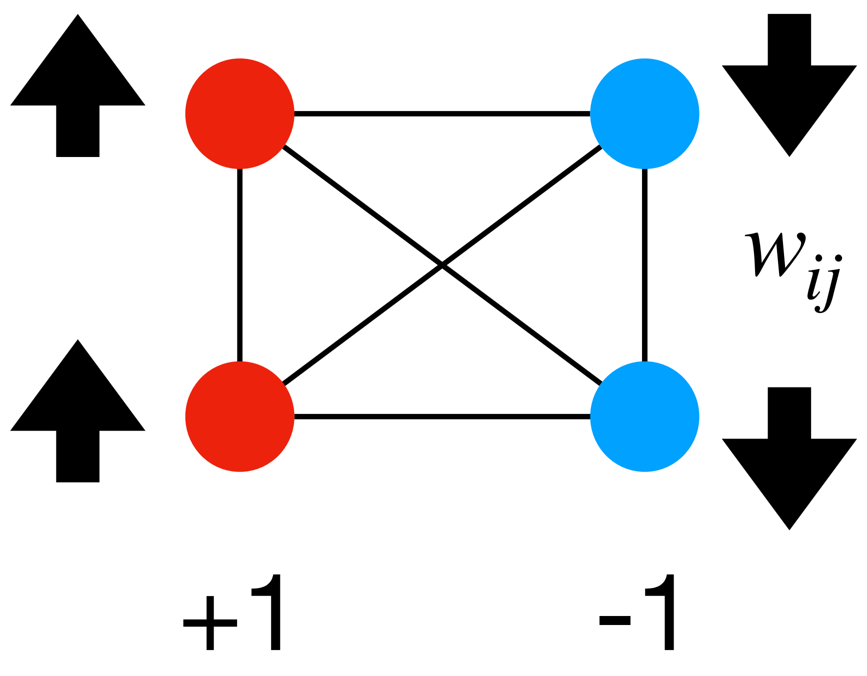
Maximally-stable Local Optima in Random Graphs and Spin Glasses: Phase Transitions and Universality

Yatin Dandi, David Gamarnik, Lenka Zdeborová

See also: concurrent work by Minzer, Sah, Sawhney (https://arxiv.org/abs/2305.03543)

Single-spin-flip stability



- Setup: weighted graphs $G = (V, W)$ with n nodes, random symmetric weights W for $i, j \in [n]$, average degree d (for dense graphs $d = \Theta(n)$).

- Hamiltonian function defined on spin configurations $\sigma \in \{+1, -1\}^n$/partitions:

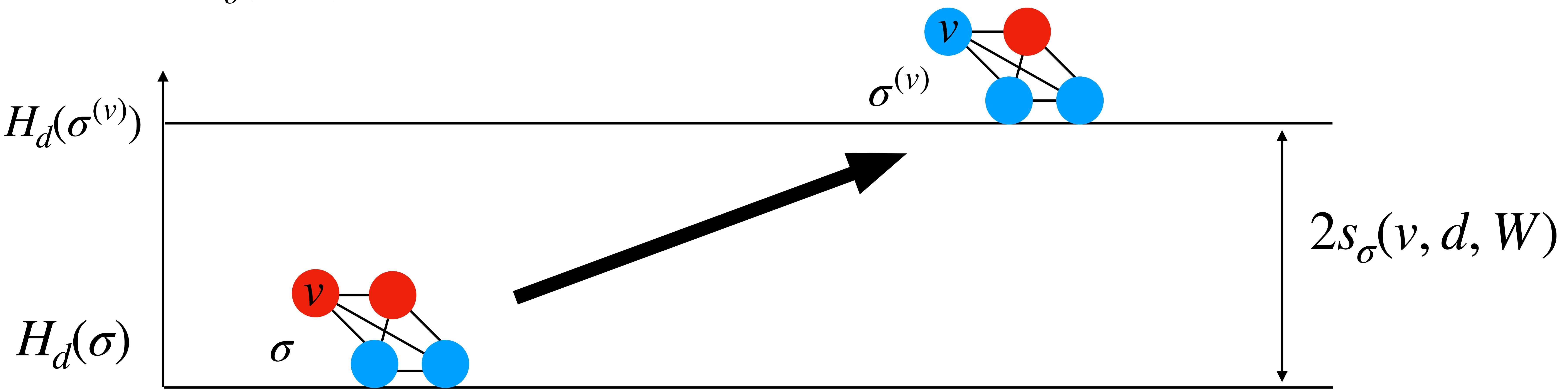
$$H_d(\sigma) := -\frac{1}{\sqrt{d}} \sum_{1 \le i < j \le n} \sigma_i \sigma_j w_{ij}$$

- $\sigma^{(v)}$: configuration obtained by flipping the v_{th} vertex.

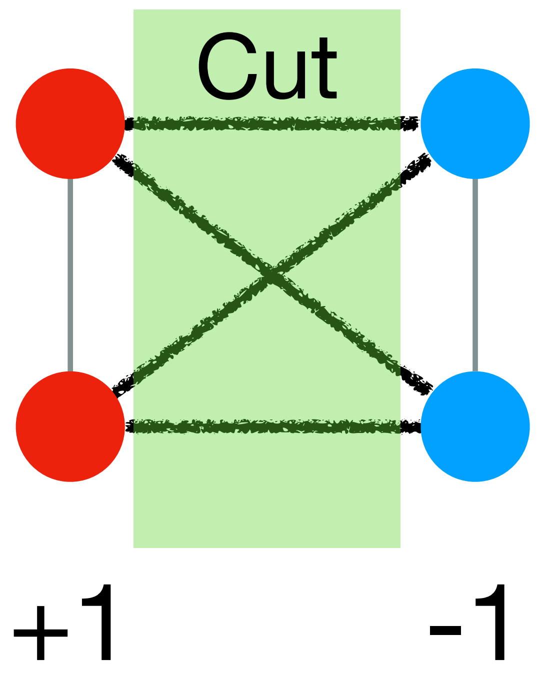
- Single-spin-flip stability for vertex v :

$$s_\sigma(v, W) = (H_d(\sigma^{(v)}) - H_d(\sigma))/2 = \frac{1}{\sqrt{d}} \sigma_v \sum_j w_{vj} \sigma_j$$

- Metastable states/local optima: $s_\sigma(v, W) \geq 0,\ \forall v \in V$.



Friendliness and Unfriendliness



- $$s_\sigma(v, W) = \frac{1}{\sqrt{d}}\Big(\sum_{i,\sigma_i=\sigma_v} w_{vi} - \sum_{i,\sigma_i\neq\sigma_v)} w_{vi}\Big),$$

- When $w_{ij} = +1$ for $(i, j) \in E$, 0 otherwise, $s_\sigma(v, W) =$ excess of neighbours within the same partition (friendliness). $H_d(\sigma) = 2(\text{cut-size})/\sqrt{d} - \sqrt{d}n/2$

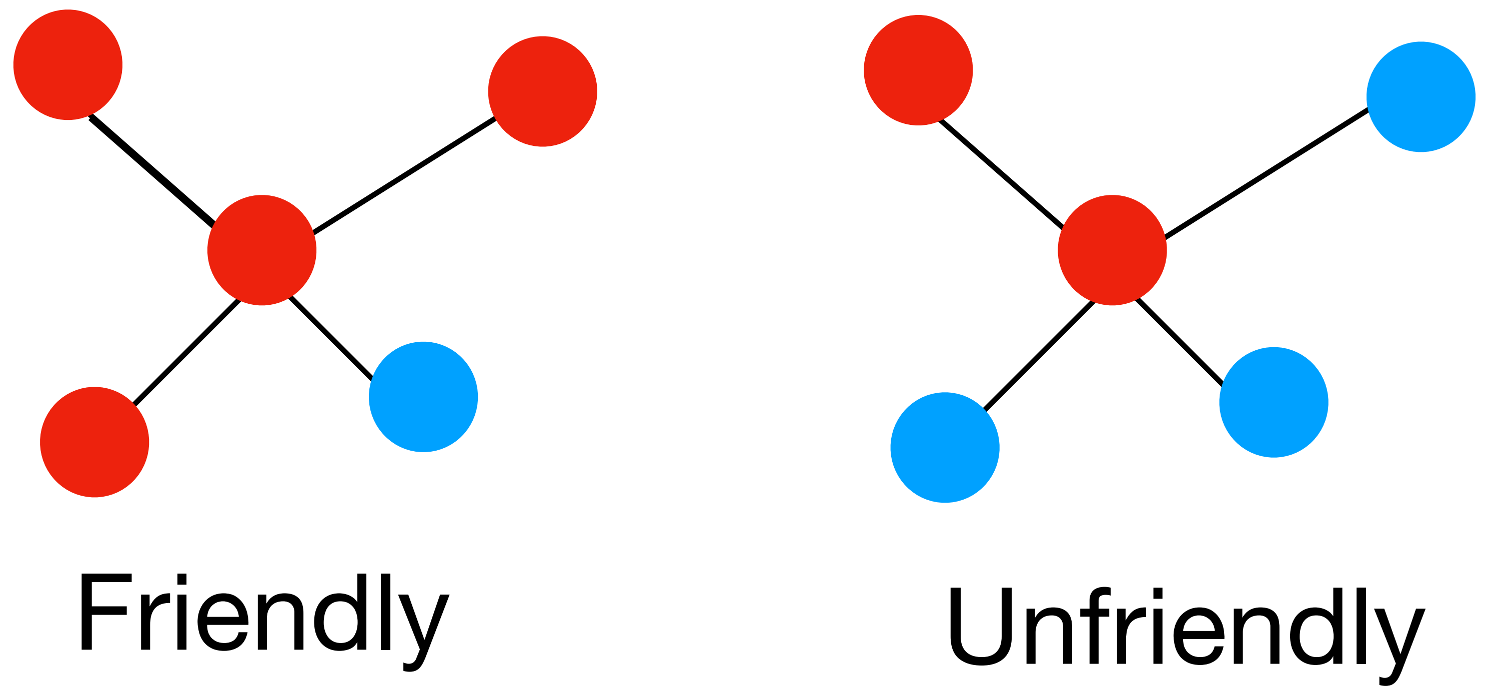
- When $w_{ij} = -1$ for $(i, j) \in E$, 0 otherwise, $s_\sigma(v, W) =$ excess of neighbours in the opposite partition (unfriendliness). $H_d(\sigma) = \sqrt{d}n/2 - 2(\text{cut-size})/\sqrt{d}$

- Equivalent to friendliness/unfriendliness in Behrens et al*. (2022), Ferber et al** (2021).

