Group Actions, the Orbit-Stabilizer Theorem, and the Rotational Symmetries of a Cube

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Group Actions

A **group action** of a group G on a set X is a function $f : G \times X \to X$ satisfying both of the following properties:

1. $f(e_G, x) = x$ for all $x \in X$
2. $f(gh, x) = f(g, f(h, x))$ for all $g, h \in G$ and $x \in X$.

Associated Terms

- A **fixed point** of an element $g \in G$ is an element $x \in X$ such that $f(g, x) = x$.
- The **stabilizer** G_x of an element $x \in X$ is the set of all elements $g \in G$ such that x is a fixed point of g.
- The **orbit** O_x of an element $x \in X$ is the set of elements $Y = \{y \in X : f(g, x) = y$ for some $g \in G\}$.

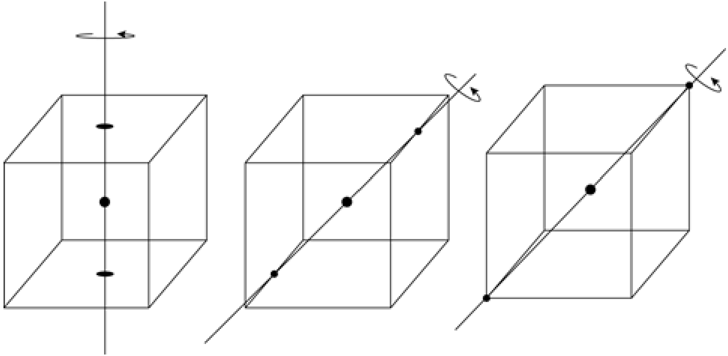
The Orbit-Stabilizer Theorem

Theorem. Let G be a group acting on a set X. Let G_x be the stabilizer of an element $x \in X$. Suppose that the orbit O_x of $x \in X$ is finite. Then the index $|G : G_x|$ is finite and equal to $|O_x|$. If G is also finite, then

$$|G_x| \cdot |O_x| = |G|.$$

Application

Example. *How many rotational symmetries does a cube have?*



Application

Example. *How many rotational symmetries does a cube have?*

- The group of rotations G acts on the set of faces F of the cube.
- Every face of the cube is in the orbit of a particular face $f \in F$, so $|O_f| = |F| = 6$.
- For any face $f \in F$, there are four rotations in G that will preserve f, so $|G_f| = 4$.
- By the Orbit-stabilizer Theorem, $|G| = |G_f| \cdot |O_f| = 4 \cdot 6 = 24$.
- Thus, a cube has **24** rotational symmetries.