CISS380: Computer Graphics
Assignment 5

Objectives

- Prove statements about vectors

I expect the proofs to be properly written in the mathematical sense. Every step must be justified. Once you have finish one proof, talk to me. I'll make suggestions and point out the mistakes, you make the corrections, and repeat until it's really done. It's best to print out a hardcopy (free printing at BUH CS Lab) so I can make comments/corrections on it. Talk to one another too.

If you need a fact to justify a deduction, but you can't prove it or find it in the axioms, let me know. If it's too difficult to prove, I'll give hints. If it's still too difficult, I might include that as a fact without proof for you to use.

There is a `latex.pdf` at my website `http://yliow.github.io`. Or just ask me if you have questions about LaTeX.

Field axioms and axioms of $\mathbb{R}$

$(F, 0, 1, +, \cdot)$ is a **field** if F is a set and $0, 1 \in F$ and satisfies the following:

- Axioms of $+$
  - Closure of $+$: If $x, y \in \mathbb{R}$, then $x + y \in \mathbb{R}$.
  - Associativity of $+$: If $x, y, z \in \mathbb{R}$, then $(x + y) + z = x + (y + z)$.
  - Inverse of $+$: If $x \in \mathbb{R}$, then there is some $x' \in \mathbb{R}$ such that $x + x' = 0 = x' + x$. (It can be shown that additive inverse of x is unique. Therefore one can denote the additive inverse of x by $-x$.)
  - Neutrality of $+$: If $x \in \mathbb{R}$, then $x + 0 = x = 0 + x$.
  - Commutativity of $+$: If $x, y \in \mathbb{R}$, then $x + y = y + x$.

  We define $x - y = x + (-y)$ if $x, y \in F$.
- Axioms of $\cdot$
  - Closure of $\cdot$: If $x, y \in \mathbb{R}$, then $x \cdot y \in \mathbb{R}$.
  - Associativity of $\cdot$: If $x, y, z \in \mathbb{R}$, then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
  - Inverse of $\cdot$: If $x \in \mathbb{R}$ and $x \neq 0$, then there is some $x' \in \mathbb{R}$ such that $x \cdot x' = 1 = x' \cdot x$. (It can be shown that multiplicative inverse of x is unique. Therefore one can denote the additive inverse of x by x^{-1}.)
  - Neutrality of $\cdot$: If $x \in \mathbb{R}$, then $x \cdot 1 = x = 1 \cdot x$.
  - Commutativity of $\cdot$: If $x, y \in \mathbb{R}$, then $x \cdot y = y \cdot x$.

  Frequently we'll write xy for $x \cdot y$. Also, if $y \neq 0$, I'll write x/y for $x \cdot y^{-1}$.
- Distributivity: If $x, y, z \in \mathbb{R}$, then $x \cdot (y + z) = x \cdot y + x \cdot z \in \mathbb{R}$.

We will assume the following axioms about $(\mathbb{R}, 0, 1, +, \cdot)$:

- $(\mathbb{R}, 0, 1, +\cdot)$ is a field, i.e. it satisfies the field axioms.
- Nontriviality: $0 \neq 1$.
- Order axioms: There is a subset $P \subseteq \mathbb{R}$ such that
  - If $x \in \mathbb{R}$, exactly one of the following is true: $x \in P$, $x = 0$, $-x \in P$
  - If $x, y \in P$, then $x + y \in P$.
  - If $x, y \in P$, then $x \cdot y \in P$.

  With the above, I can define $x > 0$ is $x \in P$ and also $x \geq 0$ if $x = 0$ or $x > 0$. Furthermore I can define $x > y$ is $x - y > 0$ and $x \geq y$ if $x - y \geq 0$.
- Least upper bound: Every nonempty set of $\mathbb{R}$ has a least upper bound. (If X is a nonempty set of $\mathbb{R}$, then $m \in \mathbb{R}$ is an **upper bound** of X if

  $$x \leq m \text{ for all } x \in X$$

  (duh). $\ell \in \mathbb{R}$ is a **least upper bound (l.u.b.)** of X if ℓ is an upper bound of X and if m is an upper bound of X, then $\ell \leq m$.)

For $x \in \mathbb{R}$, the absolute value of x is defined to be

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{otherwise} \end{cases}$$

The following facts about any field F can be proven from the field axioms of F. But we won't prove them. We will simply assume that they hold. You can try to prove them on your own. Most of these are proven in CISS451.

Proposition A. Let $a, b, c, d \in F$.

(a) $-0 = 0$
(b) $0 \cdot a = 0 = a \cdot 0$
(c) $(-1) \cdot a = -a = a \cdot (-1)$
(d) $(-1) \cdot (-1) = 1$

Here are some basic facts about the absolute value function on $\mathbb{R}$.

Proposition B. Let $a, b, c, d \in \mathbb{R}$.

(a) $|0| = 0$
(b) MULTIPLICATIVITY: $|xy| = |x||y|$
(c) TRIANGLE INEQUALITY: $|x + y| \leq |x| + |y|$

Most of the above is not difficult. The only one that might not be immediate is the triangle inequality of $\mathbb{R}$. It's a good exercise.

