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Peter M. Neumann

The mathematical writings of Évariste Galois

European Mathematical Society

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À mes chères amies

Jackie Stedall,

Sabine Rommevaux,

la Ville de Paris,

et

la Bibliothèque de l’Institut de France

A family sketch of Évariste Galois aged 15.
First published in 1896 by Paul Dupuy.

Preface

Before he died aged twenty, shot in a mysterious early-morning duel at the end of May 1832, Évariste Galois created mathematics which changed the direction of algebra. His revolutionary ideas date from around May 1829 to June 1830, the twelve to thirteen months surrounding his eighteenth birthday. An article published in June 1830 created the theory of Galois imaginaries, a fore-runner of what are now known as finite fields; his so-called *Premier Mémoire* created group theory and Galois Theory—the modern version of the theory of equations. The *Lettre testamentaire*, the letter that he wrote to his friend Auguste Chevalier on 29 May 1832, the eve of the duel, is an extraordinary summary of what he had achieved and what he might have achieved had he lived to develop and expound more of his mathematical ideas.

Although there have been several French editions of his writings, there has never until now been a systematic English translation. Translations of historical material are of little use without the originals alongside, however. What is offered here therefore is a bilingual edition. The French transcription is a new one. Following precedents set by Tannery in 1906/07 and by Bourgne & Azra in 1962 it is as close to the original manuscripts as I have been able to make it. Main text, afterthoughts, deletions, insertions, over-writings—all are recorded as faithfully as I could manage within the inevitable constraints imposed by the differences between manuscript and print.

In addition I offer three levels of commentary. First there is general contextual information; secondly there are notes on the physical state of the manuscripts and on the disposition of their content; third, there are comparisons of the various previous editions, including variant readings, in minutely pedantic and minutely printed marginal notes. Little of the commentary here is mathematical. It is focussed on the symbols on the page, on the syntax, on establishing an accurate text. Commentaries on the semantics, the meaning of what Galois wrote, would be a quite different exercise. That comes next, but must be the subject of other studies. I have neither the space nor the time. Space is a concern because the book is already substantially longer than I had anticipated in light of the shortness of Galois' productive life. Time is short because a proper modern study of his writings would take years, whereas it is planned that this book should appear on 25 October 2011 as homage to Galois on the 200th anniversary of his birth.

The book is conceived as a contribution to the history of mathematics. I hope, however, that it may bring the mathematical writings of this extraordinary genius to a wider mathematical public than has hitherto been able to appreciate them. At the very least it may serve to dispel some of the common myths that surround Galois and his understanding of mathematics. It is simply not true, for example, that he proved and used the simplicity of alternating groups. He did not need to: he was much cleverer than that; his treatment of solubility of equations is at once simpler and more elegant than what has now become textbook tradition. The details of what he did, the proper evidence of his genius, deserve to be as well understood and appreciated amongst mathematicians as amongst historians of mathematics. If this

edition extends his readership beyond the bounds presently imposed by linguistic constraints it will have succeeded.

Acknowledgements

It is a pleasure to be able to publish my very warm thanks to a large number of people without whose help and advice this book would have been greatly the poorer. First come Mme Mireille Pastoureau, Director of the Library of the Institut de France, and her staff. They have all accorded me the highest level of kindness and assistance. If I pick out two people, Mme Fabienne Queyroux and Mme Annie Chassagne, as having earned my special thanks for the many times they have given me special help, I hope it will not detract from my thanks to all their colleagues. I thank also the Commission des bibliothèques et archives de l'Institut de France, and its president, Mme Hélène Carrère d'Encausse, Secrétaire perpétuel de l'Académie française, for kind permission to include images of some of the Galois manuscripts in this work; these thanks extend to Mme Florence Greffe, Archivist of the Académie de Sciences, who has made available an image of Galois' letter of 31 March 1831 and a pamphlet by Jacques Tits. Professor Jean-Pierre Kahane, a member of that committee, has been very supportive and it is a pleasure to record my personal thanks to him. Mr F. Xavier Labrador of the Société d'Ingénierie et de Microfilmage made those high quality photographs, and I am very grateful to him. I thank also Jonathan Crayford for photographs and much enthusiasm about Galois and his work. The Governing Body of the Queen's College, Oxford, and the Research Committee of the Mathematical Institute in the University of Oxford provided financial support towards the purchase of digital images of the manuscripts and I thank them also.

Several colleagues read an early draft of the book, or parts of it, and I have benefitted from their suggestions. Two referees, Massimo Galuzzi and Caroline Ehrhardt, sent pertinent and very helpful criticisms. It is a great pleasure to be able to thank them publicly, not only for their kind comments and help, but also for agreeing to waive the conventional anonymity that usually prevents such thanks and acknowledgment. Jackie Stedall and Sabine Rommevaux also checked the whole work for me—they know already how grateful I am. I am grateful also to Catherine Goldstein and Adrian Rice for bibliographical advice, and to Tessa Shaw and Lynette Dobson of the Queen's College library for detailed and highly skilled bibliographical assistance. Professor Roger Pearson, FBA, Tutor in French at The Queen's College, offered help with the mottos and some other mysterious constructions. I am not the first, nor will I be the last, to be grateful for his great scholarship and tutorial skills.

Throughout the whole project my editors Manfred Karbe and his wife Irene Zimmermann have provided an extraordinary level of support and encouragement, freely and patiently sharing with me their technical and TEXnical expertise, experience and wisdom. One could not possibly have a better editorial team.

ΠMN Oxford, September 2011

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Chapter I
Introduction

I.1 Évariste Galois 1811–1832, revolutionary mathematician

Much has been written about Évariste Galois, who died aged 20, shot in a mysterious early-morning duel in May 1832. His extraordinary mathematical intuition and his extraordinary life have, since 1846 or 1848, attracted great publicity far beyond the mathematical world. Roughly speaking, his mathematical intuition, once it was understood, changed the theory of equations from its classical form into what is now universally known as Galois Theory, together with its associated 'abstract algebra', including the theory of groups and fields. The essentials of his short life were captured in three words on a stamp issued in France in 1984, 'révolutionnaire et géomètre'; they are taken from the title of a biographical work by a famous writer, André Dalmas (1909–1989) [Dalmas (1956/82)].

Évariste Galois: révolutionnaire et géomètre.

Over the years Galois' life has been the subject of many studies, historical, biographical, fictional, dramatic, even musical: as I have said, much has been written about him. This book, however, focusses on his mathematical work and is no place to repeat it. All I offer, for contextual purposes, is a brief *cv*:

25 October 1811: Évariste Galois born in Bourg-la-Reine, about 10km south of the centre of Paris. The second of three children born to Nicolas-Gabriel Galois and his wife Adelaïde-Marie (née Demante), his sister Nathaly-Théodore was two years older, his brother Alfred nearly three years younger.

6 October 1823: entered the Collège Louis-le-Grand. His six-year stay there started well but ended badly.

August 1828: failed to gain entrance to the École Polytechnique.

April 1829: had his first article (on continued fractions) published in Gergonne's *Annales de Mathématiques*.

25 May and 1 June 1829: submitted, through Cauchy, a pair of articles containing algebraic researches to the Académie des Sciences of the Institut de France (see [Acad (1828–31), pp. 253, 257–258]). Poinsot and Cauchy were nominated as referees. The manuscripts are now lost; in [Taton(1971)] René Taton published evidence that Galois very probably withdrew them in January 1830.

2 July 1829: suicide of Évariste's father, Nicolas-Gabriel Galois.

July or August 1829: second and final failure to gain entrance to the École Polytechnique.

November 1829: entered the École Préparatoire (as the École Normale Supérieure was briefly called at that time).

Late February 1830 (probably): re-submitted his work on equations to the Académie des Sciences in competition for the Grand Prix de Mathématiques. I have not found a record in [Acad (1828–31)], but [Taton(1971), p. 137, fn 33] includes reference to a list in the archives of the Academy that contains the name of Galois. The manuscript was lost in the academy. The prize was awarded jointly to Abel (posthumously) and Jacobi for their work on elliptic functions.

April–June 1830: had three items published in Férussac's *Bulletin*.

December 1830: another item published in Gergonne's *Annales*.

4 January 1831: official confirmation of his provisional expulsion from the École Préparatoire in December 1830.

17 January 1831: submitted his 'Mémoire sur les conditions de résolubilité des équations par radicaux', now often called the *Premier Mémoire*, to the Académie des Sciences (see [Acad (1828–31), p. 566], where, however, his name is recorded as 'Le Gallois'). It was given to Lacroix and Poisson to be examined.

10 May 1831: arrested for offensive political behaviour; acquitted and released on 15 June 1831.

4 July 1831: Poisson, on behalf of Lacroix and himself, reported back negatively on the 'Mémoire sur les conditions de résolubilité des équations par radicaux': see [Acad (1828–31), pp. 660–661]; see also Note 2 to Chapter IV on p. 146 below.

14 July 1831: arrested on the Pont-neuf during a Bastille Day republican demonstration. Held in the Sainte-Pélagie prison.

23 October 1831: convicted of carrying fire-arms and wearing a banned uniform; sentenced to six months further imprisonment.

16 March 1832: released from Sainte-Pélagie prison during an outbreak of cholera in Paris and sent to live in the 'maison de santé du Sieur Faultrier', a sort of private hospital or asylum, a safe house.

Late May 1832: mysteriously engaged to duel. There is little evidence and much contradictory conjecture as to by whom and about what.

29 May 1832: wrote his *Lettre testamentaire* addressed to his friend Auguste Chevalier and revised some of his manuscripts.

30–31 May 1832: shot in an early-morning duel; died a day later in the Côchin hospital in Paris.

For more extensive accounts of his life and relations with other people see [Chevalier (1832b)], [Anon (1848)], [Dupuy (1896)], [Taton (1947)], [Taton(1971)], [Taton (1983)], [Rigatelli (1996)]. Authors do not always agree on details. For example, the first reverse at the École Polytechnique is given as June 1828 in [Toti Rigatelli (1996), p. 34], but as August 1828 by René Taton in [APMEP (1982), p. 5]. Of the second reverse [B & A (1962), p. xxviii] has it as having taken place a few days after the suicide of Galois' father, whereas [Taton(1971), p. 130] has it a few weeks later, and [Chevalier (1832b), p. 746] dates it to the end of 1829. And [B & A (1962), p. xxviii], [Taton (1947a), p. 117] have Galois entering the École Préparatoire in October 1829, whereas [Taton(1971), p. 131] has him entering in November 1829.

Évariste Galois , mort âgé de vingt et un ans , en 1832. – Ce portrait reproduit aussi exactement que possible l'expression de la figure d'Évariste Galois. Le dessin est dû à M. Alfred Galois, qui depuis seize ans a voué un véritable culte à la mémoire de son malheureux frère.

Sketch by Alfred Galois, 1848 (see p. 389).

Galois died too young to leave much evidence about his life and career, and some of what does survive is contradictory. To take a trivial example, I have often seen authors claim that Galois died aged twenty-one. As it happens, that error is an early one. Even the contemporary death certificate and autopsy report (see [Dupuy (1896),

pp. 264–266]) make that mistake, as does the caption to the sketch published by his brother Alfred in [Anon (1848)]—although 'mort âgé de vingt et un ans, en 1832' [dead at the age of twenty-one years in 1832] could just be taken to be ambiguous and not wrong.

I would guess that the most reliable stories are those of Auguste Chevalier (1832b) and Paul Dupuy (1896). The former, though rather too heavily coloured by the sentiment proper to a close friend, and probably written mainly from memory without a great deal of checking against documents, is a contemporary account by an eye-witness. The latter, which forms the basis of most later accounts, though also coloured by sentiment, is a systematic study by a professional historian. These judgments remain, however, mere guesses on my part.

Évariste Galois, révolutionnaire et géomètre: the slogan is charmingly echoed in an ambiguity in the English title of his greatest work 'Memoir on the conditions for solubility of equations by radicals'. But in my estimation he was far more effective as a mathematician than as a revolutionary. As a *révolutionnaire* he seems a failure; in mathematics he was a *géomètre révolutionnaire*. It is the revolutionary mathematics that we celebrate in this book.

I.2 What Galois might have read

Although this book is devoted to establishing a text rather than to interpretation, there is some background on what Galois might have read that should be helpful to readers. In particular, there are references to other writers at various points in his mathematical writings, and since these are generally exiguous it may be worth surveying briefly what we know of his mathematical reading.

He seems to have first met mathematics in his fourth year at Louis-le-Grand, when he was fifteen. According to [Anon (1848)]:

> Il dévore les livres élémentaires; parmi ces livres, il y en a un, la Géométrie de Legendre, qui est l'œuvre d'un homme d'élite, qui renferme de beaux développements sur plusieurs hautes questions de mathématiques. Galois en poursuit la lecture jusqu'à ce que le sujet soit épuisé pour lui. Les traités d'algèbre élémentaire, dus à des auteurs médiocres, ne le satisfont pas, parce qu'il n'y trouve ni le cachet ni la marche des inventeurs; il a recours à Lagrange, et c'est dans les ouvrages classiques de ce grand homme, dans la *Résolution des équations numériques*, dans la *Théorie des fonctions analytiques*, dans les *Leçons sur le calcul des fonctions*, qu'il fait son éducation algébrique.
>
> [He devours the elementary books; among these books there is one, Legendre's Geometry, the work of a distinguished man, which contains

> beautiful developments on several deep questions in (higher) mathematics. Galois takes his reading of it to the point where the subject is exhausted for him. The elementary treatises on algebra, by mediocre authors, do not satisfy him because he finds in them neither the authority nor the steps of the discoverers; he turns to Lagrange and it is in the classic works of this great man, in the *Résolution des équations numériques*, in the *Théorie des fonctions analytiques*, in the *Leçons sur le calcul des fonctions*, that he acquires his algebraic education.]

In his biographical study Dupuy made a similar report [Dupuy (1896), p. 206], though most of it relies on this passage. The works referred to here are, presumably, one or other of the editions of the *Éléments de géométrie* [Legendre (1823/1799)], the *Résolution des équations numériques* [Lagrange (1798)], the *Théorie des fonctions analytiques* [Lagrange (1796/1813)], the *Leçons sur le calcul des fonctions* [Lagrange (1803)]. That Galois had read and understood this last book is confirmed by a reference to it in one of the manuscripts (**f. 189 b**, [B & A (1962), p. 413]), where he criticises and seeks to correct it.

It seems likely, as suggested in [Ehrhardt (2010a), p. 95], that Galois acquired some of his knowledge of the theory of equations from standard textbooks of the time such as the two books by Lacroix, *Élémens d'algèbre* and *Complément des Élémens d'algèbre* [Lacroix (1799)], [Lacroix (1800/1835)]; another such (see [Dhombres (1984)], [Dhombres (1985)]) was Bezout's widely read *Cours de mathématiques* [Bezout (1820/1770)]. It is not unreasonable to believe that he would also have read, or at least dipped into, such classics as the *Élémens d'Algèbre* of Clairaut [Clairaut (1801/1740)] and of Euler [Euler (1807)]. His deeper knowledge will probably have come, though, from the second or third editions of such monographs as [Lagrange (1798)] cited above on equations or Legendre's *Théorie des Nombres* [Legendre (1798/1808)] (with its supplement (1816) on numerical solution of equations), from Gauss's *Disquisitiones arithmeticae* (1801), or from Cauchy's *Cours d'analyse* (1821). Presumably he also read the issues of the main journals as they came out: Gergonne's *Annales* and Férussac's *Bulletin* in which he published; the publications of the Academy of Sciences; Crelle's *Journal*; Cauchy's *Exercices*; possibly also the publications of some of the foreign academies.

It would be interesting to know whether Galois had read the great and influential memoir 'Réflexions sur la résolution algébrique des équations' [Lagrange (1770/71)]. That is possible. But in his reaction to the note in which Poisson refers to it (see Ch. IV, Note 8, p. 153) Galois does not acknowledge the connection with his own work. I estimate it to be more likely that he acquired his deeper understanding of the algebraic solution of equations (that is to say, his understanding beyond what is to be found in texts such as those of Lacroix and Euler) from the précis of that memoir which is the content of Note XIII appended to the second and third editions of [Lagrange (1798)].

We have little detailed evidence, but it seems safe to conjecture that on elliptic functions he would surely have read Legendre's *Exercices de Calcul Intégral*

[Legendre (1811)] or his *Traité des fonctions elliptiques* [Legendre (1825–28)], and also the work of Jacobi and Abel. In respect of Jacobi there is a morsel of evidence to confirm this belief in a draft of a letter from Alfred Galois (see p. 388), though the evidential value of the passing reference there seems slight—would Alfred really have understood much of what his older brother studied? Is this claim that Évariste studied Jacobi's work deeply any more than a courtesy?

Unfortunately, because Galois had not been trained to cite his sources and, as shown in his writings, did not have an innate instinct to do so, we cannot know for certain what he had read. Nevertheless, there are some indications in what he left us. We can be sure, for example, that he had read the *Disquisitiones arithmeticae* [Gauss (1801)], not because he cites the work explicitly but because for the majority of his references to Gauss this is the only work that is relevant (see pp. 50, 62, 86, 108, 130, and see also [Neumann (2006)]).

I do not find it easy to estimate with any degree of precision what of Cauchy's writings Galois had read. We can be sure that he had read Cauchy's first article on substitutions [Cauchy (1815a)] both because it is cited explicitly at one point (p. 128) and because he discussed questions from it (see Dossier 15, p. 284). He used language from the article on substitutions and determinants [Cauchy (1815b)], but he might have got that from the *Cours d'Analyse* [Cauchy (1821), Ch. III, p. 73; Note IV, p. 521] (see Note 10 to Chapter VI, p. 296 below). But of course, having read [Cauchy (1815a)] he might naturally have let his eye stray to [Cauchy (1815b)].

There is a very interesting question as to what Galois had read of Abel's work. This will be treated briefly in Note 6 to Dossier 10 (p. 242 below).

I.3 The manuscripts

Soon after his death the manuscripts that Galois left on his desk came into the hands of his friend Auguste Chevalier, who made copies of a number of them. Some time in the summer of 1843 Chevalier gave them to Joseph Liouville (1809–1882) (see [Liouville (1843)]), who left them (included in his library of books and papers) to his son-in-law Célestin de Blignières (1823–1905). They were sorted by Mme de Blignières, daughter of Liouville and widow of de Blignières, and given to the Académie des Sciences in 1905 or 1906 (see [Tannery (1906), p. 226]). They are organised into 25 dossiers bound into one volume catalogued as Ms 2108 in the library of the Institut de France. High quality facsimiles of some of the pages have been published in, for example, [B & A (1962)] and [APMEP (1982)].

Almost all the manuscripts were written in ink, though a very few of the pages in Dossier 24 contain material in pencil. They are now very fragile and, unless one has special privileges, what one used until recently was a microfilm copy or a two-volume printout made from it. Through the efforts of Mme Sylvie Biet, Mme Annie Chassagne, and others in the library, however, very good digital images made by

Mr F. Xavier Labrador have, since 15 June 2011, been mounted on the web. They are accessible from the page listed at [Galois (2011)] (see p. 392), which makes the whole of Ms 2108 publicly available for the first time.

Each dossier has a cover sheet. The organisation into dossiers was probably done by Mme de Blignières—if such a conjecture does not read too much into Tannery's words [Tannery (1906), p. 226] "Mme de Blignières s'occupe pieusement de classer les innombrables papiers de son mari et son illustre père" [Mme de Blignières is piously busy classifying the innumerable papers of her husband and her illustrious father]. The cover sheets, however, seem to have been annotated by the librarian or perhaps by Jules Tannery himself in or after 1908. Each has a brief description of the contents forming a sort of title, together with a page-references to the 1897 Picard edition of the main works or to the 1906/07 paper by Tannery and its reissue in 1908 as a book.

The cover sheet of Dossier 1 (the First Memoir) exemplifies this, but also carries an extra explanatory note. It is inscribed as follows:

Mémoire sur les
Conditions de résolubilité des équations par radicaux.
(Oeuvres, p. 33)
(Texte autographe du Mémoire présenté à l'Académie)

Below that header and on the left of the page, as if intended as a marginal comment, is the following 9-line note:

Les renvois aux "Oeuvres"
se rapportent à l'Edition
des "Oeuvres Mathématiques
d'Evariste Galois" publiée
par la Société Mathématique
Gauthier-Villars 1907.
Les renvois marqués (M.) se
rapportent aux "Manuscrits
de Evariste Galois" Gauthier-Villars 1908.
J. T.

[The references to the "Oeuvres" refer to the edition of the mathematical works of Evariste Galois published by the Mathematical Society, Gauthier-Villars 1907. The references marked (M.) refer to the "Manuscrits de Evariste Galois" Gauthier-Villars 1908. J. T.]

The reading of the second initial here is uncertain: my reading of it as T reflects a conjecture that the cover-sheet was provided (or at least, annotated) by Jules Tannery. Whether or not that is correct, whoever it was, the writer seems to have been sufficiently familiar with the editions of Galois' works and manuscripts to have become a little blasé—to the point where he or she did not feel the need to check,

and has quoted the references somewhat imprecisely. The relevant edition of the "Œuvres Mathématiques d'Évariste Galois" is [Picard (1897)], and the reference to the manuscripts is to [Tannery (1908)]. But perhaps it was not Jules Tannery at all—perhaps the explanation was written by someone only partially familiar with the editions of Galois' works, perhaps someone such as the librarian of the Institut de France who had early charge of the Galois manuscript material. In any event, it is clear that this marginal note refers to annotations on the cover sheets of the dossiers into which the manuscripts are organised.

I.4 Publication history of Galois' mathematical writings

Five of Galois' mathematical papers were published in his lifetime. The most important works, however, are posthumous. On 29 May 1832, the eve of the fatal duel, Galois wrote his famous *Lettre testamentaire* to his friend Auguste Chevalier, a letter that summarised the mathematics he was storing in his mind and also, in effect, asked (or commanded) Chevalier to act as his literary executor. This letter was duly published, as Galois had requested, in September 1832. Liouville published his highly influential edition [Liouville (1846)] of the main works as an article in his *Journal de Mathématiques pures et appliquées*, the journal that he had founded in 1835 as a successor to Gergonne's *Annales de Mathématiques pures et appliquées* and which he edited for many years. The most influential items are

- 'Sur la théorie des nombres' published in 1830 (see [Galois (1830c)]), in which Galois produced his theory of what used to be called 'Galois imaginaries', most of what later became the theory of finite fields;
- 'Mémoire sur les conditions de résolubilité des équations par radicaux', also known as the *Premier Mémoire*, the paper that was rejected by the Académie des Sciences on 4 July 1831 and returned to its author, but which gave us what we now call Galois Theory;
- 'Des équations primitives qui sont solubles par radicaux' also known as the *Second Mémoire*;
- the letter of 29 May 1832 to Auguste Chevalier known as the *Lettre testamentaire*.

The *Œuvres* as published by Liouville were reprinted and issued in book form by Picard in 1897 for the Société Mathématique de France.

As has already been mentioned, the *Lettre testamentaire* was published (at Galois' express request) by Chevalier in September 1832 in the *Revue Encyclopédique*. It has been reprinted in all editions of Galois' works since 1846. Almost certainly (see Ch. III, Note 2, p. 101) that first publication was done from a copy made by Chevalier which was kept by the printer: no copy in Chevalier's hand is still extant, whereas the original still exists as the contents of Dossier 2 of the Galois manuscripts.

Jules Tannery published a two-part paper in the *Bulletin des Sciences Mathématiques*, 1906 and 1907. It is devoted to a comparison of the 1897 edition with the manuscripts and to the physical description and publication of some (the majority, in fact) of the lesser manuscripts.

Then in 1962 Robert Bourgne and Jean-Pierre Azra produced their great critical edition [B & A (1962)] of the *Écrits et Mémoires Mathématiques d'Évariste Galois*. Everything is collected and re-published in one volume. The manuscripts are described; crossed-out material is deciphered; Galois' insertions and afterthoughts are recorded. A second edition was published in 1976. The changes were minimal, however: simply the addition of eight unpaginated leaves (two of them blank) inserted between pages *xvi* and *xvii*, containing an *errata* list and two tables of editorial information on pagination. Minimal they may be, but they are useful and respond well to points made by Taton in his review [Taton (1964)].

The various editions discussed here are listed at the beginning of the bibliography on p. 391.

In 1889 there was a translation of most of the Liouville edition [Liouville (1846)] into German [Maser (1889)]. Much more recently there have been Italian translations of the main works [Toti Rigatelli (2000)] and of the schoolwork [De Nuccio (2003)] (I have not seen these books and owe the references to Professor Massimo Galuzzi). Translations of a few items into English have appeared from time to time. For example [Smith (1929)] contains a translation of the *Lettre testamentaire* and [Edwards (1984)] contains a translation of the *Premier Mémoire*. There are also many snippets in various source-books, and the first half of the *Second Mémoire* is translated in [Neumann (2006)]. Taken altogether, however, only about one-third of the Galois *œuvre* has been available in English up until now; moreover that one-third is to be found in a variety of disparate sources.

That is the reason for the present new edition. An English edition is, however, of little use without the original alongside for direct comparison. I had originally planned simply to use [B & A (1962)] as the French version, but it soon became clear that that would not do. For one thing, Bourgne & Azra used the facing page, which is needed here for the translation, for editorial purposes. Without that editorial material the main text of Galois' writings becomes less valuable and it therefore had to be incorporated into the French transcription. For another, comparison with the manuscripts to clear up a few points in [B & A (1962)] that had puzzled me for some forty years led me to recognise that it was somewhat less perfect than I had always believed. Therefore my French edition is not simply a re-issue of the older one; it is a new transcription. It has been as carefully checked against both the old editions and the original manuscripts as I am capable of. This is very detailed work and we must accept, I am sorry to say, a high probability that I have made mistakes. I much hope, however, that there will turn out to be few of them.

I.5 The reception of Galois' ideas

A brief account of the reception of Galois' ideas should provide some more context. Extensive treatments are offered in [Kiernan (1971)] and [Ehrhardt (2007)]. Here is a mere summary of the highlights.

After the publication of Liouville's edition [Liouville (1846)] Galois' ideas, though not easily understood in those days, spread steadily through France, Italy and Germany. Liouville had already lectured on them, perhaps in the winter of 1843–44 (see [Bertrand (1899), p. 398], [Lützen (1990), p. 131]), and Serret had attended. In [Serret (1849), p. 344, footnote] he makes clear that he had not yet understood the ideas, but when he came to publish the much enlarged third edition [Serret (1866)] he was able to include a pretty full account of the material in the *Premier Mémoire*. (Vol. 2, pp. 413–420; 607–647). I would guess that his understanding had developed through conversations with his pupil Camille Jordan, who had come to terms with Galois' ideas in the early 1860s. See [Lützen (1990), pp. 129–132, 196–197] for an excellent account of Serret and his relationships with Liouville and with Galois Theory. Jordan's writings, [Jordan (1861), Supplément], [Jordan (1865)], [Jordan (1867)], [Jordan (1869)], and especially the great *Traité des substitutions et des équations algébriques* (1870), show that he had understood Galois' ideas to the level where he could develop them, as Serret had not. As I wrote in [Neumann (2006), p. 414]:

> The *Traité* is described by its author as being nothing but a commentary on the works of Galois "[...] les Œuvres de Galois, dont tout ceci n'est qu'un Commentaire" ['... the Works of Galois, of which all this is no more than a Commentary'] (see [Jordan (1870), p. viii]). Some commentary! It is 667 quarto pages.

Meanwhile, Betti had published several papers in Italy seeking to elucidate Galois' work, of which the main ones are [Betti (1851)], [Betti (1852)]. These are not entirely successful, and their shortcomings are analysed in [Mammone (1989)] (but see my review of this paper in *Mathematical Reviews* 1991, Review 91j:01026). In Germany Kronecker wrote a little about the theory of equations, focussing more on Abel's work than on that of Galois, though he did discuss the Galois theory of irreducible equations of prime degree; Dedekind also published rather little, but he lectured on Galois Theory (see [Scharlau (1982)]); Netto's books [Netto (1882)], [Netto (1892)], heavily based on Jordan's *Traité* (in spite of some ill-feeling between Netto and Jordan in the early 1870s after the Franco–Prussian war), brought Galois Theory to a wider public both in Germany and in America; and towards the end of the century Weber's article [Weber (1893)] and his famous and very influential textbook [Weber (1895)] were published.

In the 1850s and 1870s Cayley famously tried to develop an abstract theory of groups (citing Galois for the word *groupe* in a footnote), but he did not seem to understand Galois' ideas to any depth, and Galois Theory did not take hold in Britain until the 20th century. The famous textbook [Burnside & Panton (1881)],

for example, went through many editions from 1881 until the mid-1920s (and was reprinted by Dover Publications, New York, in 1960), but does not mention Galois, or groups or any general theory of solubility of equations. Near the end of the 19th century Oskar Bolza and James Pierpont gave series of lectures that brought Galois Theory to America (see [Bolza (1890)], [Pierpont (1899)], [Pierpont (1900)]).

The above paragraphs treat the development of Galois Theory, but the theory of groups was developing not only as a part of Galois Theory but also as a subject in its own right. It came from two more-or-less independent sources, namely the publication of the *Œuvres* of Galois in 1846 and the publication by Cauchy of about 25 notes in the *Comptes rendus hebdomadaires de l'Académie des Sciences*, of which the first four are [Cauchy (1845a)], and a long article [Cauchy (1845b)] that overlaps considerably with the *CR* notes. His approach was different from that of Galois, as was his language. What was a *groupe de substitutions* in the writings of Galois was a *système de substitutions conjuguées* in those of Cauchy. The two approaches were complementary. They came together in the work of Camille Jordan who, in his thesis [Jordan (1860)], [Jordan (1861)] used the language of Cauchy to treat the Academy problem that had been announced for the Grand Prix de Mathématiques for 1860 (a problem that had come out of Cauchy's work), but who quickly came to understand and develop the ideas of Galois (see, for example, [Jordan (1865)], [Jordan (1867)], [Jordan (1869)], [Jordan (1870)], [Neumann (2006)]). There have been many studies of the development of the theory of groups in the 19th century and the reader is referred to [Wussing (1969)], [Neumann (1999)] and references cited in those works for fuller information.

I.6 Scope of this edition

Included here is everything published by Liouville and by Tannery, and a little more. Exigencies of time and space prevented me from including everything published by Bourgne & Azra. Theirs remains the only complete edition of the writings of Galois.

The material is organised as follows. First come the five mathematical articles published while Galois was alive. From this period I have excluded only the letter 'Sur l'Enseignement des Sciences: des Professeurs, des Ouvrages, des Examinateurs' published in the *Gazette des Écoles* on 2 January 1831. Although it contains Galois reflections on the study of mathematics in the colleges of Paris it is not, I find, a particularly edifying piece, and contributes little to our understanding of the mathematician in Galois.

After the published articles comes the Testamentary Letter written on 29 May 1832, the eve of the duel. I place it there for two reasons: first because it was the next to be published (in September 1832); secondly because it includes an admirable synopsis of the substance of Galois' discoveries.

Then come the manuscripts essentially in the order in which they appear in the collection in the Institut de France: the great First Memoir, which, when it was first

published by Liouville in 1846 quickly led to the development of Galois Theory and group theory; then the Second Memoir, also first published by Liouville in 1846; finally the minor manuscripts, most of which were first published by Tannery in 1906/07.

Some of the minor manuscripts, those in Dossiers 9–14, contain little mathematics. They could be described as philosophical-polemical. Nevertheless, they seem to have been intended by Galois as part of his mathematical work. He had intended to write some expositions of algebra; he apparently dreamed of publishing his First and Second Memoirs, or something like them, as a small book; and these seem to have been conceived as introductory material. This, at least, is how Auguste Chevalier seems to have interpreted them.

Each item is preceded by an introductory page giving information about previous editions, physical descriptions of the manuscripts, jottings, and other such matters. Most items are followed by notes intended to supply context. I have tried to restrain myself from exegesis. Thus the commentary is focussed on content, not meaning; on syntax, not semantics; on relationships with previous editions. I much hope to find time in the future to write articles dealing with various parts of the mathematics produced by Galois, articles similar to [Neumann (2006)], which deals with just the first few pages of the *Second Mémoire*. But for this book I have tried to suppress my mathematical instincts.

Missing from this edition are the many scraps containing scribbles and partial calculations. These are to be found in [B & A (1962), pp. 189–361] and since no translation is needed (or indeed possible) there is no point in copying them here. Also missing are the items of schoolwork published in [B & A (1962), pp. 403–458]. These would have merited inclusion had time and space permitted.

I.7 Editorial ambition and policy

Leaving aside Gergonne and Sturm (for Férrusac's *Bulletin*), the principle editors of the Galois manuscripts were Auguste Chevalier, Joseph Liouville, Jules Tannery, and Robert Bourgne & Jean-Pierre Azra.

I.7.1 Auguste Chevalier

Auguste Chevalier was a close friend of Galois, and was, in effect, appointed by him as his literary executor (see the end of the Testamentary Letter). The obituary [Chevalier (1832b)] he published in November 1832 begins

> Il y a trois ans bientôt que j'ai connu Galois; notre liaison commença à l'Ecole Normale, où il entra un an après moi.
>
> [I have known Galois for nearly three years; our relationship began at the Ecole Normale, which he entered a year after I did.]

He was the first editor of the Galois manuscripts. He had the Testamentary Letter published in the *Revue Encyclopédique* [Lettre (1832)], as Galois had commanded, and he made copies of the *Premier Mémoire*, the *Second Mémoire*, the *Discours Préliminaire* (Dossier 9), the *Préface* (Dossier 11), and the *Discussions sur les Progrès de l'Analyse pure* (Dossier 12). His manuscripts are bound in with the Galois manuscripts in the library of the Institut de France. Apart from somewhat erratic use of capital letters, Chevalier's copies are remarkably faithful and accurate. The few places where he made small corrections are indicated in my marginal notes.

I.7.2 Joseph Liouville

The Galois manuscripts came to Liouville from Chevalier some time in the summer of 1843 (see [Liouville (1843)]). He planned to publish at least the *Premier Mémoire* that same year, as is proved by the existence at the end of Dossier 3 (of the Galois manuscripts) of corrected proof sheets carrying the reference 'Tome VIII, DÉCEMBRE 1843'. The material appeared three years later as an item in the great edition [Liouville (1846)]. The delay may have been due to difficulties with the material. In his 1843 announcement to the Académie des Sciences, Liouville said:

> Le Mémoire de Galois est rédigé peut-être d'une manière un peu trop concise. Je me propose de le compléter par un commentaire qui ne laissera, je crois, aucun doute sur la réalité de la belle découverte de notre ingénieux et infortuné compatriote.
>
> [Galois' memoir is written in a style that is perhaps a little too concise. I propose to complete it with a commentary which will leave no doubt, I believe, as to the correctness of the beautiful discovery of our ingenious and unfortunate compatriot.]

No mathematical commentary accompanied the 1846 publication of the *Œuvres*. The last item in Dossier 3, however, is a manuscript by Liouville supplying a proof of Proposition II (see Ch. IV, Note 16, p. 159), to be inserted into the aborted 1843 printed version of the *Premier Mémoire* at the point where Galois had left the marginal note 'Il y a quelque chose à completer dans cette démonstration. Je n'ai pas le tems.' [There is something to be competed in this proof. I do not have the time.] Moreover, Dossier 27 contains some 25 pages of manuscript notes by Liouville working out Galois' ideas. A fine account of these notes is to be found in [Lützen (1990), pp. 567–577] .

Liouville's job as editor of the *Journal de Mathématiques pures et appliquées* was to make mathematics available to a wide mathematical public. Thus he very properly corrected errors of grammar and spelling, and even slips in the mathematics. He also had lemmas, theorems and formulae displayed as is usual in mathematical publication. He would have done the same for any author. He did his job well. See [Lützen (1990), pp. 579–580] for an analysis of his impact. The fact is that Galois'

ideas, though published in synoptic form in the *Lettre testamentaire* in September 1832, were essentially lost to the mathematical public after his death. It was as if they had been buried with Galois. Liouville not only disinterred them, he gave them the full life that they deserved.

Picard is not listed above among the principal editors because his 1897 edition followed that of Liouville pretty closely. He too was concerned to make Galois' ideas easily available, for the first time in book format, to mathematical colleagues. Since Liouville's (and Picard's) corrections are mostly routine very few have been noted in my marginal annotations.

I.7.3 Jules Tannery

Tannery was an editor of a different kind, concerned quite as much with the historical importance of the Galois material as with its mathematical content—after all, by the end of the 19th century the mathematics had been fully developed and taken a long way beyond what Galois himself had created. In [Tannery (1906), pp. 227, 229] he wrote:

> L'importance de l'œuvre de Galois sera mon excuse pour la minutie de certains détails, où j'ai cru devoir entrer, et qui va jusqu'au relevé de fautes d'impression, dont le lecteur attentif ne peut manquer de s'apercevoir. Je ne me dissimule pas ce que cette minutie, en elle-même, a de puéril. [...]
>
> J'ai collationné le manuscrit avec le texte imprimé: [...]
>
> [The importance of Galois' work will be my excuse for the extreme care over certain details that I have entered into, extending as far as the listing of printing errors which the attentive reader could not fail to notice. I am well aware that, in itself, this extreme care has a trifling element. [...]
>
> I have collated the manuscript with the printed text.]

In [Tannery (1907), p. 279] he added

> Quant aux fragments qui suivent, j'ai cru devoir les reproduire tels quels, avec une exactitude minutieuse, en conservant l'orthographe, la ponctuation ou l'absence de ponctuation, sans les quelques corrections qui se présentent naturellement à l'esprit. Cette minutie m'était imposée pour les quelques passages où la pensée de Galois n'était pas claire pour moi; sur cette pensée, les fragments informes que je publie jetteront peut-être quelque lueur. Je me suis efforcé de donner au lecteur une photographie sans retouche.
>
> [As for the fragments which follow, I have believed I should reproduce them as they are, with minute exactitude, preserving the spelling,

> the punctuation or the absence of punctuation, without the few corrections that naturally occur to one's mind. This great care was imposed upon me for the few passages where the thinking of Galois was not clear to me; the imperfect fragments which I am publishing will perhaps throw some light on this thinking. I have made great efforts to give the reader an un-retouched photograph.]

I.7.4 Robert Bourgne & Jean-Pierre Azra

Bourgne & Azra were editors of the same kind as Tannery, and guided by the same principles. Following a similar line to his, indeed, echoing some of it, Robert Bourgne [B & A (1962), p. xv] wrote:

> Ce livre rassemble tout ce que nous avons conservé d'Evariste Galois, mémoires, articles, recherches, brouillons, lettres. [...]
>
> On l'a fait pour qu'il livre au mathématicien un texte exact et complet, pour qu'il offre à l'historien de quoi préciser un grand moment de la pensée mathématique. Il ne s'agisssait que de lire scrupuleusement les manuscrits et, s'ils manquent, de revenir à la publication originale. Point de retouche. Nous livrons la copie exacte. Nous avons respecté la ponctuation de Galois et maintenu les distractions du manuscrit ou les fautes du texte original. Si la correction s'impose, on la signale en note.
>
> Nous avons déchiffré toutes les ratures. [...]
>
> [This book gathers together all that is preserved for us of Evariste Galois. [...]
>
> It has been made [written] in order to deliver to the mathematician a correct and complete text, in order to offer to the historian something to define a great moment in mathematical thought. There was nothing to be done but to read the manuscripts scrupulously and, where they are missing, to return to the original publication. Absolutely no retouching. We deliver an exact copy. We have respected Galois' punctuation and retained the slips in the manuscript or the original text. Where correction is required it is indicated in a note.
>
> We have deciphered all the crossings-out.]

I have found that most of Galois' errors of spelling and punctuation have been transcribed faithfully in Tannery's edition of the minor manuscripts, but a great many of them have been silently corrected in [B & A (1962)]. There are so many discrepancies between manuscript and print (well over 500 of them), and most of them are so trivial, that I do not have the space or the stomach to record other than a few of the more surprising and significant ones.

Although [B & A (1962)] is clearly the joint work of both the editors whose names appear on the title page, so that it would normally be correct to write such phrases

as ‘Bourgne & Azra noted that’, it is clear from various passages in their book that they themselves distinguished their contributions. The *Avertissement* [Preface] to the book, for example, is signed by Robert Bourgne alone, whereas a second *Avertissement* attached to Chapter III of the Fourth Part (‘Derniers vestiges: Brouillons et calculs inédits’ [Last remains: scraps and unpublished calculations]), [B & A (1962), pp. 189, 191], is signed by J. P. Azra alone. I shall therefore sometimes take the liberty of attaching one name or the other to various passages. Since the present book has little overlap with the part of [B & A (1962)] for which Azra was responsible, most of those references will be to Robert Bourgne.

I.7.5 The present work

In the present edition, following the precedents set by Tannery and Bourgne & Azra, the French transcription is intended as a ‘warts and all’ version of the manuscripts (I return to what this should mean below). My method was this. A few pages were transcribed directly from the manuscripts, but the majority were taken first from a printed source. Some of the pages came from [Liouville (1846)] by download from the *Gallica* digital edition, some from [Tannery (1906)], [Tannery (1907)] by download from *Gallica* (the digital library of the Bibliothèque Nationale de France), and a very little from [B & A (1962)] by transcription. All were then checked against the original publications (for the material published in Galois’ lifetime) or the manuscripts as appropriate.

My marginal annotations refer mainly to discrepancies between the manuscripts and the various editions. Although some of these are significant in relation to an ambition to establish a text, none of them is important mathematically, and small print is entirely appropriate for them.

In the notes (and sometimes in the text) the following code is used:

ms:	the manuscripts written by Galois himself;
C*ms*:	Chevalier’s manuscript copies;
C1832:	Chevalier’s edition [Lettre (1832)] of the *Lettre testamentaire*;
L1846:	Liouville’s edition [Liouville (1846)] of the main works;
P1897:	Picard’s reissue of Liouville’s edition in the reprint [P & T (2001)];
T1906/7:	Tannery’s paper of 1906–1907 in the reprint [P & T (2001)];
BA1962:	The Bourgne & Azra edition [B & A (1962)] of 1962/76.

Since P1897 is a close copy of L1846 (except in some typographical matters) it is, in fact, rarely referenced.

I.8 Translation and interpretation

The English differs from the French in some significant ways. In particular, in trying to establish an English edition I have followed the usual conventions for published mathematics. Thus, for example, italics are used where the manuscript has underlined words; italics are used (in most places—though not in Dossier 15) for the statements of propositions, theorems, lemmas, and the like; punctuation is corrected and modernised. Most, but not quite all, of the crossed-out material is incorporated into the translation. This was done mainly in the hope that some readers might find it useful, partly to help the English and French run properly in tandem on opposite pages. I can only hope that it does not make difficulties for those readers who want just the main text.

Generally I have tried to produce a rather literal translation, so as to give a fair idea in English of what Galois actually wrote. The reader will therefore find some artificial phrases and some sentences which could be re-cast into more agreeable form. Reading Galois is not difficult. He writes a legible and (mostly) pleasant hand. Understanding what he is saying is not difficult either (except where, in the polemical passages, his writing becomes unsympathetic and idiomatic). Translating, however, is harder. Many words and phrases, both in French and in English, have changed their usage and/or their meaning over these last 200 years. I can only hope that, having the original side by side with the translation, the reader will be able to make the comparisons which should lead to a good understanding of what Galois was trying to explain.

One example should give an idea of what I mean by the difference between understanding and translation. Near the beginning of the *Lettre testamentaire* (see p. 84) there is the sentence

> Le premier [mémoire] est écrit, et malgré ce qu'en a dit Poisson, je le maintiens avec les corrections que j'y ai faites.

The mention of Poisson refers to the report he made to the Académie des Sciences on 4 July 1831 (see Ch. IV, Note 2, p. 146). Although it is quite clear what Galois meant, finding a good English equivalent for his word *maintiens* presents difficulties. Dictionaries variously give 'maintain', 'keep', 'sustain', 'endorse', 'uphold', 'support', for the transitive French verb *maintenir*. But in this context none of these sounds quite right to my ear. My own translation is

> The first is written, and in spite of what Poisson has said about it I stand by it with the corrections that I have made in it.

In [Weisner (1929), p. 278] it is rendered as

> The first is written, and, despite what Poisson has said of it, I am keeping it, with the corrections I have made.

I find this an agreeable translation except that 'I am keeping it' does not quite fit the context. [Fauvel & Gray (1987), p. 503, § 15D1]—quoted in [Wardhaugh (2010), p. 21], where, however, it is mis-attributed to Weisner in [Smith (1929)]—offers

> The first is written, and despite what Poisson has said about it, I hold it aloft with the corrections that I have made.

Although 'hold it aloft' could be a version of 'uphold', this does not, I feel, present quite the picture that Galois would have had in mind. Perhaps the fact that Chevalier mistranscribed it as *soutiens* is an indication that the word *maintiens* is not particularly natural in this context.

Technical terms are a particular problem because translation involves judgments of meaning, so that translation and interpretation have to go together. Here are some notes on some of the more important words.

I.8.1 The words *analyste* and *géomètre*

The two words *analyste* and *géomètre* are fine illustrations of the problem with meanings. Their natural translations into modern English are of course 'analyst' and 'geometer' respectively (though a brass plate on an office door announcing *M. Pierre Lemesurier, Géomètre* says that Mr Pierre Lemesurier is a surveyor). In 1830, however, the two words both had two main meanings: on the one hand they were both used to mean 'pure mathematician' generically, though the former had overtones of 'algebraist'; on the other hand they indicated practitioners of the main subdivisions of pure mathematics.

They derive of course from the nouns *analyse* and *géométrie*. These were parts of *mathématiques* or *sciences mathématiques*, a broad area that included both pure and applied mathematics and more besides. Thus, for example, Vol. 21 (1830/31) of Gergonne's *Annales de mathématiques pures et appliquées* organised the material under 12 headings:

- *Analyse Algébrique;*
- *Analyse Appliquée;*
- *Analyse Élémentaire;*
- *Analyse Indéterminée;*
- *Analyse Transcendante;*
- *Arithmétique;*
- *Arithmétique Sociale;*
- *Géométrie Analytique;*
- *Géométrie des Courbes;*
- *Géométrie Élémentaire;*
- *Géométrie Transcendante;*
- *Philosophie Mathématique.*

Other volumes have similar lists, and whereas topics like *arithmétique, arithmétique sociale, astronomie, dynamique, hydrodynamique, hydrostatique, météorologie, philosophie mathématique, optique* come and go, various branches of *analyse* and *géométrie* always appear. Roughly speaking *analyse* covered algebra, number theory and calculus while *géométrie* covered spatial matters.

If the above paragraph gives an impression that *analyste* and *géomètre* were more or less synonymous then it is wrong. For one thing, as was indicated above, the former carries the suggestion of 'algebraist', if in a broad sense. For another, although pure mathematics was pretty much covered (with some overlap) by *analyse* and *géométrie*, the latter was the broader term. From 1820 to 1835, of the eleven sections of the Académie des Sciences, there was only one—as one sees when one reads the published minutes, the *Procès-Verbaux de l'Académie des Sciences de l'Institut de France*—that naturally covered pure mathematics and that was the *section de géométrie*. There were members of the *sections d'astronomie, de mécanique, de physique générale*, and perhaps even of other sections, who wrote articles that we might naturally classify as pure mathematics. These would then, however, be thought of as contributions to *géométrie*. There was no *section de mathématiques*.

During the first half of the 19th century the word *mathématiques* occurred quite rarely—hardly more than in the title of Gergonne's *Annales*, the title of Liouville's *Journal*, and in the context of the Academy prizes (*prix de mathématiques, grand prix de mathématiques*). Thus in the language of the Academy the word *géomètre* was used in the same way as we might use 'pure mathematician'. Abel, for example, contributed much more to *analyse* than to *géométrie*, but could nevertheless be referred to by Galois as a *géomètre* (see, for example, Section VI.5), meaning simply 'mathematician'. And the 1984 French postage stamp portraying Galois and describing him as '*révolutionnaire et géomètre*' is saying that he was a revolutionary and a mathematician—he was, after all, very much an algebraist and analyst, rather than a geometer in the familiar modern senses of these words.

I.8.2 The phrases *équation algébrique* and *équation numérique*

The phrases *équation algébrique* and *équation numérique* also require some thought. Their natural literal translations are 'algebraic equation' and 'numerical equation', meaning equations with literal or numerical coefficients, respectively. In almost all contexts in 18th- and 19th-century writings, however, the adjective does not in fact qualify the noun equation. It refers instead to what the writer has in mind as a strategy for solution. The former refers to the search for a formula, the latter to iterative numerical methods. Thus Lagrange's great works [Lagrange (1767)], [Lagrange (1798)], with titles involving *la résolution des équations numériques* have the adjective numerical qualifying the plural noun equations, and yet their subject is numerical methods for finding accurate approximations to the roots of polynomial equations.

A paragraph in Poinsot's preface to the 1826 edition of [Lagrange (1798)] (originally a review of the 1808 edition published in the *Magasin Encyclopédique* in 1808) explains well:

> D'abord si l'on jette un coup d'oeil général sur l'Algèbre, on voit que cette science, abstraction faite des opérations ordinaires (au nombre desquelles on peut compter l'élimination), se partage naturellement en trois articles principaux. 1°. La théorie générale des équations, c'est-à-dire l'ensemble des propriétés qui leur sont communes à toutes. 2°. Leur résolution générale, qui consiste à trouver une expression composée des coefficiens de la proposée, et qui, mise au lieu de l'inconnue, satisfasse identiquement à cette équation, en sorte que tout s'y détruise par la seule opposition des signes. 3°. La résolution des équations numériques, où il s'agit de trouver des valeurs particulières qui satisfassent d'une manière aussi approchée qu'on le voudra, à une équation dont tous les coefficiens sont actuellement connus et donnés en nombres.
>
> [If one casts a general glance over algebra, one sees first that, setting aside ordinary operations (numbered among which one can include elimination), this science is divided naturally into three principal parts. 1st. The general theory of equations, that is to say, the collection of properties which are common to all of them. 2nd. Their general solution, which consists in finding an expression composed of the coefficients of the given equation, and which, replacing the unknown, satisfies this equation identically, so that everything vanishes simply through cancellation. 3rd. The solution of numerical equations, where what matters is to find particular values which satisfy an equation, all of whose coefficients are actually known and given as numbers, to as close an approximation as is desired.]

The term *équation algébrique* has *équation littérale* ('literal equation' or 'letter equation') and *équation générale* ('general equation') as common variants. Galois used all three terms, and used them synonymously.

I.8.3 The words *permutation* and *substitution*

The words *permutation* and *substitution* are of course translated as 'permutation' and 'substitution' respectively. Straightforward and natural though that is, it involves pitfalls for the unwary modern reader.

The word *permutation* is ambiguous in French, as it is in English. In English school syllabuses the word 'permutation' in the phrase 'permutations and combinations' refers to an arrangement of symbols. In undergraduate mathematics it acquires a second (and perhaps more usual) meaning as a bijection of a set to itself. Thus it is used to mean a (static) arrangement, and also to mean an act of (dynamic) rearrangement. Lagrange in [Lagrange (1770/71)] and [Lagrange (1798), 2nd or 3rd ed., Note XIII] used the word in both senses, but more usually dynamically (in the phrase *faire une permutation*). Cauchy in [Cauchy (1815a)] used the word *permute*

as a verb in the title of his article, but used the noun *permutation* in the static sense of an arrangement in the body of his text.

The word *substitution*, on the other hand, is quite unambiguous. It always means the act of rearranging, that is to say, in modern terms a bijective mapping. This is how Cauchy defined it quite precisely in his 1815 paper cited above. Thus, referring to a function K of several variables, he wrote

> Pour indiquer cette substitution, j'écrirai les deux permutations entre parenthèses en plaçant la première au-dessus de la seconde; ainsi, par exemple, la substitution
>
> $$\begin{pmatrix} 1.2.4.3 \\ 2.4.3.1 \end{pmatrix}$$
>
> indiquera que l'on doit substituer, dans K, l'indice 2 à l'indice 1, l'indice 4 à l'indice 2, l'indice 3 à l'indice 4 et l'indice 1 à l'indice 3.
>
> [To indicate this substitution I write the two permutations between parentheses, placing the first above the second; thus, for example, the substitution
>
> $$\begin{pmatrix} 1.2.4.3 \\ 2.4.3.1 \end{pmatrix}$$
>
> will indicate that one must substitute the index 2 for the index 1, the index 4 for the index 3, the index 3 for the index 4 and the index 1 for the index 3 in K.]

In this context it should be noted that when Cauchy returned to substitutions in 1845, and wrote his many *Compte rendus* papers, of which [Cauchy (1845a)] is (or are) the first, and his long memoir [Cauchy (1845b)] (the title of which includes an interesting use of the words *arrangements*, *permutations*, and *substitutions*), he turned his two-line notation the other way up. Thus in the works of 1845 (see [Neumann (1989)] for an account of the dating of these works) we must read $\begin{pmatrix} B \\ A \end{pmatrix}$ as the substitution in which the arrangement B replaces the arrangement A.

Galois sometimes used the verb *permuter*, 'to permute' in the sense of 'to rearrange'; see, for example, Lemma II and the proofs of Lemmas III and IV of the First Memoir. Mostly, however, he used noun-forms and followed Cauchy's 1815 language, using *permutation* to mean 'arrangement' (static) and *substitution* to mean 'substitution' or 'act of rearrangement' (dynamic). Unfortunately, however, he did not use the words consistently. Sometimes he used *permutation* where he meant *substitution*—moreover, he caught himself doing this from time to time, and changed the former to the latter. The reader must be aware of the ambiguity and, where it is not immediately clear, infer the meaning from the context.

I.8.4 The word *groupe*

The word *groupe* is of course translated as 'group'. Note, however, that in the writings of Galois a group is always a group of permutations or a group of substitutions. These are not the same.

In Galois' writings a group of substitutions is a collection (non-empty goes without saying) of substitutions that is closed under composition. Since these are always substitutions of a finite number of 'letters' (the roots of a polynomial equation), closure under composition automatically implies that the identity lies in the collection and that the collection is closed under formation of inverses. Thus a *groupe de substitutions* is what we know as a group of (confusingly) permutations, a subgroup of the relevant symmetric group.

Galois also has *groupes de permutations*. A *groupe de permutations* is a collection of arrangements with the property that the collection of substitutions that change any one to any other of them is closed under composition, that is to say, is a group of substitutions. Originally these were the fundamental tools that he invented for Proposition I of the First Memoir. Gradually, though, as his thinking developed (as seen in the First Memoir, then the Second Memoir, and finally the Testamentary Letter), we see groups of substitutions becoming the principal objects of study.

Unfortunately, Galois often used the word *groupe* without specifying which kind of group he had in mind. Usually the context gives a clear indication whether the group in question is a group of permutations (arrangements) or a group of substitutions; sometimes the reader has to work quite hard to discover what was intended; and sometimes (though rarely), of course, it does not much matter.

At first Galois used the word *groupe* as an ordinary French noun meaning 'group', 'set', 'collection'. It acquired a technical meaning only through repeated use. When the academy referees read his *Premier Mémoire* they would have had to infer any special meaning of the word from the proof of Proposition I, from the first scholium (see p. 116) that follows it, and (if they were not already stymied) from the regular use of it later in the paper. At that time Galois had not explained its meaning. It was only at the final revision on the eve of the fatal duel that he added his famous explicit definition (**f.3b**, p. 114):

> Les substitutions sont le passage d'une permutation à l'autre.
>
> La permutation d'où l'on part pour indiquer les substitutions est toute arbitraire [...]
>
> [...]
>
> Donc si dans un pareil groupe on a les substitutions S et T, on est sûr d'avoir la substitution ST.

> [Substitutions are the passage from one permutation to another.
>
> The permutation from which one starts in order to indicate substitutions is completely arbitrary [...]
>
> [...]

> Therefore, if in such a group one has the substitutions S and T, one is sure to have the substitution ST.]

He added this in the margin alongside Proposition I of the *Premier Mémoire*, with the instruction to print it in the introductory section, which is where it correctly appears in all editions previous to this one.

The point made in the foregoing paragraph is illustrated by comparison with the word *ensemble* ('set', 'collection'). Galois used this word, but it did not acquire a technical meaning in his writings. It remained an informal word. Compare the words *groupe* and *ensemble* in passages where they occur together: in the deleted sentence before the last paragraph of the definition of *groupe* in the *Premier Mémoire* (**f.3b**, p. 114), in Dossier 15, **f.82a**, **f.83b** and **f.84a**, and in the first example to Proposition I of the *Premier Mémoire* (**f.3b**, p. 114, **f.54b** in Dossier 6, and **f.87a**, **f.88a** in Dossier 16).

It has been suggested to me that Galois may have acquired his word *groupe* from the lovely preface by Poinsot to the 1826 edition of Lagrange's treatise on numerical solution of equations (cited above, p. 20). Poinsot used the word informally at first, and although, as he continued to develop the ideas he was expounding there, it gradually acquired some of the characteristics of a technical term, the context is rather different from that of Galois. Poinsot's are groups of roots of an equation, not of permutations or substitutions of roots. In modern terms we can recognise them as being akin to blocks of imprimitivity of a transitive permutation group acting on the set of roots; indeed, in anachronistic terms, once the relevant Galois theory is in place, that is precisely what they are. That this is where Galois got the word is certainly a possibility. I am inclined to doubt it, though. And I doubt very much that it is where he got his concept—there is nothing like his notions of *groupe* in Poinsot's preface. Reading what Galois wrote I get a strong impression that, as I have explained above, he began by simply using the word *groupe* as an ordinary and convenient noun. He could as well have chosen some other collective noun, except that *groupe* is simple and direct and has the associated verb *grouper*. It seems extremely unlikely that he owed it to Poinsot.

I.8.5 The word *semblable*

The word *semblable* is naturally translated as 'similar'. At many points it has a technical meaning. Most often it occurs in plural form as an adjective qualifying *groupes*. Thus for example, in the discussion following the statement of Proposition VII of the First Memoir, and in **f.84a** (Dossier 15), if Γ is a group of permutations (arrangements) and S is a substitution, then ΓS is a group of permutations and Γ, ΓS are *groupes semblables*. (I have distorted the notation here, using Γ where Galois used G, because I want to emphasize the distinction between a group of permutations and a group of substitutions. Note that if G is the group of substitutions corresponding to the group Γ of permutations then the group of substitutions corresponding to ΓS is $S^{-1}GS$.)

There are a few points where the adjectival phrase *semblables et identiques* is used to qualify *groupes*. This should be understood as follows: there is a group Γ of permutations (arrangements) with group G of substitutions; this contains a group Δ of permutations with group H of substitutions; what is meant is that H is a normal subgroup of G. The point is this. Let $S_1, S_2, \ldots, S_m$ be right coset representatives for H in G, so that $G = HS_1 \cup HS_2 \cup \cdots \cup HS_m$ (a disjoint union). Then $\Gamma = \Delta S_1 \cup \Delta S_2 \cup \cdots \cup \Delta S_m$, a union of *groupes semblables*. The group ΔS_i of permutations (arrangements) corresponds to the group $S_i^{-1} H S_i$ of substitutions. Given that H is a normal subgroup of G these are identical; hence *semblables et identiques*, 'similar and identical'. The groups of permutations are similar, their groups of substitutions are identical. That this reading is likely to be robust is confirmed by a passage on **f. 55 b** (Dossier 7), where Galois changed

> Il faut donc que le groupe G se partage en p groupes H semblables et identiques
>
> [It is therefore necessary that the group G be partitioned into p similar and identical groups H.

to

> Il faut donc que le groupe G se partage en p groupes H semblables et dont les substitutions soient les mêmes
>
> [It is therefore necessary that the group G be partitioned into p groups H that are similar and of which the substitutions are the same]

The *décomposition propre* of the *Lettre testamentaire*, **f. 8 a** is the same thing. Also, in the Second Memoir, for example in **f. 37 b**, **f. 39 a**, **f. 39 b**, Galois used *groupes conjugués* ('conjugate groups') in what appears (from the context) to be a similar sense, that is to say, for groups of permutations contained in a larger group and such that their groups of substitutions are all equal and form a normal subgroup of the group of substitutions of the large one.

I.8.6 The word *primitif*

The word *primitif* is naturally translated as 'primitive'. It is a technical term for which Galois gave a definition in his 'Analyse d'un Mémoire' published in Férussac's *Bulletin*, April 1830, in **f. 8 b** of the *Lettre testamentaire*, and at one or two other points (sometimes implicitly). The paper [Neumann (2006)] devotes some 50 printed pages to the word. I do not propose to enter into such detail here. It must suffice to remind the reader that in this context a *groupe primitif* should usually be thought of in modern terms as a quasi-primitive permutation group, that is to say, a permutation group with the property that every non-trivial normal subgroup is transitive; an *équation primitive* is then an equation or polynomial whose Galois group is quasi-primitive in its action on the set of roots.

I.8.7 Other words and phrases

In addition to (semi-)technical terms discussed above there are other words and constructions that Galois used which are not easy to translate. It is easy enough to see what he meant, but finding a way of saying it in English using similar words and constructions is not easy. Here are some common examples:

- I have chosen to translate *ensemble* as 'collection' because 'set' has a modern technical meaning, and Galois used the word as an ordinary noun, not as a technical term.
- I have chosen to translate *équation proposée* literally as 'proposed equation'. Nowadays we would more naturally write 'given equation' but most writers of the time used *équation proposée*, in preference to *équation donnée* (which, however, one does see from time to time).
- Galois made extensive use of the construction *x étant*. Mostly I have used the ugly literal translation '*x* being', but more common (and more agreeable) English usage is 'where *x* is'.
- Constructions such as *Remarquons que*, *Prenons*, etc., might be translated literally as 'We note that', 'We take', etc. In French however, they indicate an imperative, and so they should be translated into the imperative mood 'Note that', 'Take', etc., that is common also in mathematical English.
- Constructions such as *on obtient* are often translated into the passive rather than 'one obtains'.
- The word *caractère* should naturally mean 'character' or 'characteristic', but Galois often used it to mean 'property' or 'condition'.

I.8.8 Glossary

For the convenience of the reader I summarise here some of the discussion above, and add a few further words to the dictionary.

Analyse is naturally translated as 'analysis', but with a meaning that is close to algebra in our context.

Analyste is naturally translated as 'analyst', but the meaning is closer to 'algebraist' or 'pure mathematician' (compare *géomètre*).

Équation is naturally translated as 'equation', but very often is used where we would use 'polynomial'.

Degré is naturally translated as 'degree'. Mostly Galois used it in exactly the same way as it is used nowadays as in degree of an equation or a polynomial, degree

of a radical (or algebraic number), degree of a substitution group (as the number n such that the group is a subgroup of $\mathrm{Sym}(n)$), degree of an algebraic surface. Note, however, that in [Cauchy (1815a), p. 13] *le degré* [*d'une*] *substitution* is defined to be what we would call its order: the degree of S is the least m such that S^m is the identity. There are a few passages where Galois seems to have this usage, or something similar to it, in mind—see **f. 84 b** and **f. 91 a**, for example (pp. 290, 314)—but there are discrepancies in that Galois could there be intending the number of letters permuted.

Géomètre is naturally translated as 'geometer', but the meaning is closer to 'pure mathematician' (compare *analyste*).

Groupe, sometimes groups of permutations (static arrangements), sometimes groups of substitutions.

Groupe partiel, is naturally translated as 'partial group'. Think of the modern terms 'subgroup' and 'coset'. In some contexts one is appropriate, in others the other.

Groupe sousmultiple, naturally translated as 'submultiple group', to be thought of as 'subgroup'. Galois also used the word *diviseur* 'divisor'.

Mémoire is naturally translated as 'memoir'. This works well in 19th century English, though 'article' or 'paper' would be more common now.

Ordre is naturally translated as 'order'. Note, however, that in the context 'la substitution sera de l'ordre p' it usually means the number of letters moved by the substitution.

Période is naturally translated as 'period'. Note, however, that in the context 'substitution dont la période sera de p termes' it means what is now called order—the substitution will have order p.

Permutation is naturally translated as 'permutation' but with the meaning of an arrangement (static).

Substitution is naturally translated as 'substitution', meaning an act of rearranging (dynamic); unfortunately, nowadays we use the word 'permutation'.

Substitution circulaire is naturally translated as 'circular substitution'. It would be very dangerous to follow one's instinct and use 'cyclic substitution' (or 'cyclic permutation') because that is not what it appears to mean. In the First Memoir, in passages about equations of prime degree p, the term refers to the whole cyclic group generated by a p-cycle. In the Second Memoir it is less clear what is meant, but at one point (see p. 178) Galois used the term to refer to the substitutions of prime order that lie in the (unique, as it happens) normal abelian subgroup of his primitive permutation group of prime-squared degree. At that point these have p p-cycles.

The usage of this term by Galois seem to differ considerably from the meaning carefully defined by Cauchy in [Cauchy (1815a), p. 17], where a *substitution circulaire* is very clearly defined as a cyclic substitution of the indices (letters, points) that it does not fix.

Transformer is translated as 'to transform'. It occurs a few times in the Second Memoir, where its meaning is what I can best describe in modern language as 'transform by conjugation'.

I.9 A 'warts and all' transcription

As has been indicated above, the French is intended as a 'warts-and-all' transcription. What this means is that I have sought to reproduce the manuscript as accurately as possible in print, with all its crossings-out, emendations, additions and quirks of writing. In particular, the crossed-out material, except where it consists of no more than one or two illegible letters, has been included in its proper place. (Most, but not all, of this was transcribed by Robert Bourgne and appears on the left hand pages of [B & A (1962)].) Some of the crossed-out material was simply abandoned as Galois proceeded to his next phrase or sentence. But sometimes a word or two here and there was retained and re-used. Thus, for example, near the bottom of **f. 39 a** in the Second Memoir there was a passage that might have read

> [...]. Ce n'est point $p^2 - p$, puisque le groupe G serait non primitif.
> Mais il faut que les substitutions [...]

Then three words from the beginning of the second paragraph were deleted, the words 'que les' were retained and incorporated into the revised text, the word 'substitutions' was changed to 'permutations', and new text was inserted at the end of the previous paragraph, so as to read

> [...]. Ce n'est point $p^2 - p$, puisque le groupe G serait non primitif. Si donc dans le groupe G on ne considère que les permutations [...]

without a paragraph break. Independently of all this the words 'dans ce cas' were inserted so that the final text reads

> [...]. Ce n'est point $p^2 - p$, puisque dans ce cas le groupe G serait non primitif. Si donc dans le groupe G on ne considère que les permutations [...]

It should be noted also that there was a false start, immediately broken off, to the second paragraph. Also, it is quite possible that at the first pass Galois stopped after 'Mais il faut'.

The following notes are intended to make the phrase 'a warts and all transcription' a little more precise.

- Misprints, mis-spellings and infelicities of punctuation have been retained; note that mis-spellings include unconventional use or non-use of diacritical marks. Often accents, especially acute accents are missing; where ‘é’ follows ‘t’, however, the accent could be swallowed up in the crossbar of the ‘t’ and where it is unclear I have assumed that it is there. Some of these may be not so much mis-spellings as dated usage, such as ‘tems’ for ‘temps’. Others may be not so much mis-spellings as haste—a circumflex accent might easily emerge like a grave accent late on a pre-duel evening.

