# Teória sietí

# Úloha vrstvy prevádzky?

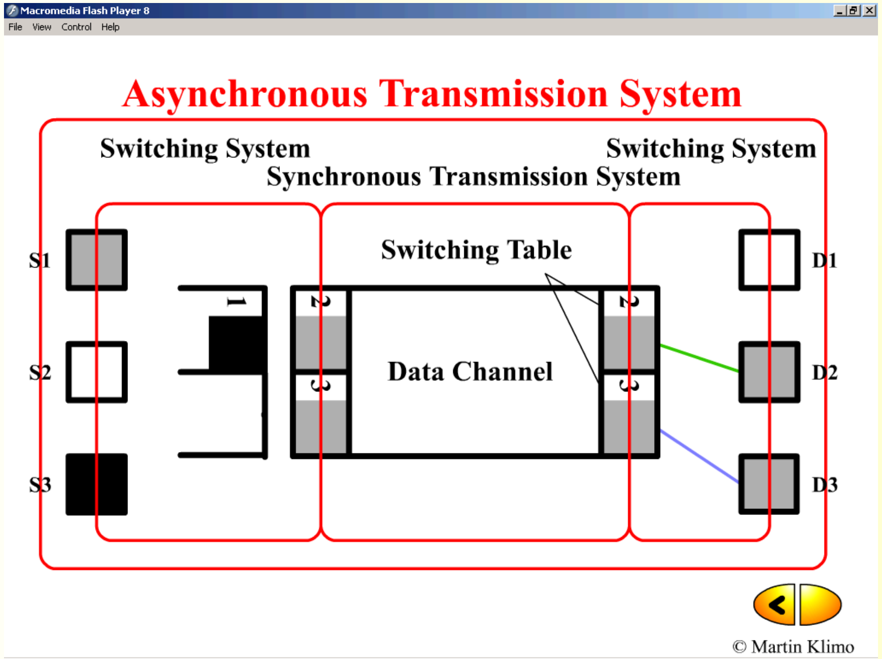
**Nájsť kompromis medzi kvalitou a efektívnosťou siete.**

1. **z ekonomických dôvodov musí byť kapacita siete menšia než sú možné požiadavky na prenos**
2. **požiadavky na prenos vznikajú náhodne**

# Riešenie ?

Policing – odmietnuť záťaž prevyšujúcu kapacitu siete

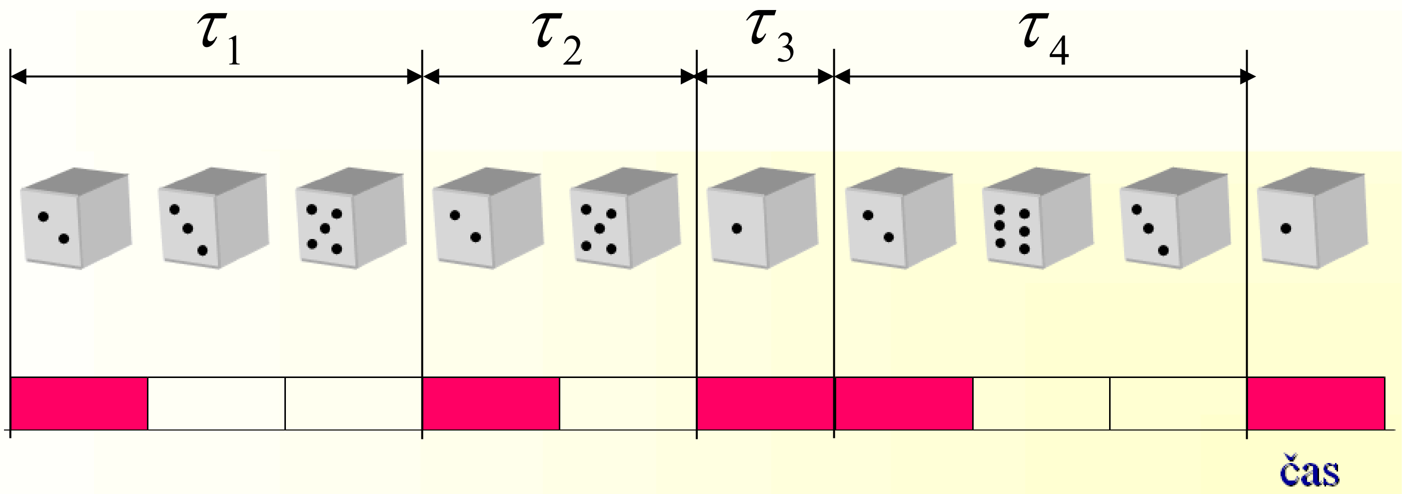
Shaping – odložiť záťaž prevyšujúcu kapacitu siete na neskôr



# Prvá úloha

**Ako popísať proces, ktorý sa v sieti odohráva?**

# Bernoulliho proces

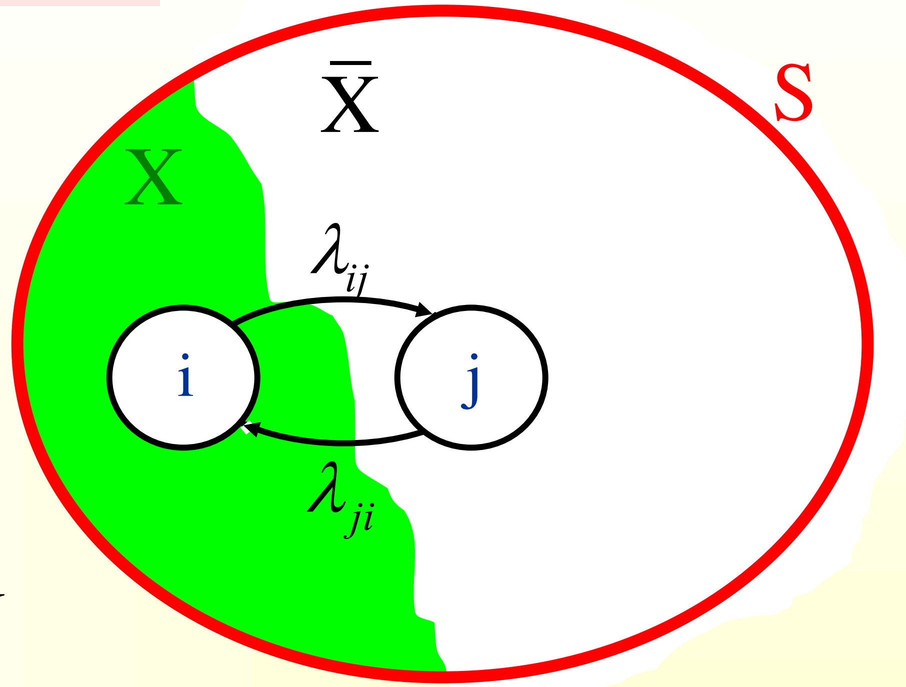



rozdelenie pravdepodobnosti

$$P\{\tau_k = n\} = P\{\tau = n\} = p(1-p)^{n-1}$$

$$\forall k,\ n = 1,2,\ldots$$

# Veta o zachovaní toku




$$\Phi_{X\bar{X}} = \Phi_{\bar{X}X}$$

$$\sum_{i\in X}\sum_{j\in \bar{X}} \pi_i \, p_{ij} = \sum_{j\in \bar{X}}\sum_{i\in X} \pi_j \, p_{ji}$$

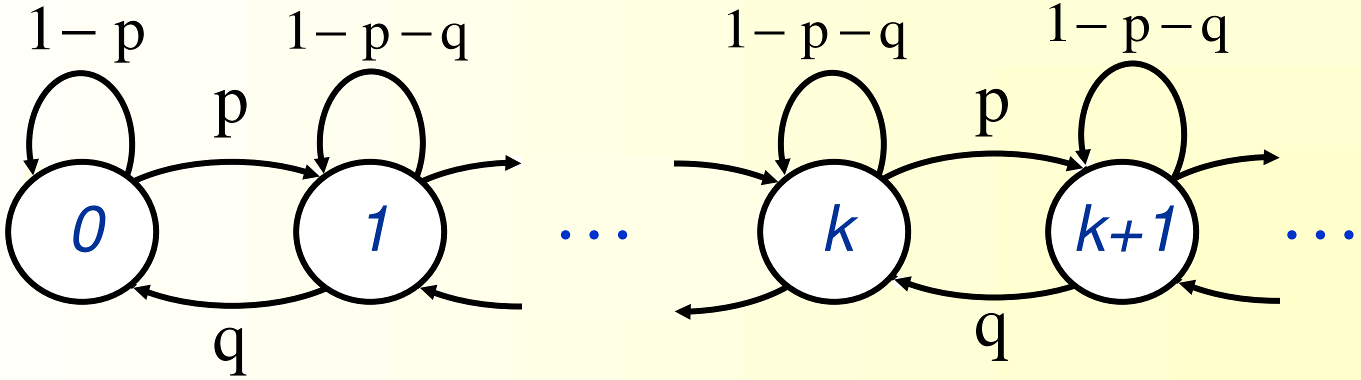
Formálny dôkaz za domácu úlohu

# Príklad

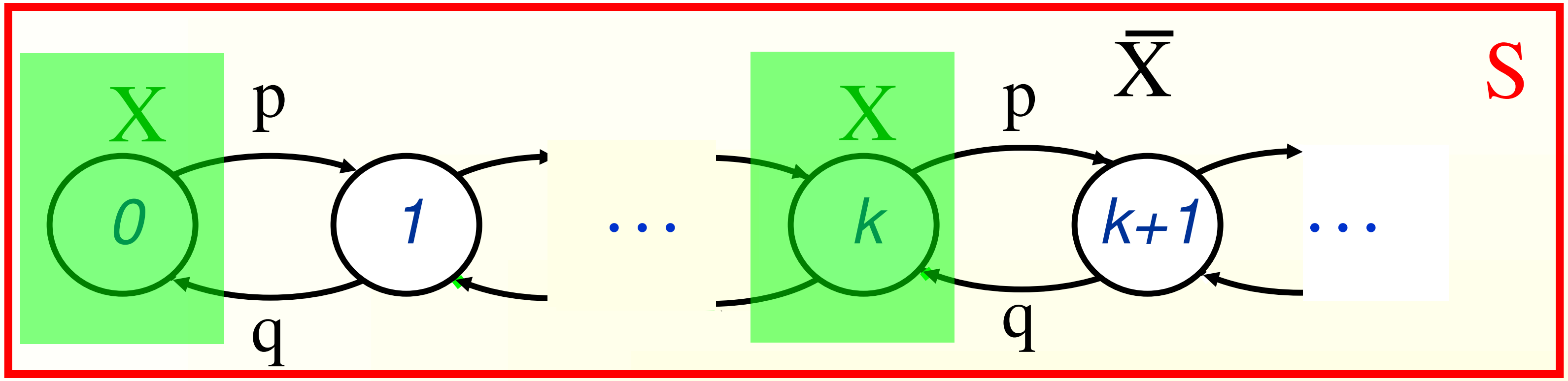
## Matica prechodov

$$\mathbf{P} = \begin{pmatrix} 1-p & p & 0 & 0 & \dots \\ q & 1-p-q & p & 0 & \dots \\ 0 & q & 1-p-q & p & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