- Friendly partitions:

$$s_\sigma(v, W) \geq 0, \ \forall v \in V$$

- We restrict to bisections (0 magnetization).

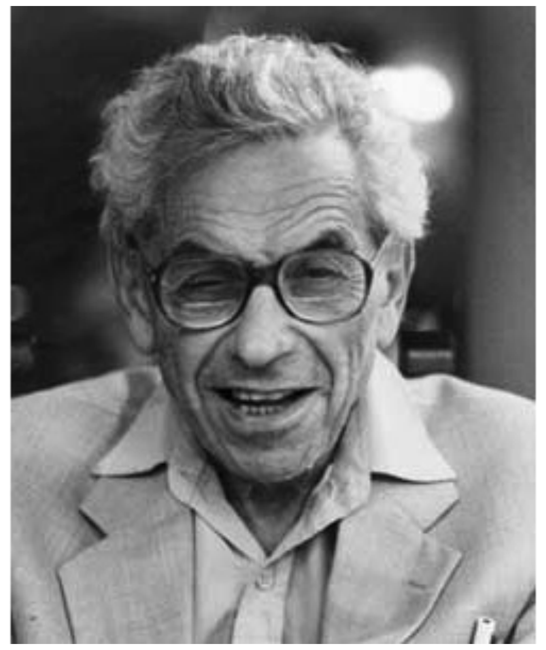


*(Dis)assortative Partitions on Random Regular Graphs: Freya Behrens, Gabriel Arpino, Yaroslav Kivva, Lenka Zdeborová: https://arxiv.org/abs/2202.10379

**Friendly bisections of random graphs: Asaf Ferber, Matthew Kwan, Bhargav Narayanan, Ashwin Sah, Mehtaab Sawhney: https://arxiv.org/abs/2105.13337

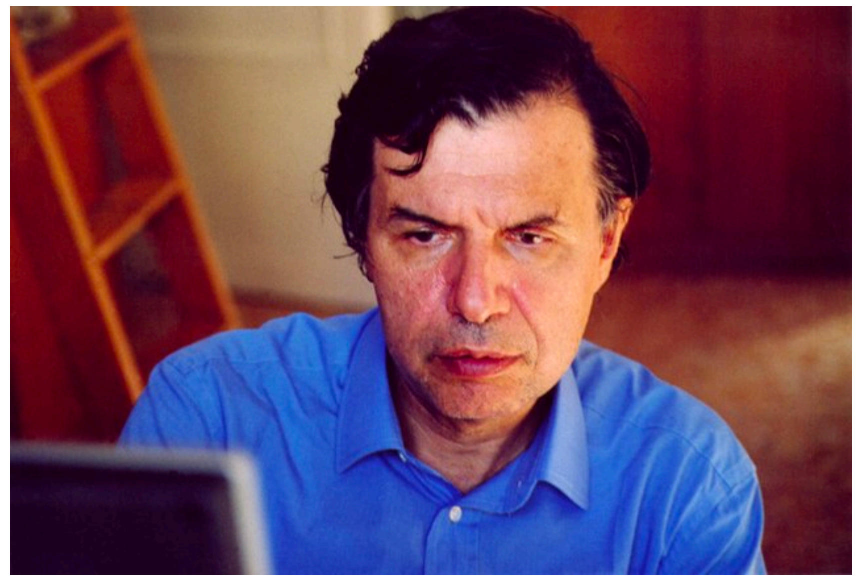
Some History

Random Graphs/TCS



- Furedi's conjecture(s) (1988): Existence of friendly bisections in Erdos-Renyi/regular random graphs.
- Local optima for Max-cut/Min Bisection: Angel et al*. (2017), Gamarnik et al. (2018).
- Related notion of "alliances" in Game theory, majority colouring, etc.

Spin Glasses



- Range of energies of Metastable states/local optima in the SK model (Bray and Moore, 1980).
- Expected number of local minima (Addario-Berry et al., 2018).
- Stable fixed-points for (symmetric) Hopfield-model (Treves and Amit, 1988, Riccardo's talk, etc.)

* Local Max-Cut in Smoothed Polynomial Time, Omer Angel, Sébastien Bubeck, Yuval Peres, Fan Wei: https://arxiv.org/abs/1610.04807

** Metastable states in asymmetrically diluted Hopfield networks, A Treves1 and D J Amit: https://iopscience.iop.org/article/10.1088/0305-4470/21/14/016/meta

Recent Conjecture (Behrens et al., 2022)

h**-stable vertex**: $s_\sigma(v, W) \geq h$

h**-stable bisection: all vertices** h**-stable**

Prediction using Cavity method: $\exists h^* \in \mathbb{R}^+$ **such that, for random sparse regular graphs, w.h.p as (**$n \to \infty, d \to \infty$**):**

- For $h > h^*$, h-stable bisections don't exist.
- For $h < h^*$, h-stable bisections exist.

$$h^* \approx 0.355$$

Universality (devil is in the details, but details don't matter!)

- h^*conjectured to be asymptotically the same for friendly/unfriendly bisections as well as for dense graphs, SK model.

- Analogous to the asymptotic universality of ground-states for sparse ferromagnetic (min-bisection), sparse anti-ferromagnetic (max-cut), spin-glass (Sherrington Kirkpatrick) model.

Conjecture on the maximum cut and bisection width in random regular graphs

Lenka Zdeborová[1], Stefan Boettcher[2]

EXTREMAL CUTS OF SPARSE RANDOM GRAPHS

AMIR DEMBO*, ANDREA MONTANARI†, AND SUBHABRATA SEN

Our strategy:

Prove for anti-ferromagnetic sparse graphs,
$w_{ij} \in \{-1,0\}, n \to \infty, d \to \infty$

Universality

Sparse ferromagnetic, dense ER graphs, SK model, etc.

Why sparse graphs?

- allows us to utilize existing results (Gamarnik et al. 2018*).
- avoids technical issues with conditional Large deviations.

***On the max-cut of sparse random graphs: David Gamarnik, Quan Li:**
https://doi.org/10.1002/rsa.20738

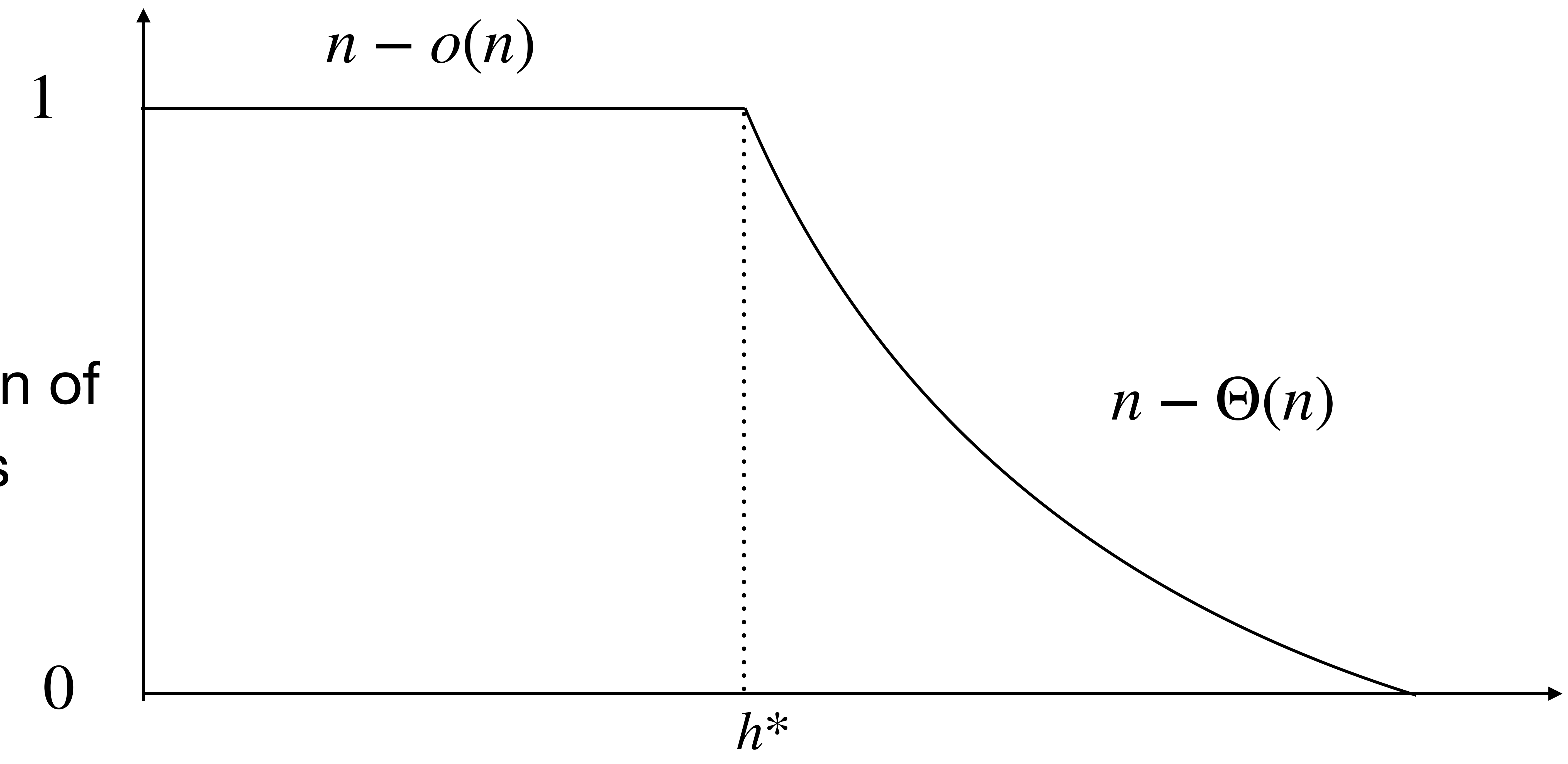
First result: (sparse graphs, $w_{ij} \in \{-1,0\}, n \to \infty, d \to \infty$)

