Chpter 1 Exercise 1.6

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1 Introduction

If the number of individuals in a population can be discribed by equation

$$\frac{dN}{dt} = aN - bN^2, \tag{1}$$

where N is the number of the individuals we consider, t is time, a and b are positive coefficients. We want to find out the time dependence of N by solving this FODE both numerically and analytically, and they'll be compared.

2 Method

Analytically, we rewrite the equation as

$$\frac{dN}{aN - bN^2} = dt,$$

and integrate both sides, using initial condition

$$N(t) = N_0, \tag{2}$$

we get

$$N(t) = \frac{aN_0 \cdot e^{at}}{a - bN_0 + bN_0 \cdot e^{at}}. \tag{3}$$

Specially, when $b = 0$, the solution becomes

$$N(t) = N_0 \cdot e^{at}. \tag{4}$$

As for numerical solution, we choose the *Euler Method* to attack it. The second order *Taylor expansion* for $N(t+\Delta t)$ is

$$\begin{aligned} N(t+\Delta t) &= N(t) + \frac{dN(t)}{dt} + O(\Delta t)^2 \\ &\approx N(t) + \frac{dN(t)}{dt} \\ &= N(t) + aN(t) - bN(t)^2. \end{aligned} \tag{5}$$

When $b = 0$, it becomes

$$N(t+\Delta t) = N(t) + aN(t) = (1+a)N(t). \tag{6}$$

3 Data & Verification

First,we choose $a = 10$, $b = 0$, $N_0 = 100$, time range $(0, 0.5)$ and the result is shown as below

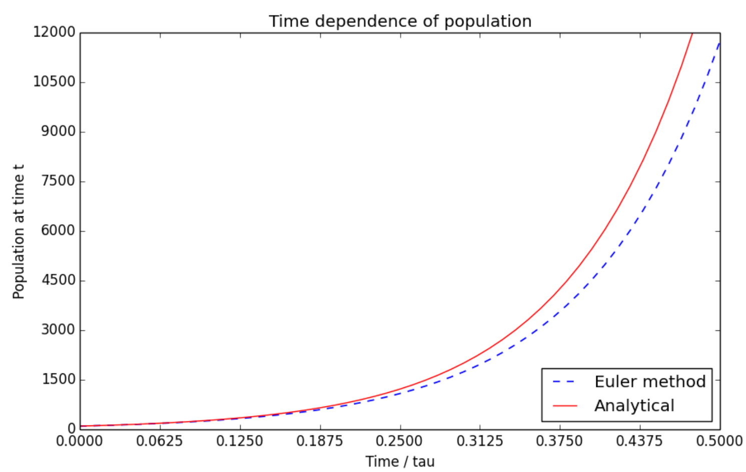

Figure 1: $b = 0$, Analytical and numerical solutions

Similar to what is shown in the text book, our numerical solution is a little bit smaller than the true result. This cannot be the whole story since the population blows up to infinity.

Next, we set $a = 10$, $b = 0.01$, $N_0 = 100$, time range $(0, 0.5)$ and plot both analytical and numerical results according to Eq.(3) and (5).

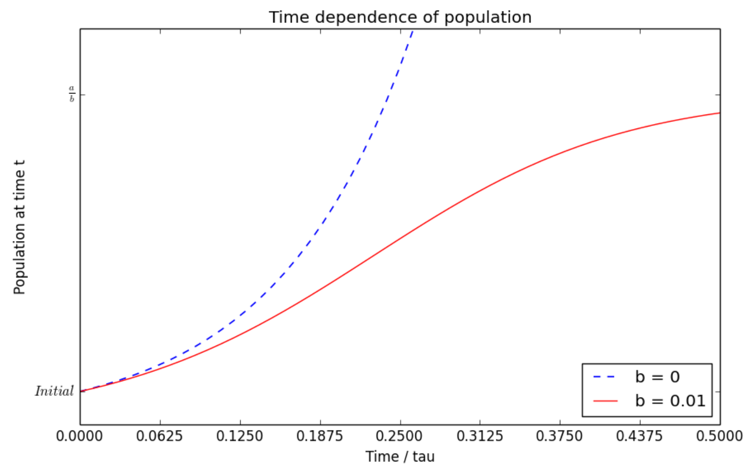

Figure 2: $b = 0.01$, Analytical and numerical solutions

This time, the analytical and numerical solutions approaches even closer. Though the real population problem is much more complicated herein the $b > 0$ ones are more reasonable.

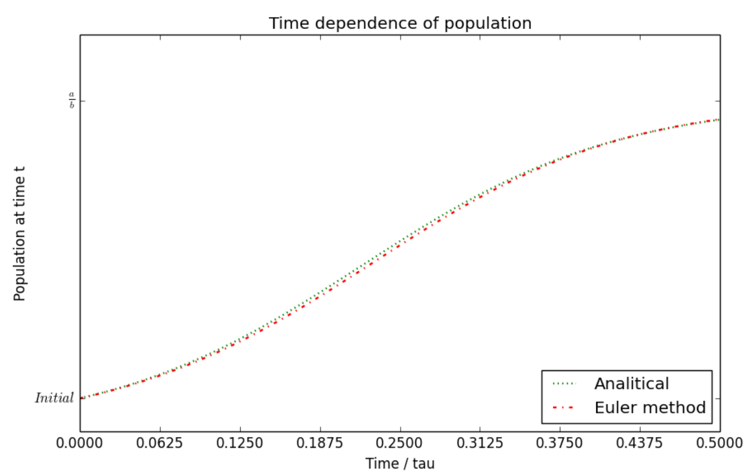

Figure 3: Comparison between $b = 0$ and $b = 0.01$

4 Interpretation & Analysis

4.1 Negative Feedback

We see from Fig.(2) or by observing Eq.(3), that there is a horizontal asymptote $N_{stable} = \frac{a}{b}$. When the *RHS* of Eq.(1) is zero, $N(t)$ becomes stable, since the change rate of $N(t)$ or say $\frac{dN}{dt}$ is zero. Moreover, the larger the N deviates from N_{stable}, the larger its change rate. Critically, the change direction always points to N_{stable}. That is when $N_0 > N_{stable}$, $N(t)$ will descend to it as long as a and b are chosen appropriately.

4.2 Numerical Error

Numerical errors are of order $O(\Delta t)^2$, which approaches 0 as Δt approaches 0.

References

[1] Nicholas J. Giordano, Hisao Nakanishi, 2007,*Computational Physics*, p17