  The lists below are unlikely to be complete—I did not start compiling them until well into the work. I hope, however, that they may give the reader some idea of the problem. Now that images of the manuscripts have (from June 2011) become available through web-publication in digitised form (see [Galois (2011)], p. 392), these and other infelicities are capable of being checked. I believe that the reader will find

  – ‘a’ for à’,
  – ‘algèbrique’ or ‘algebrique’ for ‘algébrique’,
  – ‘appêllerons’ for ‘appellerons’,
  – ‘appèle’ for ‘appelle’,
  – ‘arrèter’ for ‘arrêter’,
  – ‘bientòt’ for ‘bientôt’,
  – ‘celà’ for ‘cela’,
  – ‘complementaire’ for ‘complémentaire’,
  – ‘completer’ for ‘compléter’,
  – ‘connait’ for ‘connaît’,
  – ‘consequent’ for ‘conséquent’,
  – ‘coté’ for ‘côté’,
  – ‘dégré’ for ‘degré’,
  – ‘doît’ for ‘doit’,
  – ‘dù’ for ‘dû’,
  – ‘ecrire’ for ‘écrire’,
  – ‘equation’ for ‘équation’,
  – ‘ètre’ or ‘etre’ for ‘être’,
  – ‘eùt’ for ‘eût’,
  – ‘evidemment’ for ‘évidemment’,
  – ‘éxemple’ for ‘exemple’,
  – ‘éxige’ for ‘exige’,

  - ‘frequent’ for ‘fréquent’,
  - ‘gachis’ for ‘gâchis’,
  - ‘general’ for ‘général’,
  - ‘geometre’ for ‘géomètre’,
  - ‘guere’ for ‘guère’,
  - ‘intéret’ for ‘intérêt’,
  - ‘meme’ or ‘mème’ for ‘même’,
  - ‘numeriques’ for ‘numériques’,
  - ‘ou’ for ‘où’,
  - ‘paraitrait’ for ‘paraîtrait’,
  - ‘partageàt’ for ‘partageât’,
  - ‘periode’ for ‘période’,
  - ‘plutot’ for ‘plutôt’,
  - ‘précedemmant’ for ‘précédemmant’,
  - ‘premiérement’ for ‘premièrement’,
  - ‘prevoir’ for ‘prévoir’,
  - ‘pùt’ for ‘pût’,
  - ‘rebùte’ for ‘rebute’,
  - ‘reconnaitre’ for ‘reconnaître’,
  - ‘regles’ for ‘règles’,
  - ‘remplacant’ for ‘remplaçant’,
  - ‘reponde’ for ‘réponde’ and ‘repondre’ for ‘répondre’,
  - ‘resolues’ for ‘résolues’,
  - ‘resultat’ for ‘résultat’ (and in plural),
  - ‘siécle’ or ‘siecle’ for ‘siècle’,
  - ‘symmétrique’, ‘symmetrique’ or ‘symmettrique’ for ‘symétrique’,
  - ‘tems’ for ‘temps’ [but perhaps this is simply a case of old spelling],
  - ‘tète’ for ‘tête’,
  - ‘théoreme’ for ‘théorème’;

  note also that there is little consistency here;

- inconsistencies in hyphenation, such as

  - ‘c’est à dire’ *v.* ‘c’est-à-dire’.

  - ‘non-primitif’ *v.* ‘non primitif’,
  - ‘peut être’ *v.* ‘peut-être’.

- Unconventional but consistent usages, such as ‘de suite’ for ‘tout de suite’, have been retained.

- Often Galois clearly ran two words together, as in ‘àmoins’, ‘cequi’, ‘delà’, ‘demême’, ‘deplus’, ‘enaura’, ‘enfonction’, ‘ensorte’, ‘entout’, ‘entant’, ‘desuite’ (as in ‘ainsi desuite’), ‘lemoyen’, ‘oubien’, ‘parconséquent’, ‘quelque’ for ‘quel que’ (though of course ‘quelque’ also exists as a genuine word), ‘yavait’;

- contrariwise sometimes Galois clearly would split a word, as in ‘la quelle’ for ‘laquelle’ or ‘les quelles’ for ‘lesquelles’, ‘en suite’ for ‘ensuite’, ‘puis que’ for ‘puisque’, ‘si non’ for ‘sinon’;

Some of these infelicities are sporadic, some are more systematic, none are greatly disturbing.

The notation that Galois used was generally clear and conventional, though the Second Memoir and a few other manuscripts have some unconventional usages:

- In the Second Memoir Galois used a clearly and carefully written . (rather than ,) to separate indices, as in $a_{1.0}$, etc.; moreover, within the text a dot often stands for a comma—but this usage has not been retained as it would be too confusing, especially since the manuscripts also contain a number of random and otiose non-punctuating dots which, one may conjecture, Galois may have made by touching the paper lightly with his pen on finishing a word or formula;

- In the Second Memoir (and several other places) Galois almost always, with just a very few exceptions, wrote his 2nd order inferiors directly below the 1st order inferiors as in $a_{\substack{k \\ 1}}$, etc.;

- Galois used four or more dots $\cdot\cdot\cdot\cdot$ for ellipsis.

In the manuscripts such labels as ‘Lemme’, ‘Théorème’, ‘Démonstration’ are written in the same style as the main text, as are the statements that they label. In his copy Chevalier doubly underlined them; Liouville used SMALL CAPITALS for the main labels and *italics* for subsidiary ones; Bourgne & Azra used SMALL CAPITALS. Liouville used quotation marks to indicate the status of statements of theorems, lemmas, etc., Picard italicised them in the conventional way, Bourgne & Azra followed Galois in not distinguishing. For the present warts-and-all edition I have chosen to follow Galois in the French, but to use modern conventions in the English.

In some of his writings Galois indented the first lines of his paragraphs, in others he did not. I have not studied the phenomenon carefully, but I have an impression that it is the later writings that have indentations. If so, and if this is systematic, then it could help to date the various items. Where paragraphs are not indented

their beginnings can usually be deduced from the fact that the previous line is not full and that there is sometimes a very thin space between paragraphs. The English translation, which does have paragraph indentations, should help to clarify what is going on in the French.

Often Galois used a thin horizontal line to indicate that he had come to a full cadence or finished a passage of writing. Many of these seem to me to be significant, and I have tried to reproduce them as faithfully as possible in the transcription.

I.10 The transcription: editorial conventions

In physical descriptions of the manuscripts a cm $\times$ b cm gives width (East–West, direction of lines of writing) first, then depth (North–South). In his emendations and other adjustments to his writings Galois made heavy use of the two-dimensional nature of a page. In my transcriptions I have tried to produce something that reflects the order and the disorder of the manuscripts. The linearity of print makes that well-nigh impossible of course. There are, however, some techniques that help. The following conventions differ from, but are similar in concept to, some of those used by Bourgne & Azra (see [B & A (1962), p. xxxiii]):

- brackets ^text^ indicate emendations or afterthoughts inserted above the line;
- brackets ˬtextˬ indicate emendations or afterthoughts inserted below the line;
- brackets ⌜text⌝ indicate afterthoughts inserted on the line of text;
- asterisks used as brackets *text* indicate afterthoughts written into the margin (the left margin—there is no right margin).

Note that Galois himself used asterisks to indicate footnotes and also some of his marginal additions. These are here rendered as ✳ or ✳ .

It has been impossible to mimic the deletions in the manuscript as closely as I would have liked. Although many of them are done with a single line, sometimes Galois used a very heavy line, sometimes a wavy line, sometimes he cross-hatched. Two lines ~~thus~~ simply indicate heavier crossing-out than one ~~thus~~. Where something is crossed out and replaced (usually with a word or two above the line) it is rendered without a space thus: ~~rendered~~^transcribed^. A space, small though it may be, as in ~~rendered~~ ^A space^, indicates that the insertion was not a replacement for the deleted material. Or at least, that that is my belief.

Where I have been unable to decipher a word I have used [?], [??], [???] or [????], the number of question marks indicating the probable number of illegible syllables.

Most, but not quite all, of the crossed out material has been translated. This is partly as a service to the reader, partly to help keep the English running properly in tandem with the French on the opposite page.

Just as it is impossible to make a perfect translation, so it is impossible to transcribe a manuscript into print entirely faithfully. Choices have had to be made. I have sought to get as close to the original as I could, but it would not have been helpful, even had it been possible, to reproduce in type the exact layout on the page, such as line-breaks or replacement above (and sometimes below) the line, or in the margin, of crossed-out words and phrases. Nevertheless, I hope that this transcription will be found to be satisfactorily close to the original. If nothing else, the retention of infelicities of spelling and grammar, and the many indications of emendations and afterthoughts, should keep at the front of our mind the fact that we are dealing here with mainly unpolished manuscripts by an untrained young genius of another age.

Chapter II
The published articles

The five mathematical articles (there was also a non-mathematical article in the form of a 'letter to the editor' [Galois (1831)], reprinted in [Dalmas (1956/82), pp. 96–99], in [B & A (1962), pp. 20–25], and in [APMEP (1982), p. 16]) published in Galois' lifetime are:

(1) Démonstration d'un Théorème sur les Fractions Continues Périodiques (1829)

(2) Analyse d'un Mémoire sur la résolution algébrique des Équations (1830)

(3) Note sur la résolution des équations numériques (1830)

(4) Sur la théorie des nombres (1830)

(5) Note sur quelques points d'analyse (1830)

Galois was just seventeen years old when (1) was published, eighteen when the next three came out, and nineteen when (5) was published.

They appeared in two journals of the day. The first was *Annales de Mathématiques pures et appliquées*. Its title-page describes it as "Recueil périodique, rédigé et publié Par J. D. Gergonne, professeur à la faculté des sciences de Montpellier, membre de plusieurs sociétés savantes." [A periodic collection edited and published by J. D. Gergonne, professor at the faculty of science of Montpellier, member of several learned societies.] It appeared in monthly issues of 30–40 pages from 1801 to 1832, and was devoted to the publication of original mathematics. It was succeeded by Liouville's *Journal de Mathématiques pures et appliquées* (see the editorial by Liouville in the first issue of his journal), which started publication in 1835 and is still going strong.

The second is the *Bulletin des Sciences Mathématiques, Physiques et Chimiques* (originally the *Bulletin des Sciences Mathématiques, Astronomiques, Physiques et Chimiques*), an approximation to the title-page of which is shown overleaf. It was known familiarly as Férrusac's *Bulletin* and was very different from Gergonne's *Annales*. It was not the sort of journal in which one might nowadays expect a technical paper to appear. The great majority of its content consisted of reviews of books and articles published elsewhere. Little of what it contained was original. The note 'Sur la théorie des nombres' by Galois was one of the few exceptions. The paper [Taton (1947b)] gives a valuable account of Férussac's *Bulletin* and the mathematics it contained.

I have tried to copy the originals as faithfully as possible, bating the differences between the typography of 1830 and our time. They contain a number of misprints and other slips, some of them surely not the fault of the printer. We do not have the original manuscripts however, so although editors and compositors have intervened, these come as close as we can get to what Galois originally wrote.

The title-page of Férussac's *Bulletin*:

BULLETIN
DES SCIENCES MATHÉMATIQUES,
PHYSIQUES ET CHIMIQUES,
RÉDIGÉ PAR MM. STURM ET GAULTIER DE CLAUBRY.

1re SECTION DU BULLETIN UNIVERSEL,
PUBLIÉ
PAR LA SOCIÉTÉ
POUR LA
PROPAGATION DES CONNAISSANCES
SCIENTIFIQUES ET INDUSTRIELLES,
ET SOUS LA DIRECTION
DE M. LE BARON DE FÉRUSSAC.

TOME TREIZIÈME.

A PARIS,
1830

The five papers are very different and appear to have triggered very different responses. The first, on continued fractions, had a friendly review in Férussac's *Bulletin*, 11, pp. 254–255. It has been cited even relatively recently by Davenport, who, however, comments that its main result was 'implicit in earlier work of Lagrange' (see [Davenport (1962), p. 100]). The second, which is an abstract of what the *Second Mémoire* was probably intended to become, is an announcement containing remarkable insights, but is, unfortunately, not entirely correct, and seems to have been ignored both by mathematicians—except, perhaps, Camille Jordan (but this must be the subject of a separate article at another time)—and by historians. The third also seems to have had little impact. It has been carefully treated in [Galuzzi (2001)]. The fourth was described in the *Lettre testamentaire* as being no more than a lemma for the work in the *Second Mémoire* (see p. 86) but is a work of immense independent interest, containing, as it does, most of the salient facts in the elementary part of the theory of what are now known as finite fields. The two items in the fifth were reviewed briefly in Férussac's *Bulletin*, 15, p. 15, but seem to have had no impact and are (in my view) of little interest, except insofar as they are comparable with the essays in Dossiers 21 and 22, which seem to me to be broadly similar in scope and character if not in subject-matter.

II.1 A theorem on continued fractions

This paper was published in Gergonne's *Annales de Mathématiques Pures et Appliquées*, 19, 294–301 (April 1829), when Galois was still a pupil at the Collège Louis-le-Grand. It was reprinted in [Liouville (1846), pp. 385–392], in [Picard (1897), pp. 1–8], and in [B & A (1962), pp. 364–377]. The concluding footnote is not by Galois; it is an addition by the editor J. D. Gergonne.

The edition by Liouville in 1846 has some modifications to punctuation and typography. In particular, continued fractions are rendered there in the form

$$\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\ddots}}}}}.$$

Picard's reprint of 1897 is more or less the same (though there are a few changes of punctuation), except at one point where the clause *écrite dans un ordre inverse*, echoing the statement of the main theorem, is added. The version in [B & A (1962)] is pretty faithful to the 1829 original.

The reference to Lagrange in the opening sentence of the paper is almost certainly to passages in the third edition [Lagrange (1826)] of his great work *Traité de la Résolution des équations numériques de tous les degrés*, 'Treatise on the solution of numerical equations of all degrees' (or 'Treatise on the numerical solution of equations of any degree'—see Subsection I.8.2 above). Continued fractions appeared in his approach to the numerical solution of equations already in [Lagrange (1767)], but I doubt that Galois would have read that article; [Lagrange (1798)], which is the foundation for [Lagrange (1826)], is heavily based on that earlier work. Continued fractions are introduced in § 22 of the 1826 edition and are developed throughout the book. In [Lagrange (1826), Ch. VI, pp. 47–73], §§ 45–64 are devoted to periodic continued fractions and irrational roots of quadratic equations—§ 59 is of particular historical interest for its reference to earlier work of Euler in this area. It seems a fair conjecture that this long and careful treatment is what Galois had in mind.

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Démonstration d'un théorème sur les fractions continues périodiques;

par M. Evariste GALOIS, élève au Collége de Louis-le-Grand.

Spelling corrected in BA1962 to 'Collège'; Liouville has 'Collége' in a footnote, corrected to 'Collège' in P1897.

On sait que si, par la méthode de Lagrange, on développe en fraction continue une des racines d'une équation du second degré, cette fraction continue sera périodique, et qu'il en sera encore de même de l'une des racines d'une équation de degré quelconque, si cette racine est racine d'un facteur rationnel du second degré du premier membre de la proposée, auquel cas cette équation aura, tout au moins, une autre racine qui sera également périodique. Dans l'un et dans l'autre cas, la fraction continue pourra d'ailleurs être immédiatement périodique ou ne l'être pas immédiatement, mais, lorsque cette dernière circonstance aura lieu, il y aura du moins une des transformées dont une des racines sera immédiatement périodique.

Punctuation after 'immédiatement' changed in L1846 to semicolon.

Or, lorsqu'une équation a deux racines périodiques, répondant à un même facteur rationnel du second degré, et que l'une d'elles est immédiatement périodique, il existe entre ces deux racines une relation assez singulière qui paraît n'avoir pas encore été remarquée, et qui peut être exprimée par le théorème suivant:

THÉORÈME. Si une des racines d'une équation de degré quelconque est une fraction continue immédiatement périodique, cette équation aura nécessairement une autre racine également périodique

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que l'on obtiendra en divisant l'unité négative par cette même fraction continue périodique, écrite dans un ordre inverse.

Démonstration. Pour fixer les idées, ne prenons que des périodes de quatre termes; car la marche uniforme du calcul prouve qu'il en serait de même si nous en admettions un plus grand nombre. Soit une des racines d'une équation de degré quelconque exprimée comme il suit

$$x = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \ldots\ldots}}}}}}};$$

l'équation du second degré, à laquelle appartiendra cette racine et qui contiendra conséquemment sa corrélative, sera

$$x = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{x}}}};$$

Proof of a theorem on periodic continued fractions

by Mr Evariste GALOIS, Pupil at the College of Louis-le-Grand.

It is known that if one of the roots of an equation of the second degree is developed by the method of Lagrange as a continued fraction, this continued fraction will be periodic, and the same will be true of one of the roots of an equation of arbitrary degree if this root is a root of a rational factor of the second degree of the first member of the proposed equation, in which case this equation will have at least one other root which will equally be periodic. Furthermore, in either case the continued fraction could be periodic immediately or not immediately. But when the latter happens there will be at least one transformed equation, of which one of the roots will be immediately periodic.

Now when an equation has two periodic roots, corresponding to a rational factor of the second degree, if one of them is immediately periodic there exists a quite singular relationship between these two roots which appears not to have been noticed yet, and which may be expressed by the following theorem:

THEOREM. *If one of the roots of an equation of arbitrary degree is an immediately periodic continued fraction, the equation will necessarily have another root that is likewise periodic*

that is obtained by dividing negative unity by this same periodic continued fraction written in the reverse order.

Proof. To fix ideas, let us take only periods of four terms because the uniform progression of the calculation proves that it will be the same if we were to allow a greater number. Let one of the roots of an equation of arbitrary degree be expressed as follows:

$$x = a + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \text{.....} .$$

The equation of the second degree to which this root belongs, and which therefore contains its co-relative, will be

$$x = a + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{x} .$$

or, on tire de là successivement

$$a - x = -\frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{x}, \qquad \frac{1}{a-x} = -(b + \frac{1}{c} + \frac{1}{d} + \frac{1}{x},$$

$$b + \frac{1}{a-x} = -\frac{1}{c} + \frac{1}{d} + \frac{1}{x}, \qquad \frac{1}{b} + \frac{1}{a-x}, = -(c + \frac{1}{d} + \frac{1}{x},$$

$$c + \frac{1}{b} + \frac{1}{a-x} = -\frac{1}{d} + \frac{1}{x}, \qquad \frac{1}{c} + \frac{1}{b} + \frac{1}{a-x} = -(d + \frac{1}{x},$$

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$$d + \frac{1}{c} + \frac{1}{b} + \frac{1}{a-x} = -\frac{1}{x}, \qquad \frac{1}{d} + \frac{1}{c} + \frac{1}{b} + \frac{1}{a-x} = -x,$$

c'est-à-dire,

$$x = -\frac{1}{d} + \frac{1}{c} + \frac{1}{b} + \frac{1}{a-x};$$

c'est donc toujours là l'équation du second degré qui donne les deux racines dont il s'agit; mais en mettant continuellement pour x, dans son second membre, ce même second membre qui en est, en effet la valeur, elle donne

$$x = -\frac{1}{d} + \frac{1}{c} + \frac{1}{b} + \frac{1}{a} + \frac{1}{d} + \frac{1}{c} + \frac{1}{b} + \frac{1}{a} + \text{.....};$$

c'est donc là l'autre valeur de x, donnée par cette équation; valeur qui, comme l'on voit, est égale à -1 divisé par la première.

In P1897, though not in L1846, "écrite dans un ordre inverse" is supplied after "la première".

Dans ce qui précède nous avons supposé que la racine proposée était plus grande que l'unité; mais, si l'on avait

$$x = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \text{.....},$$

on en conclurait, pour une des valeurs de $\frac{1}{x}$,

Now from that one derives successively

$$a - x = -\frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{x}, \qquad \frac{1}{a-x} = -(b + \frac{1}{c} + \frac{1}{d} + \frac{1}{x},$$

$$b + \frac{1}{a-x} = -\frac{1}{c} + \frac{1}{d} + \frac{1}{x}, \qquad \frac{1}{b} + \frac{1}{a-x}, = -(c + \frac{1}{d} + \frac{1}{x},$$

$$c + \frac{1}{b} + \frac{1}{a-x} = -\frac{1}{d} + \frac{1}{x}, \qquad \frac{1}{c} + \frac{1}{b} + \frac{1}{a-x} = -(d + \frac{1}{x},$$

$$d + \frac{1}{c} + \frac{1}{b} + \frac{1}{a-x} = -\frac{1}{x}, \qquad \frac{1}{d} + \frac{1}{c} + \frac{1}{b} + \frac{1}{a-x} = -x,$$

that is to say,

$$x = -\frac{1}{d} + \frac{1}{c} + \frac{1}{b} + \frac{1}{a-x}.$$

This therefore is always the equation of the second degree which gives the two roots in question. But, continually putting for [in place of] x in its second member this same second member, which is, in fact its value, it gives

$$x = -\frac{1}{d} + \frac{1}{c} + \frac{1}{b} + \frac{1}{a} + \frac{1}{d} + \frac{1}{c} + \frac{1}{b} + \frac{1}{a} + \text{.....}.$$

This is therefore the other value of x given by the equation, a value which, as can be seen, is equal to -1 divided by the first [written in the reverse order].

In what precedes we have supposed that the given root was greater than unity, but if one had

$$x = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \text{.....},$$

One would conclude, for one of the values of $\frac{1}{x}$, that

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$$\frac{1}{x} = a + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \text{.....},$$

l'autre valeur de $\frac{1}{x}$ serait donc, par ce qui précède,

$$\frac{1}{x} = -\frac{1}{d} + \frac{1}{c} + \frac{1}{b} + \frac{1}{a} + \frac{1}{d} + \frac{1}{c} + \frac{1}{b} + \frac{1}{a} + \text{.....},$$

d'où l'on conclurait, pour l'autre valeur de x,

$$x = -(d + \frac{1}{c} + \frac{1}{b} + \frac{1}{a} + \frac{1}{d} + \frac{1}{c} + \frac{1}{b} + \frac{1}{a} + \text{.....},$$

ou

$$x = -\cfrac{1}{\cfrac{1}{d} + \cfrac{1}{c} + \cfrac{1}{b} + \cfrac{1}{a} + \cfrac{1}{d} + \cfrac{1}{c} + \cfrac{1}{b} + \cfrac{1}{a}};$$

ce qui rentre exactement dans notre théorème.

Soit A une fraction continue, immédiatement périodique quelconque, et soit B la fraction continue qu'on en déduit en renversant la période; on voit que, si l'une des racines d'une équa-

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tion est $x = A$, elle aura nécessairement une autre racine $x = -\frac{1}{B}$; or, si A est un nombre positif plus grand que l'unité, $-\frac{1}{B}$ sera négatif et compris entre 0 et -1; et, à l'inverse, si A est un nombre négatif compris entre 0 et -1, $-\frac{1}{B}$ sera un nombre positif plus grand que l'unité. Ainsi, lorsque l'une des racines d'une équation du second degré est une fraction continue immédiatement périodique, plus grande que l'unité, l'autre est nécessairement comprise entre 0 et -1, et réciproquement si

$$\frac{1}{x} = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \text{.....}}}}}}}}.$$

The other value of $\frac{1}{x}$ would therefore, by what precedes, be

$$\frac{1}{x} = -\cfrac{1}{d + \cfrac{1}{c + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{d + \cfrac{1}{c + \cfrac{1}{b + \cfrac{1}{a + \text{.....}}}}}}}}},$$

from which one would conclude for the other value of x that

$$x = -\left(d + \cfrac{1}{c + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{d + \cfrac{1}{c + \cfrac{1}{b + \cfrac{1}{a + \text{.....}}}}}}}}\right),$$

or

$$x = -\cfrac{1}{\cfrac{1}{d} + \cfrac{1}{c + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{d + \cfrac{1}{c + \cfrac{1}{b + \cfrac{1}{a}}}}}}}};$$

which is exactly what enters into our theorem.

Let A be an arbitrary immediately periodic continued fraction, and let B be the continued fraction that one deduces from it by reversing the period. One sees that if one of the roots of an equation

is $x = A$ it will necessarily have another root $x = -\frac{1}{B}$. Now if A is a positive number greater than unity, $-\frac{1}{B}$ will be negative and contained between 0 and -1; and conversely, if A is a negative number contained between 0 and -1, $-\frac{1}{B}$ will be a positive number greater than unity. Thus when one of the roots of an equation of the second degree is an immediately periodic continued fraction greater than unity, the other is necessarily contained between 0 and -1, and conversely, if

l'une d'elles est comprise entre 0 et -1, l'autre sera nécessairement positive et plus grande que l'unité.

On peut prouver que, réciproquement, si l'une des deux racines d'une équation du second degré est positive, est plus grande que l'unité, et que l'autre soit comprise entre 0 et -1, ces racines seront exprimables en fractions continues immédiatement périodiques. En effet, soit toujours A une fraction continue immédiatement périodique quelconque, positive et plus grande que l'unité, et B la fraction continue immédiatement périodique qu'on en déduit, en renversant la période, laquelle sera aussi, comme elle, positive et plus grande que l'unité. La première des racines de la proposée ne pourra être de la forme $x = p + \frac{1}{A}$, car alors, en vertu de notre théorème, la seconde devrait être $x = a + \frac{1}{-\frac{1}{B}} = a - B$; or, $a - B$ ne saurait être compris entre 0 et -1 qu'autant que la partie entière de B serait égale à p; auquel cas, la première valeur serait immédiatement périodique. On ne pourrait avoir davantage, pour la première valeur de x, $x = p + \frac{1}{q} + \frac{1}{A}$, car alors l'autre serait $x = p + \frac{1}{q - B}$ ou $x = p - \frac{1}{B - q}$; or, pour que cette

Corrected in L1846, BA1962 to "positive, et"; P1897 has "positive et".

Misprinted as "entre **0** et **1**" in L1846, P1897.

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valeur fût comprise entre 0 et -1, il faudrait d'abord que $\frac{1}{B - q}$ fût égal à p plus une fraction; il faudrait donc que $B - q$ fût plus petit que l'unité, ce qui exigerait que B fût égal à q, plus une fraction; d'où l'on voit que q et p devraient être respectivement égaux aux deux premiers termes de la période qui répond à B ou aux deux derniers de la période qui répond à A; de sorte que, contrairement à l'hypothèse, la valeur $x = p + \frac{1}{q} + \frac{1}{A}$ serait immédiatement périodique. On prouverait, par un raisonnement analogue, que les périodes ne sauraient être précédées d'un plus grand nombre de termes n'en faisant pas partie.

Lors donc qu'on traitera une équation numérique par la méthode de Lagrange, on sera sûr qu'il n'y a point de racines périodiques à espérer tant qu'on ne rencontrera pas une transformée ayant au moins une racine positive plus grande que l'unité, et une autre comprise entre 0 et -1; et si, en effet, la racine que l'on poursuit doit être périodique, ce sera tout au plus à cette transformée que les périodes commenceront.

Si l'une des racines d'une équation du second degré est non seulement immédiatement périodique mais encore symétrique, c'est-à-dire, si les termes de la période sont égaux à égale distance des extrêmes, on aura $B = A$; de sorte que ces deux racines seront A et $-\frac{1}{A}$; l'équation sera donc

$$Ax^2 - (A^2 - 1)x - A = 0\,.$$

one of them is contained between 0 and -1, the other will necessarily be positive and greater than unity.

One can prove that conversely, if one of the two roots of an equation of the second degree is positive and greater than unity, and if the other is contained between 0 and -1, the roots will be expressible as immediately periodic continued fractions. Indeed, let A always be an immediately periodic continued fraction, positive and greater than unity, and B the immediately periodic continued fraction that one deduces from it by reversing the period, which will likewise be positive and greater than unity. The first of the roots of the proposed equation could not be of the form $x = p + \dfrac{1}{A}$, for then, in virtue of our theorem, the second would have to be $x = a + \dfrac{1}{-\dfrac{1}{B}} = a - B$; however, $a - B$ could not be contained between 0 and -1 unless the integer part of B was equal to p; in which case the first value would be immediately periodic. Further, for the first value of x one could not have $x = p + \dfrac{1}{q} + \dfrac{1}{A}$, for then the other would be $x = p + \dfrac{1}{q - B}$ or $x = p - \dfrac{1}{B - q}$; however, in order that this value should be contained between 0 and -1 it would first be necessary that $\dfrac{1}{B - q}$ was equal to p plus a fraction; then it would be necessary that $B - q$ was smaller than unity, which requires that B was equal to q plus a fraction; from which it may be seen that q and p would have to be equal respectively to the first two terms of the period corresponding to B, or to the last two of the period corresponding to A; so that, contrary to hypothesis, the value $x = p + \dfrac{1}{q} + \dfrac{1}{A}$ would be immediately periodic. It may be proved by analogous reasoning that the periods cannot be preceded by a greater number of terms not forming a part of it.

Therefore when one treats a numerical equation by the method of Lagrange one will be sure that one cannot hope for any periodic roots as long as one does not come across a transformed equation having at least one root positive and greater than unity and another contained between 0 and -1; and if, indeed, the root one is seeking should be periodic it will be at most at this transformed one that the periods will begin.

If one of the roots of an equation of the second degree is not only immediately periodic but also symmetric, that is to say, if the terms of the period are equal at equal distances from the extremes, one will have $B = A$, so that these two roots will be A and $-\dfrac{1}{A}$. The equation will therefore be

$$Ax^2 - (A^2 - 1)x - A = 0\,.$$

Réciproquement, toute équation du second degré de la forme

$$ax^2 - bx - a = 0\,,$$

aura ses racines à la fois immédiatement périodiques et symétriques. En effet, en mettant tour à tour pour x l'infini et -1, on

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obtient des résultats positifs, tandis qu'en faisant $x = 1$ et $x = 0$, on obtient des résultats négatifs; d'où l'on voit d'abord que cette équation a une racine positive plus grande que l'unité et une racine négative comprise entre 0 et -1, et qu'ainsi ces racines sont immédiatement périodiques; de plus, cette équation ne change pas en y changeant x en $-\dfrac{1}{x}$; d'où il suit que si A est une de ses racines l'autre sera $-\dfrac{1}{A}$, et qu'ainsi, dans ce cas, $B = A$.

Appliquons ces généralités à l'équation du second degré

$$3x^2 - 16x + 18 = 0\,;$$

on lui trouve d'abord une racine positive comprise entre 3 et 4; en posant

Corrected by Liouville to $x = 3 + \frac{1}{y}$.

$$x = 3x + \frac{1}{y}$$

on obtient la transformée

$$3y^2 - 2y - 3 = 0\,,$$

dont la forme nous apprend que les valeurs de y sont à la fois immédiatement périodiques et symétriques; en effet, en posant, tour à tour,

$$y = 1 + \frac{1}{z}\,, \qquad z = 1 + \frac{1}{t}\,, \qquad t = 1 + \frac{1}{u}\,,$$

on obtient les transformées

$$2z^2 - 4z - 3 = 0\,,$$
$$3t^2 - 4t - 2 = 0\,,$$
$$3u^2 - 2u - 3 = 0\,,$$

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l'identité entre les équations en u et en y prouve que la valeur positive de y est

Misprint corrected in L1846: the minus sign is removed.

$$y = -\frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \text{.....}$$

Conversely, every equation of the second degree of the form

$$ax^2 - bx - a = 0\,,$$

will have its roots both immediately periodic and symmetric. Indeed, setting in turn x to be infinity and -1, one

gets positive results, while putting $x = 1$ and $x = 0$ one obtains negative results, from which one sees straightaway that this equation has a root that is positive and greater than unity and a root that is negative and contained between 0 and -1, and that therefore these roots are immediately periodic; moreover, this equation is not changed on changing x to $-\frac{1}{x}$, from which it follows that if A is one of its roots the other will be $-\frac{1}{A}$, and that therefore in this case $B = A$.

Let us apply these generalities to the equation of the second degree

$$3x^2 - 16x + 18 = 0\,.$$

One first finds a positive root of it contained between 3 and 4. On setting

$$x = 3 + \frac{1}{y}$$

one obtains the transformed equation

$$3y^2 - 2y - 3 = 0$$

whose form tells us that the values of y are both immediately periodic and symmetric. Indeed, setting in turn

$$y = 1 + \frac{1}{z}\,, \qquad z = 1 + \frac{1}{t}\,, \qquad t = 1 + \frac{1}{u}\,,$$

one gets the transformed equations

$$\begin{aligned} &2z^2 - 4z - 3 = 0\,,\\ &3t^2 - 4t - 2 = 0\,,\\ &3u^2 - 2u - 3 = 0\,. \end{aligned}$$

The identity of the equations in u and in y proves that the positive value of y is

$$y = \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \text{.....}$$

sa valeur négative sera donc

$$y = -\frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \ldots\ldots$$

les deux valeurs de x seront donc

$$x = 3 + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \ldots.$$

$$x = 3 - \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \ldots\ldots$$

dont la dernière, en vertu de la formule connue

$$p - \frac{1}{q} = p - 1 + \frac{1}{1} + \frac{1}{q-1},$$

devient

$$x = 1 + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \ldots \quad (*)$$

(*) On trouve diverses recherches sur le même sujet, dans le présent recueil, tom. IX, pag. 261, tom. XIV, pag. 324 et 337.

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Its negative value will therefore be

$$y = -\frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \ldots\ldots$$

The two values of x will therefore be

$$x = 3 + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \ldots. \qquad x = 3 - \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \ldots\ldots$$

of which the latter, in virtue of the known formula

$$p - \frac{1}{q} = p - 1 + \frac{1}{1} + \frac{1}{q-1},$$

becomes

$$x = 1 + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \ldots \quad (*)$$

(*) Several researches on the same subject may be found in the present collection Vol. IX, p. 261, Vol. XIV, pp. 324 and 337.

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II.2 Abstract of an article on the algebraic solution of equations

The note 'Analyse d'un Mémoire sur la résolution algébrique des équations' is extracted from *Bulletin des Sciences Mathématiques, Physiques et Chimiques* (Férussac's *Bulletin*), Vol. 13, pp. 271–272 (April 1830). It is reprinted in [Liouville (1846), pp. 395–396], in [Picard (1897), pp. 11–12], and in [B & A (1962), pp. 162–165]. A manuscript which is very probably a first draft of the first half survives as **f.94a** in Dossier 18 (see p. 323 below).

In this paper Galois announced results, some of which are treated in the *Second Mémoire*, some in the *Lettre testamentaire*. For a detailed analysis of the meaning of the word *primitif* and of the reference to Gauss see [Neumann (2006)]. The first and second assertions of the note are famous theorems of Galois; the third is wrong, as Galois himself recognised when he wrote in the *Lettre testamentaire* (**f.9a**, p. 88 below):

> La condition que j'ai indiquée dans le bulletin ferussac pour que l'équation soit soluble par radicaux est trop restreinte. Il y a peu d'exceptions, mais il y en a.
>
> [The condition that I indicated in Férussac's *Bulletin* for the equation to be soluble by radicals is too restrictive. There are few exceptions, but there are some.]

In fact it is very far from correct—the claim that there are only few exceptions does not bear close examination. I believe, however, that it was an inspiration to Camille Jordan in the 1860s—but this is the subject of a planned article that I hope to write if and when time permits.

The last paragraph of the note is expanded, corrected and explained in the *Lettre testamentaire*, **f.9b**, pp. 88, 90 below.

It was tempting to replace such notation as $a = b \pmod{c}$ with $a \equiv b \pmod{c}$ for the English version. Following the advice of a referee, I have not done so. Although the former is unusual nowadays, it is perfectly clear what it means.

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138. Analyse d'un Mémoire sur la résolution algébrique des équations; par M. E. Galois.

On appelle équations non-primitives les équations qui étant, par exemple, du degré $m\,n$, se décomposent en m facteurs du degré n, au moyen d'une seule équation du degré m. Ce sont les équations de M. Gauss. Les équations primitives sont celles qui ne jouissent pas d'une pareille simplification. Je suis, à l'égard des équations primitives, parvenu aux résultats suivans.

1° Pour qu'une équation de degré premier soit résoluble par radicaux, il faut et il suffit que deux quelconques de ses racines étant connues, les autres s'en déduisent rationnellement.

2° Pour qu'une équation primitive du degré m soit résoluble par radicaux, il faut que $m = p^\nu$, p étant un nombre premier.

3° A part les cas mentionnés ci-dessus, pour qu'une équation primitive du degré p^ν soit résoluble par radicaux, il faut que deux quelconques de ses racines étant connues, les autres s'en déduisent rationnellement.

Misprint 'ci-dessus' corrected to 'ci-dessous' in L1846 (misquoted as 'au-dessous' in BA1962).

A la règle précédente échappent les cas très-particuliers qui suivent:

1° le cas de $m = p^\nu = 9, = 25$.

2° le cas de $m = p^\nu = 4$, et généralement celui où a^α étant un diviseur de $\dfrac{p^\nu - 1}{p - 1}$ on aurait, a premier et

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$$\frac{p^\nu - 1}{a^\alpha (p-1)}\, \nu = p. \qquad (\text{mod. } a^\alpha).$$

Ces cas s'écartent toutefois fort peu de la règle générale.

Quand $m = 9, = 25$, l'équation devra être du genre de celles qui déterminent la trisection et la quintisection des fonctions elliptiques.

Dans le second cas, il faudra toujours que deux des racines étant connues, les autres s'en déduisent, du moins au moyen d'un nombre de radicaux du degré p, égal au nombre des diviseurs a^α de $\dfrac{p^\nu - 1}{p - 1}$ qui sont tels que

$$\frac{p^\nu - 1}{a^\alpha (p-1)}\, \nu = p \qquad (\text{mod. } a^\alpha) \qquad a \text{ premier.}$$

Toutes ces propositions ont été déduites de la théorie des permutations

Voici d'autres résultats qui découlent de ma théorie.

Analysis of a Memoir on the algebraic solution of equations; by Mr E. Galois.

Those equations which, being, for example of degree mn, decompose into m factors of degree n by means of a single equation of degree m, are called non-primitive equations. These are the equations of Mr Gauss. The primitive equations are those which do not enjoy such a simplification. With regard to primitive equations I have been led to the following results.

1. In order that an equation of prime degree should be soluble by radicals it is necessary and sufficient that, given any two of its roots, the others may be deduced rationally from them.

2. In order that a primitive equation of degree m be soluble by radicals, it is necessary that $m = p^\nu$, p being a prime number.

3. Apart from the cases mentioned below, in order that a primitive equation of degree p^ν be soluble by radicals, it is necessary that, given any two of its roots, the others may be deduced rationally from them.

Misprint 'above' corrected to 'below'.

The following very special cases escape from the preceding rule:

1. the case of $m = p^\nu = 9, = 25$.

2. the case of $m = p^\nu = 4$, and generally that where, a^α being a divisor of $\dfrac{p^\nu - 1}{p - 1}$, one has a prime and

$$\frac{p^\nu - 1}{a^\alpha (p-1)} \nu = p \pmod{a^\alpha}.$$

These cases deviate rather little from the general rule however.

When $m = 9, = 25$, the equation must be of the kind of those which determine the trisection and the quintisection of elliptic functions.

In the second case, it is still necessary that, given any two of the roots, the others may be deduced rationally from them, at least by means of a number of radicals of degree p equal to the number of divisors a^α of $\dfrac{p^\nu - 1}{p - 1}$ which are such that

$$\frac{p^\nu - 1}{a^\alpha (p-1)} \nu = p \pmod{a^\alpha}, \qquad a \text{ prime}.$$

All these propositions have been deduced from the theory of permutations.

Here are some other results which may be derived from my theory.

1° Soit k le module d'une fonction elliptique, p un nombre premier donné > 3; pour que l'équation du degré $p + 1$ qui donne les divers modules des fonctions transformées relativement au nombre p, soit résoluble par radicaux, *il faut* de deux choses l'une, ou bien qu'une des racines soit rationnellement connue, ou bien que toutes soient des fonctions rationnelles les unes des autres. Il ne s'agit ici, bien entendu, que des valeurs particulières du module k. Il est évident que la chose n'a pas lieu en général. Cette règle n'a pas lieu pour $p = 5$.

2° Il est remarquable que l'équation modulaire générale du 6e degré, correspondant au nombre 5, peut s'abaisser à une du 5e degré dont elle est la réduite. Au contraire, pour des degrés supérieurs, les équations modulaires ne peuvent s'abaisser.

1. Let k be the modulus of an elliptic function, p a given prime number > 3; in order that the equation of degree $p + 1$ which gives the various moduli of the transformed functions relative to the number p should be soluble by radicals, one of two things is *necessary*: either that one of the roots should be rationally known, or that all should be rational functions of each other. Of course we are concerned here only with special values of the modulus k. It is clear that the fact does not hold in general. The rule does not hold for $p = 5$

2. It is worthy of note that the general modular equation of degree 6, corresponding to the number 5, can be reduced to one of the 5th degree of which it is the reduced equation. By contrast, for higher degrees the modular equations cannot be reduced.

II.3 A note on the numerical solution of equations

We present here the original version of this note extracted from *Bulletin des Sciences Mathématiques, Physiques et Chimiques* (Férussac's *Bulletin*), Vol. 13, pp. 428–435 (June 1830). It was reprinted in [Liouville (1846), pp. 397–398], [Picard (1897), pp. 13–14], and in [B & A (1962), pp. 378–381]. One typographical idiosyncracy of the *Bulletin* that I have been unable to reproduce is the use of the letter V turned 90° one way or the other to stand for $<$ or $>$. Curiously, this was not universal. The printer had perfectly good $<$ and $>$ symbols that he used most of the time.

The reference to Legendre with which the article opens is identified by Massimo Galuzzi (see [Galuzzi (2001)]) as being to the third chapter of the supplement [Legendre (1816)] to the second edition (1808) of Legendre's great *Essai sur la théorie des nombres*. But then the first paragraph of Galois' article is pretty unfair on his readers. The only direct contact with Legendre's exposition is that his *fonction omale*, which he defines as a monotonic function, increasing or decreasing, is what for Galois is a *fonction de x qui croît constamment en même temps que x*, a function of x that grows constantly as x does. Beyond that the connections have to be made through a considerable amount of interpretation. Legendre, for example, did not put his equations into such a simple form as $\varphi x = x$. Nevertheless, Galuzzi makes a good and convincing case—and I am most grateful to him for drawing my attention to it.

In a private communication Professor Galuzzi suggests that Galois may have come across the approximation method in Cauchy's *Cours d'Analyse* rather than, or as well as, in [Legendre (1816)]. In [Cauchy (1821), Note III, p. 464 *ff*] (see also [B & S (2009), p. 312 *ff*]), there is a modification of Legendre's method for finding approximations to the roots of an equation $f(x) = 0$. In effect it requires writing $f(x) = \varphi(x) - \chi(x)$, where both φ, χ are continuous and monotonic increasing, and then using an iterative method to seek solutions of the equation $\varphi(x) = \chi(x)$. It seems likely that Galois had read this exposition by Cauchy at some time. But the 1816 supplement to Legendre (1808) was reprinted with only minimal changes as the first section, Articles (1)–(41), of the appendix to [Legendre (1830)], and although I have not been able to track down the exact publication date, it was early enough in 1830 that a substantial review could appear in the August issue of Férussac's *Bulletin*, 14 (1830), pp. 90–93. Thus, even although he may well have considered the matter at some earlier time, it seems a good possibility that Galois was stimulated to write his note by seeing this new edition of Legendre's book.

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216. Note sur la résolution des équations numériques; par M. E. Galois.

M. Legendre a le premier remarqué que, lorsqu'une équation algébrique était mise sous la forme

$$\varphi x = x$$

où φx est une fonction de x qui croît constamment en même temps que x, il était facile de trouver la racine de cette équation immédiatement plus petite qu'un nombre donné a, si $\varphi a < a$, et la racine immédiatement plus grande que a, si $\varphi a > a$.

Pour le démontrer, on construit la courbe $y = \varphi x$ et la droite $y = x$. Soit prise une abscisse $= a$, et supposons, pour fixer les idées, $\varphi a > a$, je dis qu'il sera aisé d'obtenir la racine immédiatement supérieure à a. En effet, les racines de l'équation $\varphi x = x$ ne sont que les abscisses des points d'intersection de la droite et de la courbe, et il est clair que l'on s'approchera du point le plus voisin d'intersection, en substituant à l'abscisse a l'abscisse φa. On aura une valeur plus approchée encore en prenant $\varphi\varphi a$, puis $\varphi\varphi\varphi a$, et ainsi de suite.

Soit $\mathrm{F}x = o$ une équation donnée du degré n, et $\mathrm{F}x = \mathrm{X} - \mathrm{Y}$, X et Y n'ayant que des termes positifs. M. Legendre met successivement l'équation sous ces deux formes

$$x = \varphi x = \sqrt[n]{\frac{\mathrm{X}}{\left(\frac{\mathrm{Y}}{x^n}\right)}} \qquad x = \psi x = \sqrt[n]{\frac{\mathrm{X}}{\left(\frac{x^n}{\mathrm{Y}}\right)}}$$

Massimo Galuzzi has observed that the second formula is misprinted. It should be the same as the first except for interchange of X and Y.

les deux fonctions φx et ψx sont toujours, comme on voit, l'une plus grande, l'autre plus petite que x. Ainsi à l'aide de ces deux fonctions, on pourra avoir les deux racines de l'équation les plus approchées d'un nombre donné a, l'une en plus et l'autre en moins.

Mais cette méthode a l'inconvénient d'exiger à chaque opération l'extraction d'une racine n[ième]. Voici deux formes plus commodes. Cherchons un nombre k tel, que la fonction

Misprint n for n.

$$x + \frac{\mathrm{F}x}{k\,x^n}$$

croisse avec x, quand $x > 1$. (Il suffit, en effet, de savoir trouver les racines d'une équation qui sont plus grandes que l'unité.)

Nous aurons pour la condition proposée

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$$1 + \frac{d\,\frac{\mathrm{X}-\mathrm{Y}}{kx^n}}{dx} > 0 \qquad \text{ou bien} \quad 1 - \frac{n\mathrm{X} - x\mathrm{X}'}{k\,x^{n+1}} + \frac{n\mathrm{Y} - x\mathrm{Y}'}{k\,x^{n+1}} > 0$$

216. Note on the solution of numerical equations; by Mr E. Galois.

Mr Legendre was the first to notice that when an algebraic equation was put in the form

$$\varphi x = x$$

where φx is a function of x that grows constantly at the same time as x does, it was easy to find the root of this equation immediately smaller than a given number a if $\varphi a < a$, and the root immediately greater than a if $\varphi a > a$.

To show this one constructs the curve $y = \varphi x$ and the straight line $y = x$. Let an abscissa $= a$ be taken, and suppose, to fix ideas, that $\varphi a > a$. I say that it will be easy to obtain the root immediately above a. Indeed, the roots of the equation $\varphi x = x$ are nothing other than the abscissae of the points of intersection of the straight line and the curve, and it is clear that one will approach the point which is nearest neighbour to the intersection by substituting for the abscissa a the abscissa φa. One will have a value even closer by taking $\varphi\varphi a$; then $\varphi\varphi\varphi a$; and so on.

Let $Fx = 0$ be a given equation of degree n, and $Fx = X - Y$, where X and Y have only positive terms. Mr Legendre puts the equation successively into these two forms:

$$x = \varphi x = \sqrt[n]{\frac{X}{\left(\frac{Y}{x^n}\right)}}, \qquad x = \psi x = \sqrt[n]{\frac{Y}{\left(\frac{X}{x^n}\right)}}.$$

Misprint corrected: see opposite.

As can be seen, the two functions φx and ψx are always, the one greater, the other smaller, than x. Thus with the aid of these two functions, one may get the two roots of the equation closest to a given number a, the one greater, the other less.

But this method has the disadvantage of requiring the extraction of an nth root at each step. Here are two more convenient forms. Seek a number k such that the function

$$x + \frac{Fx}{k\,x^n}$$

is increasing when $x > 1$. (Indeed, it is enough to know how to find the roots of an equation which are greater than unity.)

For the proposed condition we will have

$$1 + \frac{d\,\frac{X - Y}{kx^n}}{dx} > 0 \qquad \text{that is,} \quad 1 - \frac{nX - xX'}{k\,x^{n+1}} + \frac{nY - xY'}{k\,x^{n+1}} > 0\,.$$

Sentences merged without punctuation in the original.

or on a identiquement

$$n\mathrm{X} - x\mathrm{X}' > 0 \qquad\qquad n\mathrm{Y} - x\mathrm{Y}' > 0$$

il suffit donc de poser

$$\frac{n\mathrm{X} - \mathrm{X}'x}{k\,x^{n+1}} < 1 \qquad \text{pour } x > 1$$

et il suffit pour cela de prendre pour k la valeur de la fonction $n\mathrm{X} - \mathrm{X}'x$ relative à $x = 1$.

On trouvera de même un nombre h tel que la fonction

$$x - \frac{\mathrm{F}\,x}{h\,x^n}$$

croîtra avec x quand x sera > 1, en changeant Y en X.

Ainsi l'équation donnée pourra se mettre sous l'une des formes

$$x = x + \frac{\mathrm{F}\,x}{k\,x^n} \qquad\qquad x = x - \frac{\mathrm{F}\,x}{h\,x^n}$$

qui sont toutes deux rationnelles, et donnent pour la résolution une méthode facile.

Now one has identically

$$nX - xX' > 0\,, \qquad nY - xY' > 0\,.$$

It therefore suffices to set

$$\frac{nX - X'x}{k\,x^{n+1}} < 1 \quad \text{for } x > 1\,,$$

and for that it suffices to take k to be the value of the function $nX - X'x$ relative to $x = 1$.

Similarly, by exchanging Y for X, one will find a number h such that the function

$$x - \frac{F\,x}{h\,x^n}$$

grows with x when $x > 1$.

In this way the given equation can be put into one of the forms

$$x = x + \frac{F\,x}{k\,x^n}\,, \qquad x = x - \frac{F\,x}{h\,x^n}\,,$$

both of which are rational and give an easy method of solution.

II.4 On the theory of numbers

We present here the original version of this paper extracted from *Bulletin des Sciences Mathématiques, Physiques et Chimiques* (Férussac's *Bulletin*), Vol. 13, pp. 428–435 (June 1830). It was reprinted in [Liouville (1846), pp. 398–407], [Picard (1897), pp. 15–23], and in [B & A (1962), pp. 112–127]. As with the other published articles, it has not been easy to copy the original faithfully, even allowing for the differences between the typography of 1830 and our time. Some of the obscurities in the original are simple misprints, including confusion in formulae between *o* and 0, etc. Misprints p_ν or $p\nu$ for p^ν, are hard to distinguish because the printer uses a small letter ν set, like many Greek letters, slightly below the line of type. Moreover, random sizes of letters are used: the formula x^{p^ν}, for example, is rendered variously as $x^{p^{\nu}}$, $x^{p\nu}$, x^{p}_{ν}, and it is a moot point whether the second of these was designed by the printer to be $x^{p\nu}$, which would have been a simple misprint.

In addition to typographical obscurities and misprints there are mathematical difficulties where Galois made a slip in his treatment of the example of a primitive root in the field of size 7^3 and then continued his miscalculation. The errors were corrected without comment by Liouville in his 1846 edition. Although my transcription copies the original as closely as possible (subject to modern typographical constraints and my personal sensibilities), the English translation is based on the corrected French version.

I have not found it easy to identify all the external references in this paper. The mention of Gauss in the first paragraph is clear enough. This should be to the first four parts of *Disquisitiones* [Gauss (1801)], although I find no explicit use of the notation $F\,x \equiv 0$ there. The paragraph *Ensuite on prouvera, ..., toute la suite des autres racines* (p. 64), in which Galois explains the existence of a primitive root (a generator of the multiplicative group) in the Galois field of size p^ν, clearly refers to the fact that there is always a primitive root modulo a prime number. According to [Gauss (1801), § 56] and [Dickson (1919), p. 181] this existence had been recognised by Lambert in 1769 and by Euler in 1773. Proofs appeared in Legendre's *Essai sur la théorie des nombres* [Legendre (1798/1808), § 341–2] and in Gauss's *Disquisitiones* [Gauss (1801), §§ 52–56] and Galois would surely have been familiar with these treatments—and would have assumed that his reader was familiar with them too. The phrase *la méthode de M. Gauss* must have a very different meaning at p. 68 from its meaning in the *Lettre testamentaire* **f.8b** (p. 86), and it is not clear to me what Galois had in mind at this point. On p. 70 *la formule de Newton* is simply the Binomial Theorem (for positive integer exponents). Although I have not studied the issue carefully enough to be certain, it seems possible that the mention of Libri on p. 72 refers either to [Libri (1830)] or to [Libri (1833), pp. 18–26]. The latter bears a publication date three years later than that of Galois, but it is annotated as having been read at the Académie des Sciences on 15 June 1825, so it is entirely possible that Galois knew something of its contents.

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218. Sur la théorie des nombres; par M. Galois.

(Ce mémoire fait partie des recherches de M. Galois sur la théorie des permutations et des équations algébriques).

Quand on convient de regarder comme nulles toutes les quantités qui, dans les calculs algébriques, se trouvent multipliées par un nombre premier donné p, et qu'on cherche dans cette convention les solutions d'une équation algébrique $\mathrm{F}\,x = o$, ce que M. Gauss désigne par la notation $\mathrm{F}\,x \equiv o$, on n'a coutume de considérer que les solutions entières de ces sortes de questions. Ayant été conduit par des recherches particulières à considérer les solutions incommensurables, je suis parvenu à quelques résultatsque je crois nouveaux.

Interword space missing in original.

Soit une pareille équation ou congruence, $\mathrm{F}\,x = o$ et p le module. Supposons d'abord, pour plus de simplicité, que la congruence en question n'admette aucun facteur commensurable, c'est-à-dire qu'on ne puisse pas trouver 3 fonctions $\varphi\,x$, $\psi\,x$, $\chi\,x$ telles que

$$\varphi\,x\,.\,\psi\,x = \mathrm{F}\,x + p\,\chi\,x\,.$$

Dans ce cas, la congruence n'admettra donc aucune racine entière, ni même aucune racine incommensurable du degré inférieur. Il faut donc regarder les racines de cette congruence comme des espèces de symboles imaginaires, puisqu'elles ne satisfont pas aux questions de nombres entiers, symboles dont l'emploi dans le calcul sera souvent aussi utile que celui de l'imaginaire $\sqrt{-1}$ dans l'analyse ordinaire.

C'est la classification de ces imaginaires et leur réduction au plus petit nombre possible, qui va nous occuper.

Appelons i l'une des racines de la congruence $\mathrm{F}\,x = o$, que nous supposerons du degré ν.

Considérons l'expression générale

$$a + a_1 i + a_2 i^2 + \ldots\ldots\ldots + a_{\nu-1} i^{\nu-1} \quad \text{(A)}$$

où $a\ \ a_1\ \ a_2\ \ldots\ldots\ a_{\nu-1}$ représentent des nombres entiers. En donnant à ces nombres toutes les valeurs, l'expression (A) en acquiert p^ν, qui jouissent, ainsi que je vais le faire voir, des mêmes propriétés que les nombres naturels dans la *théorie des résidus des puissances*.

Ne prenons des expressions (A) que les $p^\nu - 1$, valeurs où $a\ a_1\ a_2\ \ldots\ a_{\nu-1}$ ne sont pas toutes nulles: soit α l'une de ces expressions.

Otiose comma in original.

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Si l'on élève successivement α aux puissances $2^{\mathrm{e}}\ 3^{\mathrm{e}}\ \ldots$, on aura une suite de quantités de même forme (parce que toute fonction de i peut se réduire au $\nu - 1^{\mathrm{e}}$ degré). Donc on devra avoir $\alpha^n = 1$, n étant un certain nombre, soit n le plus petit

On the theory of numbers; by Mr Galois.

(This memoir forms part of the research by Mr Galois on the theory of permutations and algebraic equations.)

When one agrees to treat as zero all quantities in algebraic calculations which are multiplied by a given prime number p, and under this convention one looks for the solutions of an algebraic equation $Fx = 0$, which Mr Gauss writes using the notation $Fx \equiv 0$, it is customary to consider only integer solutions to this sort of question. Having been led by a particular line of research to consider incommensurable solutions I have come across some results that I believe to be new.

Let $Fx = 0$ be such an equation or congruence, and let p be the modulus. First let us suppose for simplicity that the congruence in question does not admit any commensurable factors, that is to say, that one cannot find 3 functions φx, ψx, χx such that

$$\varphi x.\psi x = Fx + p\,\chi x\,.$$

In this case, then, the congruence will have no integer root, nor even any incommensurable roots of lower degree. One must therefore think of the roots of this congruence as a species of imaginary symbols because they do not answer questions about whole numbers, symbols whose deployment in calculations will often be just as useful as that of the imaginary $\sqrt{-1}$ in ordinary analysis.

It is the classification of these imaginaries, and their reduction to the smallest possible number, that will occupy us.

Let us call i one of the roots of the congruence $Fx = 0$, which we suppose of degree ν.

Consider the general expression

$$a + a_1 i + a_2 i^2 + \cdots + a_{\nu-1} i^{\nu-1} \quad \text{(A)}$$

where a, a_1, a_2, …, $a_{\nu-1}$ represent whole numbers. Giving to these numbers all their values, the expression (A) takes p^ν of them, which, as I shall show, enjoy the same properties as natural numbers in the theory of residues of powers.

Take only the $p^\nu - 1$ values of the expression (A) where a, a_1, a_2, …, $a_{\nu-1}$ are not all zero: let α be one of these expressions.

If one raises α successively to the 2nd, 3rd, … powers one will have a sequence of quantities of the same form (because every function of i can be reduced to the $(\nu-1)^{\text{th}}$ degree). Therefore one must have $\alpha^n = 1$, n being some number. Let n be the smallest

nombre qui soit tel que l'on ait $\alpha^n = 1$. On aura un ensemble de n expressions toutes différentes entr'elles.

$$1 \quad \alpha \quad \alpha^2 \quad \alpha^3 \quad . \quad . \quad . \quad \alpha^{n-1}$$

The original uses an old-style β that I have been unable to reproduce faithfully.

Multiplions ces n quantités par une autre expression β de la même forme. Nous obtiendrons encore un nouveau groupe de quantités toutes différentes des premières et différentes entr'elles. Si les quantités (A) ne sont pas épuisées, on multipliera encore les puissances de α par une nouvelle expression γ, et ainsi de suite. On voit donc que le nombre n divisera nécessairement le nombre total des quantités (A). Ce nombre étant $p_\nu - 1$, on voit que n divise $p\nu - 1$. De là, suit encore que l'on aura

Misprints $p_\nu - 1$ (or possibly $p\nu - 1$) and $p\nu - 1$ corrected to $p^\nu - 1$ in L1846.

$$\alpha^{p^\nu - 1} = 1 \qquad \text{ou bien} \quad \alpha^{p^\nu} = \alpha\,.$$

Ensuite on prouvera, comme dans la théorie des nombres, qu'il y a des racines primitives α pour lesquelles on ait précisément $p^\nu - 1 = n$, et qui reproduisent par conséquent, par l'élévation aux puissances, toute la suite des autres racines.

Et l'une quelconque de ces racines primitives ne dépendra que d'une congruence du degré ν, congruence *irréductible*, sans quoi l'équation en i ne le serait pas non plus, parce que les racines de la congruence en i sont toutes des puissances de la racine primitive.

On voit ici cette conséquence remarquable, que toutes les quantités algébriques qui peuvent se présenter dans la théorie, sont racines d'équations de la forme

$$x^{p^\nu} = x$$

Changed in L1846 to 'Étant donnés'.

Cette proposition énoncée algébriquement, est celle-ci. Étant donné une fonction $\mathrm{F}\,x$ et un nombre premier p, on peut poser

$$f\,x\,.\,\mathrm{F}\,x = x^{p^\nu} - x + p\,\varphi\,x$$

$f\,x$ et $\varphi\,x$ étant des fonctions entières, toutes les fois que la congruence $\mathrm{F}\,x \equiv o \pmod{p}$ sera irréductible.

Si l'on veut avoir toutes les racines d'une pareille congruence

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au moyen d'une seule, il suffit d'observer que l'on a généralement

$$\left(\mathrm{F}\,x\right)^{p^n} = \mathrm{F}\left(x^{p^n}\right)$$

et que par conséquent l'une des racines étant x, les autres seront

number with the property that one has $\alpha^n = 1$. One will have a collection of n expressions all different from one another

$$1, \quad \alpha, \quad \alpha^2, \quad \alpha^3, \quad \ldots, \quad \alpha^{n-1}.$$

Multiply these n quantities by another expression β of the same form. We will obtain again a new group of quantities all different from the first and different from each other. If the quantities (A) are not all exhausted, one will again multiply the powers of α by a new expression γ, and so on. One sees thus that the number n will necessarily divide the total number of quantities (A). This number being $p^\nu - 1$, one sees that n divides $p^\nu - 1$. It then follows that one will have

$$\alpha^{p^\nu - 1} = 1 \quad \text{or equally} \quad \alpha^{p^\nu} = \alpha .$$

Then one proves, as in the theory of numbers, that there are primitive roots α for which one has $p^\nu - 1 = n$ precisely, and which consequently reproduce the whole sequence of the other roots on being raised to powers.

And any one of these primitive roots depends only on a congruence of degree ν, which is an *irreducible* congruence otherwise the equation for i would not be either, because all the roots of the congruence for i are powers of the primitive root.

One sees now the remarkable consequence that all the algebraic quantities which can appear in the theory are roots of equations of the form

$$x^{p^\nu} = x .$$

Expressed algebraically this proposition is the following. Given a function $\mathrm{F}\,x$ and a prime number p, whenever the congruence $\mathrm{F}\,x \equiv 0 \pmod{p}$ is irreducible one can set

$$f\,x\,.\,\mathrm{F}\,x = x^{p^\nu} - x + p\,\varphi\,x$$

$f\,x$ and $\varphi\,x$ being entire functions [polynomials].

If one wants to have all the roots of such a congruence

by means of a single one, it suffices to observe that generally one has

$$(F\,x)^{p^n} = F(x^{p^n})$$

and consequently x being one of the roots, the others will be

$$x^p \qquad x^{p^2} \qquad \ldots . x^{p^{\nu-1}} \qquad (1)$$

Il s'agit maintenant de faire voir que, réciproquement à ce que nous venons de dire, les racines de l'équation ou de la congruence $x^{p^\nu} = x$ dépendront toutes d'une seule congruence du degré ν.

Soit en effet i une racine d'une congruence irréductible, et telle que toutes les racines de la congruence $x^{p^\nu} = x$ soient fonctions rationnelles de i. (Il est clair qu'ici, comme dans les équations ordinaires, cette propriété a lieu) (2).

Il est d'abord évident que le degré μ de la congruence en i

(1) De ce que les racines de la congruence irréductible de degré ν,

$$\mathrm{F}\, x = 0$$

sont exprimées par la suite

$$x \quad x^p \quad x^{p^2} \quad \ldots x^{p^{r-1}}$$

Misprint corrected to $x^{p^\nu - 1}$ in all editions.

on aurait tort de conclure que ces racines soient toujours des quantités exprimable par radicaux. Voici un exemple du contraire:

La congruence irréductible

$$x^2 + x + 1 = 0 \qquad (\text{mod. } 2)$$

donne

$$x = \frac{-1 + \sqrt{-3}}{2}$$

qui se réduit

$$\frac{0}{0} \qquad (\text{mod. } p)$$

Misprint corrected to (mod. 2) in L1846.

formule qui n'apprend rien.

(2) La proposition générale dont il s'agit ici peut s'énoncer ainsi: étant donnée une équation algébrique, on pourra trouver une fonction rationnelle θ de toutes ses racines, de telle sorte, que réciproquement chacune des racines s'exprime rationnellement en θ. Ce théorême était connu d'Abel, ainsi qu'on peut le voir par la première partie du mémoire que ce célèbre géomètre a laissé sur les fonctions elliptiques.

Misprint for 'théorème' corrected in all editions.

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ne saurait être plus petit que ν, sans quoi la conguence

$$x^{p^\nu - 1} - 1 = 0\,. \qquad (\nu)$$

aurait toutes ses racines communes avec la congruence

$$x^{p^\mu - 1} - 1 = 0$$

ce qui est absurde, puisque la congruence (ν) n'a pas de racines égales, comme on le voit en prenant la dérivée du premier membre. Je dis maintenant que μ ne peut non plus être $> \nu$.

En effet, s'il en était ainsi, toutes les racines de la congruence

Read as $x^{p^\mu} = x$ in L1846

$$x^{p\mu} = x$$

$$x^p, \quad x^{p^2}, \quad \ldots, \quad x^{p^{\nu-1}}. \quad (1)$$

Our concern now is to show the converse of what we have just said: the roots of the equation or of the congruence $x^{p^\nu} = x$ will all depend on a single congruence of degree ν.

Indeed, let i be a root of an irreducible congruence, and such that all the roots of the congruence $x^{p^\nu} = x$ are rational functions of i. (It is clear that here, as in ordinary equations, this property will hold) (2).

It is clear from the start that the degree μ of the congruence for i

(1) One would be wrong to conclude from the fact that the roots of the irreducible congruence

$$Fx = 0$$

of degree ν are expressed by the sequence

$$x^p \quad x^{p^2} \quad \ldots \quad x^{p^{\nu-1}}$$

that these roots are always quantities expressible by radicals. Here is a counterexample:

The irreducible congruence

$$x^2 + x + 1 = 0 \pmod 2$$

gives

$$x = \frac{-1 + \sqrt{-3}}{2}$$

which reduces to

$$\frac{0}{0} \pmod 2$$

a formula which says nothing.

(2) The general proposition that is relevant here can be formulated thus: given an algebraic equation, one can find a rational function θ of all its roots, of such a kind that conversely each of the roots may be expressed rationally in θ. This theorem was known to Abel as one can see in the first part of the memoir on elliptic functions which this famous geometer has left us.

could not be smaller than ν, otherwise the congruence

$$x^{p^\nu - 1} - 1 = 0 \qquad (\nu)$$

would have all its roots in common with the congruence

$$x^{p^\mu - 1} - 1 = 0,$$

which is absurd because the congruence (ν) has no equal roots, as is seen by taking the derivative of the first member. I say now that also μ cannot be $> \nu$.

Indeed, if it were so then all the roots of the congruence

$$x^{p^\mu} = x$$

devraient dépendre rationnellement de celles de la congruence

$$x^{p^\nu} = x$$

Read as $x^{p^\nu} = x$ in L1846

Mais il est aisé de voir que si l'on a

$$i^{p^\nu} = i$$

Read as $i^{p^\nu} = i$ in L1846. Paragraphing corrected in L1846. Indent removed, capitalisation retained in BA1962.

Toute fonction rationnelle $h = f\,i$ donnera encore

$$\left(f\,i\right)^{p^\nu} = f\left(i^{p^\nu}\right) = f\,i, \quad \text{d'où} \quad h^{p^\nu} = 4$$

The three exponents read as p^ν and last symbol corrected to h in L1846.

Donc toutes les racines de la congruence $x^{p^\mu} = x$ lui seraient communes avec l'équation $x^{p^\nu} = x$. Ce qui est absurde.

Nous savons donc enfin que toutes les racines de l'équation ou congruence $x^{p^\nu} = x$ dépendent nécessairement d'une *seule* congruence *irréductible* de degré ν.

Maintenant, pour avoir cette congruence irréductible d'où dépendent les racines de la conguence $x^{p^\nu} = x$, la méthode la plus générale sera de délivrer d'abord cette congruence de tous les facteurs communs qu'elle pourrait avoir avec des congruences de degré inférieur et de la forme

$$x^{p^\mu} = x$$

On obtiendra ainsi une congruence qui devra se partager en congruences irréductibles de degré ν. Et comme on sait exprimer toutes les racines de chacune de ces congruences irréductibles au moyen d'une seule, il sera aisé de les obtenir toutes par la méthode de M. Gauss.

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Le plus souvent, cependant, il sera aisé de trouver par le tâtonnement une congruence irréductible d'un degré donné ν, et on doit en déduire toutes les autres.