## Graf prechodov



# Veta o zachovaní toku

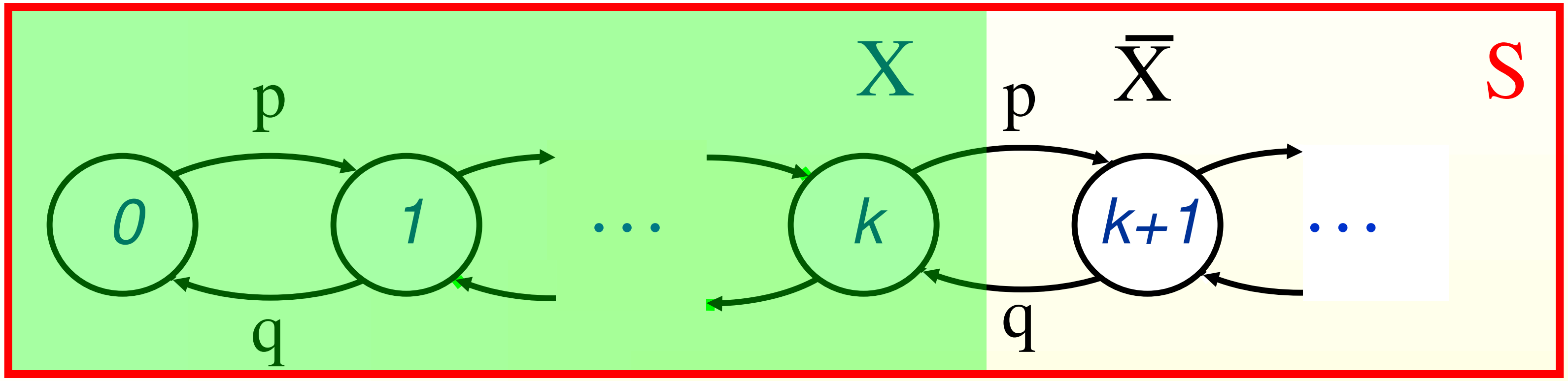



Veta o zachovaní toku pravdepodobnosti

$$\pi_0 p = \pi_1 q$$

$$\pi_k (p+q) = \pi_{k-1} p + \pi_{k+1} q, \quad k = 1,2,...$$

# Iné rozdelenie stavov?




$$\pi_k\, p = \pi_{k+1} q, \quad k = 0,1,\ldots$$

$$\pi_{k+1} = \frac{p}{q}\pi_k = \rho\pi_k, \quad k = 0,1,\ldots$$

# Invariantné rozdelenie

$$\pi_k = \rho^k \pi_0, \quad k = 0,1,\ldots$$

$$\sum_{k=0}^{\infty} \pi_k = 1$$

Riešenie

$$\sum_{k=0}^{\infty} \rho^k \pi_0 = 1 \Rightarrow \quad \pi_0 = \left( \frac{1}{1-\rho} \right)^{-1}, \quad \rho < 1$$

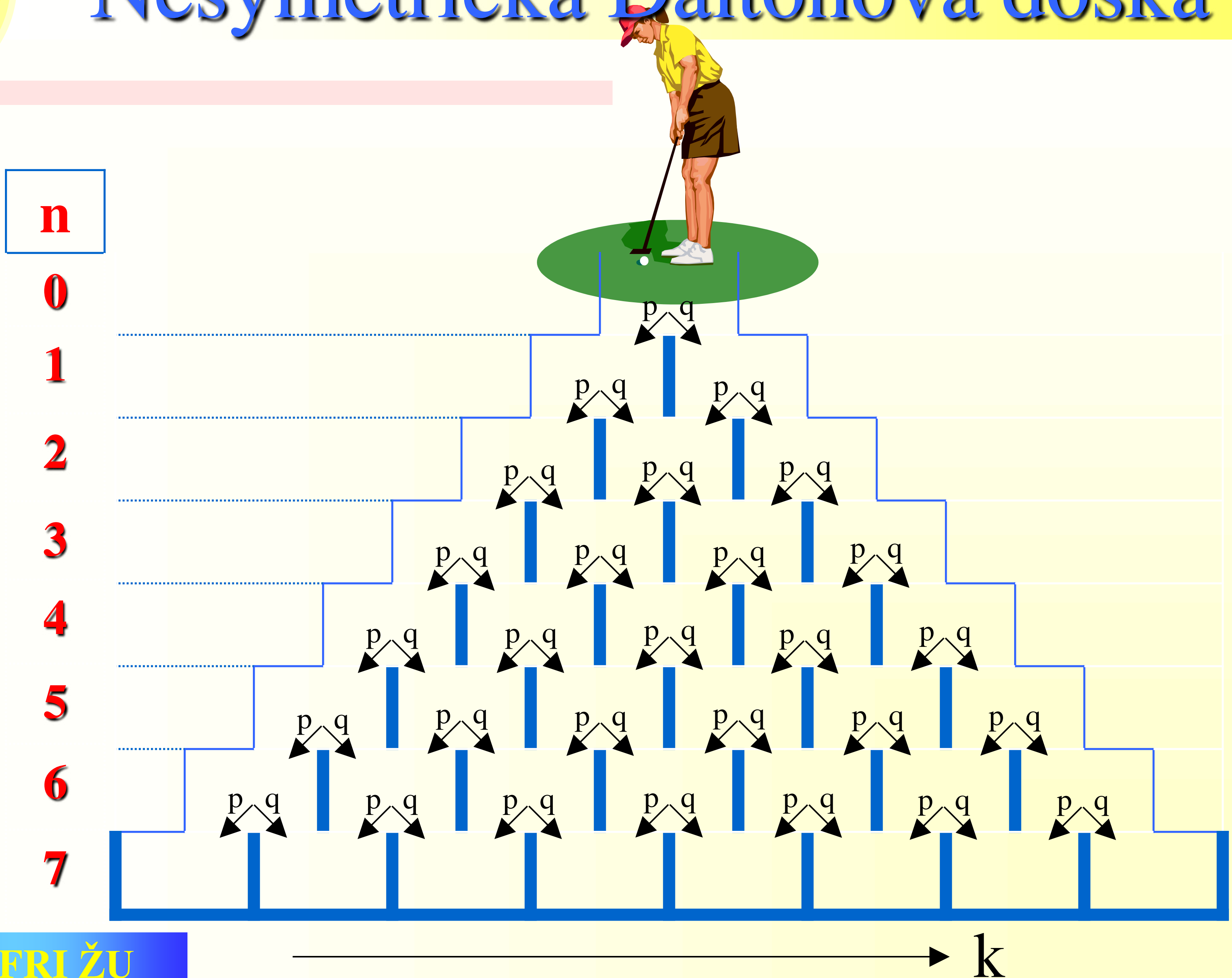
$$\boxed{\pi_k = \rho^k (1 - \rho) \quad , k = 0,1,\ldots}$$

# Invariantné rozdelenie

Postup:

1. **určenie stavov**
2. určenie rezov
3. napísanie rovníc o zachovaní toku
4. vyriešenie rovníc

# Nesymetrická Daltonova doska



# Binomické rozdelenie

$$P\{x(n)=k\}=\binom{n}{k}p^k q^{n-k}, \qquad k=0,1,\ldots,n$$

| n | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | $1p$ | $1q$ | | | | | |
| 2 | $1p^2$ | $2pq$ | $1q^2$ | | | | |
| 3 | $1p^3$ | $3p^2q$ | $3pq^2$ | $1q^3$ | | | |
| 4 | $1p^4$ | $4p^3q$ | $6p^2q^2$ | $4pq^3$ | $1q^4$ | | |
| 5 | $1p^5$ | $5p^4q$ | $10p^3q^2$ | $10p^2q^3$ | $5pq^4$ | $1q^5$ | |
| 6 | $1p^6$ | $6p^5q$ | $15p^4q^2$ | $20p^3q^3$ | $15p^2q^4$ | $6pq^5$ | $1q^6$ |

$\longrightarrow k$

# Stredná hodnota

$$P\{x(n)=k\}=\binom{n}{k}p^k q^{n-k}, \qquad k=0,1,\dots,n$$

$$E\{X(n)\}=\overline{m}=\sum_{k=0}^{n}kP\{x(n)=k\}=\sum_{k=0}^{n}k\binom{n}{k}p^k q^{n-k}$$

$$\overline{m}=\sum_{k=0}^{n}k\frac{n(n-1)\dots(n-k+1)}{k(k-1)\dots1}p^k q^{n-k}=$$

$$=\sum_{\boxed{k=1}}^{n}k\frac{n(n-1)\dots(n-k+1)}{k(k-1)\dots1}p^k q^{n-k}$$

# Stredná hodnota

$$\overline{m} = \sum_{k=1}^{n} \frac{n(n-1)\ldots(n-k+1)}{(k-1)\ldots 2} p^k q^{n-k} =$$

$r = k - 1$

$$= np \underbrace{\sum_{r=0}^{n-1} \frac{(n-1)\ldots(n-1-r+1)}{r(r-1)\ldots 1} p^r q^{n-1-r}}_{1}$$

$$\boxed{\overline{m} = np}$$

# Veľká Daltonova doska

$$P\{x(n)=k\}=\binom{n}{k}p^k q^{n-k}, \qquad k=0,1,\dots,n$$

$$\overline{m}=np \Rightarrow p=\frac{\overline{m}}{n}$$

$$P\{x(n)=k\}=\binom{n}{k}\left(\frac{\overline{m}}{n}\right)^k\left(1-\frac{\overline{m}}{n}\right)^{n-k}$$

$$P\{x(n)=k\}=\frac{\prod_{j=0}^{k-1}(n-j)}{k!}\left(\frac{\overline{m}}{n}\right)^k\left(1-\frac{\overline{m}}{n}\right)^{n-k}$$

# Veľká Daltonova doska

$$P\{x(n)=k\}=\frac{\prod_{j=0}^{k-1}(n-j)}{k!}\left(\frac{\overline{m}}{n}\right)^{k}\left(1-\frac{\overline{m}}{n}\right)^{n-k}$$

$$P\{x(n)=k\}=\frac{\overline{m}^{k}}{k!}\left(1-\frac{\overline{m}}{n}\right)^{n-k}\prod_{j=0}^{k-1}\left(1-\frac{j}{n}\right)$$

# Veľká Daltonova doska

$$n \to \infty$$

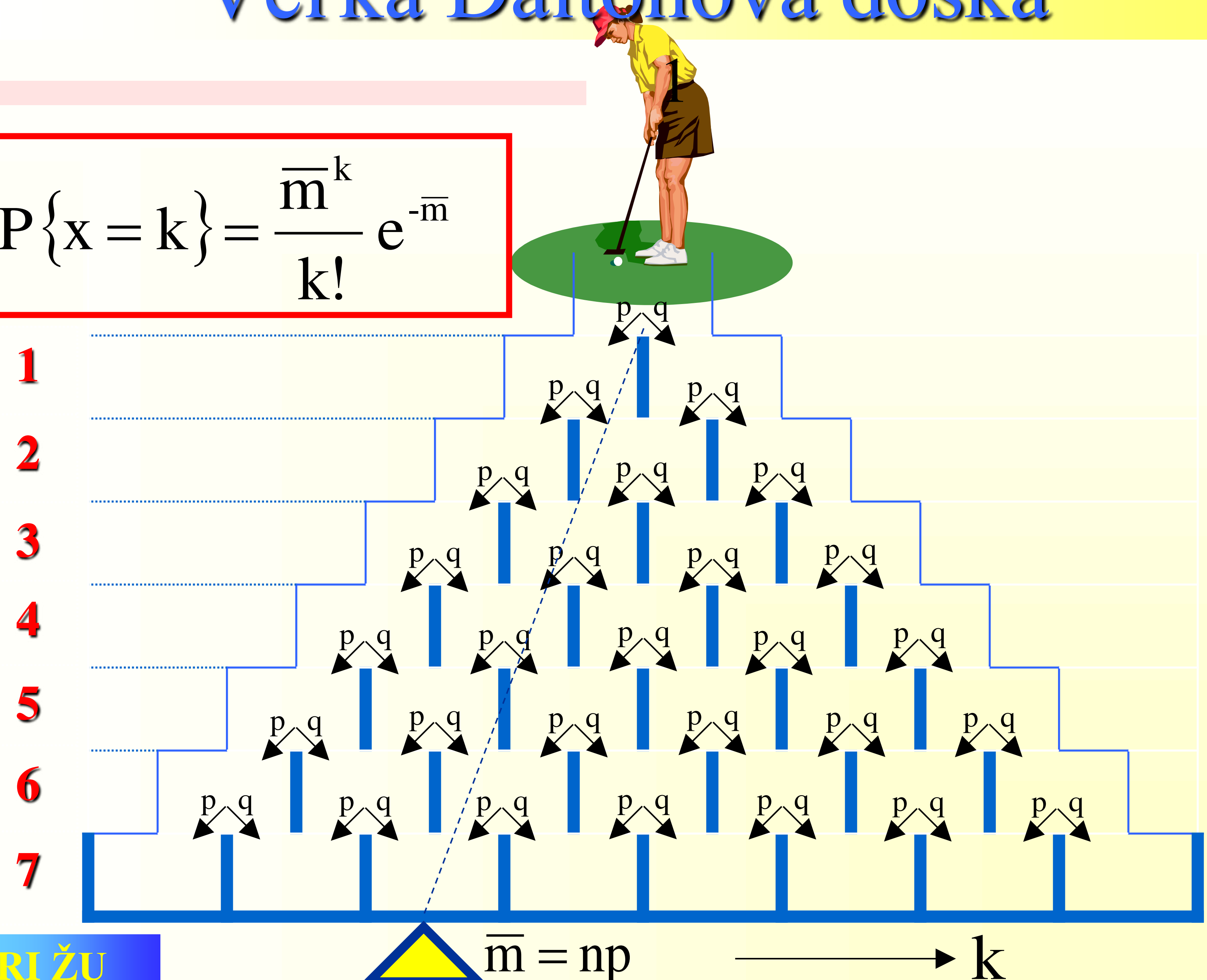
$$P\{x=k\} = \lim_{n\to\infty} \frac{\overline{m}^k}{k!}\left(1-\frac{\overline{m}}{n}\right)^{n-k} \prod_{j=0}^{k-1}\left(1-\frac{j}{n}\right)$$

