19.12.07 2

s08-043

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1. Funkciu $f(x) = x^7 - x^5 + 2x^4 - 2x^2 - x + 2$ rozviňte do Taylorovho polynómu stupňa 7 so stredom v bode $x_0 = 1$.

$f(x) = x^7 - x^5 + 2x^4 - 2x^2 - x + 2$ $\qquad f(1) = 1$

$f'(x) = 7x^6 - 5x^4 + 8x^3 - 4x - 1$ $\qquad f'(1) = 7 - 5 + 8 - 4 - 1 = 5$

$f''(x) = 42x^5 - 20x^3 + 24x^2 - 4$ $\qquad f''(1) = 42 - 20 + 24 - 4 = 42$

$f'''(x) = 210x^4 - 60x^2 + 48x$ $\qquad f'''(1) = 210 - 60 + 48 = 198$

$f^{IV}(x) = 840x^3 - 120x + 48$ $\qquad f^{IV}(1) = 840 - 120 + 48 = 768$

$f^{V}(x) = 2520x^2 - 120$ $\qquad f^{V}(1) = 2520 - 120 = 2400$

$f^{VI}(x) = 5040x$ $\qquad f^{VI}(1) = 5040$

$f^{VII}(x) = 5040$ $\qquad f^{VII} = 5040$

$f^{VIII}(x) = 0$

$T_7(x) = 1 + 5(x-1) + \frac{42(x-1)^2}{2!} + \frac{198(x-1)^3}{3!} + \frac{768(x-1)^4}{4!} + \frac{2400(x-1)^5}{5!} + \frac{-5040(x-1)^6}{6!} + \frac{5040(x-1)^7}{7!} = 1 + 5(x-1) + 21(x-1)^2 + 33(x-1)^3 + 32(x-1)^4 + 20(x-1)^5 + 7(x-1)^6 + (x-7)^7$

```
   1  0 -1  2  0 -2 -1  2
1↓ 1  1  0  2  2  0 -1
   1  1  0  2  2  0 -1 [1]
1↓ 1  2  2  4  6  6
   1  2  2  4  6  6 [5]
1↓ 1  3  5  9 15
   1  3  5  9 15 [21]
1↓ 1  4  9 18
   1  4  9 18 [33]
1↓ 1  5 14
   1  5 14 [32]
1↓ 1  6
   1  6 [20]
1↓ 1
   1 [7]
1↓
  [1]
```

2. Funkciu $f(x) = x^7 - x^5 + 2x^4 - 2x^2 - x + 2$ rozviňte do Taylorovho polynómu stupňa 7 so stredom v bode $x_0 = -1$.

$f(x) = x^7 - x^5 + 2x^4 - 2x^2 - x + 2$ $\qquad f(-1) = 3$

$f'(x) = 7x^6 - 5x^4 + 8x^3 - 4x - 1$ $\qquad f'(-1) = -3$

$f''(x) = 42x^5 - 20x^3 + 24x^2 - 4$ $\qquad f''(-1) = -42 + 20 + 24 - 4 = -2$

$f'''(x) = 210x^4 - 60x^2 + 48x$ $\qquad f'''(-1) = 210 - 60 - 48 = 102$

$f^{IV}(x) = 840x^3 - 120x + 48$ $\qquad f^{IV}(-1) = -840 + 120 + 48 = -672$

$f^{V}(x) = 2520x^2 - 120$ $\qquad f^{V}(-1) = 2520 - 120 = 2400$

$f^{VI}(x) = 5040x$ $\qquad f^{VI}(-1) = -5040$

$f^{VII}(x) = 5040$ $\qquad f^{VII}(-1) = 5040$

$f^{VIII}(x) = 0$

$T_7(x) = 3 + 3(x+1) - \frac{2(x+1)^2}{2!} + \frac{102(x+1)^3}{3!} - \frac{672(x+1)^4}{4!} + \frac{2400(x+1)^5}{5!} + \frac{5040(x+1)^6}{6!} + \frac{5040(x+1)^7}{7!} = 3 - 3(x+1) - (x+2)^2 + 17(x+1)^3 - 28(x+1)^4 + 20(x+1)^5 - 7(x+1)^6 + (x+1)^7$

```
    1  0 -1  2  0 -2 -1  2
-1↓ -1  1  0 -2  2  0 +1
    1 -1  0  2 -2  0 -1 [3]
-1↓ -1  2 -2  0  2 -2
    1 -2  2  0 -2  2 [-3]
-1↓ -1  3 -5  5 -3
    1 -3  5 -5  3 [-1]
-1↓ -1  4 -9 14
    1 -4  9 -14 [17]
-1↓ -1  5 -14
    1 -5 14 [-28]
-1↓ -1  6
    1 -6 [20]
-1↓ -1
    1 [-7]
-1↓
   [1]
```

3. Funkciu $f(x) = x^7 - x^5 + 2x^4 - 2x^2 - x + 2$ rozviňte do Taylorovho polynómu stupňa 7 so stredom v bode $x_0 = 2$.

