii) $k = \frac{1}{130}$ *and* $m = 8$ *[?, (1.1)];*

(iv) $k = \frac{7}{2640}$ *and* $m = 8$ *[?, p. 231];*

(v) $k = \frac{1}{716}$ *and* $m = 9$ *[?, p. 231];*

(vi) $k = \frac{1}{649}$ *and* $m = 9$ *[?];*

(vii) $k = \frac{7}{4540}$ *and* $m = 9$ *[?, (1.2)];*

(viii) $k = \frac{1}{615}$ *and* $m = 9$ *[?, (1.1)];*

(ix) $k = \frac{1}{915}$ *and* $m = 10$ *[?, Théorème 2].*

Recorded in literature.py as:
$|\texttt{add}_literature_exponent_pairs()|$

Theorem 5.14 (Huxley sequence). *[?, Table 17.3] For any integer $m \geq 1$, the pair*

$$\left(\frac{169}{1424 \times 2^m - 338}, 1 - \frac{169}{1424 \times 2^m - 338}\frac{712m + 1577}{712}\right)$$

is an exponent pair.

Recorded in literature.py as:
$|\texttt{add}_huxley_exponent_pairs(Constants.EXP_PAIR_TRUNCATION)|$

Theorem 5.15 (1996 Heath–Brown sequence). *[?, (6.17.4)] For any integer $m \geq 3$, the pair*

$$\left(\frac{1}{25m^2(m-2)\log m}, 1 - \frac{1}{25m^2(m-2)\log m}\right)$$

is an exponent pair.

(Currently not implemented in python due to the irrational exponents.)

Theorem 5.16 (2017 Heath–Brown sequence). *[?, Theorem 2] For any integer $m \geq 3$, the pair*

$$\left(\frac{2}{(m-1)^2(m+2)}, 1 - \frac{3m-2}{m(m-1)(m+2)}\right)$$

is an exponent pair.

Recorded in literature.py as:
$|\texttt{add}_heath_brown_exponent_pairs(Constants.EXP_PAIR_TRUNCATION)|$

Proof. This follows from Theorem 4.21 and Lemma 5.3, after some computation. □

Theorem 5.17 (Sargos C-process). *[?, Theorem 5] If (k, ℓ) is an exponent pair, then so is*

$$\left(\frac{k}{12(1+4k)}, \frac{11(1+4k)+\ell}{12(1+4k)}\right).$$

Recorded in literature.py as:
$|\texttt{C}_transform()|$

TODO: implement our new exponent pair $(1101653/15854002, 12327829, 15854002)$

Chapter 6

Growth exponents for the Riemann zeta function

Definition 6.1 (Growth rate of zeta). For any fixed $\sigma \in \mathbf{R}$, let $\mu(\sigma)$ denote the least possible (fixed) exponent for which one has the bound

$$|\zeta(\sigma + it)| \ll |t|^{\mu(\sigma)+o(1)}$$

for all unbounded t.

`One can check that for each` σ`, the set of possible candidates for` $\mu(\sigma)$ `is closed (by underspill), non-empty, and bounded from below, so that` $\mu(\sigma)$ `is well-defined as a real number. An equivalent definition without asymptotic notation, is that` $\mu(\sigma)$ `is the least real number such that for every` $\varepsilon > 0$ `there exists` $C > 0$ `such that`

$$|\zeta(\sigma + it)| \ll C|t|^{\mu(\sigma)+\varepsilon}$$

`for all` t `with` $|t| \geq C$`; equivalently, one has`

$$\mu(\sigma) = \limsup_{|t|\to\infty} \frac{\log|\zeta(\sigma + it)|}{\log|t|}.$$

`Implemented at` *`|bound`*$_m u|.py$ `as:`
`|Bound`$_m u|$

Lemma 6.2 (Trivial bound). *zeta-grow-def One has* $\mu(\sigma) = 0$ *for all* $\sigma \geq 1$.

`Implemented at` *`|bound`*$_m u|.py$ `as:`
`|apply`$_t rivial_m u_b ound()|$

Proof. Immediate from the absolute convergence of the Dirichlet series for both $\zeta(s)$ and $1/\zeta(s)$; see e.g., [**?**, Theorem 1.9]. ☐

Lemma 6.3 (Convexity). *zeta-grow-def* μ *is convex.*

Implemented at |bound$_m u|.py$ as:
|bound$_m u_c onvexity()|$

Proof. Immediate from the Phragmén–Lindelöf principle; see e.g., [**?**, §A.8]. ☐

Lemma 6.4 (Functional equation)**.** *zeta-grow-def One has* $\mu(1-\sigma) = \mu(\sigma) + \sigma - 1/2$ *for all* $0 \leq \sigma \leq 1/2$.

Implemented at |bound$_m u|.py$ as:
|apply$_f unctional_e quation()|$

Proof. Immediate from the functional equation for ζ and asymptotics of the Gamma function; see e.g., [**?**, (1.23), (1.25)]. ☐

Lemma 6.5 (Left of critical strip)**.** *zeta-grow-def One has* $\mu(\sigma) = 1/2 - \sigma$ *for* $\sigma \leq 0$.

Implemented at |bound$_m u|.py$ as:
|apply$_t rivial_m u_b ound()|$

Proof. zeta-grow-triv, zeta-functional Immediate from Lemmas 6.2, 6.4. ☐

Lemma 6.6 (Convexity bounds)**.** *zeta-grow-def One has* $\max(0, 1/2 - \sigma) \leq \mu(\sigma) \leq (1-\sigma)/2$ *for* $0 \leq \sigma \leq 1$.

Implemented at |bound$_m u|.py$ as:
|apply$_t rivial_m u_b ound()|$

Proof. zeta-grow-triv, zeta-convex, zeta-left Immediate from Lemma 6.2, Lemma 6.5, and Lemma 6.6. ☐

6.1 Connection with exponent pairs and dual exponent pairs

Lemma 6.7 (Connection with dual exponent pairs)**.** *For any* $1/2 \leq \sigma \leq 1$*, one has*

$$\mu(\sigma) \leq \sup_{0 \leq \alpha \leq 1/2} \beta(\alpha) - \alpha\sigma.$$

Proof. Let t be unbounded. From the Riemann–Siegel formula (see [**?**, Theorem 4.1]) one has

$$\zeta(\sigma + it) \ll \left| \sum_{n \leq \sqrt{t/2\pi}} \frac{1}{n^{\sigma+it}} \right| + |t|^{1/2-\sigma} \left| \sum_{n \leq \sqrt{t/2\pi}} \frac{1}{n^{1-\sigma-it}} \right| + O(1).$$

From dyadic decomposition and Definition 4.2 (and Lemma 2.1) one has for any fixed $\varepsilon > 0$ that

$$\sum_{t^\varepsilon \leq n \leq \sqrt{t/2\pi}} \frac{1}{n^{\sigma+it}} \ll |t|^{\sup_{\varepsilon \leq \alpha \leq 1/2} \beta(\alpha) - \alpha\sigma + o(1)},$$

while from the triangle inequality one has the crude bound

$$\sum_{n < t^\varepsilon} \frac{1}{n^{\sigma+it}} \ll |t|^\varepsilon.$$

Combining the bounds and using underspill, we conclude that

$$\sum_{n \leq \sqrt{t/2\pi}} \frac{1}{n^{\sigma+it}} \ll |t|^{\sup_{0 \leq \alpha \leq 1/2} \beta(\alpha) - \alpha\sigma + o(1)}.$$

A similar argument gives

$$\sum_{n \leq \sqrt{t/2\pi}} \frac{1}{n^{1-\sigma-it}} \ll |t|^{\sup_{0 \leq \alpha \leq 1/2} \beta(\alpha) - \alpha(1-\sigma) + o(1)}$$

Since $\sigma \geq 1/2$ and $\alpha \leq 1/2$, one has $(1/2 - \sigma) - \alpha(1-\sigma) \leq -\alpha\sigma$, and hence

$$\zeta(\sigma + it) \ll |t|^{\sup_{0 \leq \alpha \leq 1/2} \beta(\alpha) - \alpha\sigma + o(1)}$$

giving the claim. □

We remark that this inequality is morally an equality (indeed, it would be if one would restrict the model phases in Definition 4.2 to purely the logarithmic phase $u \mapsto \log u$).

The following form of Lemma 6.7 is convenient for applications:

Corollary 6.8 (Exponent pairs and μ). *If (k, ℓ) is an exponent pair, then*

$$\mu(\ell - k) \leq k.$$

Implemented at |bound$_m u|.py$ as:
|obtain$_m u_b ound_f rom_e xponent_p air()|$

Proof. Immediate from Lemma 6.7 and Lemma 5.3. See also [**?**, (7.57)]. □

Conjecture 6.9 (Lindelof hypothesis). *zeta-grow-def One has $\mu(1/2) = 0$.*

Implemented at |bound$_m u|.py$ as:
|bound$_m u_L indelof()|$

Proposition 6.10 (Conjectured value of μ)**.** *zeta-grow-def, LH We have the lower bound*

$$\mu(\sigma) \geq \max\left(0, \frac{1}{2} - \sigma\right) \tag{6.1}$$

for all $\sigma \in \mathbf{R}$, and equality holds everywhere in (6.1) *if and only if the Lindelöf hypothesis holds.*

`We remark that this proposition explains why there are no further lower bounds on` μ `in the literature beyond` (6.1)`; all the remaining known results revolve around upper bounds.`

Proof. zeta-grow-triv, zeta-left, zeta-convexity Clearly equality in (6.1) implies the Lindelöf hypothesis, while from the trivial bounds in Propositions 6.2, 6.5 and convexity (Lemma 6.6) one we see that the Lindelöf hypothesis implies the upper bound

$$\mu(\sigma) \leq \max\left(0, \frac{1}{2} - \sigma\right)$$

for all σ. So it suffices to establish the lower bound unconditionally. By the functional equation (Proposition 6.4) it suffices to do this for $\sigma \geq 1/2$; in fact by convexity it suffices to establish the claim when $1/2 < \sigma < 1$. In this regime, the L^2 mean value theorem (see [**?**, Theorem 1.11]) gives

$$\int_0^T |\zeta(\sigma + it)|^2 \, dt \asymp T$$

for large T, giving the claim. □

6.2 Known bounds on μ

`Recorded in literature.py as:`
`|add`$_l iterature_b ounds_m u()|$

Table 6.1: Historical bounds on $\mu(\sigma)$ for $1/2 \le \sigma \le 1$, and the exponent pair generating them (if applicable). **Longer term goal: supplement as many of these citations as possible with derivations from other exponents and relations in the database**

Reference	Results	Exponent pair
Hardy–Littlewood (1923) [?]	$\mu(1/2) \le 1/6$	(1/6, 2/3)
Walfisz (1924) [?]	$\mu(1/2) \le 193/988$	
Titchmarsh (1932) [?]	$\mu(1/2) \le 27/164$	
Phillips (1933) [?]	$\mu(1/2) \le 229/1392$	
Titchmarsh (1942) [?]	$\mu(1/2) \le 19/116$	
Min (1949) [?]	$\mu(1/2) \le 15/92$	
Haneke (1962) [?]	$\mu(1/2) \le 6/37$	
Kolesnik (1973) [?]	$\mu(1/2) \le 173/1067$	
Kolesnik (1982) [?]	$\mu(1/2) \le 35/216$	
Kolesnik (1985) [?]	$\mu(1/2) \le 139/858$	
Bombieri–Iwaniec (1985) [?]	$\mu(1/2) \le 9/56$	$(9/56, 1/2 + 9/56)$
Watt (1989) [?]	$\mu(1/2) \le 89/560$	$(89/560, 1/2 + 89/560)$
Huxley–Kolesnik (1991) [?]	$\mu(1/2) \le 17/108$	$(17/108, 1/2 + 17/108)$
Huxley (1993) [?]	$\mu(1/2) \le 89/570$	$(89/570, 1/2 + 89/570)$
Huxley (1996) [?]	$\mu(1934/3655) \le 6299/43860$	
Sargos (2003) [?]	$\mu(49/51) \le 1/204$, $\mu(361/370) \le 1/370$	
Huxley (2005) [?]	$\mu(1/2) \le 32/205$	$(32/205, 1/2 + 32/205)$
Lelechenko (2014) [?]	$\mu(3/5) \le 1409/12170$, $\mu(4/5) \le 3/71$	
Bourgain (2017) [?]	$\mu(1/2) \le 13/84$	$(13/84, 1/2 + 13/84)$
Heath-Brown (2017) [?]	$\mu(\sigma) \le \frac{8}{63}\sqrt{15}(1-\sigma)^{3/2}$ for $1/2 \le \sigma \le 1$	
Heath-Brown (2020) [?]	$\mu(11/15) \le 1/15$	

Chapter 7

Large value estimates

Definition 7.1 (Large value exponent)**.** Let $1/2 \leq \sigma \leq 1$ and $\tau \geq 0$ be fixed. We define $\mathrm{LV}(\sigma, \tau)$ to be the least fixed quantity for which the following claim is true: whenever $N > 1$ is unbounded, $T = N^{\tau+o(1)}$, $V = N^{\sigma+o(1)}$, a_n is a 1-bounded complex number for each $n \in [N, 2N]$, and $t_1, \ldots, t_R$ are a 1-separated sequence in some interval J of length T such that

$$\left| \sum_{n \in [N,2N]} a_n n^{-it_r} \right| \geq V \tag{7.1}$$

for all $r = 1, \ldots, R$, then one has

$$R \ll N^{\mathrm{LV}(\sigma,\tau)+o(1)}.$$

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`One can check that the set of possible candidates for` $\mathrm{LV}(\sigma,\tau)$ `is closed (by underspill), non-empty, and bounded from below, so` $\mathrm{LV}(\sigma,\tau)$ `is well-defined as a real number. As usual, we have an equivalent non-asymptotic definition:`

Lemma 7.2 (Asymptotic form of large value exponent)**.** *Let $1/2 \leq \sigma \leq 1$, $\tau \geq 0$, and $\rho \geq 0$ be fixed. Then the following are equivalent:*

(i) $\mathrm{LV}(\sigma, \tau) \leq \rho$.

(ii) For every (fixed) $\varepsilon > 0$ there exists $C, \delta > 0$ such that if $N \geq C$ and $N^{\tau-\delta} \leq T \leq N^{\tau+\delta}$, $N^{\sigma-\delta} \leq V \leq N^{\sigma+\delta}$, a_n is a 1-bounded complex number for each $n \in [N, 2N]$, and $t_1, \ldots, t_R$ is a 1-separated subset of an interval J of length T such that (7.1) *holds for all $r = 1, \ldots, R$, then one has*

$$R \leq CN^{\rho+\varepsilon}.$$

The proof of Lemma 7.2 is similar to that of Lemma 4.3, and is left to the reader.

Lemma 7.3 (Basic properties). *(i) (Monotonicity in σ) For any $\tau \geq 0$, $\sigma \mapsto \mathrm{LV}(\sigma,\tau)$ is upper semicontinuous and monotone non-increasing.*

(ii) (Huxley subdivision) For any $1/2 \leq \sigma \leq 1$ and $\tau' \geq \tau$ one has

$$\mathrm{LV}(\sigma,\tau) \leq \mathrm{LV}(\sigma,\tau') \leq \mathrm{LV}(\sigma,\tau) + \tau' - \tau.$$

In particular, $\tau \mapsto \mathrm{LV}(\sigma,\tau)$ is Lipschitz continuous.

(iii) ($\tau = 0$ endpoint) One has $\mathrm{LV}(\sigma,0) = 0$ for all $1/2 \leq \sigma \leq 1$, and hence by (ii) $0 \leq \mathrm{LV}(\sigma,\tau) \leq \tau$ for all $1/2 \leq \sigma \leq 1$ and $\tau \geq 0$.

TODO: implement Huxley subdivision as a way to transform a large values estimate into a better estimate

Proof. All claims are clear (using Lemma 7.2 if necessary) except perhaps for the upper bound

$$\mathrm{LV}(\sigma,\tau') \leq \mathrm{LV}(\sigma,\tau) + \tau' - \tau,$$

but this follows because any interval of length $N^{\tau'+o(1)}$ may be subdivided into $N^{\tau'-\tau+o(1)}$ intervals of length $N^{\tau+o(1)}$, so on applying Lemma 7.2 to each subinterval and summing, one obtains the claim. □

Lemma 7.4 (Lower bound). *For any $1/2 < \sigma \leq 1$ and $\tau \geq 0$, one has $\mathrm{LV}(\sigma,\tau) \geq \min(2-2\sigma,\tau)$, while for $\sigma = 1/2$ one has $\mathrm{LV}(\sigma,\tau) = \tau$.*

Proof. In view of Lemma 7.3(ii), it suffices to show that $\mathrm{LV}(\sigma, 2-2\sigma) \geq 2-2\sigma$. By definition, it suffices to find an unbounded N and a 1-bounded sequence a_n for $n \in [N,2N]$ such that $|\sum_{n\in[N,2N]} a_n n^{-it}| \geq N^{\sigma+o(1)}$ for a 1-separated set of $t = O(N^{2-2\sigma+o(1)})$ of cardinality $\gg N^{2-2\sigma-o(1)}$.

In the endpoint case $\sigma = 1$ one can achieve this by setting $a_n = 1$ for all n and taking $t = 0$, so now we assume that $1/2 < \sigma < 1$.

We use the probabilistic method. We divide $[N,2N]$ into $\asymp N^{2-2\sigma}$ intervals I of length $\asymp N^{2\sigma-1}$. On each interval I, we choose a_n to equal some randomly chosen sign $\epsilon_I \in \{-1,+1\}$, with the ϵ_I chosen independently in I. If $t = o(N^{2-2\sigma})$, then $\sum_{n\in I} a_n n^{-it}$ is equal to ϵ_I times a deterministic quantity $c_{t,I}$ of magnitude $\asymp N^{2\sigma-1}$ (the point being that the phase $t\log n$ is close to constant in this range). By the Chernoff bound, we thus see that for any such t, $\sum_{n\in[N,2N]} a_n n^{it}$ will have size $\gg N^{(2\sigma-1)+(2-2\sigma)/2} = N^{\sigma}$ with probability $\gg 1$. By linearity of expectation, we thus see that with positive probability, a $\gg 1$ fraction of integers t with $t = o(N^{2-2\sigma})$ will have this property, giving the claim.

Finally, let $\sigma = 1/2$. In this case we just take each a_n to be a random sign, then by the Chernoff bound one has for each t that $|\sum_{n\in[N,2N]} a_n n^{it}| \asymp N^{1/2}$ with positive probability, which by linearity of expectation as before gives the lower bound $\mathrm{LV}(\sigma,\tau) \geq \tau$, while the upper bound is trivial from Lemma 7.3(iii). □

We conjecturally have a complete description of the function LV:

Conjecture 7.5 (Montgomery conjecture)**.** *One has*

$$\mathrm{LV}(\sigma,\tau) \leq 2-2\sigma \tag{7.2}$$

for all fixed $1/2 < \sigma \leq 1$ *and* $\tau \geq 0$. *Equivalently (by Lemma 7.3(ii), (iii) and Lemma 7.4), one has* $\mathrm{LV}(\sigma,\tau) = \min(2-2\sigma,\tau)$ *for all* $1/2 < \sigma \leq 1$ *and* $\tau \geq 0$.

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We refer to [?] for further discussion of this conjecture, including some counterexamples to strong versions of the conjecture in which certain epsilon losses are omitted. In view of this conjecture, we do not expect any further lower bounds on $\mathrm{LV}(\sigma,\tau)$ to be proven, and the literature is instead focused on upper bounds.

In many situations, we are able to estalbish the Montgomery conjecture for certain fixed σ, and all $\tau < \tau_0$ for some fixed τ_0. By subdivision (Lemma 7.3(ii)) we then obtain the bound

$$\mathrm{LV}(\sigma,\tau) \leq \max(2-2\sigma, \tau-\tau_0+2-2\sigma) \tag{7.3}$$

for all $\tau \geq 0$.

The following basic property of $\mathrm{LV}(\sigma,\tau)$ is extremely useful in applications:

Lemma 7.6 (Raising to a power)**.** *For any* $1/2 \leq \sigma \leq 1$, $\tau \geq 0$, *and natural number* k, *one has*

$$\mathrm{LV}(\sigma,k\tau) \leq k\mathrm{LV}(\sigma,\tau).$$

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Proof. Let N be unbounded, let $T = N^{k\tau+o(1)}$ and $V = N^{\sigma+o(1)}$, let a_n be a 1-bounded sequence on $[N,2N]$, and let $t_1,\dots,t_R$ be a 1-separated sequence in an interval of length T obeying (7.1) for all $r = 1,\dots,R$. Raising to the k^{th} power, we conclude that

$$\left|\sum_{n\in[N^k,2^kN^k]} b_n n^{-it_r}\right| \geq V^k$$

where b_n is the Dirichlet convolution of k copies of a_n, and thus is bounded by $N^{o(1)}$ thanks to divisor bounds. Subdividing $[N^k,2^kN^k]$ into k intervals of the form $[N',2N']$ for $N' \asymp N^k$ and applying Lemma 7.2 (with N,T,V replaced by N',T,V^k) we conclude that

$$R \ll N^{k\mathrm{LV}(\sigma,\tau)+o(1)}$$

and the claim then follows from Lemma 7.2. ☐

7.1 Known upper bounds on $\mathrm{LV}(\sigma,\tau)$

Upper bounds on $\mathrm{LV}(\sigma,\tau)$ in the literature tend to be piecewise linear functions of σ and τ. Listed below are some examples of such bounds.

Theorem 7.7 (L^2 mean value theorem). *For any fixed $1/2 \leq \sigma \leq 1$ and $\tau \geq 0$ one has*

$$\mathrm{LV}(\sigma,\tau) \leq \max(2-2\sigma, 1+\tau-2\sigma).$$

In particular, the Montgomery conjecture holds (7.2) for $\tau \leq 1$.

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Proof. Let N be unbounded, let and let $T = N^{\tau+o(1)}$, $V = N^{\sigma+o(1)}$, let a_n be 1-bounded on $[N,2N]$, and let t_r be a 1-separated sequence in an interval of length T obeying . Let a_n and t_r be as defined in Definition 7.1. Applying [**?**, Theorem 9.4] (with N, T replaced with $2N$, $2T$ respectively and taking $a_n = 0$ for $n < N$) one has

$$RV^2 \leq \sum_{r=1}^{R} \left| \sum_{N\leq n\leq 2N} a_n n^{-it_r} \right|^2 \ll N^{1+o(1)}(T+N)$$

from which the result follows. ∎

Theorem 7.8 (Montgomery large values theorem). *If $1/2 \leq \sigma \leq 1$ and $\tau \geq 0$ is such that*

$$\sup_{1\leq\tau'\leq\tau} \beta(1/\tau')\tau < 2\sigma - 1 \tag{7.4}$$

(this condition is vacuous for $\tau < 1$) then the Montgomery conjecture (7.2) holds for this choice of parameters.

For a stronger version of this inequality, see Lemma 8.12.