$$P\{x=k\} = \frac{\overline{m}^k}{k!} \underbrace{\lim_{n\to\infty}\left(1-\frac{\overline{m}}{n}\right)^{n-k}}_{e^{-\overline{m}}} \underbrace{\lim_{n\to\infty}\prod_{j=0}^{k-1}\left(1-\frac{j}{n}\right)}_{1}$$

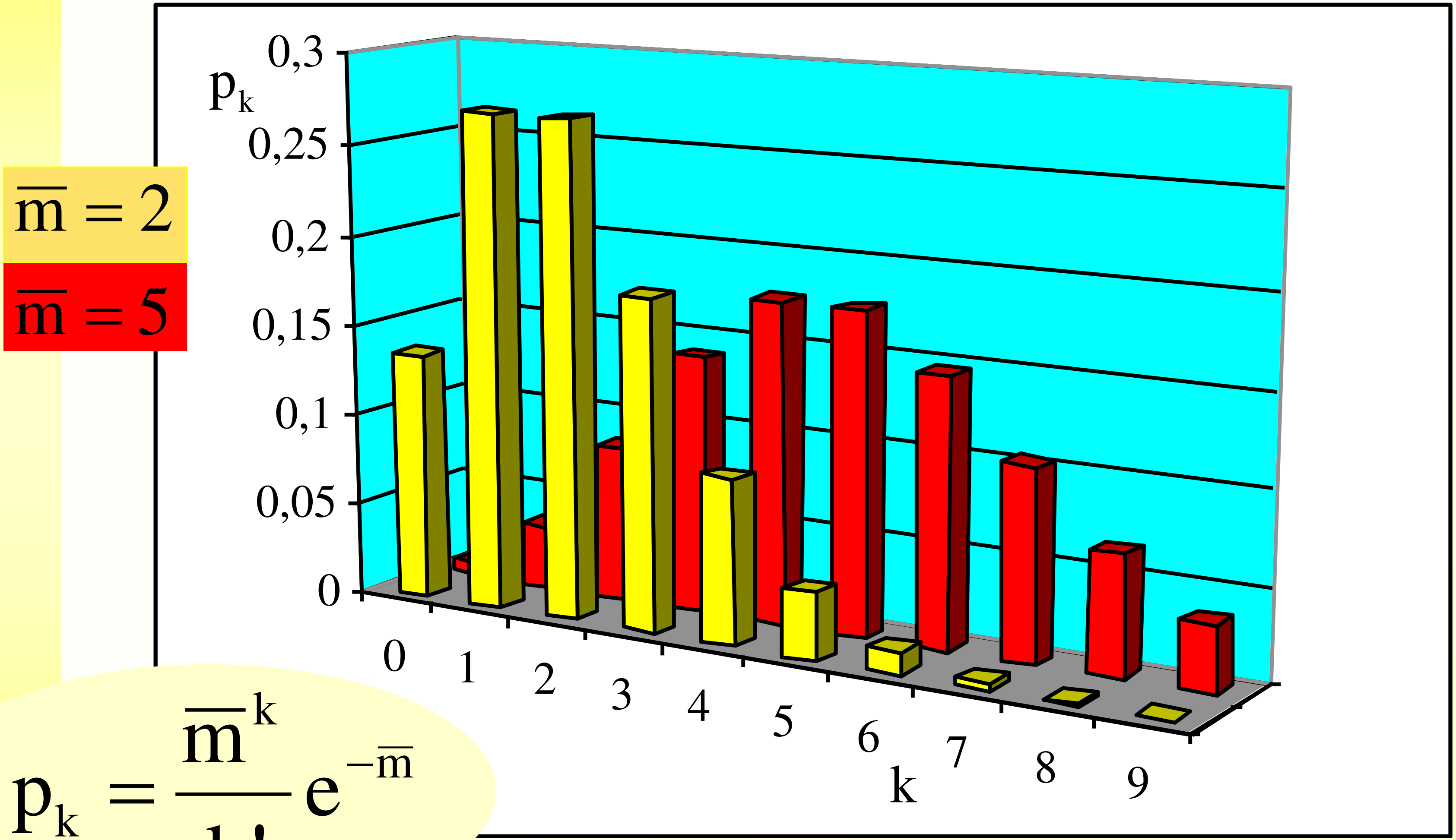
$$\boxed{P\{x=k\} = \frac{\overline{m}^k}{k!} e^{-\overline{m}}}$$

# Veľká Daltonova doska

$$P\{x = k\} = \frac{\overline{m}^{k}}{k!}\, e^{-\overline{m}}$$



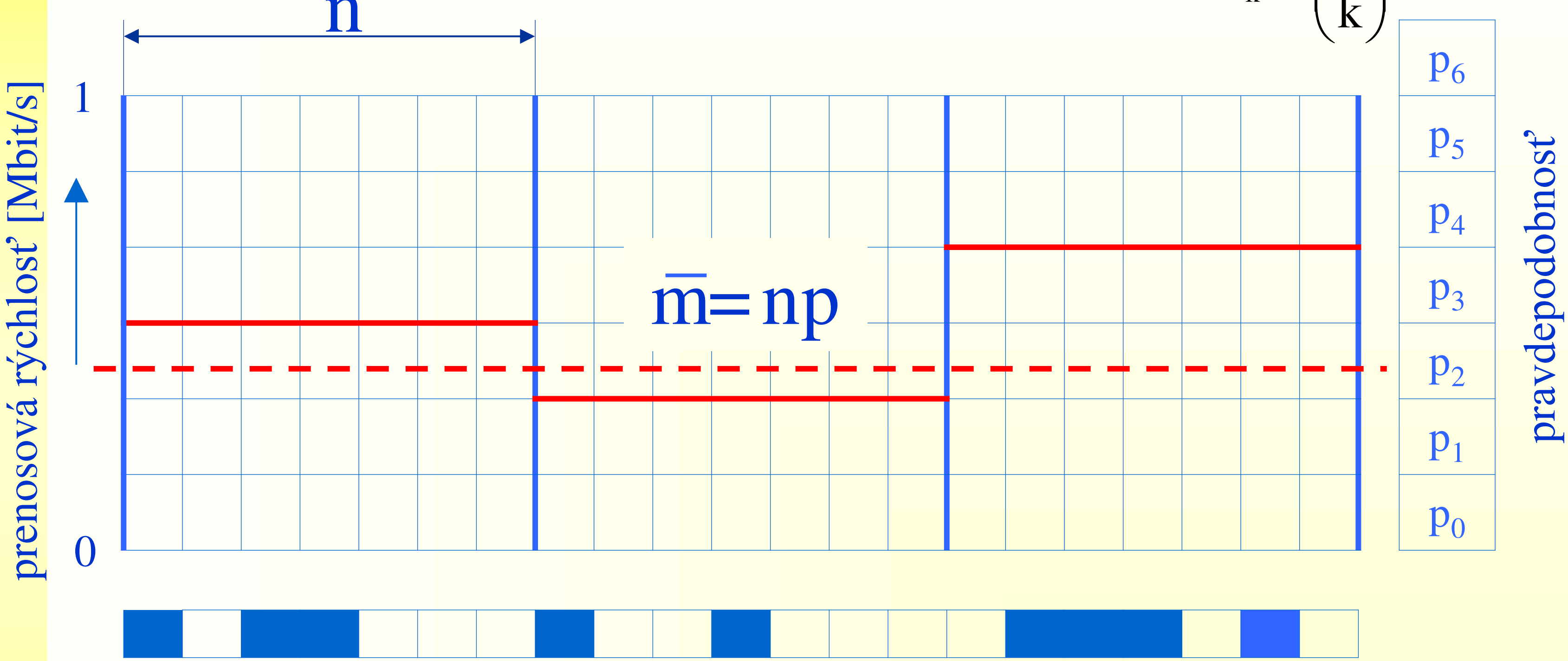
# Poissonovo rozdelenie




$$p_k = \frac{\overline{m}^k}{k!} e^{-\overline{m}}$$

# Rozdelenie prevádzky

$$p_k = \binom{n}{k} p^k q^{n-k}$$



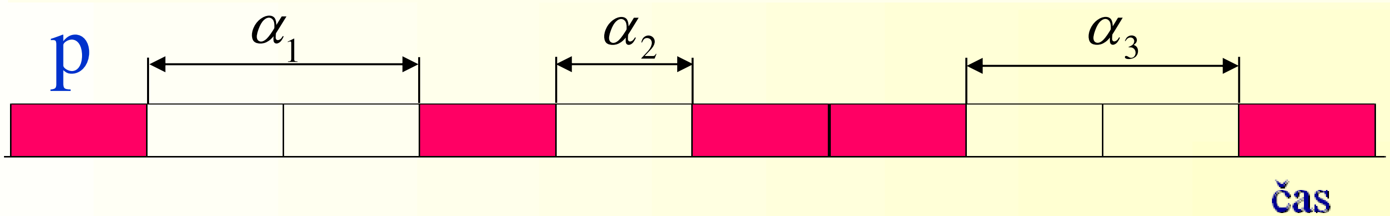

$$\bar{m} = \lambda t$$

$$\boxed{P\{x = k\} = \frac{\bar{m}^k}{k!} e^{-\bar{m}}}$$

# Proces nie Bernoulliho

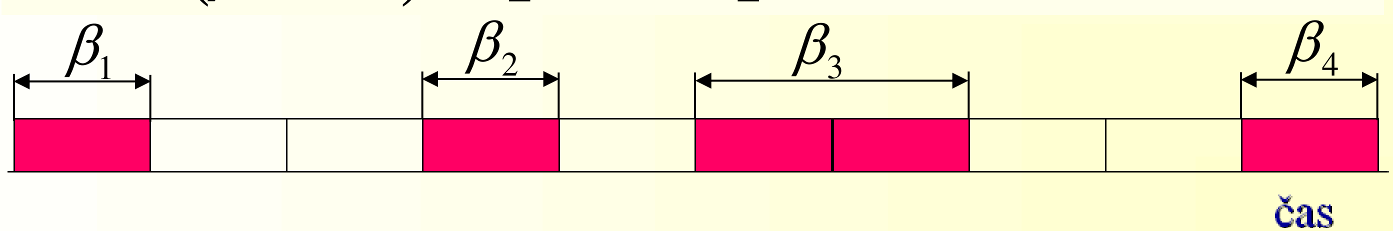
## Rozdelenie dĺžok intervalov medzi rámcami

$$P(\alpha = n) = (1-p)^n p, \qquad n = 0,1,2,\ldots$$




## Rozdelenie dĺžok zhlukov rámcov

$$P(\beta = n) = p^n (1-p), \qquad n = 0,1,2,\ldots$$



# Zovšeobecnenie

## Proces so stavmi $\{S_1,\dots,S_n\}$

počiatočné rozdelenie pravdepodobnosti

$$\mathbf{p}_0 = (p_0(1),\dots, p_0(n))$$

matica pravdepodobností prechodov

$$\mathbf{P} = \begin{pmatrix} p_{1,1} & \cdots & p_{1,n} \\ \cdots & \cdots & \cdots \\ p_{n,1} & \cdots & p_{n,n} \end{pmatrix}$$