Soient, par exemple, $p = 7 \quad \nu = 3$. Cherchons les racines de la congruence

$$x^{7^3} = x \qquad \text{(mod. 7.)} \tag{1}$$

J'observe que la congruence

$$i^3 = 2 \qquad \text{(mod. 7.)} \tag{2}$$

étant irréductible et du degré 3, toutes les racines de la congruence (1) dépendent rationnellement de celles de la congruence (2), en sorte que toutes les racines de (1) sont de la forme

$$a + a_1 i + a_2 i^2 \qquad \text{ou bien} \qquad a + a_1 \sqrt[3]{2} + a_2 \sqrt[3]{4} \tag{3}$$

Il faut maintenant trouver une racine primitive, c'est-à-dire une forme de l'expression (3) qui, élevée à toutes les puissances, donne toutes les racines de la congruence

$$x^{7^3-1} = 1 \quad \text{savoir} \quad x^{2^1.3^2.19} = 1 \quad \text{(mod. 7.)}$$

would have to depend rationally on those of the congruence

$$x^{p^\nu} = x\,.$$

But it is easy to see that if one has

$$i^{p^\nu} = i\,,$$

every rational function $h = f\,i$ would yield also

$$(f\,i)^{p^\nu} = f(i^{p\nu}) = f\,i, \quad \text{from which} \quad h^{p^\nu} = h\,.$$

Therefore all the roots of the congruence $x^{p^\mu} = x$ would be in common with those of the equation $x^{p^\nu} = x$, which is absurd.

We know finally then that all the roots of the equation or congruence $x^{p^\nu} = x$ necessarily depend on a single irreducible congruence of degree ν.

Now the most general method to get this irreducible congruence on which the roots of the congruence $x^{p^\nu} = x$ depend will be first to rid this congruence of all the factors that it could have in common with congruences of lower degree and which are of the form

$$x^{p^\mu} = x\,.$$

In this way one will get a congruence which must be partitioned [factorised] into irreducible congruences of degree ν. And since one knows how to express all the roots of each of these irreducible congruences by means of a single one it will be easy to obtain all of them by the method of Mr Gauss.

Most often, however, it will be easy to find an irreducible congruence of degree ν by trial and error, and one should deduce all the others from it.

For example, let $p = 7,\ \nu = 3$. Let us seek the roots of the congruence

$$x^{7^3} = x \pmod 7. \tag{1}$$

I observe that the congruence

$$i^3 = 2 \pmod 7 \tag{2}$$

being irreducible and of degree 3, all the roots of the congruence (1) depend rationally on those of the congruence (2), so that all the roots of (1) are of the form

$$a + a_1 i + a_2 i^2 \qquad \text{or} \qquad a + a_1 \sqrt[3]{2} + a_2 \sqrt[3]{4}\,. \tag{3}$$

Now it is necessary to find a primitive root, that is, a form of the expression (3) which, when raised to all powers, gives all the roots of the congruence

$$x^{7^3-1} = 1, \quad \text{that is} \quad x^{2^1.3^2.19} = 1 \pmod 7\,,$$

et nous n'avons besoin pour cela que d'avoir une racine primitive de chaque congruence

$$x^2 = 1 \qquad x^{3^2} = 1 \qquad x^{19} = 1$$

La racine primitive de la première est -1; celles de $x^{3^2} - 1 = 0$ sont données par les équations

$$x^3 = 2 \qquad x^3 = 4$$

en sorte que i est une racine primitive de $x^{3^2} = 1$.

Il ne reste qu'à trouver une racine de $x^{19} - 1 = 0$ ou plutôt de

$$\frac{x^{19} - 1}{x - 1} = 0$$

et essayons pour cela si l'on ne peut pas satisfaire à la question en posant simplement $x = a + a_1 i$, au lieu de $a + a_1 i + a_2 i^2$, nous devrons avoir

$$(a + a_1 i)^{19} = 1$$

ce qui, en développant par la formule de Newton et réduisant les puissances de a, de a_1 et de i par les formules

$$a^{m(p-1)} = 1 \qquad a_1^{m(p-1)} = 1 \qquad i^3 = 2$$

se réduit à

$$3\left\{a - a^4 a_1^3 + 3(a^5 a_1^2 + a^2 a_1^5) i^2\right\} = 1$$

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d'où, en séparant

Second equation corrected in L1846. See English.

$$3a - 3a^4 a^3 = 1, \quad a^5 a_1^2 + a^2 + a_1^5 = 0$$

Second equation corrected to $a_1 = 1$ in L1846; then $-1 - i$ corrected to $-1 + i$ hereafter.

Ces deux dernières équations sont satisfaites en posant $a = -1$, $a_1 = -1$. Donc

$$-1 - i$$

est une racine primitive de $x^{19} = 1$. Nous avons trouvé plus haut pour racines primitives de $x^2 - 1$ et de $x^{3^2} = 1$ les valeurs -1 et i, il ne reste plus qu'à multiplier entr'elles les 3 quantités

Misprint corrected to x^{3^2} in L1846.

$$-1, \quad i, \quad -i - 1$$

Here and below $i + i^2$ corrected to $i - i^2$ in L1846.

et le produit $i + i^2$ sera une racine primitive de la congruence

$$x^{7^3 - 1} = 1$$

Donc ici l'expression $i + i^2$ jouit de la propriété, qu'en l'élevant à toutes les puissances, on obtiendra $7^3 - 1$ expressions différentes de la forme

$$a + a_1 i + a_2 i^2$$

and for that we only need to have a primitive root of each congruence

$$x^2 = 1\,, \qquad x^{3^2} = 1\,, \qquad x^{19} = 1\,.$$

The primitive root of the first is -1; those of $x^{3^2} - 1 = 0$ are given by the equations

$$x^3 = 2\,, \qquad x^3 = 4\,,$$

so that i is a primitive root of $x^{3^2} = 1$.

It remains only to find a root of $x^{19} - 1 = 0$ or rather of

$$\frac{x^{19} - 1}{x - 1} = 0\,,$$

and for that, trying if one might answer the question by setting simply $x = a + a_1 i$ instead of $a + a_1 i + a_2 i^2$, we will need to have

$$(a + a_1 i)^{19} = 1\,,$$

which, on expanding by Newton's formula and reducing the powers of a, a_1 and i by the formulae

$$a^{m(p-1)} = 1\,, \qquad a_1^{m(p-1)} = 1\,, \qquad i^3 = 2\,,$$

reduces to

$$3\{a - a^4 a_1^3 + 3(a^5 a_1^2 + a^2 a_1^5) i^2\} = 1$$

from which, on separating,

$$3a - 3a^4 a_1^3 = 1, \qquad a^5 a_1^2 + a^2 a_1^5 = 0\,.$$

These last two equations are satisfied by setting $a = -1$, $a_1 = 1$. Therefore

$$-1 + i$$

is a primitive root of $x^{19} = 1$. We have found above the values -1 and i as primitive roots of $x^2 - 1$ and of $x^{3^2} = 1$, and what remains is only to multiply together the 3 quantities

$$-1, \quad i, \quad i - 1$$

and the product $i - i^2$ will be a primitive root of the congruence

$$x^{7^3 - 1} = 1\,.$$

Thus here the expression $i - i^2$ enjoys the property that on raising it to all powers one obtains $7^3 - 1$ different expressions of the form

$$a + a_1 i + a_2 i^2.$$

Si nous voulons avoir la congruence de moindre degré d'où dépend notre racine primitive, il faut éliminer i entre les deux équations

$$i^3 = 2 \qquad \alpha = i + i^2$$

Corrected to $\alpha = i - i^2$ in L1846.

On obtient ainsi

$$\alpha^3 + 3\alpha + 1 = 0$$

Corrected to $\alpha^3 - \alpha + 2 = 0$ in L1846.

Il sera convenable de prendre pour base des imaginaires, et de représenter par i la racine de cette équation, ensorte que

$$i^3 + 3i + 1 = 0 \qquad (i)$$

Corrected to $i^3 - i + 2 = 0$ in L1846.

et l'on aura toutes les imaginaires de la forme

$$a + a_1 i + a_2 i^2$$

en élevant i à toutes les puissances, et réduisant par l'équation (i).

Le principal avantage de la nouvelle théorie que nous venons d'exposer, est de ramener les congruences à la propriété (si utile dans les équations ordinaires) d'admettre précisément autant de racines qu'il y a d'unités dans l'ordre de leur degré.

La méthode pour avoir toutes ces racines sera très simple. Premièrement on pourra toujours préparer la congruence donnée $\mathrm{F}x = o$, de manière à ce qu'elle n'ait plus de racines égales, ou en d'autres termes, à ce qu'elle n'ait plus de facteur

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commun avec $\mathrm{F}'x = o$, et le moyen de le faire est évidemment le même que pour les équations ordinaires.

Ensuite, pour avoir les solutions entières, il suffira, ainsi que M. Libri paraît en avoir fait le premier la remarque, de chercher le plus grand facteur commun à $\mathrm{F}x = o$ et à $x^{p-1} = 1$.

Si maintenant on veut avoir les solutions imaginaires du second degré, on cherchera le plus grand facteur commun à $\mathrm{F}x = o$ et à $x^{p^2-1} = 0$ et en général, les solutions de l'ordre ν seront données par le plus grand commun diviseur à $\mathrm{F}x = o$ et à $x^{p^\nu - 1} = 1$.

Misprint corrected to $x^{p^2-1} = 1$ in L1846; no comment in BA1962.

C'est surtout dans la théorie des permutations, où l'on a sans cesse besoin de varier la forme desindices, que la considération des racines imaginaires des congruences paraît indispensable. Elle donne un moyen simple et facile de reconnaître dans quel cas une équation primitive est soluble par radicaux, comme je vais essayer d'en donner en deux mots une idée.

Interword space missing in original.

Soit une équation algébrique $f\,x = o$ de degré p^ν; supposons que les p^ν racines soient désignées par x_k, en donnant à l'indice k les p^ν valeurs déterminées par la congruence $k^{p^\nu} = k \pmod{p}$.

If we wish to have the congruence of least degree on which our primitive root depends, it is necessary to eliminate i between the two equations

$$i^3 = 2\,, \qquad \alpha = i - i^2\,.$$

In this way one obtains

$$\alpha^3 - \alpha + 2 = 0\,.$$

It will be convenient to represent by i the root of this equation, so that

$$i^3 - i + 2 = 0\,, \qquad (i)$$

and to take it as basis of the imaginaries, so that one will have all the imaginaries in the form

$$a + a_1 i + a_2 i^2$$

by raising i to all powers, and reducing by the equation (i).

The principal benefit of the new theory that we have just expounded is to carry over to congruences the property (so useful in ordinary equations) of admitting precisely as many roots as there are units in the order of their degree.

The method to get all these roots will be very simple. First one can always prepare [adjust] the given congruence $Fx = 0$ in such a way that it does not have equal roots any more, or in other words, so that it will have no factor

in common with $F'x = 0$, and the way to do this is clearly the same as for ordinary equations.

After that, to get the integer solutions it will suffice, as Mr Libri seems to have been the first to notice, to seek the greatest common factor of $Fx = 0$ and $x^{p-1} = 1$.

If one now wishes to have the imaginary solutions of the second degree one will seek the greatest common factor of $Fx = 0$ and $x^{p^2-1} = 1$, and in general the solutions of order ν will be given by the greatest common factor of $Fx = 0$ and $x^{p^\nu - 1} = 1$.

It is above all in the theory of permutations, where one forever needs to vary the form of indices, that consideration of the imaginary roots of congruences appears to be indispensable. It gives a simple and easy means of recognising the cases in which a primitive equation is soluble by radicals, of which I shall try to give an idea in two words.

Let $fx = 0$ be an algebraic equation of degree p^ν. Suppose that the p^ν roots are denoted by x_k, where the index k is given the p^ν values determined by the congruence $k^{p^\nu} = k \pmod p$.

Prenons une fonction quelconque rationnelle V des p^ν racines x_k. Transformons cette fonction en substituant partout à l'indice k l'indice $(ak+b)^{p^r}$, a, b, r étant des constantes arbitraires satisfaisant aux conditions de $a^{p^\nu-1} = 1 \quad b^{p^\nu} = b \pmod{p}$ et de r entier.

En donnant aux constantes a, b, r toutes les valeurs dont elles sont susceptibles, on obtiendra en tout $p^\nu(p^\nu-1)\nu$, manières de permuter les racines entr'elles par des substitutions de la forme $\left(x_k, \quad x_{(ak+b)^{p^r}}\right)$, et la fonction V admettra en général par ces substitutions $p^\nu(p^\nu-1)\nu$, formes différentes.

Otiose comma removed in L1846.

Otiose comma removed in L1846.

Admettons maintenant que l'équation proposée $f\,x = o$ soit telle, que toute fonction des racines invariable par les $p^\nu(p^\nu-1)\nu$ permutations que nous venons de construire, ait pour cela même une valeur numérique rationnelle.

Perhaps *permutations* should be *substitutions*.

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On remarque que dans ces circonstances, l'équation $f\,x = o$ sera soluble par radicaux, et pour parvenir à cette conséquence, il suffit d'observer que la valeur substituée à k dans chaque indice peut se mettre sous les trois formes

$$\left(ak+b\right)^{p^r} = \left(a\left\{k+b^1\right\}\right)^{p^r} = a^1k^{p^r} + b'' = a'\left(k+b'\right)^{p^r}$$

Les personnes habituées à la théorie des équations le verront sans peine.

Cette remarque aurait peu d'importance, si je n'étais parvenu à démontrer que réciproquement une équation primitive ne saurait être soluble par radicaux, à moins de satisfaire aux conditions que je viens d'énoncer. (J'excepte les équations du 9e et du 25e degré).

Ainsi, pour chaque nombre de la forme p^ν, on pourra former un groupe de permutations tel que toute fonction des racines invariable par ces permutations devra admettre une valeur rationnelle quand l'équation de degré p^ν sera primitive et soluble par radicaux.

Perhaps 2nd occurrence of *permutations* should be *substitutions*.

D'ailleurs, il n'y a que les équations d'un pareil degré p^ν qui soient à la fois primitives et solubles par radicaux.

Le théorème général que je viens d'énoncer précise et développe les conditions que j'avais données dans le *Bulletin* du mois d'avril. Il indique le moyen de former une fonction des racines dont la valeur sera rationnelle, toutes les fois que l'équation primitive de degré p^ν sera soluble par radicaux, et méne par conséquent aux caractères de résolubilité de ces équations, par des calculs sinon praticables, du moins qui sont possibles en théorie.

Misprint 'méne' corrected to 'mène' in L1846, BA1962.

Il est à remarquer que dans le cas où $\nu = 1$, les diverses valeurs de k ne sont autre chose que la suite des nombres entiers. Les substitutions de la forme $\left(x_k, \; x_{ak+b}\right)$ seront au nombre de $p(p-1)$.

La fonction qui, dans le cas des équations solubles par radicaux, doit avoir une valeur rationnelle, dépendra en général d'une équation de degré $1.2.3\ldots\ldots(p-2)$, à laquelle il faudra par conséquent appliquer la méthode des racines rationnelles.

Take an arbitrary rational function V of the p^ν roots x_k. Transform this function by substituting for the index k the index $(ak+b)^{p^r}$ throughout, a, b, r being arbitrary constants satisfying the conditions that $a^{p^\nu-1} = 1$, $b^{p^\nu} = b \pmod p$ and that r be a whole number.

Giving to the constant a, b, r all the values of which they are capable, one will obtain in all $p^\nu(p^\nu-1)\nu$ ways of permuting the roots amongst themselves by substitutions of the form $(x_k, x_{(ak+b)^{p^r}})$, and under these substitutions the function V will in general admit $p^\nu(p^\nu-1)\nu$ different forms.

Now let us accept moreover that the proposed equation $fx = 0$ is such that every function of the roots invariant under the $p^\nu(p^\nu-1)\nu$ permutations that we have just constructed will take a numerical value that is rational.

Note that under these circumstances the equation $fx = 0$ will be soluble by radicals, and to reach this conclusion it suffices to observe that the value substituted for k in each index can be put into the three forms

$$(ak+b)^{p^r} = (a\{k+b'\})^{p^r} = a'k^{p^r} + b'' = a'(k+b')^{p^r}.$$

People who are used to the theory of equations will see this without trouble.

This remark would have little importance had I not been able to prove that conversely a primitive equation cannot be soluble by radicals unless it satisfies the conditions that I have just formulated. (I except equations of the 9th and of the 25th degree.)

Thus for each number of the form p^ν, one may form a group of permutations such that every function of the roots invariant under these permutations [its substitutions] will have to admit a rational value when the equation of degree p^ν is primitive and soluble by radicals.

Moreover, it is only equations of such a degree p^ν that can be both primitive and soluble by radicals.

The general theorem that I have just formulated makes more precise and develops the conditions that I gave in the *Bulletin* for the month of April. It shows the way to form a function of the roots whose value will be rational whenever the primitive equation of degree p^ν is soluble by radicals, and consequently leads to the characteristics [conditions] for solubility of these equations by calculations which, if not practicable, at least are possible in theory.

It is worth noting that in the case where $\nu = 1$ the various values of k are none other that the sequence of whole numbers. The substitutions of the form (x_k, x_{ak+b}) will be $p(p-1)$ in number.

In the case of equations that are soluble by radicals the function which must have a rational value will depend in general on an equation of degree $1.2.3...(p-2)$, to which it will therefore be necessary to apply the method of rational roots.

II.5 On some points of analysis

This paper was published in Gergonne's *Annales de Mathématiques pures et appliquées*, 21, 183–184 (December 1830), when Galois had just started at the École Normale. It appears in a section of the journal containing material classified as *Analyse transcendante*. It was reprinted in [Liouville (1846), pp. 392–394], in [Picard (1897), pp. 9–10], and in [B & A (1962), pp. 382–385]. There are several misprints in the Gergonne original—beginning with the author's name. Most of them were corrected in the reprints but they are faithfully reproduced here. Punctuation and typography are also adjusted in those editions—I have tried to copy the original as far as modern typography permits.

I have had some difficulties with the translation here: the phrase *fonction déterminée* could be any of 'determinate function', 'well-defined function', 'certain function', 'specific function'. Indeed, the context indicates that its meaning is subtly different at each of its three occurrences. The phrase *perpendiculaire abaissée* is usually translated as 'altitude' (in the context of geometry of triangles). Here the translation 'perpendicular dropped [from ... to ...]' fits the meaning much better. Note that in modern English 'radius' is ambiguous: it can (and usually does) mean the distance from the centre of a circle to any of its points; but it can also mean, and in the past very frequently did mean, a line segment from the centre to a point on the circumference.

Galois gives us no clue as to what stimulated him to write the two little essays in this paper. It might have been passages in textbooks, it might have been articles in one or another of the journals that he had read. One would have circumstantial evidence if one identified passages in other writings that treated similar questions with similar notation, or if one could identify to what 'known theorems' of differential geometry the last sentence refers. I have not succeeded—but nor have I embarked on any systematic search.

This article, with its two separate little essays, seems to me to be of a different calibre and style from most of Galois' mathematical writing. Had I been editor, and had I been given the choice, I might have been inclined to publish either or both of the essays in Dossier 21 (on the integration of linear differential equations) and in Dossier 22 (on surfaces of the second degree) rather than this. They were written at about the same time; they seem to me to be less inconsequential; they seem to me to be more polished and more convincing as mathematics. But these are matters of taste and judgment inappropriate to an editor. My substantive point is that those two unpublished essays may be compared in very broad terms with this published article.

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Notes sur quelques points d'analyse;

par M. GALAIS, élève à l'Ecole normale

'Galais' *sic*! All editions note the misprint.

§. I.

Démontration d'un théorème d'analyse.

'Démontration' *sic*! All editions correct or overlook the misprint.

THÉORÈME. Soient Fx et fx deux fonctions quelconques données; on aura, quels que soient x et h,

$$\frac{\mathrm{F}(x+h)-\mathrm{F}x}{\mathrm{f}(x+h)-\mathrm{f}x}=\varphi(k)\,,$$

φ étant une fonction déterminée, et k une quantité intermédiaire entre x et $x+h$.

Démonstration. Posons, en effet,

$$\frac{\mathrm{F}(x+h)-\mathrm{F}x}{\mathrm{f}(x+h)-\mathrm{f}x}=P\ ;$$

on en déduira

$$\mathrm{F}(x+h)-P\mathrm{f}(x+h)=\mathrm{F}x-P\mathrm{f}x\ ,$$

d'où l'on voit que la fonction F$x-P$fx ne change pas quand on y change x en $x+h$; d'où il suit qu'à moins qu'elle ne reste constante entre ces limites, ce qui ne pourrait avoir lieu que dans des cas particuliers, cette fonction aura, entre x et $x+h$, un ou plusieurs *maxima* et *minima*. Soit k la valeur de x répondant à l'un d'eux; on aura évidemment

$$k=\psi(P)\ ,$$

ψ étant une fonction déterminée; donc on doit avoir aussi

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$$P=\varphi(k)\ ,$$

φ étant une autre fonction également déterminée; ce qui démontre le théorème.

De là on peut conclure, comme corollaire, que la quantité

$$\mathrm{Lim.}\frac{\mathrm{F}(x+h)-\mathrm{F}x}{\mathrm{f}(x+h)-\mathrm{f}x}=\varphi(x)\ ,$$

pour $h=0$, est nécessairement une fonction de x, ce qui démontre, à *priori*, l'existence des fonctions dérivées.

Printed roman in L1846, BA1962; corrected to 'a priori' in BA1962.

§. II.

Rayon de courbure des courbes dans l'espace.

Word 'dans' changed to 'de' in P1897, although L1846 is faithful.

Le rayon de courbure d'une courbe en l'un quelconque de ses points M est la perpendiculaire abaissée de ce point sur l'intersection du plan normal au point M

Notes on some points of analysis

by Mr GALOIS, student at the Ecole normale

§. I.

Proof of a theorem of analysis.

THEOREM. *Let* $\mathrm{F}x$ *and* $\mathrm{f}x$ *be two arbitrarily given functions. Whatever* x *and* h *may be one will have*

$$\frac{\mathrm{F}(x+h)-\mathrm{F}x}{\mathrm{f}(x+h)-\mathrm{f}x}=\varphi(k)\,,$$

φ being a certain function and k a quantity intermediate between x and $x+h$.

For translation of 'déterminée' see p. 77.

Proof. Indeed, set

$$\frac{\mathrm{F}(x+h)-\mathrm{F}x}{\mathrm{f}(x+h)-\mathrm{f}x}=P\,.$$

From this it may be deduced that

$$\mathrm{F}(x+h)-P\mathrm{f}(x+h)=\mathrm{F}x-P\mathrm{f}x\,,$$

from which it may be seen that the function $\mathrm{F}x-P\mathrm{f}x$ does not change when x is changed to $x+h$; from which it follows that, unless it remains constant between these limits, which could not happen except in special cases, this function will have one or more maxima and minima between x and $x+h$. Let k be the value of x corresponding to one of them. Clearly one will have

$$k=\psi(P)\,,$$

ψ being a certain function. Therefore one must also have

$$P=\varphi(k)\,,$$

φ being another, equally well defined function; which proves the theorem.

One may conclude from that, as a corollary, that the quantity

$$\mathrm{Lim}\frac{\mathrm{F}(x+h)-\mathrm{F}x}{\mathrm{f}(x+h)-\mathrm{f}x}=\varphi(x)\,,$$

for $h=0$, is necessarily a function of x, which proves *a priori* the existence of derived functions.

§. II.

Radius of curvature of curves in space.

The radius of curvature of a curve at an arbitrary one of its points M is the perpendicular dropped from this point to the intersection of the normal plane at the point M

avec le plan normal consécutif, comme il est aisé de s'en assurer par des considérations géométriques.

Cela posé, soit (x, y, z) un point de la courbe; on sait que le plan normal en ce point aura pour équation

$$(X - x)\frac{\mathrm{d}x}{\mathrm{d}s} + (Y - y)\frac{\mathrm{d}y}{\mathrm{d}s} + (Z - z)\frac{\mathrm{d}z}{\mathrm{d}s} = 0 \,. \qquad \text{(N)}$$

X, Y, Z étant les symboles des coordonnées courantes. L'intersection de ce plan normal avec le plan normal consécutif sera donnée par le système de cette équation et de la suivante

$$(X - x)\frac{\mathrm{d.}\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)}{\mathrm{d}s} + (Y - y)\frac{\mathrm{d.}\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)}{\mathrm{d}s} + (Z - z)\frac{\mathrm{d.}\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)}{\mathrm{d}s} = 1 \,, \qquad \text{(I)}$$

attendu que

$$\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)^2 = 1 \,.$$

Or, il est aisé de voir que le plan (I) est perpendiculaire au plan (N); car l'on a

$$\frac{\mathrm{d}x}{\mathrm{d}s}\,\mathrm{d.}\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right) + \frac{\mathrm{d}y}{\mathrm{d}s}\,\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right) + \frac{\mathrm{d}z}{\mathrm{d}s}\,\mathrm{d}\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right) = 0 \,;$$

Misprints: stops missing after 2nd and 3rd occurrences of d supplied in all editions.

donc la perpendiculaire abaissée du point (x, y, z) sur l'intersection des deux plans (N) et (I) n'est autre chose que la perpendiculaire abaissée du meme point sur le plan (I). Le rayon de courbure est donc la perpendiculaire abaissé du point (x, y, z) sur le plan (I). Cette considération donne, très-simplement, les théorèmes connus sur les rayons de courbure des courbes dans l'espace.

Misprints 'meme' and 'abaissé' corrected to 'même' and 'abaissée' in all editions.

with the consecutive [adjacent] normal plane, as one may easily persuade oneself by geometrical considerations.

That said, let (x, y, z) be a point of the curve. It is known that the normal plane at this point will have equation

$$(X - x)\frac{\mathrm{d}x}{\mathrm{d}s} + (Y - y)\frac{\mathrm{d}y}{\mathrm{d}s} + (Z - z)\frac{\mathrm{d}z}{\mathrm{d}s} = 0 \,, \qquad \text{(N)}$$

X, Y, Z being symbols for the usual coordinates. The intersection of this normal plane with the consecutive [adjacent] normal plane will be given by the system of this equation and the following

$$(X - x)\frac{\mathrm{d.}\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)}{\mathrm{d}s} + (Y - y)\frac{\mathrm{d.}\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)}{\mathrm{d}s} + (Z - z)\frac{\mathrm{d.}\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)}{\mathrm{d}s} = 1 \,, \qquad \text{(I)}$$

given that

$$\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)^2 = 1 \,.$$

Now it is easy to see that the plane (I) is perpendicular to the plane (N), for one has

$$\frac{\mathrm{d}x}{\mathrm{d}s}\,\mathrm{d.}\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right) + \frac{\mathrm{d}y}{\mathrm{d}s}\,\mathrm{d.}\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right) + \frac{\mathrm{d}z}{\mathrm{d}s}\,\mathrm{d.}\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right) = 0 \,.$$

Therefore the perpendicular dropped from the point (x, y, z) to the intersection of the two planes (N) and (I) is nothing other than the perpendicular dropped from this self-same point to the plane (I). The radius of curvature is therefore the perpendicular dropped from the point (x, y, z) to the plane (I). This consideration gives very easily the known theorems on radius of curvature of curves in space.

Chapter III
The Testamentary Letter of 29 May 1832

III.1 The letter

The Testamentary Letter of 29 May 1832, the eve of the fatal duel, was addressed to Auguste Chevalier. It occupies Dossier 2 of the Galois manuscripts, the cover-sheet of which is simply headed 'Lettre à Auguste Chevalier/ du 29 mai 1832', echoing the description written by Galois himself in the top left corner of the first page. It is written on two large sheets of paper, 39.7 cm × 31.3 cm folded double to make two pamphlets 19.9 cm × 31.3 cm, folios 8–11, of which it occupies all but the last side. There is a left margin which is generally about 2 cm; there are no other margins.

As Galois had requested (see p. 96), Chevalier had it published in the *Revue encyclopédique*, where it appeared in September 1832 as [Lettre (1832)]. It has been re-published many times, in, [Liouville (1846), pp. 408–415], in [Picard (1897), pp. 25–32], in [B & A (1962), pp. 172–185, 487–491], in the booklet [APMEP (1982), pp. 22–35], where an excellent facsimile is included, and probably in many other places. There is a German translation in [Maser (1889)], there is an English translation by Louis Weisner in [Smith (1929), pp. 278–285], there is an Italian translation in [Toti Rigatelli (2000)], and very probably there are translations into other languages that I do not know about.

Unlike many other Galois manuscripts the letter has generous paragraph indentations, which I have tried to emulate here. Also unlike many other important Galois manuscripts there is no extant copy by Chevalier—see Note 2, p. 101 below for some comment on this point.

8a

Lettre à Auguste Chevalier.

Paris, le 29 Mai 1832

Mon cher Ami,

J'ai fait en analyse plusieurs choses nouvelles.

Les unes concernent la théorie des Équations; les autres les fonctions Intégrales.

Dans la théorie des équations, j'ai recherché dans quels cas les équations étaient résolubles par des radicaux: ce qui m'a donné occasion d'approfondir cette théorie, et de décrire toutes les [???]⌜transformations⌝ possibles sur une équation, lors même qu'elle n'est pas soluble par radicaux.

~~Je commencerai, &c. ... que nous avons cru devoir rappeler.~~

On pourra faire avec tout celà trois mémoires.

Le premier est écrit, et malgré ce qu'en a dit Poisson, je le maintiens avec les corrections que j'y ai faites.

Le ~~troisième~~ second ~~est~~ contient des applications assez curieuses de la théorie des équations. ~~En~~ Voici le résumé des choses les plus importantes.

1º. D'après les propositions II et III du 1er Mémoire, on voit une grande différence entre adjoindre à une équation une ^des^ racine⌜s⌝ d'une équation auxiliaire ou les adjoindre toutes.

Dans les deux cas le groupe de l'équation se partage par l'adjonction ~~qui~~ en groupes ~~qui~~ tels que l'on passe de l'un à l'autre par une même substitution. Mais la condition que ~~les~~ ~~mem~~ ces groupes aient les mêmes substitutions n'a lieu certainement que dans le second cas. ~~En d'autres t~~ Celà s'appelle la décomposition propre.

En d'autres termes, quand un groupe ^G^ en contient un autre H le groupe G peut se partager en groupes ~~dans les quels à la place~~ que l'on obtient chacun en opérant sur les permutations de H une même substitution, en sorte $G = H + HS + HS' + \cdots$ et aussi il peut se décomposer en groupes qui ont tous les mêmes substitutions en sorte que $G = H + TH + T'H + \cdots$ Ces deux ^genres de^ décompositions ne coïncide pas ordinairement. Quand elles coïncident, la décomposition est dite propre.

Il est aisé de voir que quand ^le groupe d'^une équation n'est susceptible d'aucune décomposition propre, ~~cet~~ on aura beau transformer cette equation, les groupes des équations transformées auront toujours le même nombre de permutations.

Salutation suppressed in C1832; 'Ami' lower-case in L1846; 'analyse' capitalised, 'Équations', 'Intégrales' lower-case in P1897; all lower-case in C1832, L1846.

Word over-written and obliterated by 'transformations'.

Deleted line echos beginning and end of **f. 2 b** of *Premier Mémoire*.

In C1832 'M. Poisson'. Also word 'maintiens' misread as 'soutiens'.

C1832, L1846 change 1er to 'premier'.

Deletion completed to 'En d'autres termes' in BA1962. Originally it started a new paragraph. Overwritten with 'Celà ... propre'.

Missing word 'que' supplied after 'en sorte' in C1832, L1846. Equations displayed, new sentence started with 'Et aussi' in C1832, L1846.

'Ces deux' starts new paragraph in C1832, L1846, BA1962. Singular 'coïncide' corrected to 'coïncident' in C1832, L1846, BA1962.

Letter to Auguste Chevalier.

Paris, 29 May 1832

My dear friend,

I have done several new things in analysis.

Some concern the theory of equations, others integral functions.

In the theory of equations I have looked for the circumstances under which equations were soluble by radicals; this has given me occasion to deepen this theory and to describe all possible transformations on an equation even in case it is not soluble by radicals.

~~I will begin, &c. ... that we have believed we should recall.~~

Three memoirs could be made from all this.

The first is written, and in spite of what Poisson has said about it I stand by it with the corrections that I have made in it.

The ~~third~~ second ~~is~~ contains some pretty interesting applications of the theory of equations. Here is a summary of the most important things.

1° According to Propositions II and III of the first memoir one sees a great difference between adjoining to an equation one of the roots of an auxiliary equation or adjoining them all.

In both cases the group of the equation is partitioned by the adjunction into groups such that one passes from one to another by one and the same substitution; but the condition that these groups should have the same substitutions does not necessarily hold except in the second case. ~~In other w[ords]~~ That is called a proper decomposition.

In other words, when a group G contains another H, the group G can be partitioned into groups ~~in which in the place~~ each of which is obtained by operating on the permutations of H with one and the same substitution, so that $G = H + HS + HS' + \cdots$. And also it can be decomposed into groups all of which have the same substitutions, so that $G = H + TH + T'H + \cdots$. These two kinds of decomposition do not ordinarily coincide. When they coincide the decomposition is said to be proper.

It is easy to see that when the group of an equation is not susceptible of any proper decomposition one may transform the equation at will, and the groups of the transformed equations will always have the same number of permutations.

Au contraire, quand ~~un~~^le^ groupe d'une équation est susceptible d'une décomposition propre en sorte qu'il se partage en M groupes de N permutations,

8b

on pourra résoudre l'équation donnée au moyen de deux équations: l'une aura un groupe de M permutations, l'autre un de N permutations.

Lors donc qu'on aura épuisé ~~dans une équation~~ sur ~~un~~^le^ groupe ˅d'une équation˅ tout ce qu'il y a de décompositions propres possibles sur ce groupe, on arrive à des groupes qu'on pourra transformer, mais dont les permutations seront toujours en mème nombre.

In C1832, L1846 'arrive' corrected to 'arrivera'.

Si ces groupes ont chacun un nombre premier de permutations, l'équation ~~aura~~ sera soluble par radicaux. Si non, non.

C1832, L1846 have '; sinon'; in BA1962 'Si non' corrected to 'Sinon'.

Le plus petit nombre de permutations que puisse avoir un groupe ~~quand~~ indécomposable quand ce nombre ^n'est pas^ premier est 5.4.3.

~~Les d~~

2°. Les ~~substitutions~~ décompositions les plus simples sont celles qui ont lieu par la Méthode de m^r Gauss.

Printed 'M. Gauss' in all editions.

~~Toutes les fois que dans une équation étant adjointe une des racines de l'équation, l'équation devient réd~~

Comme ces décompositions sont évidentes mème dans la forme actuelle du groupe de l'équation, il est inutile de s'arrèter long-tems sur cet objet.

In C1832, 'long-temps', in L1846, BA1962, 'longtemps'. In *ms*, 'long-tems' could be old-fashioned spelling or spelling error; hyphen comes at end of line, so is ambiguous. Missing word 'pas', to read 'ne se simplifie pas par', inserted in all previous editions.

Quelles décompositions sont praticables sur une ~~gro~~ équation qui ne se ~~réduit~~ simplifie par la méthode de M. Gauss?

J'ai appelé primitives les équations qui ~~jouissent~~ ne peuvent pas se simplifier par la méthode de M. Gauss: non que ces équations soient réellement indécomposables, puisqu'elles peuvent mème se résoudre par radicaux.

Comme lemme à la théorie des équations primitives solubles par radicaux, j'ai ~~posé~~ mis en Juin 1830 dans le bulletin férussac, une analyse sur les imaginaires de la théorie des nombres.

Capitalisation of 'bulletin férussac' and punctuation variously corrected in all previous editions.

On trouvera ci-joint la démonstration des théoremes suivants.

1~~er~~. Pour qu'une équation primitive soit soluble par radicaux, elle doît ètre du dégré p^ν, p étant premier.

On 'ci-joint' see Note 3 on p. 101. For 'suivants' C1832 has 'suivans'.

2. ~~Une pareille~~ Toutes ^les^ permutations d'une pareille équation seront de la forme

$$x_{k.l.m\,\ldots.} \quad / \quad x_{ak+bl+cm+\cdots+f\,.\,a_1k+b_1l+c_1m+\cdots+g\,.\,\ldots..}$$

$k, l, m, \ldots.$ étant ν indices qui prenant chacun p valeurs

All editions have 'sont' for 'seront'.

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indiquent toutes les racines ~~,~~ ~~.~~ ~~.~~ Les indices sont pris suivant module p, c'est à dire que la racine sera la mème quand on ajoutera à l'un des indices un multiple de p.

In displayed formula letters f, g were an afterthought. Formula garbled in C1832, somewhat altered in L1846.

Le groupe qu'on obtient en opérant toutes les substitutions de cette forme linéaire, contient en tout $p^n(p^n-1)(p^n-p)\cdots(p^n-p^{n-1})$ permutations.

Formula in last line badly mangled in C1832; displayed in C1832, L1846. Slip: n for ν.

When, on the contrary, the group of an equation is susceptible of a proper decomposition, so that it is partitioned into M groups of N permutations,

one will be able to solve the given equation by means of two equations: the one will have a group of M permutations, the other one of N permutations.

Therefore once one has effected ~~in an equation~~ on the group of an equation all possible proper decompositions on this group, one will arrive at groups which one will be able to transform, but in which the number of permutations will always be the same.

If each of these groups has a prime number of permutations the equation will be soluble by radicals; if not, not.

The smallest number of permutations which can have an indecomposable group, when this number is not prime, is 5.4.3.

2° The simplest ~~substitutions~~ decompositions are those which arise by the method of Mr Gauss.

~~Whenever, on adjunction of one of the roots of an equation, the equation becomes red[ucible]~~

Since these decompositions are obvious, even in the actual form of the equation, it is useless to pause for long on this topic.

What decompositions are practicable on an equation which is not ~~reducible~~ simplifiable by the method of Mr Gauss?

I have called those equations which ~~enjoy~~ cannot be simplified by the method of Mr Gauss primitive; not that these equations will really be indecomposable because they could even be soluble by radicals.

As a lemma for the theory of primitive equations that are soluble by radicals, I ~~placed~~ published in June 1830 in Férussac's *Bulletin* an analysis of imaginaries in the theory of numbers.

The proof of the following theorems is attached:

1. In order that a primitive equation shall be soluble by radicals it must be of degree p^{ν}, p being prime.

2. ~~Such an~~ All the permutations of such an equation will be of the form

$$x_{k,l,m,\ldots} \quad / \quad x_{ak+bl+cm+\cdots+f,\, a_1k+b_1l+c_1m+\cdots+g,\ldots}$$

$k, l, m, \ldots$ being ν indices, which, taking p values each,

indicate all the roots. The indices are taken modulo p; that is to say the root will be the same when a multiple of p is added to any one of the indices.

The group that is obtained by operating with all the substitutions of this linear form contains in all $p^{\nu}(p^{\nu}-1)(p^{\nu}-p)\cdots(p^{\nu}-p^{\nu-1})$ permutations.

Formula corrected: see note opposite.

Il s'en faut que dans cette généralité les équations qui lui repondent soient solubles par radicaux.

'Bulletin de Férussac' in C1832, L1846 (italicised in L1846); 'férussac' in BA1962.

La condition que j'ai indiquée dans le bulletin ferussac pour ~~qu'elle~~ que l'équation soit soluble par radicaux est trop restreinte. Il y a peu d'exceptions, mais il y en a.

For 'Je n'ai' see Note 1 on p. 101.

~~Je n'ai~~ _____

La dernière application de la théorie des équations est relative aux équations modulaires des fonctions elliptiques.

Previous editions correct to 'de l'amplitude'; better might be plural 'des amplitudes'.

On sait que le groupe d'~~e~~ ⌜une⌝ ^l'^équation qui a pour racines les sinus de l'amplitudes des p^2 ⌜-1⌝ divisions d'une période est celui-ci:

$$x_{k.l} \qquad x_{ak+bl\,/\,ck+dl}$$

In displayed formula the sign / should be a dot. Corrected in P1897 (using commas); L1846, BA1962 are faithful to *ms*.

Parconséquent l'équation modulaire correspondante aura pour ~~gr~~ groupe,

$$x_{\frac{k}{l}} \qquad x_{\frac{ak+bl}{ck+dl}}$$

Dans la quelle $\frac{k}{l}$ peut avoir les $p+1$ valeurs $\infty \quad 0 \quad 1 \quad 2 \ldots \quad p-1$

Ainsi en convenant que k peut être infini on peut ecrire simplement

$$x_k \qquad x_{\frac{ak+b}{ck+d}}$$

Line indented as new paragraph, but starts with lower case letter e.

en donnant à $a \quad b \quad c \quad d$ toutes les valeurs, on obtient $(p+1)p(p-1)$ permutations.

Or ce groupe se décompose proprement en deux groupes, dont les substitutions sont

$$x_k \qquad x_{\frac{ak+b}{ck+d}}$$

$ad-bc$ étant un résidu quadratique de p.

Le groupe ainsi simplifié est de $(p+1)\,p\,\frac{p-1}{2}$ permutations. mais il est aisé de voir qu'il n'est plus décomposable proprement. ⌜à moins que $p=2$ ou $p=3$.⌝

Word 'quelque' changed to 'quelle' in C1832, L1846. Word 'aura', missing from after 'groupe', supplied without comment in all editions.

Ainsi de quelque manière que l'on transforme l'équation, son groupe

9b

toujours le mème nombre de permutations.

Mais il est curieux de savoir si le degré peut s'abaisser.

Deleted 'puisque' missing from BA1962. See also Note 4 on p. 102.

Et d'abord il ne peut s'abaisser ~~puisque~~ plus bas que p, puisque ~~sans celà~~ une équation de degré moindre que p, ne peut avoir p pour facteur dans le nombre des permutations de son groupe.

It is not the case that in this generality the corresponding equations are soluble by radicals.

The condition that I indicated in Férussac's *Bulletin* in order that the equation should be soluble by radicals is too restrictive; there are few exceptions, but there are some.

Probably refers to 'Analyse d'un Mémoire ...', § II.2, but could also be to 'Théorie des nombres', p. 74 above.

~~I do not have~~

The last application of the theory of equations relates to the modular equations of elliptic functions.

It is known that the group of the equation which has for its roots the sines of the amplitudes of the $p^2 - 1$ divisions of a period is this:

$$x_{k,l} \qquad x_{ak+bl,\, ck+dl} \ .$$

Consequently the corresponding modular equation will have for group

$$x_{\frac{k}{l}} \qquad x_{\frac{ak+bl}{ck+dl}} \ ,$$

in which $\frac{k}{l}$ can take the $p+1$ values ∞, 0, 1, 2, ..., $p-1$. Thus, with the convention that k may be infinite one can simply write

$$x_k \qquad x_{\frac{ak+b}{ck+d}} \ .$$

Giving to a, b, c, d all the values one obtains $(p+1)p(p-1)$ permutations.

Now this group may be *properly* decomposed into two groups, whose substitutions are

$$x_k \qquad x_{\frac{ak+b}{ck+d}} \ ,$$

$ad - bc$ being a quadratic residue of p.

Thus simplified the group has $(p+1)p.\frac{p-1}{2}$ permutations. But it is easy to see that it is not further decomposable properly unless $p = 2$ or $p = 3$.

Thus in whatever way one transforms the equation its group will

always have the same number of permutations.

But it is interesting to know if the degree can be reduced.

And to begin with, it cannot be reduced below p, because an equation of degree smaller than p cannot have p as a factor in the number of permutations of its group.

Voyons donc si l'équation de degré $p+1$ dont les ~~va~~ racines ^x_k^ ~~s'obtiennent~~ ⌄s'indiquent⌄ en donnant à k toutes les valeurs y compris l'infini et dont le groupe a pour substitutions

$$x_k \qquad x_{\frac{ak+b}{ck+d}} \qquad ad-bc \text{ étant un carré}$$

peut s'abaisser au degré p.

Or il faut ~~que~~ pour celà que le groupe se décompose (improprement, s'entend) en p groupes de $(p+1)\dfrac{p-1}{2}$ permutations chacun.

Soient 0 ^et^ ∞ deux lettres conjointes dans l'un de ces groupes. Les ~~m~~ ~~subst per~~ substitutions qui ne font pas changer 0 et ∞ de place seront de la forme

$$x_k \qquad x_{m^2k}$$

Donc si ~~K~~ ~~n'est ni nul ni infini et que~~ M ~~soit~~^est^ la lettre conjointe ^de 1^ la lettre conjointe de m^2 sera m^2M. Quand M est un carré on aura donc $M^2=1$,. ~~et la rédu~~ Mais cette simplification ne peut avoir lieu que pour $p=5$.

Deleted 'rédu' completed to 'réduction' in BA1962.

~~Dans le 7e~~ ^Pour^ $p=7$ on trouve un groupe de $(p+1)\dfrac{p-1}{2}$ permutations, ou ~~la~~ ∞ 1 2 4 ont respectivement pour lettres conjointes 0 3 6 5

The *ms* is so worded that ∞ 1 2 4 come atop 0 3 6 5.

Ce groupe ~~est~~ a ses substitutions de la forme

$$x_k \qquad x_{a\frac{k-b}{k-c}}$$

Word 'lettre' is not quite right for a; last occurrence of c should be $b-c$. Word 'tems' clear in *ms*; C1832 has 'tems'; L1846, BA1962 have 'temps'.

b étant la lettre conjointe de c, et a une lettre qui est ~~à la fois~~ résidu ou non résidu en même tems que c.

Pour $p=11$ Les mèmes substitutions auront lieu avec les mèmes notations,

	∞	1	3	4	5	9	ayant respectivement
pour conjointes	0	2	6	8	10	7	

C1832, L1846 have 'les cas'; P1897, BA1962 have 'le cas'; 's' clearly crossed out in *ms*.

~~En toute~~^Ainsi^, pour le~~s~~ cas de $p=5,\ 7,\ 11$, l'équation modulaire s'abaisse au degré p.

En toute rigueur, cette réduction n'est pas possible dans les cas plus élevés.

10a

On 3rd memoir see Dossier 24, Notes 2, 3, esp. p. 379.

Le troisième mémoire concerne les intégrales.

On sait qu'une somme de ~~fonct~~ termes d'une mème fonction elliptique se réduit toujours à un seul terme, plus des quantités algebriques ou logarithmiques.

~~Pour tou~~ Il n'y a pas d'autres fonctions pour les quelles cette propriété ait lieu.

Mais des propriétés absolument semblables y suppléent ⌜dans toutes les intégrales de fonctions algébriques.⌝

The insertion clearly came *after* the deletion of the twice started new paragraph.

~~Soit~~ ~~On traite à la fois~~

Let us see then whether the equation of degree $p+1$ whose roots x_k are ~~obtained~~ indicated by giving to k all its values, including infinity, and of which the group has for substitutions

$$x_k \qquad x_{\frac{ak+b}{ck+d}} ,$$

$ad - bc$ being a square, may be reduced to degree p.

Well, for that it is necessary that the group may be decomposed (improperly of course) into p groups each of $(p+1)\dfrac{p-1}{2}$ permutations.

Let 0 and ∞ be two letters that are linked in one of these groups. The substitutions which do not make 0 and ∞ change their places will be of the form

$$x_k \qquad x_{m^2 k} .$$

Therefore if ~~K is neither zero nor infinity~~ M is the letter linked with 1 the letter linked with m^2 will be $m^2 M$. When M is a square one will therefore have $M^2 = 1$. But this simplification cannot take place except for $p = 5$.

~~In the 7th~~ For $p = 7$ one finds a group of $(p+1)\dfrac{p-1}{2}$ permutations, in which ∞, 1, 2, 4 have respectively 0, 3, 6, 5 for linked letters.

This group has its substitutions of the form

$$x_k \qquad x_{a\frac{k-b}{k-c}} ,$$

b being the letter linked with c, and a a letter [coefficient] which is ~~simultaneously~~ residue or non-residue at the same time as $b - c$.

Correction $b - c$ incorporated. See note opposite.

For $p = 11$ the same substitutions will occur with the same notation,

	∞	1	3	4	5	9 having respectively
for their linked letters	0	2	6	8	10	7

~~In all~~ Thus for the cases $p = 5, 7, 11$ the modular equation is reducible to degree p.

In all rigour this reduction is not possible in the higher cases.

The third memoir concerns integrals.

It is known that a sum of terms of the same elliptic function always reduces to a single term plus algebraic or logarithmic quantities.

There are no other functions for which this property holds.

But absolutely analogous properties replace it in all integrals of algebraic functions.

~~Let~~ ~~Let us treat at the same time~~

On traite à la fois toutes les ~~fonctions~~ intégrales dont la différentielle est une ~~même~~ fonction de ~~l~~a variable et d'une mème fonction irrationnelle de la variable, que cette irrationnelle soit ou ne soit pas un radical, qu'elle s'exprime ou ne s'exprime pas par des radicaux.

~~Soit~~ On trouve que le nombre des périodes distinctes de ~~la fonction~~ l'intégrale la plus générale relative à une irrationnelle donnée est ~~de 2*n*~~ toujours un nombre pair.

Soit $2n$ ce nombre. On aura le théorème suivant:

Une somme quelconque de termes se réduit à n termes plus des quantités algébriques et logarithmiques.

Les fonctions de première espèce sont celles pour les quelles la partie algébrique et logarithmique est nulle.

Il y en a n distinctes.

Les fonctions de seconde espèce sont celles ~~dont~~^pour les quelles^ la partie complémentaire est purement algébrique.

Il y en a n distinctes.

~~Les fon autres fonct~~ On peut supposer que les ^différentielles des^ autres fonctions ne soient jamais infinies qu'une fois ^pour $x = a$^, et de plus que leur partie complementaire se réduise à un seul logarithme, $\log P$, P étant une quantité algébrique. En désignant par $\Pi(x, a)$ ces fonctions, on aura le théorème

$$\Pi(x, a) - \Pi(a, x) = \sum \varphi a \ \psi x$$

φa et ψx étant des fonctions de première et de seconde espèce. On en déduit ~~pour les fonctions~~ en appelant $\Pi(a)$, ~~$\varphi,$~~^et^ ψ les périodes de $\Pi(x, a)$ ~~φx~~ ^et^ ψx relatives à une mème révolution de x,

$$\Pi(a) = \psi \times \varphi a \, .$$

10b

Ainsi les périodes des fonctions de troisième espèce s'expriment toujours en fonctions de première et de seconde espèce.

C1832, L1846 have plural 'en fonctions'; P1897, BA1962 have singular. Plural is clear in *ms*.

On peut en déduire aussi des théorèmes analogues au ~~fam~~ théorème de Legendre

$$E'F'' - E''F' = \frac{\pi}{2}\sqrt{-1}$$

La réduction des fonctions de troisième espèce à ~~des une~~^des^ intégrale⌜s⌝ définie⌜s⌝, qui est la plus belle découverte de M. Jacobi, n'est pas praticable hors le cas des fonctions Elliptiques.

Reading 'premier' is plausible, but uncertain.

La multiplication des fonctions intégrales par un nombre ~~premier~~ entier est toujours possible, comme l'addition, au moyen d'une équation de degré n dont les racines sont les valeurs à substituer dans l'intégrale pour avoir les termes réduits.

Let us treat at one and the same time all ~~functions~~ integrals of which the differential is ~~such~~ a function of the variable and of an irrational function of the variable, whether this irrational is or is not a radical, whether or not it may be expressed by radicals.

One finds that the number of distinct periods of the most general integral relative to a given irrational is ~~2*n*~~ always an even number.

Letting $2n$ be this number one has the following theorem:

An arbitrary sum of terms reduces to n terms plus some algebraic and logarithmic quantities.

Functions of the first kind are those in which the algebraic and logarithmic part is null.

Of these there are n distinct ones.

The functions of the second kind are those ~~of which~~ for which the complementary part is purely algebraic.

Of these there are n distinct ones.

~~The other funct[ions]~~ One can suppose that the differentials of other functions are never infinite except once for $x = a$, and further that their complementary part reduces to a single logarithm $\log P$, P being an algebraic quantity. Denoting these functions by $\Pi(x, a)$ one will have the theorem:

$$\Pi(x, a) - \Pi(a, x) = \sum \varphi a \ \psi x \,,$$

φa and ψx being functions of the first and second kinds. It may be deduced from that ~~for the functions~~, calling $\Pi(a)$ and ψ the periods of $\Pi(x, a)$ and ψx relative to one and the same revolution of x,

$$\Pi(a) = \psi \times \varphi \ a \,.$$

Thus the periods of functions of the third kind are always expressible in terms of functions of the first and second kind.

One can also deduce from it some theorems analogous to the ~~fam[ous]~~ theorem of Legendre

$$E'F'' - E''F' = \frac{\pi}{2}\sqrt{-1}.$$

The reduction of functions of the third kind to definite integrals, which is the most beautiful discovery of Mr Jacobi, is not possible beyond the case of elliptic functions.

Multiplication of integral functions by a ~~prime?~~ whole number is always possible, like addition, by means of an equation of degree n whose roots are the values to be substituted into the integral in order to obtain the reduced terms.

L'équation qui donne la division des périodes en p parties égales est du degré $(p^{2n}-1)$. Son groupe a en tout $(p^{2n}-1)(p^{2n}-p)\cdots(p^{2n}-p^{2n-1})$ permutations.

Formula for number of permutations displayed in C1832, L1846.

L'équation qui donne la division d'une somme de n termes en p parties égales est du degré p^{2n}. Elle est soluble par radicaux.

De la transformation.

Corrected to 'Abel' in all editions.

On peut d'abord, en suivant des raisonnements analogues à ceux qu'abel a consignés dans son dernier mémoire, démontrer que ^si^ dans une même relation entre des intégrales on a les deux fonctions $\int \varphi(x,X)\,dx$, $\int \psi(y,Y)\,dy$, la dernière intégrale ayant $2n$ périodes, ~~on peut~~ ^il sera permis de^ supposer que y et Y s'expriment moyennant une seule équation de degré n enfonction de x et de X.

Mis-spelling clear in *ms*; corrected to 'prenant' in all previous editions.

D'après celà on peut supposer que les transformations aient lieu constamment entre deux intégrales seulement, puisqu'on aura evidemment, en prennant une fonction quelconque rationnelle de y et de Y

$$\sum \int f(y,Y)\,dy = \int F(x,X)\,dx + \text{une ~~quant. Alg. et log~~}.$$

Symbol f in formula reported in BA1962 as crossed out. It seems to me that f is intended (as 'fonction quelconque' of the previous line), though it appears somewhat blotted.

Il y aurait sur cette équation des réductions évidentes dans le cas où Les intégrales de l'un et de l'autre membre n'auraient pas toutes deux le mème nombre de périodes.

Ainsi nous n'avons à comparer que des intégrales qui aient toutes deux le même nombre de périodes.

In 'Les intégrales' the capital L starts a new line—and seems accidental.

On démontrera que ~~les~~^le plus petit degré^ ⌜d'⌝irrationalité de ^deux^ pareilles intégrales ne peut ètre ètre plus grand pour l'une que pour l'autre.

Word 'irrationalité' mis-spelled with 'nn' in L1846, P1897, BA1962.

11a

On fera voir ensuite qu'on peut toujours transformer une intégrale donnée en une autre dans la quelle ~~toutes les~~^une^ période de la première soit divisée par le nombre premier p, et ~~toutes~~ les $2n-1$ autres restent les mèmes.

Word 'ètre' at end of one line of *ms* repeated at start of next.

Il ne restera donc à comparer que des intégrales où les périodes seront les mèmes de part et d'autre, ~~t~~/ et telles parconséquent que n termes de l'une s'expriment sans ^autre^ équation qu'une seule du degré n, au moyen de ceux de l'autre, et réciproquement. Ici nous ne savons rien.

Tu sais, mon cher Auguste, que ces sujets ne sont pas les seuls que j'aie explorés. ~~Mais il fallait~~ Mes principales méditations, depuis quelque temps, étaient dirigées sur l'application à l'analyse transcendante de la théorie de l'ambiguité. Il s'agissait de voir à priori dans une relation entre des quantités ou ~~quantités~~ fonctions transcendantes, quel~~s~~ échanges on pouvait faire, quelles quantités on pouvait substituer aux quantités données, sans que la relation pùt cesser d'avoir lieu.

Galois' 'a priori' rendered 'a *priori*,' in C1832; 'a priori,' in L1846; '*a priori*,' in P1897; 'a priori' in BA1962.

The equation which gives the division of the periods into p equal parts is of degree $p^{2n} - 1$. Its group has $(p^{2n} - 1)(p^{2n} - p) \cdots (p^{2n} - p^{2n-1})$ permutations in all.

The equation which gives the division of a sum of n terms into p equal parts is of degree p^{2n}. It is soluble by radicals.

On transformation.

To begin with, following reasoning analogous to that which Abel wrote down in his last memoir, one can show that if in such a relation between integrals one has the two functions $\int \Phi(x, X)\, dx$, $\int \Psi(y, Y)\, dy$, the latter integral having $2n$ periods, ~~one may~~ it will be permissible to suppose that y and Y may be expressed as functions of x and of X by means of a single equation of degree n.

Accordingly one may suppose that the transformations always take place solely between two integrals because clearly, taking an arbitrary rational function of y and of Y, one will have

$$\sum \int f(y, Y)\, dy = \int F(x, X)\, dx + \text{an } \sout{\text{alg[ebraic] and log[arithmic] quant[ity]}}$$

There will be some obvious reductions of this equation in the case where the integrals of the one and the other member do not both have the same number of periods.

Thus we only have to compare integrals which both have the same number of periods.

It may be shown that the smallest degree of irrationality of two such integrals cannot be larger for the one than for the other.

One then shows that one can always transform a given integral into another in which ~~all the~~ one period of the former is divided by the prime number p and ~~all~~ the other $2n - 1$ remain the same.

It will then only remain to compare integrals where the periods will be the same on both sides, and consequently such that n terms of the one are expressible in terms of those of the other by means of only a single equation of degree n, and vice-versa. Here we know nothing.

You know my dear Auguste that these subjects are not the only ones that I have explored. ~~But it needed~~ For some time my main thinking was directed towards the application to transcendental analysis of the theory of ambiguity. It was concerned with seeing *a priori* in relations between transcendental quantities or functions what exchanges one could make, what quantities one could substitute for the given quantities, without the relation ceasing to hold.

Celà fait reconnaitre desuite l'impossibilité de ⌜beaucoup⌝ d'expressions que l'on pourrait chercher. Mais je n'ai pas le tems et mes idées ne sont pas ~~bien~~ encore bien développées sur ce terrain qui est immense.

Old spelling (or misprint?): 'tems' corrected to 'temps' in L1846, BA1962.

Tu feras imprimer cette lettre dans la revue Encyclopédique.

C1832 has REVUE ENCYCLOPÉDIQUE; L1846 has *Revue encyclopédique* and 'Lettre' capitalised.

~~En Fait~~ Je me suis souvent hasardé ^dans ma vie^ à avancer des propositions dont je n'étais pas sûr. Mais tout ce que j'ai écrit là est depuis bientòt un an dans ma tète, et ~~je p~~ il est trop de mon intéret de ne pas me tromper pour qu'on me soupçonne d'avoir énoncé des théorèmes dont je n'aurais pas la démonstration complète.

Tu ~~engageras~~ prieras publiquement Jacobi **ou** Gauss de donner leur avis non sur la vérité, mais sur l'importance des théorèmes.

Phrase 'il se trouvera' changed to 'il y aura' in C1832, L1846, P1897.

Après celà il se trouvera, j'espère, des gens qui trouveront leur profit à déchiffrer tout ce gachis.

Je t'embrasse avec effusion.

E Galois Le 29 Mai 1832.

That makes immediately recognisable the impossibility of many expressions that one could look for. But I do not have the time, and my ideas are not yet well enough developed in this area, which is immense.

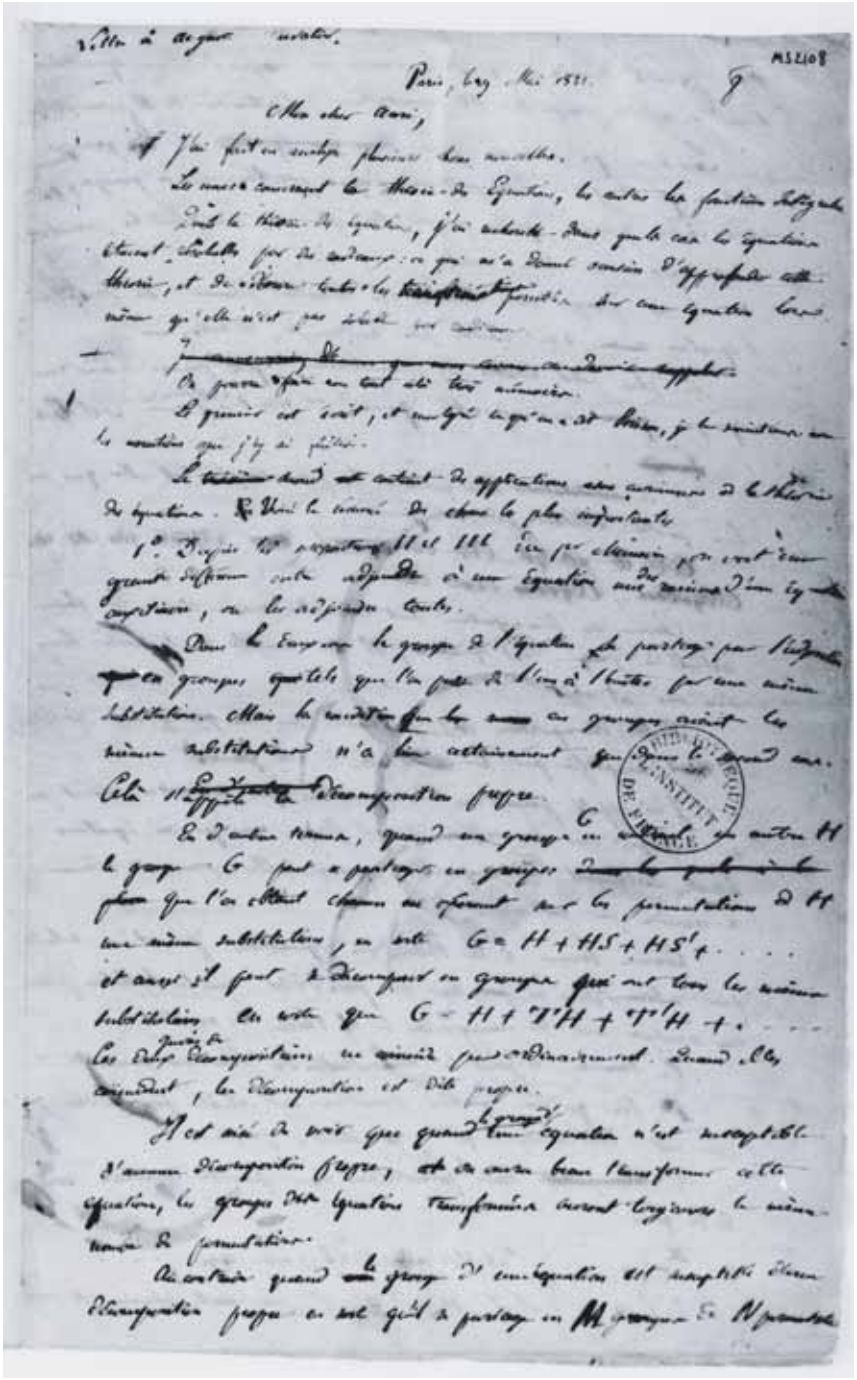
You will have this letter printed in the *Revue Encyclopédique*.

~~In fact~~ Often in my life I have risked advancing propositions of which I was not certain. But all that I have written here has been in my head for almost a year and it is not in my interest to make a mistake so that one could suspect me of having announced theorems of which I did not have the complete proof.

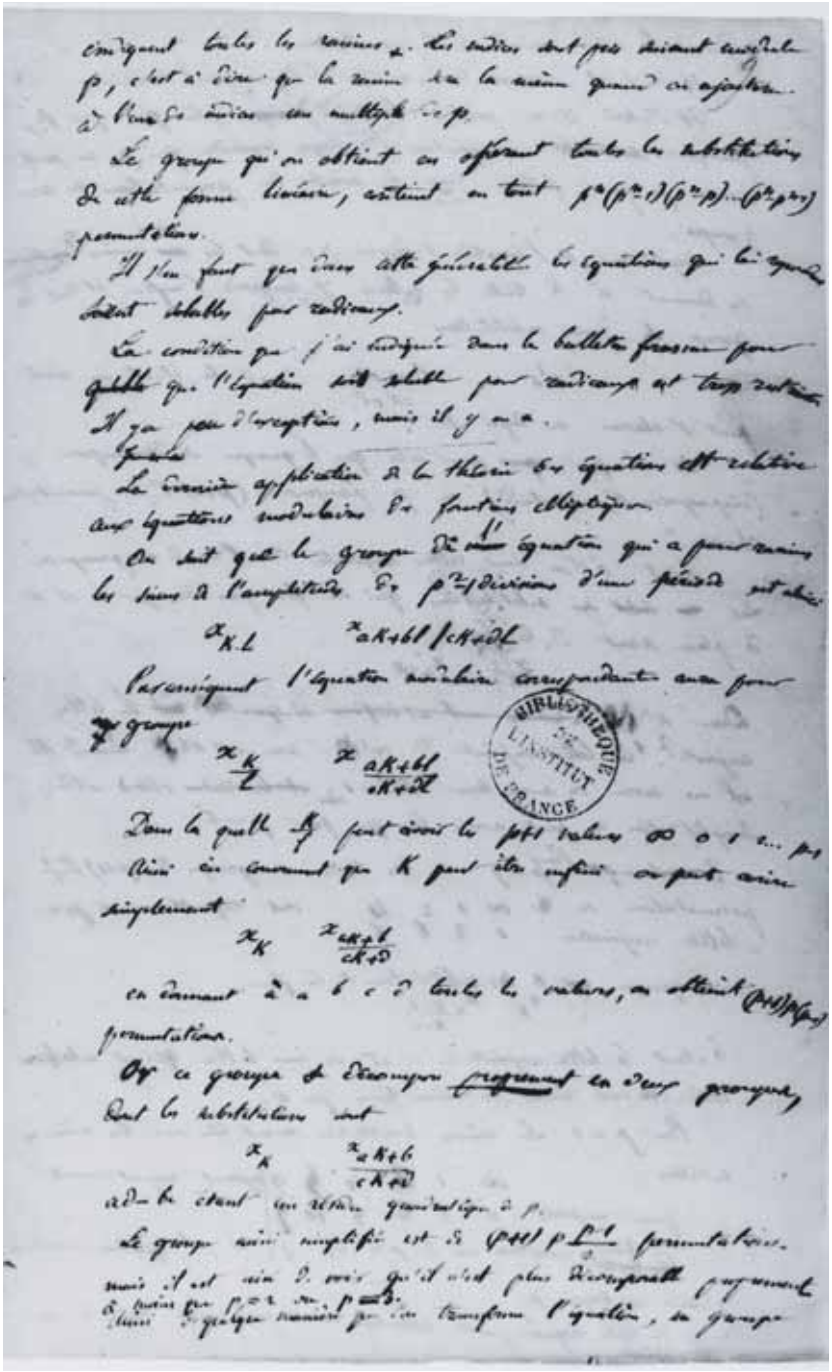
You will ~~engage~~ publicly ask Jacobi ~~and~~ or Gauss to give their opinion not on the truth but on the importance of the theorems.

After that there will, I hope, be people who will find profit in deciphering all this mess.