Associated optimization problem:

$$N(W,h,\sigma) = \sum_{i \in [n]} \mathbf{1}\left(s_\sigma(v,W) \geq h\right)$$

$$N^*(W,h) = \max_\sigma N(W,h,\sigma)$$

Maximum proportion of h-stable vertices



Threshold

* (Dis)assortative Partitions on Random Regular Graphs: Freya Behrens, Gabriel Arpino, Yaroslav Kivva, : https://arxiv.org/abs/2202.10379

Theorem 1 (sparse anti-ferromagnetic graphs)

Definition 1. *We say that a value $h^* > 0$ is a maximal stability threshold for a family of random variables $W^{(d,n)}$ denoting weighted graphs indexed by the number of nodes n and a parameter d if as $n \to \infty$, the following hold:*

1. *For any $h > h^*$, there exists an $0 < \epsilon(h) < 1$ and a $d(h)$, such that for $d > d(h)$, with high probability as $n \to \infty$, all partitions in $W^{(d,n)}$ have at least ϵn vertices violating h-stability.*

2. *For any $h < h^*$, and any $\epsilon > 0$, there exists a $d(h, \epsilon)$, such that for all $d > d(h, \epsilon)$, with high probability as $n \to \infty$, there exists a bisection in $W^{(d,n)}$ with at most ϵn vertices violating h-stability.*

$$w(h) = \sup_x (-x^2 + \log(1 + \mathrm{erf}(3x - h/\sqrt{2}))).$$

Theorem 1. *(Threshold for sparse anti-ferromagnets) Let h^* denote the unique root of the function $w(h)$. Then for sparse Erdős-Rényi graphs $\mathbb{G}(n, p = d/n)$ with average degree d, corresponding to weight matrices with independent edge weights. $w_{ij} \in \{0, -1\}$ $p(w_{ij} = -1) = \frac{d}{n}$, h^* is a maximal stability threshold according to the Definition 1.*

Proof sketch for the Sparse Anti-ferromagnetic Model

Reduction to Gamarnik et al. (2018)*

- First and second moment methods.
- The constraints $s_\sigma(v, W) \geq h$ across vertices are correlated due to the symmetry of edges.
- Conditioning on the total number of edges across partitions i.e. cut-size (and within) and using **the configuration model** "decouples" these constraints.
- For first moment, we allow $\Theta(n)$ violations.
- For second moment, further need to condition on overlap $\omega(\sigma_i, \sigma_i') = (\sum_{i=1}^{n} \sigma_i \sigma_i')/n$.
- We introduce the following random variables:

$X(z, h, r)$: number of configurations with at-least rn h-stable vertices and cut size $\approx z$n

$X_2(z, h, \omega)$: number of pairs of h-stable configurations with cut size $\approx z$n

- Conditioning on cut size will lead to another major benefit later, stay tuned!

***On the max-cut of sparse random graphs: David Gamarnik, Quan Li:**
https://doi.org/10.1002/rsa.20738

Proof sketch for the Sparse Anti-ferromagnetic Model

Reduction to Gamarnik et al. (2018)*

- $\mathbb{E}[X(z,h,r)]$ and $\mathbb{E}[X_2(z,h,\omega)]$ are exponential in n.
- Poisson and Normal approximations (Poissonization, Central Limit Theorem on lattices), Large Deviations give leading factors.
- Show that first moment vanishes for $h > h^*$ and some fraction $r(h)$ of h-stable vertices.
- Show that second moment tight for $h < h^*$ (upto sub-leading exponential terms).
- Boosting the probability (concentration).

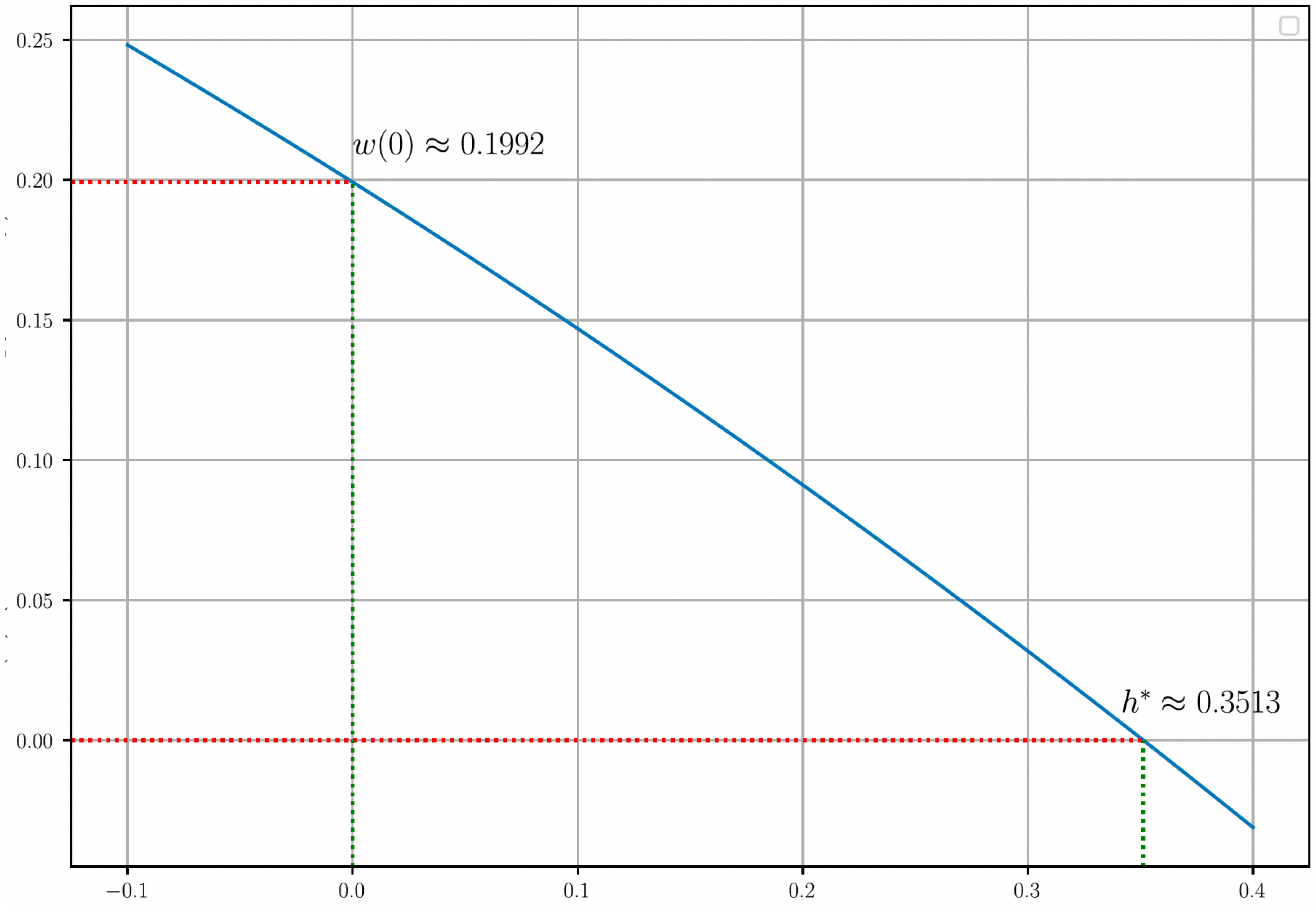
*On the max-cut of sparse random graphs: David Gamarnik, Quan Li: https://doi.org/10.1002/rsa.20738

First moment entropy density:

$$w(z,h,r) = \lim_{d\to\infty} \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}[X(z,h,r)]$$

$z^*(h)$: dominant cut-size for h $(r = 1)$

First moment entropy density
$w(z^*(h), h, 1)$



h (threshold)

First moment free entropy density < 0 for $h > h^*$.

