Chapter 3 Ex 3.12 —Lorenz Model

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1 Introduction

Edward Lorenz was interested in the predictability of the solutions to hydrodynamics equations. He was a meteorologist studying weather forecastingand the question of the fundamental limitations to this endeavor. The model he introduced can be thought of as a gross simplification of one feature of the atmosphere, namely the fluid motion driven by thermal buoyancy known as convection, although his original paper seems to use the model simply as a set of equations whose solutions afford the simplest example of a deterministic non-periodic flow of which the writer is aware.

The model is a system of three ordinary differential equations now known as the *Lorenz equations*:

$$\begin{aligned}\frac{dx}{dt} &= \sigma(y-x)\\ \frac{dy}{dt} &= -xz+rx-y\\ \frac{dz}{dt} &= xy-bz\end{aligned} \tag{1}$$

Where the parameter σ depends on the properties of the fluid (in fact the ratio of the viscous to thermal diffusivities): Lorenz took the value to be 10 in his paper, so is here. The number b = 8/3: this would be different for a different choice of horizontal wavelength or roll diameter.

2 Method

The finite difference scheme for *Lorenz equation* is

$$\begin{aligned}x_{i+1} &= x_i + \sigma(y_i - x_i)\Delta t\\ y_{i+1} &= y_i + (-x_i z_i + r x_i - y_i)\Delta t\\ z_{i+1} &= z_i + (x_i y_i - b z_i)\Delta t\end{aligned} \tag{2}$$

These equations can be directly solved by *Euler* - *Cromer* method.

3 Data & Verification

With initial condition $x = 1, y = z = 0$ and $r = 25, b = 8/3, \sigma = 10$, we get the phase diagram:

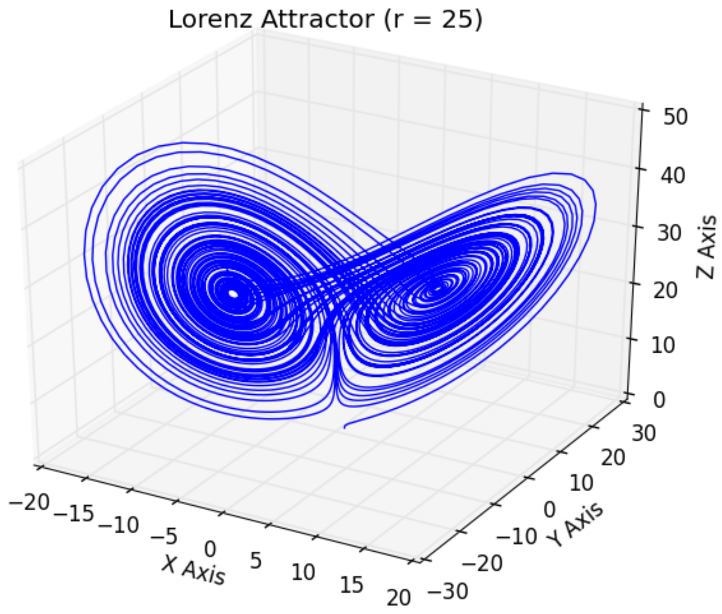

Figure 1: Lorenz attractor with $r = 25$

Then we can select the *Poincaré* section, for example (x=0):