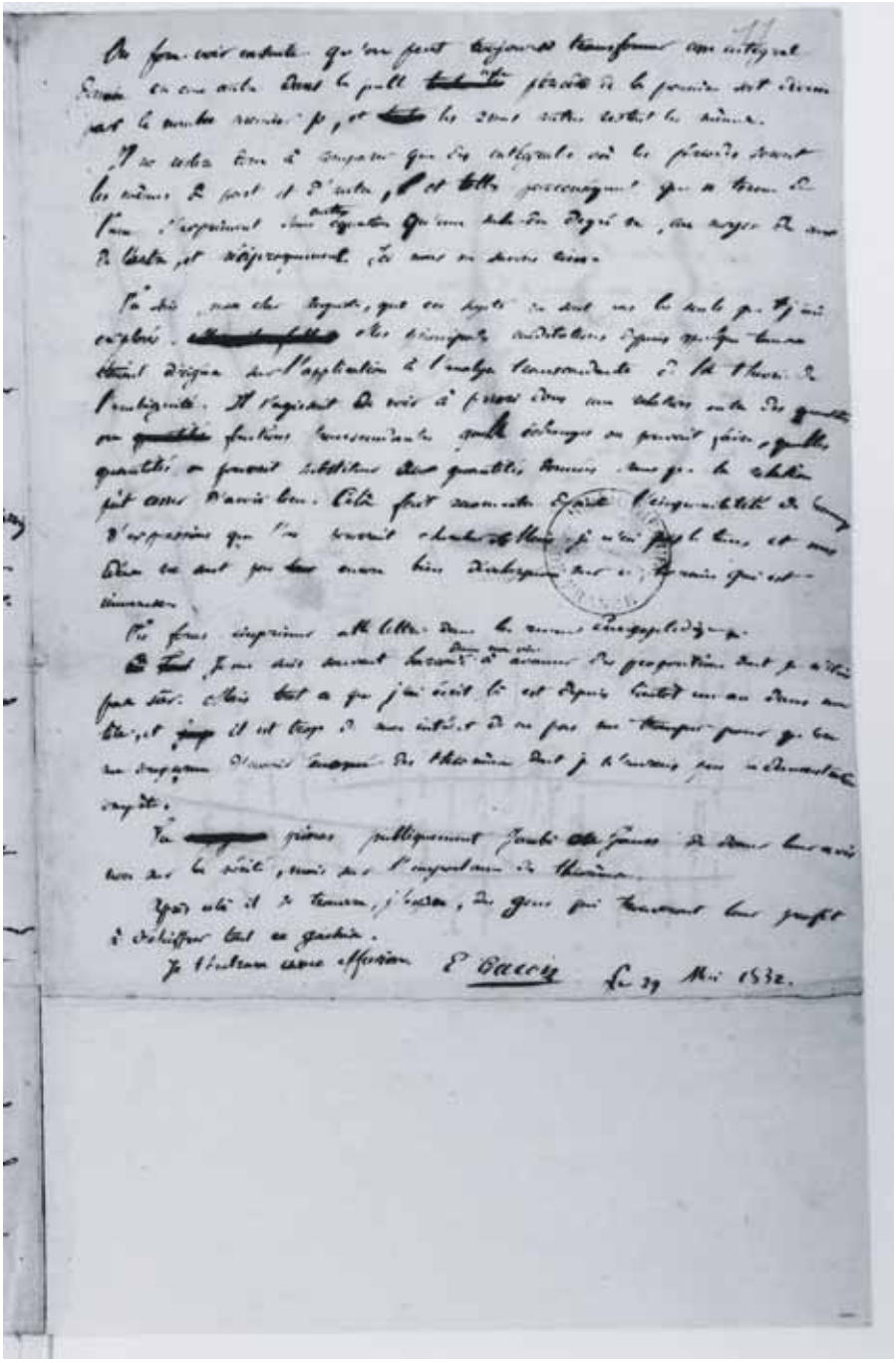
I embrace you warmly. E Galois 29 May 1832.



Testamentary Letter: Dossier 2, folio 8 recto.



Testamentary Letter: Dossier 2, folio 9 recto.



Testamentary Letter: Dossier 2, folio 11 recto.

III.2 Notes on the letter

NOTE 1: The description 'Lettre à Auguste Chevalier' is written in the top left corner of the page by Galois himself; the dateline is centred on the page; the salutation is much indented; paragraph indents are quite large, unlike in other *mss*. The writing is small and fluent.

One-third of the way down **f.9a**, the third side of his paper, Galois drew a line (a very thin line) under what he had drafted so far and wrote 'Je n'ai' (later crossed out) which, one might guess, is the start of 'Je n'ai pas le temps', echoing the note in the margin of **f.4a** of the *Premier Mémoire*. It looks very much as though he lost heart and thought of finishing there, but that his spirit returned and he continued. This is highly conjectural however.

The first page shows several semi-circular stains that seem to be the result of a wet glass having been placed on it. The colour of the stains is similar to that of the ink that Galois used, but watered down.

NOTE 2: The status of the letter within the Galois corpus is worth thinking about. All the other manuscripts belonged to Galois himself. This one ought to have 'belonged' to Chevalier. We must owe a great debt to Chevalier: first, that he did not send it to the printer when the letter was published as [Lettre (1832)], or that if he did (which seems unlikely), he managed to get it back; secondly, that he saw the need to leave it with the Galois material rather than retaining it with his own records.

There arises the question, what copies of it did he make? He wrote careful copies of all the other important Galois manuscripts. It is easy to imagine that he made copies of the letter also: to keep faith with his dead friend he probably made copies to send to Jacobi or Gauss or both; perhaps he also made a copy from which the printer of the *Revue Encyclopédique* worked and which was not returned to him as his copy of the *Premier Mémoire* was after it had been used by Liouville's printer. Such speculation is, however, of no value except in drawing attention to the very faint possibility that a copy in his hand could conceivably turn up some time in some hitherto undiscovered *Nachlaß* of Gauss or Jacobi or himself. If this possibility were to materialise it would change nothing as far as scholarship is concerned, though it might make some collector very happy.

NOTE 3: On the second page **f.8b** of his letter Galois wrote 'On trouvera ci-joint la démonstration des théoremes suivants' [The proof of the following theorems will be found attached] (see p. 86 below). On 'ci-joint':

> [Lettre (1832)] has the footnote 'Dans les manuscrits de Galois, que nous publierons' [In the manuscripts of Galois which we shall publish];
>
> [Liouville (1846)] has the footnote 'Galois parle des manuscrits, jusqu'ici inédits, que nous publions. (J. L.)' [Galois is speaking of manuscripts, hitherto unedited [unpublished], which we are publishing. (J. L.)];
>
> [B & A (1962)] has the note 'Cf. 2e *Mémoire*'.

NOTE 4: Where the word 'puisque' [since] is deleted near the top of **f.9b** (see p. 88) it looks as if Galois had intended to justify the claim that the degree could not be reduced, then realised that this was not quite true. He was dealing with the groups that are now known as $\mathrm{PSL}(2, p)$ or as $\mathrm{L}_2(p)$ for prime numbers p. He had all the salient facts, all pretty well understood by about half a century later—see [Gierster (1881)]—and now well known: that they are simple for $p \geqslant 5$; that therefore there cannot be proper subgroups of index $< p$; that they have a transitive permutation representation of degree $p + 1$ and therefore subgroups of index $p + 1$; that they have a subgroup of index p if and only if p is 5, 7 or 11. Much of the second part of the *Second Mémoire* is devoted to the groups $\mathrm{PSL}(2, p)$, but the material there falls quite a long way short of what he announced here in his final letter. Some calculations for $\mathrm{PSL}(2, 5)$, are to be seen in vestigial form in some of the jottings in the scraps in Dossier 24, in **f.123b**, **f.159a**, **f.159b**, **f.160b**, **f.176a**, **f.177a** (see [B & A (1962), pp. 299, 309, 311, 315, 327–331]).

NOTE 5: Being unfamiliar with elliptic functions and their history I have not found the reference to Legendre in **f.10b**, p. 92 easy to track down. Tannery [Tannery (1906), p. 230] suggests that the equation should have been written

$$FE' + EF' - FF' = \frac{\pi}{2}.$$

There is a theorem written like that in [Legendre (1825–28), Vol 2, Ch. XII, Art. 45] but it deals with elliptic functions of the first and second kind, whereas Galois seems to be dealing with the third kind. This requires the attention of someone with more expertise than I can offer.

NOTE 6: The last page, folio 11, is rather mysterious. One peculiarity is that, although it forms a two-page folded pamphlet with folio 10, a rectangular section about 6 cm deep has been neatly torn from the bottom and from along the fold where it would have joined folio 10. The main mystery however is what is written, and crossed out, on its verso, **f.11b**. There appear to be two letters, quite probably copies or reconstructions. Both contain gaps. Do these represent illegible material from the original (if there really was an original)? Do they represent some form of censorship? It seems unlikely that we will ever know.

The end of the first letter has three words so vigorously crossed out that I find them illegible. They are read as 'Mademoiselle' (which looks probable), 'Stephanie' (which looks plausible) and 'D...' in [B & A (1962), p. 489]. In [Infantozzi (1968)] the author goes a step further and reads 'Mademoiselle Stéphanie Du ... tel'. Although Infantozzi's paper was published some six years later than [B & A (1962)], a footnote appended by the editors (actually René Taton) to its first page makes clear that it was written in 1962 and is quite independent of [B & A (1962)].

The first letter occupies the top left quarter of the page. The second starts in the top right quarter, and continues in the bottom half but with the page turned 90° anti-clockwise, so that the right edge of the paper became the top, and the lines now read vertically downwards and from right to left. As in [Tannery (1906), pp.228, 229] and [B & A (1962), pp. 489, 490] they are transcribed here with their original

line-endings to show where the mysterious gaps occur. Here is the first, the first line of which looks as if it may, just possibly, have been inserted later with a thinner nib:

brisons là sur cette affaire je vous prie
Je n'ai pas assez d'esprit pour suivre
une correspondance de ce genre
mais je tâcherai d'en avoir assez pour
converser avec vous comme je le faisais
avant que rien soit arrivé. Voilà
M^{r} le [???]
en a qui
doit vous qu'à
moi et ne plus penser à des choses
qui ne sauraient exister et qui
n'existeront jamais.
[~~????~~] [~~???~~] [~~??~~]
14 ~~Mai 83~~

Word 'correspondance' mis-transcribed as 'conversation' in T1906/7.

Word at end of line recorded as illegible in T1906/7, missing from BA1962. It looks a little like 'retomber'.

Word 'sauraient' as read in BA1962 looks plausible but not certain. Tannery found it illegible.

[I pray you, let us break the affair off there. I do not have enough spirit to pursue a correspondence of this kind, though I shall endeavour to have enough to converse with you as I did before anything happened. That is it Sir [. . .] as to me, and not to think any more of matters which could not have existed and which will never exist.]

And here is the second letter:

J'ai suivi votre conseil et j'ai réfléchi
á ce qui s'est
passé sous quelque
dénomination que ce puisse etre font s'établir
entre nous. Au reste M^{r} soyez
persuadé qu'il n'en aurait sans doute
jamais été davantage; vous supposer
mal et vos regrets sont mal fondés.
La vraie amité n'existe guere
qu'entre des personnes de mème sèxe.
Surtout —— —— des
Amis. ——— Sans doute
le vide que ——— l'absence de
tout sentiment de ce genre . . . . . .
un confiance . . . . . mais elle a été
très blessée . . . . vous m'avais vue
triste z demandé

Beginning of 2nd line read as 'à' in T1906/7, BA1962, but accent is clearly acute, not grave. Word read as 'sous' in T1906/7and BA1962 looks more like 'sera' to me, but I defer to them.

Line 9: T1906/7, BA1962 correct 'guere' to 'guère' and 'mème sèxe' to 'même sexe'. Full stop after 'sèxe' is unclear. Capitalisation of 'Surtout' not in BA1962, though seems clear in *ms*.

Spaces here, not elsewhere, contain very thin lines. Words 'Sans doute' in T1906/7, read as 'plains dans' in BA1962. Comparison with a line above makes former more plausible.

Word 'blessée' supplied in BA1962 illegible to Tannery. And to me.

[*Here the top right quarter page finishes. The letter continues in the bottom half page, turned.*]

le motif; je vous ai répondu que
j'avais des peines; qu'on m'avait fait
éprouver. J'ai pensé que vous prendriez
celà comme toute personne devant
laquelle on laisse tomber une parole
pour ces on n'est
pas
le calme de mes idées me laisse
la liberté de juger sans beaucoup
de reflexion les personnes que je vois
habituellement; c'est ce qui fait que
j'ai rarement le regret de m'être
trompée ^ou laissée influencer^ à leur égard.
Je ne suis pas de votre avis pour
les sen plus que
les au exiger
ni se vous remercie
sincerement de tous ceux ou vous
voudriez bien descendre en ma
faveur.

T1906/7 has 'avec' but the opposite, 'sans', as in BA1962, seems clear in *ms*.

T1906/7 has 'trompé ou laissé', BA1962 has 'trompée ou laissée'. I read 'trompée ou laissé', which probably means that 'trompée ou laissée' was intended. In the context gender difference is significant.

[I have followed your advice and I have reflected on what has happened under whatever heading [name] that this could be established between us. For the rest, Sir, be persuaded that there would always have been doubt that it might have been better; your assumption is wrong and your regrets are ill-founded. True friendship does not exist at all except between persons of the same sex, above all of friends. Undoubtedly the emptiness which the absence of all feeling of this kind . . . . . my faith . . . . . but it has been very wounded . . . . . you have seen me unhappy [you have?] asked the reason; I have replied that I have struggled; that I have been tested. I thought that you would take that as any person would, before whom one lets fall a word for these one is not The calm of my ideas leaves me the freedom to judge without [*with* if T1906/7 is right] much reflection people whom I see regularly; it is this that ensures that I rarely have to regret being wrong or being influenced concerning them. I am not of your mind for the more than the [others?] to require nor to thank you sincerely for all those where you would wish to come down in my favour.]

Chapter IV
The First Memoir

IV.1 Text of the First Memoir

The memoir on the conditions for solubility of equations by radicals is undoubtedly Galois' most important work. It is here that he presented his original approach to the theory of equations which has now become known as Galois Theory. It was first published in [Liouville (1846), pp. 417–433], reprinted in [Picard (1897), pp. 33–50], and re-published in [B & A (1962), pp. 42–71]. A German translation appears in [Maser (1889)], an English translation in [Edwards (1984)], and an Italian translation in [Toti Rigatelli (2000)].

The extant manuscript represents Galois' third attempt to interest the Académie des Sciences in his ideas (see Note 1 on p. 145). It was received at the Academy in Paris on 17 January 1831 and must have been returned to Galois on or soon after 4 July 1831. It is now better known as the *Premier Mémoire*, a description that the author himself gave it in his testamentary letter to Auguste Chevalier (and also earlier in the fragment of Dossier 8, p. 220 below, and the Preface, Dossier 11, p. 246) than by its formal title

Mémoire sur les conditions de résolubilité des équations par radicaux.

It is a third draft, considerably more polished than most of Galois' work. Nevertheless, he was moved to revise it extensively, adding marginal notes, completing various arguments here and there, and replacing some faulty material. Most of these corrections were made on 29 May 1832, the awful eve of the fatal duel, and haste is evident. They fill most of the margins of the first half of the manuscript—had time not run out, later pages might presumably also have been filled with emendations.

It occupies Dossier 1 of the manuscripts. Folio 1 is a cover sheet, almost certainly provided by Galois (it shows folds that match those of the memoir) and blank when he submitted his paper to the Academy. With its preface on **f. 2 a**, the *Premier Mémoire* fills folios **f. 2 a**–**f. 7 a**, 11 pages in all. They are large pages. The paper is 50 cm × 38 cm folded double, and sewn to form a 12-page booklet with pages 25 cm × 38 cm. The small neat writing of the basic memoir is carefully confined to the right half of each page. It appears that Galois defined his left margin by a light fold in the page, and his text is neatly aligned vertically; there are no margins on the right, at the top, or at the bottom of the pages. On many pages the wide margin on the left has been filled with annotations of various kinds. These are discussed in the notes which follow the text of the memoir in this chapter.

Dossier 3 contains Auguste Chevalier's manuscript copy of the *Premier Mémoire* together with Liouville's corrected proofs of an aborted publication of the memoir in 1843 (see Notes 4, 16, pp. 151, 161).

2 a

Mémoire sur ~~la~~ ^les conditions de^ résolubilité des équations par radicaux

~~Mémoire~~

Le mémoire ci-joint est extrait d'un ouvrage que j'ai eu l'honneur de présenter à l'Académie il y a un an. Cet ouvrage n'ayant pas eté compris, les propositions qu'il renferme étant révoquées en doute, j'ai dù me contenter de donner, sous forme synthétique, les principes généraux et une seule application de ma théorie. Je supplie mes juges de lire de moins avec attention ce peu de pages.

On trouvera ici ~~la règle~~ une condition générale à laquelle satisfait toute ~~les~~ équation~~s~~ soluble~~s~~ par radicaux, et qui réciproquement assure leur résolubilité. On en fait l'application seulement aux équations dont le degré est un nombre premier. Voici le théorème donné par notre Analyse:

Pour qu'une équation de degré premier, qui n'a pas de diviseurs commensurables, soit soluble par radicaux, il faut et il suffit que ~~deux quelconques des~~^toutes les^ racines soient des fonctions rationnelles de deux quelconques d'entre elles.

Les autres applications de la théorie sont elles-mèmes autant de théories particulières. Elles nécessitent d'ailleurs l'emploi de la théorie des nombres, et d'un algorithme particulier: Nous les réservons pour une autre occasion. Elles sont en partie relatives aux équations modulaires de la théorie des fonctions elliptiques, que nous démontrons ne pouvoir se résoudre par radicaux.

Ce 16 Janvier 1831.

E. Galois

1er Mémoire

L1846, BA1962 capitalise 'mémoire' in line 1; clearly lower case in *ms*, C*ms*. L1846, BA1962 replace 'étant' in line 3 with 'ayant été'; C*ms* is faithful. In line 4 a comma introduced after 'généraux' by Chevalier, printed in L1846, does not appear in P1897, BA1962. In L1846 'seule' is italicised; in C*ms* it is underlined—but the underlining looks accidental; there is no underline in *ms*.

Colon after 'particulier' replaced with stop in BA1962; colon is clear in *ms*, C*ms*.

Last sentence and date written with a finer pen. Capitalisation of 'Janvier' clear in *ms*, but C*ms*, L1846, BA1962 correct to lower-case.

'1er Mémoire' written sloping upwards on 29 May 1832, with same ink as *Lettre testamentaire*.

Memoir on the conditions for solubility of equations by radicals

~~Memoir~~

The attached memoir is extracted from a work which I had the honour to present to the Academy a year ago. That work not having been understood, the propositions which it contained having been dismissed as doubtful, I have had to content myself with giving the general principles in synthetic form and a single application of my theory. I implore my judges at least to read these few pages with attention.

All is crossed out with vertical lines

There will be found here ~~the rule~~ a general *condition satisfied by every equation that is soluble by radicals*, and which conversely ensures their solubility. An application is made just to equations of which the degree is a prime number. Here is the theorem given by our analysis:

In order that an equation of prime degree with no commensurable divisors be soluble by radicals, it is *necessary* and it is *sufficient* that ~~any two of the~~ all the roots be rational functions of any two of them.

Other applications of the theory are as much special theories themselves. They require, however, use of the theory of numbers, and of a special algorithm; we reserve them for another occasion. In part they relate to the modular equations of the theory of elliptic functions, which we prove not to be soluble by radicals.

This 16 January 1831.

E. Galois

2 b

Principes.

Je commencerai par établir quelques définitions et une suite de lemmes qui sont tous connus. ~~J'omets les démonstrations trop faciles~~.

Définitions. Une équation est dite réductible quand elle admet des diviseurs rationnels; irréductible dans le cas contraire.

Il faut ici expliquer ce qu'on doit entendre par le mot rationnel: car il se représentera souvent.

Word 'tous' italicised in L1846; it is underlined in C*ms*, but not in *ms*.

Quand l'équation a tous ses coëfficients numériques et rationnels, celà veut dire simplement que l'équation ~~se~~ peut se décomposer en facteurs qui aient leurs coëfficients numériques et rationnel~~le~~s.

Mais quand les coefficients d'une équation ne seront pas tous numériques et rationnels, alors il faudra entendre par diviseur rationnel, un diviseur dont les coëfficients s'exprimeraient en fonction rationnelle des coëfficients de la proposée, en général, par quantité rationnelle, une quantité qui s'exprime en fonction rationnelle des coëfficients de la proposée.

The comma after 'général' in *ms* is debatable. I judge it to be very probable; Chevalier and other editors do not see it.

Il y a plus: on pourra convenir de regarder comme rationnelle toute fonction rationnelle d'un certain nombre de quantités déterminées, supposées connues à priori. ~~Ainsi~~ ^Par exemple^ on pourra choisir une certaine ~~radical et~~ racine d'un nombre entier, et regarder comme rationnelle toute fonction rationnelle de ce radical.

Chevalier has 'Par ex:'; all printed editions have comma after 'exemple'.

Lorsque nous conviendrons de regarder ainsi comme connues de certaines quantités, nous dirons que nous les adjoignons à l'équation qu'il s'agit de résoudre. Nous dirons que ces quantités sont adjointes à l'équation.

Celà posé, nous appêllerons rationnelle toute quantité qui s'exprimera en fonction rationnelle ~~de léquation~~ des coëfficients de l'équation et d'un certain nombre de quantités adjointes à l'équation et convenues arbitrairement.

Deletion missing from BA1962; probably this is 'Nous appèlerons' broken off.

~~Nous appèler~~ Quand nous nous servirons d'équations auxiliaires, elles seront rationnelles, si leurs coëfficients sont rationnels en notre sens.

On voit au surplus que les propriétés et les difficultés d'une équation peuvent ~~changer~~ ^être^ tout à fait différentes suivant les quantités qui lui sont adjointes. ~~Ainsi~~^Par exemple,^ l'adjonction d'une quantité, peut rendre réductible une équation irréductible.

~~Ainsi~~ ~~Par exemple~~ ^Ainsi^, quand on adjoint à l'équation

$$\frac{x^n - 1}{x - 1} = 0 \quad \text{où } n \text{ premier}$$

Corrected in C*ms*, L1846, BA1962 to 'où *n* est premier'.

~~l'~~une ~~des~~ racine~~s~~ d'une des équations auxiliaires de M. Gauss, ~~elle~~ cette équation se décompose en facteurs, et devient par conséquent réductible.

Principles.

I shall begin by establishing some definitions and a sequence of lemmas all of which are known. ~~I omit the very easy proofs.~~

Definitions. An equation is said to be reducible when it admits rational divisors; irreducible in the contrary case.

It is necessary to explain here what should be understood by the word rational because it will often reappear.

When the equation has all its coefficients rational numbers, it should mean simply that the equation can be decomposed into factors which have rational numbers as their coefficients.

But when the coefficients of an equation are not *all* rational numbers, then one should understand by rational divisor a divisor whose coefficients may be expressed as rational functions of the coefficients of the proposed equation; [and] in general by a rational quantity, a quantity which may be expressed as a rational function of the coefficients of the proposed equation.

Further, one could agree to regard as rational every rational function of a certain number of determined quantities, supposed known a priori. For example, one could choose a certain root of a whole number, and regard as rational every rational function of this radical.

When we thus agree to regard certain quantities as known, we shall say that we *adjoin* them to the equation which it is required to solve. We shall say that these quantities are *adjoined* to the equation.

That having been said, we shall call *rational* every quantity which can be expressed as a rational function of the coefficients of the equation together with a certain number of quantities *adjoined* to the equation and agreed arbitrarily.

~~We [shall] call~~ When we make use of auxiliary equations they will be rational if their coefficients are rational in our sense.

One sees moreover that the properties and the difficulties of an equation can ~~change~~ be quite different according to the quantities which are adjoined to it. ~~Thus~~ For example, the adjunction of a quantity can render an irreducible equation reducible.

~~Thus For example~~ Thus when one adjoins to the equation

$$\frac{x^n - 1}{x - 1} = 0 \quad \text{where } n \text{ is prime}$$

a root of one of the auxiliary equations of Mr Gauss, this equation decomposes into factors, and consequently becomes reducible.

On placement of definition of *substitution* and *groupe* see Note 12 on p. 155.

Telles sont les définitions que nous avons cru devoir rappeler~~, elles paraitront peut-être superflues. Mais nous préférons la diffusion à l'obscurité~~

---

3 a

Lemme I. ~~O~~ Une équation irréductible ne peut avoir aucune racine commune avec une ~~autre~~ équation rationnelle, sans la diviser.
Car le plus grand commun diviseur ~~sera encore rat~~ entre l'équation ^irréductible^ proposée et l'autre équation, sera encore rationnel; donc, &c.

Word 'proposée' missing from L1846.

Lemme II. Étant donnée une équation quelconque, qui n'a pas de racines égales, dont les racines sont a, b, c ... on peut toujours former une fonction V des racines, telle qu'aucune des valeurs que l'on obtient en permutant dans cette fonction les racines de toutes manières, ne soient égales.
Par exemple, on peut prendre

$$V = Aa + Bb + Cc + \cdots$$

A, B, C, ... étant des nombres entiers convenablement choisis.
Lemme III. La fonction V étant choisie comme il est indiqué dans l'article précédent, elle jouira de cette propriété, que toutes les racines de l'équation ⌜proposée⌝ s'exprimeront ^rationnellement^ en fonction de V.
En effet, soit $V = \varphi(a, b, c, d, \ldots)$ ou bien

$$V - \varphi(a, b, c, d, \ldots) = 0$$

Multiplions entre elles toutes les équations ^semblables^ que l'on obtient en permutant dans celles-ci toutes ^les^ lettres, la première seulement restant fixe ^~~à[2]~~^; il viendra [~~une~~] ^l'^expression ~~symmé~~ ^suivante^

$$\ulcorner \{V - \varphi(a, b, c, d, \ldots)\}\{V - \varphi(a, c, b, d, \ldots)\}\{V - \varphi(a, b, d, c, \ldots)\}\{V \urcorner$$

^symé^trique en b, c, d, &c, ... laquelle ~~s'exp~~ pourra parconséquent s'écrire en fonction de a. Nous aurons donc une équation de la forme

In the display $f(V, a)$ was changed by over-writing to $F(V, a)$. Similarly below.

$$F(V, a) = 0$$

Or je dis que de là on peut tirer la valeur de a. Il suffit pour celà de chercher la solution commune à cette équation et à la proposée. Cette solution est la seule commune: car on ne peut avoir, par exemple,

$$F(V, b) = 0$$

See Note 9 on p. 154.

sans quoi ^cette équation ayant un facteur commun avec l'équation semblable^ ~~la~~⌜l'une des⌝fonction⌜s⌝ $\varphi(a, \ldots.)$ serait égale à ~~la~~⌜l'une des⌝ fonction⌜s⌝ $\varphi(b, \ldots.)$. Ce qui est contre l'hypothèse.

Those are the definitions which we have believed necessary to recall. ~~They will perhaps appear superfluous, but we prefer diffuseness to obscurity.~~

Here 'diffuseness' is literal; 'verbosity' would capture the sense better.

LEMMA I. *An irreducible equation cannot have any root in common with a rational equation without dividing it.*

For the greatest common divisor of the irreducible equation and the other equation will again be rational; therefore, etc.

LEMMA II. *Given an arbitrary equation which has no equal roots, of which the roots are a, b, c, ..., one can always form a function V of the roots, such that none of the values that are obtained by permuting the roots in this function in all possible ways will be equal.*

For example, one can take

$$V = Aa + Bb + Cc + \cdots,$$

A, B, C, ... being suitably chosen whole numbers.

LEMMA III. *The function V being chosen as is indicated in the preceding article, it will enjoy the property that all the roots of the proposed equation will be rationally expressible as a function of V.*

Indeed, let $V = \varphi(a, b, c, d, \ldots)$, or

$$V - \varphi(a, b, c, d, \ldots) = 0.$$

Multiply together all the similar equations that are obtained by permuting all the letters in this one, only the first remaining fixed; the following expression will result:

$$\big(V - \varphi(a, b, c, d, \ldots)\big)\big(V - \varphi(a, c, b, d, \ldots)\big)\big(V - \varphi(a, b, d, c, \ldots)\big)\cdots,$$

symmetric in b, c, d, ..., which will consequently be expressible as a function of a. We will therefore have an equation of the form

$$F(V, a) = 0.$$

Now I say that from this one can draw the value of a. For that it suffices to look for the solution common to this equation and the proposed one. This solution is the only common one, for one cannot have, for example,

$$F(V, b) = 0,$$

this equation having a factor in common with the similar equation, without one of the functions $\varphi(a, \ldots)$ being equal to one of the functions $\varphi(b, \ldots)$; which is contrary to hypothesis.

Il suit de là que ~~à~~ a s'exprime en fonction rationnelle de V, et il en est de même des autres racines.
Cette proposition* est citée sans démonstration, par Abel dans le mémoire posthume sur les fonctions elliptiques.
Lemme IV. Supposons que l'on ait formé l'équation en V, et que l'on ait pris l'un de ses facteurs irréductibles, ensorte que V soit racine d'une équation irréductible. Soient $V \quad V' \quad V'' \quad \ldots$ les racines de cette équation irréductible. Si ~~a~~^a =^ $f(V)$ est une ~~fonct~~ des racines de la proposée, $f(V')$ ^de même^ sera une racine de la proposée.

Comma after 'En effet' in C*ms*, L1846, BA1962; none in *ms*.

In C*ms*, L1846 'dans la quelle on' changed to 'où l'on'.

~~Car~~ *En effet en multipliant entre eux tou~~te~~s les ~~équations~~ facteurs de la forme $V-\varphi(a,b,c,\ldots d)$ dans la quelle on aura opéré sur les lettres toutes les permutations possibles, on aura une équation rationnelle en V la quelle ~~sera eg~~ ^se trouvera^ nécessairement divisible par l'équation en question; ~~on aura~~ donc ~~V' égal à une~~ V'* V' doit s'obtenir par l'échange des lettres dans la fonction V. ~~Donc on aura $V' = \varphi$~~
Soit $F(V,a) = 0$ l'équation qu'on obtient en ~~permutant entre elles~~ ^permutant dans^ V toutes les lettres
~~Ces principes posés, nous allons passer à l'exposition~~ de ~~notre théorie~~
hors la première: On aura donc $F(V',b) = 0 \quad b$ pouvant etre égal à a, mais étant certainement l'une des racines de l'équation proposée. Parconséquent dememe que de la proposée et de $F(V,a) = 0$ est resultera $a = f\,V$, ⌜Donc⌝ Il résultera de la proposée et $F(V',b) = 0$ combinées, la suivante $b = f(V')$.

See Note 10 on p. 154.

For 'est resultera' see Note 11 on p. 155

3b

PROPOSITION I.

Théorème. Soit une équation donnée dont $a, b, c, \ldots$ sont les m racines. ~~On pourra~~ ^Il y aura^ toujours un groupe de permutations des lettres $a, b, c, \ldots\ldots$ qui jouira de la propriété suivante:
1° Que toute fonction des racines invariable* par les ~~permutations~~^substitutions de ce groupe^ soit rationnellement connue;

These, though marginal notes in **f. 3 a** and **f. 3 b**, are clearly intended as footnotes, and have been taken as such by all editors. The first is written with a finer nib than the surrounding text, possibly the same as the one used for the preface. In BA1962 both are dated to the original draft of the memoir.

*N. Il est remarquable, que de cette proposition on peut conclure que toute équation ~~peut résoud~~ dépend d'une équation auxiliaire telle que toutes les racines de cette nouvelle équation soient des fonctions rationnelles les unes des autres. Car l'équation auxiliaire en V est dans ce cas.
Au surplus cette remarque est purement curieuse: En effet une équation qui a cette propriété n'est pas en général plus facile à résoudre qu'une autre.

*N. Nous appelons ici invariable ~~les~~ non seulement une fonction dont la forme est invariable par les substitutions des racines entre elles, mais encore celle dont la valeur numérique ~~la serait~~ ne varierait pas par ces substitutions. Par exemple, si $Fx = 0$ est une équation, Fx est une fonction des racines qui ne varie par aucune permutation.
Quand nous disons qu'une fonction est rationnellement connue, nous voulons dire que sa valeur numérique est exprimable en fonction rationnelle des coëfficients de l'équation et des quantités adjointes.

It follows that a may be expressed as a rational function of V, and the same holds for the other roots.

This proposition* is cited without proof by Abel in the posthumous memoir on elliptic functions.

LEMMA IV. *Let us suppose that the equation in V has been formed, and that one of its irreducible factors has been taken so that V is a root of an irreducible equation. Let V, V', V'', ... be the roots of this irreducible equation. If $a = f(V)$ is one of the roots of the proposed equation, $f(V')$ will also be one of the roots of the proposed equation.*

Indeed, on multiplying together all factors of the form $V - \varphi(a, b, c, \ldots, d)$, in which all possible permutations have been applied to the letters, one will have a rational equation in V, which must necessarily be divisible by the equation in question; ~~one will therefore have V' equal to a~~ therefore V' must be obtainable by exchange of some letters in the function V. ~~Therefore one will have $V' = \varphi$~~ Let $F(V, a) = 0$ be the equation that is obtained by ~~permuting amongst themselves~~ permuting in V all the letters

~~These principles being in place, we proceed to the exposition of our theory.~~ beyond the first. One will then have $F(V', b) = 0$, b possibly being equal to a, but certainly being one of the roots of the proposed equation. Consequently, just as $a = F(V)$ results from the proposed equation and $F(V, a) = 0$, so the following will result from the proposed equation and $F(V, b) = 0$ combined: $b = f(V')$.

PROPOSITION I.

THEOREM. *Let an equation be given of which the m roots are a, b, c, ...* ~~*One will always be able*~~ *There will always be a group of permutations of the letters a, b, c, ... which will enjoy the following property:*

1. *That every function of the roots invariant* under the ~~permutations~~ substitutions of this group will be rationally known;*

---

* N[ote]. It is worth noting that from this proposition one can conclude that every equation depends on an auxiliary equation which is such that all the roots of this new equation will be rational functions of one another. For the auxiliary equation for V is of this kind.

It should be added that this remark is simply a curiosity. Indeed, an equation which has this property is, in general, not any easier to solve than another.

---

* N[ote]. Here we call invariant not only a function of which the form is invariant under the substitutions of the roots among themselves, but also one whose numerical value will not change under these substitutions. For example, if $Fx = 0$ is an equation, Fx is a function of the roots which is not changed by any permutation.

When we say that a function is rationally known, we wish to say that its numerical value is expressible as a rational function of the coefficients of the equation and of some adjoined quantities.

2° réciproquement, que toute fonction des racines déterminable rationnellement soit invariable par les ~~permutations~~ ⌜substitutions⌝

Important: for the placement of this marginal addition see Note 12 on p. 156.

* Les substitutions sont ~~les~~ le passage d'une permutation à l'autre.
La permutation d'où l'on part pour indiquer les substitutions est toute arbitraire⌜,⌝ ~~. Il n'y a d'important que la substitution du~~ quand il s'agit de fonctions. Car il n'y a aucune raison pour que dans une fonction de plusieurs lettres, une lettre occupe un rang plutot qu'un autre.

Word 'les' over-written with 'dans'

Cependant, comme on ne peut guere se former l'idée d'une substitution sans ~~se former~~ celle d'une permutation, nous ~~garderons~~ ^ferons^ ~~dans~~ le langage un emploi frequent des permutations, et nous ne considérerons les substitutions que comme le passage d'une permutation à une autre.
Quand nous voudrons grouper des substitutions nous les ferons toutes provenir d'une mème permutation.
~~Le groupe est un ensemble de lettres permutations, tell~~*~~l~~*~~e~~ que si l'on passe
Comme il s'agit toujours de questions où la disposition primitive des lettres ~~est~~ n'influe en rien, ^dans^ Les groupes que nous considérerons, on devra avoir les mèmes substitutions quelle que soit la permutation d'où l'on sera parti. Donc si ~~*S*~~ dans un pareil groupe on a les substitutions S et T, on est sûr d'avoir la substitution ST.

---

*

(Dans le cas des Équations algèbriques, ce groupe n'est autre chose que l'ensemble des 1.2.3....m permutations possibles sur les m lettres, puisque dans ce cas, les fonctions symétriques sont seules déterminables rationnellement.)

Spelling corrected to 'symétriques' in C*ms* and all print editions.

(Dans le cas de l'équation $\frac{x^n-1}{x-1}=0$, si l'on suppose $a=r \quad b=r^g$ $c=r^{g^2} \quad \ldots \quad g$ étant une racine primitive, le groupe de permutations sera simplement celui-ci

From the context it is clear that Galois had in mind *n* prime as at the foot of **f. 2 b**, p. 108.

$$\begin{array}{cccccc} a & b & c & d & \ldots\ldots & k \\ b & c & d & \ldots & k & a \\ c & d & \ldots & k & a & b \\ & & \ldots & \ldots & & \\ k & a & b & c & \ldots\ldots & i \end{array}$$

Third line from below: 'fonction rationnelle' in *ms*, C*ms*. Corrected to plural in L1846, BA1962.
Last line: no commas in lists here and below in *ms*. Commas introduced in C*ms*; reproduced in L1846, BA1962.

dans ce cas particulier, ~~il n'y a~~ le nombre des permutations est égal au degré de l'équation, et la mème chose aurait lieu dans les équations dont toutes les racines seraient des fonctions rationnelles les unes des autres.)

Démonstration. Quelle que soit l'équation donnée, on pourra trouver une fonction rationnelle V des racines telle que toutes les racines soient fonction rationnelle de V. Celà posé, considérons l'équation irréductible dont V est racine. (Lemmes III et IV.) Soient $V \;\; V \;\; V' \;\; V'' \;\; \ldots \;\; V^{(n-1)}$ les racines de cette équation.

2. *Conversely, that every function of the roots that is rationally determinable will be invariant under the ~~permutations~~ substitutions.*

* Substitutions are the passage from one permutation to another.

The permutation from which one starts in order to indicate substitutions is completely arbitrary, ~~Nothing other than the substitution is of importance~~ as far as functions are concerned; for there is no reason at all why a letter should occupy one place rather than another in a function of several letters.

Galois indicated that this famous passage, which appears in the margin of **f. 3 a**, should be moved to the definitions (p. 109). That makes excellent sense and is where it appears in all previous editions.

Nevertheless, since it is impossible to grasp the idea of a substitution without ~~grasping~~ that of a permutation, we ~~will retain~~ will make frequent use of permutations in the language, and we shall not consider substitutions other than as the passage from one permutation to another.

When we wish to group some substitutions we make them all begin from one and the same permutation.

~~The group is a collection of letters permutations such that if one passes~~

As the concern is always with questions where the original disposition of the letters has no influence, in the groups that we will consider one must have the same substitutions whichever permutation it is from which one starts. Therefore, if in such a group one has substitutions S and T, one is sure to have the substitution ST. *

(In the case of algebraic equations, this group is nothing other than the collection of the $1.2.3 \ldots m$ possible permutations on the m letters, because in this case, only the symmetric functions are rationally determinable.)

(In the case of the equation $\dfrac{x^n - 1}{x - 1} = 0$, if one sets $a = r$, $b = r^g$, $c = r^{g^2}$, …, g being a primitive root, the group of permutations will simply be this:

$$\begin{array}{cccccc} a & b & c & d & \ldots\ldots & k \\ b & c & d & \ldots & k & a \\ c & d & \ldots & k & a & b \\ & \ldots & \ldots & \ldots & & \\ k & a & b & c & \ldots\ldots & i \end{array}$$

in this particular case, ~~there is no~~ the number of permutations is equal to the degree of the equation, and the same thing will happen in the case of equations of which all the roots are rational functions of one another.)

Proof. Whatever the given equation may be, one will always be able to find a rational function V of the roots such that all the roots will be rational functions of V. That said, let us consider the irreducible equation of which V is a root (Lemmas III and IV). Let V, V', V'', …, $V^{(n-1)}$ be the roots of this equation.

Soient

$$\varphi V \quad \varphi_1 V \quad \varphi_2 V \quad \ldots\ldots \quad \varphi_{m-1} V$$

List clearly displayed in *ms*, semi-displayed in C*ms*, run on in L1846, BA1962.

les racines de la proposée.
Écrivons les n permutations suivantes des racines

The displayed table originally fitted nicely on the page. Later Galois added the labels (V), (V'), ..., $(V^{(n-1)})$. For lack of space they fall into the margin of the *ms*.

$$\begin{array}{lllllll}
(V) & \varphi V & \varphi_1 V & \varphi_2 V & \ldots\ldots & \varphi_{m-1} V \\
(V') & \varphi V' & \varphi_1 V' & \varphi_2 V' & \ldots\ldots & \varphi_{m-1} V' \\
(V'') & \varphi V'' & \varphi_1 V'' & \varphi_2 V'' & \ldots\ldots & \varphi_{m-1} V'' \\
& \ldots\ldots\ldots\ldots\ldots\ldots & & & & \\
\left(V^{(n-1)}\right) & \varphi V^{(n-1)} & \varphi_1 V^{(n-1)} & \varphi_2 V^{(n-1)} & \ldots\ldots & \varphi_{m-1} V^{(n-1)}
\end{array} .$$

Je dis que ce groupe de permutations jouit de la propriété énoncée.
En effet, 1°. toute fonction F des racines invariable par les ~~substitutions~~ ^~~permutations~~^ ˇsubstitutionsˇ de ce groupe, pourra être écrite

4a

ainsi $F = \psi V$, et l'on aura

$$\psi V = \psi V' = \psi V'' = \cdots\cdots = \psi V^{(n-1)}$$

La valeur de F pourra donc se déterminer rationnellement.

Stop and capital for 'Si' in *ms*, C*ms*; comma and lower case in L1846, BA1962.

2°. Réciproquement. Si une fonction F est déterminable rationnellement, et que l'on pose $F = \psi V$, ~~de~~ on devra avoir

$$\psi V = \psi V' = \psi V'' = \cdots\cdots = \psi V^{(n-1)}$$

puisque l'équation en V n'a pas de diviseur commensurable et que V satisfait à l'équation $F = \psi V$, F étant une quantité rationnelle. Donc la fonction F sera nécessairement invariable par les substitutions du groupe écrit ci-dessus.
Ainsi ce groupe jouit de la double propriété dont il s'agit dans le théorème ~~précédent~~ proposé. Le théorème est donc démontré.
⌜Nous appèllerons groupe de l'équation le groupe en question.⌝

Scholie. Il est évident que dans le groupe de permutations dont il s'agit ici, la disposition des lettres n'est point à considerer, mais seulement les substitutions de lettres par les quelles on passe d'une permutation à l'autre.
Ainsi l'on peut se donner arbitrairement ~~la~~ une première permutation, ~~et~~ ^pourvu que^ les autres ~~substitutions~~ permutations s'en déduisent ~~par~~ toujours par les mèmes substitutions de lettres. Le nouveau groupe ainsi formé jouira évidemment des mèmes propriétés que le premier, puisque dans le théorème précédent, il ne s'agit que des substitutions ~~que~~ de lettres que l'on peut faire dans les fonctions.

Last line: 'de lettres' missed by Chevalier and missing from all print editions.

Let

$$\varphi V, \quad \varphi_1 V, \quad \varphi_2 V, \quad \ldots, \quad \varphi_{m-1} V$$

be the roots of the proposed equation.

Let us write down the following n permutations of the roots:

(V)	φV	$\varphi_1 V$	$\varphi_2 V$	. . . . .	$\varphi_{m-1} V$
(V')	$\varphi V'$	$\varphi_1 V'$	$\varphi_2 V'$	. . . . .	$\varphi_{m-1} V'$
(V'')	$\varphi V''$	$\varphi_1 V''$	$\varphi_2 V''$	. . . . .	$\varphi_{m-1} V''$
	. . . . .				
$\left(V^{(n-1)}\right)$	$\varphi V^{(n-1)}$	$\varphi_1 V^{(n-1)}$	$\varphi_2 V^{(n-1)}$	. . . .	$\varphi_{m-1} V^{(n-1)}$.

I say that this group of permutations enjoys the specified property.

Indeed, (1) every function F of the roots that is invariant under the substitutions of this group may be written

thus: $F = \psi V$, and one will have

$$\psi V = \psi V' = \psi V'' = \cdots = \psi V^{(n-1)}.$$

The value of F will therefore be determinable rationally.

(2) Conversely, if a function F is rationally determinable, and if one sets $F = \psi V$, one will have to have

$$\psi V = \psi V' = \psi V'' = \cdots = \psi V^{(n-1)}$$

because the equation for V has no commensurable divisor and V satisfies the equation $F = \psi V$, F being a rational quantity. Therefore the function F will necessarily be invariant under the substitutions of the group written above.

Thus this group enjoys the double property stated in the ~~preceding~~ proposed theorem. The theorem is therefore proved.

We will call the group in question the group of the equation.

Scholium [1]. It is clear that in the group of permutations which is discussed here, the disposition of the letters is not at all relevant, but only the *substitutions* of letters by which one passes from one permutation to another.

Thus a first permutation may be given arbitrarily, and then the other ~~substitutions~~ permutations may always be deduced by the same substitutions of letters. The new group formed in this way will evidently enjoy the same properties as the first, because in the preceding theorem, nothing matters other than substitutions of letters that one may make in the functions.

*~~Ce qui characterise un groupe. On peut partir d'une des permutations quelconques du groupe.~~
Scholie. Les substitutions sont indépendantes mème du nombre des racines.*

PROPOSITION II.

Théorème. Si l'on adjoint à une équation ^donnée^ la racine ^r^ d'une équation auxiliaire irréductible, ~~et de degré ^p^ premier~~, ⌜1°⌝ il arrivera de deux choses l'une: ou bien le groupe de l'équation ne sera pas changé; ou bien il se partagera en p groupes appartenant ^chacun^ à l'équation proposée respectivement quand on lui adjoint chacune des racines de l'équation auxiliaire. 2°. ces groupes jouiront de la propriété remarquable, que l'on passera de l'un à l'autre en opérant dans toutes les permutations du premier une même substitution de lettres.

*Il y a quelque chose à completer dans cette démonstration. Je n'ai pas le tems.
(Note de l'A.) *

.1°. Si, après l'adjonction de r, l'équation en V, dont il est question plus haut reste irréductible, il est clair que le groupe de l'équation ne sera pas changé. Si au contraire elle se réduit, alors l'équation en V se décomposera en p facteurs ^tous^ de même degré et de la forme*

* ~~*Car si l'on élimine r entre $f(V,r) = 0$ et $Fr = 0$ Fr étant de degré premier p, il ne peut arriver que de deux choses l'une: ou le resultat de l'élimination sera demème degré en V que $f(V,r)$, ou il sera d'un degré |multiplie de| p fois plus grand.~~ *

$$f(V,r) \times f(V,r') \times f(V,r'') \times \cdots\cdots$$

$r \quad r' \quad r'' \quad \cdots\cdot$ étant ~~les diverses~~ ^d'autres^ valeurs de r. Ainsi le groupe de l'équation proposée se décomposera aussi ~~les~~ en ~~p~~ groupes chacun d'un mème nombre de permutation, puisqu'à chaque valeur de V correspond une permutation. Ces groupes seront respectivement ceux de l'équation proposée, quand on lui adjoindra successivement $r\,, \quad r'\,, \quad r''\,, \quad \ldots.$

4 b

2°. Nous avons vu plus haut que toutes les valeurs de V étaient des fonctions rationnelles les unes des autres. D'après celà, supposons que V étant une racine de $f(V,r) = 0$, $F(V)$ en soit une autre. Il est clair que de mème si V' est une racine de $f(V,r') = 0$, $F(V')$ en sera une autre.*

* *Car l'on aura $f\big(F(V),r\big)$ = une fonction divisible par $f(V,r)$
Donc, (Lemme I) $f\big(FV',r'\big)$ = une fonction divisible par $f(V',r')$.*
Celà posé, je dis que l'on obtient le groupe relatif à r' en opérant partout dans le groupe relatif à r une mème substitution de lettres.

In C*ms* Chevalier added a note to the theorem. It was printed in modified form in L1846. See Note 15 on p. 158.

The marginal note slopes upwards; probably written soon after the marginal insertion into the proof 'Car si [...] plus grand' was crossed out.// Word 'tems' transcribed correctly in T1906/7 (p. 231), BA1962 (p. 486); corrected to 'temps' in C*ms*, L1846 (p. 423), BA1962 (caption to 7th frontispiece and p. 54).

Deleted marginal insertion dated in BA1962 to original draft of paper. In 'Car si l'on élimine r' letter r misread or misprinted as N in BA1962.

Third line from end of **f. 4 a**: both occurrences of 'permutation' clearly singular in *ms*; first corrected to plural in C*ms*, L1846, BA1962. Last line: rare instance of commas in a list in *ms*.

Punctuation corrected in L1846, BA1962. FV' corrected to $F(V')$ in C*ms*, L1846, BA1962.

*~~What characterises a group. One can start from any one of the permutations of the group.~~

Scholium [2]. The substitutions are even independent of the number of roots.*

PROPOSITION II.

THEOREM. *If one adjoins to a given equation the root r of an irreducible auxiliary equation,* [*~~and of prime degree p~~*]

(1) *one of two things will happen: either the group of the equation will not be changed, or it will be partitioned into p groups each belonging respectively to the proposed equation when one adjoins to it each of the roots of the auxiliary equation;*

(2) *these groups will enjoy the remarkable property that one will pass from one to another by operating on all the permutations of the first with one and the same substitution of letters.*

There is something to be completed in this proof. I do not have the time. (Author's note)

On marginal note see Notes 15, 16, pp. 158–161.

(1). If, after the adjunction of r the equation for V discussed above remains irreducible, it is clear that the group of the equation will not be changed. If, on the other hand, it becomes reducible then the equation for V will decompose into p factors, all of the same degree and of the form

[~~For if one eliminates r between $f(V,r) = 0$ and $Fr = 0$, Fr being of prime degree p, only one of two things can happen: either the result of the elimination will be of the same degree in V as $f(V,r)$ or its degree will be p times greater.~~]

$$f(V,r) \times f(V,r') \times f(V,r'') \times \cdots,$$

r, r', r'', ... being [~~the different~~] other values of r. Thus the group of the proposed equation will also decompose into groups each with the same number of permutations, because to each value of V corresponds a permutation. These groups will be those of the proposed equation respectively when r, r', r'', ... are adjoined to it successively.

(2). We have seen above that all the values of V were rational functions of one another. In light of that, suppose that, V being a root of $f(V,r) = 0$, $F(V)$ is another one. It is clear that in the same way if V' is a root of $f(V,r') = 0$, $F(V')$ will be another one. For one will have

$$f(F(V),r) = \text{a function divisible by } f(V,r).$$

Therefore (Lemma I)

$$f(F(V'),r') = \text{a function divisible by } f(V',r').$$

That said, I say that one obtains the group relative to r' by operating throughout in the group relative to r with one and the same substitution of letters.

Subscript to first occurrence of φ changed in *ms* from m to p. But this is not p as in statement of theorem; nor is n here the number of permutations in its group: p changed to μ and n to ν in C*ms*, L1846.

En effet si l'on a par exemple $\varphi_{\not m} F(V) = \varphi_n V$ on aura encore, (Lemme I), $\varphi_p F(V') = \varphi_n V'$. Donc, pour passer de la ~~ligne~~ permutation $\big(F(V)\big)$ à la permutation $\big(F(V')\big)$, il faut faire la mème substitution que pour passer de la ~~substituti~~ permutation (V) à la permutation (V').
Le théorème est donc démontré.

PROPOSITION III.

~~Théorème. Si l'équation en r est de la forme $r^p = A$ et que les racines $p^{\text{ièmes}}$ de l'unité se trouvent au nombre des quantités précedemmant adjointes, les p groupes dont il est question dans le théorème II, jouiront |encore| deplus de cette propriété, que les substitutions de lettres par les quelles on passe d'une |substitution| permutation à l'autre dans |un même| ^chaque^ groupe soient les mèmes |dans| ^pour^ tous les groupes.~~
~~En effet, dans ce cas, |les| il revient au mème d'adjoindre à l'équation telle ou telle valeur de r. Parconséquent ses propriétés doivent etre les mèmes après l'adjonction de telle ou telle valeur. Ainsi |le| ^son^ groupe doit etre le mème quant aux substitutions. (Prop. I, scholie) Donc, &c.~~

* 1832 PROPOSITION III.

Théorème. Si l'on adjoint à une équation toutes les racines ~~de l'équ~~ d'une équation auxiliaire, les groupes dont il est question dans le théorème II jouiront deplus de cette propriété que les substitutions ~~de lettres par les quelles on passe~~ sont les mêmes dans chaque groupe.

Liouville modified a note added by Chevalier to Proposition III. See Note 17 on p. 161.

On trouvera la demonstration. *

PROPOSITION IV.

Théorème. Si l'on adjoint à une équation la valeur numérique d'une certaine fonction de ses racines, le groupe de l'équation s'abaissera de manière à n'avoir plus d'autres permutations que celles par les quelles cette fonction est invariable.

En effet, d'après la proposition I, ~~le nom~~ toute fonction connue doit être invariable par les permutations du groupe de l'équation.

PROPOSITION V.

Problème. Dans quels cas une équation est-elle soluble par de simples radicaux?

J'observerai d'abord que, pour résoudre une équation, il faut successivement abaisser son groupe jusqu'à ne contenir plus qu'une seule permutation. Car, quand une équation est résolu~~ble~~, une fonction quelconque de ses racines est connue, même quand elle n'est invariable par aucune permutation.

Indeed, if one has, for example, $\varphi_p\, F(V) = \varphi_n(V)$, one will also have (Lemma I) $\varphi_p\, F(V') = \varphi_n(V')$. Therefore to pass from the permutation $\big(F(V)\big)$ to the permutation $\big(F(V')\big)$ it is necessary to make the same substitution as to pass from the permutation (V) to the permutation (V').

The theorem is therefore proved.

PROPOSITION III.

~~THEOREM. *If the equation in r is of the form $r^p = A$ and the pth roots of unity are to be found among the quantities previously adjoined, the p groups in question in Theorem II will enjoy also furthermore the property that the substitutions of letters by which one passes from one substitution permutation to another in one and the same each group will be the same for all the groups.*~~

~~Indeed, in this case it comes to the same whichever value of r is adjoined to the equation. Consequently its properties must be the same after the adjunction of any value. Thus its group will be the same in respect of the substitutions. (Prop. I, Scholium). Therefore, etc.~~

1832 PROPOSITION III.

THEOREM. *If one adjoins to an equation **all** the roots of an auxiliary equation, the groups in question in Theorem II will enjoy the additional property that the substitutions ~~of letters by which one passes~~ are the same in each group.*

The proof is to be found.

PROPOSITION IV.

THEOREM. *If one adjoins to an equation the **numerical** value of a certain function of its roots, the group of the equation will be reduced in such a way as to not have any other permutations than those under which this function is invariant.*

Indeed, according to Proposition I, every known function must be invariant under the permutations of the group of the equation.

PROPOSITION V.

PROBLEM. *Under what circumstances is an equation soluble by simple radicals?*

I shall observe to begin with that, to solve an equation, it is necessary to reduce its group successively to the point where it does not contain more than a single permutation. For, when an equation is solved, an arbitrary function of its roots is known, even when it is not invariant under any permutation.

Celà posé, cherchons à quelle condition doit satisfaire le

5 a

groupe d'une équation, pour qu'il puisse s'abaisser ainsi par l'adjonction ~~d'u~~ de quantités radicales.

Suivons la marche des opérations possibles dans cette solution, en considérant comme ~~des~~ opérations distinctes l'extraction de chaque racine de degré premier.

Chevalier originally overlooked the clause after 'dans la solution'. When he returned to insert it in C*ms* the semicolon got changed to a stop.

Adjoignons à l'équation le premier radical extrait dans la solution; il pourra arriver deux cas: ou bien, par l'adjonction de ce radical, le groupe des permutations de l'équation sera diminué; ou bien, cette extraction de racine n'étant qu'une simple préparation, le groupe restera le mème.

Toujours sera-t-il qu'après un certain nombre fini d'extractions de racines, le groupe devra se trouver diminué, sans quoi l'équation ne serait pas soluble.

Si arrivé à ce point, il yavait plusieurs manières de diminuer le groupe de l'équation proposée par une simple extraction de racine, il faudrait, pour ce que nous allons dire, considérer seulement un radical du degré le moins haut possible parmi tous les simples radicaux qui sont tels que la connaissance de chacun d'eux diminue le groupe de l'équation.

Soit donc p le nombre premier qui représente ce degré minimum en sorte que par une extraction de racine de degré p, on diminue le groupe de l'équation.

Nous pouvons toujours supposer, du moins pour ce qui est relatif au groupe de l'équation, que parmi les quantités adjointes précédemment à l'équation se trouve une racine $p^{\text{ième}}$ de l'unité α. Car, comme cette expression s'obtient par des extractions de racines de degré inférieur a p, sa connaissance n'altérera en rien le groupe de l'équation.

Parconséquent, d'après les théorèmes II et III, le groupe de l'équation devra ~~jouir de cette double propriété~~ se décomposer en p groupes jouissant les uns par rapport aux autres de cette double propriété,⌜: 1°⌝ que l'on passe de l'un à l'autre par une ^seule et^ mème substitution; 2° que tous contiennent les mèmes substitutions.

Je dis réciproquement que si le groupe de l'équation peut se partager en p groupes qui jouissent de cette ^double^ propriété, on pourra, par ^~~l'a~~^ une simple extraction de racine $p^{\text{ième}}$, et par l'adjonction de cette racine $p^{\text{ième}}$, ~~abaisser~~^réduire^ le groupe de l'équation à l'un de ces groupes partiels.

Word 'en' before 'varie' missing from L1846. Read as 'ne' in C*ms*, BA1962. Unclear in *ms*: 'ne' looks plausible but makes no sense; 'en' looks plausible and makes some sense; 'se' looks just possible and would make excellent sense. Footnote to paragraph incorporated in text in C*ms*, L1846, but marginal in *ms* and clearly marked in same way as other notes that Galois intended as footnotes.

Prenons en effet une fonction des racines qui soit invariable pour toutes les substitutions de l'un des groupes partiels, et en varie pour toute autre substitution.⋆

Soit θ cette fonction ~~symmétrique~~ des racines.

Opérons sur la fonction θ ~~toutes~~^une^ des substitutions du groupe total qui ne lui sont pas communes avec les groupes partiels. Soit θ_1 le résultat. Opérons sur la fonction θ_1 la mème substitution, et soit θ_2 le résultat, et ainsi de suite.

⋆N. Il suffit pour celà de ~~prendre~~^choisir^ une fonction symmetrique des diverses valeurs que prend par ^toutes^ les permutations de l'un des groupes partiels, une fonction qui ~~n'est~~ n'est invariable pour aucune substitution.

That said, let us seek what condition the group of an equation must satisfy,

in order that it may be reduced in this way by the adjunction of radical quantities.

Let us follow the sequence of possible operations in this solution, considering as distinct operations the extraction of each root of prime degree.

Adjoin to the equation the first radical extracted in the solution; two cases could arise: either the group of permutations of the equation will become smaller by the adjunction of this radical; or this extraction of a root being no more than simple preparation, the group will remain the same.

It will always be the case that after a certain *finite* number of root extractions the group will have to become smaller, otherwise the equation will not be soluble.

If, having reached this point, there were several ways to diminish the group of the proposed equation by a simple root extraction, it would be necessary for what we are going to say to consider only a radical of lowest possible degree among all the simple radicals which are such that knowledge of any of them diminishes the group of the equation.

Let p then be the prime number which represents this minimum degree, so that by an extraction of a root of degree p one diminishes the group of the equation.

We can always suppose, at least for what is relative to the group of the equation, that among the quantities previously adjoined to the equation is a pth root of unity α. For, since this expression may be obtained by extraction of roots of lower degree than p, knowledge of it will not alter the group of the equation in any way.

Consequently, by Theorems II and III, the group of the equation must ~~enjoy this double property~~ decompose into p groups enjoying the following double property with respect to one another: (1) that one passes from one to another by one and the same substitution; (2) that they all contain the same substitutions.

I say conversely that if the group of the equation can be partitioned into p groups which enjoy this double property, one will be able to reduce the group of the equation to one of these partial groups by a simple extraction of a pth root and by the adjunction of this p^{th} root.

Indeed, let us take a function of the roots which is invariant under all the substitutions of one of the partial groups and changes for all other substitutions.⋆

Let θ be this ~~symmetric~~ function of the roots.

Operate on the function θ with ~~all~~ one of the substitutions of the whole group which is not one of those common to the partial groups. Let θ_1 be the result. Operate on the function θ_1 with the same substitution, and let θ_2 be the result, and so on.

⋆ N[ote]. It suffices for that to ~~take~~ choose a symmetric function of the different values taken under all the permutations of one of the partial groups by a function that is not invariant under any substitution.

Comme p est un nombre premier, cette suite ne pourra s'arrêter qu'au terme θ_{p-1}, en suite l'on aura, $\theta_p = \theta \quad \theta_{p+1} = \theta_1,$ et ainsi de suite.

Equation $\theta_p = \theta$ correct in C*ms*, but misprinted as $\theta_p = \theta_1$ in L1846.

Celà posé, il est clair que la fonction

$$\left(\theta + \alpha\theta_1 + \alpha^2\theta_2 + \cdots + \alpha^{p-1}\theta_{p-1}\right)^p$$

5 b

sera invariable par ^toutes^ les permutations du groupe total, et parconséquent sera actuellement connue.

Si l'on extrait la racine $p^{\text{ième}}$ de cette fonction et qu'on l'adjoigne à l'équation, alors par la proposition IV, le groupe de l'équation ne contiendra plus d'autres substitutions que celles des groupes partiels.

Ainsi, pour que le groupe d'une équation puisse s'abaisser par une simple extraction de racine, la condition ci-dessus est nécessaire et suffisante.

Adjoignons à l'équation le radical en question, nous pourrons raisonner maintenant sur le nouveau groupe comme sur le précédent, et il faudra qu'il se décompose lui-mème de la manière indiquée, et ainsi de suite, jusqu'à un certain groupe qui ne contiendra plus qu'une seule permutation.

In L1846 4e and 3e are rendered as 'quatrième' and 'troisième'; C*ms* has $4^{\text{ème}}$ and $3^{\text{ème}}$. C*ms*, L1846, BA1962 all correct 'elle mème' to 'elle-même'.

Scholie. Il est aisé d'observer ~~les~~ cette marche dans la résolution connue des équations générales du 4e degré. En effet, ces équations se résolvent au moyen d'une équation du 3e degré, qui éxige elle mème l'extraction d'une racine carrée. Dans la suite naturelle des idées, c'est donc par cette racine carrée qu'il faut commencer. Or en adjoignant à l'équation du 4e degré cette racine carrée, le groupe de l'équation, qui contenait en tout 24 substitutions, se décompose en deux qui n'en contiennent que douze. En désignant par $a\ b\ c\ d$ les racines, voici l'un de ces groupes.

In L1846 **24** is rendered in words; C*ms* is faithful.

$abcd$	$acdb$	$adbc$
$badc$	$cadb$	$dacb$
$cdab$	$dbac$	$bcad$
$dcba$	$bdca$	$cbda$

Slip in second entry of second row faithfully copied in C*ms*; corrected to *cabd* in L1846.

Maintenant ce groupe se partage lui mème en trois groupes, comme il est indiqué aux théorèmes II et III. Ainsi par l'extraction d'un seul radical du 3e degré il reste simplement le groupe

C*ms* has $3^{\text{ème}}$; rendered as 'troisième' in L1846, BA1962.

$$\begin{matrix} abcd \\ badc \\ cdab \\ dcba \end{matrix}$$

Ce groupe se partage de nouveau en deux groupes:

In L1846 the four displayed sets of permutations are punctuated; there is no punctuation in *ms* or C*ms*.

$abcd$	$cdab$
$badc$	$dcba$

Since p is a prime number, this sequence cannot stop except at the term θ_{p-1}, after which one will have $\theta_p = \theta$, $\theta_{p+1} = \theta_1$, and so on.

That said, it is clear that the function

$$(\theta + \alpha\theta_1 + \alpha^2\theta_2 + \cdots + \alpha^{p-1}\theta_{p-1})^p$$

will be invariant under all the permutations of the whole group, and consequently will be already known.

If one extracts the pth root of this function and adjoins it to the equation then, by Proposition IV, the group of the equation will contain no other substitutions than those of the partial groups.

Thus, in order that the group of an equation can be reduced by a simple extraction of a root, the condition above is necessary and sufficient.

Adjoin the radical in question to the equation; we may now reason on the new group as on the preceding one, and it itself has to decompose in the indicated manner, and so on, as far as a certain group which will contain no more than a single permutation.

Scholium. It is easy to observe this process in the known solution of general equations of the 4th degree. Indeed, these equations are solved by means of an equation of the 3rd degree, which itself requires the extraction of a square root. In the natural sequence of ideas, it is therefore with this square root that one must begin. Adjoining this square root to the equation of the 4th degree the group of the equation, which contains 24 substitutions in all, is decomposed into two which contain only 12 of them. Denoting the roots by a, b, c, d, here is one of these groups:

$$\begin{array}{lll} abcd, & acdb, & adbc, \\ badc, & cabd, & dacb, \\ cdab, & dbac, & bcad, \\ dcba, & bdca, & cbda. \end{array}$$

Slip (see note opposite) corrected.

Now this group is itself partitioned into three groups, as is indicated by Theorems II and III. Thus, by the extraction of a single radical of the 3rd degree, there remains simply the group

$$\begin{array}{l} abcd, \\ badc, \\ cdab, \\ dcba. \end{array}$$

This group in turn is partitioned into two groups:

$$\begin{array}{ll} abcd, & cdab, \\ badc, & dcba. \end{array}$$

Ainsi, après une simple extraction de racine carrée, il restera,

$$\begin{matrix} abcd \\ badc \end{matrix}$$

ce qui se résoudra enfin par une simple extraction de racine carrée.

With reference to Descartes and Euler see Note 18 on p. 163.

L'on obtient ainsi soit la solution de Descartes, soit celle d'Euler. Car bien qu'après ~~l'extraction~~ la résolution de l'équation auxiliaire du 3e degré, ce dernier extraye trois racines carrées, on sait qu'il suffit de deux, puis que le troisième s'en déduit rationnellement.

Nous allons maintenant appliquer cette condition aux équations irréductibles dont le degré est premier.

6a

Application aux Équations irréductibles de degré premier.

PROPOSITION VI.

Liouville corrected this lemma: see Note 18 on p. 163.

Lemme. Une équation irréductible de degré ~~p~~ ^premier^ ne peut devenir réductible par l'adjonction d'un radical.

Car si $r\ r'\ r''\ \ldots\ .$ sont les diverses valeurs ^du radical^ et $Fx = 0$ l'équation ~~F~~ proposée, il faudrait que Fx se partageat en facteurs

$$f(x,r) \times f(x,r') \times \cdots$$

The last symbol r is clear in *ms*; could be r or x in C*ms*, but x looks the more likely; printed as x in L1846.

tous de mème degré; ce qui ne se peut à moins que $f(x,r)$ ne soit du premier degré en r.

Ainsi une equation irréductible de degré premier ne peut devenir réductible, àmoins que son groupe ne ~~devienne~~ ^se réduise à^ une seule permutation.

PROPOSITION VII~~I~~.

Originally: 'Théorème. Le groupe d'une équation irréductible d'un degré premier *n*, contient toujours quelque substitution circulaire de l'ordre *n*.'

~~Théorème.~~ ^Problème.^ ^Quel est^ Le groupe d'une équation irréductible d'un degré premier n, ~~contient toujours quelque substitution circulaire de l'ordre n.~~ soluble par radicaux?

Thus after a simple extraction of a square root, there will remain

$$abcd,$$
$$badc;$$

which finally is solved by a simple extraction of a square root.

In this way one obtains either the solution of Descartes or that of Euler. For, although after the solution of the auxiliary equation of the 3rd degree, the latter would extract three square roots, it is known that two suffice, because the third may be obtained from them rationally.

We are now going to apply this condition to irreducible equations whose degree is prime.

Application to irreducible equations of prime degree.

PROPOSITION VI.

LEMMA. *An irreducible equation of prime degree cannot become reducible by the adjunction of a radical.*

For if r, r', r'', ... are the different values of the radical, and $Fx = 0$ the proposed equation, it would be necessary that Fx could be partitioned into factors

$$f(x,r) \times f(x,r') \times \ldots,$$

all of the same degree, which is not possible unless $f(x,r)$ is of the first degree in r [read x].

Thus an irreducible equation of prime degree cannot become reducible without its group being reduced to a single permutation.

PROPOSITION VII.

PROBLEM. *What is the group of an irreducible equation of prime degree n that is soluble by radicals?*

Originally: Theorem. The group of an irreducible equation of prime degree *n* always contains some circular substitution of order *n*.