Continuity w.r.t fraction r

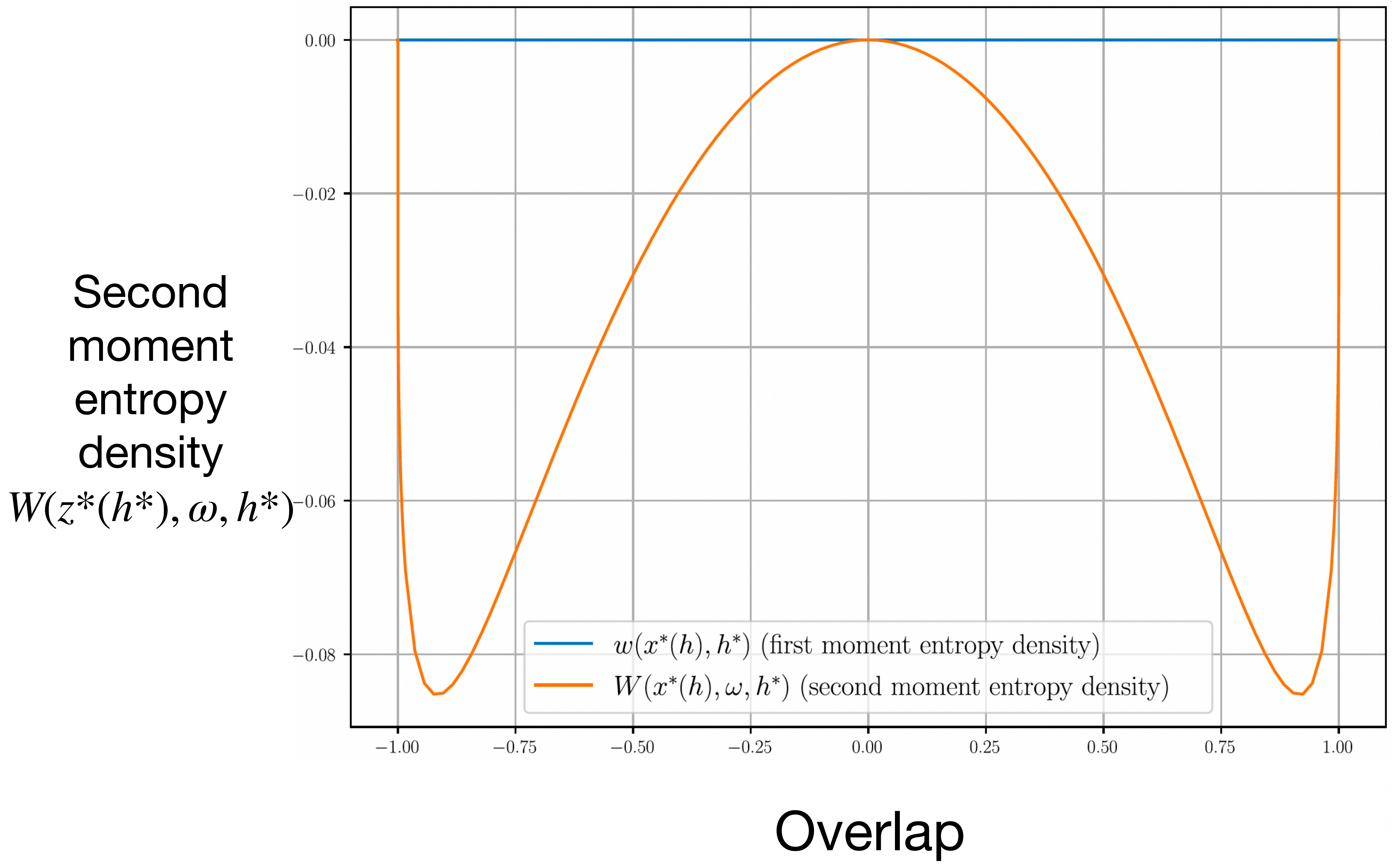
r(h) < 1 fraction of vertices violate h-stability w.h.p

At h=0, the first moment entropy density matches the one in Addario-Berry et al.* for the SK model

*Local optima of the Sherrington-Kirkpatrick Hamiltonian: Louigi Addario-Berry, Luc Devroye, Gabor Lugosi, Roberto Imbuzeiro Oliveira: https://arxiv.org/abs/1712.07775

The second moment entropy density:

$$W(z,\omega,h) = \lim_{d\to\infty}\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}[X_2(z,\omega,h)]$$



$W(z^*(h^*),\omega,h^*)$ maximized at $\omega = 0$ (uncorrelated configurations).

Paley-Zygmund

$$\mathbb{P}(\exists\ h\text{-unfriendly bisection}) \geq \exp(-o_d(1)n)$$

$W(z^*(h),\omega,h) < 0$ → Overlap-gap

Boosting the Probability

- We introduce the deficit function:

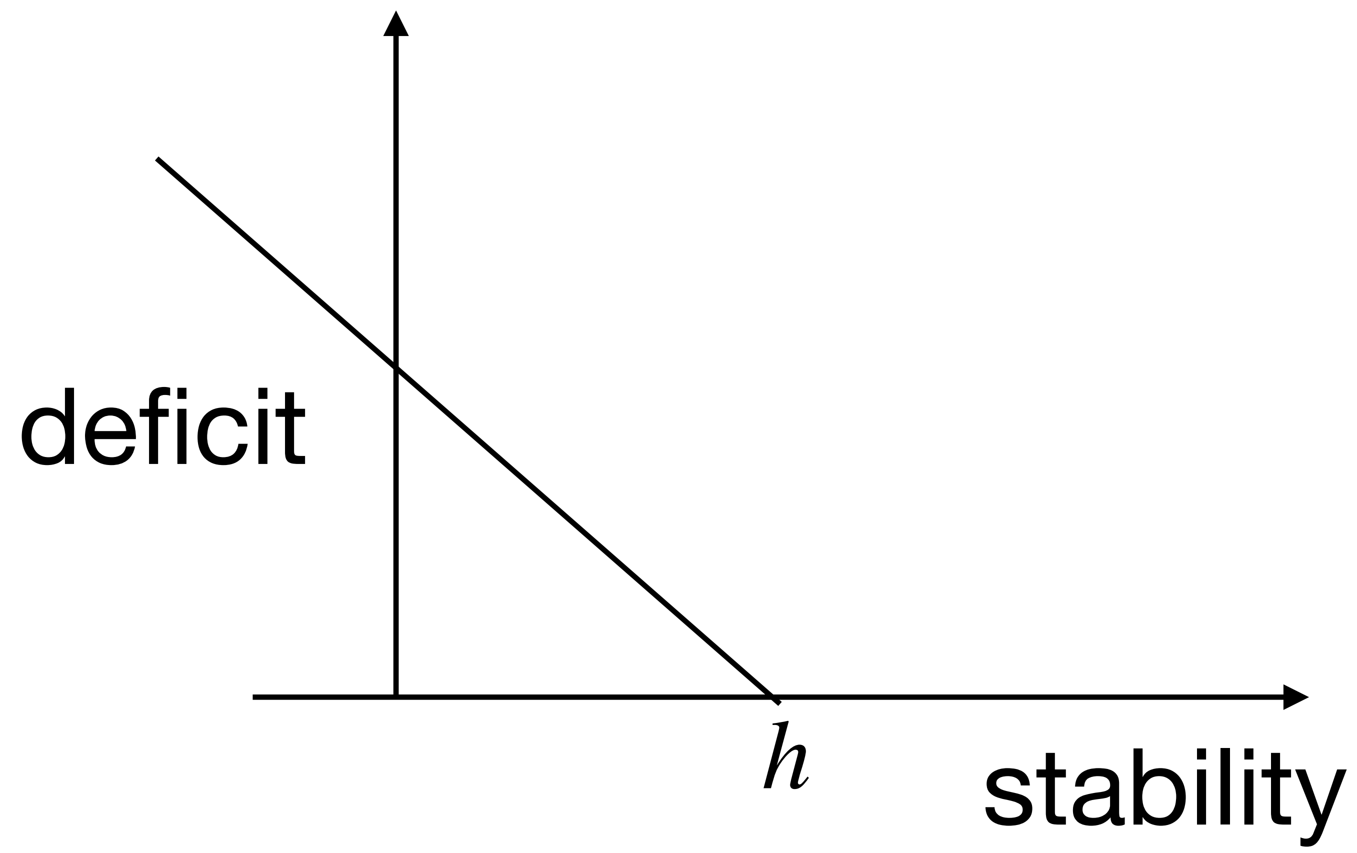
$$D(W,h,\sigma) = \sum_i \left(h - \frac{1}{\sqrt{d}} \sum_j w_{ij}\sigma_i\sigma_j \right)^+ , \quad D^*(W,h) = \min_\sigma D(W,h,\sigma)$$

- From Paley-Zygmund,
$\mathbb{P}\left[D^*(W,h,d) = 0\right] \geq \exp(-o_d(1)n)\,.$