Proof. Set $\rho := \mathrm{LV}(\sigma,\tau)$; we may assume without loss of generality that $\rho \geq 0$. Then by Lemma 7.2, we can find a sequence $N = N_i$ be a sequence going to infinity with i, let $T = N^{\tau+o(1)}$, $V = N^{\sigma+o(1)}$, a 1-bounded sequence a_n and a 1-separated sequence $t_1,\dots,t_R$ in an interval of length T with $R = N^{\rho+o(1)}$, such that (7.1) holds for all r. In particular

$$\sum_{r=1}^{R} \left| \sum_{n\in[N,2N]} a_n n^{it_r} \right| \geq RV$$

hence for some 1-bounded coefficients c_r

$$\left| \sum_{r=1}^{R} c_r \sum_{n\in[N,2N]} a_n n^{it_r} \right| \geq RV$$

We apply the Halász argument. Interchanging the summations and applying Cauchy–Schwarz, we conclude that

$$RV \le N^{1/2} \left| \sum_{1 \le r, r' \le R} c_r \overline{c_{r'}} \sum_{n \in [N,2N]} n^{i(t_r - t_{r'})} \right|^{1/2}$$

hence on squaring and using the triangle inequality

$$N^{2\rho} \ll N^{1-2\sigma+o(1)} \sum_{1 \le r, r' \le R} \left| \sum_{n \in [N,2N]} n^{i(t_r - t_{r'})} \right|.$$

In the case $|t_r - t_{r'}| \le N^{1-\varepsilon}$ for any fixed $\varepsilon > 0$, one can use Lemma 4.4 to obtain the bound

$$\sum_{n \in [N,2N]} n^{i(t_r - t_{r'})} \ll N^{o(1)} \frac{N}{1 + |t_r - t_{r'}|}.$$

The total contribution of this case can then be bounded by $N^{1+o(1)}R = N^{1+\rho+o(1)}$, thanks to the 1-separation. In the remaining cases $|t_r - t_{r'}| \ge N^{1-o(1)}$, we use Definition 4.2 to see that

$$\sum_{n \in [N,2N]} n^{i(t_r - t_{r'})} \ll N^{\sup_{1 \le \tau' \le \tau} \beta(1/\tau')\tau' + o(1)}$$

and thus

$$N^{2\rho} \ll N^{2-2\sigma+\rho+o(1)} + N^{2\rho+1-2\sigma+\sup_{1 \le \tau' \le \tau} \beta(1/\tau')\tau' + o(1)}.$$

By hypothesis, the second term on the right-hand side is asymptotically smaller than the left-hand side, and so we obtain $\rho \le 2 - 2\sigma$ as required. □

Corollary 7.9 (Converting an exponent pair to a large values theorem)**.** *If (k, ℓ) is an exponent pair, and $1/2 \le \sigma \le 1$, and $\tau \ge 0$ are fixed, then*

$$\mathrm{LV}(\sigma, \tau) \le \max\left(2 - 2\sigma, 2 - 2\sigma + \tau - \frac{2\sigma + k - \ell - 1}{k}\right).$$

In particular, the Montgomery conjecture holds for $\tau \le \frac{2\sigma+k-\ell-1}{k}$.

`One can also obtain a similar implication starting from a bound on` μ`: see`

Proof. From Lemma 5.3 one has $\beta(1/\tau')\tau' \le k\tau' + (\ell - k)$ and so the condition (7.4) holds whenever

$$\tau < \frac{2\sigma + k - \ell - 1}{k}.$$

This gives the claim in this range; the general case then follows from Lemma 7.3(ii). □

Theorem 7.10 (Huxley large values theorem)**.** *[**?**, Equation (2.9)] Let* $1/2 \leq \sigma \leq 1$ *and* $\tau \geq 0$ *be fixed. Then one has*

$$\mathrm{LV}(\sigma, \tau) \leq \max(2 - 2\sigma, 4 + \tau - 6\sigma).$$

In particular, one has the Montgomery conjecture for $\tau < 4\sigma - 2$*.*

Proof. Apply Corollary 7.9 with the pair $(k, \ell) = (1/2, 1/2)$ from Lemma 5.10. ☐

Theorem 7.11 (Heath-Brown large values theorem, preliminary form)**.** *Let* $1/2 \leq \sigma \leq 1$ *and* $\tau \geq 0$ *be fixed. If* $\mathrm{LV}(\sigma, \tau) \leq \rho$ *then*

$$\mathrm{LV}(\sigma, \tau) \leq \max\left(2 - 2\sigma, \frac{11}{12}\rho + \frac{3}{2} + \frac{\tau}{6} - 2\sigma\right)$$

Proof. Follows from [**?**, Lemma 1]. ☐

Theorem 7.12 (Heath-Brown large values theorem, optimized)**.** *Let* $1/2 \leq \sigma \leq 1$ *and* $\tau \geq 0$ *be fixed. One has*

$$\mathrm{LV}(\sigma, \tau) \leq \max(2 - 2\sigma, 10 + \tau - 13\sigma).$$

In particular, the Montgomery conjecture holds for $\tau \leq 11\sigma - 8$*.*

Proof. From the previous theorem, and setting $\rho = \mathrm{LV}(\sigma, \tau)$, we have either

$$\mathrm{LV}(\sigma, \tau) \leq 2 - 2\sigma$$

or

$$\mathrm{LV}(\sigma, \tau) \leq \frac{11}{12}\mathrm{LV}(\sigma, \tau) + \frac{3}{2} + \frac{\tau}{6} - 2\sigma.$$

The latter bound can be rearranged as

$$\mathrm{LV}(\sigma, \tau) \leq 2\tau + 18 - 24\sigma$$

and thus

$$\mathrm{LV}(\sigma, \tau) \leq \min(2 - 2\sigma, 2\tau + 18 - 24\sigma).$$

This is not quite what we wanted to show, but we can improve this bound using subdivision. The bound implies in particular that

$$\mathrm{LV}(\sigma, \tau) \leq 2 - 2\sigma \text{ when } \tau \leq 11\sigma - 8,$$

giving the claim in this range; the general case then follows from Lemma 7.3(ii). (See also the arguments in the first paragraph of [**?**, p. 226].) ☐

Lemma 7.13 (Second Heath-Brown large values theorem)**.** *If* $3/4 < \sigma \leq 1$ *and* $\tau \geq 0$ *are fixed, then*

$$\mathrm{LV}(\sigma, \tau) \leq \max(2 - 2\sigma, k\tau + k(2 - 4\sigma), 2\tau/3 + k(12 - 16\sigma)/3)$$

for any positive integer k*.*

Proof. Let $N = N_i$ be a sequence going to infinity with i, and let $T = N^{\tau+o(1)}$, $V = N^{\sigma+o(1)}$. Let a_n and t_r be as defined in Definition 7.1. By [**?**, Lemma 6] we have

$$(RV)^2 \ll T^{o(1)}(RN + R^2N^{1/2} + R^{2-1/2k}T^{1/2} + R^{2-3/8k}N^{1/2}T^{1/4k})N.$$

Thus one must have one of

$$(RN^\sigma)^2 \ll RN^{2+o(1)},$$
$$(RN^\sigma)^2 \ll R^2N^{3/2+o(1)}$$
$$(RN^\sigma)^2 \ll R^{2-1/2k}N^{\tau/2+1+o(1)},$$
$$(RN^\sigma)^2 \ll R^{2-3/8k}N^{\tau/4k+3/2+o(1)}.$$

The third estimate is not possible asymptotically since $\sigma > 3/4$. Rearranging the other estimates to solve for R, and using we conclude that

$$R \ll \max\left(N^{2-2\sigma+o(1)}, N^{k\tau+k(2-4\sigma)+o(1),2\tau/3+k(12-16\sigma)/3+o(1)}\right),$$

and the claim follows from Lemma 7.2. ∎

Theorem 7.14 (Jutila large values theorem)**.** *For any integer $k \geq 1$, one has*

$$\mathrm{LV}(\sigma,\tau) \leq \max(2-2\sigma, \tau + (4-2/k) - (6-2/k)\sigma, \tau + (6-8\sigma)k).$$

Thus for instance with $k = 2$ we have

$$\mathrm{LV}(\sigma,\tau) \leq \max(2-2\sigma, \tau + 3 - 5\sigma, \tau + 12 - 16\sigma)$$

and with $k = 3$ we have

$$\mathrm{LV}(\sigma,\tau) \leq \max(2-2\sigma, \tau + \frac{10-16\sigma}{3}, \tau + 18 - 24\sigma).$$

In particular, the Montgomery conjecture holds for

$$\tau \leq \min((4-2/k)\sigma - (2-2/k), (8k-2)\sigma - 6k + 2).$$

Proof. See [**?**, (1.4)] (setting $V = N^{\sigma+o(1)}$, $T = N^{\tau+o(1)}$, and $G \leq N$). We remark that this form is an optimized form of the inequality after (3.2) in Jutila's paper, which in our notation would read that

$$2\mathrm{LV}(\sigma,\tau)+2\sigma \leq \max\left(2+\rho, \frac{3}{2} + \left(2 - \frac{1}{k}\right)\rho + \rho + \frac{1}{2k}\max\left(k(\tau-1), \frac{\rho+\tau}{2}\right), 2\rho + 1\right)$$

whenever $\mathrm{LV}(\sigma,\tau) \leq \rho$. The optimization procedure is similar to that in Theorem 7.12. ∎

Theorem 7.15 (Guth–Maynard large values theorem)**.** *One has*

$$\mathrm{LV}(\sigma,\tau) \leq \max(2-2\sigma, 18/5 - 4\sigma, \tau + 12/5 - 4\sigma).$$

Proof. See [**?**, Theorem 1.1]. We reprove this result in Theorem 10.23 □

Theorem 7.16 (Bourgain large values theorem)**.** *[**?**] Let $1/2 < \sigma < 1$ and $\tau > 0$, and let $\rho \coloneqq \mathrm{LV}(\sigma, \tau)$. Let $\alpha_1, \alpha_2 \geq 0$ be real numbers. Then either*

$$\rho \leq \min(\alpha_2 + 2 - 2\sigma, -\alpha_2 + 2\tau + 4 - 8\sigma, 2\alpha_1 + \tau + 12 - 16\sigma)$$

or else there exists $s \geq 0$ such that

$$\max(\rho+2, 2\rho+1, 5\rho/4+\tau/2)/2+\max(s+2, 2s+1)/2 \geq \max(-2\alpha-1+2\sigma+s+\rho, -\alpha_1-\alpha_2/2+2\sigma+s/2+3\rho/2). \tag{7.5}$$

Proof. By Lemma 7.2, we can find N going to infinity, $T = N^{\tau+o(1)}$, $R = N^{\rho+o(1)}$, $V = N^{\sigma+o(1)}$, 1-bounded coefficients a_n on $[N, 2N]$, and 1-separated $t_1, \dots, t_R$ in $[T, 2T]$ such that

$$|\sum_n a_n n^{-it_r}| \geq V$$

for $r = 1, \dots, R$. Now set $\delta_1 \coloneqq N^{-\alpha_1}$, $\delta_2 \coloneqq N^{-\alpha_2}$. Applying [**?**, (4.41), (4.42), (4.55), (4.57)] (noting that as we are not imposing the restriction $N > T^{2/3+\varepsilon}$ in [**?**, Lemma 3.7], that an additional term of $R^{5/4}T^{1/2}$ must be added to the right-hand side), we conclude that either

$$R \leq \delta_2^{-1} N^2 V^{-2} + \delta_2 T^2 N^4 V^{-8} + \delta_1^2 T N^{12} V^{-16},$$

or one can bound the quantity

$$T^{-\varepsilon}\delta_1^2 V^2 |S| R + \delta_1 \delta_2^{1/2} V^2 |S|^{1/2} R^{3/2}$$

$$T^{\varepsilon}(RN^2 + R^2 N + R^{5/4} T^{1/2})^{1/2} (|S| N^2 + |S|^2 N)^{1/2}$$

for a certain non-empty set S. After passing to a subsequence, we can ensure that $|S| = N^{s+o(1)}$ for some $s \geq 1$, and the claim follows. □

Corollary 7.17 (Bourgain large values theorem, simplified version)**.** *[**?**, Lemma 4.60] Let the notation be as above, but additionally assume $\tau \leq 3/2$ and $\rho \leq 1$. Then*

$$\rho \leq \max(\alpha_2+2-2\sigma, \alpha_1+\alpha_2/2+2-2\sigma, -\alpha_2+2\tau+4-8\sigma, 2\alpha_1+\tau+12-16\sigma, 4\alpha_1+3-4\sigma).$$

Proof. With $\rho \leq 1$ and $\tau \leq 3/2$, the $2\rho+1$ and $5\rho/4+\tau/2$ terms in the previous theorem are dominated by $\rho+2$, so the inequality (7.5) simplifies to

$$(\rho+2)/2+\max(s+2, 2s+1)/2 \geq \max(-2\alpha-1+2\sigma+s+\rho, -\alpha_1-\alpha_2/2+2\sigma+s/2+3\rho/2).$$

Thus either

$$(\rho+2)/2 + (s+2)/2 \geq -\alpha_1 - \alpha_2/2 + 2\sigma + s/2 + 3\rho/2$$

or

$$(\rho + 2)/2 + (2s + 1)/2 \geq -2\alpha_1 + 2\sigma + s + \rho.$$

In both cases we may eliminate s and solve for ρ to obtain

$$\rho \leq \alpha_1 + \alpha_2/2 + 2 - 2\sigma$$

or

$$\rho \leq 4\alpha_1 + 3 - 4\sigma,$$

giving the claim. $\square$

Theorem 7.18 (Ivic large values theorem)**.** *[**?**, Lemma 8.2] If $\tau \geq 0$ and $1/2 < \sigma < \sigma' < 1$ are fixed, then*

$$\mathrm{LV}(\sigma', \tau) \leq \max(2 - 2\sigma', \tau - f(\sigma)(\sigma' - \sigma))$$

where $f(\sigma)$ is equal to

$$\begin{aligned} \frac{2}{3 - 4\sigma} &\text{ for } 1/2 < \sigma \leq 2/3; \\ \frac{10}{7 - 8\sigma} &\text{ for } 2/3 \leq \sigma \leq 11/14; \\ \frac{34}{15 - 16\sigma} &\text{ for } 11/14 \leq \sigma \leq 13/15; \\ \frac{98}{31 - 32\sigma} &\text{ for } 13/15 \leq \sigma \leq 57/62; \\ \frac{5}{1 - \sigma} &\text{ for } 57/62 \leq \sigma < 1. \end{aligned}$$

In particular, the Montgomery conjecture holds for this choice of σ' if

$$\tau \leq \sup_{1/2 < \sigma < \sigma'} f(\sigma)(\sigma' - \sigma) + 2 - 2\sigma'.$$

Chapter 8

Large value theorems for zeta partial sums

Now we study a variant of the exponent $\mathrm{LV}(\sigma, \tau)$, specialized to the Riemann zeta function.

Definition 8.1 (Large value zeta exponent)**.** Let $1/2 \leq \sigma \leq 1$ and $\tau \geq 0$ be fixed. We define $\mathrm{LV}_\zeta(\sigma, \tau) \in [-\infty, +\infty)$ to be the least (fixed) exponent for which the following claim is true: If N is unbounded, $T = N^{\tau+o(1)}$, $V = N^{\sigma+o(1)}$, I is an interval in $[N, 2N]$, and $t_1, \ldots, t_R$ is a 1-separated subset of $[T, 2T]$ such that

$$|\sum_{n \in I} n^{-it_r}| \geq V$$

for all $r = 1, \ldots, R$, then $R \ll N^{\rho+o(1)}$.

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We will primarily be interested in the regime $\tau \geq 2$ (as this is the region connected to the Riemann-Siegel formula for $\zeta(\sigma+it)$), but for sake of completeness we develop the theory for the entire range $\tau \geq 0$. (The range $0 \leq \tau \leq 1$ can be worked out exactly by existing tools, while the region $1 < \tau < 2$ can be reflected to the region $2 < \tau < \infty$ by Poisson summation.)

As usual, we have a non-asymptotic formulation of $\mathrm{LV}_\zeta(\sigma, \tau)$:

Lemma 8.2 (Asymptotic form of large value exponent at zeta)**.** *Let* $1/2 \leq \sigma \leq 1$, $\tau \geq 0$, *and* $\rho \geq 0$ *be fixed. Then the following are equivalent:*

(i) $\mathrm{LV}_\zeta(\sigma, \tau) \leq \rho$.

(ii) For every $\varepsilon > 0$ *there exists* $C, \delta > 0$ *such that if* $N > C$ *and* $N^{\tau-\delta} \leq T \leq N^{\tau+\delta}$, $N^{\sigma-\delta} \leq V \leq N^{\sigma+\delta}$, *$I$ is a subinterval of* $[N, 2N]$, *and* $t_1, \ldots, t_R$ *is*

a 1-separated subset of $[T, 2T]$ *with*

$$|\sum_{n\in I} n^{-it_r}| \geq V$$

for all $r = 1, \dots, R$*, then one has*

$$R \leq CN^{\rho+\varepsilon}.$$

The proof of Lemma 8.2 proceeds as in previous sections and is omitted.

Lemma 8.3 (Basic properties)**.** *(i) (Monotonicity in* σ*) For any* $\tau \geq 0$, $\sigma \mapsto \mathrm{LV}_\zeta(\sigma, \tau)$ *is upper semicontinuous and monotone non-increasing.*

(ii) (Trivial bound) For any $1/2 \leq \sigma \leq 1$ *and* $\tau \geq 0$*, we have* $\mathrm{LV}_\zeta(\sigma, \tau) \leq \tau$*.*

(iii) (Domination by large values) We have $\mathrm{LV}_\zeta(\sigma, \tau) \leq \mathrm{LV}(\sigma, \tau)$ *for all* $1/2 \leq \sigma \leq 1$ *and* $\tau \geq 0$*.*

(iv) (Reflection) For $1/2 \leq \sigma \leq 1$ *and* $\tau > 1$*, one has*

$$\sup_{\sigma\leq\sigma'\leq 1} \mathrm{LV}_\zeta\left(\frac{1}{2} + \frac{1}{\tau-1}(\sigma' - \frac{1}{2}), \frac{\tau}{\tau-1}\right) + \frac{1}{\tau-1}(\sigma'-\sigma) = \frac{1}{\tau-1}\sup_{\sigma\leq\sigma'\leq 1}(\mathrm{LV}_\zeta(\sigma', \tau)+\sigma'-\sigma).$$

Implemented at |zeta$_l arge_v alues|.py$ as:
|get$_t rivial_z lv()|$
We note that in practice, bounds for $\mathrm{LV}_\zeta(\sigma', \tau) + \sigma'$ are monotone decreasing[1] in σ', so the reflection property in Lemma 8.3(iv) morally simplifies[2] to

$$\mathrm{LV}_\zeta\left(\frac{1}{2} + \frac{1}{\tau-1}(\sigma - \frac{1}{2}), \frac{\tau}{\tau-1}\right) = \frac{1}{\tau-1}\mathrm{LV}_\zeta(\sigma, \tau). \tag{8.1}$$

TODO: implement a python method for reflection

Proof. The claims (i), (ii) are obvious. The claim (iii) is clear by setting $a_n = 1_I$ in Definition 7.1.

Now we turn to (iv). By symmetry it suffices to prove the upper bound. Actually it suffices to just show

$$\mathrm{LV}_\zeta\left(\frac{1}{2} + \frac{1}{\tau-1}(\sigma - \frac{1}{2}), \frac{\tau}{\tau-1}\right) \leq \frac{1}{\tau-1}\sup_{\sigma\leq\sigma'\leq 1}(\mathrm{LV}_\zeta(\sigma', \tau) + \sigma' - \sigma)$$

as this easily implies the general upper bound.

[1] This reflects the fact that large value theorems usually relate to p^{th} moment bounds for $p \geq 1$ (e.g., $p = 2, 4, 6, 12$) rather than for $0 < p < 1$.

[2] Alternatively, one can redefine LV_ζ to use smooth cutoffs in the n variable rather than rough cutoffs $1_I(n)$, in which case one can obtain the analogue of (8.1) rigorously, but we will not do so here.

Let N be unbounded, $T = N^{\frac{\tau}{\tau-1}+o(1)}$, $VN^{\frac{1}{2}+\frac{1}{\tau-1}(\sigma-\frac{1}{2})+o(1)}$, I be an interval in $[N, 2N]$, and $t_1, \dots, t_R$ be a 1-separated subset of $[T, 2T]$ such that

$$|\sum_{n \in I} n^{-it_r}| \geq V$$

for all $r = 1, \dots, R$. By definition, it suffices to show the bound

$$R \ll N^{\frac{1}{\tau-1}(\mathrm{LV}_\zeta(\sigma',\tau)+\sigma'-\sigma)+o(1)}. \tag{8.2}$$

for some $\sigma \leq \sigma' \leq 1$. By a Fourier expansion of $(n/N)^{1/2}$ in $\log n$, we can bound

$$|\sum_{n \in I} n^{-it_r}| \ll_A N^{1/2} \int_{\mathbf{R}} |\sum_{n \in I} n^{-1/2-it}|(1+|t-t_r|)^{-A}\ dt$$

and hence by the pigeonhole principle, we can find $t'_r = t_r + O(N^{o(1)})$ such that

$$|\sum_{n \in I} n^{-1/2-it'_r}| \gg N^{-1/2-o(1)}V$$

for $r = 1, \dots, R$. By refining the t_r by $N^{o(1)}$ if necessary, we may assume that the t'_r are 1-separated.