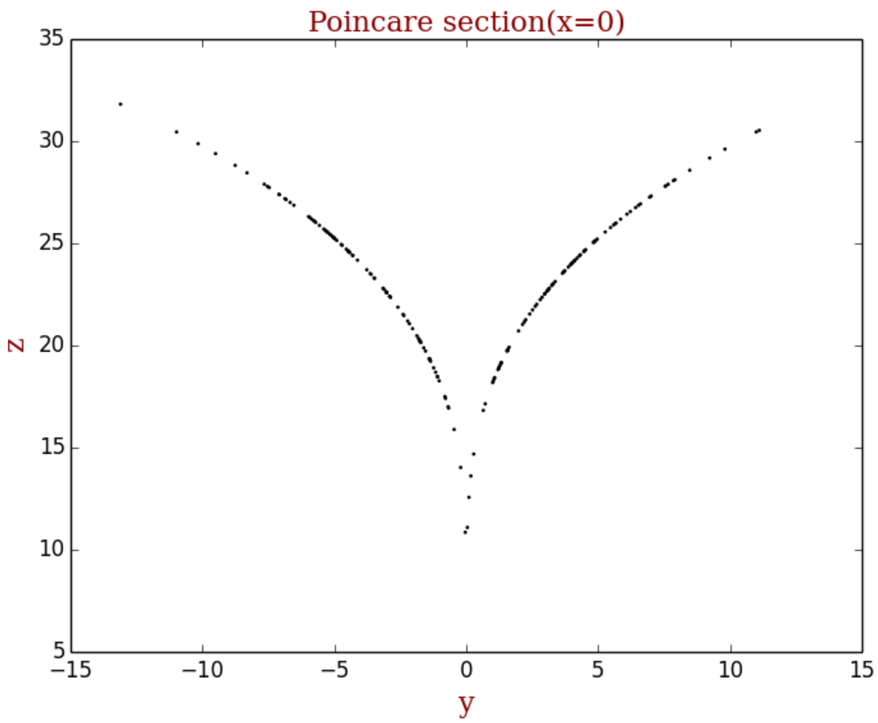

Figure 2: *Poincaré* section

In other regimes, such as $r = 22, 80, 160, 180, 200, 500$ the similar phase diagrams are:

