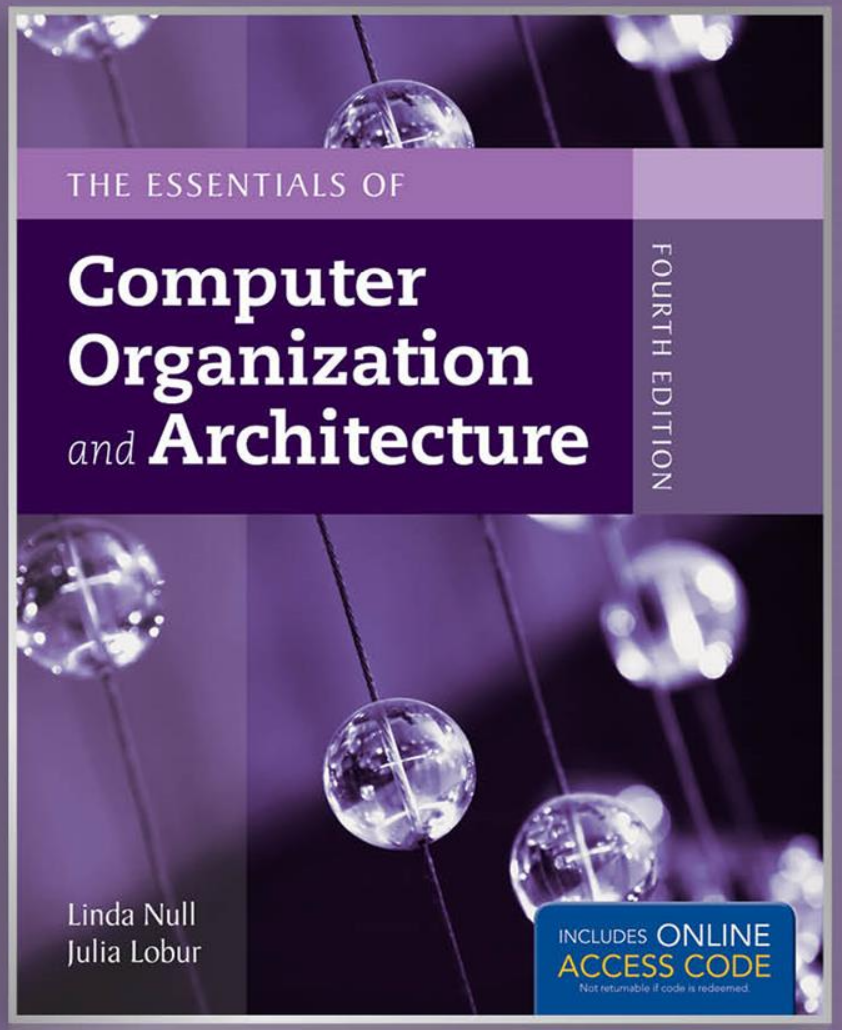
Chapter 2

Floating Point Numbers



2.5 Floating-Point Representation

- The signed magnitude, one's complement, and two's complement representation that we have just previously discussed deal with signed integer values only.
- Without modification, these formats are not useful in scientific or business applications that deal with real number values.
- Floating-point representation solves this problem.

2.5 Floating-Point Representation

- If we are clever programmers, we can perform floating-point calculations using any integer format.
- This is called *floating-point emulation*, because floating point values aren't stored as such; we just create programs that make it seem as if floating-point values are being used.
- Most of today's computers are equipped with specialized hardware that performs floating-point arithmetic with no special programming required.
  - Other than using the provided instruction set of you CPU architecture