Now we use the approximate functional equation

$$\zeta(1/2+it'_r) = \sum_{n \leq x} n^{-1/2-it'_r} + \chi(1/2+it'_r) \sum_{m \leq t'_r/2\pi x} m^{-1/2+it_r} + O(N^{-1/2}) + O((T/N)^{-1/2})$$

for $x \sim N$; see [?, Theorem 4.1]. Applying this to the two endpoints of I and subtracting, we conclude that

$$\sum_{n \in I} n^{-1/2-it'_r} = \chi(1/2+it'_r) \sum_{m \in J_r} m^{-1/2+it'_r} + O(N^{-1/2}) + O((T/N)^{-1/2})$$

where $J_r := \{m : t'_r/2\pi m \in I\}$. Since $\chi(1/2+it'_r)$ has magnitude one, we conclude that

$$|\sum_{m \in J_r} m^{-1/2-it'_r}| \gg N^{-1/2-o(1)}V.$$

Writing $M := T/N = N^{\frac{1}{\tau-1}+o(1)}$, we see that $J_r \subset [M/10, 10M]$ and

$$|\sum_{m \in J_r} (M/m)^{1/2} m^{-it'_r}| \gg M^{1/2} N^{-1/2-o(1)} V = M^{\sigma+o(1)}.$$

Performing a Fourier expansion of $(M/m)^{1/2} 1_{J_r}(m)$ (smoothed out at scale $O(1)$) in $\log m$, we can bound

$$|\sum_{m \in J_r} (M/m)^{1/2} m^{-it'_r}| \ll \int_{T/10}^{10T} |\sum_{m \in [M/10, 10M]} m^{-it}|(1+|t-t'_r|)^{-1}\ dt + T^{-10}$$

and hence

$$\int_{T/10}^{10T} |\sum_{m\in[M/10,10M]} m^{-it}|(1+|t-t_r'|)^{-1}\, dt \gg M^{\sigma+o(1)}.$$

If we let E denote the set of $t\in[T/10,10T]$ for which $|\sum_{m\in[M/10,10M]} m^{-it}| \geq M^{\sigma-o(1)}$ for a suitably chosen $o(1)$ error, then we have

$$\int_E |\sum_{m\in[M/10,10M]} m^{-it}|(1+|t-t_r'|)^{-1}\, dt \gg M^{\sigma+o(1)}.$$

Summing in r, we obtain

$$\int_E |\sum_{m\in[M/10,10M]} m^{-it}|\, dt \gg M^{\sigma+o(1)}R$$

and so by dyadic pigeonholing we can find $M^{\sigma-o(1)} \ll V'' \ll M$ and a 1-separated sequence $t_1'',\dots,t_{R''}''$ in E such that

$$|\sum_{m\in[M/10,10M]} m^{-it_r''}|\, dt \asymp V''$$

and

$$V''R'' \gg M^{\sigma+o(1)}R.$$

By passing to a subsequence we may assume that $V'' = M^{\sigma'+o(1)}$ for some $\sigma\leq\sigma'\leq 1$. Partitioning $[M/10,10M]$ into a bounded number of intervals each of which lies in a dyadic range $[M',2M']$ for some $M'\asymp M$, and using Definition 8.1, we have

$$R'' \ll M^{\mathrm{LV}_\zeta(\sigma',\tau)+o(1)}$$

and (8.2) follows. □

```
Note in comparison with LV(σ,τ), that LV_ζ(σ,τ) can be −∞, and is
indeed conjectured to do so whenever σ > 1/2 and τ ≥ 1. Indeed:
```

Lemma 8.4 (Characterization of negative infinite value)**.** *Let $1/2\leq\sigma\leq 1$ and $\tau\geq 0$ be fixed. Then the following are equivalent:*

(i) $\mathrm{LV}_\zeta(\sigma,\tau) = -\infty$.

(ii) $\mathrm{LV}_\zeta(\sigma,\tau) < 0$.

(iii) There exists a fixed $\varepsilon>0$ such that if N is unbounded and I is a subinterval of $[N,2N]$, then one has

$$\sum_{n\in I} n^{-it} \ll N^{\sigma-\varepsilon+o(1)}$$

whenever $|t| = N^{\tau+o(1)}$.

Proof. Clearly (i) implies (ii). If (iii) holds, then in the situation of (i), we see that R will vanish for sufficiently large i, giving (i). Conversely, if (i) fails, then by diagonalizing one can obtain data obeying the hypotheses of Lemma 8.2 with $R = 1$ for i large enough, contradicting (ii). $\square$

Corollary 8.5. *If $\tau \geq 0$ is fixed then $\mathrm{LV}_\zeta(\sigma, \tau) = -\infty$ whenever $\sigma > \tau\beta(1/\tau)$ is fixed. For instance, by (4.6), one has $\mathrm{LV}_\zeta(\sigma, 1) = -\infty$ whenever $\sigma > 1/2$ is fixed.*

Proof. Suppose one has data N, I obeying the hypotheses of Lemma 8.4(iii), then by (4.2) (with $\alpha = 1/\tau$) one has

$$\sum_{n \in I} n^{-it} \ll |t|^{\beta(1/\tau)+o(1)} = N^{\tau\beta(1/\tau)+o(1)}$$

and the claim follows from Lemma 8.4. $\square$

Corollary 8.6. *If $\tau > 0$ and $1/2 \leq \sigma_0 \leq 1$ are fixed, then $\mathrm{LV}_\zeta(\sigma, \tau) = -\infty$ whenever $\sigma > \sigma_0 + \tau\mu(\sigma_0)$.*

Proof. From Definition 6.1 one has

$$\zeta(\sigma_0 + it) \ll |t|^{\mu(\sigma_0)+o(1)}$$

for unbounded t. By standard arguments **give ref**, this implies that

$$\sum_{n \in I} \frac{1}{n^{\sigma_0+it}} \ll |t|^{\mu(\sigma_0)+o(1)}$$

as $N \to \infty$, if $I \subset [N, 2N]$ and $|t| = N^{\tau+o(1)}$. By partial summation this gives

$$\sum_{n \in I} n^{-it} \ll N^{\sigma_0}|t|^{\mu(\sigma_0)+o(1)} = N^{\sigma_0+\tau\mu(\sigma_0)+o(1)}.$$

The claim now follows from Lemma 8.4. $\square$

Corollary 8.7. *If (k, ℓ) is an exponent pair, then $\mathrm{LV}_\zeta(\sigma, \tau) = -\infty$ whenever $1/2 \leq \sigma \leq 1$, $\tau \geq 0$ are fixed quantities with $\sigma > k\tau + \ell - k$.*

Proof. Immediate from Corollary 8.5 and Lemma 5.3; alternatively, one can use Corollary 8.6 and Corollary 6.8. $\square$

Corollary 8.8. *Assuming the Lindelof hypothesis, one has $\mathrm{LV}_\zeta(\sigma, \tau) = -\infty$ whenever $\sigma > 1/2$ and $\tau \geq 1$.*

Proof. Apply Corollary 8.6 with $\sigma_0 = 1/2$, so that $\mu(\sigma_0)$ vanishes from the Lindelof hypothesis. $\square$

For completeness, we now work out the values of $\mathrm{LV}_\zeta(\sigma, \tau)$ in the remaining cases not covered by the above corollary.

Lemma 8.9 (Value at $\sigma = 1/2$)**.** *One has* $\mathrm{LV}_\zeta(1/2, \tau) = \tau$ *for all* $\tau \geq 1$.

Proof. The upper bound $\mathrm{LV}_\zeta(1/2, \tau) \leq \tau$ follows from Lemma 8.3(ii), so it suffices to prove the lower bound. Accordingly, let Nbe unbounded, let $T = CN$ for a large fixed constant C, and set $I := [N, 2N]$. In the case $\sigma = 1$, we see from the L^2 mean value theorem (Lemma 3.1) that the expression $\sum_{n\in I} n^{-it}$ has an L^2 mean of $\asymp N^{1/2}$ for $t \in [T, 2T]$; on other hand, from (4.6) we also have an L^∞ norm of $O(N^{1/2+o(1)})$. We conclude that $|\sum_{n\in I} n^{-it}| \gg N^{1/2+o(1)}$ for t in a subset of $[T, 2T]$ of measure $T^{1-o(1)}$, and hence on a 1-separated subset of cardinality $\gg T^{1-o(1)}$. This gives the claim $\mathrm{LV}(1/2, 1) \geq 1$.

Next, we establish the $\tau \geq 2$ case. Let N be unbounded, set $T := N^\tau$, and set $I := [N, 2N]$. From Lemma 3.1 we see that the L^2 mean of $\sum_{n\in I} n^{-it}$ is $\asymp N^{1/2}$. Also, by squaring this Dirichlet series and applying Lemma 3.1 again we see that the L^4 mean is $O(N^{1/2+o(1)})$. We may now argue as before to give the desired claim $\mathrm{LV}(1/2, \tau) \geq \tau$.

Finally we need to handle the case $1 < \tau < 2$. By Lemma 8.3(iv) with $\sigma = 1/2$ we have

$$\mathrm{LV}_\zeta\left(\frac{1}{2}, \frac{\tau}{\tau-1}\right) = \frac{1}{\tau-1} \sup_{1/2\leq\sigma'\leq 1} (\mathrm{LV}_\zeta(\sigma', \tau) + \sigma' - 1/2).$$

By the $\tau \geq 2$ case, the left-hand side is at least $\tau/(\tau - 1)$, thus

$$\sup_{1/2\leq\sigma'\leq 1} (\mathrm{LV}_\zeta(\sigma', \tau) + \sigma' - 1/2) \geq \tau.$$

On the other hand, from Theorem 7.7 and Lemma 8.3(iii) we have

$$\mathrm{LV}_\zeta(\sigma', \tau) + \sigma' - 1/2 \leq \tau + 1/2 - \sigma'.$$

We conclude that the supremum is in fact attained asymptotically at $\sigma' = 1/2$, in the sense that

$$\limsup_{\sigma'\to 1/2^+} \mathrm{LV}_\zeta(\sigma', \tau) + \sigma' - 1/2 \geq \tau.$$

By the monotonicity of LV_ζ in σ, this implies that $\mathrm{LV}_\zeta(1/2, \tau) \geq \tau$, as required. □

Lemma 8.10 (Value at $\tau < 1$)**.** *If* $0 \leq \tau < 1$, *then* $\mathrm{LV}_\zeta(\sigma, \tau)$ *is equal to* $-\infty$ *for* $\sigma > 1 - \tau$ *and equal to* τ *for* $\sigma \leq 1 - \tau$.

Proof. The first claim follows from Corollary 8.5 and Lemma 4.4. For the second claim, it suffices by Lemma 8.3(ii) to establish the lower bound $\mathrm{LV}_\zeta(\sigma, \tau) \geq \tau$. But this is clear from (4.5). □

`One can use exponent pairs to control` $\mathrm{LV}_\zeta(\sigma, \tau)$`:`

Lemma 8.11 (From exponent pairs to zeta large values estimate)**.** *[**?**, Theorem 8.2] If* (k, ℓ) *is an exponent pair with* $k > 0$, *then for any* $1/2 \leq \sigma \leq 1$ *and* $\tau \geq 0$ *one has*

$$\mathrm{LV}_\zeta(\sigma, \tau) \leq \max(\tau - 6(\sigma - 1/2), \frac{k+\ell}{k}\tau - \frac{2(1 + 2k + 2\ell)}{k}(\sigma - 1/2)).$$

Typical examples of exponent pairs that can be used here include $(1/2, 1/2)$, $(13/31, 16/31) = BAB^2A^2(1/6, 2/3)$, $(4/11, 6/11)$ (a convex combination of $(1/2, 1/2)$ and $(2/7, 4/7)$), $(2/7, 4/7) = BA(1/6, 2/3)$, $(5/24, 15/24)$ (a convex combination of $(1/6, 2/3)$ and $(4/18, 11/18)$), and $(4/18, 11/18) = BABA(1/6, 2/3)$: see [?, Corollary 8.1, 8.2].

A useful connection between large values estimates and large values estimates for the zeta function is the following strengthening of Theorem 7.8.

Lemma 8.12 (Halász–Montgomery inequality). *For any $1/2 \leq \sigma \leq 1$ and $\tau \geq 0$, we have*

$$\mathrm{LV}(\sigma, \tau) \leq \max\left(2 - 2\sigma, 1 - 2\sigma + \sup_{\substack{1 \leq \tau' \leq \tau \\ \max(1/2, 2\sigma - 1) \leq \sigma' \leq 1}} \sigma' + \mathrm{LV}_\zeta(\sigma', \tau')\right).$$

Note from Lemma 8.5 one could also impose the restriction $\sigma' \leq \tau' \beta(1/\tau')$ in the supremum if desired, at which point one recovers Theorem 7.8. Similarly, from Corollary 8.6 one could also impose the restriction $\sigma' \leq \sigma_0 + \tau' \mu(\sigma_0)$ for any fixed $1/2 \leq \sigma_0 \leq 1$.

Proof. It suffices to show that

$$\mathrm{LV}(\sigma, \tau) \leq \max\left(2 - 2\sigma, 1 - 2\sigma + \sup_{\substack{1 \leq \tau' \leq \tau \\ 1/2 \leq \sigma' \leq 1}} \sigma' + \min(\mathrm{LV}_\zeta(\sigma', \tau'), \mathrm{LV}(\sigma, \tau))\right)$$

since the terms with $\sigma' < 2\sigma - 1$ are less than the left-hand side and can thus be dropped. We repeat the proof of Lemma 7.8. Letting N, T, V, a_n, t_r be as in the proof of that lemma; we may assume without loss of generality that $R = N^{\mathrm{LV}(\sigma,\tau)+o(1)}$, and we have

$$RV \leq N^{1/2} \left| \sum_{1 \leq r, r' \leq R} c_r \overline{c_{r'}} \sum_{n \in [N, 2N]} n^{i(t_r - t_{r'})} \right|^{1/2}$$

and hence by the triangle inequality

$$RV \leq N^{1/2} R^{1/2} \sup_{r'} \left| \sum_{1 \leq r \leq R} | \sum_{n \in [N, 2N]} n^{i(t_r - t_{r'})} | \right|^{1/2}$$

which we rearrange as

$$R \leq N^{1 - 2\sigma + o(1)} \sup_{r'} \sum_{1 \leq r \leq R} | \sum_{n \in [N, 2N]} n^{i(t_r - t_{r'})} |.$$

As in the proof of Lemma 7.8, the contribution of the case $|t_r - t_{r'}| \leq N^{1-\varepsilon}$ to the right-hand side is $N^{2-2\sigma+o(1)}$, so we can restrict attention to the case $|t_r - t_{r'}| \geq N^{1-o(1)}$. By a dyadic decomposition and the pigeonhole principle, we may then assume that

$$R \leq N^{1-2\sigma+o(1)} \sum_{1 \leq r \leq R: |t_r - t_{r'}| \asymp T'} |\sum_{n \in [N,2N]} n^{i(t_r - t_{r'})}|$$

for some $N^{1-o(1)} \ll T' \ll T$ and some r'; by passing to a subsequence we may assume that $T' = N^{\tau'+o(1)}$ for some $1 \leq \tau' \leq \tau$. By further dyadic decomposition, we may also assume that $|\sum_{n\in[N,2N]} n^{i(t_r-t_{r'})}| \asymp N^{\sigma'+o(1)}$ for some $\sigma' \leq 1$; the cardinality of the sum is then bounded both by R and by $N^{\mathrm{LV}_\zeta(\sigma',\tau')+o(1)}$, hence

$$R \leq N^{1-2\sigma+\sigma'+\min(\mathrm{LV}(\sigma,\tau),\mathrm{LV}_\zeta(\sigma',\tau'))+o(1)}.$$

The case $\sigma' < 1/2$ is dominated by that of $\sigma' = 1/2$. The claim now follows. $\square$

Corollary 8.13 (Converting a bound on μ to a large values theorem)**.** *If* $1/2 \leq \sigma, \sigma' \leq 1$*, and* $\tau \geq 0$ *are fixed, then*

$$\mathrm{LV}(\sigma,\tau) \leq \max\left(2-2\sigma, 2-2\sigma+\tau-\frac{2\sigma-1-\sigma'}{\mu(\sigma')}\right).$$

In particular, the Montgomery conjecture holds for $\tau \leq \frac{2\sigma-1-\sigma'}{\mu(\sigma')}$.

Proof. By subdivision it suffices to verify the claim for $\tau < \frac{2\sigma-1-\sigma'}{\mu(\sigma')}$. The claim now follows from Lemma 8.12 and Corollary 8.6. $\square$

Theorem 8.14 (Halász-Turán large values theorem)**.** *[?, Theorem 1] On the Lindelöf hypothesis, one has the Montgomery conjecture whenever* $\sigma > 3/4$.

Proof. Immediate from Corollary 8.13, since $\mu(1/2) = 0$ in this case. $\square$

`A typical application of the Halász-Montgomery inequality is`

Lemma 8.15 (Ivic large values theorem)**.** *[?, (11.40)] For any* $1/2 \leq \sigma \leq 1$ *and* $\tau \geq 0$*, one has*

$$\mathrm{LV}(\sigma,\tau) \leq \max(2-2\sigma, \tau+9-12\sigma, 3\tau+19(3-4\sigma)/2).$$

In particular, optimizing using subdivision (Lemma 7.3(ii)) we have

$$\mathrm{LV}(\sigma,\tau) \leq \max\left(2-2\sigma, \tau+9-12\sigma, \tau-\frac{84\sigma-65}{6}\right).$$

This implies the Montgomery conjecture for

$$\tau \leq \min(10\sigma-7, 12\sigma-\frac{53}{6}).$$

Proof. Write $\rho := \mathrm{LV}(\sigma,\tau)$, and let $\varepsilon > 0$ be arbitrary. By Lemma 8.12, we may assume without loss of generality that

$$\rho \leq \max(2-2\sigma, 1-2\sigma+\sigma'+\min(\rho, \mathrm{LV}_\zeta(\sigma',\tau'))) + \varepsilon$$

for some $1/2 \leq \sigma' \leq 1$ and $1 \leq \tau' \leq \tau$. On the other hand, from Lemma 8.11 applied to the exponent pair $(2/7, 4/7)$, and bounding τ' by τ, one has

$$\mathrm{LV}_\zeta(\sigma',\tau') \leq \max(\tau - 6(\sigma'-1/2), 3\tau - 19(\sigma'-1/2))$$

and thus on taking convex combinations

$$\min(\rho, \mathrm{LV}_\zeta(\sigma',\tau')) \leq \max(\frac{5}{6}\rho + \frac{1}{6}\tau - (\sigma'-1/2), \frac{18}{19}\rho + \frac{3}{19}\tau - (\sigma'-1/2)),$$

hence ρ is bounded by either $2-2\sigma$, $1-2\sigma+\frac{5}{6}\rho+\frac{1}{6}\tau+\frac{1}{2}$, or $1-2\sigma+\frac{18}{19}\rho+\frac{3}{19}\tau+\frac{1}{2}$. The claim then follows after solving for ρ. $\square$

Chapter 9

Moment growth for the zeta function

Definition 9.1 (Zeta moment exponents)**.** For fixed $\sigma \in \mathbf{R}$ and $A \geq 0$, we define $M(\sigma, A)$ to be the least (fixed) exponent for which the bound

$$\int_T^{2T} |\zeta(\sigma + it)|^A \ dt \ll T^{M(\sigma,A)+o(1)}$$

holds for all unbounded $T > 1$.

`Again, it is not difficult to show that` $M(\sigma, A)$ `is a well-defined (fixed) real number. A non-asymptotic definition is that it is the least constant such that for every` $\varepsilon > 0$ `there exists` $C > 0$ `such that`

$$\int_T^{2T} |\zeta(\sigma + it)|^A \ dt \leq CT^{M(\sigma,A)+\varepsilon}$$

`holds for all` $T \geq C$.

Lemma 9.2 (Basic properties of $M(\sigma, A)$)**.** *(i)* $M(\sigma, A)$ *is convex in* σ.

(ii) For any σ, $a(M(\sigma, 1/a) - 1)$ *is convex non-increasing in* a.

(iii) $M(\sigma, A) = 0$ *for all* $A \geq 0$ *and* $\sigma \geq 1$.

(iv) $M(1/2, A) = 0$ *for all* $0 \leq A \leq 4$.

(v) $M(\sigma, A) \geq 0$ *for all* $1/2 \leq \sigma \leq 1$ *and* $A \geq 0$.

(vi) $M(\sigma, 0) = 1$ *for all* σ.

(vii) $M(1 - \sigma, A) = M(1 - \sigma, A) + (1/2 - \sigma)A$ *for all* $\sigma \in \mathbf{R}$ *and* $A \geq 0$.

(viii) For any σ, $a(M(\sigma, 1/a) - 1)$ *converges to* $\mu(\sigma)$ *as* $a \to 0$. *In particular,* $M(\sigma, A) \leq A\mu(\sigma) + 1$ *for all* $\sigma \geq 0$ *and* $A \geq 0$.

Proof. The claim (i) follows from the Phragmen-Lindelof principle. The claim (ii) follows from Hölder. The claim (iii) follows from standard upper and lower bounds on $\zeta(\sigma+it)$ for $\sigma \geq 1$. For the claim (iv), we have standard moment estimates

$$\int_T^{2T} |\zeta(\frac{1}{2}+it)|^2\ dt = T^{1+o(1)}$$

and

$$\int_T^{2T} |\zeta(\frac{1}{2}+it)|^4\ dt = T^{1+o(1)}$$

for any unbounded $T > 1$, and the claim follows from Hölder's inequality. The claim (v) follows from (i)-(iv), and (vi) is trivial. The claim (vii) follows easily from the functional equation.

For (viii), the bound $M(\sigma, A) \leq A\mu(\sigma) + 1$ is trivial, which implies that

$$\lim_{a\to 0} a(M(\sigma, 1/a) - 1) \leq \mu(\sigma).$$

Suppose for contradiction that

$$\lim_{a\to 0} a(M(\sigma, 1/a) - 1) < \mu(\sigma),$$

thus there is $\delta > 0$ such that

$$M(\sigma, A) \leq A(\mu(\sigma) - \delta) + 1$$

for all $A \geq 0$. By convexity, this gives

$$M(\sigma + \varepsilon, A) \leq A(\mu(\sigma) - \delta/2) + 1$$

for all sufficiently small ε, and then by the Cauchy integral formula and Hölder's inequality we can conclude that

$$|\zeta(\sigma + \varepsilon/2 + it)| \ll |t|^{\mu(\sigma)-\delta/2+O(1/A)+o(1)}$$

for unbounded $|t|$, leading to

$$\mu(\sigma + \varepsilon/2) \leq \mu(\sigma) - \delta/2 + O(1/A).$$

Sending A to infinity and ε to zero, we obtain a contradiction. □

Corollary 9.3. *If the Lindelof hypothesis holds, then* $M(\sigma, A) = \max(1/2 - \sigma, 0)A$ *for all* $\sigma \in \mathbf{R}$ *and* $A \geq 0$.

`Note from Lemma 9.2 that we always have the lower bound` $M(\sigma, A) \geq \max(1/2-\sigma, 0)A$. `Thus there are not expected to be any further lower bound results for` $M(\sigma, A)$`, and we focus now on upper bounds. From Lemma 9.2 we may restrict attention to the region` $1/2 \leq \sigma \leq 1$ `and` $A \geq 4$.

`We can relate` $M(\sigma, A)$ `to` $\mathrm{LV}_\zeta(\sigma, \tau)$`:`

Lemma 9.4. *If $1/2 \leq \sigma_0 \leq 1$ and $A \geq 1$, then*

$$M(\sigma_0, A) = \sup_{\tau \geq 2; \sigma \geq 1/2} (A(\sigma - \sigma_0) + \mathrm{LV}_\zeta(\sigma, \tau))/\tau. \tag{9.1}$$

In particular, one has

$$\mathrm{LV}_\zeta(\sigma, \tau) \leq \tau M(\sigma_0, A) - A(\sigma - \sigma_0)$$

whenever $\sigma \geq 1/2$ and $\tau \geq 2$.