D'après la proposition précédente, le plus petit groupe possible avant celui qui n'a qu'une seule permutation, ~~sera de l'ord~~ contiendra n permutations~~., et appartiendra à une équation irréductible~~. Or un groupe de permutations d'un nombre premier n de lettres ne peut se réduire à n permutations, à moins que l'une de ces permutations ne se déduise de l'autre par une substitution ^circulaire^ de l'ordre n. (Voir le mémoire de M. Cauchy, Journal de l'école, 17)

Reference amplified in L1846.

Ainsi ~~le groupe~~ l'avant dernier groupe sera

$$\left.\begin{array}{cccccccc} x_0 & x_1 & x_2 & x_3 & . & . & . & x_{n-1} \\ x_1 & x_2 & x_3 & x_4 & . & . & x_{n-1} & x_0 \\ x_2 & x_3 & . & . & . & x_{n-1} & x_0 & x_1 \\ . & . & . & . & . & . & & \\ x_{n-1} & x_0 & x_1 & . & . & . & . & x_{n-2} \end{array}\right\} \qquad \text{(G)}$$

$x_0 \quad x_1 \quad x_2 \quad \ldots \quad x_{n-1}$ étant les racines.

Comma follows 'Maintenant' in C*ms*, L1846. Comma after 'décompositions' in C*ms* edited out again in L1846.

Maintenant le groupe qui précédera immédiatement celui-ci ^dans l'ordre des décompositions^ devra se composer d'un certain nombre de groupes ayant tous les mèmes substitutions que celui-ci~~.~~. Or j'observe que ces substitutions peuvent s'exprimer ainsi: ⌜(⌝Faisons en general $x_n = x_0 \quad x_{n+1} = x_1$, ... Il est clair que chacune des substitutions ~~s'obtient~~ du groupe ~~précédent~~ ^(G)^ s'obtient en mettant partout à la place de x_k , x_{k+c} , c étant une constante.⌜)⌝

Parentheses seem an afterthought; Galois probably intended to close after the ellipsis.

Considérons l'un quelconque des groupes semblables au groupe G. D'après le théorème II, il devra s'obtenir en ~~mettant~~^opérant^ partout ~~á la place~~ dans ce groupe une mème substitution, par exemple en mettant partout dans le groupe (G) à la place de x_k , $x_{f(k)}$, f étant une certaine fonction.

Chevalier introduced comma after 'exemple'.

6b

Les substitutions de ce nouveau groupe devant être les mèmes que celles du groupe G, on devra avoir

$$f(k + c) = f(k) + C$$

Plural 'ces nouveaux groupes' in L1846; singular in *ms*, C*ms*, BA1962. G corrected to (G) in C*ms*, L1846.

C étant indépendant de k.

Donc ~~$f(k) = ak + b.$~~

$$\begin{array}{c} f(k + 2c) = f(k) + 2C \\ . \quad . \quad . \quad . \\ f(k + mc) = f(k) + mC \end{array}$$

Si $c = 1$, ^$k = 0$^ on trouvera

The comma is placed after $c = 1$ in C*ms*, after $k = 0$ in BA1962; equally plausible as reading of *ms*. Two commas in L1846.

$$\text{~~$f(k + m) =$~~ } f(m) = am + b \qquad \text{oubien}$$

$$f(k) = ak + b$$

a et b étant des constantes.

According to the preceding proposition, the smallest possible group before the one that has just a single permutation, ~~will be of ord~~[er] will contain n permutations ~~and will belong to an irreducible equation~~. Now a group of permutations of a prime number n of letters cannot reduce to n permutations unless [any] one of its permutations may be obtained from another [any other] by a circular substitution of order n. (See the memoir of Mr Cauchy, *Journal de l'École* [*Polytechnique*], 17.)

Thus the penultimate group will be

$$\left.\begin{array}{cccccccc} x_0 & x_1 & x_2 & x_3 & . & . & . & x_{n-1} \\ x_1 & x_2 & x_3 & x_4 & . & . & x_{n-1} & x_0 \\ x_2 & x_3 & . & . & . & x_{n-1} & x_0 & x_1 \\ . & . & . & . & . & . & & \\ x_{n-1} & x_0 & x_1 & . & . & . & . & x_{n-2} \end{array}\right\} \qquad \text{(G)}$$

where $x_0,\ x_1,\ x_2,\ \ldots,\ x_{n-1}$ are the roots.

Now the group which will immediately precede this one in the sequence of decompositions will have to be composed of a certain number of groups all having the same substitutions as this one. Well, I observe that these substitutions may be expressed like this: (let us put generally $x_n = x_0,\ x_{n+1} = x_1, \ldots$) it is clear that each of the substitutions ~~is obtained from the preceding group~~ of the group (G) is obtained by putting x_{k+c} in place of x_k throughout, c being a constant.

For placement of the closing parenthesis see note opposite.

Consider an arbitrary one of the groups similar to the group (G). By Theorem II, it must be obtained by operating throughout in this group with one and the same substitution; for example, by putting $x_{f(k)}$ in place of x_k everywhere in the group (G), f being some function.

The substitutions of this new group having to be the same as those of the group (G), one must have

$$f(k + c) = f(k) + C,$$

C being independent of k.

Therefore

$$\begin{gathered} f(k + 2c) = f(k) + 2C \\ . \quad . \quad . \quad . \\ f(k + mc) = f(k) + mC \end{gathered}$$

If $c = 1,\ k = 0$, one will find

$$f(m) = am + b,$$

or equally,

$$f(k) = ak + b$$

a and b being constants.

Donc le groupe qui précède immédiatement le groupe G ne devra contenir que des substitutions telles que

$$x_k \qquad x_{ak+b}$$

et ne contiendra pas, par conséquent d'autre substitution circulaire que celle du groupe G.

On raisonnera sur ce groupe comme sur le précédent, et il s'en suivra que le ^premier^ groupe dans l'ordre des décompositions, c'est à dire le groupe actuel de l'équation, ne peut contenir que des substitutions de la forme

Corrected to 'c'est-à-dire' in L1846, BA1962. No hyphens in *ms* or in C*ms*.

$$x_k \qquad x_{ak+b}$$

Donc, "Si une équation irréductible de degré premier est soluble par radicaux, le groupe de cette équation ne saurait contenir que des substitutions de la forme

$$x_k \qquad x_{ak+b}$$

a et b étant des constantes."

Réciproquement, si cette condition , a lieu je dis que l'équation sera soluble par radicaux. Considérons en effet les fonctions

Comma after 'lieu' in C*ms*, L1846; not in *ms*, BA1962.

$$\begin{aligned}
\left(x_0 + \alpha x_1 + \alpha^2 x_2 + \ldots + \alpha^{n-1} x_{n-1}\right)^n &= X_1 \\
\left(x_0 + \alpha x_a + \alpha^2 x_{2a} + \ldots + \alpha^{n-1} x_{(n-1)a}\right)^n &= X_a \\
\left(x_0 + \alpha x_{a^2} + \alpha^2 x_{2a^2} + \ldots + \alpha^{n-1} x_{(n-1)a^2}\right)^n &= X_{a^2} \\
\ldots\ldots\ldots

Therefore the group which immediately precedes the group (G) must contain only substitutions like

$$x_k \quad x_{ak+b}\,,$$

and consequently will not contain any other circular substitutions than those of the group (G).

One may reason on this group in the same way as on the preceding one, and it will follow that the first group in the sequence of decompositions, that is to say the *actual* group of the equation, can only contain substitutions of the form

$$x_k \quad x_{ak+b}\,.$$

Therefore, *If an irreducible equation of prime degree is soluble by radicals, the group of this equation must contain only substitutions of the form*

$$x_k \quad x_{ak+b}\,,$$

a and b being constants.

Conversely, if this condition holds, I say that the equation will be soluble by radicals. Indeed, consider the functions

$$\begin{aligned}
\left(x_0 + \alpha x_1 + \alpha^2 x_2 + \ldots + \alpha^{n-1} x_{n-1}\right)^n &= X_1 \\
\left(x_0 + \alpha x_a + \alpha^2 x_{2a} + \ldots + \alpha^{n-1} x_{(n-1)a}\right)^n &= X_a \\
\left(x_0 + \alpha x_{a^2} + \alpha^2 x_{2a^2} + \ldots + \alpha^{n-1} x_{(n-1)a^2}\right)^n &= X_{a^2} \\
\ldots\ldots\ldots

Si l'équation proposée a tous ses coëfficients rationnels l'équation ^auxiliaire^ qui donne cette fonction les aura tous aussi, et il suffira de reconnaitre si cette équation auxiliaire du degré $1.2.3...(n-2)$ a ou non une racine rationnelle. Ce que l'on sait faire.

~~S~~ C'est là lemoyen qu'il faudrait employer dans la pratique. Mais nous allons présenter le théorème sous une autre forme.

PROPOSITION VIII.

Théorème. Pour qu'une équation irréductible de degré premier soit soluble par radicaux, il faut et il suffit que deux quelconques des racines étant connues, les autres s'en déduisent rationnellement.

No emphasis here in *ms*, BA1962. Emphasis introduced in C*ms*, L1846.

Premièrement, il le faut: car la substitution

$$x_k \qquad x_{ak+b}$$

ne laissant jamais deux lettres à la mème place, il est clair qu'en adjoignant deux racines à l'équation, par la proposition IV, son groupe devra se réduire à une seule permutation.

En second lieu, celà suffit: car dans ce cas, aucune substitution du groupe ne laissera deux lettres aux mèmes places. Parconséquent le groupe contiendra tout au plus ~~$p(p$~~ $n(n-1)$ permutations. Donc il ne contiendra qu'une ^seule^substitution circulaire (sans quoi il y aurait au moins p^2 permutations). Donc toute substitution du groupe x_k , x_{fk} , devra satisfaire à la condition

Clearly p^2 in *ms* and C*ms*; corrected to n^2 in L1846.

$$f(k+c) = fk + C$$

Donc, &c.

Le théorème est donc démontré.

If the proposed equation has all its coefficients rational, the auxiliary equation which gives this function will also have all its coefficients rational, and it will suffice to discover if this auxiliary equation of degree $1.2.3...(n-2)$ has or has not a rational root. Which one knows how to do.

That is the method that would have to be used in practice. But we are going to present the theorem in another form.

PROPOSITION VIII.

THEOREM. *In order that an irreducible equation of prime degree should be soluble by radicals, it is necessary and sufficient that any two of its roots being known, the others may be deduced from them rationally.*

First of all, it is necessary: for the substitution

$$x_k \quad x_{ak+b}$$

never leaving two letters in the same place, it is clear that by adjoining two roots to the equation, by Proposition IV, its group must be reduced to a single permutation.

In the second place, that suffices: for in this case no substitution of the group will leave two letters in their places. Consequently the group will contain in all at most $n(n-1)$ permutations. Therefore it will contain only a single circular substitution (otherwise there will be at least p^2 [read n^2] permutations). Therefore all substitutions of the group, x_k, x_{fk}, must satisfy the condition

$$f(k+c) = fk + C.$$

Therefore, etc.

The theorem is therefore proved.

Exemple du théorème VII.

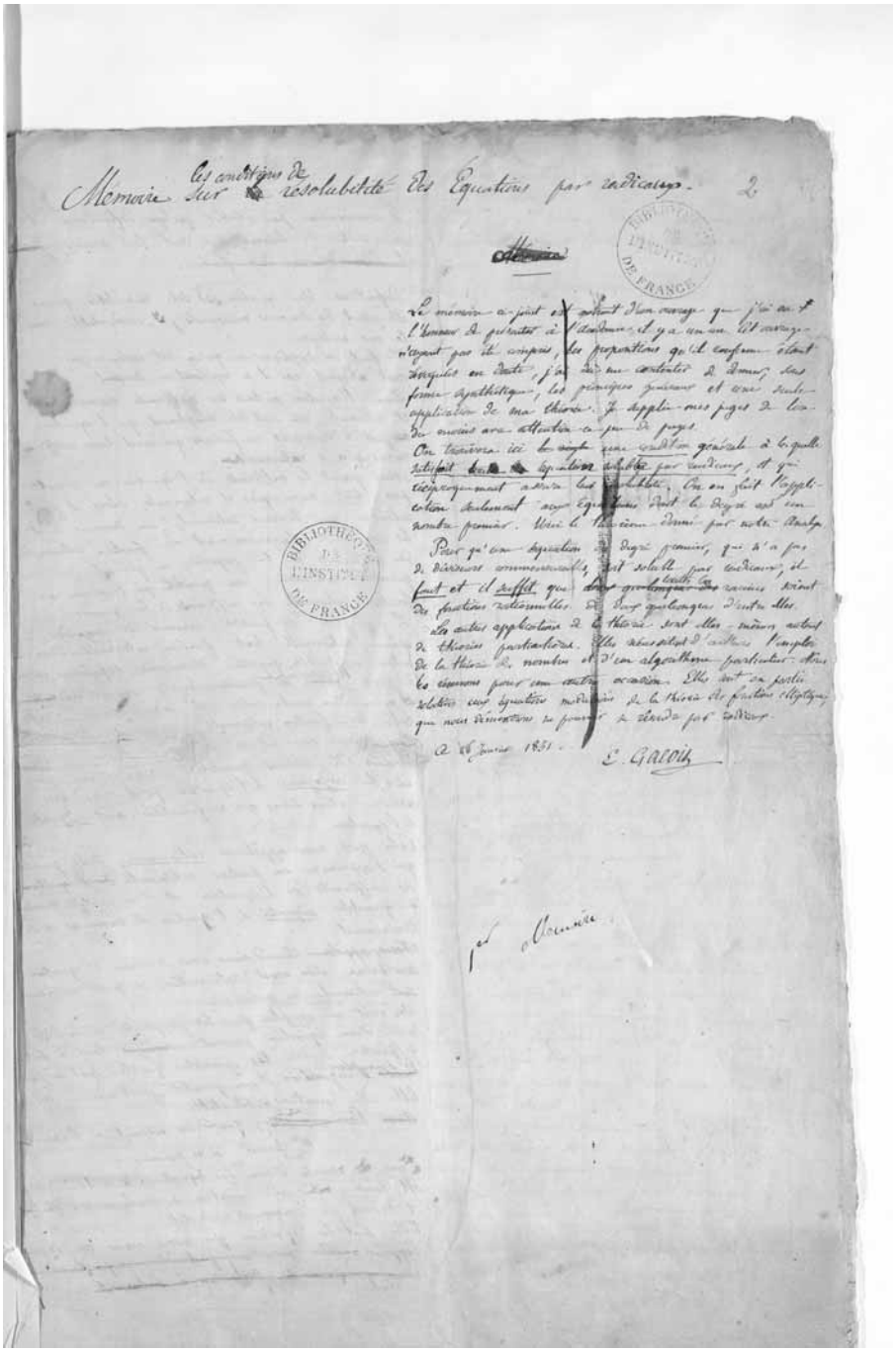
Soit $n = 5$. Le groupe sera le suivant

$$
\begin{array}{c}
abcde \\
bcdea \\
cdeab \\
deabc \\
eabcd \\
\hline
acebd \\
cebda \\
ebdac \\
bdace \\
daceb \\
\hline
aedcb \\
edcba \\
dcbae \\
cbaed \\
baedc \\
\hline
adbec \\
dbeca \\
becad \\
ecadb \\
cadbe
\end{array}
$$

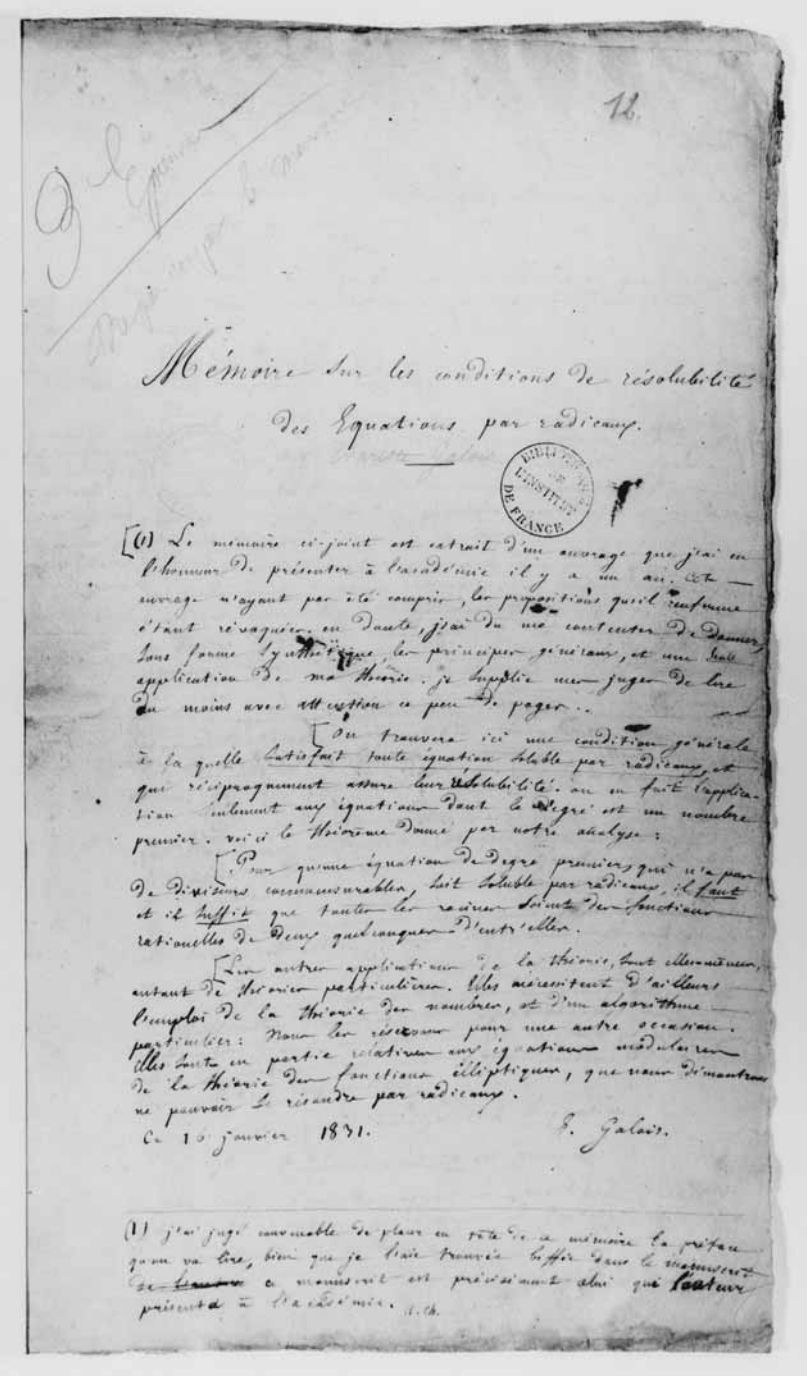
Example of Theorem VII.

Let $n = 5$; the group will be the following:

$$
\begin{array}{l}
abcde \\
bcdea \\
cdeab \\
cdeab \\
deabc \\
\hline
acebd \\
cebda \\
ebdac \\
bdace \\
daceb \\
\hline
aedcb \\
edcba \\
dcbae \\
cbaed \\
edcba \\
\hline
adbec \\
dbeca \\
becad \\
ecadb \\
cadhe.
\end{array}
$$



First Memoir: Dossier 1, folio 2 recto.



12.

Mémoire sur les conditions de résolubilité des Equations par radicaux.

[(1) Le mémoire ci-joint est extrait d'un ouvrage que j'ai eu l'honneur de présenter à l'académie il y a un an. Cet ouvrage n'ayant pas été compris, les propositions qu'il renferme étant révoquées en doute, j'ai du me contenter de donner sans forme synthétique les principes généraux, et une seule application de ma théorie. je supplie mes juges de lire du moins avec attention ce peu de pages.

[On trouvera ici une condition générale à laquelle satisfait toute équation soluble par radicaux, et qui réciproquement assure leur résolubilité. on en fait l'application seulement aux équations dont le degré est un nombre premier. voici le théorème donné par notre analyse :

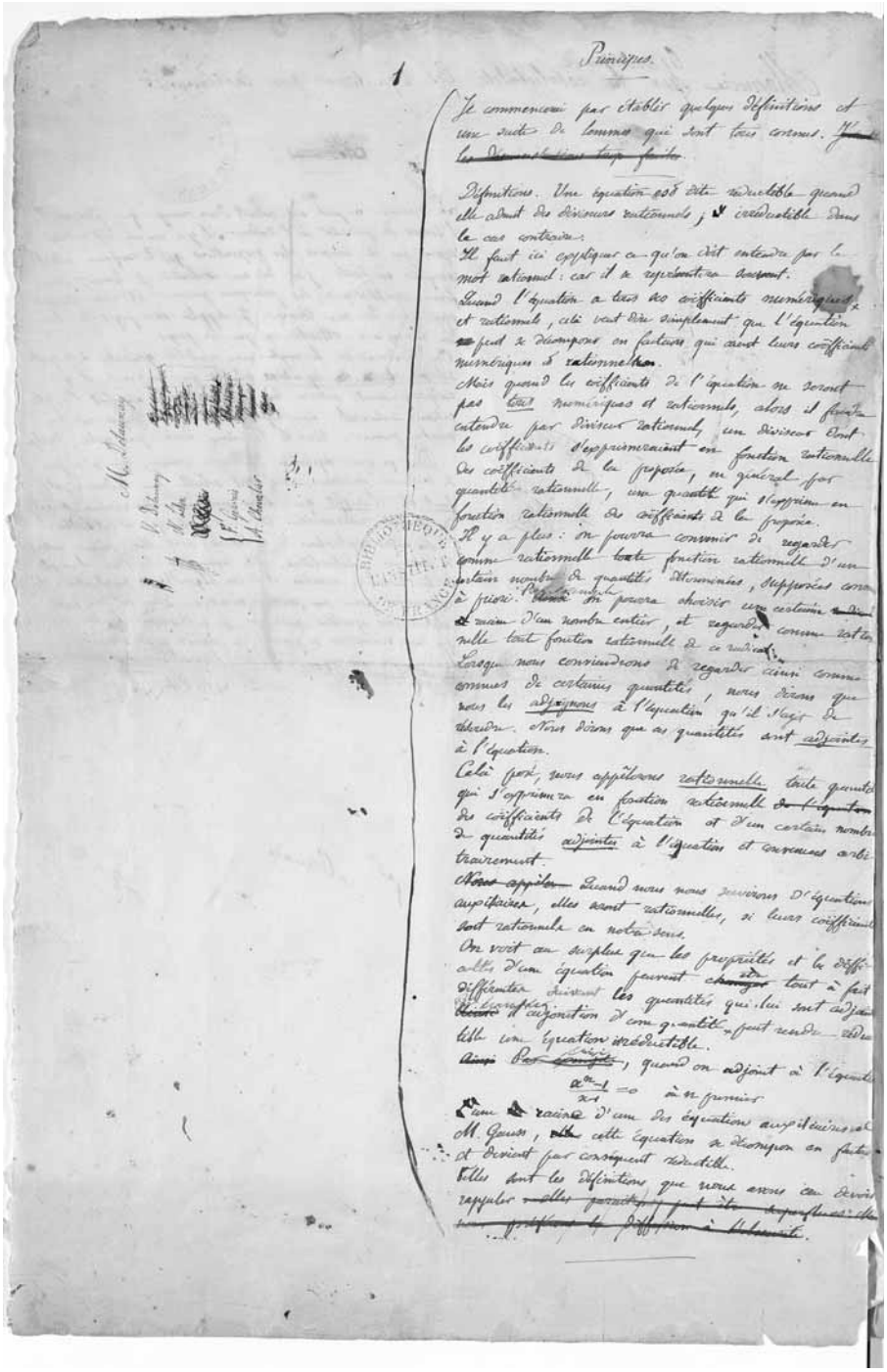
[Pour qu'une équation de degré premier, qui n'a pas de diviseurs commensurables, soit soluble par radicaux, il faut et il suffit que toutes les racines soient des fonctions rationelles de deux quelconques d'entr'elles.

[Les autres applications de la théorie, sont elles-mêmes, autant de théories particulières. Elles nécessitent d'ailleurs l'emploi de la théorie des nombres, et d'un algorithme particulier : Nous les réservons pour une autre occasion. Elles sont en partie relatives aux équations modulaires de la théorie des fonctions elliptiques, que nous démontrons ne pouvoir se résoudre par radicaux.

Ce 16 janvier 1831. E. Galois.

(1) j'ai jugé convenable de placer en tête de ce mémoire la préface qu'on va lire, bien que je l'aie trouvée biffée dans le manuscrit ~~de l'auteur~~ ce manuscrit est précisément celui qui l'auteur présenta à l'académie. A. Ch.

First Memoir: first page of copy by Chevalier (Dossier 3).

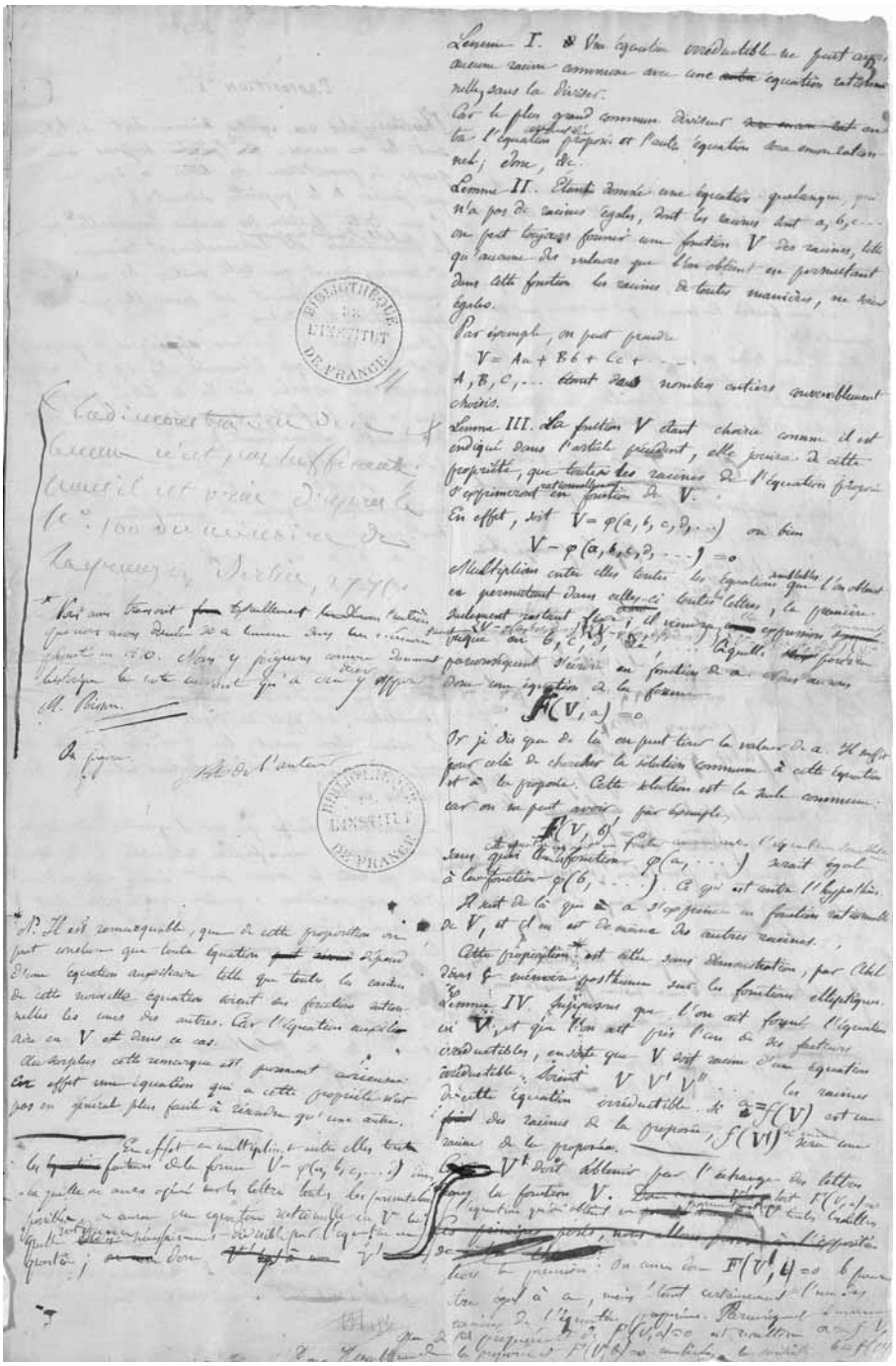


Principes.

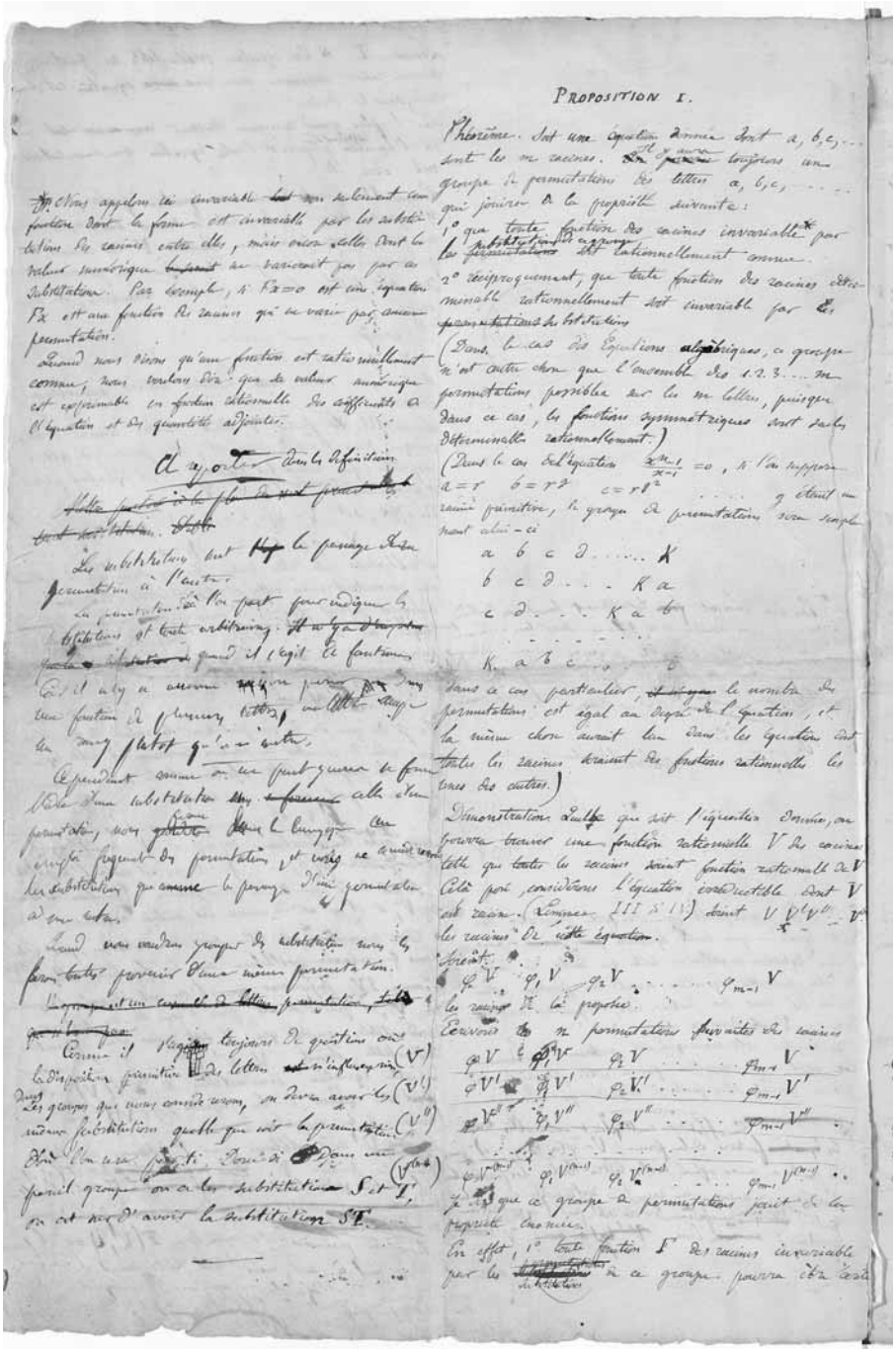
Je commencerai par établir quelques définitions et une suite de lemmes qui sont tous connus. [illegible]

Définitions. Une équation est dite réductible quand elle admet des diviseurs rationnels, et irréductible dans le cas contraire.
Il faut ici expliquer ce qu'on doit entendre par le mot rationnel : car il a quelquefois deux sens.
Quand l'équation a tous ses coefficients numériques et rationnels, cela veut dire simplement que l'équation peut se décomposer en facteurs qui aient leurs coefficients numériques et rationnels.
Mais quand les coefficients de l'équation ne seront pas tous numériques et rationnels, alors il faudra entendre par diviseurs rationnels, ces diviseurs dont les coefficients s'exprimeraient en fonction rationnelle des coefficients de la proposée, ou généralement par quantités rationnelles, une quantité qui s'exprime en fonction rationnelle des coefficients de la proposée.
Il y a plus : on pourra convenir de regarder comme rationnelle toute fonction rationnelle d'un certain nombre de quantités déterminées, supposées connues à priori. Ainsi on pourra choisir une certaine racine d'un nombre entier, et regarder comme rationnelle toute fonction rationnelle de cette racine.
Lorsque nous conviendrons de regarder ainsi comme connues de certaines quantités, nous dirons que nous les adjoignons à l'équation qu'il s'agit de résoudre. Nous dirons que ces quantités sont adjointes à l'équation.
Cela posé, nous appellerons rationnelle toute quantité qui s'exprimera en fonction rationnelle des coefficients de l'équation et d'un certain nombre de quantités adjointes à l'équation et convenues arbitrairement.
Quand nous nous servirons d'équations auxiliaires, elles seront rationnelles, si leurs coefficients sont rationnels en notre sens.
On voit au surplus que les propriétés et la différence d'une équation peuvent changer tout à fait différentes suivant les quantités que l'on leur suppose adjointes. L'adjonction d'une quantité peut rendre réductible une équation irréductible.
Par exemple, quand on adjoint à l'équation $\frac{x^m-1}{x-1}=0$ un premier
[illegible] d'une des équations auxiliaires de M. Gauss, cette équation se décompose en facteurs et devient par conséquent réductible.
Telles sont les définitions que nous avons eu besoin de rappeler [illegible]

First Memoir: Dossier 1, folio 2 verso.



First Memoir: Dossier 1, folio 3 recto.



PROPOSITION I.

Théorème. Soit une équation donnée dont a, b, c, ... sont les m racines. Il y aura toujours un groupe de permutations de lettres a, b, c, ... qui jouira de la propriété suivante:

1° que toute fonction des racines invariable par les permutations est rationnellement connue.

2° réciproquement, que toute fonction des racines déterminable rationnellement sera invariable par les permutations de ce groupe.

(Dans le cas des équations algébriques, ce groupe n'est autre chose que l'ensemble des 1.2.3. ... m permutations possibles sur les m lettres, puisque dans ce cas, les fonctions symmétriques sont seules déterminables rationnellement.)

(Dans le cas de l'équation $\frac{x^m-1}{x-1}=0$, si l'on suppose $a=r \quad b=r^2 \quad c=r^3 \ldots$ q étant une racine primitive, le groupe de permutation sera semblable à celui-ci

$$\begin{matrix} a & b & c & d & \ldots & k \\ b & c & d & \ldots & k & a \\ c & d & \ldots & k & a & b \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ k & a & b & c & \ldots & g \end{matrix}$$

dans ce cas particulier, le nombre des permutations est égal au degré de l'équation, et la même chose aurait lieu dans les équations dont toutes les racines seraient des fonctions rationnelles les unes des autres.)

Démonstration directe que soit l'équation donnée, on pourra trouver une fonction rationnelle V des racines telle que toutes les racines soient fonction rationnelle de V. Cela posé, considérons l'équation irréductible dont V est racine. (Lemme III § IV) soient $V\,V'\,V''\ldots V^{(n-1)}$ les racines de cette équation.

Soient

$$\varphi V \quad \varphi_1 V \quad \varphi_2 V \quad \ldots \quad \varphi_{m-1} V$$

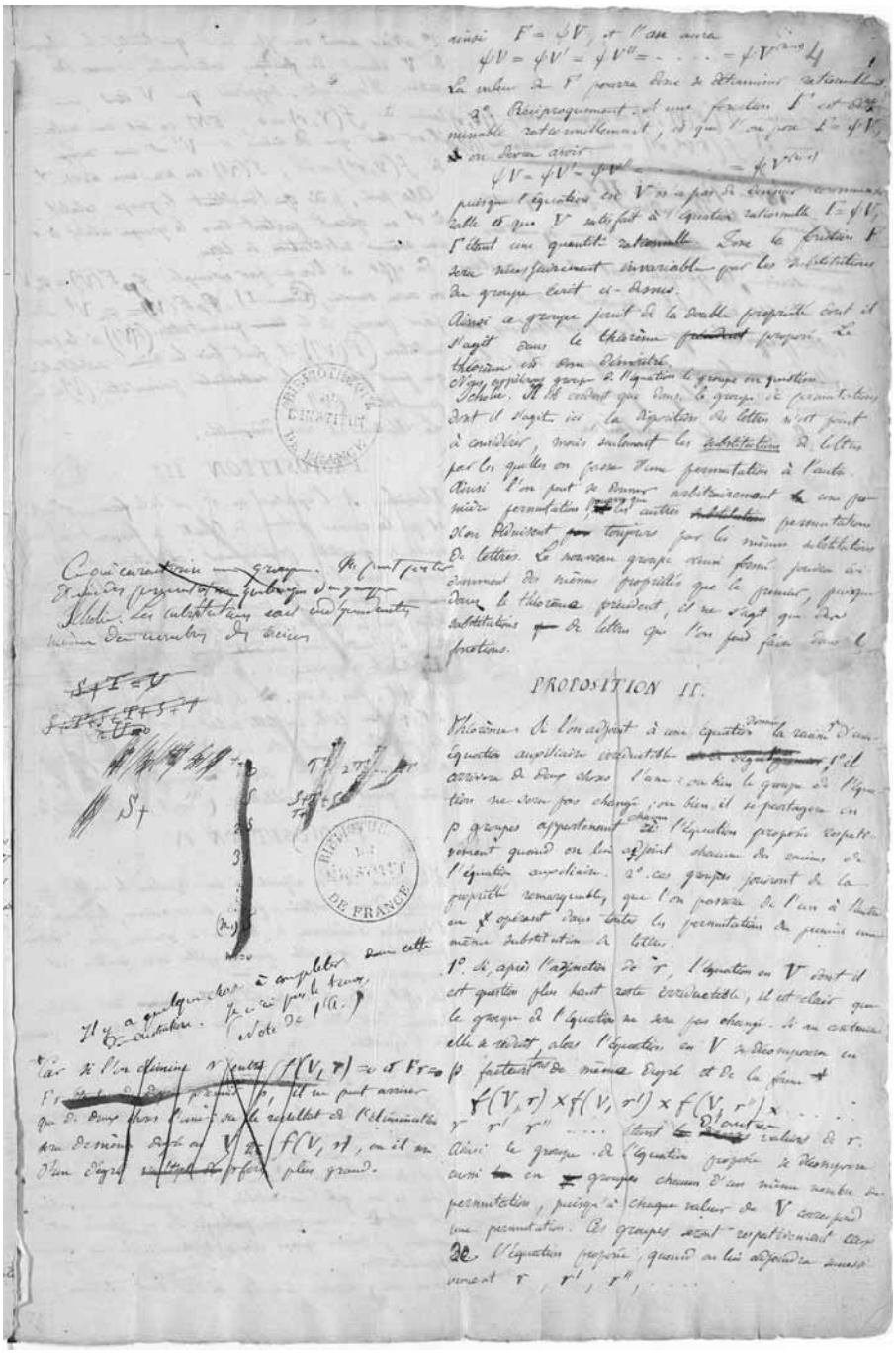
les racines de la proposée.

Ecrivons les n permutations suivantes des racines

$$\begin{matrix} \varphi V & \varphi_1 V & \varphi_2 V & \ldots & \varphi_{m-1} V \\ \varphi V' & \varphi_1 V' & \varphi_2 V' & \ldots & \varphi_{m-1} V' \\ \varphi V'' & \varphi_1 V'' & \varphi_2 V'' & \ldots & \varphi_{m-1} V'' \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \varphi V^{(n-1)} & \varphi_1 V^{(n-1)} & \varphi_2 V^{(n-1)} & \ldots & \varphi_{m-1} V^{(n-1)} \end{matrix}$$

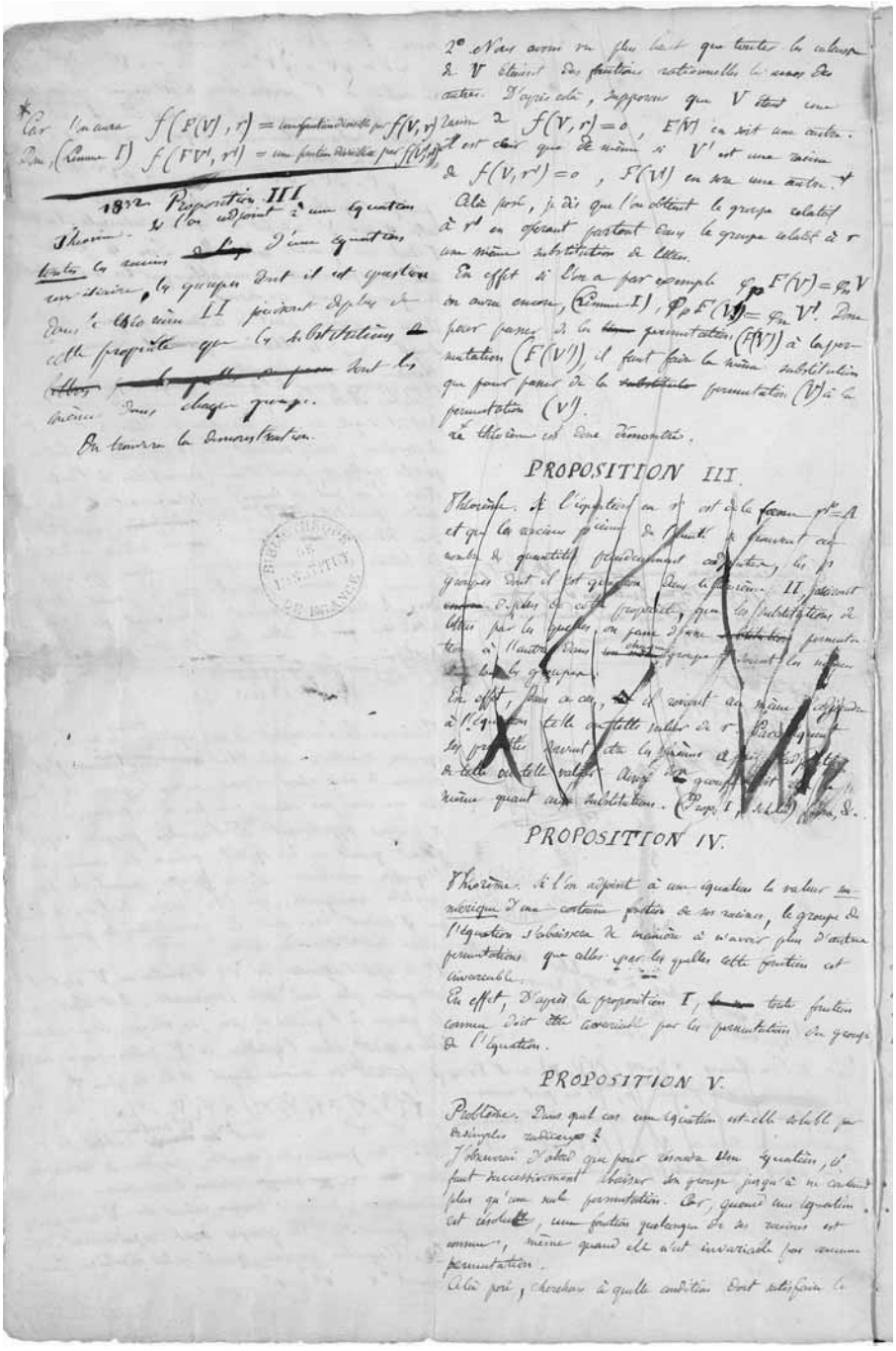
Je dis que ce groupe de permutations jouit de la propriété énoncée.

En effet, 1° toute fonction F des racines invariable par les permutations de ce groupe pourra être écrite

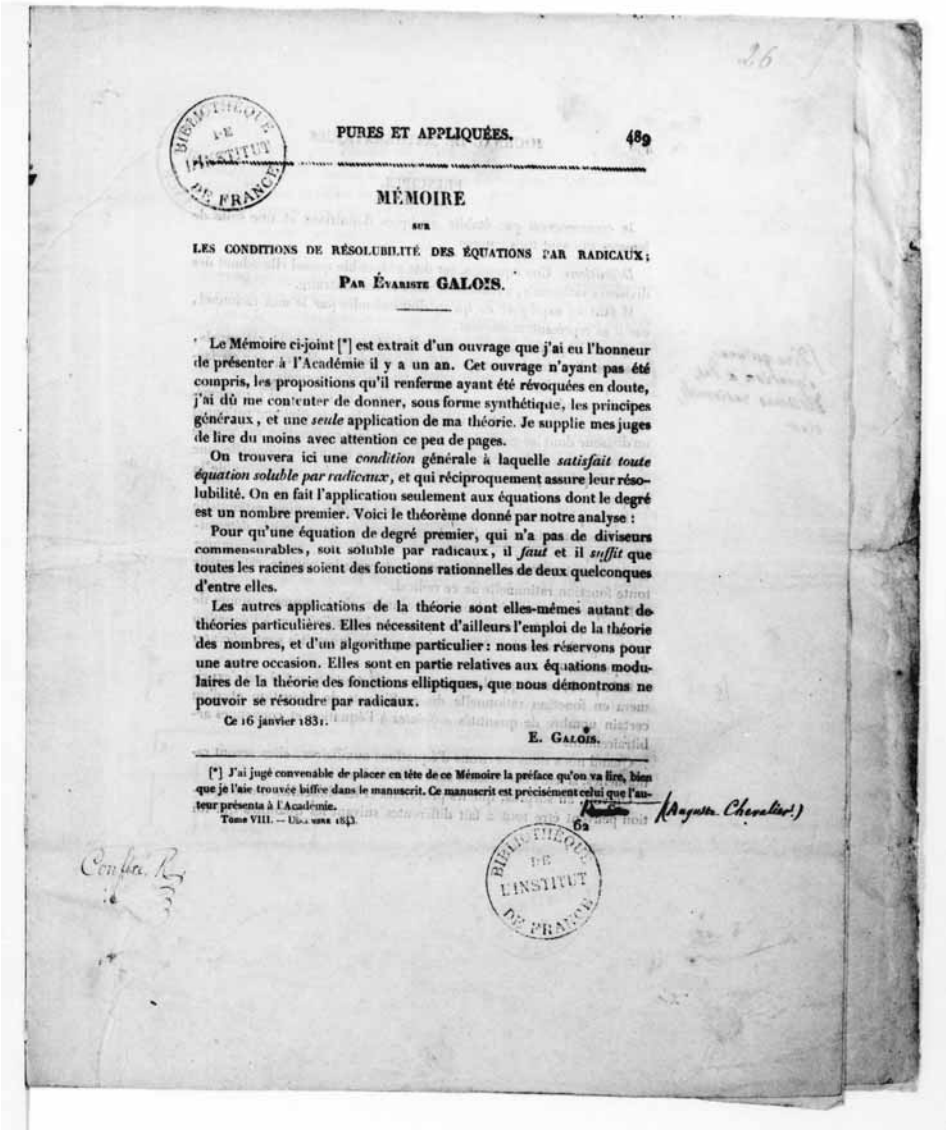


PROPOSITION II.

First Memoir: Dossier 1, folio 4 recto.



First Memoir: Dossier 1, folio 4 verso.



PURES ET APPLIQUÉES. 489

MÉMOIRE

SUR

LES CONDITIONS DE RÉSOLUBILITÉ DES ÉQUATIONS PAR RADICAUX;

PAR ÉVARISTE GALOIS.

Le Mémoire ci-joint [*] est extrait d'un ouvrage que j'ai eu l'honneur de présenter à l'Académie il y a un an. Cet ouvrage n'ayant pas été compris, les propositions qu'il renferme ayant été révoquées en doute, j'ai dû me contenter de donner, sous forme synthétique, les principes généraux, et une *seule* application de ma théorie. Je supplie mes juges de lire du moins avec attention ce peu de pages.

On trouvera ici une *condition* générale à laquelle *satisfait toute équation soluble par radicaux*, et qui réciproquement assure leur résolubilité. On en fait l'application seulement aux équations dont le degré est un nombre premier. Voici le théorème donné par notre analyse :

Pour qu'une équation de degré premier, qui n'a pas de diviseurs commensurables, soit soluble par radicaux, il *faut* et il *suffit* que toutes les racines soient des fonctions rationnelles de deux quelconques d'entre elles.

Les autres applications de la théorie sont elles-mêmes autant de théories particulières. Elles nécessitent d'ailleurs l'emploi de la théorie des nombres, et d'un algorithme particulier : nous les réservons pour une autre occasion. Elles sont en partie relatives aux équations modulaires de la théorie des fonctions elliptiques, que nous démontrons ne pouvoir se résoudre par radicaux.

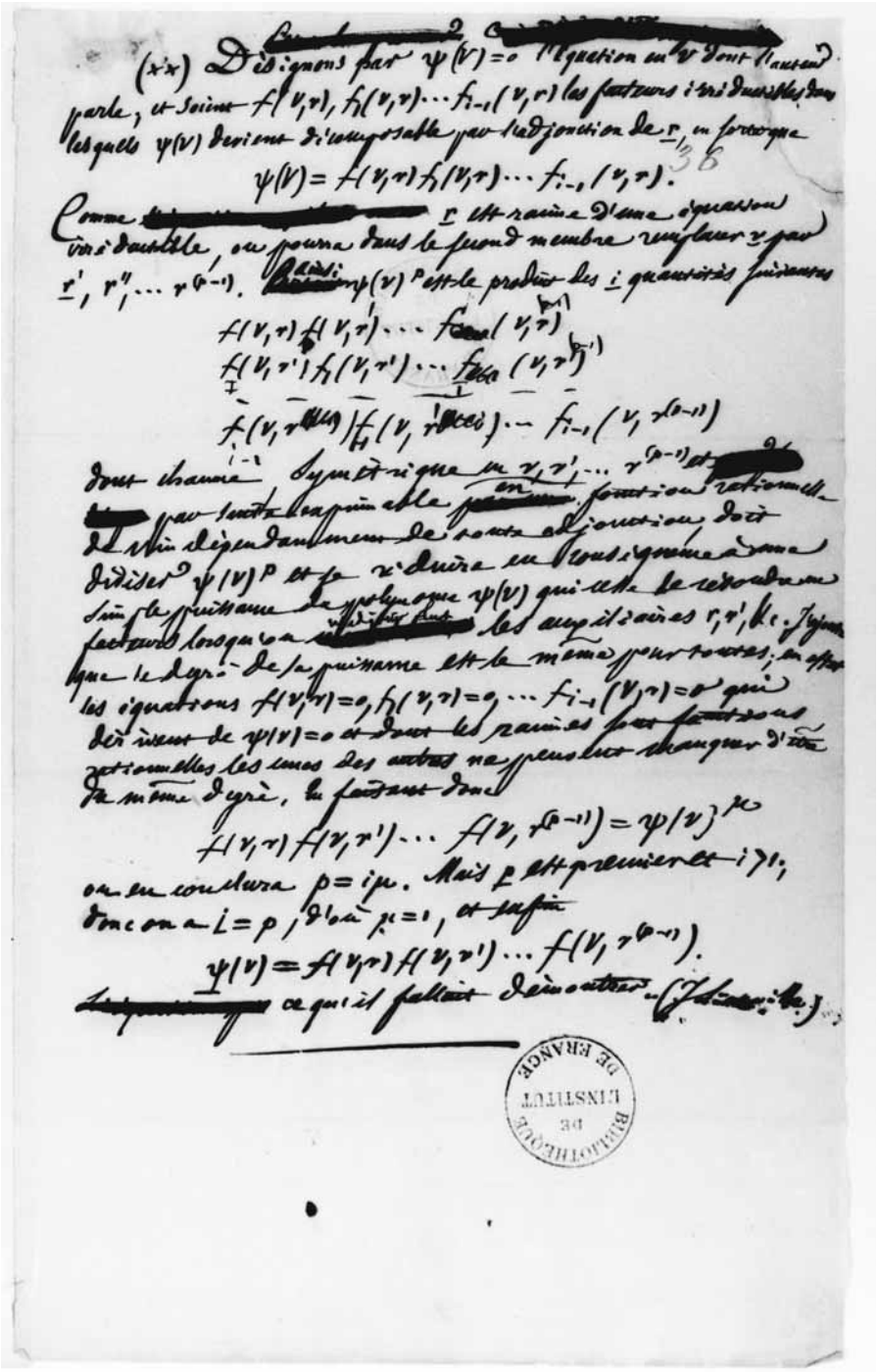
Ce 16 janvier 1831.

E. GALOIS.

[*] J'ai jugé convenable de placer en tête de ce Mémoire la préface qu'on va lire, bien que je l'aie trouvée biffée dans le manuscrit. Ce manuscrit est précisément celui que l'auteur présenta à l'Académie.

BIBLIOTHÈQUE DE L'INSTITUT DE FRANCE

First Memoir: first page of 1843 proof-sheets (Dossier 3).



First Memoir: Liouville's proof of Proposition II (Dossier 3; see Note 16, p. 159).

IV.2 Notes on the First Memoir

NOTE 1: The extant manuscript represents Galois' third attempt to interest the Académie des Sciences in his ideas. The minutes of the Academy for Monday 25 May 1829 record that

> Des *Recherches algébriques* de **M. Evariste Galois**, présentées par M. Cauchy, sont renvoyées à l'examen de MM. [Acad (1828–31), p. 253].
>
> [Algebraic researches by Mr Evariste Galois, presented by Mr Cauchy, were sent for examination by Messrs]

(The referees' names are missing.) Those of 1 June 1829 record that

> M. Cauchy présente au nom de **M. Galois** un manuscrit intitulé *Recherches sur les équations algébriques du degré premier*. MM. Poinsot et Cauchy sont nommés commissaires. [Acad (1828–31), pp. 257–258].
>
> [Mr Cauchy presents on behalf of Mr Galois a memoir entitled 'Research on algebraic equations of prime degree'. Messrs Poinsot and Cauchy are named referees.]

It is not known for certain what happened to the manuscripts, but, on the basis of strong if circumstantial evidence, René Taton has concluded that early in January 1830 Cauchy counselled Galois to revise his work in the light of what Abel had recently published and to resubmit [Taton(1971)]. Whether or not he received such advice, as Taton points out, we have no record of Galois complaining that this first version of his work was lost and he did submit a new version late in February 1830 [Taton(1971), Note (33) on p. 137].

This second version certainly was lost. Galois complained and apparently Poisson, then one of the secretaries of the academy, recommended him to rewrite and resubmit once more. According to the minutes of the meeting on 17 January 1831

> Un Mémoire de M. **Le Gallois** [*sic*] sur *les Conditions de résolubilité des équations par radicaux*, est renvoyé à l'examen de MM. Lacroix et Poisson. [Acad (1828–31), p. 566.].
>
> [A memoir by Mr Le Gallois on the conditions for solubility of equations by radicals was sent for examination by Messrs Lacroix and Poisson.]

On 31 March 1831 Galois wrote for news:

> Une réclamation de **M. Gallois** [*sic*] au sujet de son Mémoire sur les *Équations*, est renvoyée aux Commissaires, MM. Lacroix et Poisson [Acad (1828–31), p. 597].
>
> [A complaint by Mr Gallois on the subject of his memoir on equations has been sent to the referees, Messrs Lacroix and Poisson.]

His letter is treated in § IV.3 below. The referees finally reported to the meeting of the academy on 4 July 1831. Their lengthy and careful report recommending rejection is printed in the minutes [Acad (1828–31), p. 660]. For a transcription and translation of this report see the next note.

NOTE 2: Here is the referees' report on Galois' paper as recorded in the minutes of the meeting of the Académie des Sciences held on Monday 4 July 1831.

> MM. Lacroix et Poisson font le Rapport suivant sur le Mémoire de **M. Galois** relatif aux *Conditions de résolubilité par radicaux*:
>
> "Le but que l'auteur s'est proposé dans ce Mémoire est de démontrer un théorème qu'il énonce ainsi:
>
> "*Pour qu'une équation irréductible de degré premier soit soluble par radicaux, il faut et il suffit que, deux quelconques de ses racines étant connues, les autres se déduisent rationnellement.*
>
> "L'auteur entend par équation irréductible une équation dont les coefficients sont rationnels, et qui ne peut se décomposer en d'autres équations qui aient aussi leurs coefficients rationnels. D'après sa proposition, l'équation générale du 3e degré, par exemple, serait résoluble, parce que, la somme des trois racines étant égale au coefficient du second terme pris avec un signe contraire, chacune d'elles s'exprime rationnellement au moyen des deux autres. Des notes trouvées dans les papiers d'Abel et qui ont été imprimées après sa mort dans le *Journal de M. Crelle*, tome V, page 345, renferment une proposition analogue à celle de M. Galois dont voici l'énoncé:

The page reference is misprinted; it should be p. 343.

> "*Si trois racines d'une équation quelconque irréductible, dont le degré est un nombre premier, sont liées entre elles de sorte que l'une de ces racines puisse être exprimée rationnellement au moyen des deux autres, l'équation dont il s'agit sera toujours résoluble à l'aide de radicaux.*

Some minor verbal changes from the original.

> "Cet énoncé diffère de celui de M. Galois, en ce que le géomètre norvégien ne dit pas que la condition dont il s'agit soit *nécessaire*, mais seulement qu'elle *suffit* pour que l'équation soit résoluble; et il ne semble pas qu'il la regardât comme indispensable; car on trouve dans les notes citées une autre proposition relative à la résolution d'une classe nombreuse d'équations qui pourraient bien ne pas remplir cette condition. Il ne paraît pas non plus que ce soit à cette proposition qu'il ait fait allusion dans ce passage d'une lettre écrite à M. Legendre, et publiée après la mort d'Abel dans le *Journal de M. Crelle*, tome VI, page 80:
>
> "J'ai été assez heureux" dit-il "de trouver une règle sûre à l'aide de
> "laquelle on pourra reconnaître si une équation quelconque proposée
> "est résoluble ou non à l'aide de radicaux. Un corollaire de ma théo-
> "rie fait voir que généralement il est impossible de résoudre les

Some minor verbal changes from the original.

"équations supérieures au 4e degré." ([1])

"Nous ignorons si Abel a laissé un manuscrit de cette théorie; elle n'a point encore été imprimée, non plus que la démonstration du théorème analogue à celui qui fait l'objet de ce Rapport et qui appartiendrait entièrement à M. Galois, s'il parvenait à l'établir d'une manière satisfaisante. Toutefois on doit remarquer qu'il ne renferme pas, comme le titre du Mémoire le promettait, la condition de résolubilité des équations par radicaux; car en admettant comme vraie la proposition de M. Galois, on n'en serait guère plus avancé pour savoir si une équation donnée dont le degré est un nombre premier est résolue ou non par des radicaux, puisqu'il faudrait d'abord s'assurer si cette équation est irréductible, et ensuite si l'une de ces racines peut s'exprimer en fonction rationnelle de deux autres. La condition de résolubilité, si elle existe, devrait être un caractère extérieur que l'on pût vérifier à l'inspection des coefficients d'une équation donnée, ou, tout au plus, en résolvant d'autres équations d'un degré moins élevé que celui de la proposée.

"Quoiqu'il en soit, nous avons fait tous nos efforts pour comprendre la démonstration de M. Galois. Ses raisonnements ne sont ni assez clairs, ni assez développés pour que nous ayons pu juger de leur exactitude, et nous ne serions pas en état d'en donner une idée dans son Rapport. L'auteur annonce que la proposition qui fait l'objet spécial de son Mémoire est une partie d'une théorie générale susceptible de beaucoup d'autres applications. Souvent il arrive que les différentes parties d'une théorie, en s'éclairant mutuellement, sont plus faciles à saisir dans leur ensemble qu'isolément. On peut donc attendre que l'auteur ait publié en entier son travail pour se former une opinion définitive; mais dans l'état où est maintenant la partie qu'il a soumise à l'Académie, nous ne pouvons pas vous proposer d'y donner votre approbation."

Signé à la minute: **Lacroix, Poisson** Rapporteur.

L'Académie adopte les conclusions de ce Rapport.

[Messrs Lacroix and Poisson made the following report on the memoir by Mr Galois relating to conditions for the solubility of equations by radicals:

"The goal which the author proposes in this memoir is to prove a theorem which he formulates in this way:

In order that an irreducible equation of prime degree should be soluble by radicals it is necessary and sufficient that, any two of its roots being known, the others may be deduced rationally from them.

By an irreducible equation the author understands an equation in which the coefficients are rational and which cannot be decomposed into other equations which also have their coefficients rational. According to his proposition the general equation of the 3rd degree, for example, will be soluble because, the

([1]) Cette lettre est datée de Christiania le 25 Novembre 1828. Abel est mort près de cette ville le 6 Avril suivant.

sum of the three roots being equal to the coefficient of the second term taken with the opposite sign, each of them may be expressed rationally by means of the other two. Some notes found amongst the papers of Abel and which have been printed after his death in *Crelle's Journal*, Vol. V, p. 343, contain a proposition analogous to that of Mr Galois. Here is its statement:

Page reference corrected: see note above.

If three roots of an arbitrary equation of which the degree is a prime number are related to each other in such a way that any one of these roots may be expressed rationally by means of the other two, the equation in question will always be soluble with the help of radicals.

This statement differs from that of Mr Galois in that the norwegian geometer does not say that the condition in question is *necessary*, but only that it *suffices* for the equation to be soluble; it does not appear that he saw it as indispensable; for there is to be found in the cited notes another proposition relating to the solution of an extensive class of equations which could well fail to satisfy this condition. It seems not impossible that it was to this proposition that he alluded in this passage from a letter written to Mr Legendre and published after the death of Abel in *Crelle's journal*, Vol. VI, p. 80:

'I was pretty happy' he said 'to find a certain rule by means of which one might recognise if an arbitrary given equation is soluble or not with the help of radicals. A corollary of my theory shows that in general it is impossible to solve equations of degree greater than the 4th.'([1])

We do not know if Abel has left a manuscript of this theory; it has certainly not yet been published, and nor has the proof of the theorem analogous to that which is the object of this report and which would belong entirely to Mr Galois if he manages to establish it in a satisfactory way. It must be noted however that it does not contain, as the title of the memoir promised, the condition for solubility of equations by radicals; for, even accepting Mr Galois' proposition as true, one is hardly further forward in knowing whether or not an equation of prime degree is soluble by radicals, because it would first be necessary to convince oneself whether the equation is irreducible, and then whether any one of its roots may be expressed as a rational function of two others. The condition for solubility, if it exists, should be an external character which one might verify by inspection of the coefficients of a given equation, or at the worst, by solving other equations of degree lower than the one given.

Be that as it may, we have made every effort to understand Mr Galois' proof. His reasoning is neither clear enough nor well enough developed for us to have been able to judge its correctness, and we are in no position to give an idea of it in this report. The author announces that the proposition which forms the special goal of his memoir is part of a general theory susceptible of many

([1]) This letter is dated from Christiania the 25th November 1828. Abel died near this town on 6 April following.

other applications. It is often the case that the different parts of a theory, by mutually clarifying each other, are easier to grasp together than in isolation. One may therefore wait until the author will have published his work in its entirely before forming a final opinion; but given the present state of the part that he has submitted to the Academy, we cannot propose to you that you give it your approval."

Just signed: **Lacroix, Poisson** reporter.

The Academy adopts the conclusions of this report.]

The two references are to [Abel (1830a)], [Abel (1830b)]. For discussion of the context of the report see [Ehrhardt (2010a)] and some of the references cited there.

The opening paragraph of the report is not quite accurate. Although Galois did emphasize the application of his theory to the question of solubility of irreducible equations of prime degree, that remained only one application. The thrust of the memoir, as his preface (**f.2a**, p. 106) makes clear, is a general theory. This misunderstanding has, however, been repeated from time to time—see, for example, [Mammone (1989)] and my review of it in *Mathematical Reviews* (1991).

Poisson has been vigorously criticised for his report, and not just by Galois himself. Such criticism is generally along the lines that the success of Galois Theory after its publication by Liouville in 1846 proved him to have been wrong. That is of course a fair point. Equally, it is unfair: to judge Poisson in the light of hard-won later understanding rather than in the light of the context of his own times is to treat him somewhat more harshly than we would wish to be treated ourselves. Some sympathy for Poisson's position is to be found in [Taton (1983), p. 112]. And as I wrote in [Neumann (1986)],

> With hindsight one may feel that this report was wrong. But I cannot think so: it seems to me to be a model of good refereeing. Can any of us be sure that in an analogous situation today we would react differently? I doubt it: it is an admirable report, sympathetic but firm. All that is wrong with it is that it deals with the work of an exceptionally brilliant and awkward man. Galois had no research supervisor who might have shown him how his discoveries should be properly written up. Besides, Galois was not a man who took advice easily. Another young mathematician might have taken the criticism to heart, re-written his work, published it and become famous. Galois took offence, returned to political agitation, died young and became famous.

Unknown to me at the time, Jacques Tits had expressed a similar view [Tits (1982), p. 9] at a lecture delivered to the Académie des Sciences a few years earlier:

> Est-il honteux d'avouer qu'aujourd'hui encore, devant juger du mémoire présenté sous la forme que lui avait donnée Galois, je serais bien près de me rallier à l'opinion exprimée par Poisson?

[Is it shameful to admit that even today, having to judge the memoir in the form that Galois had given it, I would be pretty close to joining myself with the opinion expressed by Poisson?]

NOTE 3: As was reported in the introduction to this chapter, the *Premier Mémoire* forms the content of Dossier 1 of the Galois manuscripts, with subsidiary material in Dossier 3. The cover sheet of Dossier 1 carries the annotations described in § I.3 (p. 7) above. Below that the manuscript has its own cover sheet, presumably supplied by the secretary of the Academy when Galois submitted his manuscript there. Its first page is **f. 1 a**. In the top right corner there is written 'Rapport du 4 Juillet 1831' in a hand which Tannery identifies as that of Liouville. Below that is

M. M. Lacroix
Poisson **vu**
commissaires
le 17. j^{er}. 1831.
XX

in large and florid writing arranged on the page like that. The last line is presumably a signature, just possibly a monogram, more probably a pair of initials, which is rendered as XX above because I cannot make it out. Probably it was one of the secretaries of the Académie des Sciences who handled the paper, wrote the note, and initialled it. The word **vu** [seen] is written in blacker ink and in a different hand from the rest (though it may perhaps be the same hand that wrote "Rapport du 4 Juillet 1831", but only if Tannery's contention that this was written by Liouville is incorrect). The conjecture that it was written by Poisson himself makes good sense, but I do not have direct evidence. And just possibly it could be contradicted by the fact that the note Poisson wrote on **f. 2 a** against Lemme III is written in pencil.

The bottom half of the page, **f. 1 a**, is covered with rough work—scribbled partial calculations and formulae written at various different times. The paper was folded twice, first horizontally, then vertically. The bottom two quarters were re-used on at least two occasions. The rough work in the bottom right quarter is all aligned in the usual way from left to right and top to bottom. Some of it is written in pencil, the rest is written later to fill gaps in the page in very black ink, a little of which overwrites a little of the pencil. The bottom left quarter has been used with different orientations at different times. At one point the folded page was turned sideways so that the vertical fold came to the bottom and the left edge came to the top. In this mode the writing is mainly in pencil, with a few small additions in ink. Later the folded page was turned round so that its top edge (the horizontal fold) was at the bottom and the writing now appears upside-down. This material, fourteen very short lines of formulae, is written in a rather pale grey ink.

