Nummerical Probability

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Project 1

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Introduction

We wish to estimate Φ' i.e. the δ hedge in the framework of the Black Scholes model via 4 different methods (cf. Nummerical Probablity 3.2 and lecture 30/9):

$$\delta_1 = \varphi'(X_T^x, K, r, T) \cdot \frac{X_T^x}{x}$$

$$\delta_2 = e^{rT} \cdot \varphi'\left(x \cdot e^{rT}, K, r, T\right) + \left[\varphi'(X_T^x, K, r, T) - \varphi'\left(x \cdot e^{rT}, K, r, T\right)\right] \cdot \frac{X_T^x}{x}$$

$$\delta_3 = \phi(X_T^x, K, r, T) \cdot \frac{W_T}{x \cdot \sigma \cdot T}$$

$$\delta_4 = \left[\phi(X_T^x, K, r, T) - \phi\left(x \cdot e^{rT}, K, r, T\right)\right] \cdot \frac{W_T}{x \cdot \sigma \cdot T}$$

More specifically we want to estimate δ for call and put options and compare variances, confidence intervals and convergence for the four methods.

The framework

We will estimate the deltas use the following conventions:

$$X_T = x \cdot \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T\right),$$

where:

- x: Initial price of the asset,
- r: Risk-free interest rate,
- σ: Volatility of the asset,
- W_T: Standard Brownian motion at time T,
- T: Time to maturity.

The discounted payoff for a European call option is defined as:

$$\phi(X_T, K, r, T) = e^{-rT} \max(X_T - K, 0),$$

and for a European put option:

$$\phi(X_T, K, r, T) = e^{-rT} \max(K - X_T, 0),$$

where:

- K: Strike price of the option,
- r: Risk-free interest rate,
- T: Time to maturity.

The derivative of the payoff function with respect to the underlying price X_T is given by:

$$\varphi'(X_T, K, r, T) = \begin{cases} e^{-rT}, & \text{if } X_T > K, \\ 0, & \text{if } X_T \leq K. \end{cases}$$

For a put option:

$$\varphi'(X_T, K, r, T) = \begin{cases} -e^{-rT}, & \text{if } X_T < K, \\ 0, & \text{if } X_T \geq K. \end{cases}$$

