Příklady – Laplaceova transformace

Pomocí Laplaceovy transformace řešte počáteční úlohy:

1. $x''(t) + 4x'(t) + 3x(t) = 3e^{-2t}; \quad x(0) = 1,\ x'(0) = 0$

   Řešení

   $\mathcal{L}\{x(t)\} = X(p)$
   $\mathcal{L}\{x'(t)\} = pX(p) - 1$
   $\mathcal{L}\{x''(t)\} = p^2X(p) - p - 0$
   $\mathcal{L}\{e^{-2t}\} = \frac{1}{p+2}$

   $$\begin{aligned} p^2X - p + 4(pX - 1) + 3X &= \tfrac{3}{p+2} \\ (p^2 + 4p + 3)X &= \tfrac{3}{p+2} + p + 4 \\ X(p) &= \tfrac{p^2+6p+11}{(p+2)(p+3)(p+1)} \\ X(p) &= \tfrac{3}{p+1} - \tfrac{3}{p+2} + \tfrac{1}{p+3} \\ x(t) &= 3e^{-t} - 3e^{-2t} + e^{-3t} \end{aligned}$$

2. $x''(t) - 2x'(t) - 3x(t) = 3 - 4e^{t}; \quad x(0) = 2,\ x'(0) = 3$

   Řešení

   $\mathcal{L}\{x(t)\} = X(p)$
   $\mathcal{L}\{x'(t)\} = pX(p) - 2$
   $\mathcal{L}\{x''(t)\} = p^2X(p) - 2p - 3$
   $\mathcal{L}\{3\} = \frac{3}{p}\ \mathcal{L}\{e^t\} = \frac{1}{p-1}$

   $$\begin{aligned} p^2X - 2p - 3 - 2(pX - 2) - 3X &= \tfrac{3}{p} - \tfrac{4}{p-1} \\ (p^2 - 2p - 3)X &= \tfrac{3}{p} - \tfrac{4}{p-1} + 2p - 1 \\ X(p) &= \tfrac{2p^3-3p^2-3}{p(p-1)(p+1)(p-3)} \\ X(p) &= -\tfrac{1}{p} + \tfrac{1}{p-1} + \tfrac{1}{p+1} + \tfrac{1}{p-3} \\ x(t) &= -1 + e^t + e^{-t} + e^{3t} \end{aligned}$$

3. $x''(t) + 2x'(t) + 2x(t) = e^{-t}; \quad x(0) = 2,\ x'(0) = 1$

   Řešení

   $\mathcal{L}\{x(t)\} = X(p)$
   $\mathcal{L}\{x'(t)\} = pX(p) - 2$
   $\mathcal{L}\{x''(t)\} = p^2X(p) - 2p - 1$
   $\mathcal{L}\{e^{-t}\} = \frac{1}{p+1}$

   $$\begin{aligned} p^2X - 2p - 1 + 2(pX - 2) + 2X &= \tfrac{1}{p+1} \\ (p^2 + 2p + 2)X &= \tfrac{1}{p+1} + 2p + 5 \\ X(p) &= \tfrac{2p^2+7p+6}{(p+1)(p^2+2p+2)} \\ X(p) &= \tfrac{1}{p+1} + \tfrac{p+4}{p^2+2p+2} \\ X(p) &= \tfrac{1}{p+1} + \tfrac{p+1}{(p+1)^2+1} - \tfrac{3}{(p+1)^2+1} \\ x(t) &= e^{-t} + e^{-t}\cos t - 3e^{-t}\sin t \end{aligned}$$

4. $x''(t) + x(t) = 3\sin 2t; \quad x(0) = 1,\ x'(0) = 0$

Řešení

$\mathcal{L}\{x(t)\} = X(p)$

$\mathcal{L}\{x'(t)\} = pX(p) - 1$

$\mathcal{L}\{x''(t)\} = p^2X(p) - p$

$\mathcal{L}\{\sin 2t\} = \frac{2}{p^2+4}$

$$\begin{aligned} p^2X - p + X &= \tfrac{2}{p^2+4} \\ (p^2+1)X &= \tfrac{6}{p^2+4} + p \\ X(p) &= \tfrac{p^3+4p+6}{(p^2+1)(p^2+4)} \\ X(p) &= \tfrac{-2}{p^2+4} + \tfrac{p+2}{p^2+1} \\ X(p) &= \tfrac{-2}{p^2+4} + \tfrac{p}{p^2+1} + \tfrac{2}{p^2+1} \\ x(t) &= -\sin 2t + \cos t + 2\sin t \end{aligned}$$