Proof. We first show the lower bound, or equivalently that

$$A(\sigma - \sigma_0) + \mathrm{LV}_\zeta(\sigma, \tau) \leq \tau M(\sigma_0, A) - A(\sigma - \sigma_0)$$

whenever $\tau \geq 2$ and $\sigma \geq 1/2$. Accordingly, let N be unbounded, $T = N^{\tau + o(1)}$, $I \subset [N, 2N]$, and $t_1, \dots, t_R$ be a 1-separated subset of $[T, 2T]$ such that

$$|\sum_{n \in I} n^{-it_r}| \gg N^{\sigma + o(1)}.$$

By standard Fourier analysis, this gives

$$\int_{T/2}^{3T} |\zeta(\sigma_0 + it)| \ \frac{dt}{1 + |t - t_r|} \gg N^{\sigma - \sigma_0 + o(1)}$$

and hence by Hölder

$$\int_{T/2}^{3T} |\zeta(\sigma_0 + it)|^A \ \frac{dt}{1 + |t - t_r|} \gg N^{A(\sigma - \sigma_0) + o(1)}$$

so on summing in r

$$\int_{T/2}^{3T} |\zeta(\sigma_0 + it)|^A \ dt \gg RN^{A(\sigma - \sigma_0) + o(1)}.$$

By Definition 9.1, the left-hand side is $\ll T^{M(\sigma_0, A) + o(1)}$. Since $T = N^{\alpha + o(1)}$, we obtain

$$R \ll N^{\tau M(\sigma_0, A) - A(\sigma - \sigma_0)},$$

giving the claim.

For the converse bound, let M be the right-hand side of (9.1). From Lemma 8.9 we have $M \geq 1$. By [**?**, §8.1] it will suffice to show that for any $V > 0$ and any 1-separated $t_1, \dots, t_r \in [T, 2T]$ with

$$|\zeta(\sigma_0 + it_r)| \geq V$$

one has

$$R \ll T^{M + o(1)} V^{-A}.$$

The claim is clear if $V \geq T^C$ or $V \leq T^C$ for some sufficiently large C, so we may assume that $V = T^{O(1)}$. We also clearly can assume $R \geq 1$. Using the Riemann–Siegel formula and dyadic decomposition, we have either

$$|\sum_{n \in I} \frac{1}{n^{\sigma_0 + it_r}} \gg T^{-o(1)} V$$

or

$$T^{1/2-\sigma_0} |\sum_{n \in I} \frac{1}{n^{1-\sigma_0-it_r}}| \gg T^{-o(1)} V$$

for some $I \subset [N, 2N]$ and $1 \leq N \ll T^{1/2}$. In either case, we can perform summation by parts and conclude that

$$|\sum_{n \in I'} n^{-it_r}| \gg T^{-o(1)} V N^{\sigma_0}$$

or

$$|\sum_{n \in I'} n^{-it_r}| \gg T^{-o(1)} V N^{1-\sigma_0} T^{\sigma_0 - 1/2}$$

for some I' in $[N, 2N]$. As $\sigma_0 \geq 1/2$, the letter hypothesis is stronger than the former, so we may assume the former. If $N = T^{o(1)}$ then this would imply that $V \ll T^{o(1)}$, and we would be done from the trivial bound $R \ll T$ since $M \geq 1$. Hence, after passing to a subsequence, we can assume that $N = T^{1/\tau + o(1)}$ for some $2 < \tau < \infty$. We can also assume that $V = N^{\sigma - \sigma_0 + o(1)}$ for some $\sigma \in \mathbf{R}$. If $\sigma \leq \sigma_0$ then $V \ll T^{o(1)}$ and we are done as before, so we may assume $\sigma > \sigma_0$; in particular, $\sigma \geq 1/2$. From Lemma 8.2 we have

$$R \ll N^{\mathrm{LV}_\zeta(\sigma,\tau) + o(1)}$$

and hence by definition of M

$$R \ll N^{M\tau - A(\sigma - \sigma_0) + o(1)} = T^{M + o(1)} V^{-A}$$

as required. □

Corollary 9.5 (Fourth moment bound)**.** *One has* $\mathrm{LV}_\zeta(\sigma, \tau) \leq \tau - 4(\sigma - 1/2)$ *for all* $1/2 \leq \sigma \leq 1$ *and* $\tau \geq 2$.

Proof. Apply Lemma 9.4 with $\sigma_0 = 1/2$ and $A = 4$, using Lemma 9.2(iv). □

`We have an important twelfth moment estimate of Heath-Brown:`

Theorem 9.6 (Heath-Brown twelfth moment estimate)**.** *[**?**]* $M(1/2, 12) \leq 2$. *Equivalently (by Lemma 9.4), one has* $\mathrm{LV}_\zeta(\sigma, \tau) \leq 2\tau - 12(\sigma - 1/2)$ *for all* $\tau \geq 2$ *and* $1/2 \leq \sigma \leq 1$.

Proof. From Lemma 8.11 with the exponent pair $(1/2, 1/2)$ from Lemma 5.10 we have

$$\mathrm{LV}_\zeta(\sigma, \tau) \leq \min(\tau - 6(\sigma - 1/2), 2\tau - 12(\sigma - 1/2)).$$

If $2\tau - 12(\sigma - 1/2) \geq 0$, the claim is immediate; if instead $2\tau - 12(\sigma - 1/2) < 0$, use Lemma 8.4. □

We also have a variant bound, which is slightly better when τ is close to $6(\sigma - 1/2)$:

Theorem 9.7 (Auxiliary Heath-Brown estimate). *For $\tau \geq 2$ and $1/2 \leq \sigma \leq 1$, one has*

$$\mathrm{LV}_\zeta(\sigma,\tau) \leq \min(\tau - 6(\sigma - 1/2), 5\tau - 32(\sigma - 1/2)).$$

Proof. Let N be unbounded, let $T = N^{\tau+o(1)}$ and $V = N^{\sigma+o(1)}$, let $I \subset [N, 2N]$, and let $t_1 < \cdots < t_R$ in $[T, 2T]$ be 1-separated with

$$|\sum_{n\in I} n^{-it_r}| \geq V$$

for all r. Our task is to show that

$$R \ll T^{o(1)}(T(N^{-1/2}V)^{-6} + T^5(N^{-1/2}V)^{-32}).$$

By a Fourier analytic expansion we can bound

$$N^{-1/2}|\sum_{n\in I} n^{-it_r}| \ll T^{o(1)} \int_{T/4}^{3T} |\zeta(1/2+it)| \frac{dt}{1+|t-t_r|} + N^{-\varepsilon}$$

for some $\varepsilon > 0$, hence

$$\int_{T/4}^{3T} |\zeta(1/2+it)| \frac{dt}{1+|t-t_r|} \gg T^{-o(1)} N^{-1/2} V.$$

In particular, we can truncate to large values of $\zeta(1/2+it)$, in the sense that

$$\int_{T/4}^{3T} |\zeta(1/2+it)| 1_{|\zeta(1/2+it)| \geq T^{-o(1)}N^{-1/2}V} \frac{dt}{1+|t-t_r|} \gg T^{-o(1)} N^{-1/2} V.$$

Summing in r and using the 1-separation to bound the sum of $1/(1+|t-t_r|)$ by $T^{o(1)}$, we conclude that

$$\int_{T/4}^{3T} |\zeta(1/2+it)| 1_{|\zeta(1/2+it)| \geq T^{-o(1)}N^{-1/2}V}\ dt \gg T^{-o(1)} R N^{-1/2} V.$$

Hence by dyadic pigeonholing we have

$$V' \int_{T/4}^{3T} |\zeta(1/2+it)| 1_{|\zeta(1/2+it)| \asymp V'}\ dt \gg T^{-o(1)} R N^{-1/2} V$$

for some $V' \geq T^{-o(1)} N^{-1/2} V$, and thus

$$|\zeta(1/2+it'_r)| \asymp V'$$

for some 1-separated sequence $t'_1 < \cdots < t'_{R'}$ in $[T/4, 3T]$ with

$$R' \gg T^{-o(1)} R N^{-1/2} V / V'.$$

Applying [**?**, Theorem 2] (treating different cases using the bounds [**?**, (7), (8), (9)]), we have the bound

$$R' \ll T^{o(1)}(T(V')^{-6} + T^5(V')^{-32})$$

and thus

$$R \ll T^{o(1)}(T(N^{-1/2}V)^{-1}(V')^{-5} + T^5(N^{-1/2}V)^{-1}(V')^{-31})$$

and the claim now follows from the lower bound on V'. □

Lemma 9.8. *[**?**, Theorem 8.4] We have $M(\sigma, A) = 1$ when A is equal to*

$$\begin{aligned}
&\frac{4}{3-4\sigma} \text{ for } 1/2 < \sigma \le 5/8;\\
&\frac{10}{5-6\sigma} \text{ for } 5/8 < \sigma \le 35/54;\\
&\frac{19}{6-6\sigma} \text{ for } 35/54 < \sigma \le 41/60;\\
&\frac{2112}{859-948\sigma} \text{ for } 41/60 < \sigma \le 3/4;\\
&\frac{12408}{4537-4890\sigma} \text{ for } 3/4 \le \sigma \le 5/6;\\
&\frac{4324}{1031-1044\sigma} \text{ for } 5/6 \le \sigma \le 7/8;\\
&\frac{98}{31-32\sigma} \text{ for } 7/8 \le \sigma \le 0.91591\ldots;\\
&\frac{24\sigma-9}{(4\sigma-1)(1-\sigma)} \text{ for } 0.91591\cdots \le \sigma < 1.
\end{aligned}$$

9.1 Large values of ζ moments

It is also of interest to control large values of the moments. For fixed $1/2 \le \sigma \le 1$, $A \ge 0$, and $h \ge 0$, let $M(\sigma, A, h)$ be the least (fixed) exponent for which the bound

$$\int_{0\le t\le T: |\zeta(\sigma+it)|\ge T^h} |\zeta(\sigma+it)|^A \ dt \ll T^{M(\sigma,A,h)+o(1)}.$$

A modification of the proof of the upper bound in Lemma 9.4 reveals that

$$M(\sigma_0, A, h) \le \sup_{\tau\ge 2; \sigma\ge 1/2, h\tau} (A(\sigma-\sigma_0) + \mathrm{LV}_\zeta(\sigma,\tau))/\tau. \tag{9.2}$$

That is to say, any bound of the form

$$\mathrm{LV}_\zeta(\sigma,\tau) \le M\tau - A(\sigma-\sigma_0)$$

whenever $\tau \geq 2$ and $\sigma \geq 1/2, h\tau$, gives rise to a bound

$$M(\sigma_0, A, h) \leq M.$$

A typical result concerning $M(\sigma_0, A, h)$ is as follows.

Lemma 9.9. *[?, (8.56)] $M(1/2, 6, 11/72) \leq 1$.*

Proof. Applying Lemma 8.11 with the exponent pair $(4/18, 11/18) = BABA(1/6, 2/3)$ we have

$$\mathrm{LV}_\zeta(\sigma, \tau) \leq \min(\tau - 6(\sigma - 1/2), 15\tau/4 - 24(\sigma - 1/2)).$$

In particular, we have

$$\mathrm{LV}_\zeta(\sigma, \tau) \leq \tau - 6(\sigma - 1/2)$$

if $\sigma - 1/2 \geq 11\tau/72$, giving the claim. ☐

Lemma 9.10. *[?, Proposition 2] Suppose that $M(1/2, A, h) \leq 1$ for some $A \geq 4$ and $h \geq 0$. Then one has*

$$\mathrm{LV}(\sigma, \tau) \leq \max(\alpha + 2 - 2\sigma, -\alpha + \tau + A/2 - 2A(\sigma - 1/2))$$

whenever $1/2 \leq \sigma \leq 1$, $\tau > 0$, and $0 \leq \alpha \leq 1 - \sigma$ is such that

$$\sigma - \frac{1}{2} > \frac{\tau h}{2} + \frac{1}{4}.$$

Lemma 9.11. *[?, Proposition 5] Suppose that $M(1/2, 6, h) \leq 1$ for some $h \geq 0$. Then for any $1/2 \leq \alpha < \sigma < 1$, one has*

$$A(\sigma) \leq \max(\frac{\mu(\alpha)}{\sigma - \alpha}, \frac{3}{8\sigma - 5}, \frac{6h}{4\sigma - 3}).$$

It is remarked in [?] that this proposition could lead to some improvements in current zero density estimate bounds.

Chapter 10

Large value additive energy

10.1 Additive energy

Definition 10.1 (Additive energy)**.** Let W be a finite set of real numbers. The *additive energy* $E_1(W)$ of such a set is defined to be the number of quadruples $(t_1, t_2, t_3, t_4) \in W$ such that

$$|t_1 + t_2 - t_3 - t_4| \leq 1.$$

We remark that in additive combinatorics, the variant $E_0(W)$ of the additive energy is often studied, in which $t_1+t_2-t_3-t_4$ is not merely required to be 1-bounded, but in fact vanish exactly. However, this version of additive energy is less relevant for analytic number theory applications.

Lemma 10.2 (Basic properties of additive energy)**.** *(i) If W is a finite set of reals, then*

$$E_1(W) \asymp \int_{\mathbf{R}} |\#\{(t_1, t_2) \in W : |t_1 + t_2 - x| \leq 1\}|^2 \; dx.$$

More generally, for any $r > 0$ we have

$$E_1(W) \asymp r^{O(1)} \int_{\mathbf{R}} |\#\{(t_1, t_2) \in W : |t_1 + t_2 - x| \leq r\}|^2 \; dx.$$

(ii) If W is a finite set of reals, then

$$E_1(W) \asymp \int_{-1}^{1} |\sum_{t \in W} e(t\theta)|^4 \; d\theta.$$

(iii) If $W_1, \dots, W_k$ are finite sets of reals, then

$$E_1(W_1 \cup \dots \cup W_k)^{1/4} \ll E_1(W_1)^{1/4} + \dots + E_1(W_k)^{1/4}.$$

(iv) If W is 1-separated and contained in an interval of length $T \geq 1$, then

$$(\#W)^2, (\#W)^4/T \ll E_1(W) \ll (\#W)^3.$$

(v) If W is contained in an interval I, which is in turn split into K equally sized subintervals $J_1, \dots, J_K$, then

$$E_1(W)^{1/3} \ll \sum_{k=1}^{K} E_1(W \cap J_k)^{1/3}.$$

`Note that the lower bound of` $(\#W)^4/T$ `would be expected to be attained if the set` W `is distributed ``randomly'' and is reasonably large (of size` $\gg \sqrt{T}$`). So getting upper bounds of the additive energy of similar strength to this lower bound can be viewed as a statement of ``pseudorandomness'' (or ``Gowers uniformity'') of this set.`

Proof. For (i), we just prove the first estimate, as the second follows from the first by several applications of the triangle inequality. The right-hand side can be expanded as

$$\sum_{t_1,t_2,t_3,t_4 \in W} |\{x : |t_1 + t_2 - x|, |t_3 + t_4 - x| \leq 1\}|.$$

Every quadruple contributing to $E_1(W)$ then contributes $\gg 1$ to the right-hand side, giving the upper bound. To get the matching lower bound, note that

$$\sum_{t_1,t_2,t_3,t_4 \in W} |\{x : |t_1 + t_2 - x|, |t_3 + t_4 - x| \leq 1/2\} \leq E_1(W)$$

and hence

$$E_1(W) \gg \int_{\mathbf{R}} |\#\{(t_1,t_2) \in W : |t_1 + t_2 - x| \leq 1/2\}|^2 \, dx.$$

The upper bound then follows from the triangle inequality.

For (ii), we can upper bound the indicator function of $[-1,1]$ by the Fourier transform of a non-negative bump function φ, so that the right-hand side is bounded by

$$\sum_{t_1,t_2,t_3,t_4 \in W} \varphi(t_1 + t_2 - t_3 - t_4)$$

which is then bounded by $O(E_1(W))$ by choosing the suppport of φ appropriately. The lower bound is established similarly (using the arguments in (i) to adjust the error tolerance 1 in the constraint $|t_1 + t_2 - x| \leq 1$ as necessary.)

For (iii), first observe we may remove duplicates and assume that the W_i are disjoint, then we can use (ii) and the triangle inequality.

For (iv), the first lower bound comes from considering the diagonal case $t_1 = t_3, t_2 = t_4$ and the upper bound comes from observing that once t_1, t_2, t_3

are fixed, there are only $O(1)$ choices for t_4 thanks to the 1-separated hypothesis. Finally, observe that

$$\int_{\mathbf{R}} |\#\{(t_1,t_2) \in W : |t_1+t_2-x| \leq 1\}|\ dx = (2\#W)^2$$

hence by Cauchy–Schwarz

$$\int_{\mathbf{R}} |\#\{(t_1,t_2) \in W : |t_1+t_2-x| \leq 1\}|^2\ dx \gg (\#W)^2/T$$

and the claim follows from (i).

For (v), write $a_k \coloneqq E_1(W \cap J_k)^{1/4}$. Each tuple (t_1,t_2,t_3,t_4) that contributes to $E_1(W)$ is associated to a tuple $J_{k_1}, J_{k_2}, J_{k_3}, J_{k_4}$ of intervals with $k_1+k_2-k_3-k_4 = O(1)$. By modifying the proof of (ii), the total contribution of such a tuple of intervals is

$$\ll \int_{\mathbf{R}} \prod_{j=1}^4 |\sum_{t \in W \cap J_{k_j}} e(t\theta)|\ d\theta$$

which by Cauchy–Schwarz is bounded by

$$\ll a_{k_1} a_{k_2} a_{k_3} a_{k_4}.$$

Thus we see that

$$E_1(W) \ll \sum_{m=O(1)} a * a * \tilde{a} * \tilde{a}(m)$$

where $\tilde{a}_k \coloneqq a_{-k}$ and $*$ denotes convolution on the integers. By Young's inequality we then have

$$E_1(W) \ll \|a\|_{\ell^{4/3}}^4$$

and the claim follows.

We remark that (v) can also be proven using [?, Lemma 4.8, (4.2)]. $\square$

We will also study the following related quantity. Given a set W and a scale $N > 1$, let $S(N,W)$ denote the *double zeta sum*

$$S(N,W) \coloneqq \sum_{t,t' \in W} |\sum_{n \in [N,2N]} n^{-i(t-t')}|^2.$$

We caution that this normalization differs from the one in [?], where $n^{-1/2-i(t-t')}$ is used in place of $n^{-i(t-t')}$. This sum may also be rearranged as

$$S(N,W) = \sum_{n,m \in [N,2N]} |R_W(n/m)|^2$$

where R_W is the exponential sum

$$R_W(x) \coloneqq \sum_{t \in W} x^{it}.$$

From the first formula it is clear that $S(N,W)$ is monotone non-decreasing in W, and from the second formula one has the triangle inequality

$$S(N, \bigcup_{i=1}^k W_i)^{1/2} \leq \sum_{i=1}^k S(N, W_i)^{1/2}$$

when the W_i are disjoint, and hence also when they are not assumed to be disjoint, thanks to the monotonicity.

To relate $S(N,W)$ to $E_1(W)$, we first observe the following lemma, implicit in [?] and made more explicit in [?, Lemma 11.4].

Lemma 10.3. *Let $T \geq 1$. If a_n is a 1-bounded sequence on $[N,2N]$ for some $1 \leq N \ll T^{O(1)}$, W is 1-separated in $[-T,T]$, and*

$$|\sum_{n \in [N,2N]} a_n n^{-it}| \geq V$$

for all $t \in W$ and some $V > 0$, then

$$V^2 E_1(W) \ll T^{o(1)} \sum_{n,m \in [N,2N]} |R_W(n/m)|^3 + T^{-50}.$$

Proof. By hypothesis, we have

$$V^2 E_1(W) \leq \sum_{t_1,t_2,t_3,t_4 \in W: |t_1+t_2-t_3-t_4| \leq 1} |\sum_{n \in [N,2N]} a_n n^{-it_4}|^2.$$

By standard Fourier arguments (see [?, Lemma 11.3]), we can bound

$$|\sum_{n \in [N,2N]} a_n n^{-it_4}| \ll T^{o(1)} \int_{t: |t-t_4| \leq T^{o(1)}} |\sum_{n \in [N,2N]} a_n n^{-it}|\ dt + T^{-100}.$$

Since each t_1, t_2, t_3 generates at most $O(1)$ choices for t_4, we conclude that

$$V^2 E_1(W) \ll T^{o(1)} \sum_{t_1,t_2,t_3 \in W} \int_{s: |s| \leq T^{o(1)}} |\sum_{n \in [N,2N]} a_n n^{-i(t_1+t_2-t_3+s)}|^2\ ds + T^{-50},$$

The right-hand side can be rewritten as

$$T^{o(1)} \sum_{n,m \in [N,2N]} a_n \overline{a_m} (n/m)^{-is} \overline{R_W}(n/m)^2 R_W(n/m) + T^{-50},$$

and the claim then follows from the triangle inequality. $\square$

Thus, $S(N,W)$ involves a second moment of R_W, while the energy $E_1(W)$ is related to the third moment. Using the trivial bound $|R_W(x)| \leq |W|$ we can then obtain the trivial bound

$$V^2 E_1(W) \ll T^{o(1)} |W| S(N,W) + T^{-50} \tag{10.1}$$

It is then natural to introduce the fourth moment

$$S_4(N,W) := \sum_{n,m \in [N,2N]} |R_W(n/m)|^4$$

since from Hölder's inequality one now has

$$V^2 E_1(W) \ll T^{o(1)} S(N,W)^{1/2} S_4(N,W)^{1/2} + T^{-50} \tag{10.2}$$

(cf. [?, Lemma 3]). The quantity $S_4(N,W)$ can also be expressed as

$$S_4(N,W) = \sum_{t_1,t_2,t_3,t_4 \in W} |\sum_{n \in [N,2N]} n^{-i(t_1+t_2-t_3-t_4)}|^2.$$

One can bound this quantity by an $S(N,W)$ type expression:

Lemma 10.4. *If $W \subset [-T,T]$ is 1-separated and $1 \leq N \ll T^{O(1)}$, then one has*

$$S_4(N,W) \ll T^{o(1)} u^2 S(N,U) + T^{-100}$$

for some $1 \leq u \ll |W|$ and 1-separated subset U of $[-2T,2T]$ with

$$u|U| \ll |W|^2 \tag{10.3}$$

and

$$u^2|U| \ll E_1(W). \tag{10.4}$$

This result appears implicitly in [?, p. 229], and is made more explicit in the proof of [?, Lemma 11.6].