2.5 Floating-Point Representation

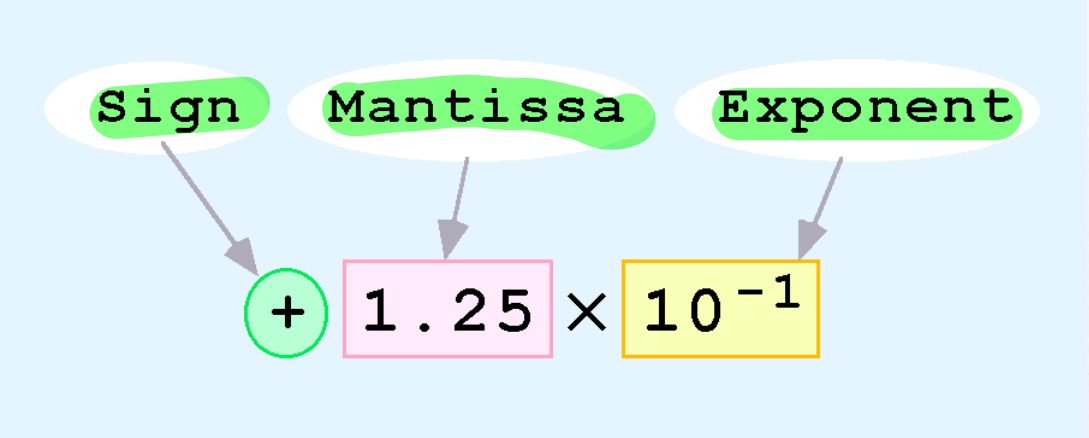
- Floating-point numbers allow an arbitrary number of decimal places to the right of the decimal point.
  - For example: $0.5 \times 0.25 = 0.125$
- They are often expressed in scientific notation.
  - For example:

    $0.125 = 1.25 \times 10^{-1}$

    $5{,}000{,}000 = 5.0 \times 10^{6}$

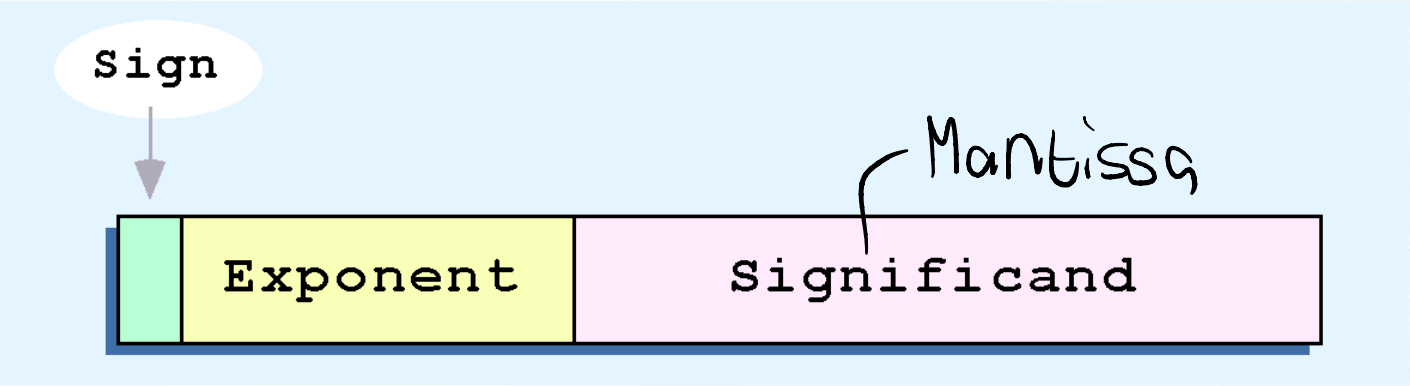
2.5 Floating-Point Representation

- Computers use a form of scientific notation for floating-point representation
- Numbers written in scientific notation have three components:



2.5 Floating-Point Representation

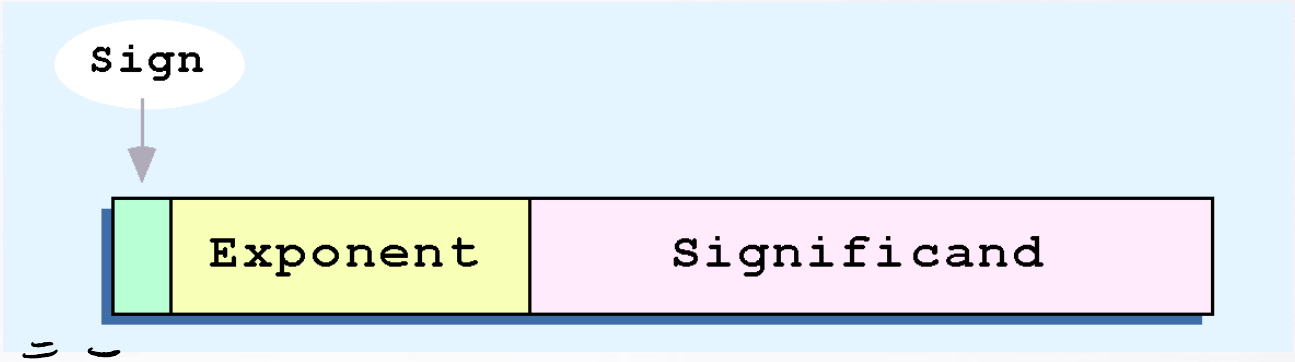
- Computer representation of a floating-point number consists of three fixed-size fields:



- This is the standard arrangement of these fields.