# Invariantné rozdelenie

Invariantné rozdelenie pravdepodobnosti

$$\boldsymbol{\pi} = (\pi(1),\ldots,\pi(n))$$

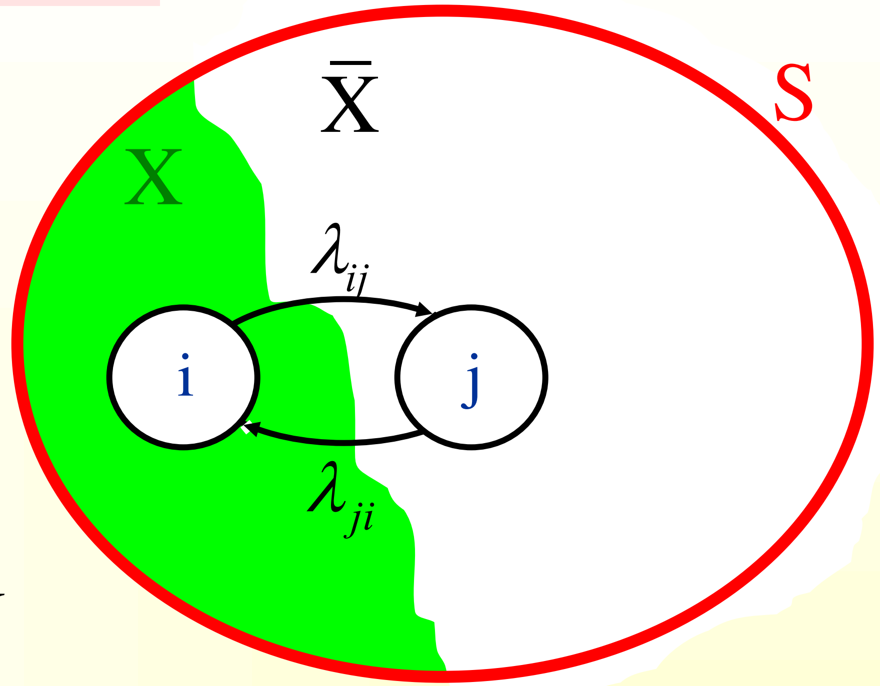
procesu so stavmi $\{S_1,\ldots,S_n\}$ a maticou pravdepodob-nosti prechodov

$$\mathbf{P} = \begin{pmatrix} p_{1,1} & \ldots & p_{1,n} \\ \ldots & \ldots & \ldots \\ p_{n,1} & \ldots & p_{n,n} \end{pmatrix}$$

nájdeme riešením sústavy lineárnych algebraických rovníc

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}\,, \quad \sum_{i=1}^{n} \pi_i = 1$$

# Veta o zachovaní toku




$$\Phi_{X\bar{X}} = \Phi_{\bar{X}X}$$

$$\sum_{i \in X} \sum_{j \in \bar{X}} \pi_i \, p_{ij} = \sum_{j \in \bar{X}} \sum_{i \in X} \pi_j \, p_{ji}$$

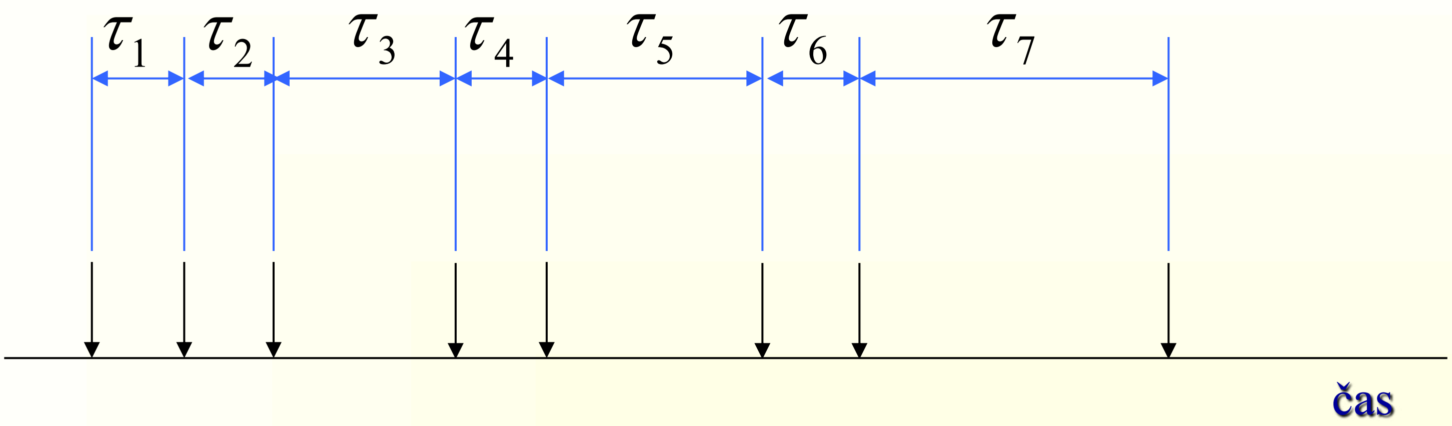
Formálny dôkaz za domácu úlohu

# Invariantné rozdelenie

Postup:

1. **určenie stavov**
2. určenie rezov
3. napísanie rovníc o zachovaní toku
4. vyriešenie rovníc