Proof. One can bound

$$S_4(N,W) \ll T^{o(1)} \sum_{t_1,t_2,t_3,t_4 \in W} \int_{t=t_1+t_2-t_3-t_4+O(T^{o(1)})} |\sum_{n \in [N,2N]} n^{-it}|^2 \, dt + T^{-100},$$

and hence

$$S_4(N,W) \ll T^{o(1)} \sum_{t_1,t_2 \in [-2N,2N] \cap \mathbf{Z}} f(t_1) f(t_2) \int_{t=t_1-t_2+O(T^{o(1)})} |\sum_{n \in [N,2N]} n^{-it}|^2 \, dt + T^{-100}$$

where f is the counting function

$$f(t) := |\{(t_1,t_2) \in W : |t - t_1 - t_2| \leq 1\}|.$$

Note that f is integer valued and bounded above by $|W|$. By dyadic decomposition, one can then find $1 \leq u \ll |W|$ and a subset U of $[-2N,2N] \cap \mathbf{Z}$ such that $f(t) \asymp u$ for $t \in U$, and

$$S_4(N,W) \ll T^{o(1)} \sum_{t_1,t_2 \in U} u^2 \int_{t=t_1-t_2+O(T^{o(1)})} |\sum_{n \in [N,2N]} n^{-it}|^2 \, dt + T^{-100}$$

which we can rearrange as

$$S_4(N,W) \ll T^{o(1)} u^2 \int_{s=O(T^{o(1)})} \sum_{n,m\in[N,2M]} (n/m)^{is} |R_U(n/m)|^2 \; ds + T^{-100}$$

and hence by the triangle inequality

$$S_4(N,W) \ll T^{o(1)} v^2 S(N,V) + T^{-100}.$$

Also, by double counting one easily verifies the claims (10.3), (10.4). The claim follows. □

10.2 Large value additive energy region

Because the cardinality $|W|$ and additive energy $E_1(W)$ of a set W are correlated with each other, as well as with the double zeta sum $S(N,W)$, we will not be able to consider them separately, and instead we will need to consider the possible joint exponents for these two quantities. We formalize this via the following set:

Definition 10.5 (Large value energy region)**.** The *large value energy region* $\mathcal{E} \subset \mathbf{R}^5$ is defined to be the set of all fixed tuples $(\sigma,\tau,\rho,\rho^*,s)$ with $1/2 \le \sigma \le 1$, $\tau,\rho,\rho' \ge 0$, such that there exists an unbounded $N > 1$, $T = N^{\tau+o(1)}$, $V = N^{\sigma+o(1)}$, a 1-bounded sequence a_n on $[N,2N]$, and a 1-separated subset W of cardinality $N^{\rho+o(1)}$ in an interval J of length T such that

$$\left|\sum_{n\in[N,2N]} a_n n^{-it}\right| \ge V \tag{10.5}$$

for all $t \in W$, and such that $E_1(W) = N^{\rho^*+o(1)}$ and $S(N,W) = N^{s+o(1)}$.

We define the *large value energy region for zeta* $\mathcal{E}_\zeta \subset \mathbf{R}^5$ similarly, but now the interval J is required to be of the form $[T,2T]$, and the sequence a_n is required to be of the form $1_I(n)$ for some interval $I \subset [N,2N]$. Thus, in order for $(\sigma,\tau,\rho,\rho^*,s)$ to lie in $\mathcal{E}_\zeta$, there must exist an unbounded $N > 1$, $T = N^{\tau+o(1)}$, $V = N^{\sigma+o(1)}$, an interval I in $[N,2N]$, and $W = W$ is a 1-separated subset of cardinality $N^{\rho+o(1)}$ in $[T,2T]$ such that

$$\left|\sum_{n\in I} n^{-it}\right| \ge V \tag{10.6}$$

for all $t \in W$, and such that $E_1(W) = N^{\rho^*+o(1)}$ and $S(N,W) = N^{s+o(1)}$.

Clearly we have

Lemma 10.6 (Trivial containment)**.** *We have $\mathcal{E}_\zeta \subset \mathcal{E}$.*

These region is related to $\mathrm{LV}(\sigma,\tau)$ and as follows:

Lemma 10.7. *For any fixed $1/2 \leq \sigma \leq 1, \tau \geq 0$, we have*

$$\mathrm{LV}(\sigma,\tau) = \sup\{\rho : (\sigma,\tau,\rho,\rho^*,s) \in \mathcal{E}\}$$

and

$$\mathrm{LV}_\zeta(\sigma,\tau) = \sup\{\rho : (\sigma,\tau,\rho,\rho^*,s) \in \mathcal{E}_\zeta\}$$

In particular, we have $\rho \leq \mathrm{LV}(\sigma,\tau)$ for all $(\sigma,\tau,\rho,\rho^*,s) \in \mathcal{E}$, and $\rho \leq \mathrm{LV}_\zeta(\sigma,\tau)$ for all $(\sigma,\tau,\rho,\rho^*,s) \in \mathcal{E}_\zeta$.

Proof. Clear from definition. □

Inspired by this, we can define

Definition 10.8. For any fixed $1/2 \leq \sigma \leq 1, \tau \geq 0$, we define

$$\mathrm{LV}^*(\sigma,\tau) \coloneqq \sup\{\rho^* : (\sigma,\tau,\rho,\rho^*,s) \in \mathcal{E}\}$$

and

$$\mathrm{LV}^*_\zeta(\sigma,\tau) \coloneqq \sup\{\rho^* : (\sigma,\tau,\rho,\rho^*,s) \in \mathcal{E}_\zeta\}.$$

Thus these exponents are upper bounds for the additive energy of large values of Dirichlet polynomials which may or may not be of zeta function type.

As usual, we have an equivalent non-asymptotic definition of the large value energy region:

Lemma 10.9 (Non-asymptotic form of large value energy region)**.** *Let $1/2 \leq \sigma \leq 1$, $\tau \geq 0$, $\rho, \rho^* \geq 0$, and $s \in \mathbf{R}$ be fixed. Then the following are equivalent:*

(i) $(\sigma,\tau,\rho,\rho^*) \in \mathcal{E}$.

(ii) For every $\varepsilon > 0$ and $C > 0$, there exists $N \geq C$, $N^{\tau-\delta} \leq T \leq N^{\tau+\delta}$, $N^{\sigma-\delta} \leq V \leq N^{\sigma+\delta}$, a 1-bounded a_n for each $n \in [N,2N]$, and a 1-separated subset W of an interval J of length T such that (10.5) *holds for all $t \in W$, with*

$$N^{\rho-\varepsilon} \leq |W| \leq N^{\rho+\varepsilon},$$
$$N^{\rho^*-\varepsilon} \leq E_1(W) \leq N^{\rho^*+\varepsilon}$$
$$N^{s-\varepsilon} \leq S(N,W) \leq N^{s+\varepsilon}.$$

Similarly with $\mathcal{E}$ replaced by $\mathcal{E}_\zeta$, J replaced by $[N,2N]$, a_n replaced by an interval I in $[N,2N]$, and (10.5) *replaced by* (10.6)*.*

This lemma is proven by a routine expansion of the definitions, and is omitted.

Lemma 10.10 (Basic properties)**.**

(i) (Monotonicity in σ) If $(\sigma,\tau,\rho,\rho^*,s) \in \mathcal{E}$, then $(\sigma',\tau',\rho,\rho^*,s) \in \mathcal{E}$ for all $1/2 \leq \sigma' \leq \sigma$ and $\tau' \geq \tau$.

(ii) (Subdivision) If $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ *and* $0 \le \tau' \le \tau$*, then there exists* $(\sigma, \tau', \rho', (\rho')^*, s') \in \mathcal{E}$ *such that*

$$\rho' \le \rho \le \rho' + \tau - \tau'$$

and

$$(\rho')^* + (\rho - \rho') \le \rho^* \le \rho' + 3(\rho - \rho').$$

and

$$s' \le s \le s' + 2(\rho - \rho').$$

(iii) (Trivial bounds) If $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$*, one has*

$$2\rho, 4\rho - \tau \le \rho^* \le 3\rho.$$

Proof. ... □

Lemma 10.11 (Raising to a power)**.** *If* $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$*, then* $(\sigma, \tau/k, \rho/k, (\rho^*)/k, s/k) \in \mathcal{E}$ *for any integer* $k \ge 1$*.*

Proof. ... □

10.3 Known relations for the large value energy region

Theorem 10.12 (Reflection principle)**.** *[**?**, §11.5] If* $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ *with* $\sigma \ge 3/4$ *and* $\tau > 1$*, then for any integer* $k \ge 1$*, either* $\rho \le 2 - 2\sigma$*, or there exists* $0 < \alpha \le k(\tau - 1)$ *and* $(\sigma, \tau/\alpha, \rho/\alpha, \rho^*/\alpha, s'/\alpha) \in \mathcal{E}$ *such that*

$$\rho \le \min(2 - 2\sigma, k(3 - 4\sigma)/2 + s' - 1).$$

Proof. By definition, there exists an unbounded $N > 1$, $T = N^{\tau + o(1)}$, a 1-bounded sequence a_n on $[N, 2N]$, and a 1-separated subset W of an interval of length T (which we can normalize without loss of generality to be $[0, T]$) such that

$$|\sum_{n \in [N,2N]} a_n n^{-it}| \ge N^{\sigma + o(1)}$$

for all $t \in W$, with $|W| = N^{\rho + o(1)}$, $E_1(W) = N^{\rho^* + o(1)}$ and $S(N, W) = N^{s + o(1)}$. By [**?**, (11.58)], one has

$$|W|^2 \ll T^{o(1)}(|W|N^{2-2\sigma} + N^{1-2\sigma}|W|^2 + N^{(3-4\sigma)/2} \int_{v = O(T^{o(1)})} \sum_{t,t' \in W} |\sum_{n \le 4T/N} n^{-1/2+it-it'+iv}|\, dv).$$

Since $\sigma > 1/2$, the $N^{1-2\sigma}|W|^2$ term can be dropped. Applying Hölder's inequality and dyadic pigeonholing as in [**?**, (11.59)], we conclude that

$$|W| \ll T^{o(1)}(N^{2-2\sigma} + N^{k(3-4\sigma)/2}(\sum_{t,t' \in W} |\sum_{n \in [N', 2N']} b_n n^{-1/2+it-it'+iv}|^2)^{1/2}$$

for some $v = O(T^{o(1)})$ and coefficients $b_n = O(T^{o(1)})$, and some $N' \ll (4T/N)^k$. After passing to a subsequence if necessary, we may assume that $N' = N^{\alpha+o(1)}$ for some $0 \leq \alpha \leq k(\tau - 1)$. If $\alpha = 0$ then the second term here is negligible compared to the first and we obtain $\rho \leq 2 - 2\sigma$, so suppose that $\alpha > 0$. Using [**?**, Lemma 11.1] to eliminate the $b_n n^{-1/2+iv}$ coefficients, we conclude that

$$|W| \ll T^{o(1)}(N^{2-2\sigma} + N^{k(3-4\sigma)/2-1}S(N', W).$$

By construction, we have $S(N', W) = (N')^{s'/\alpha+o(1)} = N^{s'+o(1)}$ for some tuple $(\sigma, \tau/\alpha, \rho/\alpha, \rho^*/\alpha, s'/\alpha) \in \mathcal{E}$. The claim follows. ☐

Heuristically one expects $s \leq \max(\rho+1, 2\rho)+1$ (see [**?**, (11.63)]). There is one easy case in which this is true:

Lemma 10.13. *If $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ with $\tau < 1$, then $s \leq \max(\rho + 1, 2\rho) + 1$.*

Proof. See the discussion after [**?**, (11.63)]. ☐

Another bound is

Lemma 10.14. *[**?**, Lemma 11.2] If (k, ℓ) is an exponent pair with $k > 0$, and $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$, then*

$$s \leq \max(\rho + 1, 5\rho/3 + \tau/3, \frac{2 + 3k + 4\ell}{1 + 2k + 2\ell}\rho + \frac{k + \ell}{1 + 2k + 2 \ll}\tau) + 1.$$

Finally, we have the useful

Lemma 10.15 (Heath-Brown bound on double sums)**.** *If $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$, then*

$$s \leq \max(\rho + 1, 2\rho, 5\rho/4 + \tau/2) + 1.$$

Note that if $\tau \leq 3/2$, the $5\rho/4+\tau/2$ term is bounded by the convex combination $(3/4)(\rho + 1) + (1/4)(2\rho)$ and may therefore be omitted.

Proof. See [**?**, Theorem 1] or [**?**, Lemma 11.5]. ☐

Lemma 10.4 can be formulated in terms of the large value energy region as follows.

Lemma 10.16. *If $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$, then there exists $(\sigma, \tau, \rho', (\rho')^*, s') \in \mathcal{E}$ and $0 \leq \kappa \leq \rho$ such that*

$$\kappa + \rho' \leq 2\rho$$

$$2\kappa + \rho' \leq \rho^*$$

and

$$\rho^* + 2\sigma \leq \kappa + (s + s')/2.$$

Proof. By definition, there exists an unbounded $N \geq 1$, $T = N^{\tau+o(1)}$, $V = N^{\sigma+o(1)}$, 1-bounded coefficients a_n on $[N, 2N]$, and a 1-separated subset W of an interval of length T such that

$$|\sum_{n\in[N,2N]} a_n n^{-it}| \geq V$$

for all $t \in W$, with $|W| = N^{\rho+o(1)}$, $|W^*| = N^{\rho^*+o(1)}$, and $S(N, W) = N^{s+o(1)}$, then from (10.2) we have

$$V^2 E_1(W) \ll T^{o(1)} S(N, W)^{1/2} S_4(N, W)^{1/2} + T^{-50}.$$

By Lemma 10.4, there exists $1 \leq u \ll |W|$ and a 1-separated subset U of $[-2T, 2T]$ such that such that

$$V^2 E_1(W) \ll T^{o(1)} u S(N, W)^{1/2} S(N, U)^{1/2} + T^{-50}$$

with (10.3), (10.4) holding. Since W is non-empty, $E_1(W) \geq 1$ and $V \geq N^{1/2} \geq 1$, so the T^{-50} error here may be discarded. Passing to a subsequence, we may assume that $u = N^{\kappa+o(1)}$ for some $0 \leq \kappa \leq \rho$, and that $|U| = N^{\rho'+o(1)}$ for some $\rho' \geq 0$. Then we have $S_2(N, U) = s'$ for some $(\sigma, \tau, \rho', (\rho')^*, s') \in \mathcal{E}$, and the claim follows. □

`These bounds on the double zeta sums can be used to control additive energies:`

Theorem 10.17 (Heath-Brown relation)**.** *[?, (33)] If $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$, then one has*

$$\rho^* \leq 1 - 2\sigma + \max(\rho + 1, 2\rho, 5\rho/4 + \tau/2)/2 + \max(\rho^* + 1, 4\rho, 3\rho^*/4 + \rho + \tau/2)/2.$$

Proof. By Lemma 10.16 followed by we have

$$\rho^* + 2\sigma \leq \kappa + (\max(\rho + 1, 2\rho, 5\rho/4 + \tau/2) + \max(\rho' + 1, 2\rho', 5\rho'/4 + \tau/2))/2 + 1$$

for some $0 \leq \kappa \leq \rho$ with

$$\kappa + \rho' \leq 2\rho$$

$$2\kappa + \rho' \leq \rho^*$$

In particular,

$$2\kappa + 5\rho'/4 \leq 3\rho^*/4 + \rho$$

and the claim follows after moving the κ inside the second maximum and performing some algebra. □

Corollary 10.18 (Simplified Heath-Brown relation)**.** *If $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ and $\tau \leq 3/2$, then*

$$\rho^* \leq \max(3\rho + 1 - 2\sigma, \rho + 4 - 4\sigma, 5\rho/2 + (3 - 4\sigma)/2).$$

`This result essentially appears as [?, Lemma 3].`

Proof. Apply the previous result. For $\tau \leq 3/2$ we observe that $5\rho/4 + \tau/2$ is less than $5\rho/4 + 3/4$, which is a convex combination of $\rho + 1$ and 2ρ. Similarly $3\rho^*/4 + \rho + \tau/2$ is less than $3\rho^*/4 + \rho + 3/4$, which is a convex combination of $\rho^* + 1$ and 4ρ. We conclude that

$$\rho^* \leq 1 - 2\sigma + \max(\rho + 1, 2\rho)/2 + \max(\rho^* + 1, 4\rho)/2.$$

Thus ρ^* is less than one of

$$1-2\sigma+(\rho+\rho^*+2)/2, 1-2\sigma+(5\rho+1)/2, 1-2\sigma+(2\rho+\rho^*+1)/2, 1-2\sigma+(6\rho)/2;$$

solving for ρ^*, we conclude

$$\rho^* \leq \max(4 - 4\sigma + \rho, (3 - 4\sigma)/2 + 5\rho/2, 3 - 4\sigma + 2\rho, 1 - 2\sigma + 3\rho).$$

But since $\sigma \geq 1/2$, $3 - 4\sigma + 2\rho$ is less than $5/2 - 3\sigma + 2\rho$, which is the mean of $4 - 4\sigma + \rho$ and $1 - 2\sigma + 3\rho$. Thus

$$\rho^* \leq \max(4 - 4\sigma + \rho, (3 - 4\sigma)/2 + 5\rho/2, 1 - 2\sigma + 3\rho),$$

which gives the claim. □

Lemma 10.19 (Second Heath-Brown relation)**.** *If If $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ then*

$$\rho \leq \max(2 - 2\sigma, \rho^*/4 + \max(\tau/4 + k(3 - 4\sigma)/4, k/4 + k(1 - 2\sigma)/2))$$

for any positive integer k.

Proof. See [**?**, Lemma 4]. □

Lemma 10.20 (Guth-Maynard relation)**.** *If $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ then*

$$\rho \leq \max(2 - 2\sigma, 1 - 2\sigma + \max(S_1, S_2, S_3)/3)$$

where S_1, S_2, S_3 are real numbers with

$$S_1 \leq -10,$$

$$S_2 \leq \max(2 + 2\rho, \tau + 1 + (2 - 1/k)\rho, 2 + 2\rho + (\tau/2 - 3\rho/4)/k)$$

for any positive integer k and

$$S_3 \leq 2\tau + \rho/2 + \rho^*/2$$

and also

$$S_3 \leq \max(2\tau + 3\rho/2, \tau + 1 + \rho/2 + \rho^*/2).$$

Proof. This follows from [**?**, Propositions 4.6, 5.1, 6.1, 8.1, 10.1, (5.5)]. □

Lemma 10.21 (Second Guth-Maynard relation). *[?, Lemma 1.7] If $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ then*

$$\rho^* \leq \rho + s - 2\sigma.$$

In particular, from Lemma 10.15 we see for $\tau \leq 3/2$ that

$$\rho^* \leq \max(3\rho + 1 - 2\sigma, 2\rho + 2 - 2\sigma).$$

Proof. By definition, we can find an unbounded $N > 1$, $T = N^{\tau+o(1)}$, $V = N^{\sigma+o(1)}$, a 1-bounded sequence a_n on $[N, 2N]$, and a 1-separated subset W of an interval of length T such that

$$|\sum_{n \in [N,2N]} a_n n^{-it}| \geq V$$

for all $t \in W$, with $|W| = N^{\rho+o(1)}$, $E_1(W) = N^{\rho^*+o(1)}$, and $S(N, W) = N^{s+o(1)}$. From (10.1) one has

$$V^2 E_1(W) \ll T^{o(1)} |W| S(N, W) + T^{-50}.$$

Since W is non-empty, $E_1(W) \geq 1$, and $V \gg 1$, so the T^{-50} error can be discarded. The claim then follows. □

Lemma 10.22 (Third Guth-Maynard relation). *If $(\sigma, \tau, \rho, \rho^*) \in \mathcal{E}$ and $1 \leq \tau \leq 4/3$, then*

$$\rho^* \leq \max(\rho + 4 - 4\sigma, 21\rho/8 + \tau/4 + 1 - 2\sigma, 3\rho + 1 - 2\sigma).$$

Proof. See [?, Proposition 11.1]. □

`We can put this all together to prove the Guth--Maynard large values theorem (Theorem 7.15):`

Theorem 10.23 (Guth–Maynard large values theorem, again). *One has*

$$\mathrm{LV}(\sigma, \tau) \leq \max(2 - 2\sigma, 18/5 - 4\sigma, \tau + 12/5 - 4\sigma).$$

Proof. For $\sigma \leq 7/10$ this follows from Lemma 7.7, and for $\sigma \geq 8/10$ it follows from Lemma 7.10. Thus we may assume that $7/10 \leq \sigma \leq 8/10$. By subdivision (Lemma 7.3(ii)) it then suffices to treat the case $\tau = 6/5$, that is to say to show that

$$\rho \leq 18/5 - 4\sigma$$

whenever $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ with $\tau = 6/5$ and $7/10 \leq \sigma \leq 8/10$.

Applying Lemma 10.20 and discarding the very negative S_1 term, we have

$$\rho \leq \max(2 - 2\sigma, 1 - 2\sigma + \max(S_2, S_3)/3)$$

where S_2, S_3 are real numbers with

$$S_2 \leq \max(2 + 2\rho, \tau + 1 + (2 - 1/k)\rho, 2 + 2\rho + (\tau/2 - 3\rho/4)/k)$$

for any positive integer k and

$$S_3 \leq 2\tau + \rho/2 + \rho^*/2$$

and also

$$S_3 \leq \max(2\tau + 3\rho/2, \tau + 1 + \rho/2 + \rho^*/2).$$

From the latter bound and Lemma 10.22, one has

$$S_3 \leq \max(2\tau + 3\rho/2, \tau + \rho + 3 - 2\sigma, \tau + 2\rho + 3/2 - \sigma, 9\tau/8 + 29\rho/16 + 3/2 - \sigma).$$

Inserting this and the S_2 bound (with $k = 4$) into the bound for ρ and simplifying (using $\tau = 6/5$), we eventually obtain the desired bound $\rho \leq 18/5 - 4\sigma$. □

Chapter 11

Zero density theorems

Definition 11.1 (Zero density exponents)**.** For $\sigma \in \mathbf{R}$ and $T > 0$, let $N(\sigma, T)$ denote the number of zeroes ρ of the Riemann zeta function with $\mathrm{Re}(\rho) \geq \sigma$ and $|\mathrm{Im}(\rho)| \leq T$.

If $1/2 \leq \sigma < 1$ is fixed, we define the zero density exponent $\mathrm{A}(\sigma) \in [-\infty, \infty)$ to be the infimum of all (fixed) exponents A for which one has

$$N(\sigma - \delta, T) \ll T^{A(1-\sigma)+o(1)}$$

whenever T is unbounded and $\delta > 0$ is infinitesimal.

`The shift by` δ `is for technical convenience, it allows for` $\mathrm{A}(\sigma)$ `to control (very slightly) the zeroes to the left of` $\mathrm{Re}s = \sigma$`. In non-asymptotic terms:` $\mathrm{A}(\sigma)$ `is the infimum of all` A `such that for every` $\varepsilon > 0$ `there exists` $C, \delta > 0$ `such that`

$$N(\sigma - \delta, T) \leq CT^{A(1-\sigma)+\varepsilon}$$

`whenever` $T \geq C$`.`

Lemma 11.2 (Basic properties of A)**.** *(i)* $\sigma \mapsto (1-\sigma)\mathrm{A}(\sigma)$ *is non-increasing and left-continuous, with* $\mathrm{A}(1/2) = 2$*.*

(ii) If the Riemann hypothesis holds, then $\mathrm{A}(\sigma) = -\infty$ *for all* $1/2 < \sigma \leq 1$*.*

Proof. The claim (i) is clear using the Riemann-von Mangoldt formula [**?**, Theorem 1.7] and the functional equation. The claim (ii) is also clear. $\square$

Remark 11.3. One can ask what happens if one omits the δ shift. Thus, define $\mathrm{A}_0(\sigma)$ to be the infimum of all fixed exponents A for which $N(\sigma, T) \ll T^{A(1-\sigma)+o(1)}$ for unbounded T. Then it is not difficult to see that

$$\lim_{\sigma' \to \sigma^+} \mathrm{A}(\sigma) \leq \mathrm{A}_0(\sigma) \leq \mathrm{A}(\sigma)$$

for any fixed $1/2 < \sigma < 1$; thus A_0 is basically the same exponent at A, except possibly at jump discontinuities of the left-continuous function A, in which case it could theoretically take on a different value. (But we do not expect such discontinuities to actually exist.) Thus there is not a major difference between $\mathrm{A}(\sigma)$ and $\mathrm{A}_0(\sigma)$, but the former has some very slight technical advantages (such as the aforementioned left continuity).