The verso, **f.1b**, is in two halves. The top is blank. The bottom was used after the first fold (horizontal) and before the second, vertical fold which is so very influential on **f.1a**. The exclamation

Oh! Cherubins

(not 'chérubins' as in [Tannery (1906), p. 231], nor 'Chérubins' as in [B & A (1962), pp. x, 482]) is centred on this page (not, as suggested in [B & A (1962)], on **f.1a**), somewhat below the middle. It is written neatly in a small hand with a fine pen. A little above and to the right are the words

ou bien

written in the same style with what looks like the same pen but darker ink. Mathematical scribbles and partial calculations surround these phrases. Mostly they are in a slightly larger handwriting, made with a larger nib. They are transcribed in [B & A (1962), p. 482].

NOTE 4: The extant original material of the *Premier Mémoire* consists of the original manuscript that Galois wrote and submitted to the Académie des Sciences on 17 January 1831; a manuscript copy that was written by Auguste Chevalier; and the page-proofs of a postponed project to print the *Premier Mémoire* in the issue for December 1843, Volume VIII of the *Journal de Mathématiques Pures et Appliquées* (Liouville's *Journal*). The original manuscript was heavily annotated and corrected by Galois on a number of occasions after it was returned to him on or soon after 4 July 1931. It is the sole occupant of Dossier 1; Chevalier's copy, the proof-sheets for December 1843, and a manuscript page by Liouville together form Dossier 3.

The first page, **f.12a**, of Chevalier's copy has pencilled annotations from which we can deduce that it was this, not Galois' original manuscript, that was given by Liouville to his printer. Diagonally across the top left corner we read '3 Epreuves' and below that, separated by a line, 'Ne pas couper le memoire' all in large handwriting [3 Proofs; Do not cut the memoir]. Also 'par Evariste Galois' appears in pencil below the title. Somewhere between manuscript and page-proof Évariste has acquired his diacritic.

NOTE 5: On **f.2a** the title, probably written after the memoir was completed, spreads across the whole top of the page, before the (wide) margin becomes operative. At some point it was corrected from

Mémoire sur la résolubilité des Équations par radicaux.

to

Mémoire sur les conditions de résolubilité des Équations par radicaux.

Below that and centred over the preface is the heavily deleted header

~~Mémoire~~

At the bottom of **f. 2 a**, several centimetres below the preface, which occupies only half the page, is '1er Memoire' sloping slightly upwards. It resonates with the words 'Second memoire' written in similar sloping orientation on **f. 37 a** at the start of the piece headed 'Des équations primitives qui sont solubles par radicaux'. It corresponds with the wording of the letter to Chevalier of 29 May 1832 and was written by Galois at the same time.

NOTE 6: For some reason the preface on **f. 2 a** is crossed out. A vertical line has been drawn in very black ink with a medium to broad nib, near the middle of the text. The pen was run about two-thirds of the way down through the writing, but the ink ran out after the first four lines. Another vertical line is drawn from bottom to about two-thirds of the way up, and is considerably smudged. These lines are similar to the vertical line that crosses out the material above the start of the *Second Mémoire* on **f. 37 a**; the ink and the pen appear to be the same. It seems probable that these two deletions were made on 29 May 1832 when Galois was editing his papers before going out to the fatal duel in the morning. If an analysis of the ink were possible it might decide the matter.

In his copy of the *Premier Mémoire* Chevalier added a footnote, gloriously free of capital letters (except his own initials), attached to the first line of Galois' preface:

> j'ai jugé convenable de placer en tête de ce mémoire la préface qu'on va lire, bien que je l'aie trouvée biffée dans le manuscrit ~~de l'auteur~~ ce manuscrit est précisement celui que l'auteur présenta à l'académie. A. Ch.
>
> [I have judged it appropriate to place at the head of this memoir the preface which is to be read here, even although I found it crossed out in the ~~author's~~ manuscript. This manuscript is precisely the one that the author submitted to the Academy. A. Ch.]

This footnote is reproduced (with appropriate capitalisation) in the 1846 Liouville edition of the *Œuvres*. Mysteriously, however, the second sentence does not appear. Part of the mystery lies in this, that in the page-proofs of Liouville's *Journal* which form the second item, ff. 26–35 (with **f. 34 b**, **f. 35 a** and **f. 35 b** blank) of Dossier 3, the second sentence is included in the printed footnote and is not crossed out; moreover, there is a request, carefully written in black ink, that the initials 'A. CH' be changed to '(Auguste Chevalier)'. Those page-proofs were, however, prepared for Vol. VIII of the *Journal de Mathématiques Pures et Appliquées*, with the date Décembre 1843 shown at the bottom left of each 8-page section. The *Œuvres d'Évariste Galois* were in fact not published until three years later, and these particular page-proofs were, presumably, ignored.

NOTE 7: The wide margins of the first few pages of the manuscript of the *Premier Mémoire* contain much material. Some of this consists of emendations and these have been incorporated into my transcription. The first page, **f.2b**, is quite different. First, the numeral **1** is written in very clear black ink in the top right corner of the margin. It is possible that this was written by Galois, and was intended as something like a section number, but it is not matched by similar numbers later.

Secondly, just to the left of the text is a vertical line that curves in at the top and the bottom like an enormously long left parenthesis bracket.

Third, to the left of that and a little above the middle of the page, sideways, there is a double list of names:

M. Delaunay

	[~~??~~]
V. Delaunay	~~Flaugergues~~
	~~Mathé~~
N. Lebon	~~Duchatelet~~
	[~~??~~]
~~E. Galois~~	[~~??~~]
	~~Duchatelet~~
F. Gervais	[~~??~~]
	~~Blanqui~~ [~~?~~]
A. Chevalier	~~Raspail~~
	[~~??~~]

The sixth entry of the second column is read by Robert Bourgne as ~~Mathé~~, the eighth as ~~Bussière~~, and the last as ~~Gal~~ [B & A (1962), p. 483]. The paper was turned 90° clockwise, to have its left edge at the top, so that the list appears sideways. One name appears on its own, and the remainder are in two columns as shown in my transcription. Most of the names are vigorously, perhaps furiously, deleted; two that are not are bracketed, as shown.

NOTE 8: The famous comment by Poisson is written in pencil in the margin of **f.3a**, alongside the statement of Lemme III:

> La démonstration de ce lemme n'est pas suffisante; mais il est vrai d'après le N^{O}. 100 du mémoire de Lagrange, Berlin, 1771.
>
> [The proof of this lemma is insufficient; but it is true according to N^{O}. 100 of the memoir by Lagrange, Berlin, 1771.]

Immediately below this Galois wrote

> Nous avons transcrit [~~??~~] textuellement la démonstration que nous avons donnée de ce lemme dans un mémoire présenté en 1830. Nous y joignons

In the *ms* the double lines that terminate the note by Galois slope a little upwards from left to right.

comme document historique le note suivante qu'a cru ^devoir^ y apposer M. Poisson.

On jugera.

Note de l'auteur

[We have faithfully transcribed the proof of this lemma that we have given in a memoir presented in 1830. We append as a historical document the following note which Mr Poisson believed he should add.

Posterity will judge.

Note by the author]

In [B & A (1962), p. 49] Poisson's note appears to be attached to the end of the explanation of Lemma II, and a passage in [B & A (1962), p. 486] confirms that this is where Bourgne thinks it belongs. Their reading probably recognises the fact that there is a pair of short diagonal lines drawn in pencil in the margin above the note and near to the last part of the proof of Lemma II. But I am sure that it refers to Lemma III. There is also a cross pencilled in against Lemma III; the note is aligned with Lemma III; and, most telling of all, §100 of Lagrange's Berlin memoir [Lagrange (1770/71)] has nothing to do with Lemma II (which is relatively elementary) but certainly does treat matters related to Lemma III. Harold Edwards [Edwards (1984), p. 43, fn] has come to the same conclusion. It is confirmed by Chevalier, who, in a footnote on **f. 14 a** attached to his transcription of Lemme III wrote 'J'ai trouvé en marge dans le manuscrit de Galois la note suivante tracée au crayon par M. Poisson' [I have found in the margin of the Galois manuscript the following note written in pencil by Mr Poisson].

Personally, to transgress my self-imposed bounds and move briefly away from syntax into matters of semantics, I find myself with Poisson in this matter. Section 100 of Lagrange's great memoir is perhaps not as clear as it might be, but it seems to me that it does offer what Galois needed.

NOTE 9: On **f. 3 a** (see p. 110) the sentence 'Cette solution est la seule commune ...' originally continued after '$F(V, b) = 0$' with 'sans quoi la fonction $\varphi(a, \ldots.)$ serait égale à la fonction $\varphi(b, \ldots.)$'. It was changed by insertion of 'cette équation ayant un facteur commun avec l'équation semblable' and alteration of 'la fonction' to 'l'une des fonctions' (both occurrences). In his copy Chevalier placed the insertion before 'sans quoi' and that is where it comes in [Liouville (1846)]. In [B & A (1962)] it is placed after 'sans quoi,' which looks a better rendering of what Galois wrote, though there is no comma after 'sans quoi' in the manuscript.

NOTE 10: The second page **f. 3 a** of the memoir has been reworked extensively. In particular, it is worth clarifying what happened with Lemme IV. Originally the statement of the lemma was followed simply by the following:

Car V' doit s'obtenir par l'échange des lettres dans la fonction V.

Ces principes posés, nous allons passer à l'exposition de notre théorie.

[For V' must be obtained by exchange of letters in the function V.

These principles being in place we proceed to the exposition of our theory.]

Indeed it is possible that not even that exiguous explanation of the lemma was there—at the end of the statement of the lemma there is a horizontal line of the kind that Galois used in various places as indication of a sort of cadence.

Be that as it may, the proof of Lemme IV was supplied hastily on 29 May 1832 when Galois was going through all his manuscripts and writing to Chevalier. He used the space in the margin below his footnote; he then incorporated the minimal explanation he had already supplied and he deleted the cadence 'Ces principes posés, nous allons passer à l'exposition de notre théorie' presumably because he needed the line above it and the little space below it for cramming in the remainder of his proof. Curiously, that deletion is not recorded as such in [B & A (1962)].

NOTE 11: At the end of **f.3a** Galois seems to have written ungrammatically '[...] est resultera $a = f\ V$, Il resultera [...]'. The last sentence was squeezed into the last line of the page; then 'Donc' was added in before its beginning. This was rendered by Chevalier as '[...] est résulté $a = f(V)$, de même il résultera [...]' and this is what appears in all previous print editions.

NOTE 12: The margin of **f.3b** is completely filled with adjustments. The first is the passage linked with the symbol $\star$ to the word 'invariable' in clause 1° of the theorem of Proposition I. Although it is written in the margin to the left of the theorem it is clearly intended as a footnote and looks as if it was written at the same sitting as the main text. Indeed, this is confirmed by Bourgne in [B & A (1962), p. 484].

Below it is the instruction 'À reporter dans les definitions' [To be moved back to the definitions] written later, in a larger, hastier hand and finished with a stroke of the pen that looks like an underline of 'À reporter' but was probably not intended as such. This instruction has been taken by all editors to refer to the passage that follows, not the note that it itself immediately follows. Indeed, it seems almost certain that this was the last addition to the page, and appears where it does simply because there was no space anywhere else.

Below the instruction is the deleted memorandum (perhaps intended for Galois himself) 'Mettre partout à la place du mot permutation le mot substitution' [Replace the word permutation with the word substitution throughout]. This was written later. It looks as if it was written at the time of the last corrections of the manuscript, presumably immediately after Galois had made this change in 1° and 2° of his theorem.

And below that comes the famous passage

> Les substitutions sont le passage d'une permutation à l'autre.
> [...]
> Donc si dans un pareil groupe on a les substitutions S et T on est sur d'avoir la substitution ST.
>
> [Substitutions are the passage from one permutation to another.
> [...]
> Therefore if one has the substitutions S and T in such a group one is sure to have the substitution ST.]

which tells his readers about the essential properties of permutations, substitutions and groups.

The instruction 'À reporter dans les definitions' has been taken by all editors to refer to this. I have no doubt that they are correct—even although there was enough space in the margin of **f.1b** that the passage could have been placed there, alongside the definitions. Nevertheless, in order to try to remain faithful to the manuscript, I have chosen to place it in the midst of Proposition I, treating it in the same way as others of Galois' marginal afterthoughts.

It is interesting to note that in the 1843 proof sheets of the aborted publication of the *Premier Mémoire* (folios 26–34 of the Galois manuscripts) the passage is placed at the end of the definitions, just as it is in L1846 and BA1962, but is neatly crossed out with diagonal lines drawn with the same pen and ink as Liouville used for other corrections. Why he deleted it (without asking for its inclusion elsewhere) and then reinstated it three years later is a mystery.

There is a line drawn in the original manuscript between the second and third paragraphs of this passage. It is neither long nor heavy, but seems to suggest that Galois originally stopped there, leaving it as a simple comment about substitutions, and returned only later to fill out the explanation and complete it to include what he meant by a group.

NOTE 13: In the margin of **f.4a**, opposite the last few lines of the first (originally the only) scholium, are four lines:

> Ce qui characterise un groupe. On peut partir
> d'une des permutations quelconques du groupe.
>
> Scholie. Les substitutions sont indépendantes
> mème du nombre des racines.
>
> [What characterises a group. One can start from
> an arbitrary permutation of the group.
>
> Scholium. The substitutions are independent
> even of the number of roots.]

The first two of these lines are crossed out in a cavalier fashion with two diagonal lines that delete the middle one-third. They read like a reminder to the author to explain about groups: note the resonance with the late addition to **f.3b** (treated in the preceding note) which does precisely that. Referring to these four lines Bourgne wrote that the writing is hasty and ill-formed but different from that of 29 May 1832, and he saw this as confirmation that Galois had re-worked his memoir in the meantime [B & A (1962), p. 486].

Below the four lines are some odd formulae that are reported in [B & A (1962), p. 54]. They mainly involve the letters S and T; U occurs also, but only twice. In [B & A (1962)] there is a cross-reference to the late addition explaining about groups, where the letters S and T play an important role in the last sentence. The letters S, T are used also on **f.8a**, the first page of the Testamentary Letter, to denote substitutions. Here, though, in the margin of **f.4a**, the formulae make no sense, even if Galois was using the symbol $+$ to denote composition of substitutions—which is just barely possible, but would conflict with the notation used in the two cited places.

Below these mysterious doodles (if that is what they are) is the oft-quoted note

> Il y a quelque chose à completer dans cette démonstration. Je n'ai pas le tems. (Note de l'A.)
>
> [There is something to be completed in this proof. I do not have the time. (Author's note)]

The reference is to the proof of Proposition II.

Finally, below this poignant *cri de coeur* is a crossed-out five-line addition to the proof of Proposition II. We return to this proposition, and to the commentaries on it written in turn by Chevalier, Liouville and Bourgne in the next three notes.

NOTE 14: There is a line drawn in pencil vertically down through the middle of Proposition II and the first part of its proof on **f.4a**; a similar line is drawn through the continuation of the proof on **f.4b** as far as the end of Proposition III and its proof (original version); four or five lines are drawn very faintly to the left of the main one and parallel to it on **f.4b**. It is a fairly safe conjecture that these were drawn by Poisson and indicate material with which he had difficulty. Indeed, it is quite possible that this is as far as Poisson was able (or prepared) to read the manuscript in the form that it was submitted to the academy in January 1831. Here is how Bourgne saw it [B & A (1962), p. 486]:

> La proposition II, sa démonstration qui se poursuit au verso sont barrées, ainsi que la proposition III, d'un trait de crayon long et sinueux, qui descend au milieu des paragraphes. On y reconnaît la main légère de Poisson qui annota du même crayon le lemme II. Notre géomètre a certainement percu l'insuffisance qui arrêtait Poisson. Il hachure vivement une note de la rédaction primitive et supprime dans l'énoncé après: "Si l'on adjoint à une

An error, I believe, for Lemme III. See Note 6 above.

équation donnée la racine r d'une équation irréductible", la condition: "et de degré premier". Il est trop tard. Il ajoute rapidement: "Il y a quelque chose à compléter dans cette démonstration, je n'ai pas le tems. (Note de l'A.)". Ce sont ses lignes posées en travers dans la marge qui ont persuadé A. Chevalier que son ami avait revu son Mémoire la nuit qui précéda le duel. Cela est certain. Nous renvoyons aux notes **26–36** (p. 492) où l'on jugera du soin avec lequel Liouville étudia ce passage.

[Proposition II and its proof which continues on the verso of the page, as also Proposition III, are crossed through with a long, sinuous pencil line that goes down the middle of the paragraphs. The light hand of Poisson can be recognised there, who annotated Lemma II [read Lemma III] with the same pencil. Our geometer has certainly perceived the inadequacy which halted Poisson. He vigorously crosses out a note from the original drafting and suppresses the condition "and of prime degree" in the statement after "If one adjoins to a given equation the root r of an irreducible equation". It is too late. He rapidly adds "There is something to be completed in this proof, I do not have the time. (Author's note.)". These are the lines which persuaded A. Chevalier that his friend had reviewed his memoir during the night that preceded the duel. That is certain. We refer to the notes [to folios] 26–36 (p. 492), where the care with which Liouville studied this passage is judged.]

NOTE 15: The Liouville edition [Liouville (1846), p. 423] has the following footnote attached to the theorem of Proposition II:

Dans l'énoncé du théorème, après ces mots "la racine r d'une équation auxiliaire irréductible," Galois avait mis d'abord ceux-ci: "de degré p premier," qu'il a effacés plus tard. De même, dans la démonstration, au lieu de "r, r', r'', ... étant d'autres valeurs de r," la rédaction primitive portait "r, r', r'', ... étant les diverses valeurs de r." Enfin on trouve à la marge du manuscrit la note suivante de l'auteur:

"Il y a quelque chose à compléter dans cette démonstration. Je n'ai pas le temps."

Cette ligne a été jetée avec une grande rapidité sur le papier; circonstance qui, jointe aux mots: "Je n'ai pas le temps," me fait penser que Galois a relu son Mémoire pour le corriger avant d'aller sur le terrain.

A. CH.

[In the statement of the theorem, after the words "the root r of an irreducible auxiliary equation", Galois had originally put these: "of prime degree p", which he later deleted. Likewise, in the proof, in place of "r, r', r'', ... being some other values of r", the original draft had "r, r', r'', ... being the different values of r." Finally, in the margin of the manuscript is to be found the following note by the author:

"There is something to be completed in this proof. I do not have the time."

This line has been thrown onto the paper with great speed; a circumstance which, together with the words "I do not have the time", makes me think that Galois had re-read his memoir in order to correct it before going to the duelling field.

A. CH.]

In fact, this footnote is as much Liouville as Chevalier. The corresponding footnote on **f.18a** of Chevalier's manuscript copy of the memoir is this:

vis à vis la démonstration de ce théorème dans le manuscrit, j'ai trouvé ceci:

"il y a quelque chose à compléter dans cette démonstration. je n'ai pas le temps. " (note de l'auteur)"

cette ligne a été jetée avec une grande rapidité sur le papier; circonstance ⌜qui⌝jointe à ces mots: "je n'ai pas le temps" me fait penser que galois a relu son mémoire pour le corriger avant d'aller sur le terrain.

[Opposite the proof of this theorem in the manuscript I have found this:

"There is something to be completed in this proof. I do not have the time. (Author's note)"

This line has been thrown onto the paper with great speed; a circumstance which, together with the words "I do not have the time", makes me think that Galois had re-read his memoir in order to correct it before going to the duelling field.]

Unlike his other editorial notes, this one does not, in fact, carry his initials; like most of them it is devoid of capital letters.

NOTE 16: As was mentioned in the introduction to this chapter, Dossier 3 includes corrected proof-sheets of an aborted 1843 publication by Liouville of the *Premier Mémoire*. These include a page of manuscript containing the proof he intended to offer for Proposition II, where Galois, after replacing the original with a new version in the margin of **f.4b**, had run out of time and written 'Il y a quelque chose . . . Je n'ai pas le tems' [. . . I do not have the time]. In those proof sheets there is an editorial note in black ink attached with the marker $(\times\times)$ to the clause "$r, r', r'', \ldots$ étant d'autres valeurs de r" (see p. 118). At the foot of the page Liouville has written "Ceci mérite d'être expliqué avec quelque détail" [This merits explanation in some detail]. This same marker then appears again on a manuscript page, **f.36a**, after the end of the proof sheets, in Liouville's hand:

$(\times\times)$ Désignons par $\psi(V) = 0$ l'équation en V dont l'auteur parle; et soient $f(V, r)$, $f_1(V, r) \ldots f_{i-1}(V, r)$ les facteurs irréductibles dans lesquels

$\psi(V)$ devient décomposable par l'adjonction de r, en sorte que

$$\psi(V) = f(V,r)\, f_1(V,r) \cdots f_{i-1}(V,r)\,.$$

Comme r est racine d'une équation irréductible, on pourra dans le second membre remplacer r par r', r'', $\ldots$ $r^{(p-1)}$. Ainsi $\psi(V)^p$ est le produit des i quantités suivantes

Originally Liouville had factors $f(V,r)$, ..., $f_{i-1}(V,r)$ in the first line of this display, $f(V,r')$, ..., $f_{i-1}(V,r')$ in the second, etc. The slip $f_1(V,r')$ for $f_1(V,r)$ as the first factor in the second line is a leftover from that.

$$\begin{array}{l} f(V,r)\, f(V,r') \cdots f(V,r^{(p-1)}) \\ f_1(V,r')\, f_1(V,r') \cdots f_1(V,r^{(p-1)}) \\ \quad - \quad - \quad - \quad - \quad - \quad - \\ f_{i-1}(V,r)\, f_{i-1}(V,r') \cdots f_{i-1}(V,r^{(p-1)}) \end{array}$$

dont chacune, symétrique en r, r', $\ldots$ $r^{(p-1)}$ et par suite exprimable en fonction rationnelle de V indépendamment de toute adjonction, doit diviser $\psi(V)^p$ et se réduire en conséquence à une simple puissance du polynome $\psi(V)$ qui cesse de résoudre en facteurs lorsqu'on ne adjoint plus les auxiliaires r, r', &c. J'ajoute que le degré de la puissance est le même pour toutes; en effet les équations $f(V,r) = 0$, $f_1(V,r) = 0, \ldots f_{i-1}(V,r) = 0$ qui dérivent de $\psi(V) = 0$ et dont les racines sont fonctions rationnelles les unes des autres ne peuvent manquer d'être du même degré. En faisant donc

$$f(V,r)\, f(V,r') \cdots f(V,r^{(p-1)}) = \psi(V)^{\mu}$$

on en conclura $p = i\mu$. Mais p est premier et $i > 1$; donc on a $i = p$, d'où $\mu = 1$, et enfin

$$\psi(V) = f(V,r)\, f(V,r') \cdots f(V,r^{(p-1)}).$$

ce qu'il fallait démontrer. (J Liouville)

[Denote the equation in V of which the author speaks by $\psi(V) = 0$; and let $f(V,r)$, $f_1(V,r) \ldots f_{i-1}(V,r)$ be the irreducible factors into which $\psi(V)$ becomes decomposable on adjunction of r, so that

$$\psi(V) = f(V,r)\, f_1(V,r) \cdots f_{i-1}(V,r)\,.$$

Since r is root of an irreducible equation, in the second member one could replace r by r', r'', $\ldots$ $r^{(p-1)}$. Thus $\psi(V)^p$ is the product of the following i quantities

$$\begin{array}{l} f(V,r)\, f(V,r') \cdots f(V,r^{(p-1)}) \\ f_1(V,r)\, f_1(V,r') \cdots f_1(V,r^{(p-1)}) \\ \quad - \quad - \quad - \quad - \quad - \quad - \\ f_{i-1}(V,r)\, f_{i-1}(V,r') \cdots f_{i-1}(V,r^{(p-1)}) \end{array}$$

each of which, being symmetric in r, r', ... $r^{(p-1)}$ and consequently expressible as a rational function of V independently of any adjunction, must divide $\psi(V)^p$ and as a consequence reduce simply to a power of the polynomial $\psi(V)$, which ceases to resolve into factors when one does not adjoin the auxiliaries r, r', &c. any more. I add that the degree of the power is the same for all; indeed, the equations $f(V,r) = 0$, $f_1(V,r) = 0$, ... $f_{i-1}(V,r) = 0$ which derive from $\psi(V) = 0$, and of which the roots are rational functions of each other cannot fail to be of the same degree. Therefore setting

$$f(V,r)\, f(V,r') \cdots f(V,r^{(p-1)}) = \psi(V)^{\mu}$$

one deduces that $p = i\mu$. But p is prime and $i > 1$; therefore one has $i = p$, from which $\mu = 1$, and finally

$$\psi(V) = f(V,r)\, f(V,r') \cdots f(V,r^{(p-1)}).$$

as was to be proved. (J Liouville)]

Presumably, since the proof sheets carry the date DÉCEMBRE 1843 this note was written towards the end of that year. No trace of it remained, however, when Liouville published his edition of the works of Galois in November 1846. One might conjecture that it was intended to respond to Galois' marginal *cri de cœur* 'Il y a quelque chose à completer dans cette démonstration. Je n'ai pas le tems.'

NOTE 17: In **f. 4b** there are to be found two versions of Proposition III. The first is heavily crossed out; the second is in the margin immediately below a heavily drawn line beneath the two lines 'Car l'on aura [...] divisible par $f(V', r')$' that form a marginal addition to the proof of Proposition II. Liouville (L1846, p. 425; P 1897, pp. 41–42) has the following footnote:

Dans le manuscrit, l'énoncé du théorème qu'on vient de lire se trouve en marge et en remplace un autre que Galois avait écrit avec sa démonstration sous le même titre: *Proposition III*. Voici le texte primitif: THÉORÈME. "Si l'équation en r est de la forme $r^p = A$, et que les racines $p^{ièmes}$ de l'unité se trouvent au nombre des quantités précédemment adjointes, les p groupes dont il est question dans le théorème II jouiront de plus de cette propriété, que les substitutions de lettres par lesquelles on passe d'une permutation à une autre dans chaque groupe soient les mêmes pour tous les groupes." En effet, dans ce cas, il revient au même d'adjoindre à l'équation telle ou telle valeur de r. Par conséquent, ses propriétés doivent être les mêmes après l'adjonction de telle ou telle valeur. Ainsi son groupe doit être le même quant aux substitutions (Proposition I, scolie). Donc, etc.

Liouville's closing quotation marks appear in the wrong place; Picard perpetuated the error. Corrected in translation below.

Tout cela est effacé avec soin; le nouvel énoncé porte la date 1832, et montre, par la manière dont il est écrit, que l'auteur était extrêmement pressé, ce qui confirme l'assertion que j'ai avancée dans la note précédente. A. CH.

[In the manuscript the statement of the theorem that the reader has just seen is to be found in the margin and replaces another that Galois had written with its proof under the same title, *Proposition III*. Here is the original text: THEOREM. "If the equation for r is of the form $r^p = A$, and the pth roots of unity are to be found among the quantities previously adjoined, the p groups in question in Theorem II will further enjoy the property that the substitutions of letters by which one passes from one permutation to another in each group will be the same for all the groups. Indeed, in this case it it comes to the same whichever value of r is adjoined to the equation. Consequently its properties must be the same after the adjunction of whichever value. Thus its group must be the same in respect of the substitutions (Proposition I, Scholium). Therefore, etc."

All that is carefully deleted; the new statement carries the date 1832, and shows, in the manner in which it is written, that the author was extremely pressed for time, which confirms the assertion that I have advanced in the preceding note. A. CH.]

Liouville has, in fact, expanded what Chevalier supplied as a footnote to his manuscript copy of the *Premier Mémoire*, **f.19a**:

dans le manuscrit de galois ~~la ^démonstration de la même^~~ proposition 3 ^l'énoncé du théorème^ qu'on vient de lire se trouve en marge et vis à vis ~~d'une autre~~ de la démonstration qu'il n'avait écrite d abord [*sic*]. celle-ci est effacée avec soin; ~~l'autre~~ ^l'énoncé précédent^ porte la date 1832 et montre par la manière dont il est écrit que l'auteur était extrèmement pressé; ce qui confirme l'assertion que j'ai avancée dans la note précédente.

A. CH.

[In the manuscript of Galois the statement of the theorem that the reader has just seen is to be found in the margin opposite the proof that he had written previously. The latter is carefully deleted; the statement of the former carries the date 1832 and shows, in the way in which it is written, that the author was extremely pressed for time, which confirms the assertion that I advanced in the preceding the note.

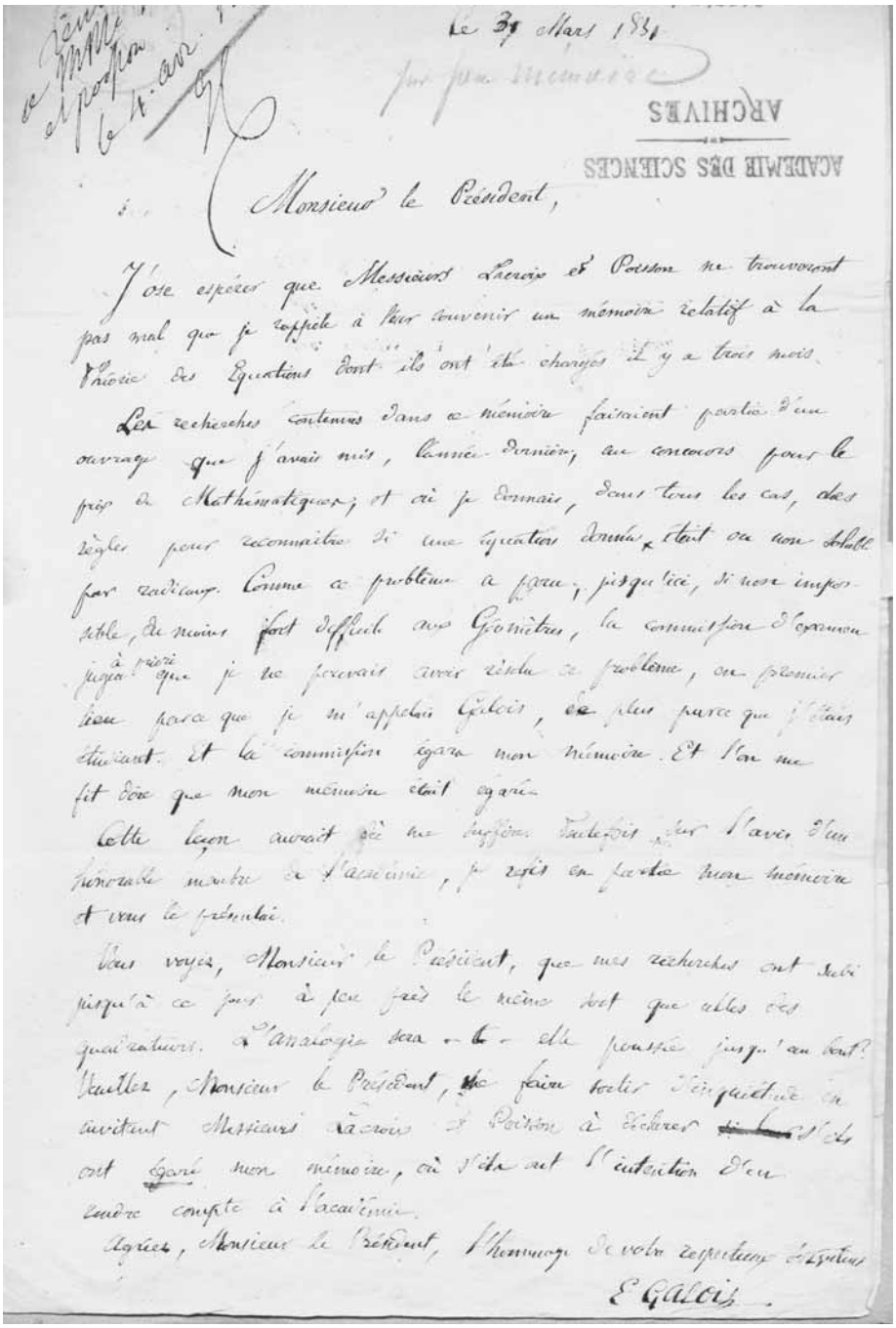
A. CH.]

In **f.30a** of the aborted 1843 proof sheets, against the sentence 'On trouvera la démonstration' Liouville has placed the same marker ($\times\times$) as he used on the previous page with reference to the proof of Proposition II, and it is accompanied at the foot of the page by the same note 'Ceci mérite d'être expliqué avec quelque détail' [This merits explanation in some detail]. Although there is no accompanying manuscript proof of Proposition III at the end of this dossier, there is a line of argument that completes the proof in the Liouville manuscripts in Dossier 27 (see [Lützen (1990), p. 573]).

NOTE 18: In his treatment of the solution of equations of degree 4 Galois referred towards the end of **f.5b** to known solutions credited to Descartes and Euler (see p. 126). Presumably these are the solutions that Lagrange analysed in some detail in his great Berlin memoir [Lagrange (1770/71), §§ 33–37, 46–47]. For context, and for a modern account both them and of what Lagrange says about them see [Stedall (2011)], especially pages 47–48, 173, 122–125, 174–175. It is just possible that Galois acquired his understanding of these methods from [Lagrange (1770/71)]. But I estimate that it is more likely that he learned them (or of them—it is entirely possible that he learned them by re-creating them for himself) either from one of the textbooks of the time, or from §§ 33–37 of the précis of Lagrange's memoir which is the content of Note XIII appended to the second and third editions of [Lagrange (1798)]. As far as textbooks go [Euler (1807), Section IV] must be a prime candidate in this context, though I have found no mention of Descartes' method there—Section 4, Chapter 14 is 'De la Règle de BOMBELLI, pour réduire la résolution des Équations du quatrième degré à celle des Équations du troisième degré' [On Bombelli's rule for reducing the solution of equations of the fourth degree to that of equations of the third degree], and Section 4, Chapter 15 is 'D'une nouvelle méthode de résoudre les Équations du quatrième degré' [On a new method for solving equations of the fourth degree]. It is just possible that he learned them from the *Cours d'analyse* [Cauchy (1821), Ch. X, § 3], but the names Descartes and Euler do not appear there. (See Section I.2 above.)

NOTE 19: The correction 'dont l'indice serait autre que le degré même de l'équation' is added in pencil to **f.22a**, Chevalier's copy of the lemma in Proposition VI (see p. 126), and is printed in [Liouville (1846), p. 429]. I am no hand-writing expert, but I see many similarities between this and some annotations of the 1843 proof sheets. Those, such as 'dire qu'une équation a des diviseurs rationnels, c'est' which occurs on **f.26b**, or 'Ceci mérite d'être expliqué avec quelque détail' which occurs on **f.29b**, are in ink, not in pencil, but a number of letters are formed in the same way, and such idiosyncracies as the writing of acute accents backwards (so that they read as grave accents) and the writing of circumflex as a curvy line a little like a large tilde, are common to all three passages. The presumption that the proof-sheet annotations are Liouville's work must be pretty robust; and therefore I am reasonably confident in attributing to Liouville the pencilled annotation of Chevalier's manuscript, which we know was used as the printer's copy.

That this correction to the lemma is a contribution by Liouville is overlooked by Tannery, even although he focusses much attention on Liouville's interventions elsewhere (see [T1906/7, pp. 234, 235]). [B & A (1962)] also is silent on this point.



Le 31 Mars 1831

Pour un mémoire

ARCHIVES

ACADEMIE DES SCIENCES

Monsieur le Président,

J'ose espérer que Messieurs Lacroix et Poisson ne trouveront pas mal que je rappele à leur souvenir un mémoire relatif à la théorie des Equations dont ils ont été chargés il y a trois mois.

Les recherches contenues dans ce mémoire faisaient partie d'un ouvrage que j'avais mis, l'année dernière, au concours pour le prix de Mathématiques; et où je donnais, dans tous les cas, des règles pour reconnaître si une équation donnée, était ou non soluble par radicaux. Comme ce problème a paru, jusqu'ici, si peu susceptible, de moins fort difficile aux Géomètres, la commission d'examen jugea à propos que je ne pouvais avoir résolu ce problème, en premier lieu parce que je m'appelais Galois, et plus parce que j'étais étudiant. Et la commission égara mon mémoire. Et bien me fut dû que mon mémoire était égaré.

Cette leçon aurait été me suffisante toutefois pour m'avis d'un honorable membre de l'Académie, je refis en partie mon mémoire et vous le présentai.

Vous voyez, Monsieur le Président, que mes recherches ont suivi jusqu'à ce jour à peu près le même sort que celles des quadratures. L'analogie sera-t-elle poussée jusqu'au bout? Veuillez, Monsieur le Président, me faire sortir l'inquiétude en invitant Messieurs Lacroix et Poisson à déclarer s'ils ont égaré mon mémoire, ou s'ils ont l'intention d'en rendre compte à l'Académie.

Agréez, Monsieur le Président, l'hommage de votre respectueux Egalois

E Galois.

Letter to the president of the Paris Academy, 31 March 1831.

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IV.3 A letter of complaint and enquiry sent on 31 March 1831 to the President of the Academy

Galois wrote to the president of the Académie des Sciences on 31 March 1831 about the paper he had submitted on 17 January. His letter is to be found in the archives of the Academy. Its receipt and a decision to send it to Lacroix and Poisson were recorded in the Minutes of the Academy for Monday 4 April 1831 (*Procès-Verbaux*, p. 597, with the name mis-spelled there as Gallois), and it rests in the dossier of items from that meeting.

It was first published by Joseph Bertrand in *Journal des Savants*, July 1899, pp. 396–397; then again in [Taton (1947a), p. 118], in [Dalmas (1956/82), pp. 100, 101] (where, however, it is mis-described as being addressed to the president of the Institut de France), in [B & A (1962), p. 465], and in [APMEP (1982), p. 17]. There may well be other editions, but I estimate that others are likely to be based on one or other of these five. They all differ from each other and from the original. Mostly the differences are minor matters of spelling or punctuation, but some are more substantial. Robert Bourgne's version is described in [B & A (1962), p. 525] as following that of Bertrand, but it does not seem to do so; in fact it follows the Taton version a little more closely, as can be understood from my marginal notes.

Galois wrote on a large sheet of paper, 39.7 cm × 31.3 cm folded to make a pamphlet 19.9 cm × 31.3 cm. The letter is on the first page, the inside is, of course, blank, and the back page contains the spaciously written address as shown below, with an accent missing from 'Président' and with Galois' characteristic thin sign-off line below.

Monsieur

le President de l'Académie des Sciences

de l'Institut de France.

———

Before the address was written the letter was folded twice horizontally into thirds, twice vertically into approximate thirds, then tucked in and sealed with a wax seal. Thus the final letter as received at the Institute was 12 cm × 8 cm.

A transcription and a translation of the letter, with some further commentary, follow overleaf. A number of mis-spellings have been variously corrected in the five previous editions. Where these are not uncommon in Galois' writing, such as 'rappele' for 'rappelle', 'mème' for 'même', 'où' for 'ou', 'reconnaitre' for 'reconnaître' I have followed Galois silently; the same applies to punctuation; the comma after 'Toutefois' is rather faint in the manuscript, and occurs much nearer to the following word, but this is common with Galois and I am reasonably confident in

it, especially since it is reproduced by Bertrand and Dalmas (though not Taton or Bourgne & Azra). Where there are more major discrepancies they have been noted in the margin. In the marginal notes B1899 is an abbreviation for [Bertrand (1899)], T1947 for [Taton (1947a)], and D1956 for [Dalmas (1956/82)].

The date is moved to the bottom in BA1962.

Le 31 Mars 1831

Monsieur le Président,

'Messieurs' rendered 'MM.' in B1899, D1956, BA1962.

J'ose espérer que Messieurs Lacroix et Poisson ne trouveront pas mal que je rappele à leur souvenir un mémoire relatif à la Théorie des Equations dont ils ont été chargés il y a trois mois.

T1947, BA1962 insert 'grand' before 'prix'.

Les recherches contenues dans ce mémoire faisaient partie d'un ouvrage que j'avais mis, l'année dernière, au concours pour le prix de Mathématiques, et où je donnais, dans tous les cas, des règles pour reconnaitre si une équation donnée, était ou non soluble par radicaux. Comme ce problème a paru, jusqu'ici, si non impossible, du moins fort difficile aux Géomètres, la commission d'examen jugea ^à priori^ que je ne pouvais avoir résolu ce problème, en premier lieu parce que je m'appelais Galois, et en plus parce que j'étais étudiant. Et la commission égara mon mémoire. Et l'on me fit dire que mon mémoire était égaré.

Lines 6, 7: word 'des' overwritten onto something like 'la'. B1899, BA1962 have 'les', T1947, D1956, APMEP have 'des'. Word 'donnée' missing from B1899, BA1962; 'soluble' changed to 'résoluble' in B1899.

Cette leçon aurait dù me suffire. Toutefois, sur l'avis d'un honorable membre de l'académie, je refis en partie mon mémoire et vous le présentai.

Lines 8, 9: 'du moins' changed to 'au moins' in B1899, BA1962; 'd'examen' missing from T1947.

Vous voyez, Monsieur le Président, que mes recherches ont subi jusqu'à ce jour à peu près le mème sort que celles des quadrateurs. L'analogie sera-t-elle poussée jusqu'au bout? Veuillez, Monsieur le Président, me faire sortir d'inquiétude en invitant Messieurs Lacroix et Poisson à déclarer ~~si leur~~ s'ils ont égaré mon mémoire, où s'ils ont l'intention d'en rendre compte à l'académie.

Line 11: sentence 'Et ... mémoire' missing from B1899, T1947, BA1962. Word 'dire' changed to 'savoir' in T1947, BA1962.

Line 14: 'Académie' in all editions. But same initial letter as 'analogie' in Line 16.

Line 15: 'subi' missing from B1899.

'Messieurs' rendered 'MM.' in all editions.

Line 19: 'mémoire' changed to 'manuscrit' in D1956.

Agréez, Monsieur le Président, l'hommage de votre respectueux serviteur

E. Galois .

31 March 1831

Mr President,

I dare to hope that Messrs Lacroix and Poisson will not take it badly that I recall to their memory a memoir relating to the theory of equations with which they were charged three months ago.

The research contained in this memoir formed part of a work which I submitted last year in competition for the prize in mathematics, and in which I gave, for all cases, rules to recognise whether a given equation was or was not soluble by radicals. Since, until now, this problem had appeared to geometers to be, if not impossible, at least very difficult, the examining committee judged *a priori* that I could not have solved this problem, in the first place because I was called Galois, and further because I was a student. And the committee lost my memoir. And someone had me told that my memoir was lost.

This lesson should have been enough for me. All the same, on the advice of an honorable member of the Academy, I reconstructed part of my memoir and presented it to you.

You see Mr President that my research has suffered up to now almost the same fate as that of the circle squarers. Will the analogy be pushed to its conclusion? Be so kind Mr President as to relieve my disquiet by inviting Messrs Lacroix and Poisson to declare whether they have *lost* my memoir or whether they have the intention to make a report of it to the academy.

Be assured Mr President of the homage of your respectful servant

E. Galois .

The letter itself fits neatly onto the first page. Galois wrote the date at the top right some 2 cm below the top of the page; he then left plenty of space before writing the salutation. Surrounding the date is written in pencil “m. Galois” (above and corrected from “m. Gallois”) and “sur son mémoire” (below). In the top left corner is written in ink: “renvoyer/ à MM. Lacroix/ et Poisson./ le 4. avr. 1831.” in four lines below which is a flourished initial that is presumably that of the secretary

or the president of the academy. The date is underlined in purple wax crayon—I conjecture that this is a note by a librarian to indicate the Archive file into which the item should go. Then across the top right, above the date and its pencilled surround, is written "Rapport du 4 Juillet 1831" in black ink and quite a large, robust hand.

Galois' final signature is in large, bold writing completed with a flourish that I cannot reproduce in print. The letter is carefully written and bears very few changes: there are only two deletions (a comma and 'si leur'), and there are only three places where Galois made immediate changes by over-writing ('des' on 'la' (conjecturally), 'de' on 'en', 'me' on 'f'). Presumably it was copied from a draft.

The letter invites much comment. I restrict myself here to two matters of fact. The first is that the "honorable membre de l'académie" (third paragraph) is identified by Bertrand (and others) as having been Poisson. A deleted passage in Dossier 11, **f. 72 a** (see p. 246) can probably be taken as direct evidence.

The second concerns chronology. Galois' "il y a trois mois" ("three months ago") is a little premature in that the manuscript he was concerned about was received by the Academy on 17 January 1831, two months and two weeks before he wrote this letter. He was not the only one to get temporal facts wrong, however. In [Bertrand (1899), p. 396], introducing the letter, Bertrand wrote:

> Galois [...] envoya une seconde rédaction. Poisson et Lacroix furent commissaires. Un accusé de réception fut addressé à Galois, et plusieurs mois s'écoulèrent sans qu'il entendît parler de l'Académie. Enfermé à Sainte-Pélagie, il ne pouvait aller voir les commissaires; il écrivit au président de l'Académie le 31 mars 1831. [...]. C'est Poisson qui reçut sa lettre.
>
> [Galois [...] sent a second draft. Poisson and Lacroix were referees. An acknowledgement of receipt was sent to Galois, and several months flowed by without him hearing a word from the Academy. Imprisoned in Sainte-Pélagie he could not go to see the referees; he wrote to the president of the Academy on 31 March 1831. [...]. It was Poisson who received his letter.]

This is quite wrong. Since it was not until later in 1831 that Galois was imprisoned, the reference to Sainte-Pélagie is nonsense. Already in his previous paragraph Bertrand showed confusion about the manuscripts that Galois had submitted to the Academy. He conflated the submissions of May/June 1829 with that of February 1830 (see § IV.2 above). Hence the reference to "une seconde rédaction" is also wrong. The manuscript in question was, in fact, Galois' third attempt to interest the Academy.

Chapter V
The Second Memoir

V.1 Text of the Second Memoir

The *Second Mémoire*, 'Des équations primitives qui sont solubles par radicaux' (On primitive equations which are soluble by radicals), occupies Dossier 4, folios 37–41 of the manuscripts; Dossier 5, folios 42–53, is a manuscript copy by Auguste Chevalier. It was first published by Liouville in 1846, reprinted by Picard in 1897, and it appears on pp. 128–147 of [B & A (1962)], with commentary on pp. 493–496. There is a German translation in [Maser (1889)]. An English translation of the first part, together with detailed commentary, appears in [Neumann (2006)].

The manuscript is written on separate sheets of paper $22.5\,\text{cm} \times 35\,\text{cm}$. They show clear signs of having been torn carefully, but not completely regularly, from sheets twice the size. As elsewhere Galois created a vertical fold to give himself a guideline for his left margin; it varies a little but is always $6\,\text{cm} \pm 0.5\,\text{cm}$, wide enough to accommodate a few afterthoughts, one of which (on **f. 40 a**) is quite substantial. The folios are numbered 1–5 in the top left corner of their rectos. The numbering is in black ink that matches the words 'Second Mémoire' on **f. 37 a**, the thick vertical line that crosses out the first part of that page, and the writing of the *Lettre testamentaire*. That this is an old, unfinished piece of work which Galois had intended to amplify, polish and publish seems clear. Robert Bourgne [B & A (1962), p. 494] identifies the paper, the ink and the writing as being the same as those of folio 54 (Dossier 6, an early version of Proposition I of the *Premier Mémoire*), and dates it to June 1830. For further description of the manuscript see Note 1, p. 195.

Galois made several references to the *Second Mémoire* elsewhere in his writings. The best known is on **f. 8 a**, the first page of the *Lettre testamentaire* (see p. 84), and in the following passage of that letter. On **f. 8 b** he described the article 'Sur la théorie des nombres' (June 1830; see § II.4) as providing a lemma for it. On **f. 9 a** there is a reference to the article 'Analyse d'un mémoire sur la résolution algébrique des équations' (April 1830; see § II.2), which makes clear that that abstract is an announcement of results that Galois intended to explain here. There are also less well known and less clear references elsewhere, for example on **f. 58 b** in Dossier 8 (see p. 220) and in Dossier 11 (see p. 245).

37a

soient representées par

$$\varphi_1 x \quad \varphi_2 x \quad \varphi_3 x \quad \ldots\ldots \quad \varphi_{n-1} x$$

List $\varphi_1 x, \ldots$ not displayed in BA1962; punctuation inserted in T1906/7, BA1962.

Je dis que P et p étant des nombres quelconques on aura

$$(P-p)(P-\varphi_1 p)(P-\varphi_2 p)(P-\varphi_3 p)\ldots.(P-\varphi_{n-1} p) \equiv F(P) \text{ (mod. } Fp)$$

Crossed-out words omitted in BA1962.

~~En effet~~

Démonstration. ^Il suit^ De l'hypothèse ~~il suit~~ que l'on pourra par des opérations entières et rationnelles déduire de l'équation

$$Fx = 0$$

celle-ci

$$(P-x)(P-\varphi_1 x)(P-\varphi_2 x)\cdot\cdots\cdot(P-\varphi_{n-1} x) = F(P)$$

Quelque soit P.

Or on a évidemment

$$Fp = 0 \qquad (\text{mod. } Fp)$$

T1906/7, p. 239 has 'égalités' but 'équations' is clear in *ms*.

Donc en substituant les congruences aux équations ~~aux congruences~~ on aura le théorème énoncé.

Corollaire. Soit $Fx = \dfrac{x^\nu - 1}{x-1}$, ν étant premier, nous aurons $\varphi_1 x = x^2$ $\varphi_2 x = x^3$, ... Donc on aura en général si ν est un nombre premier

Equations $\varphi_1 = x^2, \ldots$ displayed in BA1962, not in *ms*.

$$(P-p)(P-p^2)(P-p^3)\cdot\cdots\cdot(P-p^{\nu-1}) = \frac{P^\nu - 1}{P-1} \quad \left(\text{mod. } \frac{p^\nu - 1}{p-1}\right).$$

Si enfin on fait $P = p^\nu$ on aura le théorème suivant

$$(p^\nu - p)(p^\nu - p^2)(p^\nu - p^3)\cdot\cdots\cdot(p^\nu - p^{\nu-1}) = \nu \quad \left(\text{mod. } \frac{p^\nu - 1}{p-1}\right),$$

T1906/7 has 'fin du Mémoire', BA1962 has 'Fin de Mémoire'; 'fin du mémoire' is clear in the *ms* margin.

~~qui~~ ν étant premier. *fin du mémoire*

Des équations primitives qui sont solubles par radicaux
Second Mémoire

First clause omitted in L1846. Originally a sentence on its own. L1846 has commas round 'en général'.

Revenons maintenant à notre objet, ⌜et⌝ Cherchons en général dans quel cas une équation primitive est soluble par radicaux. Or nous pouvons de suite établir un caractère général fondé sur le degré ^même^ de ces équations. Ce caractère est celui-ci: pour qu'une équation primitive soit résoluble par radicaux, il faut que son degré

be represented by

$$\varphi_1 x \quad \varphi_2 x \quad \varphi_3 x \quad \ldots\ldots \quad \varphi_{n-1} x$$

I say that P and p being arbitrary numbers one will have

$$(P-p)(P-\varphi_1 p)(P-\varphi_2 p)(P-\varphi_3 p)\ldots.(P-\varphi_{n-1} p) \equiv F(P) \text{ (mod. } Fp)$$

~~Indeed~~ Proof. It follows from the hypothesis that, using rational integer operations, one could deduce from the equation

$$Fx = 0$$

this:

$$(P-x)(P-\varphi_1 x)(P-\varphi_2 x)\cdot\cdots\cdot(P-\varphi_{n-1} x) = F(P)$$

whatever P may be.

Now evidently

$$Fp = 0 \qquad (\text{mod } Fp)\,.$$

Therefore on substituting the congruences for the equations one gets the stated theorem.

Corollary. Let $Fx = \dfrac{x^\nu - 1}{x-1}$, ν being prime. We will have $\varphi_1 x = x^2$ $\varphi_2 x = x^3, \ldots\ldots$ Therefore in general, if ν is a prime number one will have

$$(P-p)(P-p^2)(P-p^3)\cdot\cdots\cdot(P-p^{\nu-1}) = \frac{P^\nu - 1}{P-1} \quad \left(\text{mod. } \frac{p^\nu - 1}{p-1}\right).$$

Finally if one sets $P = p^\nu$ one will have the following theorem.

$$(p^\nu - p)(p^\nu - p^2)(p^\nu - p^3)\cdot\cdots\cdot(p^\nu - p^{\nu-1}) = \nu \quad \left(\text{mod. } \frac{p^\nu - 1}{p-1}\right),$$

ν being prime. *End of memoir*

On primitive equations which are soluble by radicals.
Second Memoir

Let us now return to our goal and seek the general circumstances in which a primitive equation is soluble by radicals. Now we can immediately establish a general condition based simply on the degree of these equations. The condition is this: for an equation to be soluble by radicals it is necessary that its degree

soit de la forme p^ν, p étant premier. Et delà suivra ~~de suite~~ ^immédiatement^ que lorsqu'on aura ^à résoudre par radicaux^ une équation irréductible dont ^le^ degré admettrait des facteurs premiers inégaux, ~~elle devra être ou non-primitive~~ on ne pourra le faire que par la Méthode de décomposition due à Mr Gauss. Si non l'équation sera insoluble.

BA1962 has 'il suivra'.

L1846 has 'M. Gauss; sinon'. BA1962 has 'M. Gauss. Si non'

Pour établir la propriété générale que nous venons d'énoncer relativement aux équations primitive qu'on peut résoudre par radicaux, nous pouvons supposer que l'équation ~~dont~~ [?]

37b

^que^ l'on veut résoudre, soit primitive, mais cesse de l'être par ~~une simple extraction de~~ ^l'adjonction d'un simple^ radical. En d'autres termes, nous pouvons supposer que ~~le~~ n étant premier, le groupe de l'équation se partage en n groupes ^irréductibles^ conjugués, mais non-primitifs. Car àmoins que le degré de l'équation soit premier, un pareil groupe se présentera toujours dans la suite des décompositions.

Soit N le degré de l'équation, et supposons qu'après une extraction de ~~degr radical~~ ^racine^ de degré premier n, elle devienne non primitive, et se partage en Q équations ~~de degré~~ ^primitives de degré^ P, au moyen d'une seule équation de degré Q.

Si nous appelons G le groupe de l'équation, ce groupe devra se partager en n groupes conjugués non-primitifs, ~~ou les~~ ^dans lesquels^ les lettres se ~~parta~~ rangeront en systèmes ~~de~~ composés de P lettres conjointes chacun. ⌜Voyons de combien de manières celà pourra se faire.⌝

Hyphen clear in *ms*, missing from BA1962.

Soit H l'un des groupes conjugués non primitifs. Il est aisé de voir que dans ce groupe, deux lettres quelconques prises à volonté feront partie d'un ^certain^ système de P lettres conjointes, et ne feront partie que que ^d'^un seul.

Word 'que' repeated in *ms*.

Car 1° s'il y avait deux lettres qui ne pussent faire partie d'un mème système de P lettres conjointes, le groupe G qui est tel que l'une quelconque de ses substitutions, transforme les unes dans les autres toutes les substitutions du groupe H, serait non-primitif: cequi est contre l'hypothèse.

En second lieu, si deux lettres faisaient partie de plus⌜ieurs⌝ ~~d'un~~ système⌜s⌝ ~~conjoint~~ ^différents^, il s'ensuivrait que ~~[?] et les~~ ^les groupes qui répondent aux divers^ systèmes de ^P^ lettres conjointes ne ~~répondrait~~ ^seraient^ pas ~~à de groupes~~ primitifs, cequi est encore contre l'hypothèse.

In *ms* 'plus d'un système conjointe' changed to 'plusieurs systèmes différents'; 'ne répondrait pas à de groupes primitifs' changed to 'ne seraient pas primitifs'.

Celà posé, soient

$$\begin{array}{cccccc} a_0 & a_1 & a_2 & \ldots. & a_{P-1} \\ b_0 & b_1 & b_2 & \ldots. & b_{P-1} \\ c_0 & c_1 & c_2 & \ldots. & c_{P-1} \\ & & \ldots. & & \end{array}$$

Les ~~diverses~~^N^ lettres : supposons que chaque ligne ^horizontale^ représente

be of the form p^ν, p being prime. And from that it will follow ~~at once~~ immediately that when one has to solve by radicals an irreducible equation whose degree admits unequal prime factors, ~~it will be primitive or not~~ it will not be possible to do so except by the method of decomposition due to Mr Gauss. If not, the equation will be insoluble.

To establish the general property that we have just stated in relation to primitive equations that one can solve by radicals, we may suppose that the equation ~~of which~~

to be solved is primitive, but ceases to be so by ~~a simple extraction of~~ the adjunction of a simple radical. In other words, we can suppose that, n being prime, the group of the equation is partitioned into n conjugate irreducible, but non-primitive, groups. For unless the degree of the equation is prime, such a group will always present itself in the sequence of decompositions.

Let N be the degree of the equation, and suppose that, after an extraction of a ~~radical~~ root of prime degree n, the equation becomes non-primitive, and splits into Q primitive equations of degree P by means of a single equation of degree Q.

If we call the group of the equation G, this group will have to be partitioned into n conjugate non-primitive groups in which the letters are arranged in systems each composed of P related letters. Let us see in how many ways that may be done.

Let H be one of the conjugate non-primitive groups. It is easy to see that any two letters chosen at will in this group will form part of a system of P related letters, and will not form part of more than one.

For 1st if there were two letters which could not form part of such a system of P related letters, the group G, which is such that any one of its substitutions transforms the substitutions of the group H among themselves, would be non-primitive: this is contrary to hypothesis.

In the second place, if two letters formed part of several different systems, it would follow that the groups which correspond to the various systems of P related letters would not be primitive, which is again contrary to hypothesis.

That said, let

$$\begin{array}{ccccc} a_0 & a_1 & a_2 & \ldots. & a_{P-1} \\ b_0 & b_1 & b_2 & \ldots. & b_{P-1} \\ c_0 & c_1 & c_2 & \ldots. & c_{P-1} \\ & . & . & . & \end{array}$$

be the ~~various~~ N letters: let us suppose that each horizontal line represents

un système de lettres conjointes. Soient

$$a_0 \quad a_{0.1} \quad a_{0.2} \quad \cdots \quad a_{0.P-1}$$

P lettres conjointes toutes situées dans la première colonne verticale, (il est clair que nous pouvons faire qu'il en soit ainsi, en intervissant ~~de même~~ l'ordre des lignes horizontales.)

38a

Soient demème

$$a_{1.0} \quad a_{1.1} \quad a_{1.2} \quad a_{1.3} \quad \cdots \quad a_{1.P-1}$$

P lettres conjointes toutes situées dans la seconde colonne verticale ~~en sorte~~ ensorte que

$$a_{1.0} \quad a_{1.1} \quad a_{1.2} \quad a_{1.3} \quad \cdots \quad a_{1.P-1}$$

apartiennent respectivement aux mèmes lignes horizontales que

$$a_{0.0} \quad a_{0.1} \quad a_{0.2} \quad a_{0.3} \quad \cdots \quad a_{0.P-1}$$

soient demème les ~~*n grou*~~ systèmes de lettres conjointes

$$\begin{array}{ccccccc} a_{2.0} & a_{2.1} & a_{2.2} & a_{2.3} & \cdots & a_{2.P-1} \\ a_{3.0} & a_{3.1} & a_{3.2} & a_{3.3} & \cdots & a_{3.P-1} \\ & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & & & & & \end{array}$$

Nous ~~aurons~~ ^obtiendrons ainsi^ en tout P^2 lettres. Si le nombre total des lettres n pas épuisé, on prendra ^un^ troisième indice, en sorte que

Corrected silently to 'n'est pas' in C*ms*, L1846.

$$a_{m.n.0} \quad a_{m.n.1} \quad a_{m.n.2} \quad a_{m.n.3} \cdots \quad a_{m.n.P-1}$$

soit en général un système de lettres conjointes. Et l'on parviendra ainsi à cette conclusion que $N = P^{\mu}$, μ étant un certain nombre égal à celui des indices différents dont on aura eu besoin. La forme générale des lettres sera

$$a_{k_1.k_2.k_3 \ldots k_{\mu}}$$

$k_1, k_2, k_3, \ldots k_{\mu}$ étant des indices qui peuvent prendre chacun les P valeurs 0, 1, 2, 3, …, $P-1$.

On voit aussi ^par la manière dont nous avons procédé^ que dans le groupe H, toutes les substitutions seront de la forme

$$\left(a_{k_1.k_2.k_3 \ldots k_{\mu}} \quad , \quad a_{\varphi(k_1).\psi(k_2).\chi(k_3). \ldots .\sigma(k_{\mu})} \right)$$

a system of related letters. Let

$$a_0, \quad a_{0,1}, \quad a_{0,2}, \quad \ldots, \quad a_{0,P-1}$$

be P related letters all situated in the first vertical column. (It is clear that we can arrange that it should be like this by interchanging the order of the horizontal lines.)

Similarly, let

$$a_{1,0}, \quad a_{1,1}, \quad a_{1,2}, \quad a_{1,3}, \quad \ldots., \quad a_{1,P-1}$$

be P related letters all situated in the second vertical column and such that

$$a_{1,0}, \quad a_{1,1}, \quad a_{1,2}, \quad a_{1,3}, \quad \ldots, \quad a_{1,P-1}$$

belong respectively to the same horizontal lines as

$$a_{0,0}, \quad a_{0,1}, \quad a_{0,2}, \quad a_{0,3}, \quad \ldots, \quad a_{0,P-1}.$$

Similarly, let

$$\begin{array}{llllll} a_{2,0}, & a_{2,1}, & a_{2,2}, & a_{2,3}, & \ldots, & a_{2,P-1} \\ a_{3,0}, & a_{3,1}, & a_{3,2}, & a_{3,3}, & \ldots, & a_{3,P-1} \\ & . & . \; . & . \; . & & \\ a & & & & & \end{array}$$

be systems of related letters. ~~We will have~~ In this way we will obtain P^2 letters in all. If the total number of letters is not exhausted a third index will be taken in such a way that

$$a_{m,n,0}, \quad a_{m,n,1}, \quad a_{m,n,2}, \quad a_{m,n,3}, \ldots, \quad a_{m,n,P-1}$$

shall in general be a system of related letters. And in this way one will eventually come to the conclusion that $N = P^\mu$, μ being a certain number, equal to the number of different indices which will have been needed. The general form of the letters will be

$$a_{k_1, k_2, k_3, \ldots, k_\mu}$$

k_1, k_2, k_3, $\ldots$, k_μ being indices, each of which can take the P values 0, 1, 2, 3, $\ldots$, $P-1$.

From the way in which we have proceeded it may also be seen that in the group H all the substitutions will be of the form

$$\left(a_{k_1, k_2, k_3, \ldots, k_\mu}, \quad a_{\phi(k_1), \psi(k_2), \chi(k_3), \ldots, \sigma(k_\mu)}\right)$$

puisqu⌜e⌝'~~on chaque ind~~ ~~obtient chaque système conjoint en ne laissant qu'un seul~~ chaque indice correspond à un système de lettres conjointes.

Si P n'est pas un nombre premier, on raisonnera sur le groupe de permutations de l'un quelconque des systèmes de lettres conjointes, comme sur le groupe G *en remplaçant chaque indice par un ^certain^ nombre de nouveaux indices*, et l'on trouvera $P = R^{\alpha}$, et ainsi desuite, d'où enfin $N = p^{\nu}$, p étant ⌜un nombre premier.⌝

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~~un nombre premier.~~

It seems that 'On voit qu'ainsi' was written, then crossed out and changed to 'Ainsi le dernier groupe élément'; this was then crossed out and 'Ainsi' re-used. In the *ms* 'élément' is clear but the phrase breaks off; possibly 'élémentaire' incomplete.

~~On voit qu'A⌜a⌝insi le dernier groupe élément ^on pourra, dans la suite^ des décompositions parvenir à un groupe ^irréductible^ dont les substitutions seront toutes exprimées comme il suit~~

Des Équations primitives de degré p^2.

~~$\left(a_{k.k.k\ldots k}\ ;\ a_{k+\alpha.k+\alpha.k+\alpha,\ldots.k+\alpha}\right)$~~

~~Les ν indices étant pris relativement au module p, et $\alpha_1, \alpha_2 \cdots \alpha_\nu$ ^étant^ des nombres constants.~~

~~Nous allons chercher maintenant quelles peuvent être les autres substitutions d'un groupe primitif qui répond à une équation soluble par radicaux.~~

In *ms* lines are shorter and the aborted header 'Des Equations' appears in mid-line following 'par radicaux'.

~~*Des Equations*~~

~~C'est~~ Arrètons-nous un moment, pour ~~éclaircir par des exemples ce que nous venons de dire, et traitons~~ ^~~de suite~~^ traiter de suite les équations primitives ~~des~~⌜'un⌝ ~~degrés~~ p^2 ~~et p^3~~, p étant premier impair. ⌜(Le cas de $p = 2$ a été examiné.)⌝

Si une équation du degré p^2 est soluble par radicaux, supposons⌜-la⌝ d'abord ~~qu'elle soit~~ telle, qu'elle devienne non primitive par une extraction de radical.

Soit donc G ~~le~~^un^ group primitif de p^2 lettres qui se partage en n groupes non primitifs ~~de degré~~ conjugués à H.

Les lettres devront nécessairement, dans le groupe H, se ranger ainsi,

$$\begin{array}{cccccc} a_{0.0} & a_{0.1} & a_{0.2} & a_{0.3} & \cdots\cdots & a_{0.p-1} \\ a_{1.0} & a_{1.1} & a_{1.2} & a_{1.3} & \cdots\cdots & a_{1.p-1} \\ a_{2.0} & a_{2.1} & a_{2.2} & a_{2.3} & \cdots\cdots & a_{2.p-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{p-1.0} & a_{p-1.1} & a_{p-1.2} & a_{p-1.3} & \cdots\cdots & a_{p-1.p-1} \end{array}$$

Chaque ligne ~~de lettres~~ horizontale et ~~de~~ chaque ligne verticale étant un système de lettres conjointes.

~~Le groupe de permutations des lettres de la première ligne~~ ^Si l'on permute entre elles les lignes horizontales, le groupe ~~premier~~ que l'on obtiendra^ étant primitif et de degré premier, ne devra contenir que des substitutions de la forme

$$\left(a_{k.k}\ ,\ a_{mk+n.k}\right)$$

because ~~each related system is obtained by letting only a single~~ each index corresponds to a system of related letters.

If P is not a prime number, one will reason on the group of permutations of any one of the systems of related letters in the same way as on G, replacing each index by a certain number of new indices, and one will find that $P = R^{\alpha}$, and so on, from which ultimately $N = p^{\nu}$, p being a prime number.

~~One sees that in this way Thus the last element[ary?] group~~ Thus in the sequence ~~of decompositions one could reach an irreducible group of which the substitutions will all be expressible as follows:~~

~~$$\left(a_{k_1,k_2,k_3,\ldots,k_\nu},\quad a_{k_1+\alpha_1,k_2+\alpha_2,k_3+\alpha_3,\ldots,k_1+\alpha_\nu},\right),$$~~

~~the ν indices being taken relative to the modulus p, and $\alpha_1, \alpha_2, \ldots, \alpha_\nu$ being constants.~~

~~We shall now seek what the other substitutions can be in a primitive group which corresponds to an equation that is soluble by radicals~~

On primitive equations of degree p^2.

Let us pause for a moment to ~~clarify by examples what we have just said, and let us treat straightaway~~ treat straightaway primitive equations of degree p^2 ~~and p^3~~, where p is an odd prime. (The case of $p = 2$ has been examined.)

