Numerical Probability

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Project 2

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Introduction We wish to compute the price of an Asian option via 3 methods; naive-, partitions-, and adaptive method and observe the development of the variance of the option price wrt. The strike price. Furthermore, we wish to price an Asian option under the framework of the Heston model and with the Kemna Vorst control variate. For simplicity, we will only consider an Asian call option throughout this project.

Framework

We work within the frames of the Black-Scholes model, where the dynamic of the asset price is defined as:

$$X_t = x \cdot \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_T\right),$$

where:

- x: Initial price of the asset,
- r: Risk-free interest rate,
- σ: Volatility of the asset,
- W_T: Standard Brownian motion at time T,
- T: Time to maturity.

The discounted payoff for an Asian call option is defined as:

$$\phi(X_T, K, r, T) = e^{-rT}\left(\frac{1}{T - T_0}\int_0^T S_t dt - K\right)^+,$$

To start with, we wish to implement the three methods and consider the convergence of the option price for a given strike. To start with, we wish to implement the three methods and consider the convergence of the option price for a given strike.

The Naive Method To implement the naive method, we simply calculate the mean of the simulated payoffs:

$$\hat{\phi}_{\text{naive}}(X_T, K, r, T) = \frac{1}{N}\sum_{i=1}^{N}\phi_i(X_T, K, r, T),$$

The Partitions Method

For the partitions method, we use a control variate to reduce the variance of the estimator. The adjusted payoff is computed as:

$$\hat{X}^\lambda = X - \lambda\left(\Xi - \mathbb{E}(\Xi)\right),$$

where:

- X is the original payoff of the option,
- $\Xi = e^{-rT}\frac{1}{N}\sum_{i=1}^{N} X_i$ is the control variate, with X_i being the arithmetic average of the stock prices for the i-th simulation,
- $\mathbb{E}(\Xi)$ is calculated under the Black-Scholes framework:

$$\mathbb{E}(\Xi) = e^{-rT} x \frac{1 - e^{-rT}}{rT},$$

- λ is the regression coefficient that minimizes the variance of $\hat{X}^{\lambda}$:

$$\lambda = \frac{\mathrm{Cov}(X, \Xi)}{\mathrm{Var}(\Xi)}.$$

To estimate λ in practice, we compute it using a subset of $N_{sample} = 5\%$ of simulations, typically the first paths.

$$\lambda_{\text{start}} = \frac{\frac{1}{N_{sample}}\sum_{i=1}^{N_{sample}} X_i \Xi_i}{\frac{1}{N_{sample}}\sum_{i=1}^{N_{sample}} X i_i^2},$$

Hence the payoff of the Asian option using the partitions method is estimated as:

$$\hat{\phi}_{\text{partitions}}(X_T, K, r, T) = \frac{1}{N}\sum_{i=1}^{N} \phi_i(X_T, K, r, T) - \lambda\left(\Xi - \mathbb{E}(\Xi)\right).$$

The Adaptive Method The adaptive method calculates the payoff similarly to the partitions method, using the same control variate. However, the key difference is that the regression coefficient λ is dynamically updated during the simulation process. The adaptive λ is calculated as described in *Numerical Probability (3.5)*:

$$V_M = \sum_{i=1}^{M} \Xi_i^2, \quad C_M = \sum_{i=1}^{M} X_i \Xi_i,$$

$$\lambda_M = \frac{C_M}{V_M}, \quad \lambda_0 = 0.$$

To ensure numerical stability, we implement an adaptive constraint on λ as:

$$\tilde{\lambda}_i = \max\left(-i, \min\left(\lambda_i, i\right)\right).$$

The adjusted estimator for the i-th path becomes:

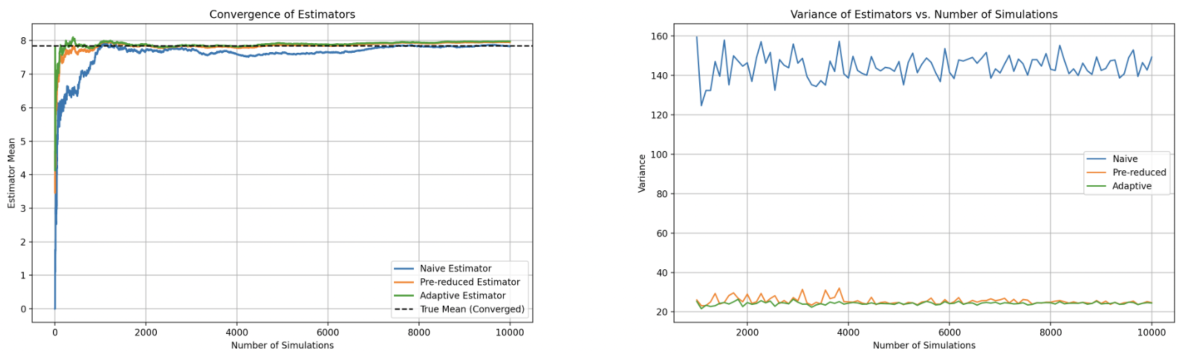
$$\tilde{X}_i^{\tilde{\lambda}} = X_i - \tilde{\lambda}_i \Xi_i.$$

The final estimator for the option price is the average over M simulations:

$$\tilde{\tilde{X}}_M^{\tilde{\lambda}} = \frac{1}{M}\sum_{i=1}^{M} \tilde{X}_i^{\tilde{\lambda}}.$$

Results

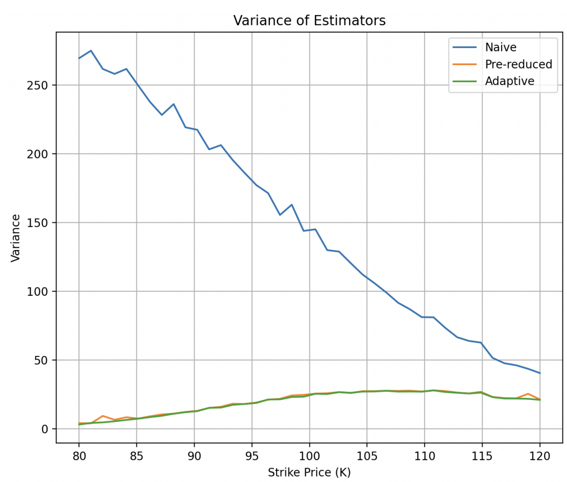
First we wish to get an overview of the convergence of the the methods. So we compute the mean and the variance for a fixed strike and observe how the estimators converge



Figur 1: Mean and variance convergence of an Asian call with K=100

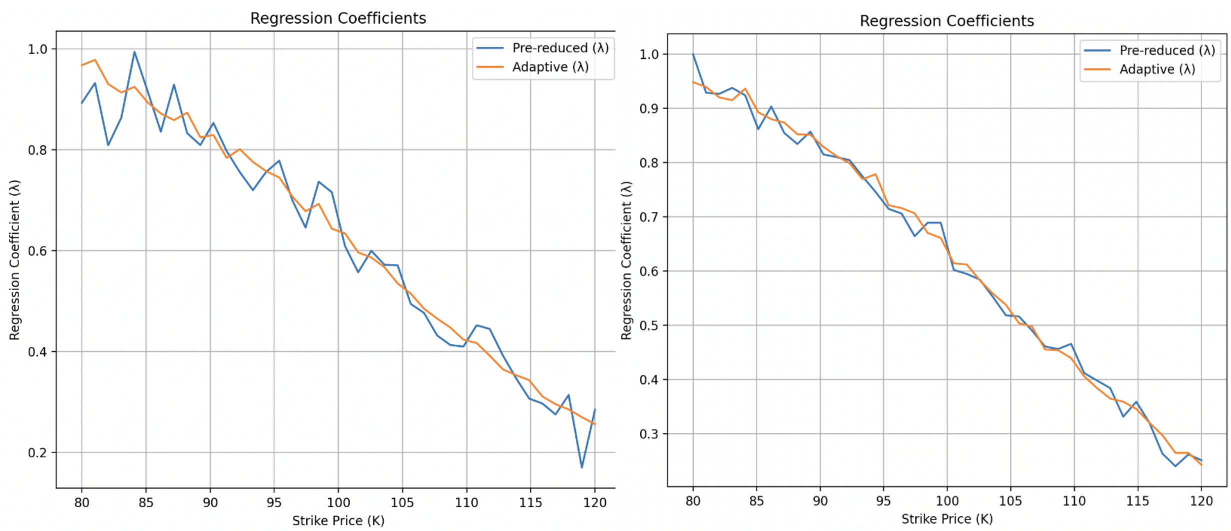
From the figure we see that for an Asian call with a strike price of 100 the naive method has a significantly slower convergence than the other estimators and a higher variance while both the partitions and the adaptive method has lower variance with the latter being the most stable.