$f(x) = x^7 - x^5 + 2x^4 - 2x^2 - x + 2$ $\qquad f(2) = (2)^7 - (2)^5 + 2\cdot(2)^4 - 2(2)^2 - 2 + 2 = 128 - 32 + 32 - 8 = 120$

$f'(x) = 7x^6 - 5x^4 + 8x^3 - 4x - 1$ $\qquad f'(2) = 7\cdot(2)^6 - 5(2)^4 + 8(2)^3 - 4\cdot 2 - 1 = 448 - 80 + 64 - 8 - 1 = 423$

$f''(x) = 42x^5 - 20x^3 + 24x^2 - 4$ $\qquad f''(2) = 42\cdot(2)^5 - 20\cdot(2)^3 + 24(2)^2 - 4 = 1344 - 160 + 96 - 4 = 1276$

$f'''(x) = 210x^4 - 60x^2 + 48x$ $\qquad f'''(2) = 210(2)^4 - 60\cdot(2)^2 + 48\cdot 2 = 3360 - 240 + 96 = 3216$

$f^{IV}(x) = 840x^3 - 120x + 48$ $\qquad f^{IV}(2) = 840(2)^3 - 120\cdot 2 + 48 = 6720 - 240 + 48 = 6528$

$f^{V}(x) = 2520x^2 - 120$ $\qquad f^{V}(2) = 2520\cdot(2)^2 - 120 = 9960$

$f^{VI}(x) = 5040x$ $\qquad f^{VI}(2) = 5040\cdot 2 = 10080$

$f^{VII}(x) = 5040$ $\qquad f^{VII}(2) = 5040$

$f^{VIII}(x) = 0$

$T_7(x) = 120 + 423(x-2) + \frac{1276(x-2)^2}{2!} + \frac{3216(x-2)^3}{3!} + \frac{6528(x-2)^4}{4!} + \frac{9960(x-2)^5}{5!} + \frac{10080(x-2)^6}{6!} + \frac{5040(x-2)^7}{7!} = 120 + 423(x-2) + 638(x-2)^2 + 536(x-2)^3 + 272(x-2)^4 + 83(x-2)^5 + 14(x-2)^6 + (x-2)^7$

```
   1  0 -1  2  0 -2 -1  2
2↓ 2  4  6 16 32 60 118
   1  2  3  8 16 30 59 [120]
2↓ 2  8 22 60 152 364
   1  4 11 30 76 182 [423]
2↓ 2 12 46 152 456
   1  6 23 76 228 [638]
2↓ 2 16 78 308
   1  8 39 154 [536]
2↓ 2 20 118
   1 10 59 [272]
2↓ 2 24
   1 12 [83]
2↓ 2
   1 [14]
2↓
  [1]
```

—pokračovanie—

$^{6}* \; +\frac{x^7}{120}-\frac{x^9}{720}+\frac{x^{11}}{120^2}-\frac{x^9}{720}+\frac{x^{11}}{36\cdot120}+\frac{x^{13}}{720\cdot120}+\frac{x^{11}}{120^2}-\frac{x^{12}}{120\cdot720}+\frac{x^{15}}{120^3}$

$^{5}* \; \frac{x^7}{120}+\frac{x^7}{36}+\frac{x^9}{720}+\frac{x^7}{120}-\frac{x^9}{720}+\frac{x^{11}}{120^2}+\frac{x^7}{120}-\frac{x^5}{6}+\frac{x^7}{36}-\frac{x^9}{720}+\frac{x^7}{120}-\frac{x^9}{720}+\frac{x^{11}}{120^2}-\frac{x^5}{6}+\frac{x^7}{36}+\frac{x^9}{720}-$

$-\frac{x^7}{36}-\frac{x^9}{216}+\frac{x^9}{4320}+\frac{x^{11}}{6\cdot720}+\frac{x^{12}}{6\cdot120^2}+\frac{x^7}{120}-\frac{x^9}{720}+\frac{x^{11}}{120^2}-\frac{x^9}{720}+\frac{x^{11}}{36\cdot120}+\frac{x^{13}}{720\cdot120}+\frac{x^{11}}{120^2}-\frac{x^{12}}{120\cdot720}+$