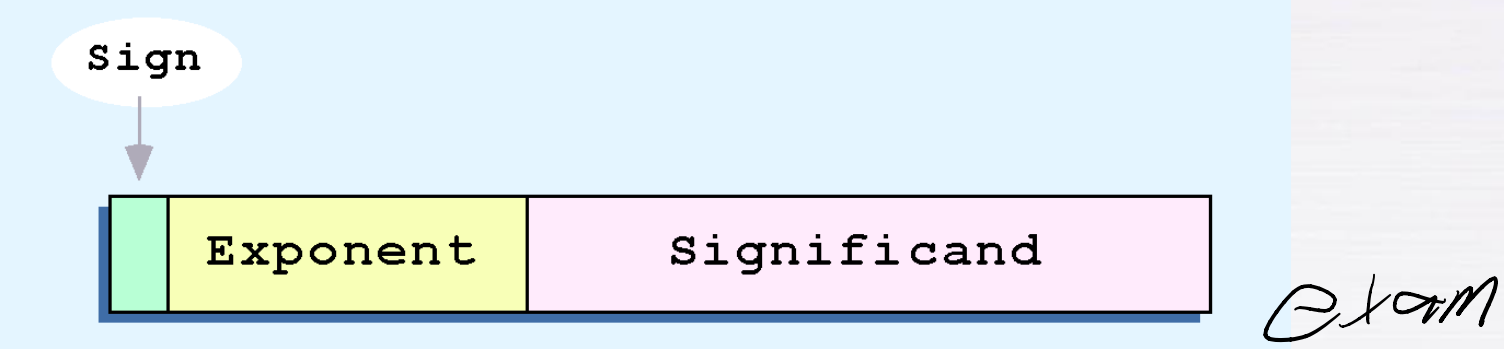
Note: Although "significand" and "mantissa" do not technically mean the same thing, many people use these terms interchangeably. We use the term "significand" to refer to the fractional part of a floating point number.

2.5 Floating-Point Representation



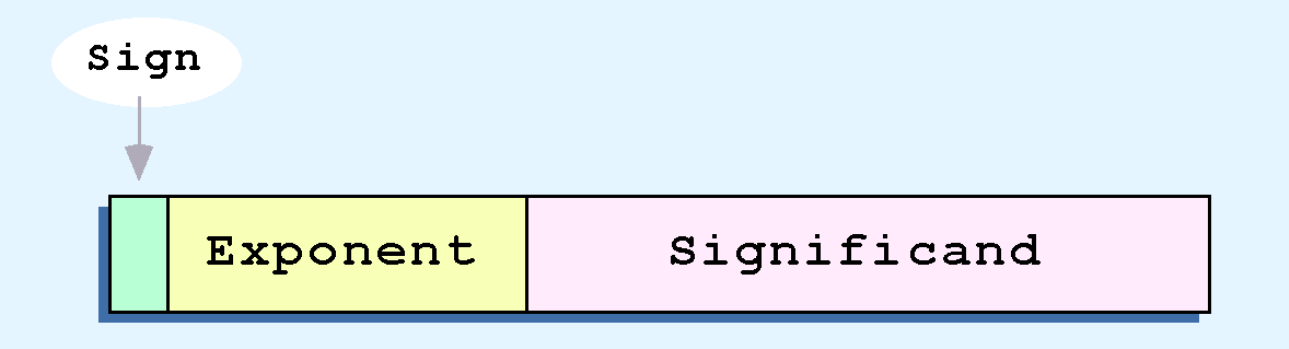
- The one-bit sign field is the sign of the stored value.
- The size of the exponent field determines the **range** of values that can be represented.
- The size of the significand determines the **precision** of the representation.