The quantity $\|\mathrm{A}\|_\infty := \sup_{1/2 \le \sigma < 1} \mathrm{A}(\sigma)$ is of particular importance to the theory of primes in short intervals; see Section 13. From Lemma 11.2 we have $\|\mathrm{A}\|_\infty \ge 2$. It is conjectured that this is an equality.

Conjecture 11.4 (Density hypothesis)**.** *One has* $\|\mathrm{A}\|_\infty = 2$. *Equivalently,* $\mathrm{A}(\sigma) \le 2$ *for all* $1/2 \le \sigma < 1$.

Indeed, the Riemann hypothesis implies the stronger assertion that $\mathrm{A}(\sigma) = -\infty$ for all $12 < \sigma < 1$. However, for many applications to the prime numbers in short intervals, the density hypothesis is almost as powerful; see Section 13.

Upper bounds on $\mathrm{A}(\sigma)$ can be obtained from large value theorems via the following relation.

Lemma 11.5 (Zero density from large values)**.** *Let* $1/2 < \sigma < 1$. *Then*

$$\mathrm{A}(\sigma)(1-\sigma) \le \max(\sup_{\tau \ge 2} \mathrm{LV}_\zeta(\sigma,\tau)/\tau, \limsup_{\tau\to\infty} \mathrm{LV}(\sigma,\tau)/\tau).$$

Proof. Write the right-hand side as B, then $B \ge 0$ (from Lemma 7.3(iii)) and we have

$$\mathrm{LV}_\zeta(\sigma,\tau) \le B\tau \tag{11.1}$$

for all $\tau \ge 1$, and

$$\mathrm{LV}(\sigma,\tau) \le (B+\varepsilon)\tau \tag{11.2}$$

whenever $\varepsilon > 0$ and τ is sufficiently large depending on ε (and σ). It would suffice to show, for any $\varepsilon > 0$, that $N(\sigma - o(1), T) \ll T^{B+O(\varepsilon)+o(1)}$ as $T \to \infty$.

By dyadic decomposition, it suffices to show for large T that the number of zeroes with real part at least $\sigma - o(1)$ and imaginary part in $[T, 2T]$ is $\ll T^{B+O(\varepsilon)+o(1)}$. From the Riemann-von Mangoldt theorem, there are only $O(\log T)$ zeroes whose imaginary part is within $O(1)$ of a specified ordinate $t \in [T, 2T]$, so it suffices to show that given some zeroes $\sigma_r + it_r$, $r = 1, \dots, R$ with $\sigma - o(1) \le \sigma_r < 1$ and $t_r \in [T, 2T]$ 1-separated, that $R \ll T^{B+O(\varepsilon)+o(1)}$.

Suppose that one has a zero $\sigma_r + it_r$ of this form. Then by a standard approximation to the zeta function [**?**, Theorem 1.8], one has

$$\sum_{n \le T} \frac{1}{n^{\sigma_r + it_r}} \ll T^{-1/2}.$$

Let $0 < \delta_1 < \varepsilon$ be a small quantity (independent of T) to be chosen later, and let $0 < \delta_2 < \delta_1$ be sufficiently small depending on δ_1, δ_2. By the triangle inequality, and refining the sequence t_r by a factor of at most 2, we either have

$$\left| \sum_{T^{\delta_1} \le n \le T} \frac{1}{n^{\sigma_r + it_r}} \right| \gg T^{-\delta_2}$$

for all r, or

$$\sum_{n \leq T^{\delta_1}} \frac{1}{n^{\sigma_r + it_r}} \ll T^{-\delta_2} \tag{11.3}$$

for all r.

Suppose we are in the former ("Type I") case, we perform a smooth partition of unity, and conclude that

$$\left| \sum_{T^{\delta_1} \leq n \leq T} \frac{\psi(n/N)}{n^{\sigma_r + it_r}} \right| \gg T^{-\delta_2 - o(1)}$$

for some fixed bump function ψ supported on $[1/2, 1]$, and some $T^{\delta_1} \ll N \ll T$.

We divide into several cases depending on the size of N. First suppose that $N \ll T^{1/2}$. The variable n is restricted to the interval $I \coloneqq [\max(N/2, T^{\delta_1}), N]$. We have

$$\left| \sum_{n \in I} \psi(n/N)(n/N)^{-\sigma_r} n^{-it_r} \right| \gg N^{\sigma} T^{-\delta_2 - o(1)}.$$

Performing a Fourier expansion of $\psi(n/N)(n/N)^{-\sigma_r}$ in $\log n$ and using the triangle inequality, we can bound

$$\sum_{n \in I} \psi(n/N)(n/N)^{-\sigma_r} n^{-it_r} \ll_A \int_{\mathbf{R}} \left| \sum_{n \in I} \frac{1}{n^{it}} \right| (1 + |t - t_r|)^{-A} \, dt$$

for any $A > 0$, so by the triangle inequality we conclude that

$$\left| \sum_{n \in I} n^{-it'_r} \right| \gg N^{\sigma} T^{-\delta_2 - o(1)}$$

for some $t'_r = t_r + O(T^{o(1)})$. By refining the t_r by a factor of $T^{o(1)}$ if necessary, we may assume that the t'_r are 1-separated, and by passing to a subsequence we may assume that $T = N^{\tau + o(1)}$ for some $2 \leq \tau \leq 1/\delta_1$, then we conclude that

$$\left| \sum_{n \in I} \frac{1}{n^{it'_r}} \right| \gg N^{\sigma - \delta_2/\delta_1 + o(1)}$$

for all remaining r. By Definition 8.1 we then have (for δ_2 small enough)

$$R \ll N^{\mathrm{LV}_\zeta(\sigma,\tau) + \varepsilon + o(1)} \ll T^{\mathrm{LV}_\zeta(\sigma,\tau)/\tau + \varepsilon + o(1)}$$

and the claim follows in this case from (11.1).

In the case $N \asymp T$, a standard summation by parts argument **cite** gives

$$\sum_{T^{\delta_1} \leq n \leq T} \frac{\psi(n/N)}{n^{\sigma_r + it_r}} \ll T^{-\sigma_r}$$

which leads to a contradiction. So the only remaining case is when $T^{1/2} \ll N \ll o(T)$. Here we can ignore the cutoffs on n and write

$$\sum_n \psi(n/N)(n/N)^{\sigma_r} n^{-it_r} \gg N^{\sigma} T^{-\delta_2 - o(1)}.$$

Applying the van der Corput B-process **give cite**, we have

$$\sum_m \psi(2\pi t_r/mN)(2\pi t_r/Nm)^{\sigma_r} m^{-it_r} \gg M^{1/2} N^{\sigma-1/2} T^{-\delta_2-o(1)};$$

where $M := 2\pi T/N \ll N^{1/2}$. In particular

$$\sum_{m\in[M/10,10M]} \psi(2\pi t_r/mN)(2\pi t_r/Nm)^{\sigma_r} m^{-it_r} \gg M^{\sigma} T^{-\delta_2-o(1)};$$

since $N \gg T^{1/2}$ and $\sigma \geq 1/2$. Performing a Fourier expansion as before, we conclude that

$$\sum_{m\in[M/10,10M]} m^{-it'_r} \ll M^{\sigma} T^{-\delta_2-o(1)}$$

for some $t'_r = t_r + O(T^{o(1)})$, and one can argue as in the $N \ll T^{1/2}$ case (partitioning $[M/10, 10M]$ into $O(1)$ intervals each contained in some $[M', 2M']$ with $M' \ll T^{1/2}$).

Now suppose instead we are in the latter ("Type II") case (11.3). We multiply both sides of (11.3) by the mollifier $\sum_{m\leq T^{\delta_2/2}} \frac{1}{m^{\sigma_r+it_r}}$ to obtain

$$\left|1 + \sum_{T^{\delta_2/2}\leq n\leq T^{\delta_1+\delta_2/2}} \frac{a_n}{n^{\sigma_r+it_r}}\right| = o(1)$$

where a_n is some sequence with $a_n \ll T^{o(1)}$. By dyadic decomposition and the pigeonhole principle, and refining the t_r by a factor of $O(T^{o(1)})$ as needed, we can then find an interval I in $[N, 2N]$ with $T^{\delta_2/2} \ll N \ll T^{\delta_1+\delta_2/2}$ such that

$$\left|\sum_{n\in I} \frac{a_n}{n^{\sigma_r+it_r}}\right| \gg T^{-o(1)}$$

and hence by Fourier expansion of $\frac{1}{n^{\sigma_r}}$ in $\log n$

$$\left|\sum_{n\in I} \frac{a_n}{n^{it'_r}}\right| \gg N^{\sigma_r} T^{-o(1)}$$

for some $t'_r = t_r + O(T^{o(1)})$; by refining the t_r by a further factor of $T^{o(1)}$ we may assume that the t'_r are also $O(1)$-separated; we can also pigeonhole so that $T = N^{\tau+o(1)}$ for some $\frac{1}{\delta_1+\delta_2/2} \leq \tau \leq \frac{1}{\delta_2/2}$. Applying Lemma 7.2, we conclude that

$$R \ll N^{\mathrm{LV}(\sigma,\tau)+o(1)} = T^{\mathrm{LV}(\sigma,\tau)/\tau+o(1)}$$

and the claim follows in this case from (11.2). □

Recently, a partial converse to the above lemma was established:

Lemma 11.6 (Large values from zero density)**.** *[?, Theorem 1.2] If $\tau > 0$ and $1/2 \leq \sigma \leq 1$ are fixed, then*

$$\mathrm{LV}_\zeta(\sigma,\tau)/\tau \leq \max(\frac{1}{2}, \sup_{\sigma \leq \sigma' \leq 1} \mathrm{A}(\sigma')(1-\sigma') + \frac{\sigma'-\sigma}{2}).$$

Proof. Let N go to infinity, $T = N^{\tau+o(1)}$, and $I \subset [N,2N]$ be an interval, and $t_1, \dots, t_R \in [T,2T]$ be 1-separated with

$$\left|\sum_{n\in I} \frac{1}{n^{it_r}}\right| \gg N^{\sigma-o(1)}$$

uniformly for all r. By [**?**, Theorem 1.2], we have for any fixed $\delta > 0$ that

$$R \ll T^\delta \sup_{\sigma-\delta \leq \sigma' \leq 1} T^{\frac{\sigma'-\sigma}{2}} N(\sigma', O(T)) + T^{\frac{1-\sigma}{2}+\delta}.$$

Using Definition 11.1, we conclude that

$$R \ll T^{\max(\frac{1}{2}, \sup_{\sigma-\delta\leq\sigma'\leq 1} \mathrm{A}(\sigma')(1-\sigma') + \frac{\sigma'-\sigma}{2}) + O(\delta)}$$

and thus

$$\mathrm{LV}_\zeta(\sigma,\tau) \leq \tau \max(\frac{1}{2}, \sup_{\sigma-\varepsilon\leq\sigma'\leq 1} \mathrm{A}(\sigma')(1-\sigma') + \frac{\sigma'-\sigma}{2}) + O(\delta).$$

Here the implied constant in the $O()$ notation is understood to be uniform in δ. Letting δ go to zero, and using left-continuity of A, we obtain the claim. □

The suprema in Lemma 11.5 require unbounded values of τ, but thanks to the ability to raise to a power, we can reduce to a bouned range of τ. Here is a basic such reduction, suited for machine-assisted proofs:

Corollary 11.7. *Let $1/2 < \sigma < 1$ and $\tau_0 > 0$. Then*

$$\mathrm{A}(\sigma)(1-\sigma) \leq \max\left(\sup_{2\leq\tau<\tau_0} \mathrm{LV}_\zeta(\sigma,\tau)/\tau, \sup_{\tau_0\leq\tau\leq 2\tau_0} \mathrm{LV}(\sigma,\tau)/\tau\right)$$

with the convention that the first supremum is $-\infty$ if it is vacuous (i.e., if $\tau_0 < 2$).

Proof. Denote the right-hand side by B, thus

$$\mathrm{LV}(\sigma,\tau) \leq B\tau$$

for all $\tau_0 \leq \tau \leq 2\tau_0$, and

$$\mathrm{LV}_\zeta(\sigma,\tau) \leq B\tau \tag{11.4}$$

whenever $2 \le \tau < 2\tau_0$. From Lemma 7.6 we then have

$$\mathrm{LV}(\sigma,\tau) \le B\tau$$

for all $k\tau_0 \le \tau \le 2k\tau_0$ and natural numbers k. Note that the intervals $[k\tau_0, 2k\tau_0]$ cover all of $[\tau_0, \infty)$, hence we have

$$\mathrm{LV}(\sigma,\tau) \le B\tau$$

for all $\tau \ge \tau_0$. In particular

$$\limsup_{\tau\to\infty} \mathrm{LV}(\sigma,\tau)/\tau \le B.$$

Also, combining the previous estimate with (11.4) using Lemma 8.3(iii) we have

$$\mathrm{LV}_\zeta(\sigma,\tau) \le B\tau \tag{11.5}$$

for all $\tau \ge 2$. By Lemma 8.3(iv), this implies that

$$\mathrm{LV}_\zeta\left(\frac{1}{2} + \frac{1}{\tau-1}(\sigma - \frac{1}{2}), \frac{\tau}{\tau-1}\right) \le B\frac{\tau}{\tau-1}$$

for $\tau \ge 2$. Thus

$$\sup_{\tau\ge 2} \frac{\mathrm{LV}_\zeta(\sigma,\tau)}{\tau} \le B.$$

The claim now follows from Lemma 11.5. □

For machine assisted proofs, one can simply take τ_0 to be a sufficiently large quantity, e.g., $\tau_0 = 3$ for σ not too close to 1, and larger for σ approaching 1, to recover the full power of Lemma 11.5. However, the amount of case analysis required increases with τ_0. For human written proofs, the following version of Corollary 11.7 is more convenient:

Corollary 11.8. *Let $1/2 < \sigma < 1$ and $\tau_0 > 0$. Then*

$$\mathrm{A}(\sigma)(1-\sigma) \le \max\left(\sup_{2\le\tau<4\tau_0/3} \mathrm{LV}_\zeta(\sigma,\tau)/\tau, \sup_{2\tau_0/3\le\tau\le\tau_0} \mathrm{LV}(\sigma,\tau)/\tau\right).$$

Proof. Applying Corollary 11.7 with τ replaced by $4\tau_0/3$, it suffices to show that

$$\sup_{4\tau_0/3\le\tau\le 8\tau_0/3} \mathrm{LV}(\sigma,\tau)/\tau \le \sup_{2\tau_0/3\le\tau\le\tau_0} \mathrm{LV}(\sigma,\tau)/\tau.$$

But this follows from Lemma 7.6, since the intervals $[2k\tau_0/3, k\tau_0]$ for $k = 2, 3$ cover all of $[4\tau_0/3, 8\tau_0/3]$. □

The following special case of the above corollary is frequently used in practice to assist with the human readability of zero density proofs:

Corollary 11.9. *Let $1/2 < \sigma < 1$ and $\tau_0 > 0$. Suppose that one has the bounds*

$$\mathrm{LV}(\sigma, \tau) \leq (3 - 3\sigma)\frac{\tau}{\tau_0} \tag{11.6}$$

for $2\tau_0/3 \leq \tau \leq \tau_0$, and

$$\mathrm{LV}_\zeta(\sigma, \tau) \leq (3 - 3\sigma)\frac{\tau}{\tau_0} \tag{11.7}$$

for $2 \leq \tau < 4\tau_0/3$. Then $\mathrm{A}(\sigma) \leq \frac{3}{\tau_0}$.

The reason why this particular special case is convenient is because the inequality

$$2 - 2\sigma \leq (3 - 3\sigma)\frac{\tau}{\tau_0} \tag{11.8}$$

obviously holds for $\tau \geq 2\tau_0/3$. That is to say, we automatically verify (11.6) in regimes where the Montgomery conjecture holds. In fact, we can do a bit better, thanks to subdivision:

Corollary 11.10. *Let $1/2 < \sigma < 1$ and $\tau_0 > 0$. Suppose that one has the bound (11.7) for $2 \leq \tau < 4\tau_0/3$, and the Montgomery conjecture $\mathrm{LV}(\sigma, \tau) \leq 2 - 2\sigma$ whenever $0 \leq \tau \leq \tau_0 + \sigma - 1$. Then $\mathrm{A}(\sigma) \leq \frac{3}{\tau_0}$.*

Proof. We may assume that $\tau_0 \geq 3 - 3\sigma$, since otherwise the claim follows from the Riemann–von Mangoldt bound

$$\mathrm{A}(\sigma)(1 - \sigma) \leq \mathrm{A}(1/2)(1 - 1/2) = 1.$$

By Lemma 7.3(ii) we have

$$\mathrm{LV}(\sigma, \tau) \leq \max(2 - 2\sigma, 3 - 3\sigma + \tau - \tau_0)$$

for all $\tau \geq 0$. But both expressions on the right-hand side are bounded by $(3 - 3\sigma)\frac{\tau}{\tau_0}$ for $2\tau_3 \leq \tau \leq \tau_0$ and $\tau_0 \geq 3 - 3\sigma$, so the claim follows from Corollary 11.9. ☐

Let us see some examples of these corollaries in action.

Theorem 11.11. *The Montgomery conjecture implies the density hypothesis.*

Proof. Apply Corollary 11.9 with $\tau_0 = 3/2$ (so that (11.7) is vacuously true). ☐

Theorem 11.12. *The Lindelof hypothesis implies the density hypothesis, and also that $\mathrm{A}(\sigma) \leq 0$ for $3/4 < \sigma \leq 1$.*

Proof. The first result is proved in [**?**], and the second result is due to [**?**]. We will apply Corollary 11.8. From Corollary 8.8 we see that for any choice of τ_0 we have

$$\sup_{2\tau_0/3 \leq \tau \leq \tau_0} \mathrm{LV}(\sigma, \tau)/\tau \leq 0.$$

From Theorem 7.7 and Lemma 7.6 we have

$$\mathrm{LV}(\sigma, \tau) \leq \max((2-2\sigma)k, \tau + (1-2\sigma)k) \tag{11.9}$$

for any natural number k and $\tau \geq 1$; setting k to be the integer part of τ we conclude in particular that

$$\mathrm{LV}(\sigma, \tau) \leq (2-2\sigma)\tau + O(1),$$

and hence by taking τ_0 large enough, we can make $\sup_{2\tau_0/3 \leq \tau \leq \tau_0} \mathrm{LV}(\sigma, \tau)/\tau$ bounded by $2 - 2\sigma + \varepsilon$ for any $\varepsilon > 0$. This already gives the density hypothesis bound $\mathrm{A}(\sigma) \leq 2$. For $\sigma > 3/4$, we may additionally apply Lemma 8.14 to make $\sup_{2\tau_0/3 \leq \tau \leq \tau_0} \mathrm{LV}(\sigma, \tau)/\tau$ arbitrarily small, giving the bound $\mathrm{A}(\sigma) \leq 0$. □

`There are similar results assuming weaker versions of the Lindelof hypothesis. For instance, we have`

Theorem 11.13 (Ingham's first bound)**.** *[?] (See also [?]) For any $1/2 < \sigma < 1$, we have*

$$\mathrm{A}(\sigma) \leq 2 + 4\mu(1/2).$$

Proof. We give here a proof (somewhat different from the original proof) that passes through Corollary 11.7. We apply Corollary 11.7 with τ_0 chosen so that $\mu(1/2)\tau_0 < \sigma - 1/2$. From Corollary 8.6 we then have

$$\mathrm{A}(\sigma)(1-\sigma) \leq \sup_{\tau_0 \leq \tau \leq 2\tau_0} \mathrm{LV}(\sigma, \tau)/\tau.$$

For any integer $k \geq 0$ and $k \leq \tau \leq k+1$, we see from (11.9) that

$$\mathrm{LV}(\sigma, \tau) \leq (2-2\sigma)(k+1)$$

and

$$\mathrm{LV}(\sigma, \tau) \leq \tau + (1-2\sigma)k;$$

multiplying the first inequality by $2\sigma - 1$, the second by $2 - 2\sigma$, and summming, we conclude that

$$\mathrm{LV}(\sigma, \tau) \leq (\tau + 2\sigma - 1)(2 - 2\sigma);$$

inserting this bound we have

$$\mathrm{A}(\sigma) \leq 2 + \frac{2\sigma - 1}{\tau_0}.$$

Optimizing in τ_0, we obtain the claim. □

Theorem 11.14 (Ingham's second bound)**.** *[?] For any $1/2 < \sigma < 1$, one has $\mathrm{A}(\sigma) \leq \frac{3}{2-\sigma}$.*

Proof. We apply Corollary 11.10 with $\tau_0 := 2 - \sigma$. Here we have $4\tau_0/3 < 2$ since $\sigma > 1/2$, so the claim (11.7) is automatic; and the Montgomery conjecture hypothesis follows from Theorem 7.7. □

Either of Theorem 11.13 or Theorem 11.14 implies an older result of Carlson [?] that $\mathrm{A}(\sigma) \leq 4\sigma$ for $1/2 < \sigma < 1$.

Theorem 11.15 (Huxley bound)**.** *[?] For any $1/2 < \sigma < 1$, one has $\mathrm{A}(\sigma) \leq \frac{3}{3\sigma-1}$. (In particular, the density hypothesis holds for $\sigma \geq 5/6$.)*

Proof. We apply Corollary 11.10 with $\tau_0 := 3\sigma - 1$. The Montgomery conjecture hypothesis follows from Theorem 7.10. So it remains to show that (11.7) holds for $2 \leq \tau < 4\tau_0/3$. For $\sigma \leq 5/6$ we have $4\tau_0/3 \leq 2$, so the claim is vacuously true in this case. For $\sigma > 5/6$ we use Corollary 8.6 and the bound $\mu(1/2) \leq 1/6$ from Table 6.2 to conclude that $\mathrm{LV}_\zeta(\sigma, \tau) = -\infty$ whenever $\sigma > 1/2 + \tau/6$, but this is precisely $\tau < 6\sigma - 3$. Since $6\sigma - 3 > 4\tau_0/3$ when $\sigma > 5/6$, we obtain the claim. □

Theorem 11.16 (Guth–Maynard bound)**.** *For any $1/2 < \sigma < 1$, one has $\mathrm{A}(\sigma) \leq \frac{15}{3+5\sigma}$.*

Proof. We may assume that $7/10 < \sigma < 8/10$, since the bound follows from the Ingham and Huxley bounds otherwise. We apply Corollary 11.9 with $\tau_0 := \frac{3+5\sigma}{5}$. We have $4\tau_0/3 < 2$, so the claim (11.7) is vacuous and we only need to establish (11.6). We split into the subranges $13/5 - 2\sigma \leq \tau < \tau_0$ and $2\tau_0/3 \leq \tau \leq 13/5 - 2\sigma$. In the former range we use Theorem 7.15 (and (11.8)), and reduce to showing that

$$18/5 - 4\sigma \leq (3 - 3\sigma)\frac{\tau}{\tau_0},$$

and

$$\tau + 12/5 - 4\sigma \leq (3 - 3\sigma)\frac{\tau}{\tau_0}$$

for $13/5 - 2\sigma \leq \tau < \tau_0$. The first inequality follows from

$$18/5 - 4\sigma \leq (3 - 3\sigma)\frac{13/5 - 2\sigma}{\tau_0} \tag{11.10}$$

which one can numerically check holds in the range $7/10 < \sigma < 8/10$. Finally, the third inequality is obeyed with equality when $\tau = \tau_0$ and the right-hand side has a larger slope in τ than the left (since $\tau_0 \geq 3 - 3\sigma$), so the claim follows as well.