$+\frac{x^{15}}{120^3}$

5. $T_n(x)=f(0)+\frac{f'(0)x}{1!}+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}+\dots$

$\hookrightarrow 0+0+0+\frac{6x^3}{3!}+0-\frac{60x^5}{5!}+\dots=x^3-\frac{3x^5}{6}+$ zvyšok $\frac{x^7}{120}+\frac{x^7}{36}+\frac{x^9}{720}+\frac{x^7}{120}-\frac{x^9}{720}+\frac{x^{11}}{120^2}+\frac{x^7}{120}-\frac{x^5}{6}+\frac{x^7}{36}$

$\sin^3 x=(\sin x)^3=\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\dots\right)^3=\dots$ t.j. $x^3-\frac{3x^5}{6}+$ zvyšok $*$

$\hookrightarrow=\left(x^2-\frac{x^4}{6}+\frac{x^6}{120}-\frac{x^4}{6}+\frac{x^6}{36}+\frac{x^8}{720}+\frac{x^6}{120}-\frac{x^8}{720}+\frac{x^{10}}{120^2}\right)\cdot\left(x-\frac{x^3}{6}+\frac{x^5}{120}\right)=x^3-\frac{x^5}{6}+\frac{x^7}{120}-\frac{x^5}{6}+\frac{x^7}{36}+\frac{x^9}{720}+\frac{x^7}{120}-\frac{x^9}{720}+$

$+\frac{x^{11}}{120^2}-\frac{x^5}{6}+\frac{x^7}{120}-\frac{x^5}{6}+\frac{x^7}{36}+\frac{x^9}{720}+\frac{x^7}{120}-\frac{x^9}{720}+\frac{x^{11}}{120^2}-\frac{x^5}{6}+\frac{x^7}{36}+\frac{x^9}{720}-\frac{x^7}{36}-\frac{x^9}{216}+\frac{x^9}{4320}+\frac{x^{11}}{6\cdot720}+\frac{x^{13}}{6\cdot120^2}+*$

6. $T_n(x)=f(0)+\frac{f'(0)x}{1!}+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}+\frac{f^{IV}(0)x^4}{4!}+\dots$

$\hookrightarrow 0+0+0+\frac{6x^3}{3!}+0+0+0+\dots=x^3+\dots$

$\sin t=t-\frac{t^3}{3!}+\frac{t^5}{5!}$

$\sin x^3=x^3-\frac{(x^3)^3}{3!}+\frac{(x^3)^5}{5!}=x^3-\frac{x^9}{3!}+\frac{x^{15}}{5!}\dots \Rightarrow$ t.j. $\underline{\underline{x^3}}$ + zvyšok $-\frac{x^9}{6}+\frac{x^{15}}{120}$

4. Určte Maclaurinov polynóm stupňa n pre funkciu $f(x) = \sin x$, $n \in N$.

$f'(x) = \cos x \quad f'(0) = 1$

$f''(x) = -\sin x \quad f''(0) = 0$

$f'''(x) = -\cos x \quad f'''(0) = -1$

$f^{IV}(x) = \sin x \quad f^{IV}(0) = 0$

$f^{V}(x) = \cos x \quad f^{V}(0) = 1$

$f^{VI}(x) = -\sin x$

$T_n(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots$

$\to 0 + x + \frac{0 \cdot x^2}{2!} + \frac{-1x^3}{3!} + \frac{0 \cdot x^4}{4!} + \dots = x - \frac{x^3}{6} + \frac{x^5}{120} + \dots + \frac{(-1)^k x^n}{(2k)!}$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + R_n(x)$ $\quad k = 0,1,2$ $\quad n = 2k$

5. Určte Maclaurinov polynóm stupňa 6 pre funkciu $f(x) = \sin^3 x$. [Úlohu najprv riešte priamo pre funkciu $f(x)$ a potom pomocou Maclaurinovho radu funkcie $g(x) = \sin x$. Oba výsledky porovnajte (musia byť rovnaké).]