2.5 Floating-Point Representation



- We introduce a hypothetical “Model” to explain the concepts, after which we will discuss the IEEE-754 one.
- In this model:
  - A floating-point number is 14 bits in length
  - The exponent field is 5 bits
  - The significand field is 8 bits

2.5 Floating-Point Representation



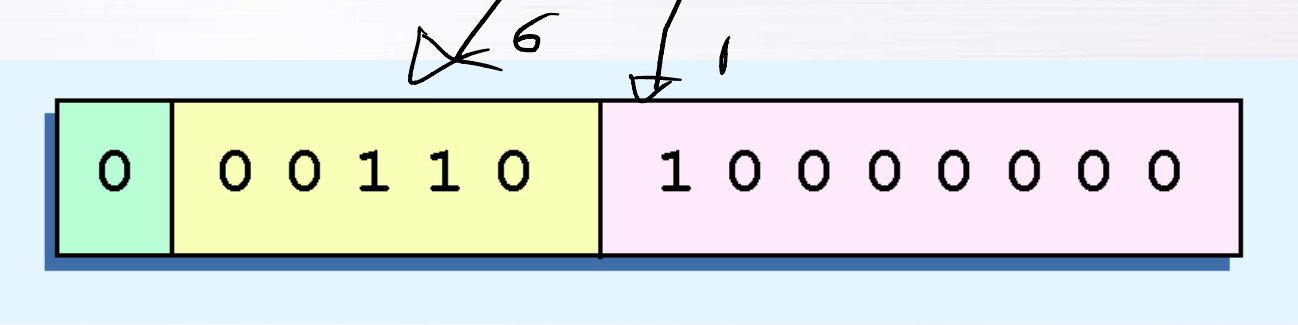
- The significand is always preceded by an implied binary point.
- Thus, the significand always contains a fractional binary value.
- The exponent indicates the power of 2 by which the significand is multiplied.

2.5 Floating-Point Representation

- Example:
  - Express 32_{10} in the simplified 14-bit floating-point model. (1-bit sign, 5-bit exponent, 9-bit significand)
- We know that 32 is 2^5. So in (binary) scientific notation $32 = 100000_2 = 0.1_2 \times 2^6$.

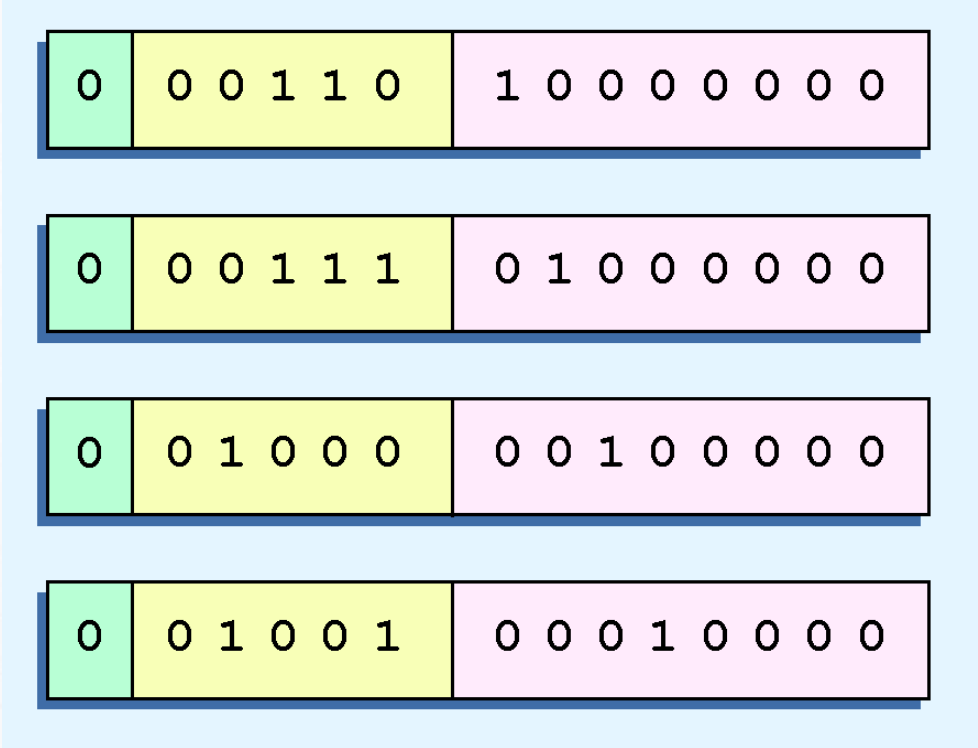
Always make it a fraction '

- Using this information, we put 110 (= 6_{10}) in the exponent field and 1 in the significand as shown.