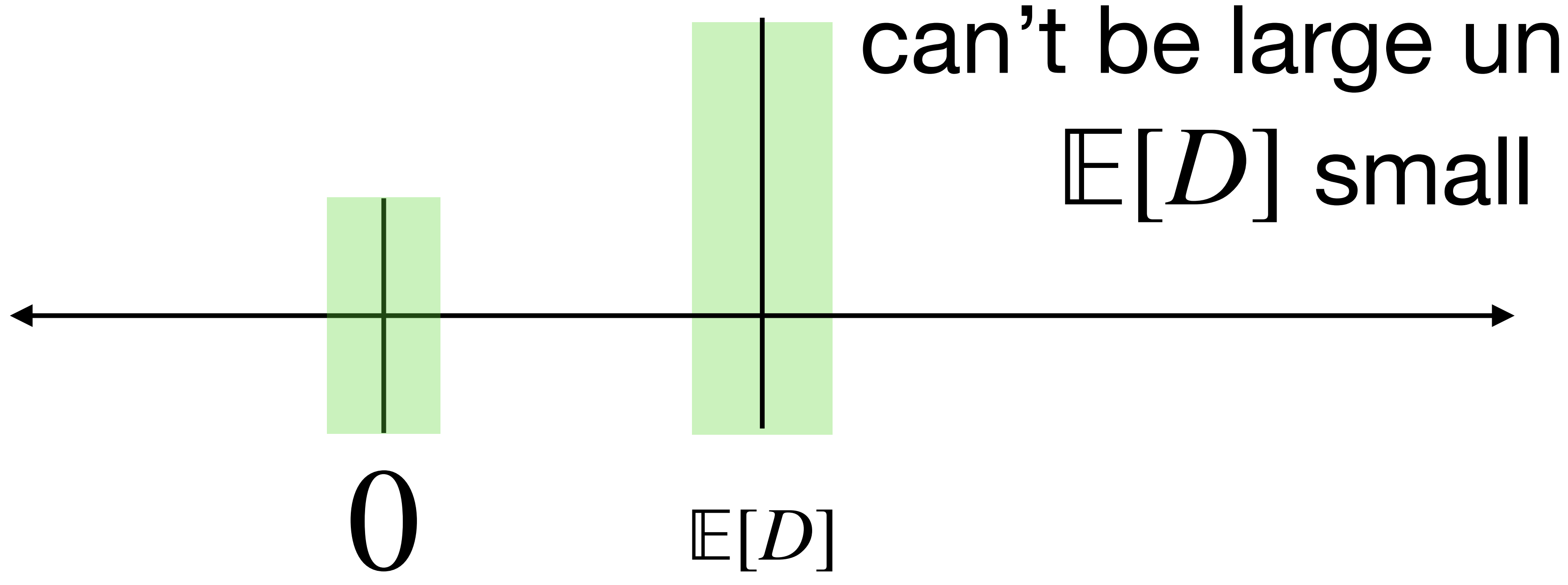
- We boost this using the strong concentration of D. Inspired by A. Frieze, 1990* and Gamarnik et al. (2018)**.

- But small minimal deficit doesn't imply the existence of nearly h-stable partitions.

- Another trick: perturb h-slightly, a small deficit for a larger $h' > h$ implies large number of h-stable vertices.



$\mathbb{P}\left[D^*(W,h,d) = 0\right]$
can't be large unless
$\mathbb{E}[D]$ small



Small deficit for $h' > h$

Large number of h-stable vertices

*A. Frieze, On the independence number of random graphs, Discrete Mathematics 81 (1990), 171–175.
** On the max-cut of sparse random graphs: David Gamarnik, Quan Li: https://doi.org/10.1002/rsa.20738

Universality of the Threshold (for bisections):

- Threshold is universal for any $W^{(d,n)}$ satisfying:

1. w_{ij} *are independent and identically distributed with mean* μ.
2. $\mathbb{E}[|w_{ij} - \mu|^2] = \frac{d}{n}(1 - \frac{d}{n})$.
3. $\mathbb{E}[|w_{ij} - \mu|^3] = O_n(\frac{d}{n})$.

→ **Weighted max/min bisection on sparse ER-graphs**

- And for dense-weighted graphs satisfying:

1. w_{ij} *are independent and identically distributed with mean* μ.
2. $\mathbb{E}[|w_{ij} - \mu|^2] = 1$.
3. $\mathbb{E}[|w_{ij} - \mu|^3] = O_n(1)$.

→ **The Sherrington-Kirkpatrick Model and dense ER graphs (Furedi's conjecture)*.**

*Proven without o(n) in Minzer, Sah, Sawhney (https://arxiv.org/abs/2305.03543)

Proving Universality

- Lindeberg's method: Show that $\mathbb{E}[f(a_1, a_2, \cdots, a_n)]$ approximates $\mathbb{E}[f(b_1, b_2, \cdots, b_n)]$ by iterative swapping and Taylor's theorem (Chatterjee et al. 2005*, Sen et al. 2018**).
- Directly proving universality of h^* appears infeasible.
- $D^*(W, h)$ is not-differentiable and involves a minimization over the set of configurations.
- Approximate $D^*(W, h, d)$ using the free energy for a Hamiltonian defined using a smooth approximation of D. $\mathscr{H}_d(W, g, \sigma) = \sum_{1 \le i \le n} g\left(h - \frac{1}{\sqrt{d}} \sum_j w_{ij}\sigma_i\sigma_j\right)$ g : smooth approximation of $()^+$
- Zero temperature limit.

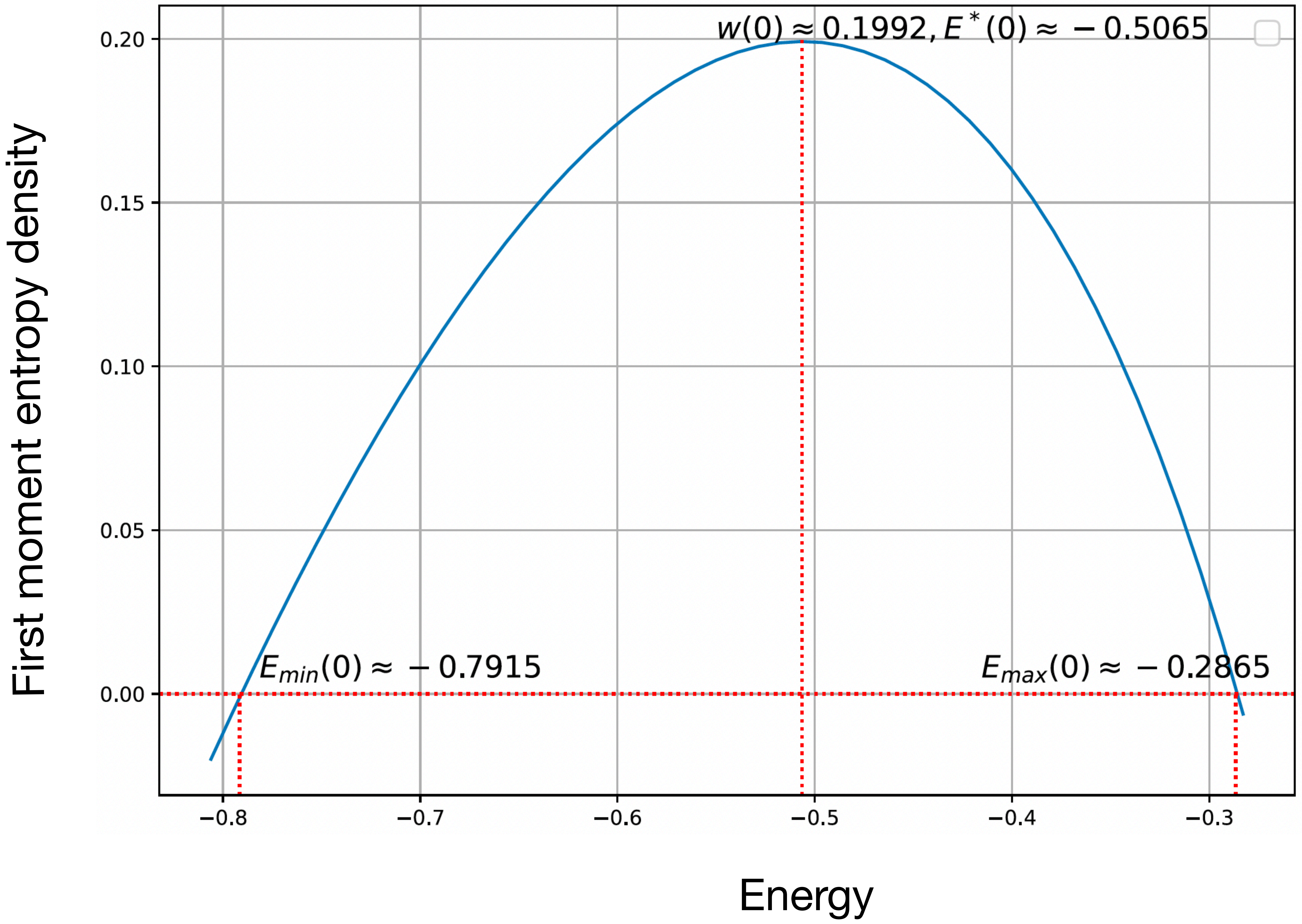
*A generalization of the Lindeberg principle: Sourav Chatterjee, Ann. Probab. DOI: 10.1214/009117906000000575

**Optimization on sparse random hypergraphs and spin glasses: Subhabrata Sen, Random Structures & Algorithms

Energy of h-stable optima

- Cut size zn related to the normalised energy E: $z = d/4n + En\sqrt{d}/2$.
- We already obtained first, second moments conditioned on cut-sizes.
- For what range of E do h-local optima exist?
- By convexity, first moment free entropy density has two roots.
- But second moment is not always tight!