$f(x) = \sin^3 x$

$f'(x) = 3\sin^2 x \cdot (\cos x)$

$f''(x) = 6\sin x \cdot (\cos x)(\cos x) + 3\sin^2 x \cdot (-\sin x) = 6\sin x \cdot \cos^2 x - 3\sin^3 x$

$f'''(x) = 6\cos x \cdot \cos^2 x + 6\sin x \cdot 2\cos x(-\sin x) - 9\sin^2 x(\cos x) = 6\cos^3 x + 21\sin^2 x \cdot \cos x$

$f^{IV}(x) = 18\cos^2 x(-\sin x) - 42\sin x \cdot \cos x \cdot \cos x - 21\sin^2 x \cdot \cos x = 21\sin^3 x - 60\cos^2 x \sin x$

$f^{V}(x) = 63\sin^2 x \cdot \cos x - [120\cos x \cdot (-\sin x) \cdot \sin x + \cos^3 x \cdot 18] = 183\sin^2 x \cos x - 18\cos^3 x$

$f^{VI}(x) = 183 \cdot 2\sin x \cdot \cos^2 x - \sin^3 x \cdot 183 + 18 \cdot 3\cos^2 x \cdot \sin x = 420\cos^2 x \cdot \sin x - 183\sin^3 x$

$f(0) = 0$

$f'(0) = 0$

$f''(0) = 0$

$f'''(0) = 6$

$f^{IV}(0) = 0$

$f^{V}(0) = -60$

$f^{VI}(0) = 0$

$T_n(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots$

$\to 0 + 0 + 0 + \frac{6x^3}{3!} + 0 - \frac{60x^5}{5!} = x^3 - \frac{18x^5}{5!} + \dots + \frac{(-1)^k x^{2k}}{(2k)!}$ $\quad k = 0,1,2$ $\quad n = 2k$

iba po 6!

$\sin^3 x = (\sin x)^3 = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)^3 = \dots$

Zoberieme iba po exponent 6

6. Určte Maclaurinov polynóm stupňa 6 pre funkciu $f(x) = \sin(x^3)$. [Úlohu najprv riešte priamo pre funkciu $f(x)$ a potom pomocou Maclaurinovho radu funkcie $g(x) = \sin x$. Oba výsledky porovnajte (musia byť rovnaké).]

$f(x) = \sin(x^3)$

$f'(x) = \cos(x^3) \cdot 3x^2$

$f''(x) = -\sin(x^3) \cdot 3x^2 \cdot 3x^2 + \cos(x^3) \cdot 6x = -9x^4 \sin x^3 + 6x \cos x^3$

$f'''(x) = -36x^3 \cdot \sin x^3 - 9x^4(\cos x^3) \cdot 3x^2 + 6\cos x^3 - 6x \sin x^3 \cdot 3x^2 = -54x^3 \sin x^3 + 6\cos x^3 - 27x^6 \cos x^3$

$f^{IV}(x) = -162x^2 \sin x^3 - 54x^3 \cdot \cos x^3 \cdot 3x^2 - 6\sin x^3 \cdot 3x^2 - 162x^5 \cos x^3 + 27x^6 \sin x^3 \cdot 3x^2 = -180x^2 \sin x^3 - 324x^5 \cos x^3 + 81x^8 \sin x^3$

$f^{V}(x) = -360x \sin x^3 - 180x^2 \cdot \cos x^3 \cdot 3x^2 - 972x^2 \cdot \cos x^3 + 324x^3 \sin x^3 + 648x^7 \sin x^3 + 81x^8 \cdot \cos x^3 \cdot 3x^2 = -360x \sin x^3 - 540x^4 \cos x^3 - 972x^2 \cdot \cos x^3 + 324x^3 \sin x^3 + 648x^7 \sin x^3 + 243x^{10} \cos x^3$

$f^{VI}(x) = -360 \sin x^3 - 360x \cos x^3 \cdot 3x^2 - 2160x^3 \cos x^3 + 540x^4 \sin x^3 \cdot 3x^2 - 1944x \cdot \cos x^3 + 972x^2 \sin x^3 \cdot 3x^2 + 972x^2 \cdot \sin x^3 + 324x^3 \cdot \cos x^3 \cdot 3x^2 + 4536x^6 \cdot \sin x^3 + 648x^7 \cdot \cos x^3 \cdot 3x^2 + 2430x^9 \cos x^3 - 243x^{10} \cdot \sin x^3 \cdot 3x^2 =$
$= -360 \sin x^3 - 3240x^3 \cos x^3 + 6156x^6 \sin x^3 - 1944x \cdot \cos x^3 + 2916x^4 \sin x^3 + 972x^2 \sin x^3 + 972x^5 \cos x^3 + 4374x^9 \cos x^3 - 729x^{12} \sin x^3$