2.5 Floating-Point Representation

- The illustrations shown at the right are *all* equivalent representations for 32 using our simplified model.
- Not only do these synonymous representations waste space, but they can also cause confusion.



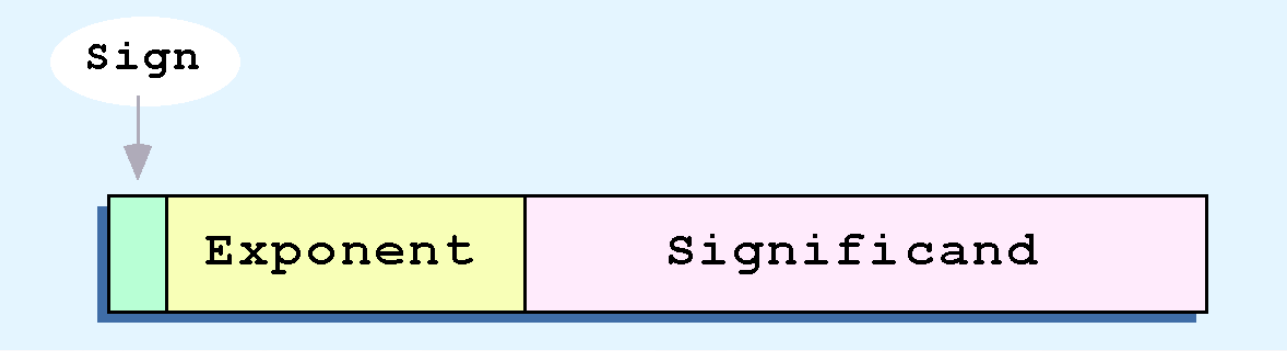
Force the digit to be a one

2.5 Floating-Point Representation

- To resolve the problem of synonymous forms, we establish a rule that the first digit of the significand must be 1, with no ones to the left of the radix point.
- This process, called *normalization*, results in a unique pattern for each floating-point number.
  - In our simple model, all significands must have the form 0.1xxxxxxxx
  - For example, $4.5 = 100.1 \times 2^0 = 1.001 \times 2^2 = 0.1001 \times 2^3$. The last expression is correctly normalized.

In our simple instructional model, we use no implied bits.

2.5 Floating-Point Representation



- Another problem with our system is that we have made no allowances for negative exponents. We have no way to express 0.25! (Notice that there is no sign in the exponent field.)

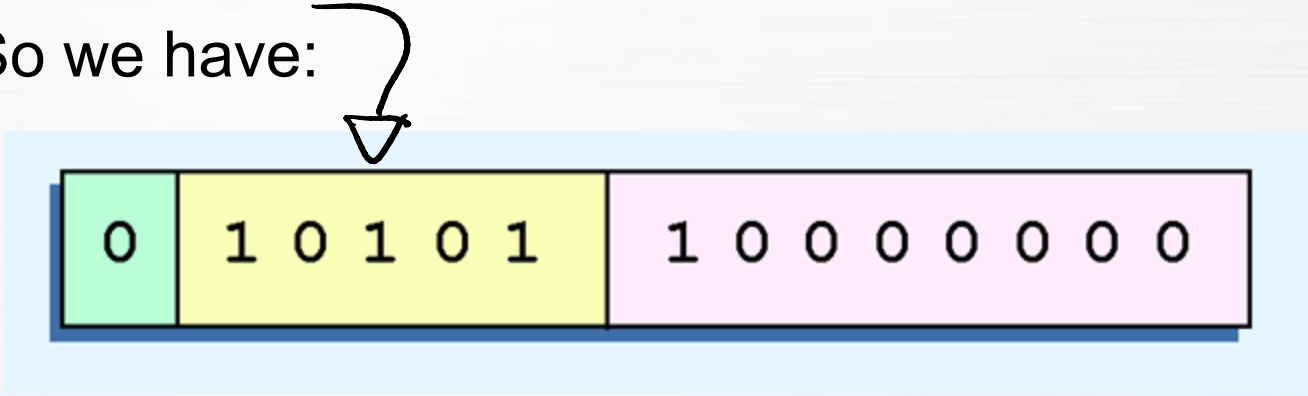
All of these problems can be fixed with no changes to our basic model.

2.5 Floating-Point Representation

- To provide for negative exponents, we will use a *biased exponent*.
  - In our case, we have a 5-bit exponent.
  - $2^{5-1} - 1 = 2^4 - 1 = 15$
  - Thus will use 15 for our bias: our exponent will use *excess-15* representation.
- In our model, exponent values less than 15 are negative, representing purely fractional numbers.