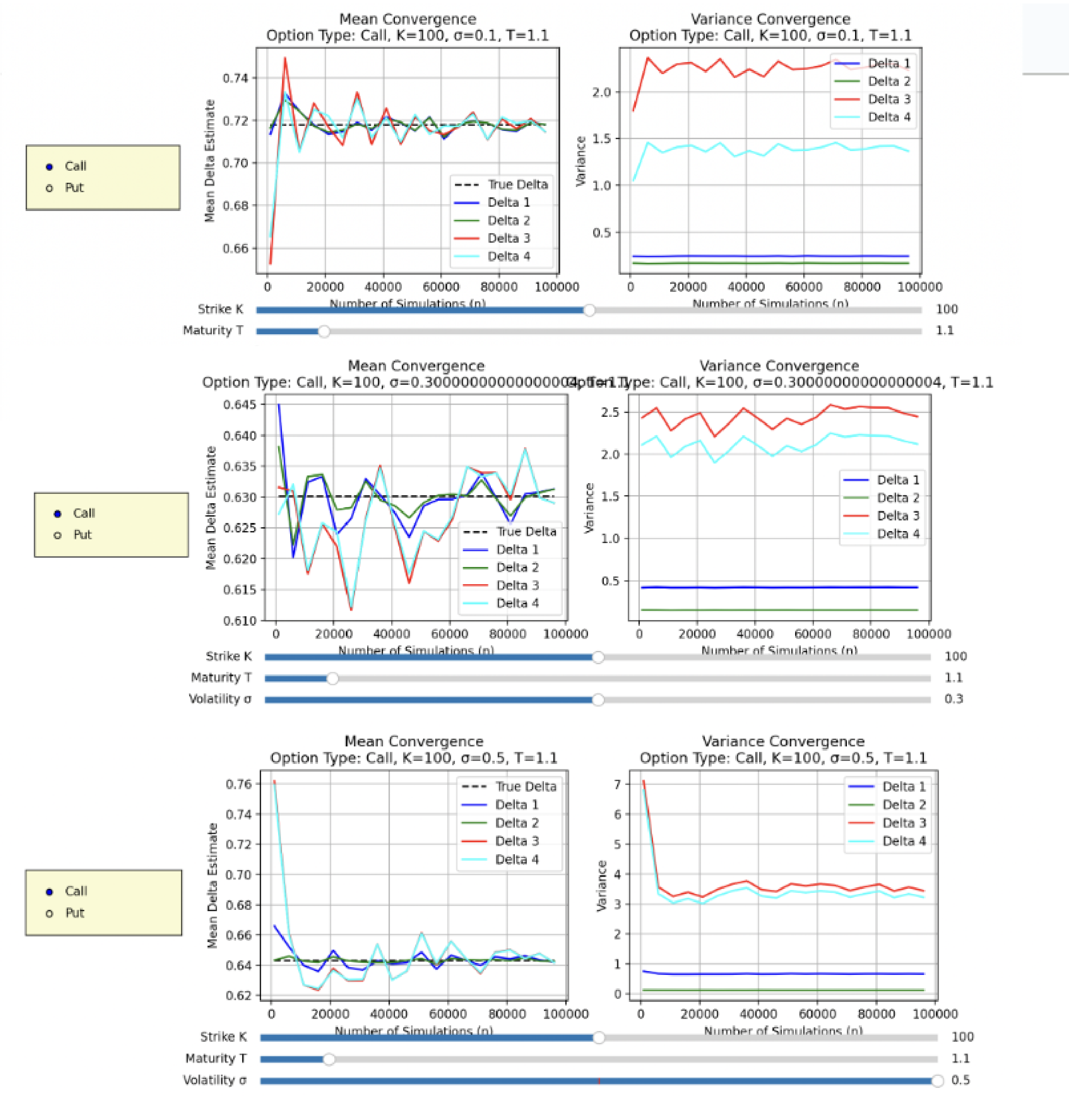
A call option

First we consider a call option. To get an understanding of the strength and weaknesses of the estimators we visualize the convergence of the mean and variance of each estimator over the number of simulations and how the estimators react to changes in strike, volatility and time to maturity.

Figur 1: Mean and variance convergence of a call with varying strikes

The illustration shows us that for a call estimator 3 and 4 have a slower rate of convergence and higher variance compared to estimator 1 and 2. Furthermore we observe a big decrease in the variance as the strike price rises. Another interesting observation is how that delta 3 has a lot higher variance than delta 4 when the strike price is low.

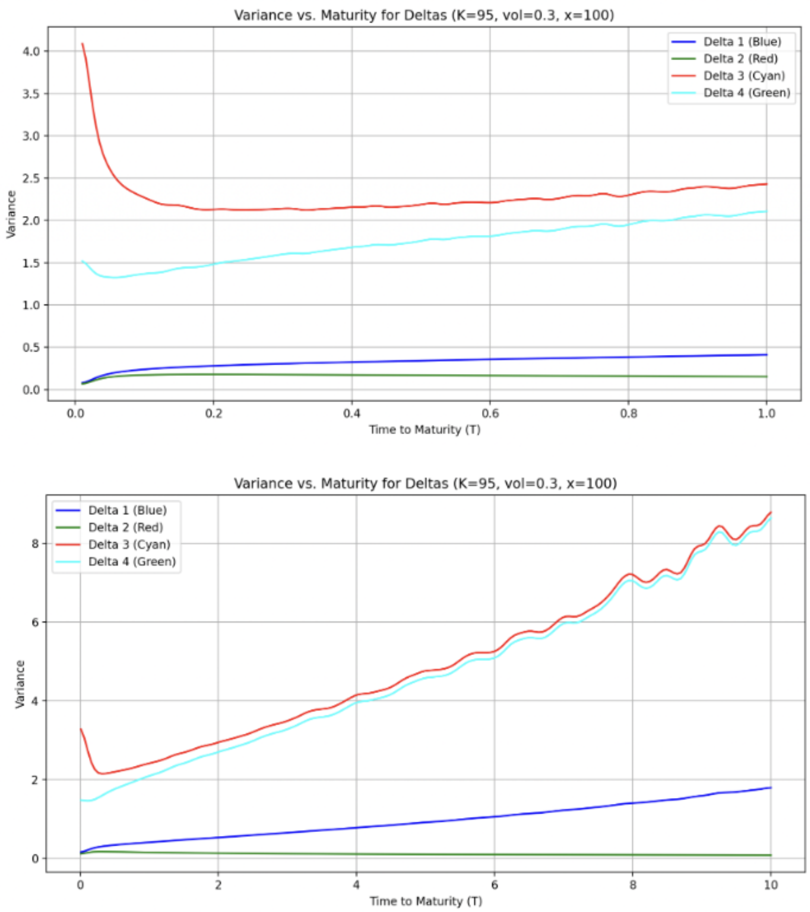
Next let's look at how the estimators react to changes in the volatility:



Figur 2: Mean and variance convergence of a call with varying volatility

Changing the volatility shows us first of all how the variance increases with higher volatility, while estimator 3 and 4 get more similar when volatility rises since the control variate used in estimator 4 gets smaller when volatility increases, while estimator 1 has a significant increase in variance in comparison to estimator 2 which is the least sensible estimator wrt. volatility changes.

Now let's turn our heads to the effect of "time to maturity". We wish to illustrate the effect of T with a plot of the variance over a continuous interval of maturities. To paint a nuanced picture of the effect of T we show a plot of the variance development for short maturities and one for longer maturities (we set $r = 0$ for simplicity):

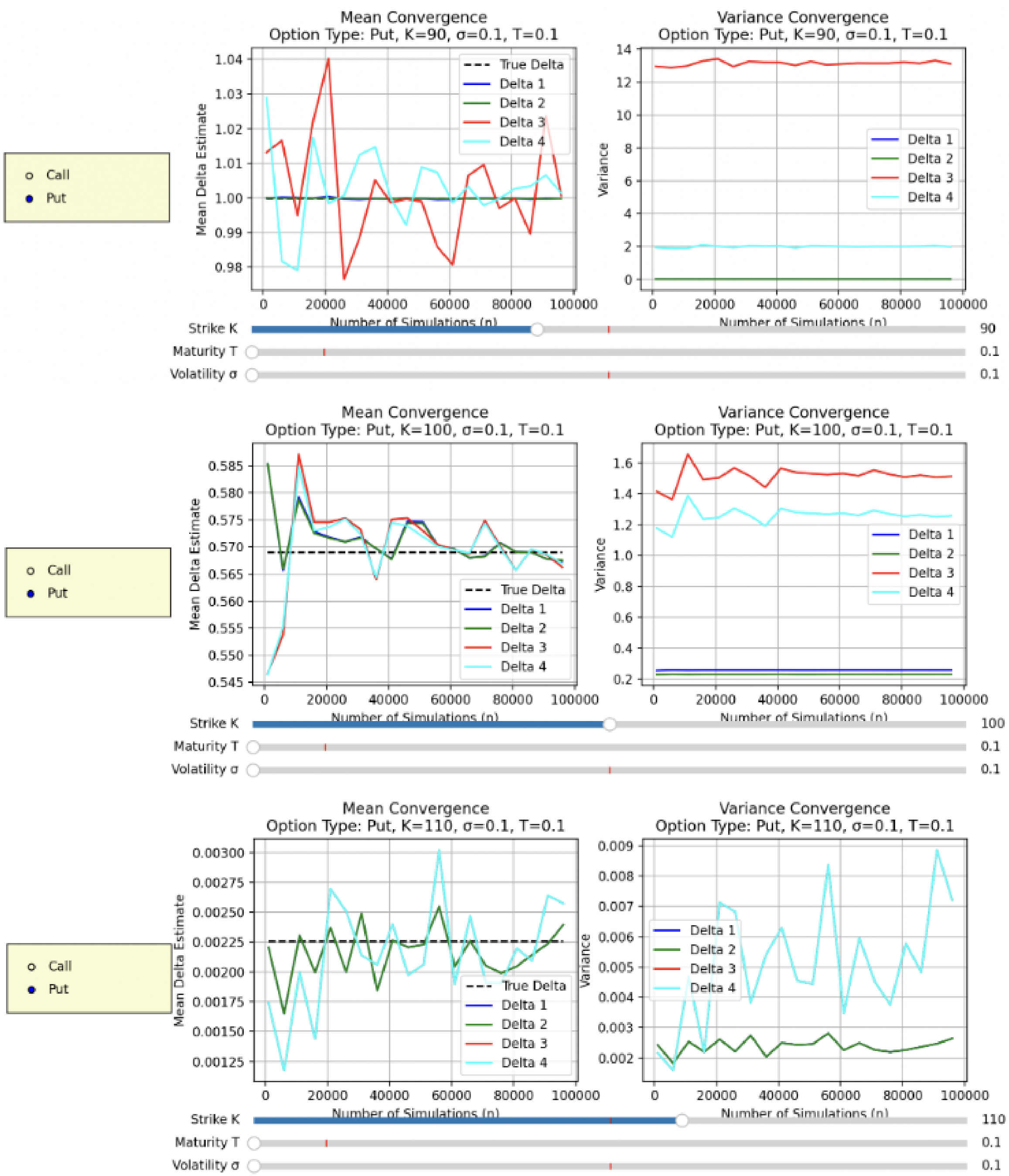


Figur 3: Variance of a call wrt. T

From this figure we see for very short maturities estimator 3 and 4 have high variances. Although when $T \in [0.1, 1]$ estimator 3 and 4 have rather low variances. Although when we observe the estimators outside of this interval the variance of estimator 3 and 4 grows significantly faster than estimator 1 and 2.

A put option

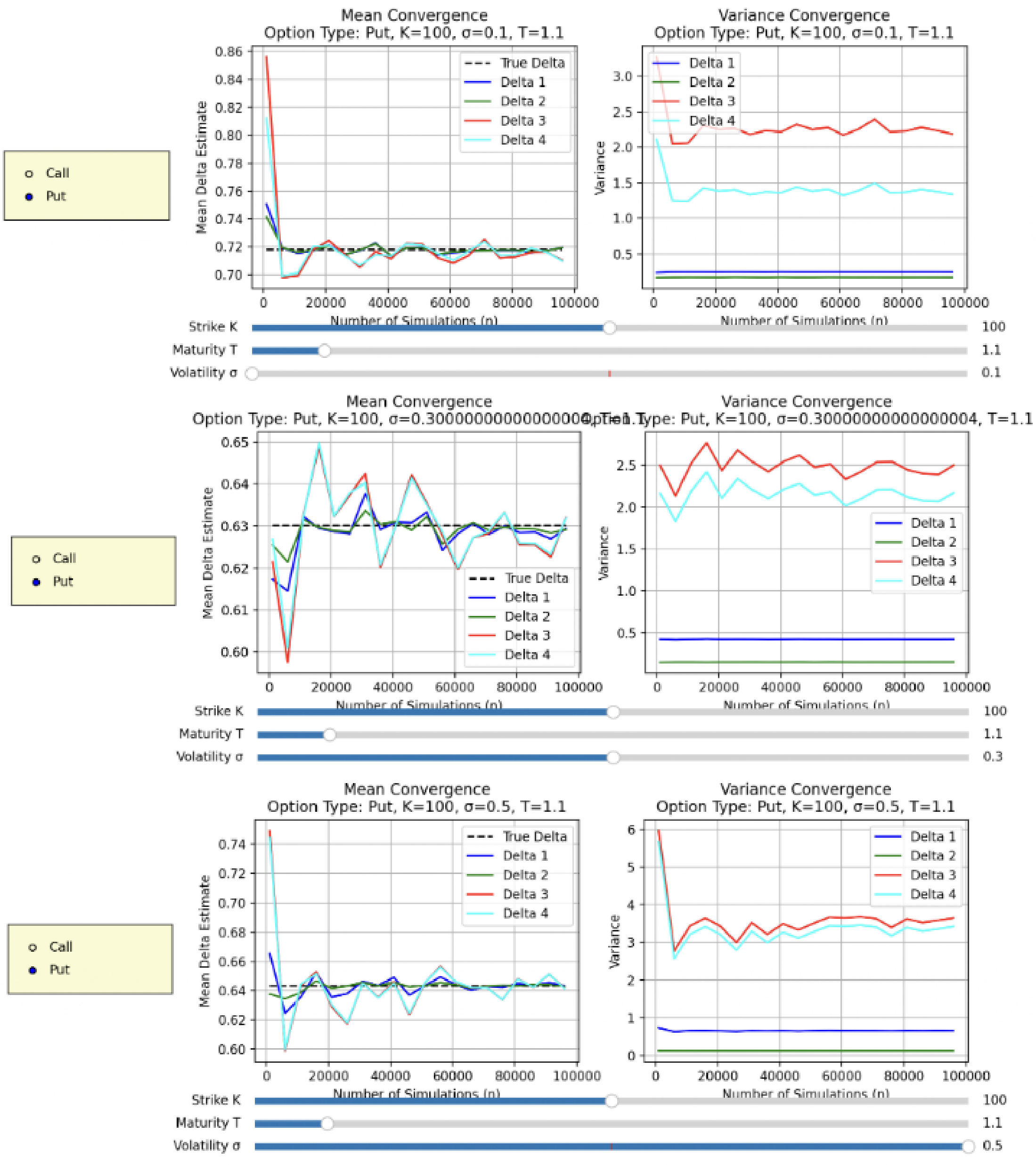
We now wish to investigate how the delta estimators react to changes in strike, volatility and time to maturity when we consider a put option. To do so we proceed the same way as we did with the call and start with visualizing how a put option reacts to changes in the strike:



Figur 4: Mean and variance convergence of a put with varying strikes

Here we see the same patters as with the call option where the variance of estimator 3 and 4 decrease as the strike price increases. Especially estimator 3 has a big reduction in variance as the strike price rises. While estimator 2 sees a slight increase in variance when the strike price moves from 90 to 100.

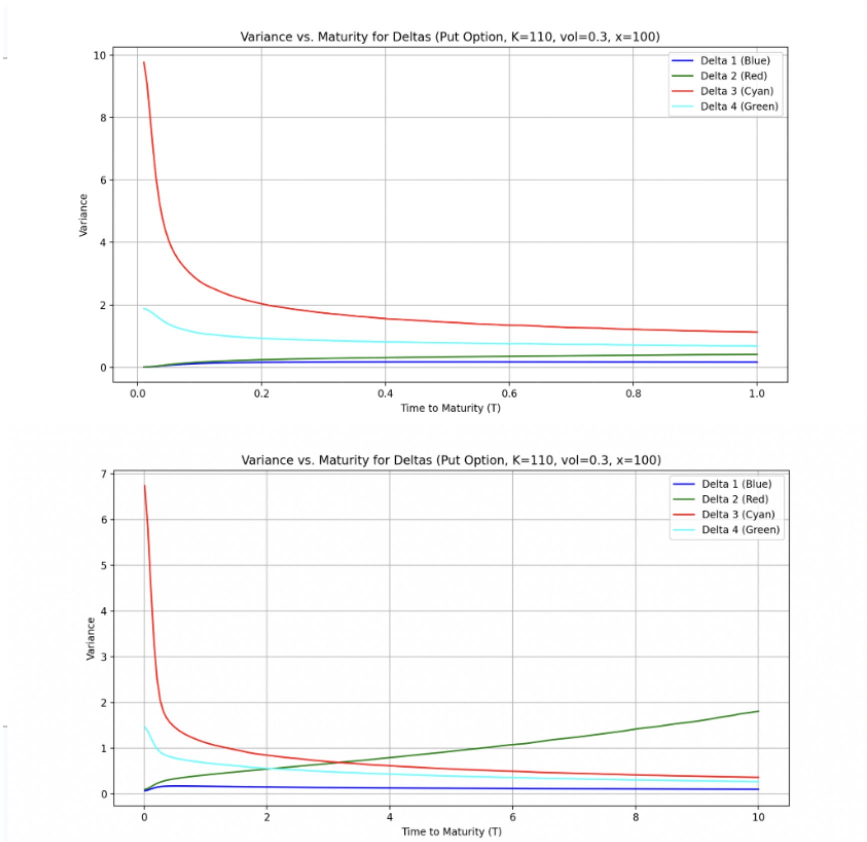
Next we look at how the estimators react to volatility changes:



Figur 5: Mean and variance convergence of a put with varying Volatility

From the figure we see as the volatility rises the variance of all estimators go up, with estimator

4 being the most sensible to changes in volatility and therefore seeing the biggest increase as the control variate gets smaller as volatility increases. Furthermore we note that estimator 1 has a greater increase in variance compared to estimator 2 when volatility increases. Lastly we plot the variance of the estimators over a continuous interval of maturities like in the case with a call option (we again set $r = 0$):