If an equation of degree p^2 is soluble by radicals, let us suppose to begin with that it is such that it becomes non-primitive by an extraction of a radical.

Thus let G be a primitive group of p^2 letters which may be partitioned into n non primitive groups conjugate to H.

In the group H the letters must necessarily be arranged thus:

$$\begin{array}{ccccccc}
a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} & \ldots & a_{0,p-1} \\
a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} & \ldots & a_{1,p-1} \\
a_{2,0} & a_{2,1} & a_{2,2} & a_{2,3} & \ldots & a_{2,p-1} \\
 & \cdot & \cdot & \cdot & \cdot & \\
a_{p-1,0} & a_{p-1,1} & a_{p-1,2} & a_{p-1,3} & \ldots & a_{p-1,p-1}
\end{array}$$

each horizontal line and each vertical line being a system of related letters.

~~The group of permutations of the letters of the first line~~ If one permutes the horizontal lines amongst themselves, the ~~first~~ group that is obtained, being primitive and of prime degree, must contain only substitutions of the form

$$\left(a_{k_1,k_2},\ a_{mk_1+n,\,k_2}\right),$$

Although ‘~~et~~’ is at beginning of line ‘Il en sera ...’, it looks as if ‘et’ originally directly followed the preceding displayed formula.

^Les indices étant pris relativement au module p.^

~~et~~ Il en sera de même pour les ~~lettres~~ lignes verticales, qui ne pourront donner que des substitutions de la forme

$$\left(a_{\underset{1}{k}.\underset{2}{k}} \quad , \quad a_{\underset{1}{k}.\underset{2}{qk+r}}\right)$$

Donc enfin toutes les substitutions du groupe H seront de la forme

$$\left(a_{\underset{1}{k}.\underset{2}{k}} \quad , \quad a_{\underset{1}{mk+n}.\underset{2}{mk+n}}\right)$$

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Si un groupe G se partage en n groupes conjugués à celui que nous venons de décrire, toutes les substitutions du groupe G devront transformer ^les^ ~~l'~~unes dans ^les^ autres les substitutions circulaires du groupe H, qui sont toutes écrites comme il suit

$$\left(a_{\underset{1}{k}.\underset{2}{k}.\cancel{\underset{3}{k}/\cdots/\underset{\nu}{k}}} \quad , \quad a_{\underset{1}{k+\alpha}.\underset{2}{k+\alpha}.\cancel{\underset{3}{k+\alpha}/\cdots/\underset{\nu}{k+\alpha}}}\right) \qquad (a)$$

Supposons donc que l'une des substitutions du groupe G ~~soit celle-ci~~ se forme en remplacant respectivement

$$\begin{array}{ccc}
\underset{1}{k} & & \varphi_1\left(\underset{1}{k}\ ,\ \underset{2}{k}\ \ulcorner\urcorner\ \cancel{\underset{3}{k}/\cdots/\underset{\nu}{k}}\right) \\
 & \text{par} & \\
\underset{2}{k} & & \varphi_2\left(\underset{1}{k}\ ,\ \underset{2}{k}\ \ulcorner\urcorner\ \cancel{\underset{3}{k}/\cdots/\underset{\nu}{k}}\right) \\
\cancel{\underset{3}{k}} & \cancel{\text{par}} & \cancel{\varphi_3\left(\underset{1}{k}/\underset{2}{k}/\cdots/\underset{3}{k}/\cdots/\underset{\nu}{k}\right)} \\
\cancel{//} & & \cdot\quad\cdot\quad\cdot\quad\cdot \\
\cancel{\underset{\nu}{k}} & & \cancel{\varphi_\nu\left(\underset{1}{k}/\underset{2}{k}/\cdots/\underset{3}{k}/\cdots/\underset{\nu}{k}\right)}
\end{array}$$

Si dans les fonctions $\varphi_1 \quad \varphi_2 \quad \cancel{\varphi_3/\cdots/\varphi_\nu}$ on substitue pour $\underset{1}{k}$ ⌜et⌝ $\underset{2}{k}\ \cancel{,\underset{3}{k}/}\ \ldots$ les valeurs $\underset{1}{k}+\underset{1}{\alpha}\ ,\ \underset{2}{k}+\underset{2}{\alpha}\ \cancel{\underset{3}{k}+\underset{3}{\alpha},\ldots,k}$ il devra venir des resultats de la forme

$$\varphi_1+\beta_1 \qquad \varphi_2+\beta_2\ ,$$

et delà il est aisé de conclure immédiatement que les substitutions du groupe G doivent ètre toutes comprises dans la formule

$$\left(a_{\underset{1}{k}.\underset{2}{k}} \quad , \quad a_{\underset{1}{mk+n}\underset{2}{k+\alpha}.\underset{2}{mk+n}\underset{1}{k+\alpha}}\right) \qquad (A)$$

Slip: for n_2k_1 in (A) read n_2k_2. (Overlooked in L1846, noted in BA1962.)

On missing reference see Note 4, p. 196.

Or nous savons par le n° que les substitutions du groupe G ne peuvent embrasser que p^2-1 ou p^2-p lettres. Ce n'est point p^2-p, puisque ^dans ce cas^ le groupe G serait non primitif. ⌜Si donc dans le groupe G on ~~fait ab~~ ne considère⌝ ~~P[?] Mais~~

A mysterious cross to read ‘ce × cas’ could possibly turn ‘ce’ into ‘cet’ but more probably is a mark deleted.

the indices being taken relative to the modulus p.

The same will be the case for the ~~letters~~ vertical lines, which can only give substitutions of the form

$$\left(a_{k_1,k_2}\ ,\ a_{k_1,qk_2+r}\right)\ .$$

The outcome is therefore that all the substitutions of the group H will be of the form

$$\left(a_{k_1,k_2}\ ,\ a_{m_1k_1+n_1\,,\,m_2k_2+n_2}\right)\ .$$

If a group G is partitioned into n groups conjugate to the one we have just described, all the substitutions of the group G will have to transform amongst themselves the circular substitutions of the group H, which can all be written as follows:

$$\left(a_{k_1,k_2}\ ,\ a_{k_1+\alpha_1\,,\,k_2+\alpha_2}\right)\ . \qquad (a)$$

Suppose, then, that one of the substitutions of the group G ~~is this~~ is formed by replacing, respectively

$$\begin{matrix} k_1 \\ \\ k_2 \end{matrix} \quad \text{by} \quad \begin{matrix} \varphi_1\,(k_1\ ,\ k_2) \\ \\ \varphi_2\,(k_1\ ,\ k_2) \end{matrix}\ .$$

If in the functions φ_1, φ_2 one substitutes the values $k_1+\alpha_1$, $k_2+\alpha_2$ for k_1 and k_2 the outcome must be results of the form

$$\varphi_1+\beta_1\,, \qquad \varphi_2+\beta_2\,,$$

and from that it is easy to conclude immediately that the substitutions of the group G must all be comprised in the formula

$$\left(a_{k_1,k_2}\ ,\ a_{m_1k_1+n_1k_2+\alpha_1\,,\,m_2k_1+n_2k_2+\alpha_2}\right) \qquad (A)$$

Formula (*A*) amended: see note opposite.

Now we know from no. that the substitutions of the group G can involve only p^2-1 or p^2-p letters. It cannot be p^2-p because in this case the group G would be non-primitive. If then in the group G one considers only ~~But~~

~~il faut~~ que les ~~substitutions~~ ^permutations^ où la lettre $a_{0.0}$, par exemple, conserve toujours la mème place, on n'aura que des substitutions ~~cir~~ de l'ordre $p^2 - 1$ entre les $^p^2 - 1^$ autres lettres.

In BA1962 the syllable 'cir' is (very plausibly) completed to 'circulaires'.

Mais rappelons-nous ici que ~~dans~~ c'est simplement pour la démonstration, que nous avons supposé que le groupe ^primitif^ G se partageàt en groupes conjugués non-primitifs. ~~Mais~~[?] Comme cette condition n'est nullement nécessaire, les groupes seront souvent beaucoup plus composés.

Il s'agit donc de reconnaître dans quel cas ces groupes pourront admettre des substitutions où $p^2 - p$ lettres seulement varieraient, et cette recherche va nous

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retenir quelque temps.

Soit donc G un groupe qui contienne quelque substitution de l'ordre $p^2 - p$, ~~mais qui se décompose en p un nombre premier de groupes conjugués ^à H^, et supposons que ces groupes conjugués ne contiennent pas de pareilles substitutions~~. ~~Il est clair~~ ^Je dis d'abord^ que toutes les substitutions de ce groupe seront linéaires, c'est à dire de la forme (A); ~~puisque ^toutes^ les seules substitutions circulaire de H sont de la forme (a).~~

Galois first wrote 'Puisque les seules' or perhaps 'Puisque les seules substitutions'; then replaced 'seules' with 'toutes'; later all was crossed out.

La chose est ^reconnue^ vraie pour les substitutions de l'ordre $p^2 - 1$. ^Il suffit donc de la démontrer pour celles de l'ordre $p^2 - p$.^ Ne considérons donc que⌜'un⌝ ~~les substitutions~~ groupe où les substitutions seraient toutes ou de l'ordre p^2 ou de l'ordre $p^2 - p$. (Voyez l'endroit cité).

First occurrence of 'ou' misread as ***m*** in C*ms*, **f. 46 a**.

Alors les p lettres qui dans ~~les~~ ^une^ substitution de l'ordre $p^2 - p$ ne varieront pas, devront ètre des lettres conjointes. Supposons que ces lettres conjointes soient

$$a_{0.0} \quad a_{0.1} \quad a_{0.2} \quad \cdots \quad a_{0.p-1}$$

~~Celà posé, il y aura ou il n'y aura pas de substitutions de l'ordre $p^2 - p$ dont la période soit de p termes.~~

Where I read 'nécessairement' BA1962 has 'respectivement'; reading apart, the former makes more sense.

~~S'il ~~y~~ n'y en a pas, on pourra partager ~~le~~ toutes les substitutions de l'ordre $p^2 - p$ seront nécessairement de la forme~~

$$\left(a_{k.k} \quad , \quad a_{mk.\,mk+n}\right)$$

~~et parconséquent linéaires.~~

~~Si maintenant il y avait des substitutions de l'ordre $p^2 - p$ dont la période fut de p termes~~

Word 'les' crossed out, 'ces' inserted, then changed back to 'les'.

Nous pouvons ~~faire dép~~ déduire toutes ~~les~~^ces^ substitutions où ces p lettres ne changent pas de place, nous pouvons les déduire de substitutions de la forme

$$\left(a_{k.k} \quad , \quad a_{k.\varphi k}\right)$$

~~it is necessary that~~ the ~~substitutions~~ permutations where the letter $a_{0,0}$, for example, always stays in the same place, one will only have substitutions of order $p^2 - 1$ among the $p^2 - 1$ other letters.

But let us recall at this point that it is simply for the demonstration that we have supposed that the primitive group G is partitioned into conjugate non-primitive groups. Since this condition is in no way necessary, the groups will often be much more complicated.

It is a matter therefore of recognising the cases in which these groups may admit some substitutions where only $p^2 - p$ letters vary, and this investigation will detain us a little.

Let then G be a group which contains some substitution of order $p^2 - p$ ~~but which decomposes into a prime number of groups conjugate to H, and suppose that these conjugate groups contain no such substitutions. It is clear~~. I say first that all the substitutions of this group will be linear, that is to say, of the form (A) ~~because the only~~ all the circular substitutions of H ~~are of the form (a).~~

This is known to be true for substitutions of order $p^2 - 1$. It is sufficient therefore to prove it for those of order $p^2 - p$. Consider, then, only ~~the substitutions~~ a group where the substitutions will all either be of order p^2 or of order $p^2 - p$. (See the cited place).

Now the p letters which do not vary in a substitution of order $p^2 - p$ must be related letters. Suppose that these related letters are

$$a_{0,0}\,, \quad a_{0,1}\,, \quad a_{0,2}\,, \quad \ldots\,, \quad a_{0,p-1}\,.$$

~~That said, there will or will not be substitutions of order $p^2 - p$ of which the period is p terms.~~

~~If there are none one could partition the all the substitutions of order $p^2 - p$ will necessarily be of the form~~

$$\left(a_{k_1,k_2}\,, \quad a_{m_1k_1,m_2k_2+n_2}\right)$$

~~and consequently linear~~

~~If now there were substitutions of order $p^2 - p$ of which the period were p terms~~

We can deduce all the substitutions where these p letters do not change their place, we can deduce them from substitutions of the form

$$\left(a_{k_1,k_2}\,, \quad a_{k_1,\varphi k_2}\right)$$

et de substitution de l'ordre $p^2 - p$ dont la periode serait de p termes. (Voyez encore l'endroit cité.)

Word 'substitution' singular in *ms*, C*ms*; corrected to plural in L1846, BA1962.

Les premiers doivent nécessairement, pour que le groupe jouisse de la propriété voulue, se réduire à la forme

$$\left(a_{k.\,k}{}_{1\ 2} \quad , \quad a_{k.\,m\,k}{}_{1\ \ 2}\right)$$

d'après ce qu'on a vu pour les équations de degré p.

Quant aux substitutions ~~qui~~ ^dont la période^ seraient de p termes, ~~il nous~~ comme elles sont conjuguées aux précédentes, nous pouvons supposer un groupe qui ^les^ contienne sans contenir celles-ci. ~~, et par conséquent~~ ^Donc^ elles devront transformer les substitutions circulaire (a) les unes

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dans les autres. Donc elles seront aussi linéaires.

Nous sommes donc arrivés à cette conclusion que le groupe primitif de permutations de p^2 lettres doit ne contenir que des substitutions de la forme (A).

Maintenant, prenons le groupe total que l'on obtient en opérant sur l'expression

$$a_{k.\,k}{}_{1\ 2}$$

toutes les substitutions linéaires possibles, ~~qui sont au nombre de $p^2(p^2-1)(p^2-p)$~~, et cherchons quels sont les diviseurs de ce groupe qui peuvent jouir de la propriété voulue pour la résolubilité des équations.

~~Soit~~ ~~Ne considerons~~ ~~Si donc~~ Quel est d'abord le nombre total des substitutions linéaires ? Premiérement, il est clair que toute transformation de la forme

$$k_1.k_2 \qquad m_{11}k_1 + n_{12}k_2 + \alpha_1 \,.\, m_{21}k_1 + n_{22}k_2 + \alpha_2$$

ne sera pas pour celà une substitution, car il faut dans une substitution, qu'à chaque lettre de la première permutation il ne reponde qu'une seule lettre de la seconde et réciproquement.

Si donc on prend une lettre quelconque $a_{l.\,l}{}_{1\ 2}$ de la seconde permutation, et que l'on veut ~~rep~~ remonter à la lettre correspondante dans la première, on devra trouver une lettre $a_{k.\,k}{}_{1\ 2}$ où les indices k_1 . k_2 seront parfaitement déterminés. Il faut donc que quels que soient l_1 et l_2 on ait par les deux équations

On word 'veut' see Note 5, p. 196.

$$m_{11}k_1 + n_{12}k_2 + \alpha_1 = l_1 \qquad m_{21}k_1 + n_{22}k_2 + \alpha_2 = l_2$$

des valeurs de k_1 et k_2 finies et déterminées. ~~Donc on doit avoir~~ Ainsi la condition pour qu'une pareille transformation soit réellement une substitution que $m_{12}n_{21} - m_{21}n_{12}$ ne soit ni nul ni divisible par le module p, ce qui est la mème chose.

Word 'est' missing before 'que' in *ms*, C*ms* is supplied in L1846.

and from substitutions of order $p^2 - p$ whose period will be of p terms. (See the cited place again.)

In order that the group should enjoy the desired property, the first must necessarily reduce to the form

$$\left(a_{k_1,k_2}\ ,\quad a_{k_1,mk_2}\right)$$

by what we have seen for equations of degree p.

As for the substitutions whose period will be of p terms, since they are conjugate to the preceding ones, we can suppose a group which contains them without containing the latter. Thus they must transform the circular substitutions (a) amongst each other.

Thus they too will be linear.

We have reached the conclusion therefore that the primitive group of permutations of p^2 letters must contain only substitutions of the form (A).

Now let us take the complete group that is obtained by operating on the expression

$$a_{k_1,k_2}$$

with all possible linear substitutions, ~~which are $p^2(p^2-1)(p^2-p)$ in number,~~ and let us seek the divisors of this group which can have the property desired for solubility of equations.

~~Let~~ ~~We consider only~~ ~~Then if~~ To begin with, what is the total number of linear substitutions? First, it is clear that not every transformation of the form

$$k_1, k_2 \qquad m_1k_1 + n_1k_2 + \alpha_1\ , m_2k_1 + n_2k_2 + \alpha_2$$

will be a substitution, because in a substitution it is necessary that to each letter of the first permutation there corresponds just one single letter of the second and conversely.

If then one takes an arbitrary letter a_{l_1,l_2} of the second permutation and wishes to go up to the corresponding letter in the first, one must find one letter a_{k_1,k_2} in which the indices $k_1,\ k_2$ are perfectly determined. It is necessary therefore that whatever l_1 and l_2 may be one should get from the two equations

$$m_1k_1 + n_1k_2 + \alpha_1 = l_1\,, \qquad m_2k_1 + n_2k_2 + \alpha_2 = l_2$$

finite and well defined values of k_1 and k_2. ~~Therefore one must have~~ Thus the condition that such a transformation should really be a substitution is that $m_1n_2 - m_2n_1$ should be neither zero nor divisible by the modulus p, which come to the same thing.

*Je dis maintenant que bien que ce groupe à substitutions linéaires ~~ne jouisse~~ ^n'appartienne^ pas toujours, comme on le verra, à des ~~gr~~ équations solubles par radicaux, ~~on pourra~~ il jouira toutefois de ~~la~~ ^cette^ propriété, que si dans une ^quelconque^ de ses substitutions il y a n lettres de fixes, n divisera le nombre des lettres. Et en effet, quel que soit le nombre des lettres qui restent fixes on pourra exprimer cette circonstance par ~~un ⌈certain⌉ nombre~~ ^des^ équations linéaires, qui donneront tous les indices ^de l'une des lettres fixes^ au moyen d'un ~~seule~~ certain nombre ~~de l'au~~ d'entre eux. Donnant à chacun de ces indices, restés arbitraires, p valeurs, on aura p^m systèmes de valeurs, m étant un certain nombre. Dans le cas qui nous occupe m est nécessairement < 2 et est par conséquent 0 ou 1. Donc le nombre des ~~valeurs~~ ^substitutions^ ne saurait ètre plus grand que

$$p^2(p^2-1)(p^2-p) \qquad *$$

Galois wrote 'si nous trouvons le nombre des permutations que l'on peut avoir dans ce cas au mo[ins]' but broke off to alter the sentence.

Ne considérons ~~que~~ maintenant que les substitutions linéaires où la lettre $a_{0.0}$ ne varie pas; si ^dans ce cas^ nous trouvons le nombre ^total^ des permutations ~~que l'on peut avoir dans ce cas au mo~~ du groupe qui contient toutes les substitutions linéaires possibles, il nous suffira de multiplier ce nombre par p^2.

Or, premiérement, ~~en supposant p lettres fixes~~ ^ne touchant pas à l'indice k_2^, toutes les substitutions de la forme

That what is so severely crossed out is m_2k_1 is conjectural—this is unreadable.

$$\left(a_{\underset{1}{k}.\underset{2}{k}} \quad , \quad a_{\underset{1}{m}\underset{1}{k}.\cancel{\underset{2}{m}\underset{1}{k}+}\underset{2}{k}}\right)$$

donneront en tout ~~p^2-p~~ ^$p-1$^ substitutions. On enaura ^p^2-p^ en ajoutant au terme k_2 le terme m_2k_1 ainsi qu'il suit:

In BA1962: the page-break is misplaced after the formula; label (m) is missing; the crossed-out material that follows is overlooked.
In *ms* the label could be (m'), but the dash is an accidental mark.

40 b

$$\left(\underset{1}{k}.\underset{2}{k} \quad , \quad \underset{1}{m}\underset{1}{k}\,.\underset{2}{m}\underset{1}{k}+\underset{2}{k}\right) \qquad (m)$$

^~~Si dans~~^ ~~chacune des substitutions (m) on ajoute au premier terme. Il suffit maintenant de chercher combien de [?]~~

D'un autre coté, il est aisé de trouver un groupe linéaire ~~où~~ ^de p^2-1 permutations^ tel que dans chacune de ses substitutions, toutes les lettres à l'exception ~~d'un~~ ^de^ $a_{0.0}$ varient. Car en remplaçant ~~par~~ le double indice $\underset{1}{k}.\underset{2}{k}$ par l'indice simple k_1+ik_2, i étant une ~~sol~~ racine primitive de la congruence

$$x^{p^2-1}-1=0 \quad (\text{mod. } p)$$

il est clair que toute substitution de la forme

The very severe crossing out of the formula in top left deletes also the first occurrence of a. But that seems unintended.

$$\cancel{(\underset{1}{m}+\underset{2}{m}i}$$

$$\left(a_{\underset{1}{k}+\underset{2}{k}i} \quad , \quad a_{(\underset{1}{m}+\underset{2}{m}i)\cdot(\underset{1}{k}+\underset{2}{k}i)}\right)$$

sera une substitution linéaire; mais dans ces substitutions aucune lettre ne reste à

I say now that, although this group of linear substitutions ~~does not enjoy~~ does not always, as we shall see, belong to equations soluble par radicals, ~~it will be possible~~ it will always enjoy this property, that if in an arbitrary one of its substitutions there are n letters fixed, n will divide the number of letters. And indeed, whatever the number of letters that stay fixed may be, one will be able to express this fact by ~~a certain number~~ some linear equations which will give all the indices of one of the fixed letters by means of a ~~single~~ certain number from among them. Giving to each of these indices, which remain arbitrary, p values, one will have p^m systems of values, m being some number. In the case we are treating m is necessarily < 2 and is consequently 0 or 1. Therefore the number of ~~values~~ substitutions cannot be greater than

$$p^2(p^2-1)(p^2-p).$$

Now consider just linear substitutions in which the letter $a_{0,0}$ does not change; if in this case we find the total number of permutations ~~that one can have in this case, at le[ast]~~ of the group which contains all possible linear substitutions it will suffice to multiply this number by p^2.

Well, firstly, ~~supposing p letters fixed~~ leaving the index k_2 untouched, all substitutions of the form

$$\left(a_{k_1,k_2}\ ,\quad a_{m_1k_1,k_2}\right)$$

will give $p-1$ substitutions in all. One will have p^2-p of them by adding to the term k_2 the term m_2k_1 as follows:

$$\left(k_1,k_2\ ,\quad m_1k_1, m_2k_1+k_2\right). \qquad (m)$$

~~If in each of the substitutions (m) one adds to the first term It now suffices to seek how many of these~~

On the other hand, it is easy to find a linear group of p^2-1 permutations such that in each of its substitutions, all the letters with the exception ~~of one~~ of $a_{0,0}$ move. For, on replacing the double index k_1, k_2 by the simple index k_1+ik_2, i being a primitive root of the congruence

$$x^{p^2-1}-1=0 \pmod p,$$

it is clear that every substitution of the form

$$\left(a_{k_1+k_2i}\ ,\quad a_{(m_1+m_2i),(k_1+k_2i)}\right)$$

will be a linear substitution; but in these substitutions no letter stays in

la mème place, et elles sont en nombre de $p^2 - 1$. Nous avons donc un système de $p^2 - 1$ permutations tel que dans chacune de ses substitutions toutes les lettres varient à l'exception de $a_{0.0}$. Combinant ces substitutions avec les $p^2 - p$ dont il est parlé plus haut nous aurons

$$(p^2 - 1)(p^2 - p) \quad \text{^substitutions^}$$

~~subs~~ Or, nous avons vu à priori que le nombre ^des substitutions où $a_{0.0}$ reste fixe^ ne pouvait ètre plus grand que $(p^2 - 1)(p^2 - p)$. Donc il est précisément égal à $(p^2 - 1)(p^2 - p)$, et le groupe linéaire total aura entout

$$p^2(p^2 - 1)(p^2 - p)$$

No clear reason for capital P. With 'Il reste à chercher' there is a change of pen from broad nib to fine.

Permutations. Il reste à chercher les diviseurs de ce groupe qui peuvent jouir de la propriété d'ètre solubles par radicaux. Pour cela nous allons faire une transformation qui a pour but d'abaisser autant que possible les équations générales de degré p^2 ~~qui auraient pour~~ ^dont le groupe^ serait linéaire.

Premièrement, comme les substitutions ~~linéaires~~ ^circulaires^ d'un pareil groupe sont telles, qu'⌜e⌝ ~~on peut y substit~~ toute ^autre^ substitution ~~linéaire~~ ^du groupe^ les transforme les unes dans les autres on pourra ~~les~~ abaisser ^l'équation^ d'un degré, et considérer une ~~équation~~ ^~~groupe~~^ ˅équation˅ ^du degré $p^2 - 1$^ dont le groupe n'aurait que des substitutions de la forme

C*ms*, L1846, BA1962 all have 'de degré', but 'du degré' clear in *ms*.

The variables were originally a, then over-written with b.

$$\left(\boldsymbol{b}_{k_1 . k_2} \quad , \quad \boldsymbol{b}_{\underset{1\,1}{mk} + \underset{1\,2}{nk} . \underset{2\,1}{mk} + \underset{2\,2}{nk}}\right)$$

41 a

les ~~n~~ ^$p^2 - 1$^ lettres étant

$$\begin{array}{ccccc} & b_{0.1} & b_{0.2} & b_{0.3} & \cdots \\ b_{1.0} & b_{1.1} & b_{1.2} & b_{1.3} & \cdots \\ b_{2.0} & b_{2.1} & b_{2.2} & b_{2.3} & \cdots \\ & \cdot & \cdot & \cdot & \cdot \end{array}$$

J'observe maintenant que ce groupe est non primitif ensorte que toutes les lettres où le rapport des deux indices est le mème, sont des lettres conjointes. ~~En effet toutes ces sub~~ Si l'on remplace par une seule lettre chaque système de lettres conjointes, on aura un groupe ~~de la forme~~ dont toutes les substitutions seront de la forme

Slip: $n_2 k_1$ should be $n_2 k_2$. Overlooked in L1846, corrected in P1897, noted in BA1962.

$$\left(b_{\frac{k_1}{k_2}} \quad , \quad b_{\frac{m_1 k_1 + n_1 k_2}{m_2 k_1 + n_2 k_1}}\right)$$

$\frac{k_1}{k_2}$ étant les nouveaux indices. En remplacant ce rapport par une seule ~~lettre~~

the same place, and the number of them is $p^2 - 1$. We therefore have a system of $p^2 - 1$ permutations such that in each of its substitutions all the letters vary with the exception of $a_{0,0}$. Combining these substitutions with the $p^2 - p$ which were discussed above we will have

$$(p^2 - 1)(p^2 - p) \quad \text{substitutions.}$$

Now we have seen *a priori* that the number of substitutions in which $a_{0,0}$ stays fixed cannot be larger than $(p^2 - 1)(p^2 - p)$. Therefore it is exactly equal to $(p^2 - 1)(p^2 - p)$, and the whole linear group will have in all

$$p^2(p^2 - 1)(p^2 - p)$$

permutations. It remains to seek the divisors of this group which can enjoy the property of being soluble by radicals. For that we are going to make a transformation, the point of which is to reduce as far as possible the general equations of degree p^2 ~~which will have for~~ of which the group will be linear.

First, since the ~~linear~~ circular substitutions of such a group are such that ~~one can subst[itute] there~~ every other ~~linear~~ substitution of the group transforms them amongst each other, one will always be able to reduce the equation by one degree, and consider ~~a group~~ an equation of degree $p^2 - 1$ whose group will have only substitutions of the form

$$\left(b_{k_1,k_2}\ ,\quad b_{m_1k_1+n_1k_2,\,m_2k_1+n_2k_2}\right),$$

the $p^2 - 1$ letters being

$$\begin{array}{ccccccc} & b_{0,1} & b_{0,2} & b_{0,3} & \dots \\ b_{1,0} & b_{1,1} & b_{1,2} & b_{1,3} & \dots \\ b_{2,0} & b_{2,1} & b_{2,2} & b_{2,3} & \dots \\ & . & . & . & . & . & . \end{array}$$

I observe now that this group is not primitive in that all the letters where the ratio of the two indices is the same are related letters. ~~Indeed, all these sub[stitutions]~~ If one replaces each system of related letters by a single letter one will have a group ~~of the form~~ all of the substitutions of which will be of the form

$$\left(b_{\frac{k_1}{k_2}}\ ,\quad b_{\frac{m_1k_1+n_1k_2}{m_2k_1+n_2k_2}}\right),$$

Formula amended. See note opposite.

$\frac{k_1}{k_2}$ being the new indices. On replacing this ratio by a single ~~letter~~

^indice^ k, on voit que les p ~~+~~ 1 lettres seront

$$b_0 \quad b_1 \quad b_2 \quad b_3 \quad \ldots . \quad b_{p-1} \quad b_{\frac{1}{0}}$$

et les substitutions seront de la forme

$$\left(k \quad , \quad \frac{mk+n}{rk+s}\right)$$

[~~2~~]Cherchons combien ^de lettres^ dans chacune de ces substitutions, restent à la mème place; il faut pour celà résoudre l'équation

Equation misprinted in all editions as $(rk+s)k - m(mk+n) = 0$. The otiose m is not in *ms*, nor in C*ms*.

$$(rk+s)k - (mk+n) = 0$$

qui aura deux ou ~~aucune~~ ^une^ ou aucune racine, suivant que $(m-s)^2 + 4nr$ sera résidu quadratique, nul, ou non-résidu quadratique. ⌜Suivant ces trois cas la substitution sera de l'ordre $p-1$, ou p, ou $p+1$.⌝

On peut prendre pour type des deux premiers ^cas^ les substitutions de la forme

$$(k \ , \quad mk+n)$$

ou la seule lettre $b_{\frac{1}{0}}$ ne varie pas. ~~Et delà on voit que le nombre~~ *et delà on voit que le nombre total des substitutions du groupe réduit est $(p+1)p(p-1)$.* ~~Quand le groupe contient~~ ^C'est après avoir ainsi réduit le groupe, que nous allons le traiter généralement.^

BA1962 has 1er. It is illegible in *ms*. I prefer 1o because it is what Galois often has elsewhere, e.g. in **f. 41 b**.

~~Maintenant~~ ~~N~~Nous ~~distinguerons deux cas. 1o Quand le~~ ^~~un pareil~~^ ~~groupe~~ ^~~soluble~~^ ~~con~~ ^chercherons d'abord dans quel cas^ ˬun diviseur de ce groupeˬ ⌜qui⌝ contiendrait des substitutions de l'ordre p, ⌜pourrait [~~22~~] ^appartenir^ à une équation soluble par radicaux.⌝

Dans ce cas, ~~il faudrait d'apres~~ l'équation serait primitive et elle ne pourrait ètre soluble par radicaux, àmoins que l'on n'eùt $p+1 = 2^n$, n étant un certain nombre.

Nous pouvons supposer que le groupe ne contienne que des substitutions de l'ordre p et de l'ordre $p+1$. Toutes les substitutions de l'ordre $p+1$ seront parconsequent semblables, et leur période sera [donc?] [de?] 2.

Page end, 'de 2' or 'donc 2'? See Note 6, p. 196.

41 b

Prenons donc l'expression

$$\left(k \ , \quad \frac{mk+n}{rk+s}\right)$$

et voyons dans quel cas cette substitution peut avoir une période de deux termes. Il faut pour celà que la substitution inverse se confonde avec elle. La substitution inverse est:

$$\left(k \ , \quad \frac{-sk+n}{rk-m}\right)$$

index k, one sees that the $p+1$ letters will be

$$b_0\,,\quad b_1\,,\quad b_2\,,\quad b_3\,,\quad \ldots\,,\quad b_{p-1}\,,\quad b_{\frac{1}{0}}$$

and the substitutions will be of the form

$$\left(k\,,\quad \frac{mk+n}{rk+s}\right).$$

Let us find how many letters in each of these substitutions stay in the same place; for that it is necessary to solve the equation

$$(rk+s)k-(mk+n)=0\,,$$

which will have two, or ~~no~~ one, or no root according as $(m-s)^2+4nr$ is a quadratic residue, zero, or a quadratic non-residue. According to these three cases the substitution will be of order $p-1$, or p, or $p+1$.

One may take as type of the two first cases the substitutions of the form

$$(k\,,\quad mk+n)$$

where only the letter $b_{\frac{1}{0}}$ does not change, and from that one sees that the total number of substitutions of the reduced group is $(p+1)p(p-1)$. ~~When the group contains~~ It is after having reduced the group in this way that we shall treat it in general.

~~Now we shall distinguish two cases. When such a soluble group con[tains?]~~ First we shall seek those cases in which a divisor of this group which contains some substitutions of order p might belong to an equation soluble by radicals.

In this case ~~it would be necessary by~~ the equation will be primitive and it could not be soluble by radicals unless one had $p+1=2^n$, n being some number.

We may suppose that the group contains only substitutions of order p and of order $p+1$. As a consequence all the substitutions of order $p+1$ will be similar and their period will [therefore?] be 2.

Take, then, the expression

$$\left(k\,,\quad \frac{mk+n}{rk+s}\right)$$

and let us see under what circumstances this substitution can have a period of two terms. For that it is necessary that the inverse substitution coincides with it. The inverse substitution is

$$\left(k\,,\quad \frac{-sk+n}{rk-m}\right).$$

Donc on doit avoir $m = -s$, et toutes ~~nos~~ les substitutions en question ~~devront être~~ seront

$$\left(k\ ,\quad \frac{mk+n}{k-m}\right)$$

ou encore

$$\left(k\ ,\quad m+\frac{N}{k-m}\right),$$

N étant un certain nombre *qui est le mème pour toutes les substitutions, puisque ces substitutions doivent ètre transformées les unes dans les autres par toutes les substitutions de l'ordre p,

$$(k\ ,\ k+m)\quad *.$$

Or ces substitutions doivent deplus ~~satisfaire à deux conditions. 1º elles doivent ^ètre^ de l'ordre $p+1$ et parconséquent la congruence~~

$$\cancel{K = m + \frac{N}{k-m}}$$

~~ne doit jamais etre soluble. Ce qui exige que N soit non-résidu quadratique. 2º elles doivent etres~~ ^ètre^ conjuguées les unes desautres. Si donc

$$\left(k\ ,\quad m+\frac{N}{k-m}\right)\quad ,\qquad \left(k\ ,\quad n+\frac{N}{k-n}\right)$$

sont deux pareilles substitutions, il faut que l'on ait

$$n+\frac{N}{\dfrac{N}{k-m}+m-n} = m+\frac{N}{\dfrac{N}{k-n}+n-m}$$

savoir $\cancel{m^2 = n^2}\quad \cancel{N+}(m-n)^2 = 2N.\qquad \cancel{m+m^2}$

Donc la différence entre ~~m~~ deux valeurs de m ne peut acquérir que deux valeurs différentes. Donc ~~il~~ m ne peut avoir plus de trois valeurs. Donc enfin $p = 3$. Ainsi c'est seulement dans ce cas que le groupe réduit pourra contenir des substitutions de l'ordre p.

Et en effet la réduite sera alors du 4ᵉ degré, et parconséquent soluble par radicaux.

Words 'appartient' and 'groupe' crossed out before completion

Chevalier added a footnote. See Note 8 on p. 197.

Nous savons ^par là^ qu'en général, ~~les~~ parmi les substitutions ~~qui appart~~ ^de notre^ groupe ^réduit^, il ne devra pas se trouver de ~~group~~ substitutions de l'ordre p. Peut-il y en avoir de l'ordre $p-1$? C'est ^ce^ que je vais rechercher⌜.⌝~~, et je ferai voir que d'abord qu'il ne~~

Therefore one must have $m = -s$, and all ~~our~~ the substitutions in question ~~must be~~ will be

$$\left(k\ ,\quad \frac{mk+n}{k-m}\right),$$

or again

$$\left(k\ ,\quad m+\frac{N}{k-m}\right),$$

N being a certain number, which is the same for all the substitutions because these substitutions must be transformed amongst each other by all the substitutions of order p,

$$(k\ ,\ k+m).$$

~~Now these substitutions must, moreover, satisfy two conditions. 1st they must be of order $p+1$ and consequently the congruence~~

$$\overline{K = m + \frac{N}{k-m}}$$

~~must never be soluble, which requires that N be a quadratic non-residue. 2nd they must be~~ Now these substitutions must, moreover, be conjugate to one another. If therefore

$$\left(k\ ,\quad m+\frac{N}{k-m}\right),\qquad \left(k\ ,\quad n+\frac{N}{k-n}\right)$$

are two such substitutions, it is necessary that one have

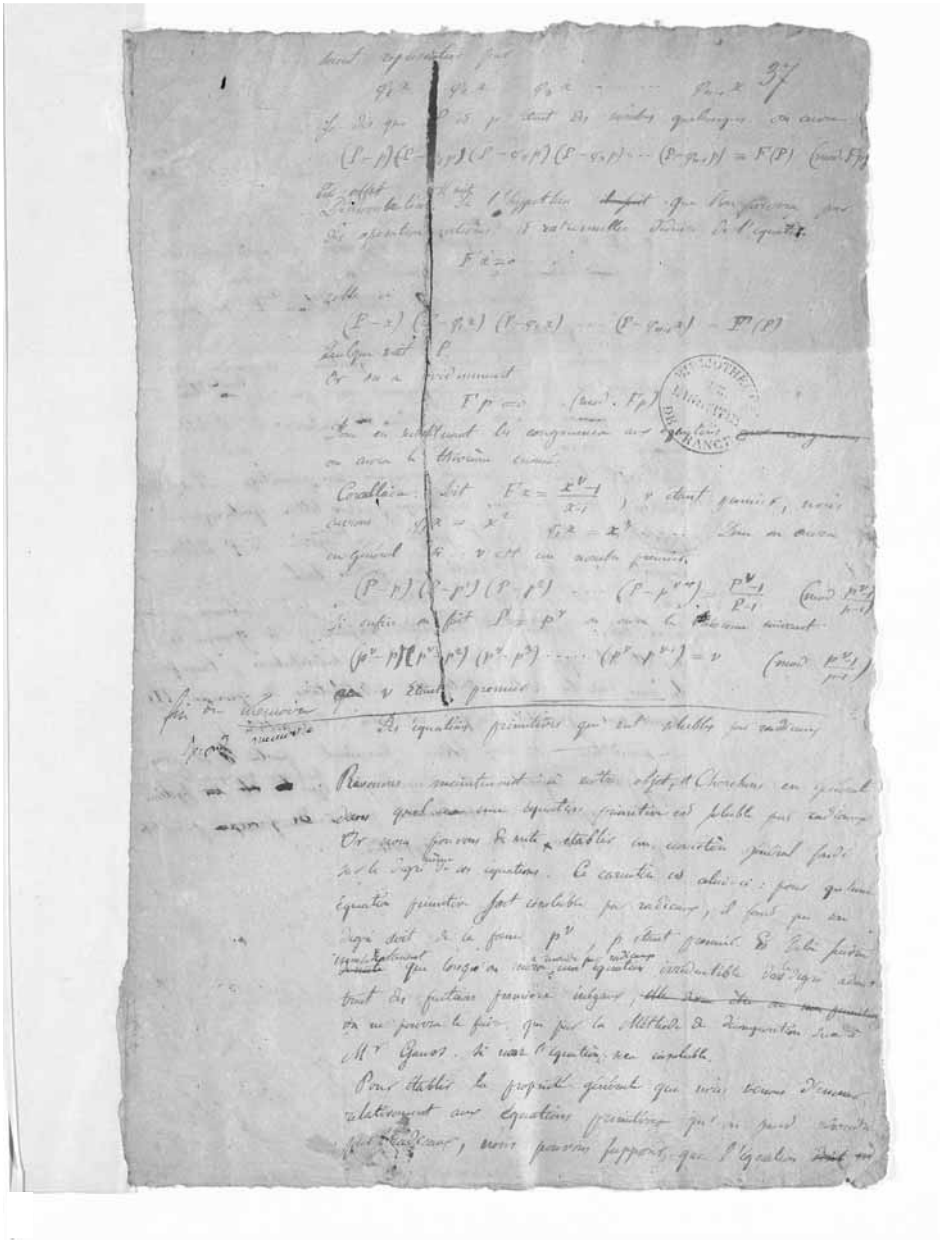
$$n+\frac{N}{\frac{N}{k-m}+m-n} = m+\frac{N}{\frac{N}{k-n}+n-m},$$

that is, $(m-n)^2 = 2N$.

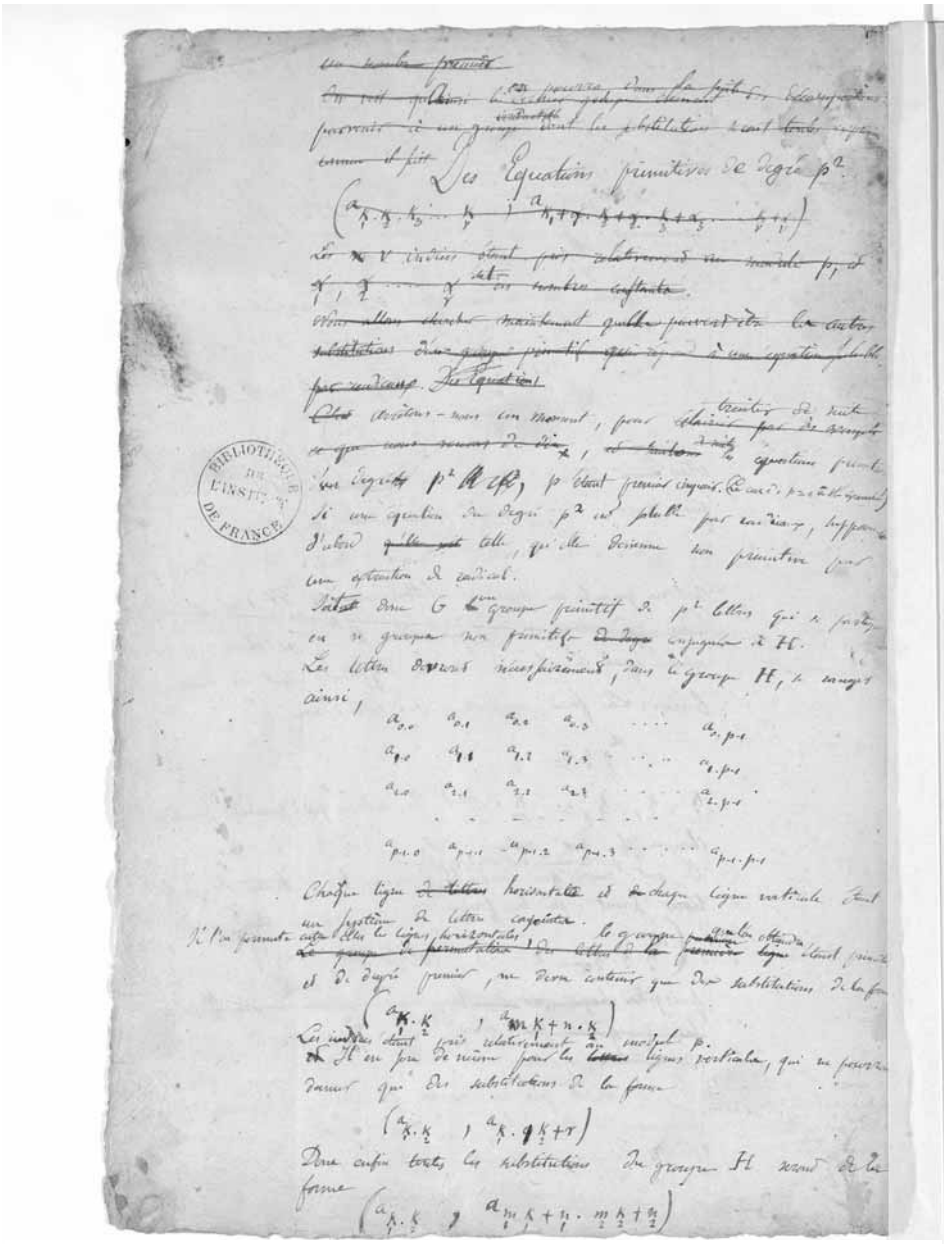
Therefore the difference between two values of m can take only two different values. Therefore m cannot have more than three values. Therefore finally $p = 3$. Thus it is only in this case that the reduced group could contain substitutions of order p.

And indeed, the reduced equation will then be of the 4th degree, and consequently soluble by radicals.

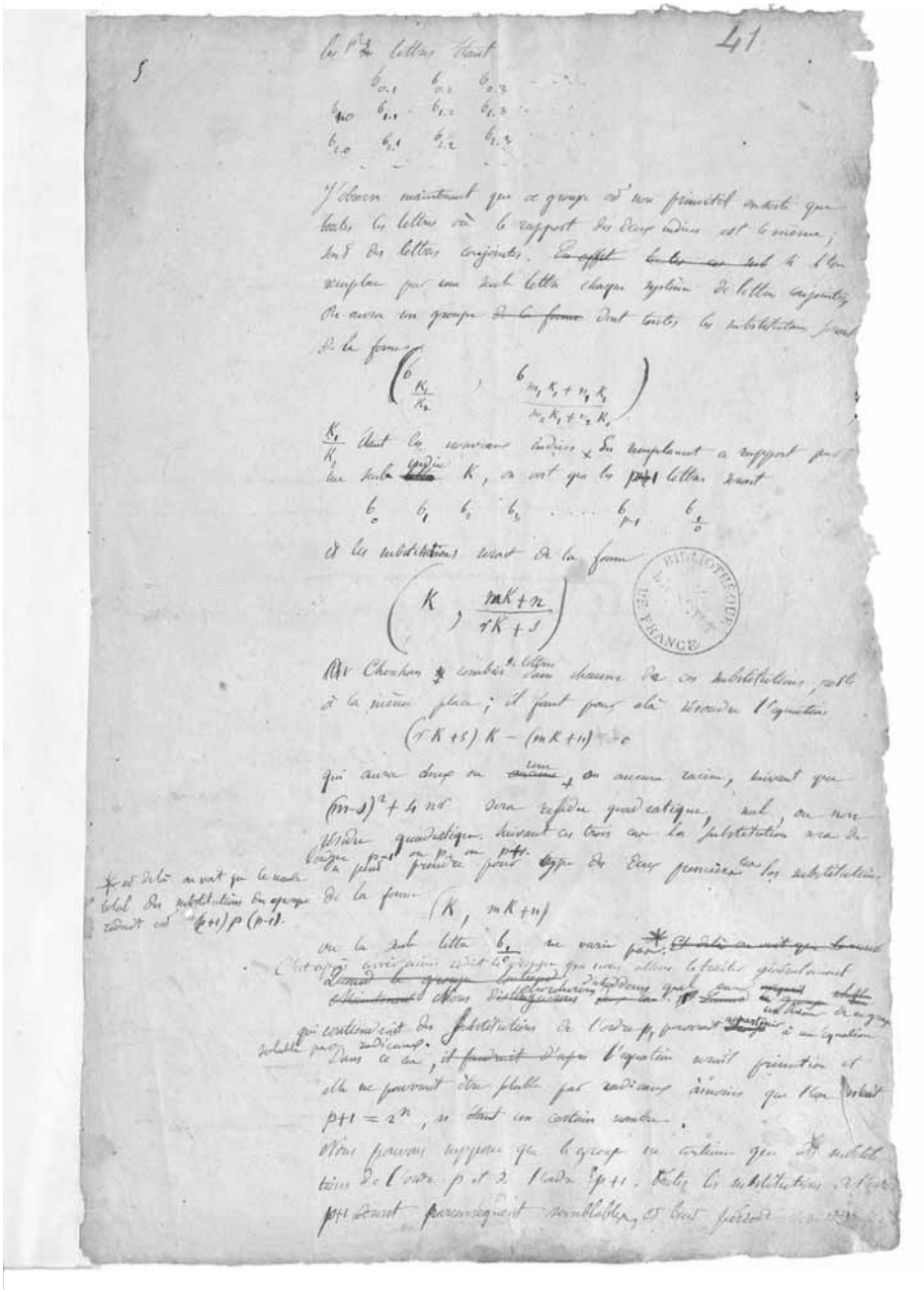
From that we know that in general, among the substitutions ~~which belong~~ of our reduced group there cannot be any ~~group~~ substitutions of order p. Can there be any of order $p-1$? That is what I shall investigate ~~and I propose to show first that ...~~



Second Memoir: Dossier 4, folio 37 recto.



Second Memoir: Dossier 4, folio 38 verso.



Second Memoir: Dossier 4, folio 41 recto.

V.2 Notes on the Second Memoir

NOTE 1: The *Second Mémoire* begins just over halfway down **f.37a**. Above it a horizontal line is drawn right across the page, which rules off 21 lines of crossed-out material (transcribed in T1906/7, pp. 238, 239 and BA1962, p. 128). The words 'fin de mémoire' appear just above the line in the left margin, which is wide enough for them to appear on one line. Just below the line the main text continues smoothly in exactly the same hand using the same pen and ink. The line appears to have been drawn later, though perhaps only a little later. And just below the line, over in the left margin just below the words 'fin de mémoire', appear the words 'Second mémoire', written in black ink much like what Galois was using on the eve of his duel, ink that is very different from the grey ink of the main text.

Ignoring that long line for the moment, below the first 21 lines of crossed-out text and formulae is a thin and quite short line. It is very much like the line that Galois used many times as a sort of 'sign-off' or cadence. Below it comes the sub-header

'Des équations primitives qui sont solubles par radicaux'

and then another thin and quite short line. After that the text 'Revenons maintenant à notre objet ...' continues very much as if all this, including the words 'fin de mémoire', the whole page in fact, was written at one sitting and without pause.

As has already been mentioned, just below the long line, but over in the left margin and just below the words 'fin de mémoire', are the words 'Second mémoire'. They slope slightly upwards, but otherwise are pretty well aligned with the sub-header 'Des équations primitives [...]'. The ink is much blacker than any other on the page. From the context and from the fact that the ink looks the same as (or very similar to) that of the letter of 29 May 1832 one might infer that these words were written on the eve of the duel when Galois wrote his letter to Chevalier with the famous paragraphs:

> On pourra faire avec tout cela trois mémoires.
> Le premier est écrit, [...].
> Le second contient des applications assez curieuses de la théorie des équations. [...].

Without detailed analysis of the ink we cannot be certain, but the inference has a very good chance of being correct. Certainly Bourgne [B & A (1962), p. 493] shows confidence in it.

NOTE 2: As has already been mentioned, the material that is above the line on **f.37a** (21 lines of text and formulae) is crossed out. The crossing-out is very different from most of Galois' deletions in that it is effected with a single thick vertical line, and in much blacker ink than one sees on the rest of the page. It is very similar to the crossing out of the address to members of the Institute, the preface to

the *Premier Mémoire* (**f. 2 a**, p. 106 above). It seems very probable that these two deletions were made on 29 May 1832 when Galois was editing his papers before going out to the fatal duel in the morning. Again, if an analysis of the ink were possible it should decide the matter.

NOTE 3: The copy made by Chevalier occupies Dossier 5 of the manuscript collection, folios 42–51 (folios 52, 53 are blank). It is remarkably faithful to the original (except that the opening words "Revenons maintenant à notre objet, et" are not included) and is neatly and legibly written in a booklet made from six very large pieces of paper folded double and sewn. It seems likely that this was the copy from which Liouville's printer worked in 1846, but, unlike Chevalier's copy of the *Premier mémoire*, this one carries no pencilled instructions from Liouville. Of course that proves nothing—it could be that this was not what the printer worked from, but equally possible is that the instructions on the first item were intended also to apply to the second.

NOTE 4: Referring to the reference missing from **f. 39 a** (and therefore also **f. 39 b**), Chevalier has a footnote (in the margin) of **f. 46 a** of his manuscript copy:

> Note. ce mémoire faisant suite à un travail de Galois que je ne possède pas, il m'est impossible d'indiquer le numéro cité ici et plus bas.
>
> A. CH.
>
> [Note. This memoir forming a sequel to a work by Galois that I do not possess, it is impossible for me to indicate the number cited here and below.
>
> A. CH.]

NOTE 5: In **f. 39 b** (see p. 182) the word 'veut' may be crossed out, but it looks more as if Galois had trouble with his pen there, or as if the cross-bar of the letter t was written a little too enthusiastically. All of C*ms*, L1846, and BA1962 have 'remonte' to read 'et que l'on remonte à ...', but 'remonter' is clear in the manuscript, preceded by a crossed-out syllable, which in turn is preceded by the word 'veut' that might possibly be crossed out, but much more probably is not. And the reading 'et que l'on veut remonter' makes good sense: if Galois thought of his substitutions in the two-line notation of [Cauchy (1815a)] then his assertion about starting from a letter a_{l_1,l_2} of the second permutation (arrangement) and going up to the corresponding letter in the first would make very good sense.

NOTE 6: The line at the end of **f. 41 a** (p. 188) perhaps finishes with 'de 2'. Chevalier's copy has 'de 2 termes', Liouville printed 'de deux termes'. The phrase is almost illegible in the manuscript, but there is no room at the line-end for the word 'termes' after '2', which appears to be followed simply by a full-stop. The phrase 'de deux termes' occurs near the top of **f. 41 b** however, and the construction 'de p termes' occurs elsewhere in the *Second Mémoire*, so Chevalier's reconstruction (or emendation) makes good sense.

If the word before '2' is 'de' then it is overwritten onto a faintly written and illegible word that is longer than 'de'. Bourgne supplies 'donc' [B & A (1962), p. 145], which makes excellent sense. I am inclined to favour his reading over Chevalier's.

NOTE 7: The manuscript of the Second Memoir is different from all others in that it carries no jottings. Even the First Memoir has a few—such as a list of names in the margin of **f.1b** and some odd formulae involving S, T, U in the margin of **f.4a**—but the Second Memoir has none. Parts of it are crossed out and rewritten, and there are a few marginal corrections, but it is otherwise remarkably clean.

The most interesting corrections are those that involve the cadence which originally came near the top of **f.38b**, and which, after the corrections had been effected, comes at the bottom of **f.38a**. I propose to write about the content and meaning of this material elsewhere. Here it is only appropriate to record that Galois seems at this point (the first ten lines of **f.38b**) to have had some trouble continuing with his general analysis of primitive soluble equations of degree p^ν, that after re-starting the passage two or three times he finally decided to specialise to the case $\nu = 2$, but, as the corrections in the passage in the top half of **f.39a** indicate, he still had the general case somewhere near the front of his mind.

NOTE 8: From its last sentence we see clearly that the Second Memoir is incomplete and Galois had intended to continue it. In **f.51a** of Chevalier's copy there is a footnote:

> j'ai cherché inutilement ^dans les papiers de galois^ la continuation de ce qu'on vient de lire. (Note de M. A. Chevalier)
>
> [I have searched without success in the papers of Galois for the continuation of what the reader has just seen. (Note by Mr A. Chevalier)]

Here the note is by Chevalier, the parenthetic attribution is in a different hand, that of Liouville, who corrected the lower-case to capital letters in 'J'ai' and 'Galois' when he had the footnote printed in [Liouville (1846), p. 444].

Chapter VI
The minor mathematical manuscripts

As was explained above (§ I.3) the manuscripts are organised into dossiers. This chapter is devoted to minor material to be found in Dossiers 6–24. The items are presented in the same order as they appear in the bound volume of manuscripts. Some of them were published first by Tannery in 1906 and 1907, and this is the order in which he presented them. The ordering in [B & A (1962)] is based on content, but I am not convinced that that has been entirely successful—hence my decision to follow the dossiers. The headings of the sections are adapted from the descriptions carried on the front cover-sheets of the dossiers.

Not all the minor material is reproduced here. I have omitted all the material in Dossiers 23 and 24 that consists of unexplained formulae and jottings, graffiti. All this is to be found in [B & A (1962), pp. 191–361], edited by J. P. Azra. There are few passages of text in amongst the crowds of symbols, and those few can perfectly well remain untranslated for now. I have no regrets over these omissions. I do regret not including the early essays and the school work to be found in Dossiers 25 and 26, transcribed in [B & A (1962), pp. 403–458]. They are omitted mainly on account of shortage of time, partly on account of shortage of space—this book has grown much longer than I had originally envisaged already, and the addition of 100 further pages is hardly justified by their content.

The dossiers presented here may be seen as falling into distinctive, but overlapping, groups.

Group I: Dossiers 6, 7, 16, 17 contain material directly related to the *Premier Mémoire*.

Group II: Dossiers 15, 18, 19 contain material related to the *Second Mémoire*.

Group III: Dossiers 8, 9, 11, 12, 13, 14 contain quasi-philosophical and polemical material.

Group IV: Dossiers 8, 10 contain catalogues of Galois' writings or plans.

Group V: Dossier 10 contains a note protesting Galois' independence of Abel.

Group VI: Dossiers 20–24 contain essays on various topics.

The material in Group III seems intended partly as prefatory to the mathematics, partly for publication in the *Revue Encyclopédique*. It was highly regarded by Chevalier, who made copies of the essays in Dossiers 9, 11, 12, and who cited it in his obituary of Galois [Chevalier (1832b)]. It seems that Galois at one time intended

to publish a memoir, presumably the First Memoir, in one form or another, and at another time intended to publish two memoirs together as one volume. Those two might have been the First Memoir and the Second Memoir, once it had been completed, or they might have been the First Memoir and the Third Memoir, if it was ever written (which is open to doubt); see Note 1 to Dossier 8, and Note 1 to Dossier 24. Thus Dossier 9 contains a sort of preface or introduction for the former; Dossier 11 consists of a preface for the latter. From internal evidence, 'Nous exposerons donc dans quelques articles' [We therefore expound in a few articles] (see p. 264 below), the essay in Dossier 12 was almost certainly intended as part of a series of encyclopedic articles conceived for publication, probably in the *Revue Encyclopédique* or Férussac's *Bulletin*.

Some of the essays in Group VI seem to me to be similar in character to the published papers 'A theorem on continued fractions', 'A note on the numerical solution of equations', 'On some points of analysis'. There is no indication, however, that they were ever submitted for publication.

Many, indeed most, of the pages in Dossiers 6–24 carry jottings of various kinds, fragments of calculations, odd formulae, numbers, names, little sketches and simple scribbles. These are not unlike the material recorded by Azra in [B & A (1962), pp. 191–361]. I have noted the more interesting ones in the notes to each dossier. One name occurs more frequently than any other: 'Galois' occurs at least 60 times, sometimes with 'Evariste' or Évariste', sometimes just in the form 'E G', and sometimes Evariste alone. Many of the instances are very elegantly written.

VI.1 Dossier 6: An 1830 version of Proposition I of the First Memoir

The sole content of Dossier 6 is one page, 22.5 cm × 34.5 cm. The cover sheet of the dossier carries the annotation 'Feuille contenant une première rédaction de la proposition I du Mémoire sur les conditions de résolubilité des équations par radicaux'. It is treated in [B & A (1962), pp. 88–93, 496–498]. When the manuscripts were sorted, numbered and bound this page was reversed. Thus it begins on **f. 54 b** and continues on **f. 54 a**. Galois left margins just over 6 cm wide on the left of his pages; in addition, **f. 54 b** has a 2 cm margin at the top; other sides, and the bottoms, of the pages have no margins. A corner, approximately triangular, 13 cm high and 4 cm along its base has been torn away from the bottom right of the page. Because of the wide margins this hardly affects **f. 54 a** (where, because of the page-reversal, it appears on the left), but on **f. 54 b** a small (and reconstructable) amount of the table that follows the words 'Ecrivons les *m* permutations' is lost, as is perhaps a quarter to a third of each of the last three lines.

It was Tannery (T1906/7, pp. 236, 237) who first described this page and its contents, his attention having been drawn to it by Paul Dupuy, the historian and biographer. The format of the paper, the colour of the ink and the form of the handwriting led him to suggest that it was originally part of the booklet in which the *Second Mémoire* was written. Bourgne, quoting Tannery at some length [B & A (1962), pp. 496, 497], confirms this conjecture and dates the material to somewhere around June 1830. I do not have the expertise to demur—nor would I wish to, except for one small but puzzling detail, namely that, according to my measurements the page is 0.5 cm shorter than those of the *Second Mémoire*.

54b

Théorème. Soit une équation donnée avec tant de quantités adjointes que l'on voudra. Soient $a\ \ b\ \ c\ \ \ldots.$ les m racines. On pourra toujours former un groupe de permutations des lettres $a\,b\,c\,\ldots$ tel que toute fonction des racines invariable par les substitutions du groupe, est rationnellement connue, et tel réciproquement que toute fonction déterminable rationnellement par les coëfficients de l'équation et par les quantités adjointes est invariable par les substitutions du groupe. (Dans le cas des Equations Algebriques ce ~~système~~ groupe n'est autre chose que l'ensemble des $1.2.3\ldots m$ permutations possibles sur les m lettres, puisque dans ce cas les fonctions symetriques sont seules connues à priori.)

BA1962 has 'soit' in place of 'soient'.

BA1962 has singular 'de lettre'; plural is clear in *ms*.

Spelling corrected to 'symétriques' in BA1962.

~~Considérons d'abord un cas particulier.~~ ~~Supposons que l'équation donnée soit irréductible~~ ^n'ait aucun diviseur rationnel^ et soit telle que toutes ses racines se déduisent rationnellement de l'une quelconque d'entre elles. ~~La proposition sera, dans ce cas,~~ facile à démontrer.

Three attempts to start this paragraph

~~Comme [??]~~ ^En effet^ Dans ~~notre hypothese, toute fonction des racines ne pourrait être que~~ ^s'exprimera en^ fonction d'une seule ^racine^ et sera de la forme ϕx, x étant une racine. Soient

$$x \qquad x_1 = f_1 x \qquad x_2 = f_2 x \qquad \ldots\ldots \qquad x_{m-1} = f_m\langle_{-1} x\rangle$$

les m racines. Ecrivons les m permutations

$$\begin{array}{ccccccc}
x & f_1 x & f_2 x & \ldots. & f_{m-1} x \\
x_1 & f_1 x_1 & f_2 x_1 & \ldots. & f_{m-1}\langle x_1\rangle \\
x_2 & f_1 x_2 & f_2 x_2 & \ldots. & f_m\langle_{-1} x_2\rangle \\
_ & _ & _ & _ & _ & _ & _ \\
x_{m-1} & f_1 x_{m-1} & f_2 x_{m-1} & \ldots. & \langle f_{m-1} x_{m-1}\rangle
\end{array}$$

Words supplied where page is torn off are conjectural, based on the cognate passage in *Premier Mémoire*.

Je dis que ce groupe de permutations d⟨es racines⟩ jouit de la propriété énoncée. En ef⟨fet toute⟩ fonction ^F^ des racines invariable par les sub⟨stitutions de ce⟩
A reporter plus bas

54a

groupe ~~deviendra~~ ~~ϕx~~ ⌜pourra être ecrite ainsi $F = \psi V$, et l'on aura⌝

$$F = \psi V = \psi V_1' = \psi V_2'' = \psi V_{\_}''' \_\ \_ = \psi V_{m-1} \_\ \_\ \_$$

et sera parconséquent connue.

THEOREM. *Let an equation be given with as many adjoined quantities as desired. Let* $a, b, c, \ldots$ *be the* m *roots. A group of permutations of the letters* $a, b, c, \ldots$ *can always be formed such that every function of the roots invariant under the substitutions of the group is rationally known, and such that conversely, every function that is rationally determinable in terms of the coefficients of the equation and the adjoined quantities is invariant under the substitutions of the group.*

(In the case of algebraic equations this ~~system~~ group is nothing other than the collection of the $1.2.3 \ldots m$ permutations possible on the m letters, because in this case the symmetric functions are the only ones known *a priori*.)

~~Let us first consider a particular case. Suppose that the given equation is irreducible ^has no rational divisor^ and is such that all its roots may be deduced rationally from any one among them. The proposition will , in this case, be easy to prove.~~

~~As [??] Indeed, under our hypothesis, every function of the roots cannot be other ^is expressible as a^ function of a single ^root^ and will be of the form ϕx, x being a root. Let~~

~~$x \quad x_1 = f_1 x \quad x_2 = f_2 x \quad \ldots\ldots \quad x_{m-1} = f_m \langle {}_{-1} x \rangle$~~

~~be the m roots. Write the m permutations~~

~~$x \quad f_1 x \quad f_2 x \quad \ldots \quad f_{m-1} x$~~

~~$x_1 \quad f_1 x_1 \quad f_2 x_1 \quad \ldots \quad f_{m-1} \langle x_1 \rangle$~~

~~$x_2 \quad f_1 x_2 \quad f_2 x_2 \quad \ldots \quad f_m \langle {}_{-1} x_2 \rangle$~~

~~$x_{m-1} \quad f_1 x_{m-1} \quad f_2 x_{m-1} \quad \ldots \quad \langle f_{m-1} x_{m-1} \rangle$~~

I say that this group of permutations of the roots will enjoy the stated property. Indeed, every function F of the roots that is invariant under the substitutions of this

To be moved lower down

group ~~will become ϕx~~ will be expressible as $F = \psi V$, and one will have

~~$F = \psi V = \psi V_1' = \psi V_2'' = \psi V''' \ldots = \psi V_{m-1}$~~

and will consequently be known.

2°. Réciproquement si une fonction ~~φx~~ ^F^ est déterminable rationnellement ~~on devra avoir~~ et que l'on ~~eut~~^pose^ $F = \psi V$, on devra avoir

$$F = \psi V = \psi V_1 = \psi V_2 = \psi V''' \ldots = \varphi x_{m-1} \ldots$$

~~Expri~~ puisque l'équation ^en V^ n'a pas de diviseur commensurable. ~~Donc~~ ^~~et~~^ ^Donc ^la fonction F sera nécessairement ^invariable^ par les substitutions du groupe ecrit ci-dessus.

For the placing of the marginal instructions see Note 1(ii). This one is in two lines rising diagonally from left to right.

À reporter plus bas

~~Le théorème sera ^est^ donc démontré dans l'hypothèse particulier que nous avons établie.~~

~~Venons au cas général~~

*Démonstration*Quelle que soit l'équation , donnée, on pourra trouver une fonction ~~des racines~~ ^rationnelle V des racines^ telle que réciproquement toutes les racines soient fonctions rationnelles de V. (~~Toute~~^Une^fonction linéaire des racines jouira en général de cette propriété.)

Celà posé considérons l'équation du moindre degré ^possible^ dont V est racine. ~~Cette équation tombera dans le cas particulier examiné plus haut.~~ ✳ Soient ~~donc~~

$$\varphi V \quad \varphi_1 V \quad \varphi_2 V \quad \ldots . \quad \varphi_{m-1} V$$

The last item of the V list is illegible under the crossing out. Possibly $V^{(n-1)}$ as in *Premier Mémoire*; possibly V with an inadvertent ✳.

les racines de la proposée. ✳ Soient $V \quad V' \quad V'' \quad V''' \ldots$ les ~~autres valeurs de V~~ ^diverses racines de cette équation^. ✳

~~en~~ Écrivons le groupe suivant

φV	$\varphi_1 V$	$\varphi_2 V$	. . . . .	$\varphi_{m-1} V$
$\varphi V'$	$\varphi_1 V'$	$\varphi_2 V'$	. . . . .	$\varphi_{m-1} V'$
$\varphi V''$	$\varphi_1 V''$	$\varphi_2 V''$	. . . . .	$\varphi_{m-1} V''$
$\varphi V'''$	$\varphi_1 V'''$	$\varphi_2 V'''$	. . . . .	$\varphi_{m-1} V'''$

. . . . . .

Mettre ici la partie sautée

~~On reconnaitra, comme ci dessus, que~~ ^Ainsi^ ce groupe jouit de la ^double^ propriété dont il s'agit dans le théorème proposé. Le théorème est donc démontré.