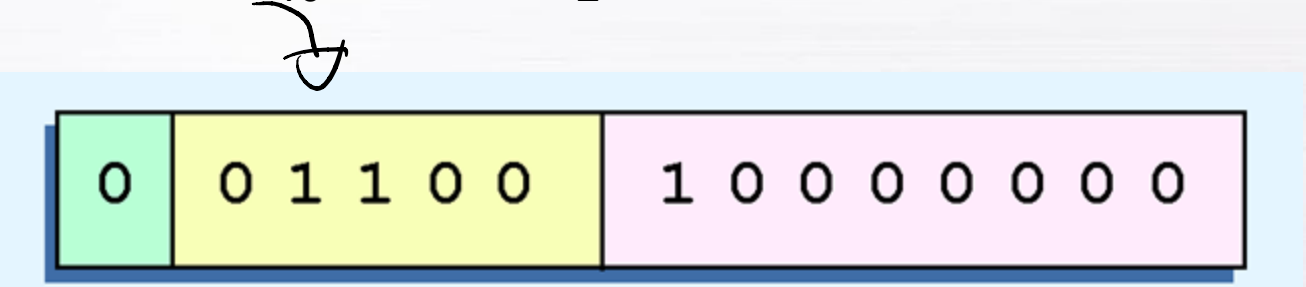
2.5 Floating-Point Representation

- Example: 0.1×2^{6} $15 + 6 = 21$
  - Express 32_{10} in the revised 14-bit floating-point model.
- We know that $32 = 1.0 \times 2^5 = 0.1 \times 2^6$.
- To use our excess 15 biased exponent, we add 15 to 6, giving 21_{10} (=10101_2).
- So we have:



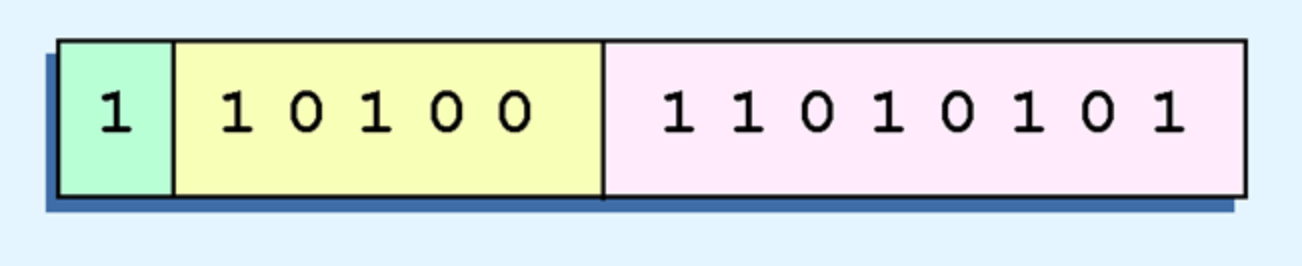
2.5 Floating-Point Representation

- Example:
  - Express 0.0625_{10} in the revised 14-bit floating-point model.
- We know that 0.0625 is 2^{-4}. So in (binary) scientific notation $0.0625 = 0.0001_2 = 1.0 \times 2^{-4} = 0.1 \times 2^{-3}$.
- To use our excess 15 biased exponent, we add 15 to -3, giving 12_{10} ($=01100_2$).



2.5 Floating-Point Representation

- Example:
  - Express -26.625_{10} in the revised 14-bit floating-point model.
- We find $26.625_{10} = 11010.101_2$. Normalizing, we have: $26.625_{10} = 0.11010101 \times 2^5$.
- To use our excess 15 biased exponent, we add 15 to 5, giving 20_{10} (=10100_2). We also need a 1 in the sign bit.



2.5 Floating-Point Representation

- The IEEE has established a standard for floating-point numbers
- The IEEE-754 *single precision* floating point standard uses an 8-bit exponent (with a bias of 127) and a 23-bit significand.
- The IEEE-754 *double precision* standard uses an 11-bit exponent (with a bias of 1023) and a 52-bit significand.