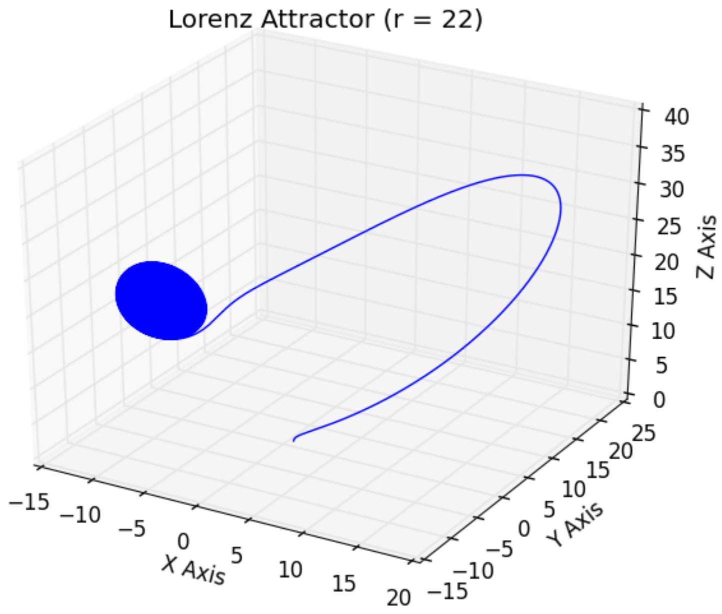

Figure 3: r=22

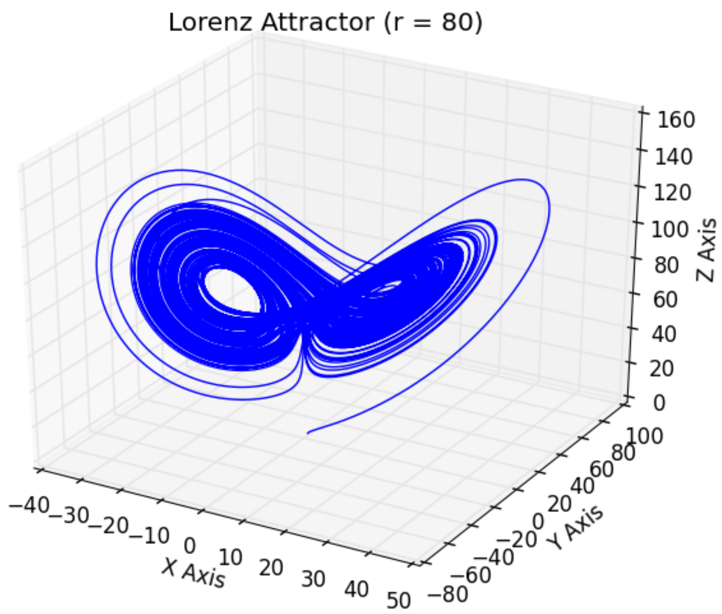

Figure 4: r=80

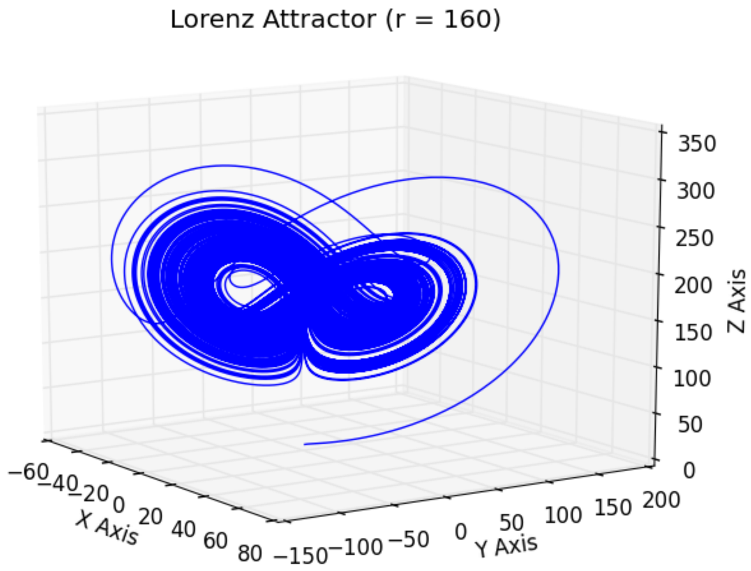

Figure 5: r=160

Its *Poincaré* section is

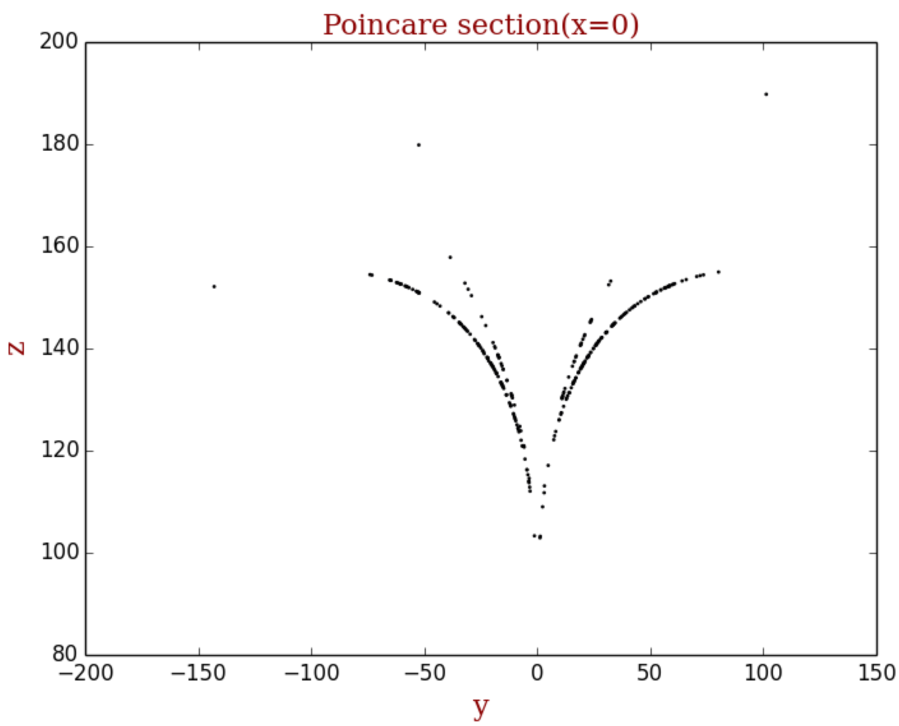

Figure 6: *Poincaré* section

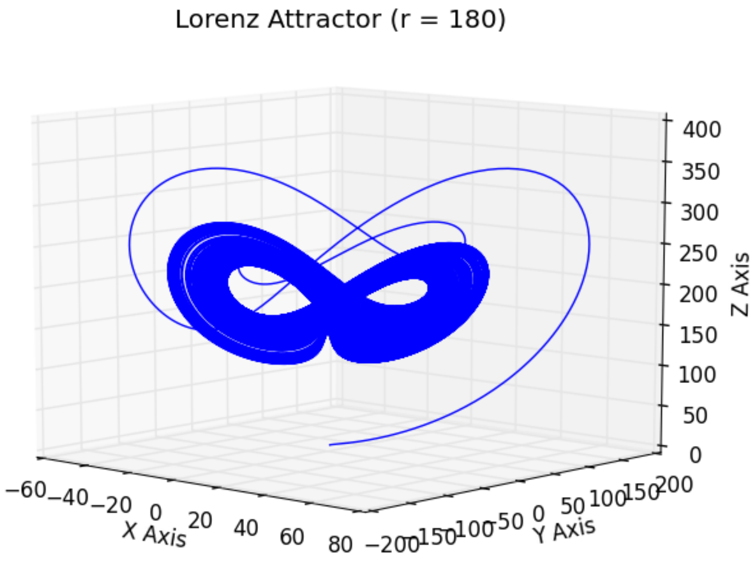

Figure 7: r=180

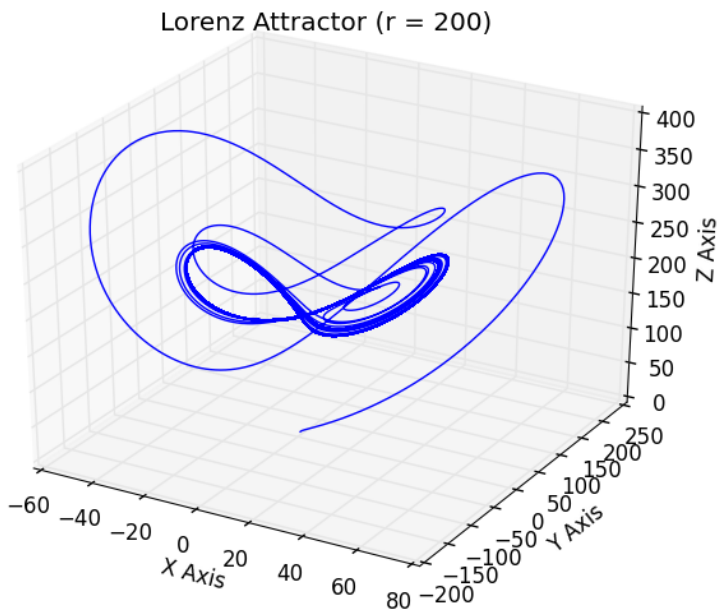

Figure 8: r=200

Its *Poincaré* section is

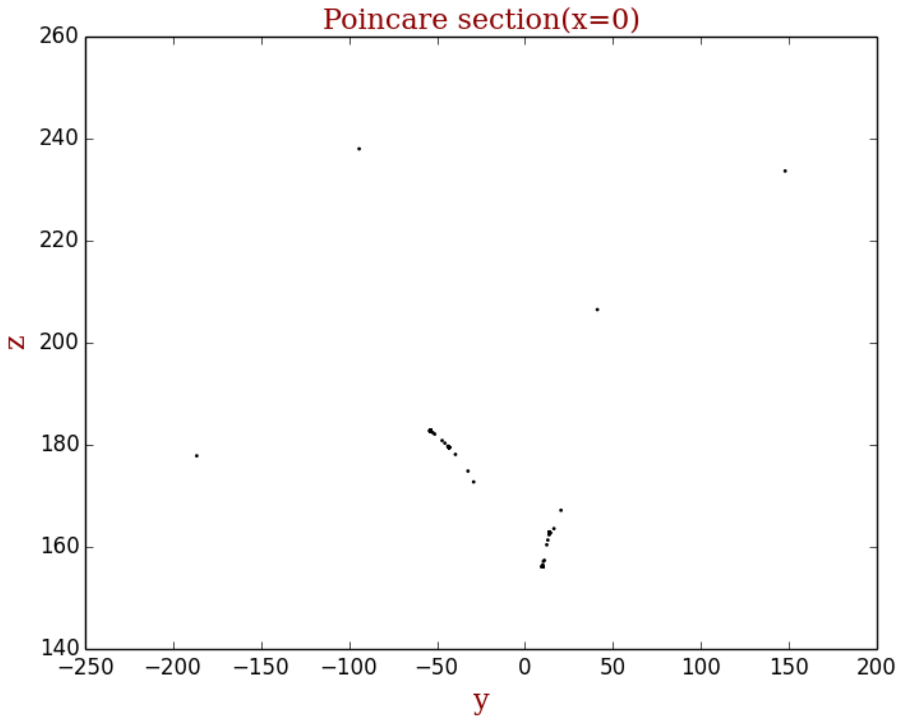

Figure 9: *Poincaré* section

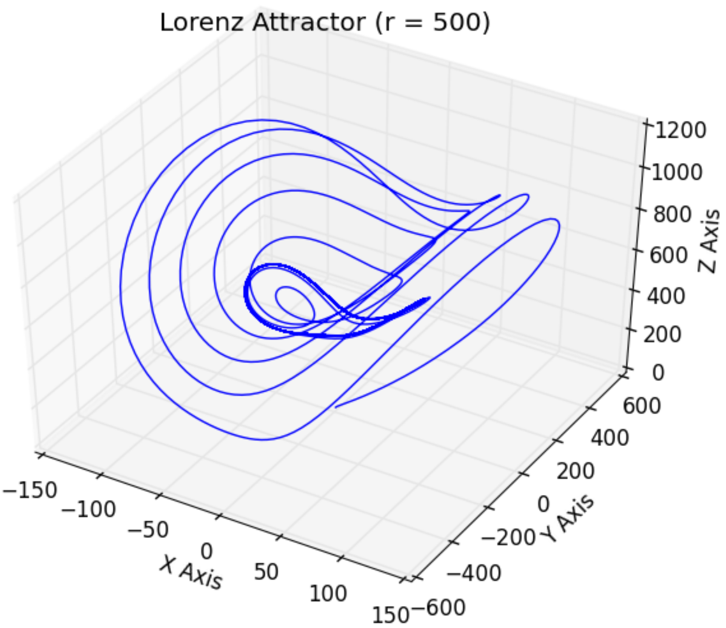

Figure 10: r=500

We see the limit ring shrinks as r rises to a high value.

4 Interpretation & Analysis