# Popis procesu v čase



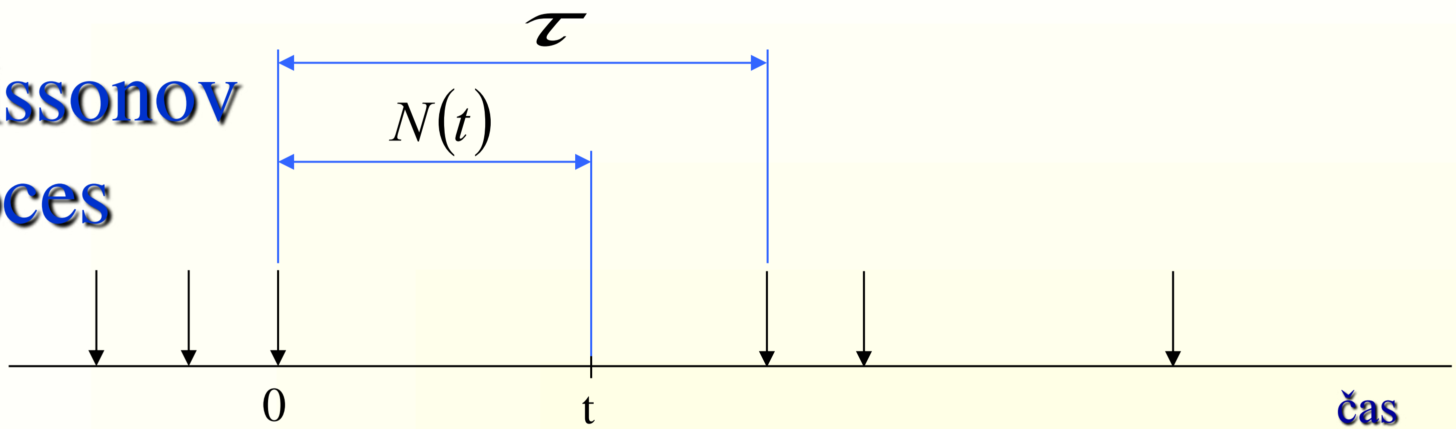

Distribučná funkcia

$$F_k(t) = P\{\tau_k < t\} = F(t),\ \forall k$$

Proces je homogénny

# Interval medzi príchodmi

**Poissonov proces**

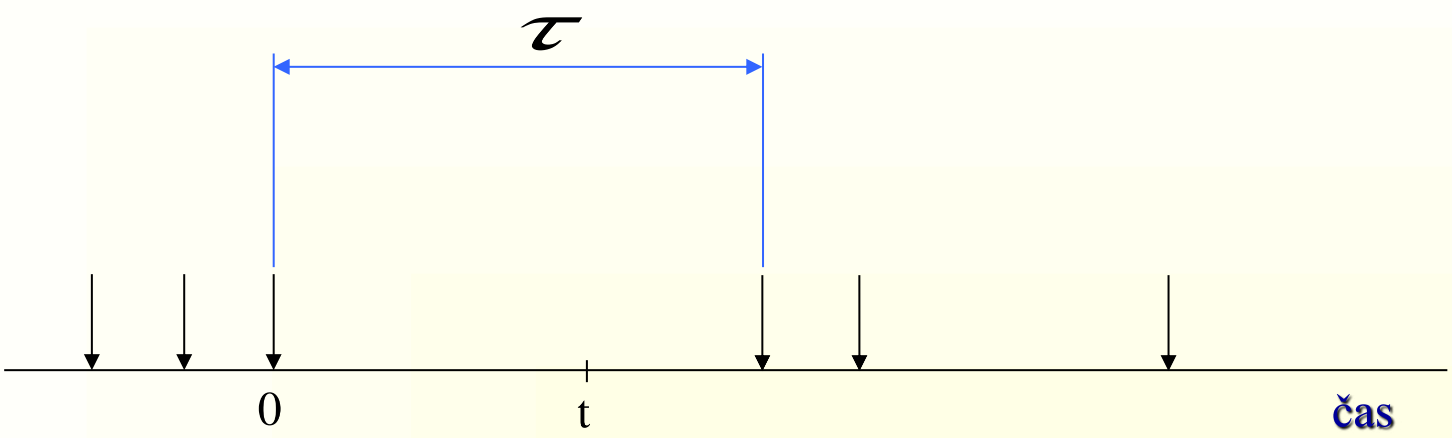



Distribučná funkcia

$$F(t) = P\{\tau < t\} = 1 - \boxed{P\{\tau \geq t\} = ?}$$

$$= 1 - P\{N(t) = 0\} = 1 - \frac{(\lambda t)^0}{0!} e^{-\lambda t}$$

# Interval medzi príchodmi




Distribučná funkcia

$$F(t) = P\{\tau < t\} = 1 - e^{-\lambda t}$$

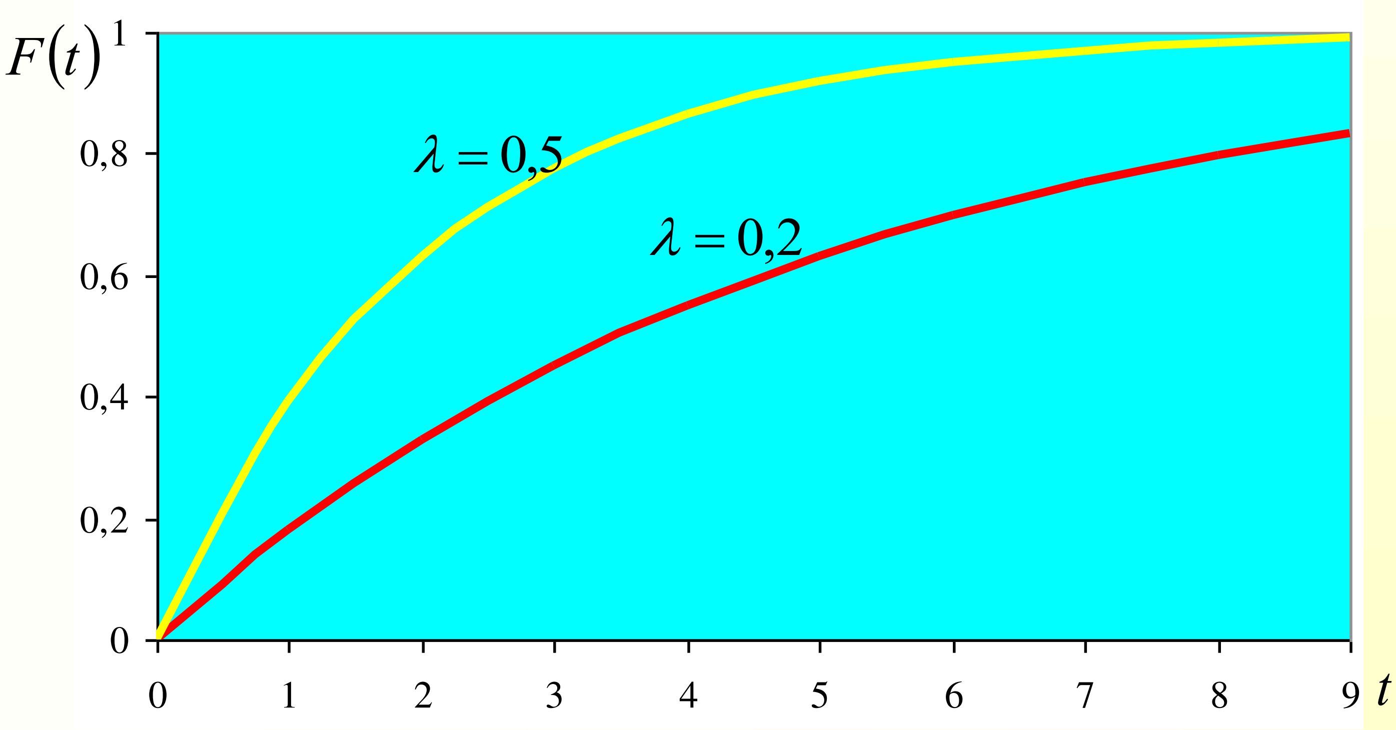
Hustota rozdelenia pravdepodobnosti

$$f(t) = F'(t) = \lambda e^{-\lambda t}$$

# Interval medzi príchodmi

## Distribučná funkcia

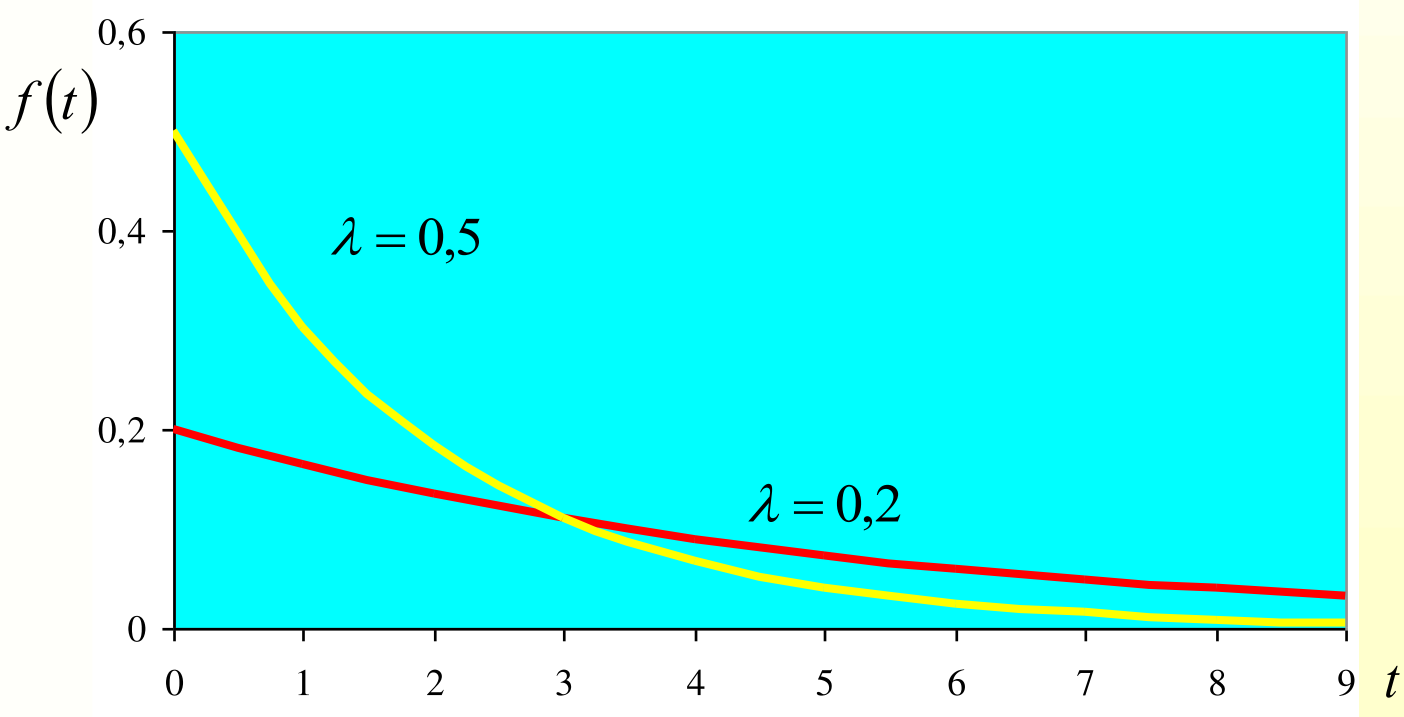
$$F(t) = 1 - e^{-\lambda t},\ t \geq 0, \lambda > 0$$