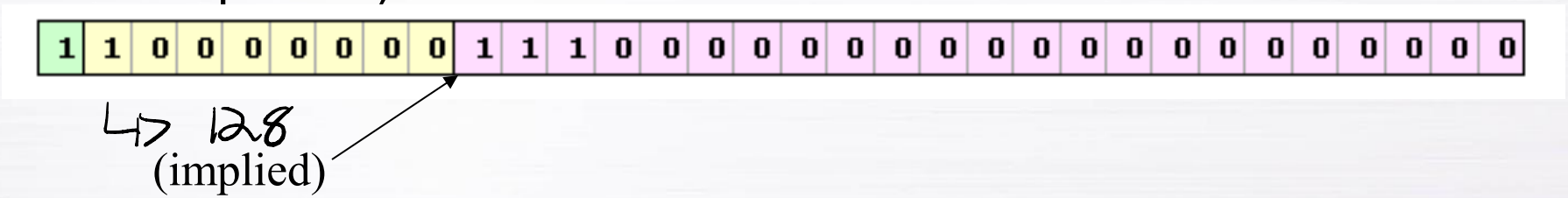
largest smallest Exam question

2.5 Floating-Point Representation

- In both the IEEE single-precision and double-precision floating-point standard, the significant has an implied 1 to the LEFT of the radix point.
  - The format for a significand using the IEEE format is: 1.xxx…
  - For example, $4.5 = .1001 \times 2^3$ in IEEE format is $4.5 = 1.001 \times 2^2$. The 1 is implied, which means is does not need to be listed in the significand (the significand would include only 001).

2.5 Floating-Point Representation

- Example: Express -3.75 as a floating point number using IEEE single precision.
- First, let's normalize according to IEEE rules:
  - $-3.75 = -11.11_2 = -1.111 \times 2^1$
  - The bias is 127, so we add 127 + 1 = 128 (this is our exponent)



  - Since we have an implied 1 in the significand, this equates to

$$-(1).111_2 \times 2^{(128-127)} = -1.111_2 \times 2^1 = -11.11_2 = -3.75.$$

2.5 Floating-Point Representation

- Using the IEEE-754 single precision floating point standard:
  - An exponent of 255(after adding the bias (all 1's)) indicates a special value.
    - If the significand is zero, the value is ± infinity.
    - If the significand is nonzero, the value is NaN, "not a number," often used to flag an error condition.
- Using the double precision standard:
  - The "special" exponent value for a double precision number is 2047, instead of the 255 used by the single precision standard.

2.5 Floating-Point Representation

- Both the 14-bit model that we have presented and the IEEE-754 floating point standard allow two representations for zero.
  - Zero is indicated by all zeros in the exponent and the significand, but the sign bit can be either 0 or 1.
- This is why programmers should avoid testing a floating-point value for equality to zero.
  - Negative zero does not equal positive zero.