First moment for $h = 0$



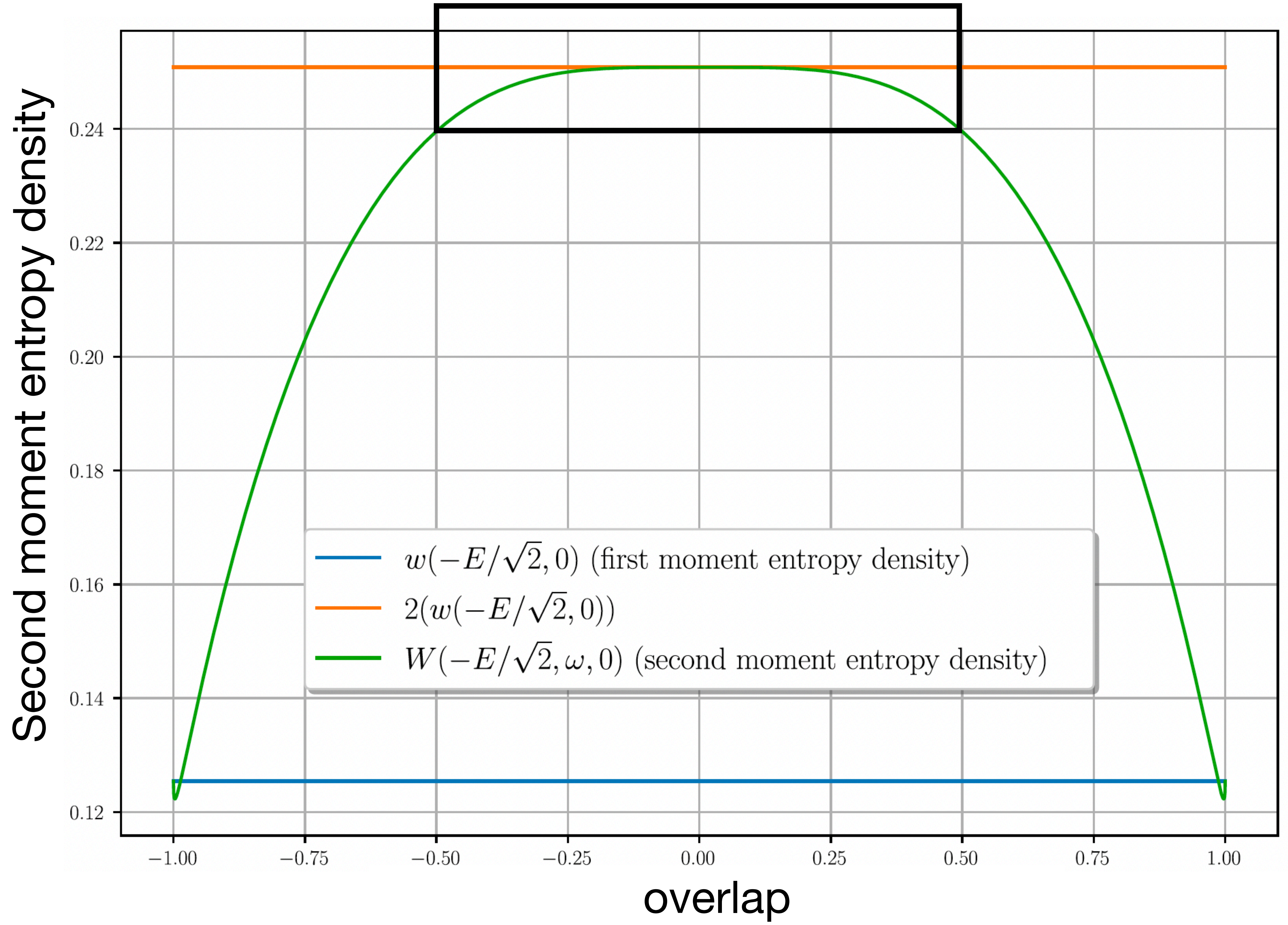
Failure of second moment for $h = 0$

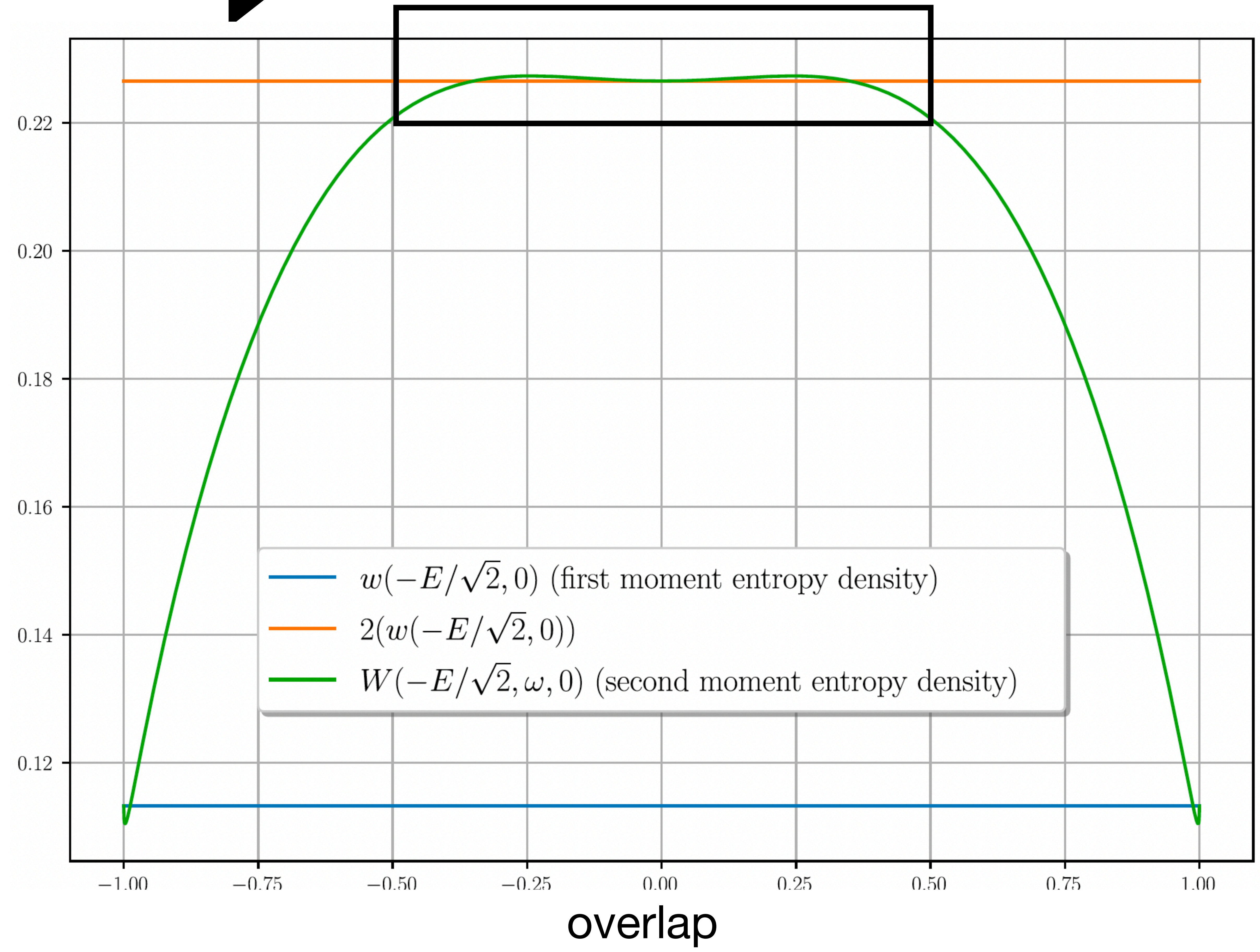
Local maxima

$E = -0.6725$

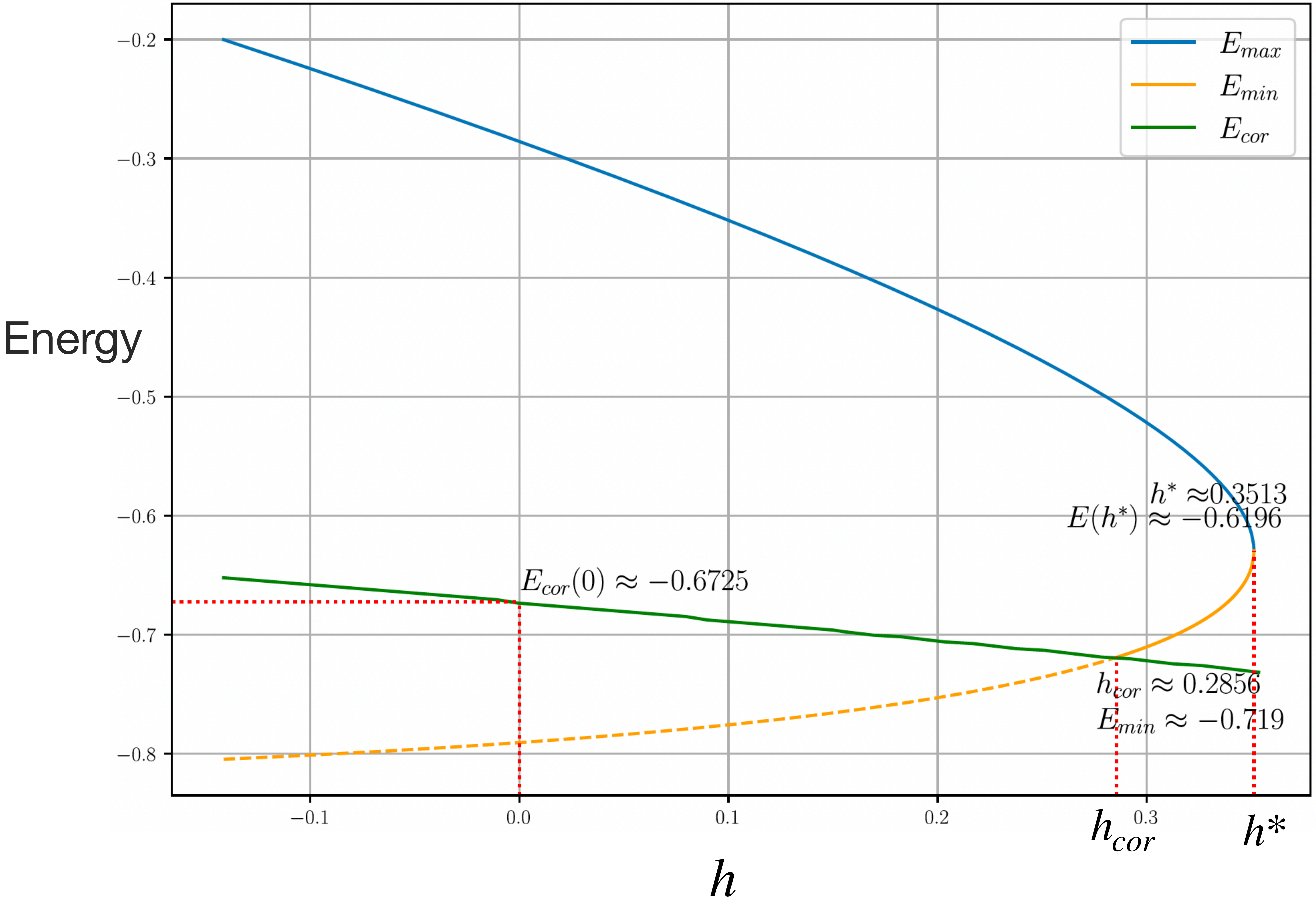
Local minima

$E = -0.6725 - \epsilon$



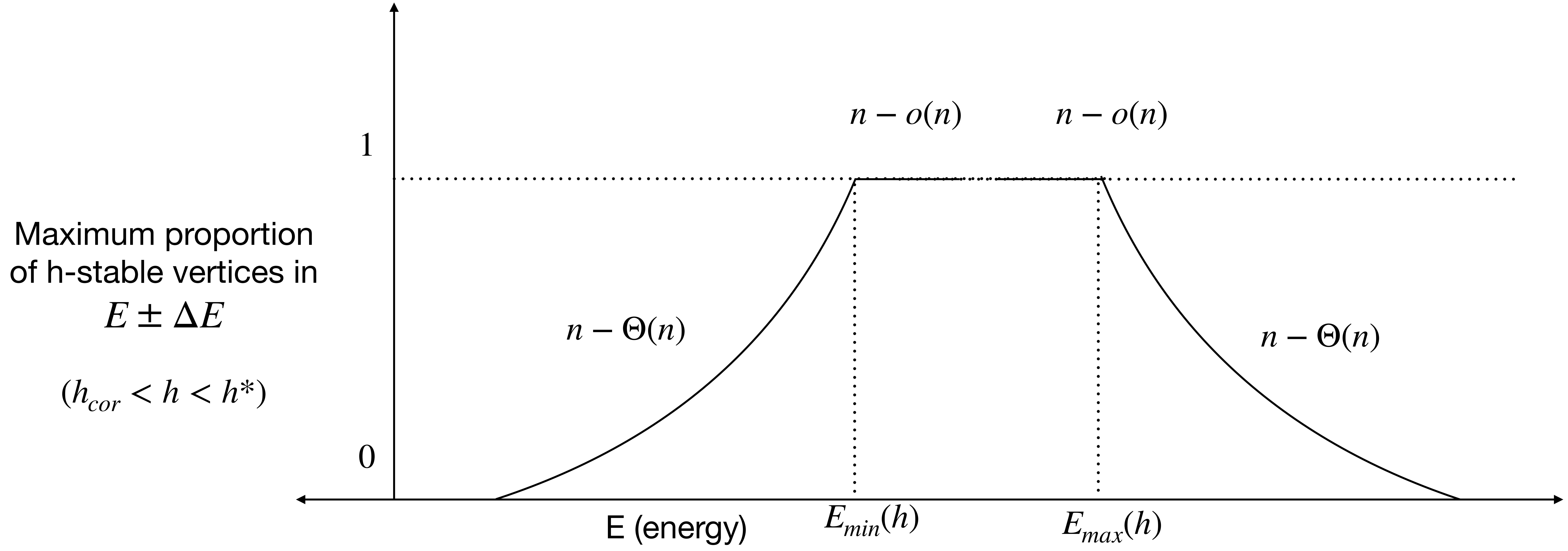


Another phase transition!



- For $h \in [h_{cor}, h^*]$, phase transitions around both E_{max} and E_{min}.
- For $h < h_{cor}$, the minima become correlated at sufficiently low energy.
- Even lower energies could lead to phenomenon such as Replica-Symmetry Breaking as we know for $h = 0$.

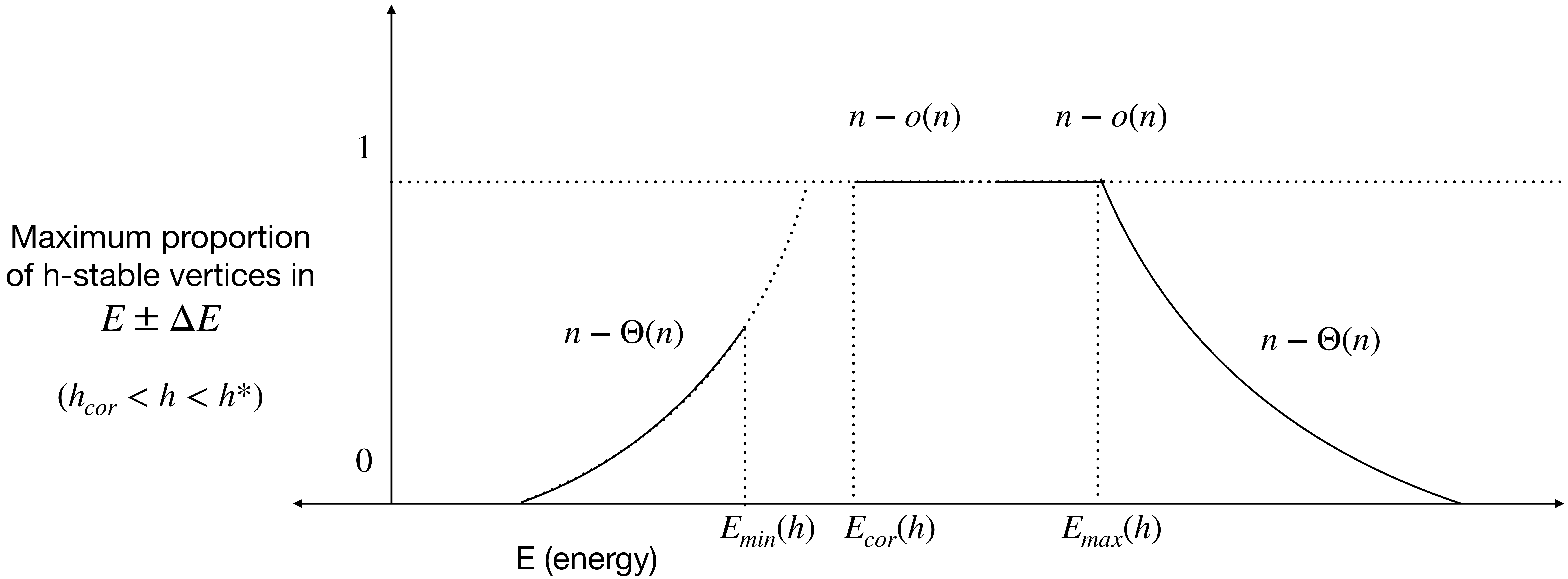
Energy of h-stable Configurations



Proof of phase transition and universality through a constrained deficit function:

$$D_{E_1,E_2}(W,h,\sigma) = \sum_i \left(h - \frac{1}{\sqrt{d}} \sum_j w_{ij}\sigma_i\sigma_j \right)^+ + (nE_1 - H_d(\sigma))^+ + (H_d(\sigma) - nE_2)^+$$

Energy of h-stable Configurations $(h < h_{cor})$



Matches free energy computations from Statistical Physics!

- We obtain $E_{max}(0) \approx -0.2865$, matching computations in Bray and Moore, 1980*.
- Furthermore, the values of $E_{cor}(0)$ and the free energy match our analysis.
- This suggests stronger concentration and universality results.