It can be proved[3] that If $r < 1$ then there is only one equilibrium point, which is at the origin. This point corresponds to no convection. All orbits converge to the origin, which is a global attractor, when $r < 1$.

A pitchfork bifurcation occurs at $r = 1$, and for $r > 1$ two additional critical points appear at

$$\left(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1\right)$$

These correspond to steady convection. This pair of equilibrium points is stable only if

$$r < \sigma\frac{\sigma+b+3}{\sigma-b-1},$$

which can hold only for positive r if $\sigma > b+1$. At the critical value, both equilibrium points lose stability through a Hopf bifurcation.

When $r = 28$, $\sigma = 10$, and $b = 8/3$, the Lorenz system has chaotic solutions (but not all solutions are chaotic). Almost all initial points will tend to an invariant set the Lorenz attractor - a strange attractor and a fractal. Its Hausdorff dimension is estimated to be 2.06 ± 0.01, and the correlation dimension is estimated to be 2.05 ± 0.01

References

[1] Nicholas J. Giordano, Hisao Nakanishi, 2007,*Computational Physics.*

[2] E. Lorenz, J. Atmos. Sciences, **20**, 130 (1963)

[3] Hirsch, Morris W.; Smale, Stephen; Devaney, Robert (2003) *Differential Equations, Dynamical Systems, and An Introduction to Chaos* , Boston, ISBN 978-0-12-349703-1.