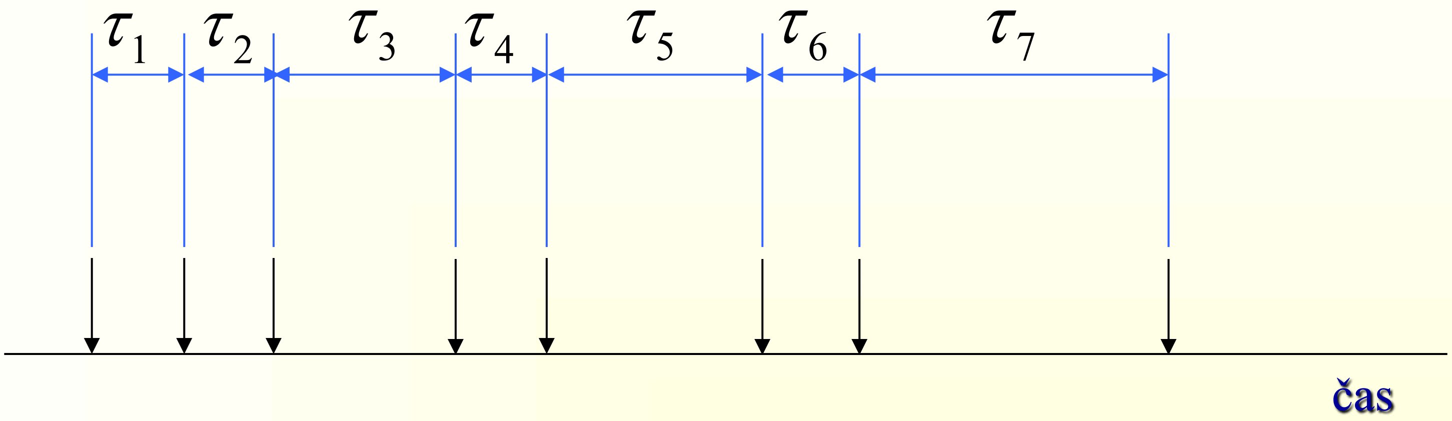
# Interval medzi príchodmi

## Hustota rozdelenia pravdepodobnosti

$$f(t) = F'(t) = \lambda e^{-\lambda t}$$

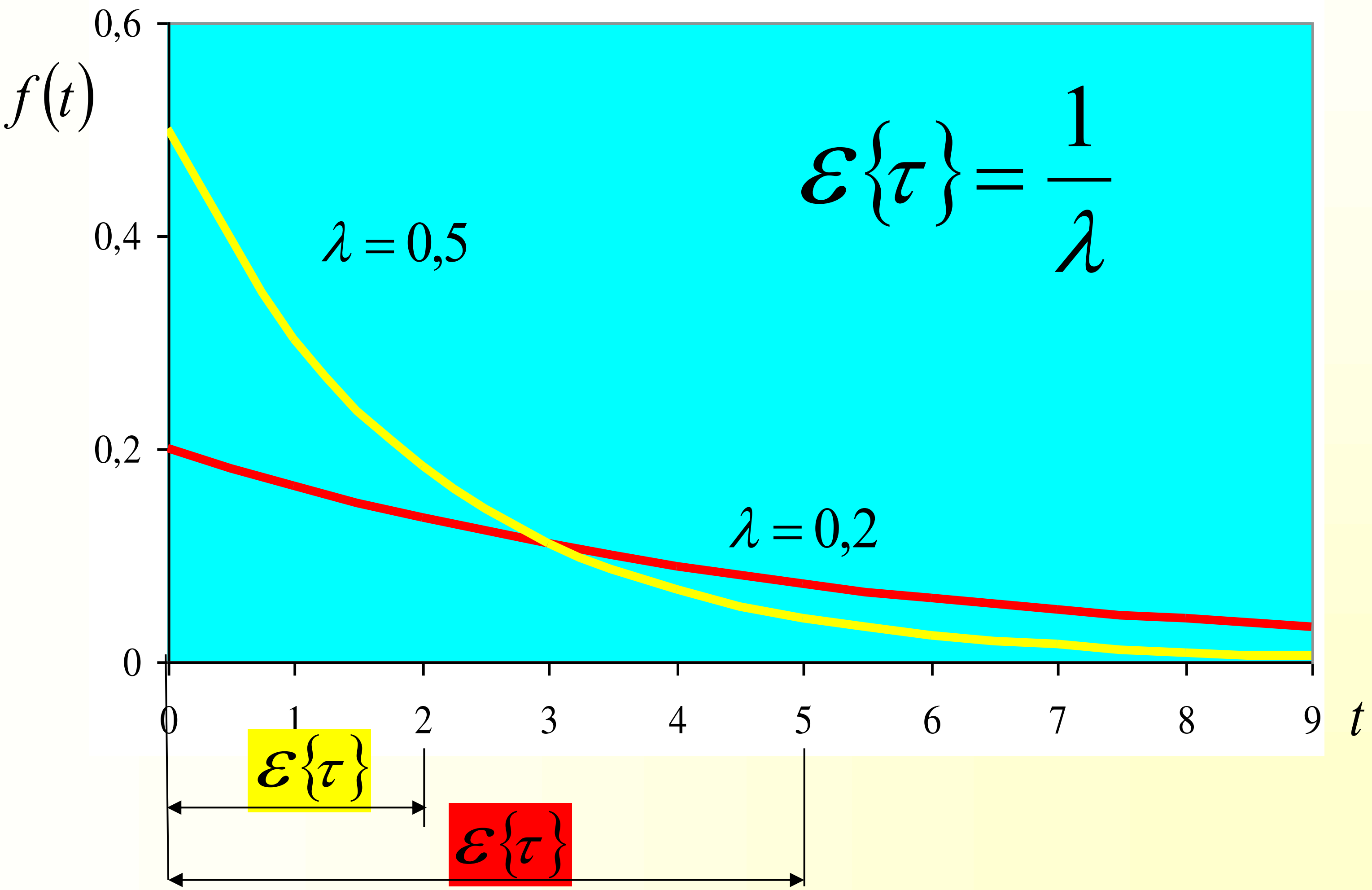


# Stredná hodnota intervalu

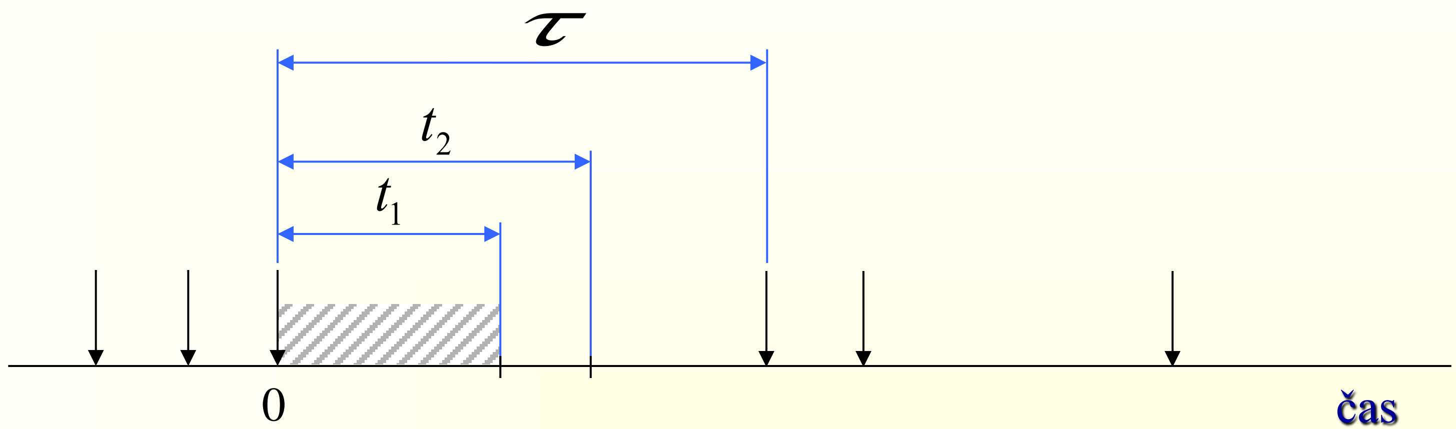



$$\varepsilon\{\tau\} = \int_0^\infty t f(t)\,dt = \int_0^\infty t\lambda e^{-\lambda t}\,dt = \frac{1}{\lambda}$$

# Interval medzi príchodmi



# Neexistencia pamäte




$$F(t_2 / \tau > t_1) = P\{\tau < t_2 / \tau > t_1\} =$$

$$= 1 - P\{\tau \geq t_2 / \tau > t_1\}$$

$$P\{\tau \geq t_2 / \tau > t_1\} = \frac{P\{(\tau \geq t_1) \wedge (\tau \geq t_2)\}}{P\{\tau > t_1\}} =$$

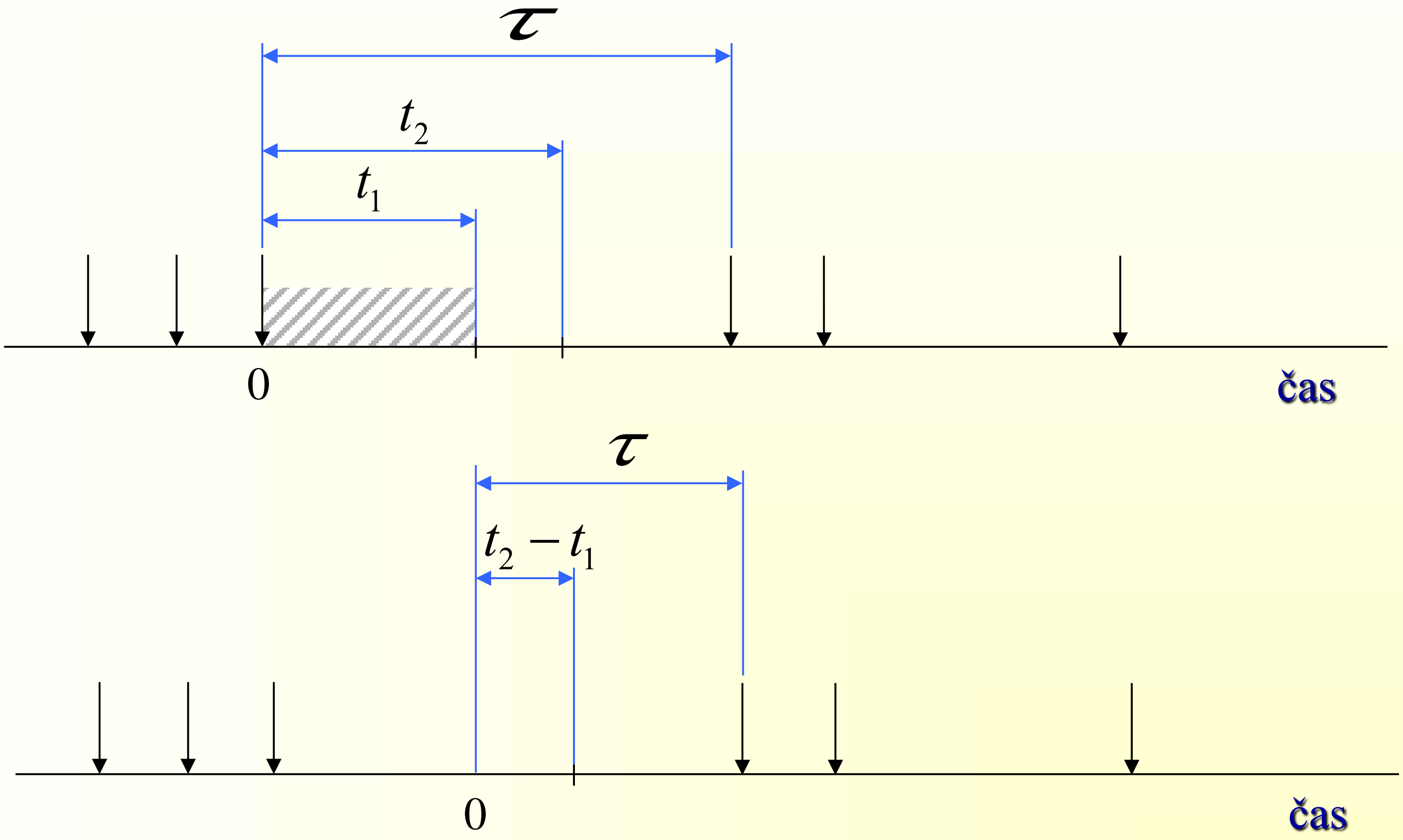
# Neexistencia pamäte

$$P\{\tau \geq t_2 / \tau > t_1\} = \frac{P\{\tau \geq t_2\}}{P\{\tau > t_1\}} = \frac{e^{-\lambda t_2}}{e^{-\lambda t_1}} = e^{-\lambda(t_2 - t_1)}$$