Plot from Bray and Moore, 1980*:

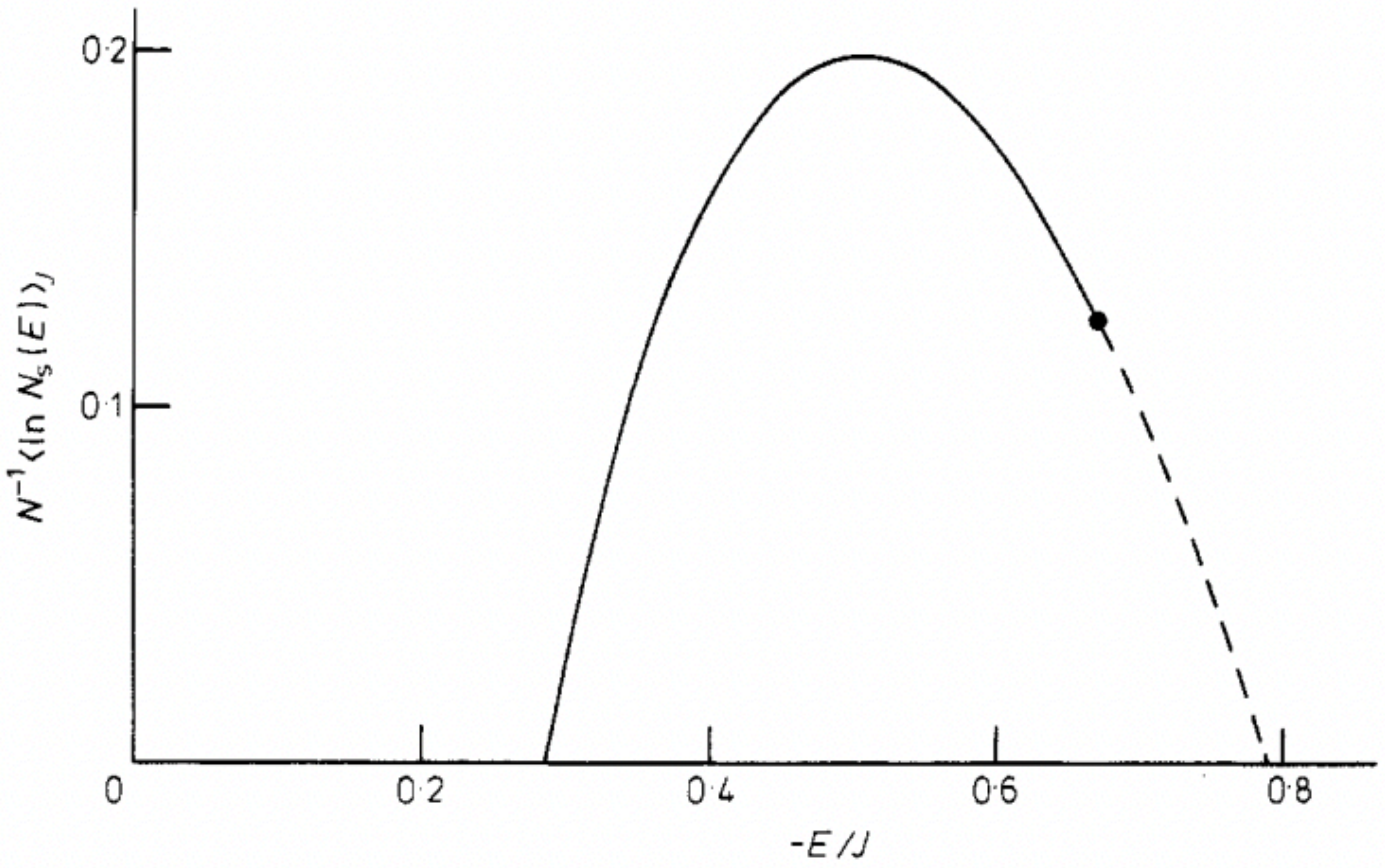


Figure 2. Logarithm of the density of metastable (i.e. one spin-flip stable) states, divided by N, for the SK model at $T = 0$. The broken curve corresponds to the range of energies for which the 'direct average' is unstable against off-diagonal fluctuations.

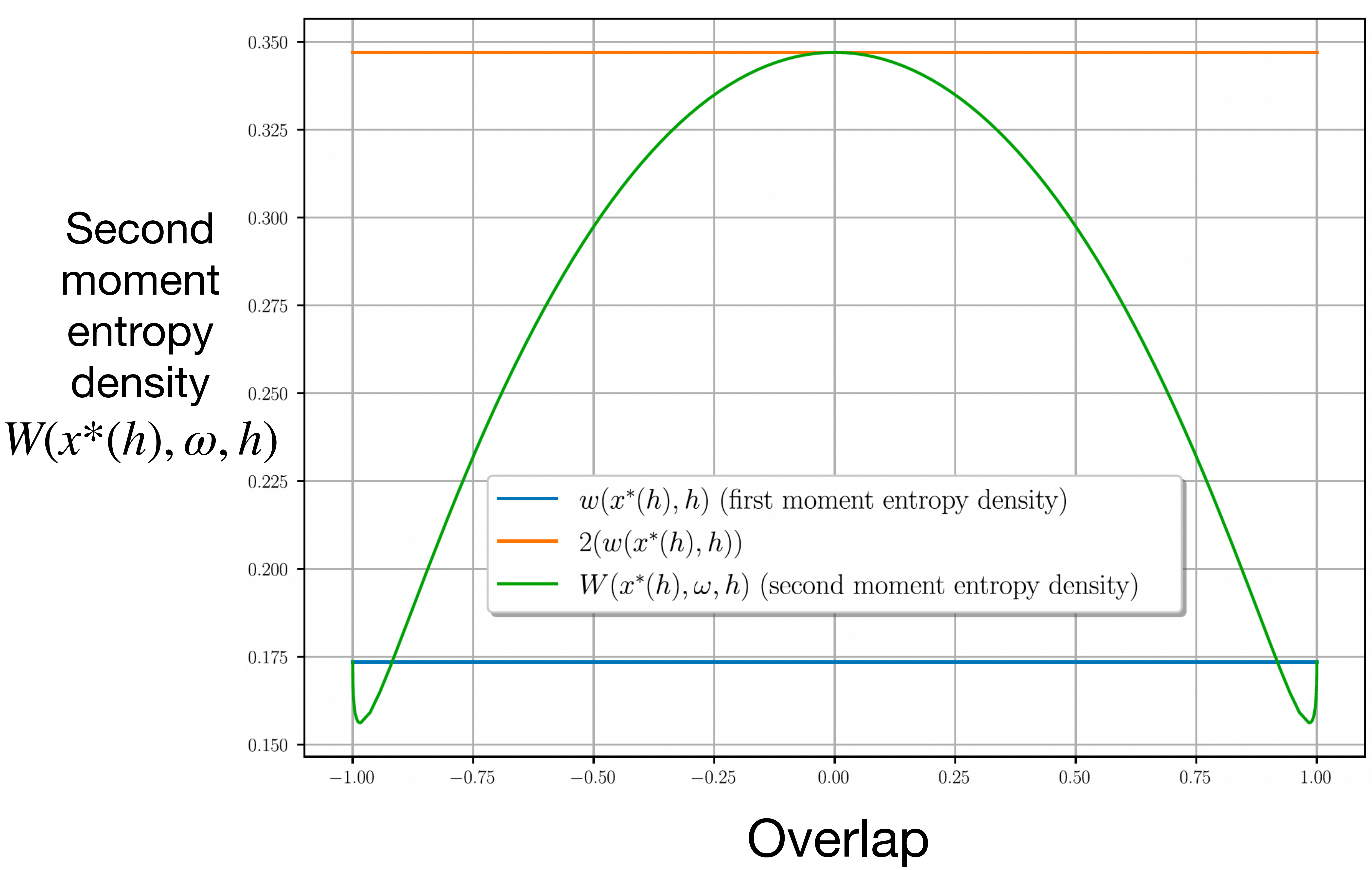
*Metastable states in spin glasses: A J Bray[1] and M A Moore:
10.1088/0022-3719/13/19/002

Overlap Gap Property

- For large $h < h*$, second moment entropy density proves OGP for approximate solutions.

- However, for $h \approx 0$, second moment is inconclusive.

- Based on the result for sparse regular graphs (Behrens et al.*), overlap gap might not be extensive for all $h > 0$.



* (Dis)assortative Partitions on Random Regular Graphs: Freya Behrens, Gabriel Arpino, Yaroslav Kivva, Lenka Zdeborová: https://arxiv.org/abs/2202.10379

Further progress and open questions

- Removing the $o(n)$ violations (achieved for dense ER-graphs by Minzer, Sah, Sawhney*, but universality still remains open!).
- Related result for all but finitely many regular graphs (Anastos et al., 2023).
- Geometry of solutions: Overlap Gap property, etc.
- Universality of first, second moment free entropy densities, etc.
- Much more appears to be universal than what we've proven!

* On Perfectly Friendly Bisections of Random Graphs: Dor Minzer, Ashwin Sah, Mehtaab Sawhney: https://arxiv.org/abs/2305.03543

** (Dis)assortative Partitions on Random Regular Graphs: Freya Behrens, Gabriel Arpino, Yaroslav Kivva, Lenka Zdeborová: https://arxiv.org/abs/2202.10379