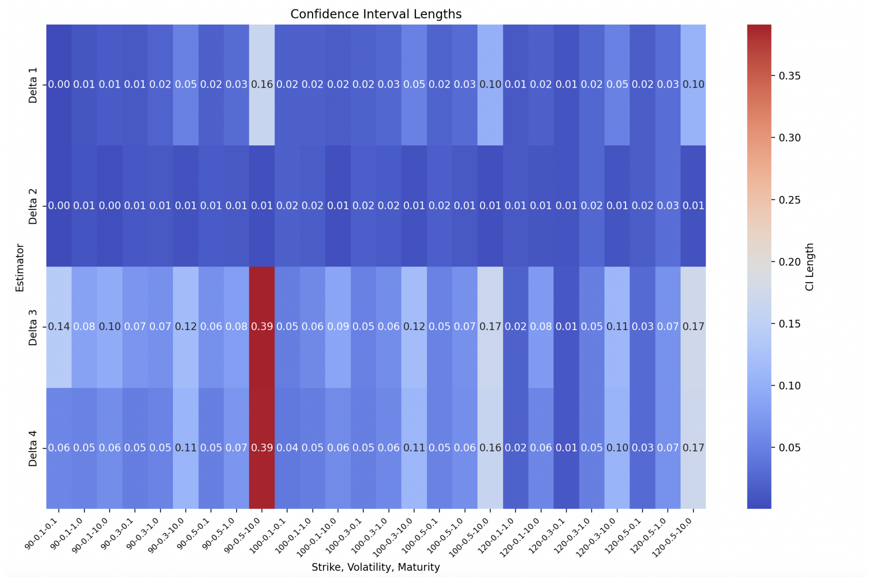
Figur 6: Variance of a put wrt. T

We see that for short maturities estimator 1 and 2 have the lowest variance, but at the maturity of the put gets higher estimator 3 and 4 get lower variance while estimator 2 sees an increase in variance.

To map out the strengths and weaknesses of each estimator we can create a table with length of confidence intervals for each estimator for different values of strike, volatility and maturity. We calculate an $\alpha = 0,95\%$ confidene interval according to

$$I_M^{\alpha} = \left[\bar{X}_M - q_{\alpha}\sqrt{\frac{\bar{V}_M}{M}}, \bar{X}_M + q_{\alpha}\sqrt{\frac{\bar{V}_M}{M}} \right], \quad \textit{cf. p 32 Nummerical Probability}$$

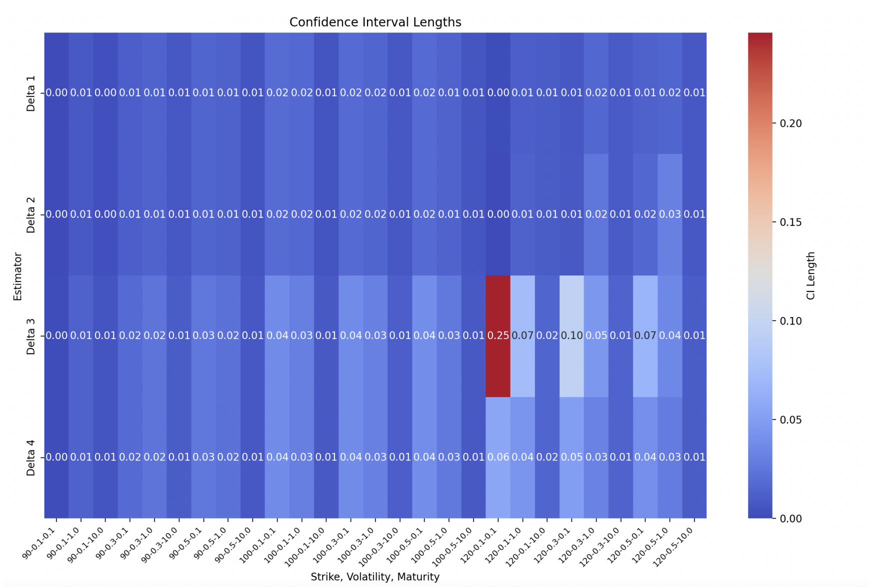
First we look at the confidence intervals for a call option:



Figur 7: Length of confidence intervals for a call with different values of strike, Volatility and time to maturity

From the plot we quickly see that estimator 1 and 2 give us more a accurate result for the delta in comparison to 3 and 4. Especially when the time to maturity and volatility is high estimator 1, 3 and 4 have significantly bigger confidence intervals than delta 2.

Next we look at a put option



Figur 8: Length of confidence intervals for a put with different values of strike, Volatility and time to maturity

Here we see that all estimators give a tight confidence interval for options with higher maturities, while estimator 3 and 4 give larger confidence intervals for options with shorter maturities.

The biggest weakness