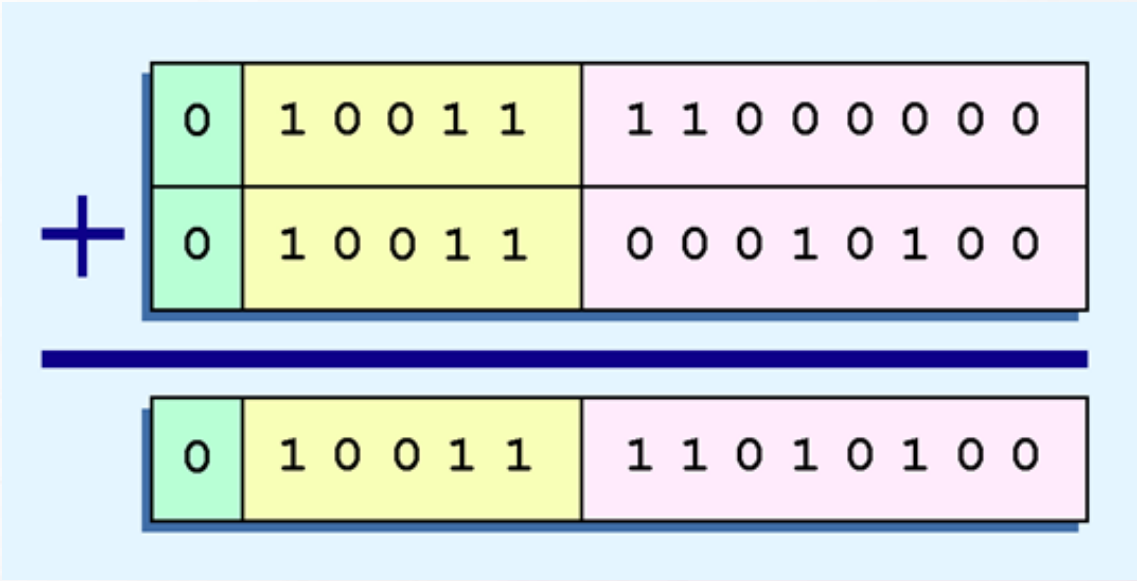
2.5 Floating-Point Representation

Subtract 15 else you wil make it Basie of 30

- IEEE Floating-point **addition** and **subtraction** are done using methods **analogous** to how we perform calculations using pencil and paper.
- The first thing that we do is express both operands in the same exponential power, then add the numbers, preserving the exponent in the sum.
- If the exponent requires adjustment, we do so at the end of the calculation.

2.5 Floating-Point Representation

- Example:
  - Find the sum of 12_{10} and 1.25_{10} using the 14-bit "simple" floating-point model.
- We find $12_{10} = 0.1100 \times 2^4$. And $1.25_{10} = 0.101 \times 2^1 = 0.000101 \times 2^4$.
- Thus, our sum is 0.110101×2^4.



2.5 Floating-Point Representation

- Floating-point multiplication is also carried out in a manner akin to how we perform multiplication using pencil and paper.
- We **multiply** the two operands and **add** their exponents.
- If the exponent requires adjustment, we do so at the end of the calculation.

2.5 Floating-Point Representation

- No matter how many bits we use in a floating-point representation, our model must be finite.
- The real number system is, of course, infinite, so our models can give nothing more than an approximation of a real value.
- At some point, every model breaks down, introducing errors into our calculations.
- By using a greater number of bits in our model, we can reduce these errors, but we can never totally eliminate them.

2.5 Floating-Point Representation

- Consider
- 0.1 in decimal.
- It cannot be perfectly represented in binary..
- $0.1_{10} = 0.0001100110011001100110011 \ldots \ldots_2$

2.5 Floating-Point Representation

- Our job becomes one of reducing error, or at least being aware of the possible magnitude of error in our calculations.
- We must also be aware that errors can compound through repetitive arithmetic operations.
- For example, our 14-bit model cannot exactly represent the decimal value 128.5. In binary, it is 9 bits wide:

$$10000000.1_2 = 128.5_{10}$$

2.5 Floating-Point Representation

- When we try to express 128.5_{10} in our 14-bit model, we lose the low-order bit, giving a relative error of:

$$\frac{128.5 - 128}{128.5} \approx 0.39\%$$

- If we had a procedure that repetitively added 0.5 to 128.5, we would have an error of nearly 2% after only four iterations.

2.5 Floating-Point Representation

- Floating-point errors can be reduced when we use operands that are similar in magnitude.
- If we were repetitively adding 0.5 to 128.5, it would have been better to iteratively add 0.5 to itself and then add 128.5 to this sum.
- In this example, the error was caused by loss of the low-order bit.
- Loss of the high-order bit is more problematic.

2.5 Floating-Point Representation

- Floating-point overflow and underflow can cause programs to crash.
- Overflow occurs when there is no room to store the high-order bits resulting from a calculation.
- Underflow occurs when a value is too small to store, possibly resulting in division by zero.

Experienced programmers know that it's better for a program to crash than to have it produce incorrect, but plausible, results.

2.5 Floating-Point Representation

- When discussing floating-point numbers, it is important to understand the terms *range, precision,* and *accuracy*.
- The range of a numeric integer format is the difference between the largest and smallest values that can be expressed.
- Accuracy refers to how closely a numeric representation approximates a true value.
- The precision of a number indicates how much information we have about a value

2.5 Floating-Point Representation

- Most of the time, greater precision leads to better accuracy, but this is not always true.
  - For example, 3.1333 is a value of pi that is accurate to two digits, but has 5 digits of precision.
- There are other problems with floating point numbers.
- Because of truncated bits, you cannot always assume that a particular floating point operation is commutative or distributive.

2.5 Floating-Point Representation

- This means that we cannot assume:

  $(a + b) + c = a + (b + c)$ or

  $a*(b + c) = ab + ac$

- Moreover, to test a floating point value for equality to some other number, it is best to declare a "nearness to x" epsilon value. For example, instead of checking to see if floating point x is equal to 2 as follows:

  if x = 2 then …

  it is better to use:

  if (abs(x - 2) < epsilon) then ...

  (assuming we have epsilon defined correctly!)