In the remaining region $2\tau_0/3 \leq \tau \leq 13/5 - 2\sigma$, we use Theorem 7.7 and (11.8) to reduce to showing that

$$\tau + 1 - 2\sigma \leq (3 - 3\sigma)\frac{\tau}{\tau_0}$$

in this range. This follows again from (11.10) which guarantees the inequality at the right endpoint $\tau = 13/5 - 2\sigma$. □

Theorem 11.17 (Jutila zero density theorem)**.** *[?] The zero density hypothesis is true for $\sigma \geq 11/14$.*

Proof. We apply Corollary 11.8 with $\tau_0 \coloneqq 3/2$, then it suffices to show that

$$\mathrm{LV}(\sigma, \tau) \leq (2 - 2\sigma)\tau$$

for all $1 \leq \tau \leq 3/2$.

From the $k = 3$ case of Theorem 7.14 we have

$$\mathrm{LV}(\sigma, \tau) \leq \max\left(2 - 2\sigma, \tau + \frac{10 - 16\sigma}{3}, \tau + 18 - 24\sigma\right).$$

But all terms on the right-hand side can be verified to be less than or equal to $(2 - 2\sigma)\tau$ when $1 \leq \tau \leq 3/2$ and $\sigma \geq 11/14$, giving the claim. $\square$

`In fact, we can do better:`

Theorem 11.18 (Heath-Brown zero density theorem)**.** *[?] For $11/14 \leq \sigma < 1$, one has $\mathrm{A}(\sigma) \leq \frac{9}{7\sigma - 1}$ (in particular, this recovers Theorem 11.17 range). For any $3/4 \leq \sigma \leq 1$, one has $\mathrm{A}(\sigma) \leq \max(\frac{3}{10\sigma - 7}, \frac{4}{4\sigma - 1})$ (which is a superior bound when $\sigma \geq 20/23$).*

Proof. For the first estimate, we apply Corollary 11.9 with $\tau_0 \coloneqq \frac{7\sigma - 1}{3}$. To verify (11.6), we apply the $k = 3$ version of Theorem 7.14, which gives

$$\mathrm{LV}(\sigma, \tau) \leq \max\left(2 - 2\sigma, \tau + \frac{10 - 16\sigma}{3}, \tau + 18 - 24\sigma\right).$$

When $\sigma \geq 11/14$ one has $18 - 24\sigma \leq \frac{10 - 16\sigma}{3}$, so by (11.8) we need to show that

$$\tau + \frac{10 - 16\sigma}{3} \leq (3 - 3\sigma)\frac{\tau}{\tau_0}$$

for $2\tau_0/3 \leq \tau \leq \tau_0$. This holds with equality at $\tau = \tau_0$, hence holds for $\tau \leq \tau_0$ as well since $\tau_0 \geq 3 - 3\sigma$. As for (11.7), we invoke Theorem 9.6 and reduce to showing that

$$2\tau + 6 - 12\sigma \leq (3 - 3\sigma)\frac{\tau}{\tau_0}$$

for $2 \leq \tau \leq 4\tau_0/3$. Since $6 - 12\sigma$ is negative, the ratio of the left-hand side and right-hand side is increasing in τ, so it suffices to verify this claim at the endpoint $\tau = 4\tau_0/3$. The claim then simplifies to $\tau_0 \leq \frac{3}{4}(4\sigma - 1)$, which one can verify from the choice of τ_0 and the hypothesis $\sigma \geq 11/14$.

For the second estimate, we take $\tau_0 \coloneqq \min(10\sigma - 7, \frac{3}{4}(4\sigma - 1))$. To verify (11.6), we now use Theorem 7.12 and (11.8), and reduce to showing that

$$\tau + 10 - 13\sigma \leq (3 - 3\sigma)\frac{\tau}{\tau_0}$$

for $2\tau_0/3 \leq \tau \leq \tau_0$. The inequality holds at $\tau = \tau_0$ since $\tau_0 \leq 10\sigma - 7$, and hence for all smaller τ since $\tau_0 \geq 3 - 3\sigma$. As for (11.7), we can repeat the previous arguments since $\tau_0 \leq \frac{3}{4}(4\sigma - 1)$. $\square$

`With the aid of computer assistance, we were able to strengthen the second claim here. We first need a lemma:`

Lemma 11.19. $(3/40, 31/40)$ *is an exponent pair. In particular, by Corollary 6.8,* $\mu(7/10) \leq 3/40$.

Proof. This can be derived from the Watt exponent pair $W := (89/560, 1/2 + 89/560)$ from Theorem 5.11 as well as the A and B transforms and convexity (Lemmas 5.4, 5.8, 5.9) after observing that

$$(3/40, 31/40) = xyAW + (1-x)yABAW + (1-y)W$$

with $x = 37081/40415$ and $y = 476897/493711$. (One could of course also use more recent exponent pairs that are stronger, such as the Bourgain exponent pair $(13/84, 1/2 + 13/84)$.) We remark that one could also obtain this result from Lemma 5.3, after observing that the required bound $\beta(\alpha) \leq 3/40 + 7\alpha/10$ can be derived from Theorem 4.14 (as well as the classical bounds in Corollary 4.7). We also note that the corollary $\mu(7/10) \leq 3/40 = 0.075$ is immediate from [**?**, Theorem 2.4], which in fact gives the slightly stronger bound $\mu(7/10) \leq 218/3005 = 0.07254\dots$. $\square$

Theorem 11.20 (Improved Heath-Brown zero density theorem)**.** *For any* $1/2 \leq \sigma \leq 1$, *one has* $\mathrm{A}(\sigma) \leq \frac{3}{10\sigma - 7}$.

Proof. We apply Corollary 11.10 with $\tau_0 := 10\sigma - 7$. The claim (11.6) again follows from Theorem 7.12 and (11.8) as in the proof of Theorem 11.18. Meanwhile, from Lemma 11.19 and Corollary 8.7 we have $\mathrm{LV}_\zeta(\sigma, \tau) = -\infty$ whenever $\sigma > \frac{3}{40}\tau + \frac{7}{10}$, or equivalently $\tau < \frac{4}{3}(10\sigma - 7)$, which then immediately gives (11.7). $\square$

Theorem 11.21 (Bourgain result on density hypothesis)**.** *The density hypothesis holds for* $\sigma > 25/32$.

Note: it appears the proof here may be optimized to go beyond the density hypothesis. This will be a good test case of our python machinery.

Proof. The arguments below are a translation of the original arguments of Bourgain [**?**] to our notational framework.

In view of Theorem 11.17 (or Theorem 11.18), we may assume that $25/32 < \sigma < 11/14$. Set $\rho := \mathrm{LV}(\sigma, \tau)$. As in the proof of Theorem 11.17, it suffices to show that

$$\rho \leq (2 - 2\sigma)\tau \tag{11.11}$$

for all $1 \leq \tau \leq 3/2$.

From the $k = 3$ case of Theorem 7.14 we have

$$\rho \leq \max\left(2 - 2\sigma, \tau + \frac{10 - 16\sigma}{3}, \tau + 18 - 24\sigma\right)$$

which in the $\sigma < 11/14$ regime simplifies to

$$\rho \leq \max(2 - 2\sigma, \tau + 18 - 24\sigma) \tag{11.12}$$

and this already suffices unless

$$\tau \geq \frac{24\sigma - 18}{2\sigma - 1}. \tag{11.13}$$

In the regime $\sigma > 25/32$ and $\tau \leq 3/2$, the bound (11.12) certainly implies

$$\rho \leq 1$$

so we may invoke Corollary 7.17 to conclude that

$$\rho \leq \max(\alpha_2+2-2\sigma, \alpha_1+\alpha_2/2+2-2\sigma, -\alpha_2+2\tau+4-8\sigma, 2\alpha_1+\tau+12-16\sigma, 4\alpha_1+3-4\sigma) \tag{11.14}$$

for any $\alpha_1, \alpha_2 \geq 0$.

We now divide into cases. First suppose that $\tau \leq \frac{4(1+\sigma)}{5}$. In this case we set $\alpha_1 := \frac{\tau}{3} - \frac{2}{3}(7\sigma - 5)$ (which can be checked to be nonnegative using (11.13) and $\sigma \geq 25/32$) and $\alpha_2 = 0$, and one can check that (11.14) implies (11.11) in this case (with some room to spare).

Now suppose that $\tau > \frac{4(1+\sigma)}{5}$. In this case we choose $\alpha_1 = \frac{\tau}{8} - \frac{9\sigma-7}{2}$ and $\alpha_2 = \frac{5\tau}{4} - (1+\sigma)$, which can be checked to be nonnegative using the hypotheses on σ, τ. In this case one can again check that (11.14) implies (11.11). ☐

Theorem 11.22 (Bourgain zero density theorem)**.** *[?, Proposition 3] Let (k, ℓ) be an exponent pair with $k < 1/5$, $\ell > 3/5$, and $15\ell + 20k > 13$. Then, for any $\sigma > \frac{\ell+1}{2(k+1)}$, one has*

$$\mathrm{A}(\sigma) \leq \frac{4k}{2(1+k)\sigma - 1 - \ell}$$

assuming either that $k < \frac{11}{85}$, or that $\frac{11}{85} < k < \frac{1}{5}$ and $\sigma > \frac{144k-11\ell-11}{170\kappa-22}$.

Corollary 11.23. *[?, Corollary 4] One has*

$$\mathrm{A}(\sigma) \leq \frac{4}{30\sigma - 25}$$

for $\frac{15}{16} \leq \sigma \leq 1$ and

$$\mathrm{A}(\sigma) \leq \frac{2}{7\sigma - 5}$$

for $\frac{17}{19} \leq \sigma \leq \frac{15}{16}$.

Proof. Apply Theorem 11.22 with the classical pairs $(\frac{1}{14}, \frac{11}{14})$ and $(\frac{1}{6}, \frac{1}{3})$ respectively from Proposition 5.10. ☐

`It is remarked in [?] that further zero density estimates could be obtained by using additional exponent pairs, such as the Huxley--Watt exponent pair` $(9/56, 37/56)$ `from Theorem 5.11.`

Lemma 11.24. *[?, Theorem 11.2] We have*

$$\mathrm{A}(\sigma) \leq \frac{4}{2\sigma+1}$$

for $17/18 \leq \sigma \leq 1$, *and*

$$\mathrm{A}(\sigma) \leq \frac{24}{30\sigma - 11}$$

for $155/174 \leq \sigma \leq 17/18$.

Proof. From Lemma 8.15 we have

$$\mathrm{LV}(\sigma, \tau) \leq \max(2 - 2\sigma, \tau + 9 - 12\sigma, \tau - \frac{84\sigma - 65}{6})$$

for all $\tau \geq 0$. Meanwhile, applying Lemma 8.11 with the exponent pair $(2/7, 4/7)$ we have

$$\mathrm{LV}_\zeta(\sigma, \tau) \leq \max(\tau + (3 - 6\sigma), 3\tau + 19(1/2 - \sigma)).$$

We apply Corollary 11.9 with $\tau_0 := \max(\frac{30\sigma - 11}{8}, \frac{6\sigma+3}{4})$, and reduce to showing that (11.6) for $2\tau_0/3 \leq \tau \leq \tau_0$ and (11.7) for $2 \leq \tau < 4\tau_0/3$. But this follows from the preceding estimates after routine calculations. □

`One can also use bounds on` μ `to obtain zero density theorems:`

Lemma 11.25 (Zero density from μ bound)**.** *[?, Theorem 12.3] If* $1/2 \leq \alpha \leq 1$ *and* $\frac{\alpha+1}{2} \leq \sigma \leq 1$, *then*

$$\mathrm{A}(\sigma) \leq \mu(\alpha) \frac{2(3\sigma - 1 - 2\alpha)}{(2\sigma - 1 - \alpha)(\sigma - \alpha)}.$$

Corollary 11.26. *[?, Theorem 11.3] For any* $9/10 \leq \sigma \leq 1$ *and* $1/2 \leq \alpha \leq 1$ *one has*

$$\mathrm{A}(\sigma)(1 - \sigma) \leq \frac{7}{6}\mu(5\sigma - 4).$$

In particular, for $152/155 \leq \sigma \leq 1$, *one has*

$$\mathrm{A}(\sigma) \leq \min(35/36, 1600(1 - \sigma)^{1/2}).$$

Proof. Apply the previous lemma with $\alpha = 5\sigma - 4$. □

Lemma 11.27 (Preliminary zero density estimate)**.** *If* $m \geq 2$ *is an integer,* $3/4 < \sigma \leq 1$, *and* (k, ℓ) *is an exponent pair, then*

$$\mathrm{LV}(\sigma, \tau) \leq \max(2 - 2\sigma, m(2 - 4\sigma) + m\tau, \min(X, Y))$$

where

$$X := 2\tau/3 + 4m(3 - 4\sigma)/3$$

and

$$Y := \max(\tau + 3m(3 - 4\sigma), (k + \ell)\tau/k + k(1 + 2k + 2\ell)(3 - 4\sigma)/k).$$

Proof. See [**?**, (11.74)]. ☐

Lemma 11.28 (General zero density estimate)**.** *[**?**, (11.76), (11.77)] If (k,ℓ) is an exponent pair, and $m \geq 2$ an integer, then*

$$\mathrm{A}(\sigma) \leq \frac{3m}{(3m-2)\sigma + 2 - m}$$

whenever

$$\sigma \geq \min\left(\frac{6m^2-5m+2}{8m^2-7m+2}, \qquad max\left(\frac{9m^2-4m+2}{12m^2-6m+2}, \frac{3m^2(1+2k+2\ell)-(4k+2\ell)m+2k+2\ell}{4m^2(1+2k+2\ell)-(6k+4\ell)m+2k+2\ell}\right)\right).$$

Proof. With the hypothesis on σ, one sees from Lemma 11.27 that

$$\mathrm{LV}(\sigma,\tau) \leq \max(2-2\sigma, \tau - \frac{(4m-2)\sigma+2-2m}{m} + 2 - 2\sigma)$$

for $0 \leq \tau < \frac{(4m-2)\sigma+2-2m}{m}$, and hence for all $\tau \geq 0$ by by Lemma 7.3(ii). Meanwhile, from Theorem 9.6 one has

$$\mathrm{LV}_\zeta(\sigma,\tau) \leq 2\tau - 12(\sigma - 1/2)$$

for all $\tau \geq 2$. The claim then follows from Corollary 11.9 with $\tau_0 := \frac{(3m-2)\sigma+2-m}{m}$ after a routine calculation. ☐

Corollary 11.29. *[**?**, Theorem 11.4] One can bound $\mathrm{A}(\sigma)$ by*

$$\frac{3}{2\sigma} \textit{ for } \frac{3831}{4791} \leq \sigma \leq 1;$$
$$\frac{9}{7\sigma-1} \textit{ for } \frac{41}{53} \leq \sigma \leq 1;$$
$$\frac{6}{5\sigma-1} \textit{ for } \frac{13}{17} \leq \sigma \leq 1;$$

Proof. Apply Lemma 11.28 with $m = 2$ and $(k,\ell) = (\frac{97}{251}, \frac{132}{251})$ for the first claim; $m = 3$ and arbitrary (k,ℓ) for the second claim; and $m = 4$ and arbitrary (k,ℓ) for the third claim. ☐

Lemma 11.30 (Preliminary large values theorem)**.** *If $1/2 \leq \sigma \leq 1$ and $\tau < 8\sigma - 5$, then*

$$\mathrm{LV}(\sigma,\tau) \leq \max(2-2\sigma, 6\tau/5 + (20-32\sigma)/5).$$

Proof. See [**?**, (11.95)]. ☐

Corollary 11.31 (Zero density estimates for σ close to 3/4)**.** *[**?**, Theorem 11.5] One has $\mathrm{A}(\sigma) \leq \frac{3}{7\sigma-4}$ for $3/4 \leq \sigma \leq 10/13$, and $\mathrm{A}(\sigma) \leq \frac{9}{8\sigma-2}$ for $10/13 \leq \sigma \leq 1$.*

Proof. For $3/4 \leq \sigma \leq 10/13$, we see from Lemma 11.30 that the bound

$$\mathrm{LV}(\sigma, \tau) \leq \max(2 - 2\sigma, \tau + 7 - 10\sigma)$$

holds for $0 \leq \tau < 8\sigma - 5$, and hence for all $\tau \geq 0$ by Lemma 7.3(ii). Meanwhile, from Lemma 9.5 we have

$$\mathrm{LV}_\zeta(\sigma, \tau) \leq \tau - 4(\sigma - 1/2)$$

for all $1/2 \leq \sigma \leq 1$ and $\tau \geq 2$. The claim then follows from Corollary 11.9 with $\tau_0 \coloneqq 7\sigma - 4$ after a routine calculation. Similarly, for $10/13 \leq \sigma \leq 1$, we have

$$\mathrm{LV}(\sigma, \tau) \leq \max(2 - 2\sigma, \tau + \frac{11 - 17\sigma}{3})$$

for $0 \leq \tau < \frac{11\sigma - 5}{3}$, hence for all $\tau \geq 0$ by Lemma 7.3(ii); the claim then follows from Corollary 11.9 with $\tau_0 \coloneqq \frac{8\sigma - 2}{3}$ after a routine calculation. □

Theorem 11.32 (Pintz zero density theorem)**.** *[**?**, Theorem 1] If $k \geq 4$, $\ell \geq 3$ are integers and $\sigma = 1 - \eta$ is such that*

$$\frac{1}{k(k+1)} \leq \eta \leq \frac{1}{k(k-1)} \tag{11.15}$$

and

$$\frac{1}{2\ell(\ell+1)} \leq \eta \leq \frac{1}{2\ell(\ell-1)} \tag{11.16}$$

then

$$\mathrm{A}(\sigma) \leq \max(\frac{3}{\ell(1 - 2(\ell - 1)\eta)}, \frac{4}{k(1 - (k-1)\eta)}).$$

Proof. We apply Corollary 11.9 with

$$\tau_0 \coloneqq \min(\ell(1 - 2(\ell - 1)\eta), \frac{3}{4}(k(1 - (k-1)\eta))) - \varepsilon \tag{11.17}$$

for an arbitrarily small ε. It then suffices to show that (11.6) holds for $2\tau_0/3 \leq \tau \leq \tau_0$ and (11.7) holds for $2 \leq \tau < 4\tau_0/3$.

To prove (11.7), it suffices by Lemma 8.5 to show that $\sigma > \tau\beta(1/\tau)$ for all $2 \leq \tau < 4\tau_0/3$. By (11.17) one has $2 \leq \tau < k(1 - (k-1)\eta)$. Meanwhile, from Lemma 4.21 one has

$$\tau\beta(1/\tau) \leq 1 + \max\left(\frac{\tau - r}{r(r-1)}, -\frac{1}{r(r-1)}, -\frac{2\tau}{r^2(r-1)}\right) \tag{11.18}$$

for any $r \geq 3$. So by (11.15) it suffices to find $3 \leq r \leq k$ such that

$$\frac{r - \tau}{r(r-1)}, \frac{2\tau}{r^2(r-1)} \geq \eta$$

or equivalently

$$\tau \in [\frac{r^2(r-1)\eta}{2}, r(1 - (r-1)\eta)].$$

But one can check that these intervals for $3 \leq r \leq k$ cover the entire range $2 \leq \tau < 4\tau_0/3$, as required.

To prove (11.6), it suffices by Lemma 7.8 and (11.8) to show that

$$\sup_{1 \leq \tau \leq \tau_0} \beta(1/\tau)\tau < 2\sigma - 1 = 1 - 2\eta.$$

Using (11.18), (11.16) we obtain the claim whenever

$$\tau \in [r^2(r-1)\eta + \varepsilon, r(1 - 2(r-1)\eta) - \varepsilon]$$

for some $3 \leq r \leq \ell$. These cover the range $[18\eta + \varepsilon, \tau]$. For the remaining range $[1, 18\eta + \varepsilon]$ we use the van der Corput bound

$$\tau\beta(1/\tau) \leq \frac{\tau}{2} \leq 9\eta$$

from Corollary 4.7, which suffices since $\eta \leq \frac{1}{k(k-1)} \leq \frac{1}{12}$. $\square$

we should be able to do better with subdivision!

Chapter 12

Zero density energy theorems

Definition 12.1 (Zero density exponents)**.** For $1/2 \leq \sigma \leq 1$ and $T > 0$, let $N^*(\sigma, T)$ denote the additive energy $E_1(\Sigma)$ of the imaginary parts of the zeroes ρ of the Riemann zeta function with $\mathrm{Re}(\rho) \geq \sigma$ and $|\mathrm{Im}(\rho)| \leq T$. For fixed $1/2 \leq \sigma \leq 1$, the zero density exponent $A^*(\sigma) \in [-\infty, \infty)$ is the infimum of all exponents A^* for which one has

$$N^*(\sigma - \delta, T) \ll T^{A^*(1-\sigma)+o(1)}$$

for all unbounded T and infinitesimal $\delta > 0$.

The exponent $\mathrm{A}^*(\sigma)$ is also essentially referred to as $B(\sigma)$ in [?] (though without the technical shift by δ in that reference).

Lemma 12.2 (Basic properties of A^*)**.**

(i) We have the trivial bounds

$$2\mathrm{A}(\sigma), 4\mathrm{A}(\sigma) - \frac{1}{1-\sigma} \leq \mathrm{A}^*(\sigma) \leq 3\mathrm{A}(\sigma)$$

for any $1/2 \leq \sigma \leq 1$.

(ii) $\sigma \mapsto (1-\sigma)\mathrm{A}^*(\sigma)$ *is non-increasing, with* $\mathrm{A}^*(1/2) = 6$ *and* $\mathrm{A}^*(1) = -\infty$.

(iii) If the Riemann hypothesis holds, then $\mathrm{A}^*(\sigma) = -\infty$ *for all* $1/2 < \sigma \leq 1$.