$$F(t_2 / \tau > t_1) = 1 - e^{-\lambda(t_2 - t_1)}$$

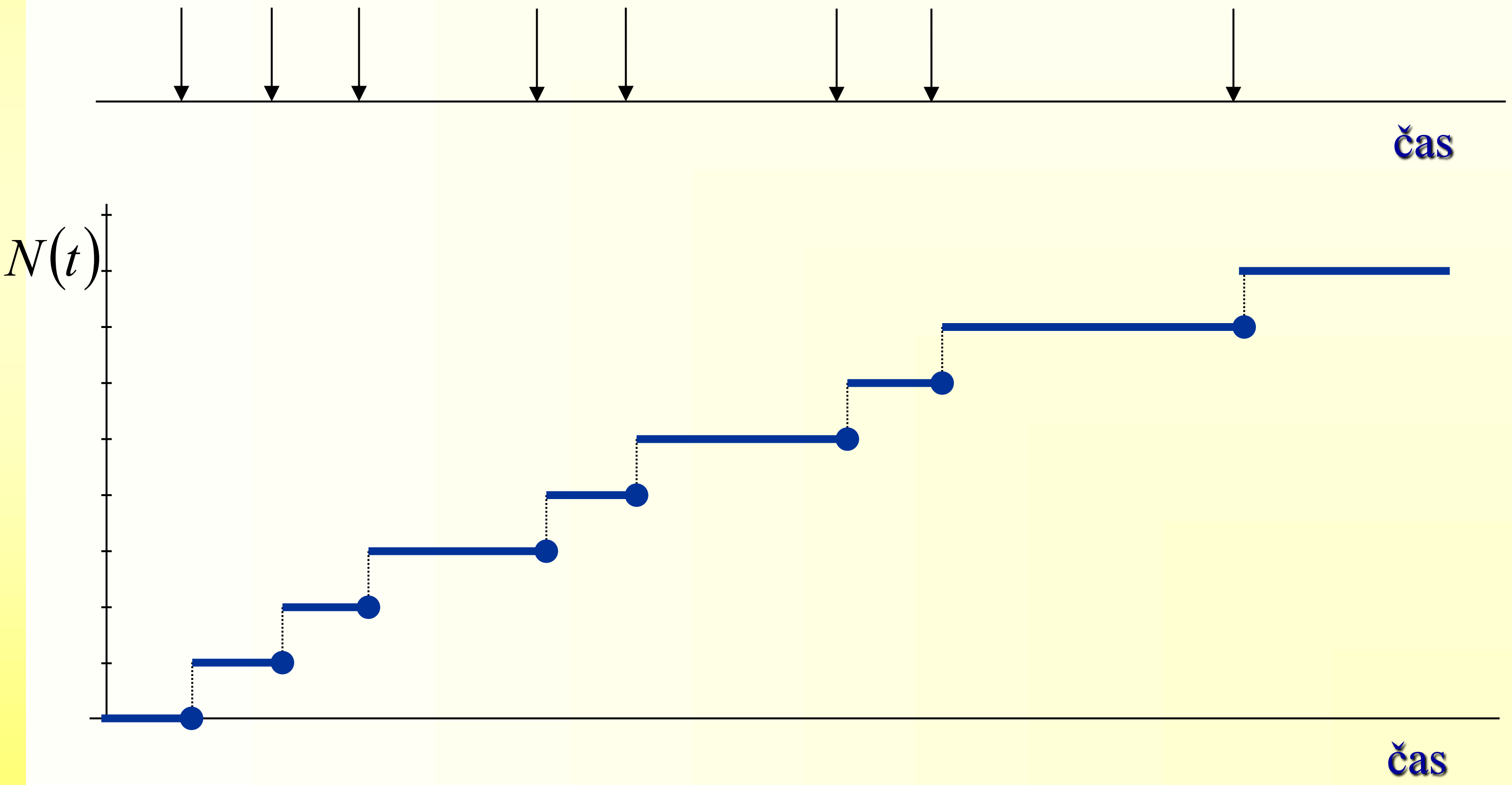
$$F(t_2 / \tau > t_1) = F(t_2 - t_1)$$

# Neexistencia pamäte



# Stav procesu

## Poissonov proces

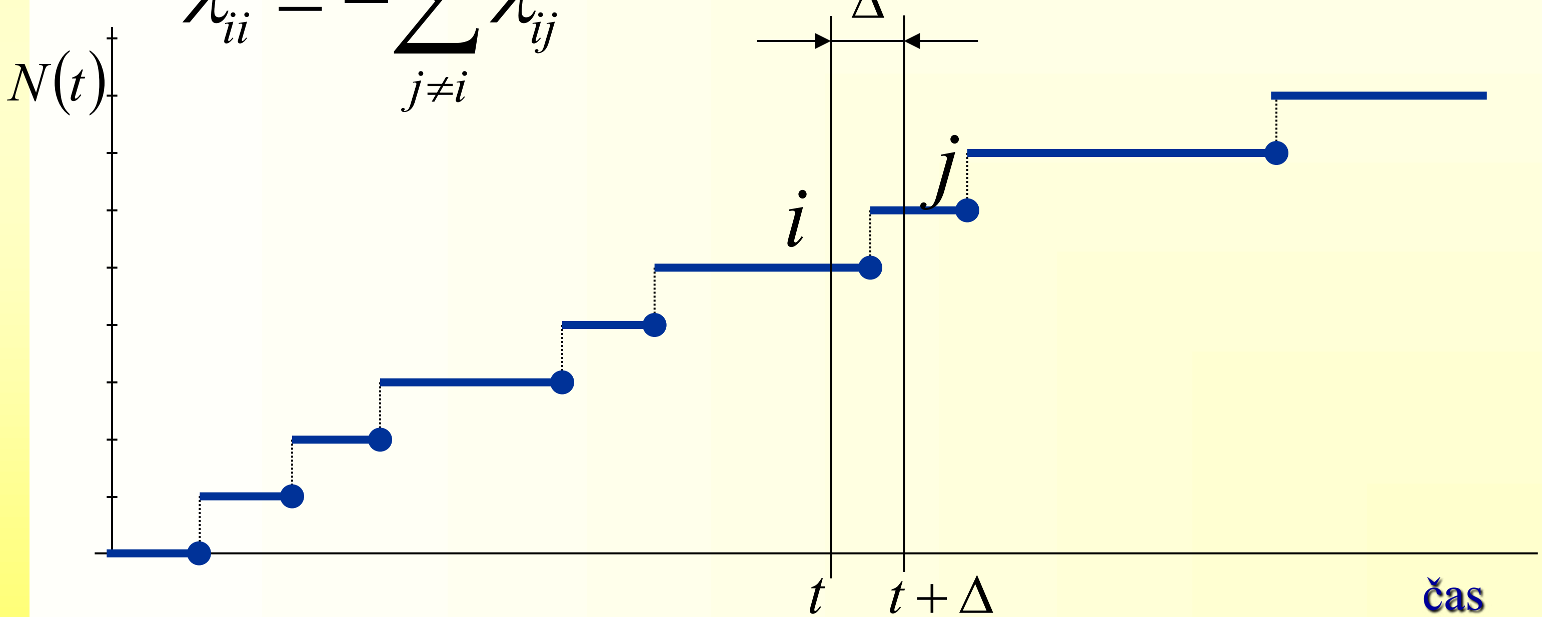


# Intenzity prechodov

$$i = 0,1,\dots,$$

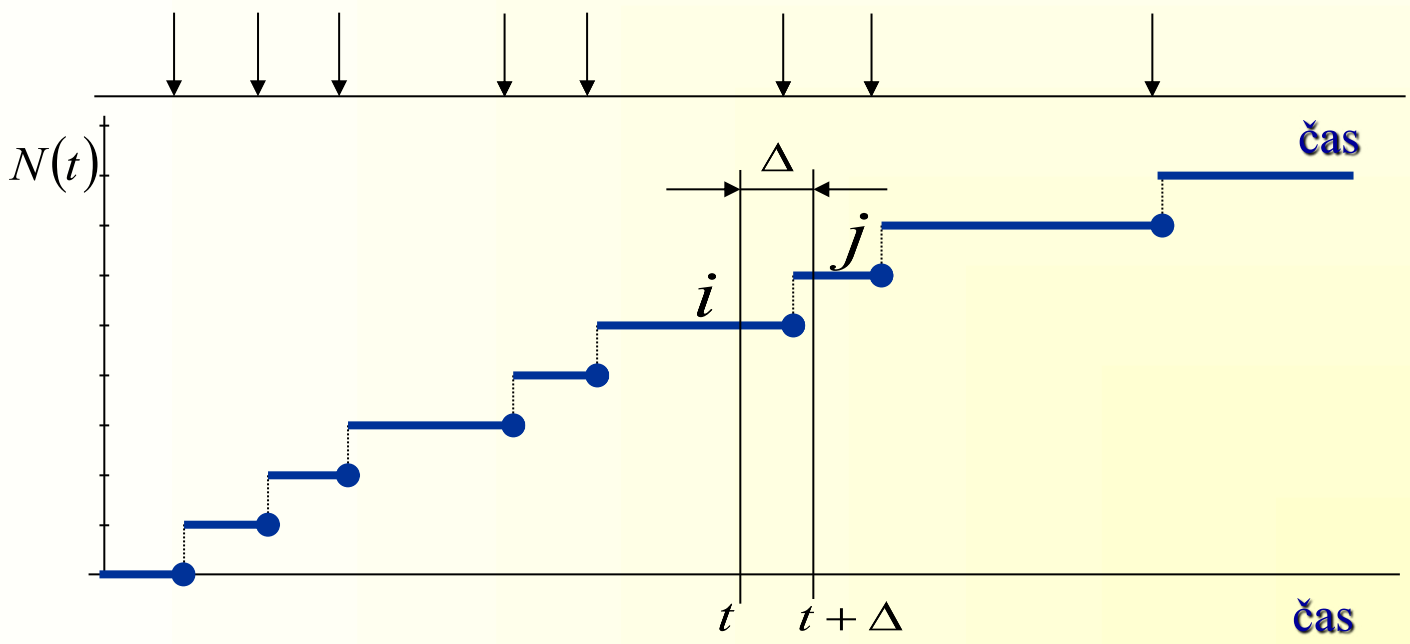
$$\lambda_{ij} = \lim_{\Delta \to 0^+} \frac{1}{\Delta} P\{N(t+\Delta) = j \,/\, N(t) = i\}, \quad j \neq i$$

$$\lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}$$



# Poissonov proces

$$\lambda_{ij} = \begin{cases} \lambda, & j = i+1 \\ -\lambda, & j = i, \qquad\qquad i = 0,1,\ldots, \\ 0, & j - ostatné \end{cases}$$



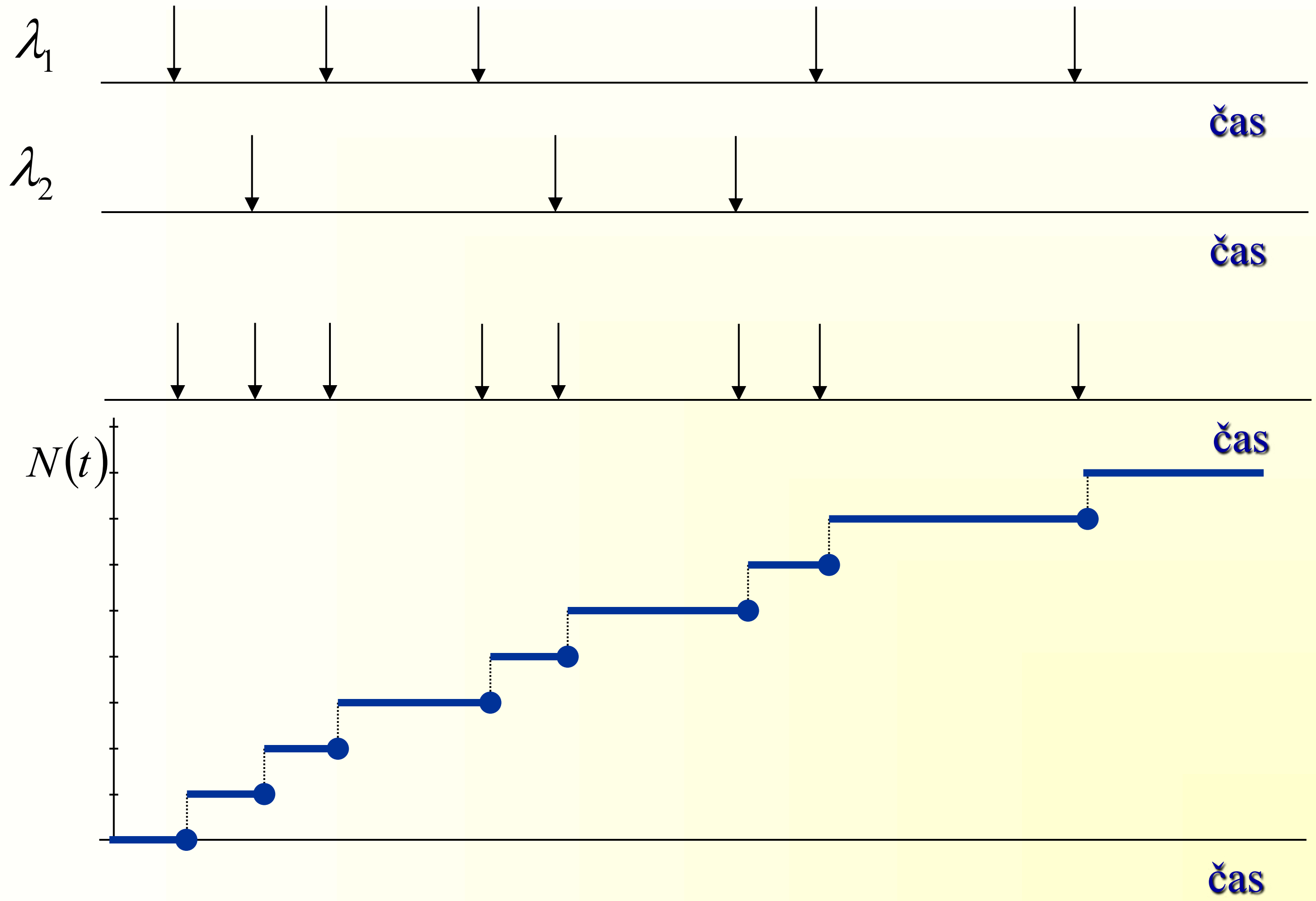
# Matica intenzít

$$\mathbf{\Lambda} = \left(\lambda_{ij}\right) = \begin{pmatrix} \lambda_{00} & \lambda_{01} & \dots \\ \lambda_{10} & \lambda_{11} & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