Now we see how the variance of the methods compare when the strike changes.



Figur 2: Variance of the methods with changes in strike price

The figure tells us that when the strike increases the adaptive method sees a decrease in variance while the other increase for $K \in [80, 100]$ although still being significantly than the naive. like the prior plot both the pre-reduced and the adaptive have almost identical values of variance while the adaptive methods still has the more stabile variance. In order to understand the difference in the computation of the control variate based methods we plot the computations of λ wrt. differing strike prices. In the following plot we calculate the pre reduced where λ is computed with 5% of the simulations on the left and 20% on the right:



Figur 3: λ wrt. differing strike prices

We observe that the regression coefficients follow eachother, with the pre reduced having bigger fluctuations than the adaptive. Although these fluctuations get smaller when we compute λ_{start} with 20% of the simulations in comparison to 5%. All in all we observe how the adaptive method shows to be the most accurate when varying the strike prices. Although having the highest computational cost.

We now introduce the Kemna Vorst pseudo control variate.

$$k_T^{KV} := \varphi\left(x \cdot e^{-\left(\frac{r}{2}+\frac{\sigma^2}{12}\right)T} \cdot \exp\left(r - \frac{1}{2}\sigma^2 - \frac{\sigma^2}{3}\right)T + \sigma \cdot \frac{1}{T}\int_0^T W_t\,dt\right)$$

We calculate this via the Black Scholes framework. The Brownian Motion integrals are approximated by.

$$\frac{1}{T}\int_0^T W_s ds \approx \frac{1}{N}\sum_{i=0}^{N} W_{t_i}$$

And the payoff is calculated with

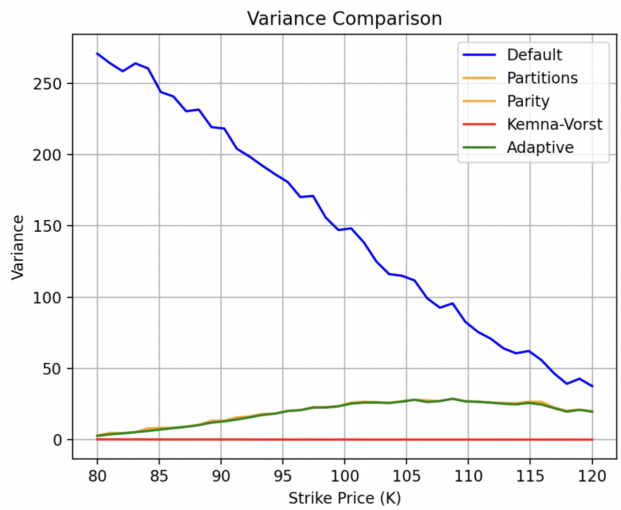
$$X_{\mathrm{KV}} = x_{\mathrm{adjusted}} \cdot \exp\left(\sigma_{\mathrm{KV}} \cdot W_T\right), \quad \sigma_{\mathrm{KV}} = \frac{\sigma}{\sqrt{3}}, \quad x_{\mathrm{adjusted}} = x \cdot \exp\left((r - \frac{\sigma^2}{6})T\right).$$

$$\text{Payoff} = e^{-rT}\left(X_{\text{KV}} \cdot N(d_1) - K \cdot N(d_2)\right),$$

where $N(\cdot)$ is the cumulative distribution function of the standard normal distribution and.