Proof. The claim (i) follows from Lemma 10.2(iv), and the remaining claims then follow from Lemma 11.2. □

Upper bounds on $A^*(\sigma)$ can be obtained from large value energy theorems via the following relation.

Lemma 12.3 (Zero density energy from large values energy)**.** *Let* $1/2 < \sigma < 1$*. Then*

$$\mathrm{A}^*(\sigma)(1-\sigma) \leq \max(\sup_{\tau \geq 1} \mathrm{LV}^*_\zeta(\sigma,\tau)/\tau, \limsup_{\tau\to\infty} \mathrm{LV}^*(\sigma,\tau)/\tau).$$

Proof. Write the right-hand side as B, then $B \geq 0$ (from Lemma 10.10(iii)) and we have

$$\mathrm{LV}^*_\zeta(\sigma,\tau) \leq B\tau \tag{12.1}$$

for all $\tau \geq 1$, and

$$\mathrm{LV}^*(\sigma,\tau) \leq (B+\varepsilon)\tau \tag{12.2}$$

whenever $\varepsilon > 0$ and τ is sufficiently large depending on ε (and σ). It would suffice to show, for any $\varepsilon > 0$, that $N^*(\sigma, T) \ll T^{B+O(\varepsilon)+o(1)}$ as $T \to \infty$.

By dyadic decomposition, it suffices to show for large T that the additive energy of imaginary parts of zeroes in $[T, 2T]$ is $\ll T^{B+O(\varepsilon)+o(1)}$. As in the proof of Lemma 11.5, we can assume the imaginary parts are 1-separated (here we take advantage of the triangle inequality in Lemma 10.2(iii)).

Suppose that one has a zero $\sigma' + it$ of this form. Then by standard approximations to the zeta function, one has

$$\sum_{n \leq T} \frac{1}{n^{\sigma'+it}} \ll T^{-1}.$$

Let $0 < \delta_1 < \varepsilon$ be a small quantity (independent of T) to be chosen later, and let $0 < \delta_2 < \delta_1$ be sufficiently small depending on δ_1, δ_2. By the triangle inequality, and refining the sequence t' by a factor of at most 2, we either have

$$\left| \sum_{T^{\delta_1} \leq n \leq T} \frac{1}{n^{\sigma'+it}} \right| \gg T^{-\delta_2}$$

for all zeroes, or (11.3) for all zeroes.

Suppose we are in the former ("Type I") case, we can dyadically partition and conclude from the pigeonhole principle that

$$\left| \sum_{n \in I} \frac{1}{n^{\sigma'+it}} \right| \gg T^{-\delta_2 - o(1)}$$

for some interval I in some $[N, 2N]$ with $T^{\delta_1} \ll N \ll T$, with at most $O(\log T)$ different choices for I. Performing a Fourier expansion of $n^{\sigma'}$ in $\log n$ and using the triangle inequality one can then deduce that

$$\left| \sum_{n \in I} \frac{1}{n^{it'}} \right| \gg N^{\sigma'} T^{-\delta_2 - o(1)}$$

for some $t' = t + O(T^{o(1)})$; refining the t by a factor of $T^{o(1)}$ if necessary, we may assume that the t' are 1-separated and that the interval I is independent of t', and by passing to a subsequence we may assume that $T = N^{\tau+o(1)}$ for some $1 \leq \tau \leq 1/\delta_1$, then

$$\left|\sum_{n\in I} \frac{1}{n^{it'}}\right| \gg N^{\sigma-\delta_2/\delta_1+o(1)}$$

for all t'. If we let Σ' denote the set of such t', then by Definition 10.8 we then have (for δ_2 small enough) we have

$$E_1(\Sigma') \ll N^{\mathrm{LV}^*_\zeta(\sigma,\tau)+\varepsilon+o(1)} \ll T^{\mathrm{LV}^*_\zeta(\sigma,\tau)/\tau+\varepsilon+o(1)}.$$

By Lemma 10.2(i) this implies that the set Σ of real parts of zeroes under consideration also obeys the bound

$$E_1(\Sigma) \ll T^{\mathrm{LV}^*_\zeta(\sigma,\tau)/\tau+\varepsilon+o(1)}.$$

and the claim follows in this case from (12.1).

The Type II case similarly follows from Lemma 10.9 and (12.2) exactly as in the proof of Lemma 11.5. □

Corollary 12.4. *Let $1/2 < \sigma < 1$ and $\tau_0 > 0$ be fixed. Then*

$$\mathrm{A}^*(\sigma)(1-\sigma) \leq \max\left(\sup_{2\leq\tau<\tau_0} \mathrm{LV}^*_\zeta(\sigma,\tau)/\tau, \sup_{\tau_0\leq\tau\leq 2\tau_0} \mathrm{LV}^*(\sigma,\tau)/\tau\right)$$

Proof. Repeat the proof of Corollary 11.7. □

Theorem 12.5 (Heath-Brown's additive energy bound)**.** *[**?**, Theorem 2] Let $1/2 \leq \sigma \leq 1$ be fixed. Then one can bound $A^*(\sigma)$ by*

$$\frac{10-11\sigma}{(2-\sigma)(1-\sigma)} \text{ for } 1/2 \leq \sigma \leq 2/3; \frac{18-19\sigma}{(4-2\sigma)(1-\sigma)} \quad \text{for } 2/3 \leq \sigma \leq 3/4; \frac{12}{4\sigma-1} \text{ for } 3/4 \leq \sigma \leq 1.$$

Proof. We first suppose that $\sigma \leq 3/4$. Here we apply Corollary 12.4 with $\tau_0 = 2$. The LV^*_ζ supremum is now trivial, so it suffices to show that

$$\rho^* \leq \max\left(\frac{10-11\sigma}{2-\sigma}, \frac{18-19\sigma}{4-2\sigma}\right)\tau \tag{12.3}$$

whenever $(\sigma,\tau,\rho,\rho^*) \in \mathcal{E}$ with $2 \leq \tau \leq 3$. Let k be the first integer for which $1 \leq \tau/k \leq 3/2$, thus $k = 2, 3$ and also $\tau/(k+1) \leq 1$. By Lemma 10.11, we have

$$(\sigma, \tau/k, \rho/k, (\rho^*)/k), (\sigma, \tau/(k+1), \rho/(k+1), (\rho^*)/(k+1)) \in \mathcal{E}. \tag{12.4}$$

From applying Corollary 10.18 to the former quadruple, we have

$$\rho^*/k \leq \max(3\rho/k + 1 - 2\sigma, \rho/k + 4 - 4\sigma, 5\rho/2k + (3-4\sigma)/2).$$

Write $\tau' := \tau/k$. From Lemma 7.7 one has

$$\rho/k \leq \tau' + 1 - 2\sigma$$

and also

$$\rho/k = \frac{k+1}{k}\rho/(k+1) \leq \frac{k+1}{k}(2-2\sigma) = 3-3\sigma$$

and thus

$$\rho/k \leq \min(\tau'+1-2\sigma, 3-3\sigma) \tag{12.5}$$

and

$$\rho^*/k \leq \max(3\min(\tau'+1-2\sigma,3-3\sigma)+1-2\sigma, \min(\tau'+1-2\sigma,3-3\sigma)+4-4\sigma, 5\min(\tau'+1-2\sigma,3-3\sigma)/2+(3-4\sigma)/2)$$

A tedious calculation shows that for $1 \leq \tau' \leq 3/2$, we have

$$3\min(\tau'+1-2\sigma,3-3\sigma)+1-2\sigma \leq \frac{10-11\sigma}{2-\sigma}\tau',$$

$$\min(\tau'+1-2\sigma,3-3\sigma)+4-4\sigma \leq \max(\frac{7-7\sigma}{2-\sigma}, 6-6\sigma)\tau'$$

and

$$5\min(\tau'+1-2\sigma,3-3\sigma)/2+(3-4\sigma)/2 \leq \frac{18-19\sigma}{4-2\sigma}\tau';$$

since

$$\max(\frac{7-7\sigma}{2-\sigma}, 6-6\sigma) \leq \max(\frac{10-11\sigma}{2-\sigma}, \frac{18-19\sigma}{4-2\sigma})$$

we obtain the claim.

Now suppose that $\sigma > 3/4$. From Theorem 11.18 and Lemma 12.2(i) we are already done when $\sigma \geq 25/28$, so we may assume $\sigma < 25/28$.

Here we apply Corollary 12.4 with $\tau_0 = 4\sigma - 1$. To control the LV^*_ζ term, we need to establish

$$\rho^* \leq \frac{12(1-\sigma)}{4\sigma-1}\tau \tag{12.6}$$

whenever $(\sigma,\tau,\rho,\rho^*) \in \mathcal{E}_\zeta$ and $2 \leq \tau < 4\sigma-1$. We use Lemma 8.3(ii) followed by Lemma 9.6 to give

$$\rho^* \leq 3\rho \leq 3(2\tau - 12(\sigma-1/2))$$

so the claim reduces to verifying

$$3(2\tau - 12(\sigma-1/2)) \leq \frac{12(1-\sigma)}{4\sigma-1}\tau.$$

This holds with equality when $\tau = 4\sigma-1$, and the slope in τ is higher on the left-hand side for $\sigma > 1/2$, so the claim (12.6) follows.

It remains to establish

$$\rho^* \leq \frac{12(1-\sigma)}{4\sigma-1}\tau \tag{12.7}$$

whenever $(\sigma,\tau,\rho,\rho^*) \in \mathcal{E}_\zeta$ and $4\sigma-1 \leq \tau \leq 2(4\sigma-1)$. Let k be the first integer for which $(4\sigma-1)/2 \leq \tau/k \leq 3(4\sigma-1)/4$, thus $k = 2,3$ and also

$\tau/(k+1) \leq 4\sigma - 1$. By Lemma 10.11, we have (12.4). From Theorem 7.10 we have

$$\rho/k \leq \max(2-2\sigma, \tau' + 4 - 6\sigma)$$

and also

$$\rho/k = \frac{k+1}{k}\rho/(k+1) \leq \frac{k+1}{k}(2-2\sigma) = 3 - 3\sigma$$

and hence

$$\rho/k \leq \min(\max(2-2\sigma, \tau'+4-6\sigma), \tau'+4-6\sigma, 3-3\sigma). \tag{12.8}$$

Among other things, this implies that $\rho/k \leq 1$.

From Theorem 10.17, we have

$$\rho^*/k \leq 1-2\sigma+\max(\rho/k+1, 2\rho/k, 5\rho/4k+\tau'/2)/2+\max(\rho^*/k+1, 4\rho/k, 3\rho^*/4k+\rho/k+\tau'/2)/2 \tag{12.9}$$

where $\tau' := \tau/k$. This expression is complicated, so we divide into cases. First suppose that $\rho/k + 1 \geq 5\rho/4k + \tau'/2$. In this case the first maximum in the above expression is $\rho/k + 1$, and we simplify to

$$\rho^*/k \leq 3/2 - 2\sigma + \rho/2k + \max(\rho^*/k+1, 4\rho/k, 3\rho^*/4k + \rho/k + \tau'/2)/2,$$

which after solving for ρ^*/k gives

$$\rho^*/k \leq \max(\rho/2k + 4 - 4\sigma, 5\rho/2k + (3-4\sigma)/2, 8\rho/5k + 2\tau'/5 + (12-16\sigma)/5).$$

Inserting (12.8), one can verify after a tedious analysis (using the hypothesis $3/4 \leq \sigma < 25/28$) that

$$\rho^*/k \leq \frac{12(1-\sigma)}{4\sigma-1}\tau' \tag{12.10}$$

as required.

It remains to treat the case where $\rho/k + 1 > 5\rho/4k + \tau'/2$. Using (12.8) one can check that this forces

$$4\sigma - 2 \leq \tau' \leq \frac{3}{4}(4\sigma - 1), \tag{12.11}$$

so that (12.8) now becomes

$$\rho/k \leq 3 - 3\sigma. \tag{12.12}$$

The bound (12.9) becomes

$$\rho^*/k \leq 1 - 2\sigma + (5\rho/4k + \tau'/2)/2 + \max(\rho^*/k+1, 4\rho/k, 3\rho^*/4k + \rho/k + \tau'/2)/2$$

which simplifies to

$$\rho^*/k \leq \max(5\rho/4k+\tau'/2+3-4\sigma, 21\rho/8k+\tau'/4+1-2\sigma, 9\rho/5k+4\tau'/5+(8-16\sigma)/5).$$

Inserting (12.12) and (12.11), one can eventually show (again using the hypothesis $3/4 \leq \sigma < 25/28$) that (12.10) holds as required. □

Chapter 13

Applications to the primes

Recall that Λ is the von Mangoldt function, and that the prime number theorem asserts that

$$\sum_{n \leq x} \Lambda(n) = x + o(x)$$

for unbounded x. If p_n denotes the n^{th} prime, the prime number theorem is also equivalent to

$$p_n = (1 + o(1)) n \log n$$

for unbounded n.

We now consider local versions of the prime number theorem.

Definition 13.1 (Prime number theorem in short interval exponents)**.** (i) We let θ_{PNT} denote the least exponent with the following property: if $\varepsilon > 0$ is fixed, and x is unbounded, then

$$\sum_{x \leq n < x+y} \Lambda(n) = y + o(y)$$

whenever $x^{\theta_{\mathrm{PNT}}+\varepsilon} \leq y \leq x^{1-\varepsilon}$.

(ii) We let $\theta_{\mathrm{PNT-AA}}$ denote the least exponent with the following property: if $\varepsilon > 0$ is fixed, and X is unbounded, then we have

$$\int_X^{2X} |\sum_{x \leq n < x+y} \Lambda(n) - y| \, dx = o(Xy)$$

whenever $X^{\theta_{\mathrm{PNT-AA}}+\varepsilon} \leq y \leq X^{1-\varepsilon}$.

(iii) We let θ_{gap} denote the least exponent such that, if p_n denotes the n^{th} prime, that

$$p_{n+1} - p_n \ll n^{\theta_{\mathrm{gap}}+o(1)} = p_n^{\theta_{\mathrm{gap}}+o(1)}$$

as $n \to \infty$.

(iv) We let $\theta_{\mathrm{gap},2}$ denote the least exponent such that

$$\sum_{p_n \le x} (p_{n+1} - p_n)^2 \ll x^{\theta_{\mathrm{gap},2}+o(1)}$$

as $x \to \infty$.

Lemma 13.2 (Trivial bounds)**.** *We have*

$$0 \le \theta_{\mathrm{PNT-AA}}, \theta_{\mathrm{gap}} \le \theta_{\mathrm{PNT}} \le 1$$

and $1 \le \theta_{\mathrm{gap},2} \le 1 + \theta_{\mathrm{gap}}$.

Proof. These are all immediate, after noting from the prime number theorem that $\sum_{p_n \le x} p_{n+1} - p_n = x^{1+o(1)}$. □

The Cramér random mdoel **give cite** predicts

Conjecture 13.3 (Prime gap conjecture)**.** $\theta_{\mathrm{PNT}} = 0$, *and hence (by Lemma 13.2)* $\theta_{\mathrm{PNT-AA}} = \theta_{\mathrm{gap}} = 0$ *and* $\theta_{\mathrm{gap},2} = 1$.

We note that the results of Maier **give cite** show that there is some deviation from the prime number theorem at very small scales (of order $\log^{O(1)} x$), but this does not directly affect the exponents discussed here due to the epsilons in our definitions.

A basic connection with zero density exponents is

Proposition 13.4 (Zero density theorems and prime gaps)**.** *Let*

$$\|\mathrm{A}\|_\infty := \sup_{1/2 \le \sigma \le 1} A(\sigma). \tag{13.1}$$

Then

$$\theta_{\mathrm{PNT}} \le 1 - \frac{1}{\|\mathrm{A}\|_\infty}$$

and

$$\theta_{\mathrm{PNT-AA}} \le 1 - \frac{2}{\|\mathrm{A}\|_\infty}.$$

Proof. See for instance [**?**, §13.2]. □

Corollary 13.5 (Ingham-Huxley bound)**.** *We have* $\theta_{\mathrm{PNT}} \le \frac{7}{12}$ *and* $\theta_{\mathrm{PNT-AA}} \le \frac{1}{6}$.

Proof. From Theorem 11.14 and Theorem 11.15 one as $\|\mathrm{A}\|_\infty \le 12/5$, and the claim now follows from Proposition 13.4. □

Corollary 13.6 (Ingham-Guth-Maynard bound)**.** *[**?**] We have* $\theta_{\mathrm{PNT}} \le \frac{17}{30}$ *and* $\theta_{\mathrm{PNT-AA}} \le \frac{2}{15}$.

These are currently the best known upper bounds on θ_{PNT} and $\theta_{\mathrm{PNT-AA}}$.

Proof. From Theorem 11.14 and Theorem 11.16 one as $\|\mathrm{A}\|_\infty \leq 30/13$, and the claim now follows from Proposition 13.4. ☐

Corollary 13.7. *The density hypothesis implies that* $\theta_{\mathrm{PNT}} \leq 1/2$ *and* $\theta_{\mathrm{PNT-AA}} = 0$.

The current unconditional best bound on θ_{gap} is

Theorem 13.8 (Baker-Harman-Pintz theorem). *[?] We have* $\theta_{\mathrm{gap}} \leq 0.525$.

Historical bounds on θ_{gap} are summarized in the following table:

Table 13.1: Historical upper bounds on θ_{gap}.

Reference	Upper bound
Hoheisel (1930) [?]	$1 - \frac{1}{33000}$
Heilbronn (1933) [?]	$1 - \frac{1}{250}$
Ingham (1937) [?]	$\frac{5}{8}$
Montgomery (1969) [?]	$\frac{3}{5}$
Huxley (1972) [?]	$\frac{7}{12}$
Iwaniec–Jutila (1979)[?]	$\frac{13}{23}$
Heath-Brown–Iwaniec (1979) [?]	$\frac{11}{20}$
Pintz (1981) [?]	$\frac{17}{31}$
Iwaniec–Pintz (1984) [?]	$\frac{23}{42}$
Baker–Harman–Pintz (2001) [?]	$\frac{21}{40}$

The following general bound on $\theta_{\mathrm{gap},2}$ is known:

Proposition 13.9. *We have*

$$\theta_{\mathrm{gap},2} \leq \max(2 - \frac{2}{\|\mathrm{A}\|_\infty}, \sup_{1/2 \leq \sigma \leq 1} \max(\alpha(\sigma), \beta(\sigma)))$$

where

$$\alpha(\sigma) := 4\sigma - 2 + 2\frac{A^*(\sigma)(1-\sigma) - 1}{A^*(\sigma) - A(\sigma)}$$

and

$$\beta(\sigma) := 4\sigma - 2 + \frac{A^*(\sigma)(1-\sigma) - 1}{A(\sigma)}.$$

Proof. See [?, Lemma 2]. We remark that this lemma allows σ to range over $0 \leq \sigma \leq 1$ rather than $1/2 \leq \sigma \leq 1$, but it is easy to see that the contributions of the $0 \leq \sigma < 1/2$ cases are dominated by the $\sigma = 1/2$ case. ☐

This proposition recovers several known bounds (both conditional and unconditional) on $\theta_{\mathrm{gap},2}$:

Corollary 13.10.

(i) *Assuming the Riemann hypothesis,* $\theta_{\mathrm{gap},2} = 1$. **Selberg**

(ii) *Unconditionally, one has* $\theta_{\mathrm{gap},2} \leq 4/3$. *[**?**]*

(iii) *Assuming the Lindelof hypothesis,* $\theta_{\mathrm{gap},2} \leq 7/6$. *[**?**]*

(iv) *Unconditionally,* $\theta_{\mathrm{gap},2} \leq 1 + 2 \times \frac{173}{1067}$. *[**?**, §7]*

(v) *Unconditionally,* $\theta_{\mathrm{gap},2} \leq 23/18$. *[**?**, Theorem 12.14].*

Proof. **give details** □

Chapter 14

The generalized Dirichlet divisor problem

Definition 14.1 (Divisor sum exponents)**.** Let $k \geq 1$. Then α_k is the best exponent for which one has the asymptotic

$$\sum_{n \leq x} d_k(n) = xP_{k-1}(\log x) + O(x^{\alpha_k + o(1)})$$

as $x \to \infty$, where P is an explicit polynomial of degree $k-1$ and $d_k(n) := \sum_{n_1 \dots n_k = n} 1$ is the k^{th} divisor function.

Lemma 14.2 (d_1 exponent)**.** *One has $\alpha_1 = 0$.*

Lemma 14.3. *[**?**, Theorem 13.1] One has $\alpha_2 \leq 35/108$.*

Lemma 14.4. *[**?**] One has $\alpha_3 \leq 43/96$.*

Lemma 14.5 (Lower bound)**.** *$\alpha_k \geq \frac{1}{2} - \frac{1}{2k}$ for all k.*

Proof. See [**?**, **?**, **?**]. $\square$

`It is conjectured that this lower bound is in fact an equality.`

Lemma 14.6. *Let $k \geq 2$ be an integer. If $M(\sigma, k) = 1$ then $\alpha_k \leq \sigma$.*

Proof. See [**?**, §13.3]. $\square$

`Using Lemma 14.6, the following bounds were obtained:`

Theorem 14.7. *[?, Theorem 13.12] One can bound* α_k *by*

$$\begin{aligned}
&(3k-4)/4k \textit{ for } 4 \le k \le 8\\
&35/54 \textit{ for } k = 9\\
&41/60 \textit{ for } k = 10\\
&7/10 \textit{ for } k = 11\\
&(k-2)/(k+2) \textit{ for } 12 \le k \le 25\\
&(k-1)/(k+4) \textit{ for } 26 \le k \le 50\\
&(31k-98)/32k \textit{ for } 51 \le k \le 57\\
&(7k-34)/7k \textit{ for } k \ge 58.
\end{aligned}$$

list known bounds on α_k

Chapter 15

The number of Pythagorean triples

Definition 15.1 (Pythagorean triple exponent)**.** Let θ_{Pythag} be the least exponent for which one has

$$P(N) = cN^{1/2} - c'N^{1/3} + N^{\theta_{\mathrm{Pythag}}+o(1)}$$

for unbounded N and some fixed c, c', where $P(N)$ is the number of primitive Pythagorean triples of area no greater than N.

Lemma 15.2. *One has* $\theta_{\mathrm{Pythag}} \leq 1/4$*.*

Proof. See [**?**, **?**]. The previous bound $\theta_{\mathrm{Pythag}} \leq 1/3$ was obtained in [**?**]. □

Lemma 15.3. *If* (k, ℓ) *is an exponent pair, and RH holds, then*

$$\theta_{\mathrm{Pythag}} \leq \max(\frac{1}{3} - \frac{5}{6}\frac{k+\ell-3/2}{4(k+\ell)-7}, \frac{1}{2} - \frac{3}{2}\frac{k+\ell-3/2}{4(k+\ell)-7})$$

Proof. See [**?**] and and [**?**, Section 5.10]. □

Lemma 15.4. *Assuming RH, one has* $\theta_{\mathrm{Pythag}} \leq 71/316$*.*

Proof. See [**?**, Section 5.10]. □