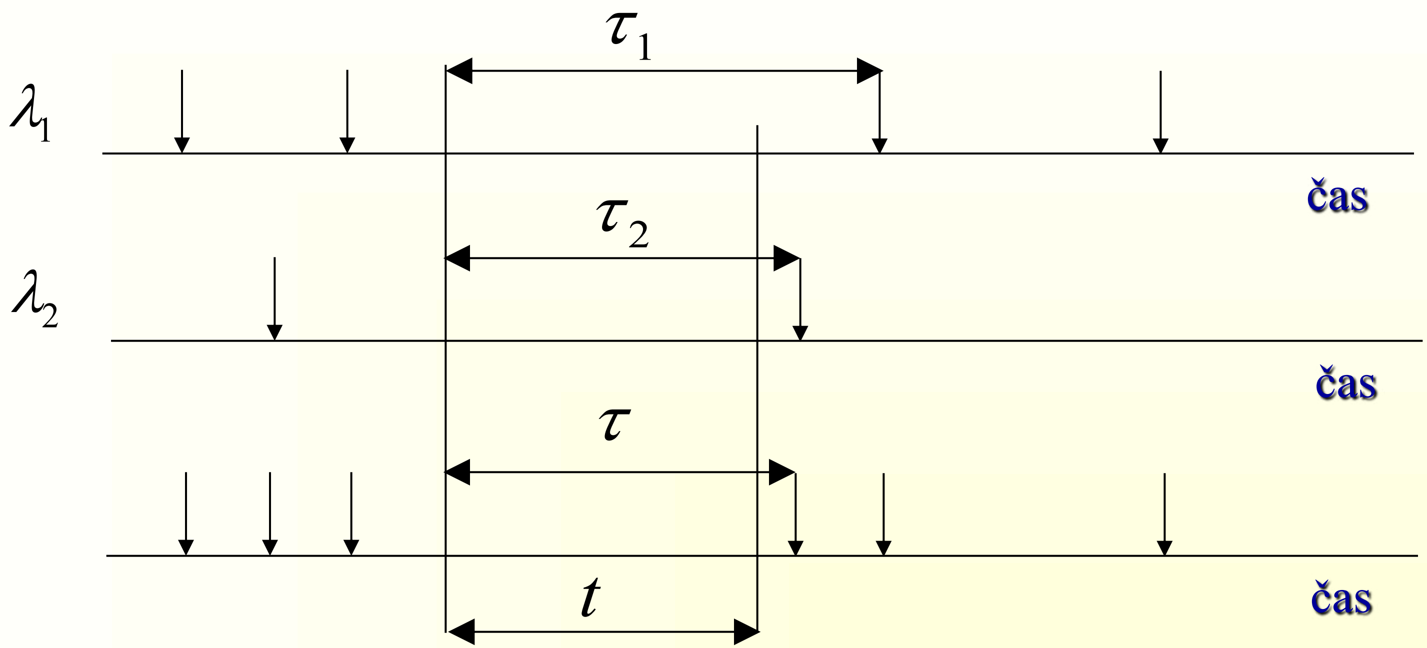
Poissonov proces s parametrom $\lambda$

$$\mathbf{\Lambda} = \left(\lambda_{ij}\right) = \begin{pmatrix} -\lambda & \lambda & 0 & \dots \\ 0 & -\lambda & \lambda & \dots \\ 0 & 0 & -\lambda & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

# Súčet Poissonových procesov



# Súčet Poissonových procesov

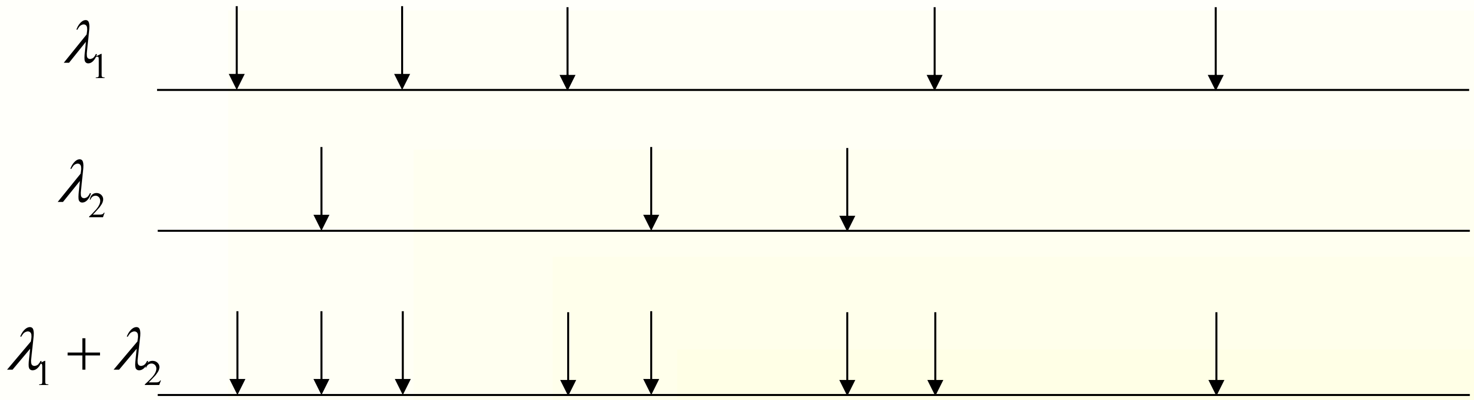



$$1-F(t)=P\{\tau>t\}=$$

$$=P\{(\tau_1>t)\wedge(\tau_2>t)\}=P\{\tau_1>t\}P\{\tau_2>t\}=$$

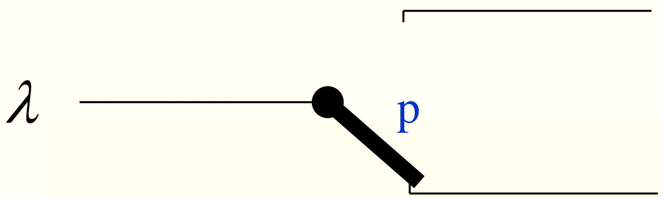
$$=e^{-\lambda_1 t}e^{-\lambda_2 t}=e^{-(\lambda_1+\lambda_2)t}$$

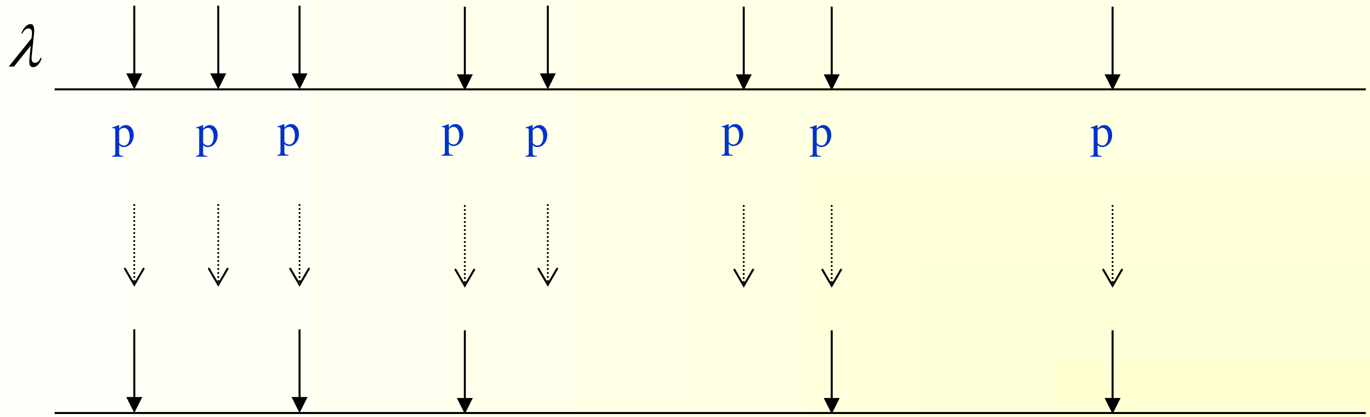
# Súčet Poissonových procesov



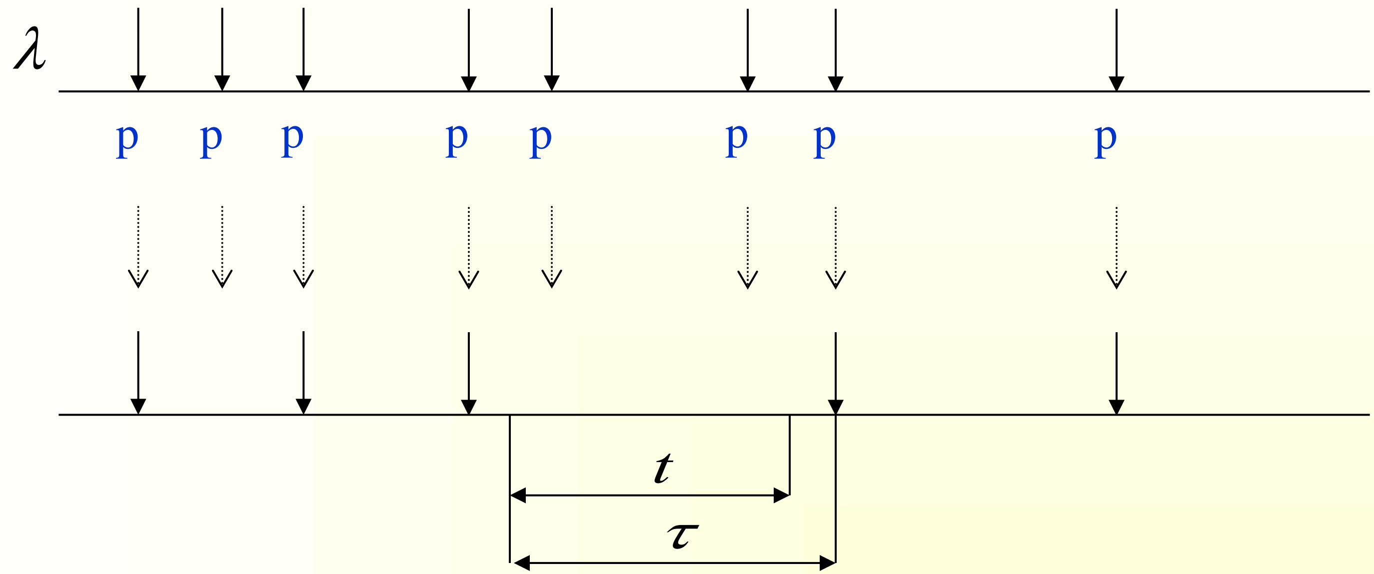

Súčet Poissonových procesov s parametrami $\lambda_1$ a $\lambda_2$ je Poissonovým procesom s parametrom $\lambda = \lambda_1 + \lambda_2$

# Náhodné smerovanie



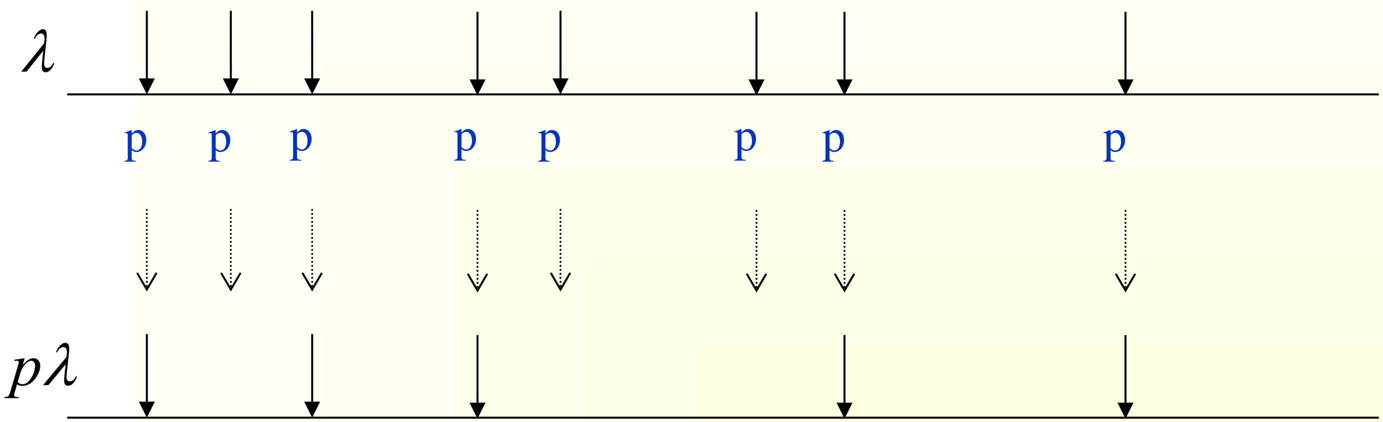
# Náhodné smerovanie




$$1 - F(t) = P\{\tau > t\} = \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} (1-p)^k =$$

$$= e^{-p\lambda t} \sum_{k=0}^{\infty} \frac{[\lambda t (1-p)]^k}{k!} e^{-[\lambda t(1-p)]} = \quad e^{-p\lambda t}$$

# Náhodné smerovanie




Náhodný výber udalostí s pravdepodobnosťou $p$ z Poissonovho procesu s parametrom $\lambda$ je Poissonovým procesom s parametrom $p\lambda$

# Ďakujem za pozornosť