$$d_1 = \frac{\log\left(\frac{X_{\text{KV}}}{K}\right) + \left(r + 0.5\sigma_{\text{KV}}^2\right)T}{\sigma_{\text{KV}}\sqrt{T}},$$
$$d_2 = d_1 - \sigma_{\text{KV}}\sqrt{T}.$$

We now view how the Kemna Vorst method compares to the three prior by plotting the variance wrt. different strike prices:



Figur 4: Varaiance comparison with changing stikes including Kemna Vorst

We see that for options with $K > 85$ Kemna Vorst comes out with significantly lower variance than three other methods earlier discussed. Although for $K < 85$ both the partitions and the adaptive method has a variance so close to the Kemna Vorst. Next we introduce the Heston model:

$$dS_t = (r - 0.5v_t)S_t dt + \sqrt{v_t}S_t dW_t^S,$$
$$dv_t = k(a - v_t)dt + \theta\sqrt{v_t}dW_t^v,$$

Where we set the parameters as in nummerical probability: $k = 2.0$, $a = 0.01$, $v_0 = 0.1$, $\theta = 0.2$, $\rho = 0.5$.

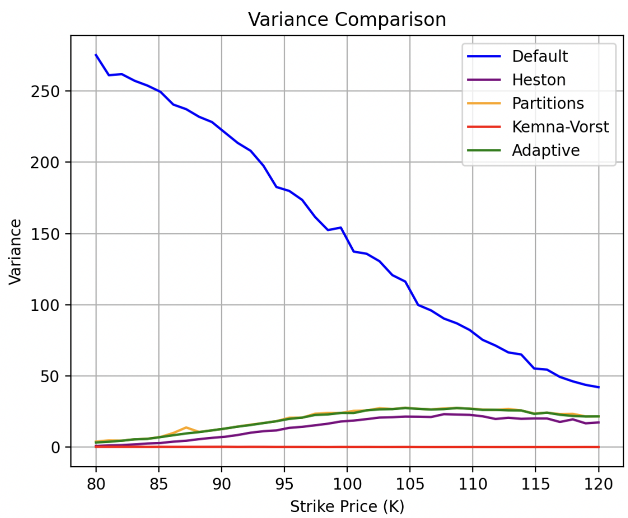
We simulate the correlated Brownian motions:

$$dW_t^S = \sqrt{\Delta t} \cdot Z_1,$$
$$dW_t^v = \rho\sqrt{\Delta t} \cdot Z_1 + \sqrt{1-\rho^2}\sqrt{\Delta t} \cdot Z_2.$$

At each time step t, the variance and asset price are updated and the option price is calculated with the arithmetic average like in the other methods. We use the price as a control variate to compute the option price. The adjusted payoff is calculated as:

$$X_{\text{cv}} = X - \lambda \left(X_{Heston} - \mathbb{E}(X_{Heston}) \right), \tag{1}$$

where lambda is calculated with the same method as in the partitions method. With these parameters we plot the variance pf the estimator in comparison to the others with varying strikes:



Figur 5: Varaiance comparison with changing stikes including Heston

We see from the plot that the Heston model with the above parameters has variance that changes parallel to the partitions and adaptive method when changing the strike price. Although having a slightly lower variance.

Finally we take a look at how the methods price the option for different strikes to get an idea of how the methods differ in the way they price the option. Here we plot the Asian option price in comparison to the strike price for the option.

Figur 6: Option pricing for varying strikes

This plot tells us that the Kemna Vorst and Heston method has a tendency to price the options closer to the naive method hence having higher prices than the partitions and adaptive method. This gap in pricing seems to be increasing when the options ate deep in the money while the gap gets smaller as the strike price increases.