In the proofs below, you can quote Proposition A and Proposition B. You have to specify which part. For instance

$$\begin{aligned} x + ((-1) \cdot (-1)) \cdot y &= x + 1 \cdot y && \text{by Proposition A(d)} \\ &= x + y && \text{by neutral axiom of } \cdot \end{aligned}$$

Definitions of vectors over field F

Let F be a field. A 2D **vector** over F is

$$\langle a, b \rangle$$

where $a, b \in F$. Define vector equality by

$$\langle a, b \rangle = \langle c, d \rangle \text{ if } a = c, b = d$$

We define the **zero vector** to be

$$\vec{0} = \langle 0, 0 \rangle$$

The **length** or **norm** or **magnitude** of a vector $\langle a, b \rangle$ is defined to be

$$\| \langle a, b \rangle \|$$

A **unit vector** is a vector of length 1. Define the following operators on the 2D vectors over F. Let $a, b, c, d \in F$.

- Vector addition: $\langle a, b \rangle + \langle c, d \rangle = \langle a + c, b + d \rangle$
- Scalar product: $c \cdot \langle a, b \rangle = \langle c \cdot a, c \cdot b \rangle$
- Negative vector: $-\langle a, b \rangle = \langle -a, -b \rangle$
- Vector subtraction: $\langle a, b \rangle - \langle c, d \rangle = \langle a, b \rangle + (-\langle c, d \rangle)$
- Vector dot product: $\langle a, b \rangle \cdot \langle c, d \rangle = a \cdot c + b \cdot d$

Q1. Prove the following proposition from our notes. Proofs in later question can quote this proposition.

Proposition 1. Let $\vec{u}, \vec{v}, \vec{w}$ be vectors and $\alpha, \beta \in \mathbb{R}$. Then

(a) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
(b) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
(c) $\vec{u} + \vec{0} = \vec{u} = \vec{0} + \vec{u}$
(d) $\alpha \cdot (\vec{u} + \vec{v}) = \alpha \cdot \vec{u} + \alpha \cdot \vec{v}$
(e) $(\alpha + \beta) \cdot \vec{u} = \alpha \cdot \vec{u} + \beta \cdot \vec{u}$
(f) $\vec{u} - \vec{u} = \vec{0}$
(g) $1 \cdot \vec{u} = \vec{u}$
(h) $-1 \cdot \vec{u} = -\vec{u}$
(i) $0 \cdot \vec{u} = \vec{0}$
(j) $\alpha \cdot (\beta \cdot \vec{u}) = (\alpha \cdot \beta) \cdot \vec{u}$

(Put your proofs in `q01s.tex`.)

Proof. In all the proofs below, by definition of 2D vectors, let $\vec{u} = \langle a, b \rangle$, $\vec{v} = \langle c, d \rangle$, and $\vec{w} = \langle e, f \rangle$ where $a, b, c, d, e, f \in F$.

(a) We have

$$\begin{aligned} \vec{u} + \vec{v} &= \langle a, b \rangle + \langle c, d \rangle \\ &= \langle a + c, b + d \rangle && \text{by definition of vector addition} \\ &= \langle c + a, d + b \rangle && \text{by commutativity of } + \text{ of } F \\ &= \langle c, d \rangle + \langle a, b \rangle && \text{by definition of vector addition} \\ &= \vec{v} + \vec{u} \end{aligned}$$

(b) We have

$$\begin{aligned} (\vec{u} + \vec{v}) + \vec{w} &= \\ &= \\ &= \end{aligned}$$

(c)

(d)

(e)

(f)

(g)

(h)

(i)

(j) □

Q2.

I'll state this proposition, but you don't have to provide proofs:

Proposition 2. Let $\vec{u}, \vec{v}, \vec{w}$ be vectors and $c, d \in \mathbb{R}$. Then

(a) $\|\vec{u}\| \geq 0$
(b) $\|\vec{u}\| = 0$ iff $\vec{u} = \vec{0}$.
(c) $\|c\vec{u}\| = |c| \cdot \|\vec{u}\|$
(d) Triangle inequality: $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

The proof of the triangle inequality for 2D vectors is in the notes. Study it carefully to the point that you can produce the proof on your own.

For this question, you'll prove Proposition 3:

Proposition 3. Let $\vec{u}$, $\vec{v}$ be 2D vectors over $\mathbb{R}$ and let $\alpha \in \mathbb{R}$.

(a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
(b) $(\alpha\vec{u}) \cdot \vec{v} = \alpha(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (\alpha\vec{v})$
(c) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
(d) $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$
(e) $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$ where θ is the angle between $\vec{u}$ and $\vec{v}$.
(f) $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\|\|\vec{u}\|$
(g) If $\vec{u}$ and $\vec{u}$ are nonzero, then $\vec{u}$ and $\vec{v}$ are perpendicular iff $\vec{u} \cdot \vec{v} = 0$.
(h) If $\vec{u} \neq \vec{0}$, then $(1/\|\vec{u}\|) \cdot \vec{u}$ is a unit vector.

(Edit `q02.tex`.)

Proof. By definition, $\vec{u} = \langle a, b \rangle$, $\vec{v} = \langle c, d \rangle$, $\vec{w} = \langle e, f \rangle$ for some $a, b, c, d, e, f \in \mathbb{R}$.

(a) We have

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \langle a, b, \rangle \cdot \langle c, d \rangle \\ &= \\ &= \end{aligned}$$

(b)

(c)

(d)

(e)

(f)

(g) □