$f(0) = 0$

$f'(0) = 0$

$f''(0) = 0$

$f'''(0) = 6$

$f^{IV}(0) = 0$

$f^{V}(0) = 0$

$f^{VI}(0) = 0$

$T_n(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{IV}(0)x^4}{4!} + \dots$

$\to 0 + 0 + 0 + \frac{6x^3}{3!} + 0 + 0 + 0 + \dots = x^3 + \frac{(-1)^k x^{2k}}{(2k)!}$

iba po 6. člen

$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!}$

$\sin x^3 = x^3 - \frac{(x^3)^3}{3!} + \frac{(x^3)^5}{5!}$

$x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \dots$ 2.5. Iba x^3

$$^{6}* + \frac{x^7}{120} - \frac{x^9}{720} + \frac{x^{11}}{120^2} - \frac{x^9}{720} + \frac{x^{11}}{36\cdot 120} + \frac{x^{13}}{720\cdot 120} + \frac{x^{11}}{120^2} - \frac{x^{12}}{120\cdot 720} + \frac{x^{15}}{120^3}$$

$$^{5}* \ \frac{x^7}{120} + \frac{x^7}{36} + \frac{x^9}{720} + \frac{x^7}{120} - \frac{x^9}{720} + \frac{x^{11}}{120^2} + \frac{x^7}{120} - \frac{x^5}{6} + \frac{x^7}{36} - \frac{x^9}{720} + \frac{x^7}{120} - \frac{x^9}{720} + \frac{x^{11}}{120^2} - \frac{x^5}{6} + \frac{x^7}{36} + \frac{x^9}{720} -$$

$$- \frac{x^7}{36} - \frac{x^9}{216} + \frac{x^9}{4320} + \frac{x^{11}}{6\cdot 720} + \frac{x^{12}}{6\cdot 120^2} + \frac{x^7}{120} - \frac{x^9}{720} + \frac{x^{11}}{120^2} - \frac{x^9}{720} + \frac{x^{11}}{36\cdot 120} + \frac{x^{13}}{720\cdot 120} + \frac{x^{11}}{120^2} - \frac{x^{12}}{120\cdot 720} +$$

$$+ \frac{x^{15}}{120^3}$$

5. $T_n(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots$

$\hookrightarrow 0+0+0+\frac{6x^3}{3!}+0-\frac{60x^5}{5!}+\dots = x^3 - \frac{3x^5}{6}$ + zvyšok $\frac{x^7}{120} + \frac{x^7}{36} + \frac{x^9}{720} + \frac{x^7}{120} - \frac{x^9}{720} + \frac{x^{11}}{120^2} + \frac{x^7}{120} - \frac{x^5}{6} + \frac{x^7}{36}$

$\sin^3 x = (\sin x)^3 = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)^3 = \dots$ tj. $\overline{x^3 - \frac{3x^5}{6}}$ + zvyšok $*$

$\hookrightarrow = \left(x^2 - \frac{x^4}{6} + \frac{x^6}{720} - \frac{x^4}{6} + \frac{x^6}{36} + \frac{x^8}{720} + \frac{x^6}{120} - \frac{x^8}{720} + \frac{x^{10}}{120^2}\right)\cdot\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right) = x^3 - \frac{x^5}{6} + \frac{x^7}{120} - \frac{x^5}{6} + \frac{x^7}{36} + \frac{x^9}{720} + \frac{x^7}{120} - \frac{x^9}{720} +$

$+ \frac{x^{11}}{120^2} - \frac{x^5}{6} + \frac{x^7}{120} - \frac{x^5}{6} + \frac{x^7}{36} + \frac{x^9}{720} + \frac{x^7}{120} - \frac{x^9}{720} + \frac{x^{11}}{120^2} - \frac{x^5}{6} + \frac{x^7}{36} + \frac{x^9}{720} - \frac{x^7}{36} - \frac{x^9}{216} + \frac{x^9}{4320} + \frac{x^{11}}{6\cdot 720} + \frac{x^{12}}{6\cdot 120^2} + *$

6. $T_n(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{IV}(0)x^4}{4!} + \dots$

$\hookrightarrow 0+0+0+\frac{6x^3}{3!}+0+0+0+\dots = x^3 + \dots$

$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!}$

$\sin x^3 = x^3 - \frac{(x^3)^3}{3!} + \frac{(x^3)^5}{5!} = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} \dots \Rightarrow$ tj. $\underline{\underline{x^3}}$ + zvyšok $-\frac{x^9}{6} + \frac{x^{15}}{120}$