(Je dois observer que j'avais d'abord démontré le théorème autrement, sans penser à me servir ~~pour la démonstration~~ de ~~la~~⌜cette⌝ propriété très simple des équations ~~rapportée plus haut.~~, ~~et qui~~ ^propriété que^ Je ~~ne [??]ais~~ ^regardais^ ~~cette~~ ^~~[??]~~^ comme une conséquence du théorème. C'est la lecture d'un mémoire de qui m'a suggéré ~~[??] [?]~~ ^la^ ^~~[???]~~^ [??]

On missing name see Note 3.

2. Conversely, if a function ~~φx~~ F is rationally determinable ~~one must have~~ and if one ~~had~~ sets $F = \psi V$, one will necessarily have

$$F/\!/\!/\psi x V = \psi x V_1' = \psi x V_2'' = \psi V''' \ldots = \varphi x_{m-1} \ldots$$

because the equation for V has no commensurable divisor. Therefore the function F will necessarily be invariant under the substitutions of the group written above.

To be moved lower down

~~The theorem will be ^is^ therefore proved under the special hypothesis that we have assumed.~~

~~Let us come to the general case~~

Proof Whatever the given equation may be, one will always be able to find a rational function V of the roots such that conversely all the roots will be rational functions of V. (~~Every~~ In general a linear function of the roots will enjoy this property.)

That said, consider the equation of least possible degree of which V is a root. ~~This equation will fall into the particular case examined above.~~ ✳ Let ~~then~~

$$\varphi V, \quad \varphi_1 V, \quad \varphi_2 V, \quad \ldots, \quad \varphi_{m-1} V$$

be the roots of the proposed equation. ✳ Let V, V', V'', V''', … be the ~~other values of V~~ various roots of this equation. ✳

Write the following group

φV	$\varphi_1 V$	$\varphi_2 V$	. . . . .	$\varphi_{m-1} V$
$\varphi V'$	$\varphi_1 V'$	$\varphi_2 V'$	. . . . .	$\varphi_{m-1} V'$
$\varphi V''$	$\varphi_1 V''$	$\varphi_2 V''$	. . . . .	$\varphi_{m-1} V''$
$\varphi V'''$	$\varphi_1 V'''$	$\varphi_2 V'''$	. . . . .	$\varphi_{m-1} V'''$

. . . . . .

Put here the part to be moved.

~~As above it may be seen that~~ Thus this group enjoys the double property in question in the stated theorem. The theorem is therefore proved.

(I should say that I had originally proved the theorem differently, not having thought of exploiting ~~for the proof~~ this very simple property of equations ~~reported above, and which~~, a property which I thought of as a consequence of the theorem. It was reading a paper by that suggested to me [illegible—but presumably something like 'this line of argument)'].

Notes on Dossier 6

NOTE 1: The page has been much revised. There are Galois' usual on-the-hoof amendments, made by crossings-out and insertions. Over and above those, however, three major changes can be seen.

(i) The passage beginning 'Considérons d'abord' and ending with the last line of the table of permutations $f_i x_j$ was crossed out. This was done with sixteen diagonal lines through the whole block (text and table) and by crossing the last line of the table through additionally with a horizontal line.

(ii) A marginal bracket was drawn (starting near the bottom of **f.54b** and continuing overleaf) to the left of the passage starting 'Je dis que ce groupe de permutations' and ending 'du groupe écrit ci-dessus'. At the same time instructions to move the bracketed material down so as to come immediately after the table of permutations $\varphi_i V^{(j)}$ were inserted. These instructions are the marginal notes 'á reporter plus bas' that occur in the margin to the left of the bracket at the bottom of **f.54b** and the top of **f.54a**, and the note '⟨Me⟩ttre ici la partie sautée' which starts well over in the margin but continues into the text-space just below the $\varphi_i V^{(j)}$ table.

(iii) The first few lines of **f.54a**, from 'groupe ~~deviendra~~' to 'du groupe écrit ci-dessus' were originally written in terms of the roots φx_i, where φ is the function introduced in the crossed-out passage just before the displayed equations that express the roots x_i as $f_i x$. Later the equations

$$F = \varphi x = \varphi x_1 = \varphi x_2 = \cdots = \varphi x_{m-1}$$

were overwritten to read

$$\psi V = \psi V' = \psi V'' = \psi V''' = \cdots .$$

It is possible that this was all done at one time. Certainly what is suggested by Tannery [Tannery (1906), pp. 236, 237], namely that Galois had the proof of the special case, and only later realised that he could simply adapt it to the general case, seems plausible. But it is also possible that the process was more long-drawn out and complicated than that. This is a matter that merits further research because the relationship between the special case (of Galois groups that act regularly on the set of roots) and the general case sheds considerable light on the development of Galois' thoughts and understanding.

NOTE 2: Just below the middle of **f.54a** the symbol $*$ appears three times. It seems clear that the intention was to move the sentence enclosed by the 2nd and 3rd occurrences to the place marked with the 1st occurrence. That would make excellent sense and accords with the corresponding passage in the *Premier Mémoire*.

NOTE 3: The last few lines are much corrected and suffer also from being very cramped. They are squeezed in at the bottom of **f.54a** and appear to contain an afterthought. In [Tannery (1906), p. 237] Tannery suggested that the missing name might be that of Abel. It seems possible, however, that the paper referred to in this last paragraph is [Libri (1830)]. That treats irreducible equations with the property that every root may be expressed as a rational function of any other (or in modern terms, that the Galois group acts regularly on the set of roots). This would accord with, and amplify, a conjecture in [B & A (1962), pp. 92, 498] where the names of both Abel and Libri are mentioned.

NOTE 4: The margin of **f.54b** contains a few scribbles and some jottings that precede the tearing-off of the corner. They occur on three lines written with the page turned a quarter-turn anti-clockwise so that its left edge came to the bottom and the writing appears sideways:

$$F(V) = 0 \qquad V = \varphi(a, b, c, \ldots)$$
$$\begin{cases} V - \varphi(a, \ldots) & Aa + Bb + Cc + \\ \varphi(V, a) = 0 \qquad fa = 0 \end{cases}$$

As was pointed out by Tannery these formulae have to do with Lemmas II and III of the *Premier Mémoire*. The second equation appears originally to have read

$$V = \varphi(x, x_1, x_2, \ldots, x_{m-1}),$$

which is how it is read in [B & A (1962), p. 498], but it is clear that the first three variables were overwritten with a, b, c and the final x_{m-1} was scribbled out. In this form it matches the expression in the statement of Lemma III of the *Premier Mémoire*. The expression $Aa + Bb + Cc +$ probably continued with ellipsis as in Lemma II of the *Premier Mémoire* but was curtailed by the tearing off of the corner of the page. In the last line it is not clear whether or not there is a symbol φ preceding (V, a).

VI.2 Dossier 7: An 1830 draft of Proposition V of the First Memoir

Dossier 7 contains a single sheet of paper, folded to make a two-page booklet approximately 17.5 cm × 22.5 cm. Unfolded, the paper is 34.5 cm wide and varies from 22 cm to 23 cm deep, very similar in size, and of the same make, as the paper of the *Second Mémoire*, turned through a quarter-turn before folding. It is clearly torn along its top edge from a larger sheet.

The four sides, **f. 55 a**–**f. 56 b**, are densely written, with narrow margins and with many amendments. The paper is thin, the ink shows through from the other side; it is not easy to read. The cover sheet of the dossier carries the annotation 'Rédaction primitive de la proposition V du Mémoire sur les conditions de résolubilité ...,' [Original draft of Proposition V of the memoir on solubility ...]. Tannery [Tannery (1906), p. 235] described the piece as 'une sorte de brouillon de la proposition V; ce brouillon a passé en grande partie dans la rédaction du Mémoire' [a sort of rough draft of Proposition V; this rough draft has mostly been incorporated into the compilation of the memoir], and in a footnote he wrote that he did not think it of enough interest to be worth publishing.

It appears on pp. 94–101 of BA1962. Bourgne writes [B & A (1962), p. 498]:

> On datera cette rédaction à l'entour juin-septembre 1830. Le texte en est repris assez fidèlement dans la proposition V du *Premier Mémoire*. Aussi l'interêt ici est-il surtout dans les hésitations, les tâtonnements qui trahissent encore la recherche et l'effort de mise au point.
>
> [This draft may be dated to around June–September 1830. Its text is repeated pretty faithfully in Proposition V of the *Premier Mémoire*. Accordingly its interest here is primarily in the hesitation, the groping around which again betrays the research and the effort to perfect it.]

It is true that the first few paragraphs of Proposition V (see p. 120 above) have much in common with the material in **f. 55 a**, and the discussion of quartic equations at the end is much the same. But the middle passage, which contains the substance of the argument, is presented very differently in the two versions.

55a

Condition pour qu'une équation soit soluble par radicaux

⌜J'observerai d'abord, &c.⌝

Soit G le groupe d'une équation soluble par radicaux. ^Cherchons à quelles conditions satisfera ce groupe.^ Suivons ^pour celà^ la marche des opérations possibles dans cette solution, ⌜en considérant comme des opérations distinctes l'extraction de chaque racine de degré premier.⌝

Adjoignons à l'équation ~~chaque~~ ^le premier^ radical extrait ^dans cette solution;^ ~~en commencant par les radicaux~~ ~~racines des quantités données~~ ^il pourra arriver^ de deux choses l'une. Ou bien, par l'adjonction de ce radical, le groupe de l'équation sera diminué; ou bien, ce⌜tte⌝ ~~radical~~ extraction de racine n'étant qu'une simple préparation, le ~~rad~~ groupe restera le mème.

~~Toujours sera-t-il qu'après un certain nombre d'extractions de racines, le groupe de l'équation devra se trouver diminué, sans quoi elle ^l'équation^ ne serait pas soluble.~~ ~~Donc~~

~~Considérons donc le premier radical~~ radical qui adjoint à l'équation ~~diminuera son groupe.~~

~~Donc après avoir adjointe ^à l'équation^ s'il est nécessaire un certain nombre de quantités radicales qui ne diminuent pas son groupe, [??] on arrivera à un certain point où l'extraction d'une simple radical diminuera son groupe~~.

Word 'radical' changed to 'racine'.

~~Or il se pourra~~

~~Raisonnons d'abord dans la première hypothèse.~~

Si, arrivé à ce point, il y avait plusieurs manières de diminuer le groupe de l'équation par une simple extraction de racine, ~~nous fais ne considérerons~~ ^il ~~faudrait~~^, pour ce que nous allons dire, considérer seulement un radical du degré le moins haut possible ~~et dont la connaissance di~~ ^parmi tous les simples radicaux^ qui sont tels que la connaissance de chacun d'eux diminue le groupe de l'équation.

Soit donc p le nombre premier qui exprime ce degré minimum, en sorte que par une ~~équat~~ extraction de racine de degré p, on diminue le groupe de l'équation.

55 b

Nous pouvons toujours supposer, ^du moins pour ce qui est relatif au groupe de l'équation^ que parmi les quantités adjointes précédemment à l'équation se trouve une racine $p^{\text{ième}}$ de l'unité α; car, comme ~~dans~~ cette expression s'obtient par ~~de simples~~ des extractions de racines de degré inférieur a p, ~~elle ne dim~~ ^sa connaissance^ n'altérera en rien le groupe G de l'équation. Soit donc r une des valeurs du radical.

~~De cette manière~~ ~~En considérant comme rationnelle~~ ~~On voit que l'une quelconque des valeurs du radical de degré p s'exprimera en fonction d'une autre quelconque d'entre elles. Donc le groupe ^de substitutions^ que l'on obtiendra en adjoignant à l'équation ^par l'adjonction des^ telle quantité que l'on voudra~~

Condition for an equation to be soluble by radicals

⌜To begin I will observe, &c.⌝

Let G be the group of an equation soluble by radicals. Let us seek what conditions this group will satisfy. To do that we follow the progression of the possible operations in this solution, considering as distinct operations each extraction of a root of prime degree.

Adjoin to the equation ~~each~~ the first radical extracted in this solution; ~~beginning with the radicals~~ roots of given quantities one of two things could happen. Either the group of the equation will be diminished by the adjunction of this radical, or this root extraction being no more than a simple preparation, the group will remain the same.

~~It will always happen that after a certain number of root extractions the group of the equation must be found to be diminished, otherwise the equation would not be soluble~~ ~~Therefore~~

~~Consider then the first radical~~ radical which, adjoined to the equation, ~~will diminish its group.~~

~~Therefore after having adjoined to the equation if necessary a certain number of radical quantities which do not diminish its group [ ] one will arrive at a certain point where the extraction of a simple radical root [radical] will diminish its group~~.

~~Well it could happen~~

~~We begin by reasoning under the first hypothesis.~~

If, having got to this point there were several ways of diminishing the group of the equation by a simple root extraction, ~~we make do not consider~~ for what we are about to say it would be necessary to consider only a radical of the least high degree possible ~~and the knowledge of which [would diminish]~~ amongst all the simple radicals which are such that the knowledge of any one of them diminishes the group of the equation.

Let then p be the prime number which is this minimum degree such that by a root extraction of degree p the group of the equation is diminished.

We can always suppose, at least for what relates to the group of the equation, that among the previously adjoined quantities is a pth root of unity α; for since ~~in~~ this expression may be obtained by ~~some simple~~ root extractions of degree lower than p, knowledge of it in no way alters the group G of the equation. Let r then be one of the values of the radical.

~~In this way~~ ~~By considering as rational~~ ~~It may be seen that any one of the values of the radical of degree p may be expressed as a function of any other one of them~~ ~~Therefore the group of substitutions that one obtains by adjoining to the equation by the adjunction of whatever quantity one wishes~~

^Soit H le groupe de l'équation quand on lui adjoint cette quantité^toute fonction ~~$\varphi(H)$~~ invariable par les substitutions des groupes H ~~$H + HS +$~~ sera connue.^
Il faut donc que le groupe G se ~~décompose~~ ^partage^ en p groupes H semblables et ~~identiques~~ ^dont^ ⌜les substitutions soient les mèmes.⌝
~~Considérant l'équation~~
Telle est la condition pour que le radical en question abaisse le groupe de l'équation ^à un groupe sousmultiple de l'ordre p^. Adjoignons donc à l'équation ce radical, et nous aurons une équation dont le groupe sera simplement H.
Nous pourrons raisonner sur ce groupe comme sur le précédent, et ^continuant ainsi^ nous trouverons ^enfin^ cette condition générale.
Le groupe G d'une équation soluble par radicaux, doit se ~~décomposer~~^partager^ en un nombre premier de groupes H ~~egaux~~ ^semblables^ et identiques. Le groupe H a un nombre premier de groupes K semblables et identiques, et ainsi desuite, jusqu'à un certain groupe M qui ne contiendra plus qu'un nombre premier de ~~substitutions~~ ^~~perms~~substitutions^.
Réciproquement si ~~cette~~ le groupe G satisfait à la condition précédente l'équation sera soluble par radicaux.

It seems that Galois intended to change 'd'une équation' to 'dont l'équation est', but the amendment is unfinished.

56 a

En effet ^supposons que le groupe [partiel?] H soit contenu p fois dans G.^ prenons une fonction rationnelle φ des racines de l'équation qui ne soit pas ^numériquement^ invariable par toutes les substitutions du groupe G mais ^qui ~~peut~~ d'après sa forme le soit^ par toutes celles du ^premier^ divisant H, ~~et supposons que H soit contenu p fois dans G.~~ ^On pourra toujours avoir une pareille fonction rationnelle sans quoi^
Opérons sur la ~~substitution~~ fonction φ ~~toutes~~ ^une^ des substitutions du groupe G ^qui ne lui soit pas commune au groupe H^, nous aurons en tout p ~~valeurs~~ ⌞fonctions⌟ différentes, ~~suivant~~ φ φ_1 φ_2 φ_3 ... ^qui répondront aux système de permutations AH, AHS, AHS^2^ Ces fonctions ~~qui~~ par ~~la propriété du groupe H~~ ^l'une des hypothèse^ ~~auront~~^seront^ toutes invariables par les substitutions du même groupe H. Alors en appelant α une racine $p^{\text{ième}}$ de l'unité la fonction

$$\left(\varphi + \alpha\varphi_1 + \alpha^2\varphi_2 + \cdots\right)^p$$

sera invariable par toutes les substitutions du groupe G, et par conséquent connue. Extrayant la racine $p^{\text{ième}}$ de cette fonction, et adjoignant le radical à l'équation proposée, on n'aura qu'une ~~le~~ ^équation relative^ groupe H, et ainsi desuite; ~~Jusqu'à ce qu'on arrive à ^l'équation^ une certaine équation dont le groupe ne contienne qu'un nombre premier de substitutions~~.
Il est aisé de vérifier cette marche essentielle, dans la résolution des équations ^générales^ du quatrième degré. En effet, cette équation se resolvent au moyen d'une

Word after 'groupe' is hard to read. It is omitted from BA1962. Reading 'partiel' comes from corresponding passage in *Premier Mémoire*.

Word 'au' missing after 'relative' noted in BA1962.

Slip by Galois: singular 'cette équation' does not match plural 'se resolvent' (for 'se résolvent').

Let H be the group of the equation when one adjoins this quantity to it. Every function ~~$\varphi(H)$~~ invariant under the substitutions of the groups H will be known.

It is necessary therefore that the group G is ~~decomposed~~ partitioned into p groups H that are similar and ~~identical~~ of which the substitutions are the same.

~~Considering the equation~~

That is the condition for the radical in question to reduce the group of the equation to a submultiple group of order p. Therefore adjoin this radical to the equation, and we will have an equation whose group will simply be H.

We may reason on this group as on the previous one, and continuing in this way, in the end we will find this general condition:

The group G of an equation soluble by radicals must ~~decompose~~ be partitioned into a prime number of groups H that are ~~equal~~ similar and identical, the group H has a prime number of groups K that are similar and identical, and so on, as far as a certain group M which will contain no more than a prime number of substitutions.

Conversely, if ~~this~~ the group G satisfies the preceding condition the equation will be soluble by radicals.

Indeed, suppose that the partial group H is contained p times in G. We take a rational function φ of the roots of the equation which is not numerically invariant under all the substitutions of the group G but which, ~~can~~ on account of its form, will be so under all those of the first divisor H ~~and suppose that H is contained p times in G~~. One will always be able to get such a rational function, otherwise

Hard to translate. Possibly 'divisant' was a slip. Was 'diviseur' intended?

Operate on the function φ with ~~all~~ one of the substitutions of the group G that is not also in the group H. We will have p ~~values~~ different functions in all ~~following~~ $\varphi,\ \varphi_1,\ \varphi_2,\ \varphi_3,\ \ldots$ which correspond to the system of permutations $AH,\ AHS,\ AHS^2$. These functions, ~~which the property of the group H~~ by one of the hypotheses, ~~will have~~ will all be invariant under the substitutions of that same group H. Then, denoting by α a pth root of unity, the function

$$\left(\varphi + \alpha\varphi_1 + \alpha^2\varphi_2 + \ \cdots\right)^p$$

will be invariant under all the substitutions of the group G, and consequently known. Extracting the pth root of this function, and adjoining the radical to the proposed equation, one will have only an equation relative to the group H, and so on; ~~until one gets to a certain equation whose group contains only a prime number of substitutions~~.

It is easy to verify this essential progression in the solution of general equations of the fourth degree. Indeed, this equation is solved by means of an

équation du troisième degré, qui éxige elle mème ~~la résolution d'une équation du second degré~~ ^l'extraction d'une racine carrée^. Dans la suite naturelle des idées, c'est donc par cette ~~équation du second degré~~ ^racine carrée^ qu'il faut commencer. ~~Or~~^En effet^ en adjoignant à l'équation du 4eme degré ~~cette une~~^cette^racine carrée, le groupe ~~///~~ de l'équation ^qui^ ~~en~~ contenait ^en tout^ 24 substitutions, ~~n'en contient plus que~~ ^se décompose en deux groupes semblables et identiques qui n'en contiennent que^12. En désignant par $a\ b\ c\ d$ les racines, voici ce groupe en partant de la permutation arbitraire $abcd$

$$\begin{pmatrix} abcd \\ badc \\ cdab \\ dcba \end{pmatrix} \quad \begin{pmatrix} acdb \\ cabd \\ dbac \\ bdca \end{pmatrix} \quad \begin{pmatrix} adbc \\ dacb \\ bcad \\ cbda \end{pmatrix}$$

et l'on voit que les trois groupes dans lesquels nous les avons partagées sont semblables et identiques

56 b

Ensuite ~~l'équat~~ par l'extraction du troisième degré ~~on a~~^il reste^simplement le groupe

$$\begin{matrix} abcd \\ badc \\ cdab \\ dcba \end{matrix}$$

Ce groupe ^se^ partage de nouveau en deux groupes semblables et identiques

$$\begin{matrix} abcd & badc \\ cdab & dcba \end{matrix}$$

Ainsi par une ~~nouvelle~~ extraction de racine carrée, on n'aura plus ~~que~~ ^qu'une équation dont^ le groupe sera

$$\begin{matrix} abcd \\ badc \end{matrix}$$

~~Le quel~~ ce qui se resoudra ~~aussi~~ ^enfin^ par une ~~simple extraction de~~ ^~~troisième~~ [simple?]^ racine carrée.

Nous connaissons donc la condition à~~ux~~^la^ quelle satisfait le groupe d'une equation Algebriquement résoluble. Il reste ra à developper ces conditions, afin de resserer la classe des équations résolubles dans ses vérit~~ables limites. Nous commencerons pour cela par les equations les plus simples.~~

~~1er Cas.~~ §1 ~~Dans quel cas une equation dont toutes ses racines sont fonctions de deux quelconques d'entre elles, peut elle se resoudre par radicaux~~

equation of the third degree, which itself requires ~~the solution of an equation of the second degree~~ the extraction of a square root. In the natural sequence of ideas it is necessary therefore to begin with this ~~equation of the second degree~~ square root. ~~Well~~ Indeed, on adjunction of this square root to the equation of the 4th degree, the group of the equation, which contains 24 substitutions in all, ~~will contain no more of them than~~ is decomposed into two similar and identical groups which will contain only 12 of them. Denoting the roots by a, b, c, d here is the group starting from the arbitrary permutation $abcd$:

$$\begin{pmatrix} abcd \\ badc \\ cdab \\ dcba \end{pmatrix} \quad \begin{pmatrix} acdb \\ cabd \\ dbac \\ bdca \end{pmatrix} \quad \begin{pmatrix} adbc \\ dacb \\ bcad \\ cbda \end{pmatrix}$$

and one sees that the three groups into which we have partitioned them are similar and identical.

Then after the extraction [of a root] of the third degree ~~one has~~ there remains simply the group

$$\begin{matrix} abcd \\ badc \\ cdab \\ dcba \end{matrix}$$

This group is partioned in its turn into two similar and identical groups

$$\begin{matrix} abcd & badc \\ cdab & dcba \end{matrix}$$

Then by an ~~new~~ extraction of a square root one will have an equation of which the group will be no more than

$$\begin{matrix} abcd \\ badc \end{matrix}$$

~~Which~~ which may finally be solved ~~also~~ by a ~~simple extraction of a third~~ simple square root.

Thus we know the condition that is satisfied by the group of an algebraically soluble equation. It remains to develop these conditions in order to capture the class of soluble equations within its ~~true bounds. For that we begin with the simplest equations.~~

~~1st Case.~~ §1 ~~In what circumstance is an equation, all of whose roots are functions of any two among them, soluble by radicals~~

Notes on Dossier 7

NOTE 1: The one-sentence paragraph 'Soit H le groupe ... sera connue' appears between the first few lines of a passage that has been much re-written and suffered much crossing out. It seems to be intended to be the final version of that material. After two false starts (missing from [B & A (1962), p. 96]) '~~De cette manière~~' [~~In this way~~] and '~~En considérant comme rationnelle~~' [~~Considering as rational~~], there appear to be at least three layers here, each with its own local corrections. They are difficult to decipher but the following readings look plausible. The first attempt probably was terminated before completion and read as follows:

> On voit que l'une quelconque des valeurs du radical de degré p s'exprimera en fonction d'une ~~autre~~ quelconque d'entre elles. Donc le groupe que l'on obtiendra en adjoignant à l'équation telle quantité que l'on voudra
>
> [It can be seen that any one of the values of the radical of degree p may be expressed as a function of any one of them. Therefore the group that one obtains by adjoining to the equation whichever quantity one wishes]

The second had a line added in at the start, then re-used the first sentence, but continued differently:

> Soit H le groupe de l'équation quand on lui adjoint cette quantité. On voit que l'une quelconque des valeurs du radical de degré p s'exprimera en fonction d'une ~~autre~~ quelconque d'entre elles. Donc les ~~groupes que l'on obtient~~ ^substitutions de H obtenues^ par l'adjonction de ces valeurs seront les memes pour toutes ces valeurs, en sorte que tous les groupes semblables à H qui diviseront G seront identiques.
>
> [Let H be the group of the equation when one adjoins this quantity to it. It can be seen that any one of the values of the radical of degree p may be expressed as a function of any one of them. Therefore the ~~groups that one obtains~~ substitutions of H obtained by the adjunction of these values will be the same for all these values, so that all the groups similar to H which divide G will be identical.]

The third, the paragraph that survives, has all but the first sentence of this second version deleted and continues with two new lines inserted between deleted lines near the top of the abandoned material (several lines of it, a little more than a quarter of **f. 55 b**).

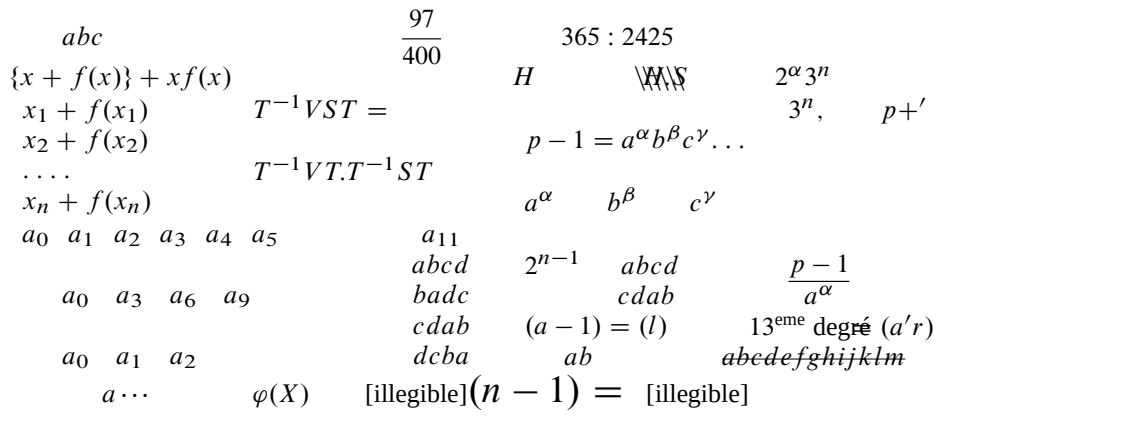
The outcome is rather mysterious. The deleted $\varphi(H)$ was probably a misprint and should have simply been φ. The deleted group $H + HS+$ could perhaps have been $H,\ HS,\ \ldots$, which would make sense, except that the substitution S has not been introduced. What I read as 'des groupes H ...' was read by Robert Bourgne [B & A (1962), p. 97] as 'du groupe H'. It is quite unclear but it looks as if it may have been changed from the latter to the former, or perhaps vice-versa.

NOTE 2: There is a marginal line to the left of the three paragraphs from 'Telle est la condition ...' to '... contiendra plus qu'un nombre premier de substitutions'. It curves round top and bottom like a parenthesis, the bottom continuing for a couple of centimeters between two lines of text, and seems to be intended to bracket those three paragraphs for some reason. Between that bracket and the beginning of the second of these paragraphs 'Nous pourrons raisonner ... cette condition générale' there is a mark that looks like a large tick. If ticks had for Galois the same positive connotation as they do for schoolchildren now, perhaps this was an indication to remind Galois that he wanted to use these paragraphs again later. I have found only vestigial traces of some of their phrases elsewhere, however.

NOTE 3: There are many marginal and other additions that are irrelevant to the mathematics. In **f.55b** the narrow left margin contains a few scribbles, and at the bottom right of the page are two signatures, É Galois and Galois Evariste, written in a careful, almost copperplate hand, one below the other. They are close to a similar signature, Galois Evariste, squeezed into the bottom of the narrow left margin of **f.56a**. The left margin of **f.56b** contains the name Galois, written four times in a similarly elegant hand, one after another, sideways with the page turned a quarter turn clockwise so that they follow each other up the page. They were clearly inserted some time after the page been written.

There are ink marks on **f.56a** which look as if they may have been produced by wiping the pen.

The bottom one-third of **f.56b** was originally blank. It has been used as rough paper and contains jottings that start well over in the left margin, intermingled with a few scribbles.



In the middle of all this there are various jottings in a very small hand, such as $x_1 = f(x)$ with $x = f(x_1)$ written immediately below it, and a scribbled-over long-multiplication of 97 by 25 to get 2425.

VI.3 Dossier 8: A torn fragment

The cover sheet of Dossier 8 is inscribed simply ‘Fragment déchiré’ [Torn fragment]. The content is a single sheet of paper, 34.5 cm × 21.5 cm, of medium weight but heavy enough that writing (even a few thick black lines of crossing out) hardly shows through to the other side. It is folded to make folios 57, 58, each 17.2 × 21.5 cm; at one time it had been folded a second time, lengthways. The lower half of the first page has been torn off so that folio 57 is half a page whereas folio 58 is almost complete, missing only a small triangle from its bottom left. I have searched the manuscripts for the missing piece, but without success.

The material is treated briefly in [Tannery (1906), pp. 241–242], and extensively in [B & A (1962), pp. 30–33, 499–501].

It seems likely that the various sides were written within a few months of each other. If so then, since one of the dates it contains is Février 1832, the material dates from around, or perhaps some of it from a little before, February–May 1832.

This manuscript, or perhaps just **f. 57 a**, is unusual in that it has clear paragraph indentation.

57 a

C'est aujourd'hui une vérité vulgaire que les équations générales de degré supérieure au 4e ne peuvent ~~pas~~ se résoudre par radicaux, c'est à dire que leurs racines ne peuvent ~~pas~~ s'exprimer~~, en~~ ^par des^ fonctions des coëfficients ~~par des~~ qui ne contiendraient d'autres irrationnelles que des radicaux.

Cette vérité est ^devenue^ vulgaire, ^en quelque sorte par ouï-dire^ et quoique la plupart des géomètres en ignorent les démonstrations présentées par Ruffini, Abel, &c. démonstrations fondées sur ce qu'une telle solution est déjà impossible au cinquième degré. ~~Les géomètres depuis long tems ont perdu l'usage des ??~~

BA1962 has 'longtemps'. Two words without letter 'p' clear in *ms*. A hyphen, if there, is submerged in the deletion line.

Il paraît au premier abord, que là ~~doit s'arrêter~~ se termine la de la résolution par radicaux

Bourgne suggests 'théorie' for what was lost when the page was torn.

58 b

1er Mémoire sur les ^conditions de^ résolubilité des equations par radicaux
(Janvier 1830)

2e Mémoire sur le mème sujet.
(Juin 1830)

Mémoire sur les ~~conditions~~ Equations modulaires des fonctions Elliptiques
⌜(février1832)⌝

Word 'Janvier' overwritten with 'février'.

Mémoire sur les ~~intégrales~~ fonctions de la forme $\int Xdx$, X étant une fonction ^quelconque^ Algebrique de x.
(Septembre 1831)

Word 'Octobre' over- written with 'Septembre'. Most of the months are written with lower-case initial letters in BA1962, but capitals are clear in *ms*, except for 'février'.

It is today a commonly known truth that general equations of degree greater than the 4th cannot be solved by radicals, that is, that their roots cannot be expressed as functions of the coefficients that contain no other irrationals than radicals.

This truth has become commonly known to some extent by hearsay and even although most geometers do not know the proofs of it given by Ruffini, Abel, etc., proofs founded upon the fact that such a solution is already impossible for the fifth degree. ~~Since long ago geometers have lost the habit of~~

In the first instance it would seem that the [theory] of solution of equations by radicals would end there

1st Memoir on the conditions for solubility of equations by radicals
(January 1830)

2nd Memoir on the same subject.
(June 1830)

Memoir on the ~~conditions~~ modular equations of elliptic functions
⌜(February 1832)⌝

Memoir on the ~~integrals~~ functions of the form $\int X dx$, where X is an arbitrary algebraic function of x.
(September 1831)

Notes on Dossier 8

NOTE 1: There are some mysteries in the catalogue on **f.58b**. The '1er Mémoire' is clear enough, though one might have expected it to be dated either to February 1830, referring to the lost version that had been submitted for the grand prix de Mathématiques 1830, or January 1831, referring to the version that survives. Likewise, the '2e Mémoire' is clear enough. Of the third item, on the modular equations of elliptic functions, we have no trace, though there is some information in the Testamentary Letter, and perhaps what is referred to here is the second part of the Second Memoir, which Galois might well have intended to complete and write up as an independent piece. If he had done so in February 1832 however then we should surely see some trace of it in what he left behind—whereas I myself have not recognised any. As for the memoir on integrals of algebraic functions, this is listed also in Dossier 10, **f.62b**, and its title resonates with that of **f.112a** in Dossier 24 (see Section VI.19). If it was ever written it could well be the Third Memoir referred to in the Testamentary Letter. That it ever existed, however, I am inclined to doubt—see Note 2 to Dossier 24, pp. 377–379.

NOTE 2: The inside of the folded sheet is filled with jottings. Most of them are arithmetical or algebraic.

The page **f.57b** seems to have been used on two occasions. First, with the paper turned through a quarter-turn clockwise, so that the left edge became the top, Galois wrote

~~18 = 2.3^2~~ ——
24 = 2^3.3 —
~~30 = 2.3.5~~
36 = 2^2.3^2 —
~~42 = 2.3.7~~
45 = 3^2.5 —
48 = 2^4.3 ——
54 = 2.3^3 ——
60 = 2^2.3.5

40 = 2^3.5 —
56 = 2^3.7 —

$M\ p$

8.3

6.5.

3

6.4

60 = 30.2 = 20.3 = 12.5

NOTE 3: I do not see here what is reported in [B & A (1962), pp. 499–500]. Robert Bourgne wrote

> Décomposition des multiples de six qui montrent du moins qu'Evariste Galois reprend à cette époque ses recherches sur la théorie des nombres.
>
> [Decomposition of multiples of six which at least show that at this time Évariste Galois took up again his research in the theory of numbers.]

To the left of a version of what I have transcribed above, [B & A (1962)] has two columns, the first of which contains simply the prime factorisations of the numbers from 2 to 16, and the second the prime factorisations of the multiples of six from 6 to 60. I do not see those two columns in the manuscript. Indeed, I have not found them anywhere in the Galois material—where Bourgne got them from is a mystery. But even if another, even more careful, search were to reveal them somewhere in Galois' writings I do not see that such elementary material provides evidence for a renewal of research in the theory of numbers.

NOTE 4: With the page turned the other way Galois wrote a column of numbers that now proceeds from right to left along the bottom of what remains of the page. Some disappeared when the bottom half of the page was torn off. There remain vestiges of 14 and 15 (presumably) and the numbers 16 to 39 (or just possibly 40) in a slightly wavy column. The last four numbers are crossed out; 17 and 18 are followed with a dot; the numbers 20 to 28, except for 25, are followed with a horizontal line, a sort of extended dash.

NOTE 5: At the start of **f.58a** there are a few jottings, including the word "Libertine", a monogram which I have been unable to decipher, and another word which is thoroughly deleted but which is read as "Marsouin" in [B & A (1962), p. 500] (a reading which looks very plausible to me). This is all written in a careful and elegant script. The rest of the page is filled with elementary algebraic jottings. For example, near the top left of the page one reads $\alpha^n = \beta^n$ and below that to its right $\left(\frac{\alpha}{\beta}\right)^n = 1$, and below that to its left $x^2 - ax - b = 0$. The middle of the page contains

$$\begin{array}{ll} \alpha^2 = \beta^2 & \alpha = -\beta \\ \alpha^4 = \beta^4 & \alpha = \beta\sqrt{-1} \\ \alpha^3 = \beta^3 & \alpha = \beta\,\dfrac{-1+\sqrt{-3}}{2} \\ \alpha^6 = \beta^6 & \alpha = \beta\,\dfrac{1+\sqrt{-3}}{2} \end{array}$$

written rather spaciously. Other formulae, from the looks of them written later, surround this. The right of the page has various quadratic equations such as: $x^2 - x + 1 = 0$, $x^2 - 2x + 2 = 0$, $x^2 - 3x + 3 = 0$, $x^2 - (a+d)x + (ad - bc) = 0$, $x^2 + 2ax + 2a^2 = 0$, $x^2 + ax + a^2 = 0$ (twice), $x^2 + 3ax + 3a^2 = 0$, $x^2 - b = 0$.

It also has material related to these such as various quadratic irrationalities and the expression $(a+d)^2-4(ad-bc)$. The lower one-third of the page has jottings related to fractional linear functions. One reads: $= \dfrac{ax+b}{cx+d}$, an equation whose first member has been lost with the tearing of the page; $a - cy = \dfrac{ad-bc}{cx+d}$, $\dfrac{a}{c} + \dfrac{ad-bc}{c(cx+d)}$, $(cy+d) = a + d - \dfrac{ad-bc}{cx+d}$, and a number of other expressions, a few of which are crossed out.

NOTE 6: The catalogue on **f.58b** is also followed by algebraic jottings. Mostly they are algebraic equations in two variables u, v; but the last few lines seem to be something to do with continued fractions involving the letters a, b. I owe to Massimo Galuzzi the observation that similar jottings occur on **f.159a**; I have not made a systematic search for other related material elsewhere.

VI.4 Dossier 9: Preliminary discussion

Dossier 9, whose cover sheet carries the simple description 'Discours préliminaire', contains just two items: folio 59 is the Galois manuscript written on both sides of the paper; folios 60, 61 are a one-sided copy by Chevalier. The piece has been published three times before: in [Tannery (1906), pp. 245–246]; in an appendix to [Dalmas (1956/82)]; in [B & A (1962), pp. 38–41, 501, 502].

Galois wrote, maintaining a left margin of 6.5 cm, in black ink on large (22.5 cm × 35 cm), thin paper. The writing shows through on the other side. Mostly it shows only faintly, but vigorous crossings-out come through more strongly and can make the reading difficult.

Chevalier (see Note 1 below) dates this piece to September 1830; Bourgne likewise [B & A (1962), p. 501]. This dating, which is based upon what Galois wrote, including what he deleted, in his first paragraph, seems very plausible, but perhaps not as robust as one would like. There are questions in my mind stemming from the fact that Galois wrote 'Le mémoire qui suit' and 'cet ouvrage'. To what could these refer? The actual memoir submitted in February 1830 in competition for the mathematical prize of the Academy of Sciences had been lost. Had Galois kept a copy? I doubt it. There are no other papers of which Galois made and kept copies. There is no extant copy of the work he submitted to the academy in 1829; neither is there a second copy of the *Premier Mémoire* submitted in 1831; nor are any of the minor manuscripts accompanied by duplicates except for a small amount of the version of Proposition I to be found in Dossier 16. Had he, then, already written it anew? I doubt that too. Although there is no conclusive proof, it looks as if the extant manuscript of the *Premier Mémoire* was written in January 1831, as the dating of its foreword (p. 106) indicates; if he had rewritten his work already by September 1830 would he not have submitted it at that time? Why would he have waited until January 1831?

These questions do not, of course, undermine the dating as September 1830. They do however, indicate that this is an instance, by no means the only one, where Galois may have been dreaming, mentally transforming the mathematics that he had in his mind into potential but not actual written form.

Although the last paragraph could be read as supplying a satisfactory conclusion to the piece, with 'Je citerai encore les équations modulaires' as its final cadence, the page ends with a comma, not a full stop. Therefore it seems likely that Galois had intended to continue. Indeed it is possible that Galois did continue and that the continuation is lost. Chevalier's copy ends with a sequence of dots making clear that he judged the piece to be unfinished.

59 a

Discours préliminaire

In BA1962 ‘pour conc’ completed to ‘pour concourir au grand prix de mathématiques’.

Colon is clear in *ms*. C*ms* has comma, T1906/7 has no punctuation here, BA1962 had colon and comma in 1st edition, reduced to comma in 2nd edition (*cf* BA1976 *errata*).

‘Analyse’ lower case in C*ms*, T1906/7; capital is clear in *ms*; capital in BA1962.

Corrected in C*ms* and BA1962 to ‘ne rebute dès’; T1906/7 has ‘ne rebûte des’.

Le mémoire qui suit a été ~~envoyé~~ ^adressé^ ^il y a ~~environ~~ ^~~plus de~~^ ^environ^ sept mois^ ~~par l’auteur~~ ~~pour conc~~ à l’académie des sciences de Paris, et égaré par les commissaires qui devaient l’examiner. ~~Rien d~~ Cet ouvrage ~~n’a donc~~ ~~est donc privé de toute recommandation~~ n’a donc, pour le faire lire, acquis aucune autorité: et cette raison n’était pas la dernière qui retenait l’auteur dans sa publication. S’il s’y décide, c’est par crainte que des geometres plus habiles, en s’emparant du meme ~~sujet~~^champ^, ne lui fassent perdre ^entièrement^ les fruits d’un long travail.

~~La matière~~ Le but que l’on s’est proposé est de déterminer des caractères pour la résolubilité des équations par radicaux. ~~La matière est tellement~~ Nous pouvons affirmer qu’il n’existe pas dans l’Analyse pure de matière plus obscure et peut être plus isolée de~~s autres parties~~^tout le reste^. La nouveauté ~~du sujet~~^de cette matière^ a exigé l’emploi de nouvelles dénominations, de nouveaux caractères. Nous ne doutons pas que cet inconvénient ne rebùte des les premiers pas le lecteur qui pardonne à peine aux auteurs ^mèmes^ qui ont tout son crédit, de lui parler un nouveau langage. Mais enfin, force nous a été de nous conformer à la nécessité du sujet, ~~dont l’importance vaincra sans doute les repugnances~~ ~~et~~ dont l’importance mérite sans doute quelque attention.

~~On nous demandera ici~~ ~~si nous~~ ~~regardons~~^~~croyons avoir résolu completement~~^ ~~le problème~~. ~~S’il s’agit~~ ~~On conçoit ici que notre bùt n’a pas été de trouver des méthodes expéditives~~

Etant donnée une équation algébrique, à coefficients quelconques, numeriques~~,~~ ^ou^ litteraux, ~~&c~~ ~~déterminer~~ ⌜reconnaitre⌝

59 b

si les racines peuvent s’exprimer ~~par~~ ~~en fonction~~ ^en^ radicaux, telle est la question dont nous offrons une solution complète. ~~Quant~~

The word ‘radicaux’ is incomplete in tiny writing at the end of the inserted line; it may have been complete before the manuscripts were bound by the library. Chevalier’s copy is clear.

Si maintenant vous ~~nous~~^me^ donnez une équation que vous aurez choisie à votre gré, ^et que vous désiriez connaitre si elle est ou non résoluble par radicaux^ ~~je nous~~^je^ n’aurai rien à y faire que de vous indiquer le moyen ~~de reconnaitre~~ de repondre à votre question, sans ~~char~~ vouloir charger ni moi ni personne de le faire. En un mot les calculs sont impraticables.

~~Maint~~ Il paraitrait d’après celà ~~celà~~ qu’il n’y a aucun fruit a tirer de la solution que nous proposons. ~~Mais il faut bien remarquer ici que traiter une équation~~ ^~~qu’ici comme partout ailleurs~~^ ~~qu’il est beaucoup plus difficile de traiter une équation~~ ^~~numérique~~^ ~~abst~~ dégagée

Preliminary discussion

The memoir that follows was ~~sent~~ addressed ~~by the author to comp[ete for the great prize in mathematics]~~ to the Academy of Science in Paris ~~more than~~ about seven months ago, and was lost by the referees who should have examined it. ~~nothing~~ This work ~~has therefore not therefore comes without any recommendation~~ has therefore acquired no authority to make one read it: and this is not the last reason that has held the author back from its publication. If he decides to do so, it is in a belief that more experienced geometers, harvesting the same field, may not make him entirely lose the fruits of a long labour.

~~The material~~ The proposed goal is to determine the characteristics for the solubility of equations by radicals. ~~The material is so~~ We can affirm that in pure analysis there does not exist any material that is more obscure and perhaps more isolated ~~from other parts~~ from all the rest. The novelty ~~of the subject~~ of this material has required the use of new terminology, of new symbols. We have no doubt that, after the first steps, this inconvenience will not deter the reader who will hardly pardon even those authors in whom he believes for speaking a new language to him. But in the end we have been forced to conform to the needs of the subject ~~whose importance will no doubt overcome repugnance~~ whose importance no doubt merits attention.

~~We will be asked here if we see~~ believe we have completely solved the problem ~~If it were a matter of~~ It is understood here that our goal has not been to find effective methods

Given an algebraic equation with arbitrary coefficients, numerical or literal, ~~etc. to determine~~ to recognise

whether the roots may be expressed ~~as a function~~ by radicals: that is the question to which we offer a complete solution. ~~As for~~

If you now give ~~us~~ me an equation that you have chosen at will, and you wish to know whether or not it is soluble by radicals, I will have nothing to do other than to indicate to you the way ~~to recognise~~ to respond to your question, without wishing to charge either myself or anyone else with doing it. In a word, the calculations are impracticable.

It would appear from that, that there is nothing fruitful in the solution that we propose. ~~But it is as well to note here that to deal with an equation that here as everywhere else that it is much more difficult to deal with a numerical equation disengaged~~

⌜En effet⌝ Il en serait ainsi si la question se présentait ordinairement sous ce point de vue. Mais, la plupart du temps, ~~on n'est conduit~~ dans les applications de l'Analyse Algebrique, on est conduit à des équations dont on connait d'avance toutes les propriétés: propriétés au moyen des quelles il sera toujours aisé de repondre à la question ~~au moyen~~^par^ des regles que nous exposerons. Il existe en effet pour ces sortes de questions un certain ordre de ~~propriétés~~^considérations^ Métaphysiques qui planent sur tous les calculs, et qui souvent les rendent inutiles. Je citerai pour exemple les équations qui donnent la division des fonctions Elliptiques, et que ~~M~~^le celebre^ Abel a resolues⌜.⌝ ~~sans bien que ne connaisse~~ ^~~sans qu'on~~^ ~~sentait meme~~^~~pourtant~~^ ~~calculé ces équations~~

Ce n'est certainement pas d'après leur forme numérique que ce geometre y est parvenu. ~~Et certes si on lui eut donné un de ces équations sans lui dire à quelle classe de probleme elle app~~ Car ~~tout la m~~ tout ce qui fait la beauté et à la fois ~~la mérite~~ ^difficulté^ de cette théorie, c'est ~~qu'il faut~~ ^qu'on^ ~~n'y a que des~~ ^a ^sans^ cesse à indiquer ~~des~~ la marche des^ ~~calculs~~ ^l'analyse^ et à en~~voir~~ ^prevoir^ les resultats sans jamais pouvoir les effectuer. Je citerai encore les equations modulaires,

Word 'temps' somewhat hastily written and under-formed. Probably the p is there. C*ms*, BA1962 have 'temps', T1906/7 has 'tems'.

T1906/7, BA1962 have 'par exemple'; in *ms* 'pour' is clear. In C*ms*, Chevalier first wrote 'par' then changed it to 'pour'.

In BA1962, 'app' is completed to 'appartenait'.

Terminated with ellipsis in C*ms*. BA1962 has comma and the note 'La suite manque' [The continuation is missing]. Terminated with full stop in T1906/7.

Indeed this would be the case if the question were ordinarily presented from this point of view. But most of the time ~~one is not led~~ in applications of algebraic analysis one is led to equations all of whose properties one knows beforehand: properties by means of which it will always be easy to answer the question ~~by means~~ by the rules which we shall expound. Indeed, for these kinds of questions there exist considerations of a metaphysical nature which hover over all the calculations, and often make them useless. I will cite, for example, the equations which give the division of elliptic functions and which the celebrated Abel has solved [Fragments along the lines: ~~even although one did not know how to calculate these equations~~.]

It was certainly not from their numerical form that this geometer got his result. ~~And indeed, if one had given him these equations without telling him to which class of problems they be[longed]~~ For, all that creates the beauty and at the same time the ~~value~~ difficulty of this theory is ~~that it is necessary~~ that one ~~has nothing but~~ has ceaselessly to indicate the path of the ~~calculations~~ analyses and to anticipate the results, without ever being able to carry them out. I cite the modular equations again,

Notes on Dossier 9

NOTE 1: In the (wide) margin at the top left of the first page (**f. 60 a**) of his copy Chevalier wrote 'Discours préliminaire/ fait en 7bre 1830' on two lines (as indicated by the slash) and encircled by a neat oval. This was crossed out to leave the piece without a header. The copy then begins with a marker (1) that refers to a marginal note. In the margin we see '(1) en note' (and the words 'en note' are encircled by a neat oval just as the original header had been, perhaps to indicate instruction to a printer), followed by the note itself:

> ce qui ~~est~~ suit est un fragment du discours préliminaire ^destiné^ par galois à être placé en tête du mémoire sur la théorie des équations qu'il avait résolu de publier– ce projet ^formé en septembre 1830^ n'a pas de suite; des obstacles de tous genres s'y sont opposés.
>
> [What follows is a fragment of the preliminary discussion destined by Galois to be placed at the head of the memoir on the theory of equations that he had resolved to publish. This project, formed in September 1830, has no continuation; obstacles of all kinds opposed it.]

This note has suffered much amendment. At one time it read

> ce qui ~~est~~ suit est un fragment du discours préliminaire que galois à voulait placer en tête du mémoire sur la théorie des équations qu'il ^[devait?]^ écrire de nouveau pour le publier– l'auteur avait l'intention d'en ^faire la^ publication– ~~le l'aut–;~~ ~~mais~~ ce projet [...]
>
> [What follows is a fragment of the preliminary discussion that Galois wanted to place at the head of the memoir on the theory of equations that he had to re-write in order to publish it—the author intended to make a publication of it—this project [...]]

but deletions and corrections were made at several reprises, and it is not entirely clear what follows what. Moreover, the word after 'des équations qu'il', which is inserted above 'qu'il' and later substantially obliterated by a word inserted at the last revision, is essentially illegible—my reading of it as 'devait' is conjectural.

The lack of capitalisation is typical of Chevalier, and is corrected in the published editions [Tannery (1906), p. 245], [B & A (1962), p. 38]. In the latter some, but not all, of the deleted material ('d'écrire de nouveau pour' and 'mais') is reinstated and incorporated into the note. Tannery supplied '[eu]' to read 'n'a pas [eu] de suite'; also he changed plural 'tous genres' to singular 'tout genre' although plural is clear in Chevalier's manuscript.

On his second page, in the top left corner of **f. 61 a**, Chevalier has written 'Disc. préli.', underlined it, surrounded it with a neat oval, and then mysteriously crossed it out.

NOTE 2: This manuscript carries relatively few jottings. There are a few meaningless diagrammatic doodles in the margin of the recto **f. 59 a**, confined to the bottom left. The bottom left margin of the verso **f. 59 b** contains the name 'Stéphanie' overwritten with 'Évariste', and the initial E intertwined with the initial S. The names occur three times, the initials twice. All of this is written in a careful, rather beautiful calligraphic hand. It is possible that it was written at the same sitting as the *Discours préliminaire* itself; slightly more probable is that it was written at some later time—the pen and ink look just a little different from that of the main text. Also on the verso, in the upper half of the right-hand side are a few words, mostly incomplete, and certainly not forming any sort of phrase or sentence. They are written sideways, with the page turned so that its right-hand side becomes the top. The pen had a fine nib and the writing is thin—it hardly interferes with the text at all. One can just make out 'Mons', 'Monsieur' and 'La', but the remaining bits (four in all) seem to consist of odd letters. It is all rather spaced out. Quite possibly this was on the page *before* Galois used it for his *Discours préliminaire*.

VI.5 Dossier 10: Publication project and note on Abel

The cover sheet of the dossier is inscribed 'Chemise pour "Deux Mémoires d'Analyse pure ..." ' [Cover-sheet for two memoirs on pure analysis]. Bourgne described it [B & A (1962), p. 502] as 'Projet de Publication et Note sur Abel'. The content is folios 62 and 63, each 19.5 cm × 30.5 cm. They are formed by folding a sheet of thin paper 39 cm × 30.5 cm, similar to the material of the Testamentary Letter of 29 May 1832. The paper is thin enough, and the ink is black enough, that some of the writing and most of the crossings out show through to the other side—and add to one's reading difficulties. The material was published as Item B of [Tannery (1906), pp. 247, 248] and it is treated in [B & A (1962), pp. 28–31, 34, 35, 502–504].

In [B & A (1962), pp. 28, 34] the manuscript is dated to the end of the year 1831, but without explanation. Tannery [Tannery (1906), p. 241] wrote

> Cette pièce ne porte pas de date; je pense, d'après l'avis de M. Dupuy, qu'elle a été écrite à la prison Sainte-Pélagie
>
> [This piece carries no date; following the advice of Mr Dupuy I think that it was written in the Sainte-Pélagie prison.]

Certainly the most likely date is late 1831 as suggested by Bourgne; but early 1832 seems just possible too.

It is clear that originally it was intended as a cover sheet for three articles by Galois. Of these, the 'Deux mémoires d'analyse pure' were, presumably, versions of the First and Third Memoirs (as described in the Testamentary Letter), and the 'Dissertation sur la classification des problèmes' may have been conceived as an essay along the lines of the fragments to be found in Dossiers 12 and 13 (compare also the catalogue in **f.58b** in Dossier 8). That the Third Memoir ever existed outside of Galois' mind is, however, doubtful—see Note 2 to Dossier 24. The title page **f.62a**, and the contents page **f.62b** were written at the same time. The lists of names on **f.63a** may have been written at that same sitting, but it looks a little more likely that they were written later. The note protesting Galois' independence from Abel was almost certainly written somewhat later. Perhaps the back page of the cover sheet just happened to be around when Galois needed a blank sheet. This note is one of the pages that does have paragraph indentations.

After the note on Abel was written about one-quarter of the whole sheet was torn from the bottom. This does not affect folio 62, and probably nothing was lost from the lists on **f.63a**. Indeed, the shape of the tear seems to be designed to avoid the last item on the first list while capturing some of the blank page below the second. Or quite possibly the lists were written after the part was torn away—in which case they were certainly written after the note on Abel. It was found by Tannery, given the folio number 64 (then or later), and identified by him as matching **f.63b** exactly.

62 a

Deux mémoires de l'Analyse pure

suivis

d'une dissertation sur la classification des Problèmes

par Évariste Galois.

A rare instance where Galois used the acute accent on his first initial.

62 b

Table des matières

Galois stopped and rewrote after getting to 'la Résolubilité d' (incomplete 'des'). He did not change 'la' to 'les' however, though that is what is in T1906/7, BA1962.

Memoire sur la ~~Résolubilité~~ ^conditions^ pour qu'une equation soit soluble par radicaux.

Mémoire ~~pour qu'une~~ ^sur les ~~intégrales~~^ fonctions de la forme $\int X\,dx$, X étant une fonction algébrique de x.

Dissertation sur la classification des problèmes de Mathématiques, et sur la nature des quantités et des fonctions transcendantes.

Two papers on pure analysis

followed by

a dissertation on the classification of problems

by Évariste Galois.

Table of contents

Memoir on the ~~solubility~~ conditions for an equation to be soluble by radicals

Memoir ~~so that a~~ on ~~integrals~~ functions of the form $\int X\,dx$, X being an algebraic function of x.

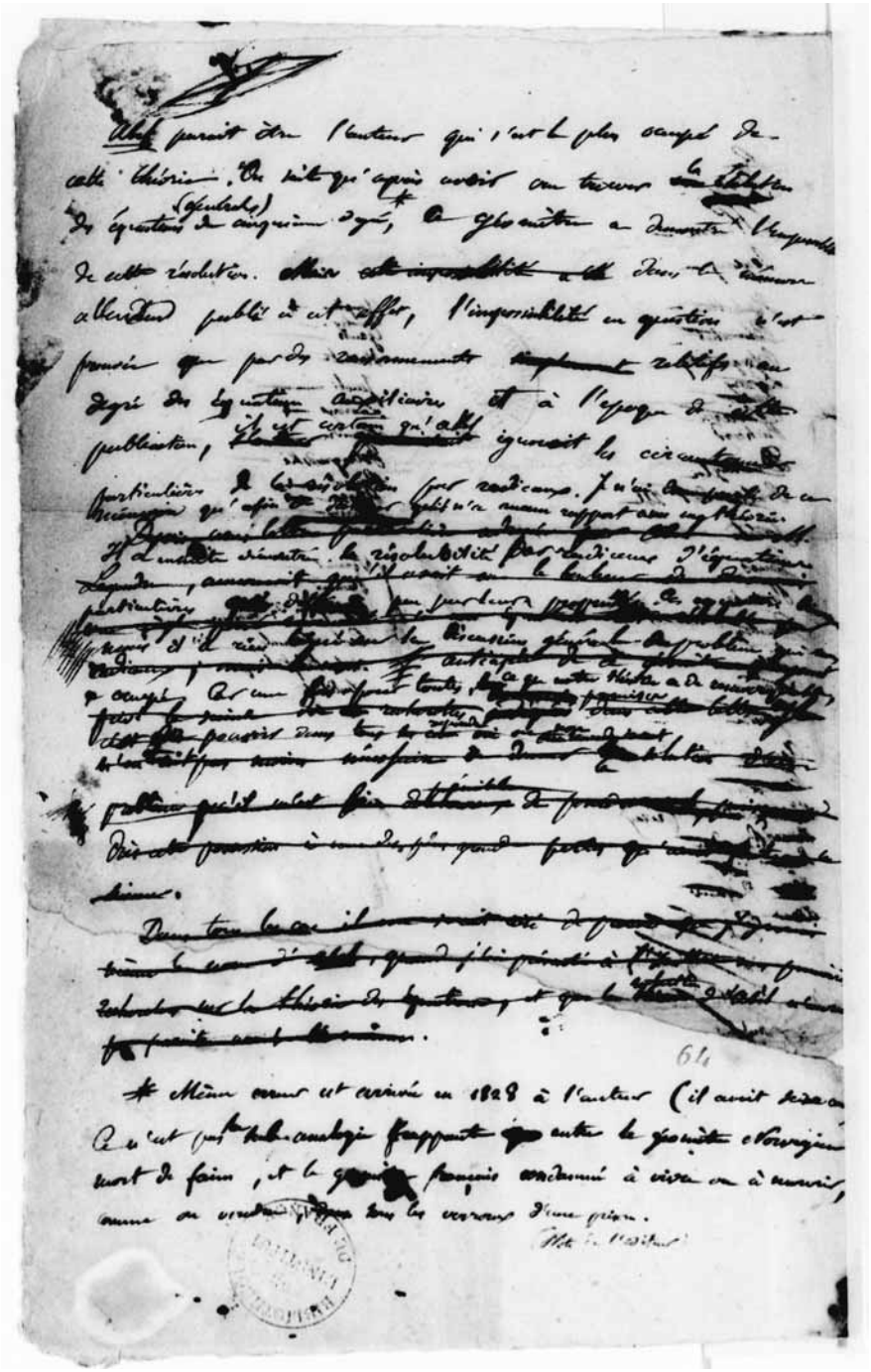
Dissertation on the classification of mathematical problems and on the nature of transcendental quantities and functions.

63 a

~~Ra~~
~~Cauchy~~
Sturm ——
Ségeay ——
~~Hachette~~ ——
~~Blanchet~~ ——
Ostrogradsky ——
~~Leroy~~
~~Vernier~~ ——
~~Navierre~~
Richard
~~Duchayla~~
~~Poullet de l'Isle~~
~~Jacobi~~
~~Gauss~~
Poinsot
~~Cauchy~~
Poisson
~~Lacroix~~
~~Legendre~~
~~Ampère~~
~~Francoeur~~
~~[?]Lefebvre de Fourcy~~
~~[??]Bourdon~~
~~[??] Dinet~~
~~[??] Reynaud~~
~~[??] École Normale~~
~~École Polytechnique~~
~~Faculté de Sciences~~
~~Institut~~

Ampère —~~+~~
Cauchy ———
Gauss ———
Hachette ———
Jacobi ———
Lacroix —~~+~~
Legendre —~~+~~
Ostrogradsky ———
Poinsot —~~+~~
Poisson —~~+~~
Sturm ———

Vernier ——
Richard ——
Bulletin des Sciences ——
École Normale ——
École Polytechnique ——
Institut ——



Note on Abel: Dossier 10, folios 63 verso and 64 recto.

ff. 63 b, 64

Abel parait être l'auteur qui s'est le plus occupé de cette théorie. On sait qu'après avoir cru trouver ~~une~~^la^ ⌜ré⌝solution des équations ^(générales)^ du cinquième degré, * ce géomètre a demontré l'impossibilité de cette résolution. Mais ~~cette impossibilité a été~~ dans le mémoire allemand publié à cet effet, l'impossibilité en question n'est prouvée que par des raisonnements ~~simplement~~ relatifs au degré des équations auxiliaires et à l'époque de cette publication, ~~l'auteur parraissait~~ ^il est certain qu'Abel^ ignorait les circonstances particulières de la résolution par radicaux. ⌜Je n'ai donc parler de ce mémoire qu'afin de déclarer qu'il n'a aucun rapport avec ma théorie.⌝

The second deleted word is almost illegible; 'parraissait' is plausible but fragile; missing from BA1962.

~~Depuis une lettre particulière addressée par Abel à M. Legendre, annoncait qu'il avait eu le bonheur de découvrir une règle pour reconnaitre si une équation était résoluble par radicaux, mais la mort f anticipée de ce géomètre ayant privé la science de⌈s⌉ la recherches indiquées^annoncées promises^ dans cette lettre, il n'en était pas moins nécessaire de donner ^au monde savant^ les la⌄ solution d'un problème qu'il m'est bien douloureux^pénible^ de posséder la s, puisque je dois cette possession à une des plus grandes pertes qu'aura faite la science~~.

Inserted phrase 'au monde savant' missing from BA1962.

This paragraph follows the next one in *ms* (see Note 3). The tear separating **f. 64** cuts it in two diagonally, top left to bottom right.

~~Dans tous les cas il me serait aisé de prouver que j'ignorais même le nom d'Abel, quand j'ai présenté à l'Institut mes premières recherches sur la théorie des équations, et que la théorie^solution^d'Abel n'aura pu paraitre avant la mienne~~.

⌜Il a ensuite démontré la résolubilité par radicaux d'équations particulières qui diffèrent peu par leurs propriétés des equations binomes mais il a rien laissé sur la discussion générale du problème qui nous a occupé. Car une fois pour toutes, ~~la~~ ce que notre théorie a de remarquable, ~~c'est de pouvoir dans tous les cas~~ ^répondre^ ~~oui ou non~~⌝

The footnote lies entirely on **f. 64**.

* Mème erreur est arrivée en 1828 à l'auteur (il avait seize ans). Ce n'est pas ^la^ seule analogie frappante ~~que~~ entre le géomètre Norvegien mort de faim, et le géomètre français condamné à vivre ou à mourir, comme on voudra, ~~dans~~ sous les verroux d'une prison.

(Note de l'éditeur)

The 'éditeur' was Galois himself. See Note 5.

ABEL appears to be the author who has been most concerned with this theory. It is well known that after having believed that he had found ~~a~~the solution of general equations of the fifth degree, * this geometer proved the impossibility of this solution. But ~~this impossibility has been~~ in the German memoir published to this effect, the impossibility in question is proved only by reasoning ~~simply~~ related to the degree of auxiliary equations and at the time of this publication ~~the author appeared~~ it is certain that Abel did not know the particular circumstances of solution by radicals. Thus I have mentioned this memoir only in order to declare that it has no relationship with my theory.

~~According to a certain letter addressed by Abel to Mr Legendre, he announced that he had the pleasure of finding a rule for recognising if an equation is soluble by radicals, but the premature death of this geometer having deprived science of the research indicated announced promised in this letter it became no less necessary to give to the world of learning the solution of a problem which is the more sad painful to me to possess, because I owe this possession to one of the greatest losses that science has suffered.~~

~~In any event it would be easy for me to prove that I did not even know the name of ABEL when I presented my first research on the theory of equations to the Institute and that Abel's theorysolution could not have appeared before mine.~~

Further, he proved the solubility by radicals of some particular equations which differ little in their properties from binomial equations, but he has left nothing on the general discussion of the problem which has occupied us. Once and for all, ~~the~~ what is remarkable in our theory ~~is to be able to answer yes or no in all cases~~.

* The author fell into the same error in 1828 (he was sixteen years old). This is not the only striking analogy between the Norwegian geometer who died of hunger and the French geometer condemned to live or die, who cares, under [behind] the walls of a prison.

(Editor's note)

Notes on Dossier 10

NOTE 1: The table of contents on **f. 62 b** should be compared with the catalogue in Dossier 8, **f. 58 b**, that was probably written a month or two later. Compare also Dossier 11, p. 245.

NOTE 2: The lists of names on **f. 63 a** are somewhat mysterious. Robert Bourgne [B & A (1962), pp. 28, 502] assumed without discussion that they are the people and institutions to which Galois planned to send the material. That seems probable but it raises some questions. Did Galois plan to write out 17 copies, or would he have undergone the expense of pamphlet publication? Why is Libri missing from the list? He had arrived in Paris by this time and, although he was not yet a member of the Academy of Sciences, his work was known to Galois. Why is the list restricted to people living and working in Paris? Gergonne, for example, is missing. He lived and worked in Montpellier but he was editor of the *Annales de mathématiques pures et appliquées* in which Galois had had two articles published.

The first list seems to have been written in random order and 17 of the names are transcribed to the second, which is more systematic. Almost all the names in the first are crossed out. Most of the crossings out are different from Galois' usual style; the great majority are made with a diagonal line as though the names had been crossed through to record their use in the definitive list. Only the first two entries, the mysterious unfinished syllable and Cauchy, are deleted with the usual horizontal line. Five names originally at the bottom of the list are thoroughly scribbled out. Those five names were replaced by Lefebvre de Fourcy, etc. They are so thoroughly inked over that I have found them impossible to read, though it is just possible that the third was Binet and the fourth Libri. Perhaps the crossings out at the top of the list occurred before Galois proceeded with the rest of the list. That could account for the repetition of Cauchy's name lower down.

The second list comes in two parts. First come the names from Ampère to Sturm in alphabetical order. Then there is a line so thick that the ink has corroded the thin paper and there is now a neat and very thin 2cm long cut in the paper along the middle of the line. After that comes quite a substantial gap followed by Vernier, Richard (two of Galois' former teachers) and the four institutional members of the list.

Both lists are embellished with various lines. Seven of the names near the head of the first list are followed by dashes; Ampère is underlined. In the second list the items are all followed by horizontal lines of varying lengths; the lines against five of the names are crossed through near their ends with a short and thick vertical cross-line. These, as Bourgne notes [B & A (1962), p. 503], are the five academicians who appear. Possibly most of the marks, including the diagonal crossings out and the underline of Ampère in the first list, were part of the book-keeping required for the transcription and reduction of the first to create the second list.

Tannery missed Ostrogradsky from the second list. He wrote in a footnote [Tannery (1906), pp. 247–248] that all the names in the list on the left of the page

are crossed out excepting those of Sturm, of Richard and of another which he had not been able to decipher—could this perhaps have been Ostrogradsky? But in fact there are six, not just three, undeleted names: Sturm, Ségeay, Ostrogradsky, Richard, Poinsot and Poisson.

Bourgne [B & A (1962), p. 503] identified the people as follows (corrections from the list of *errata* [B & A (1976), p. xviii] have been incorporated):

> Il n'est pas utile de parler de Cauchy, Gauss ou Jacobi. On remarque que les noms des membres de l'Institut: Ampère, Lacroix, Legendre, Poinsot, Poisson, sont marqués d'une croix, hors Cauchy en exil. Vernier et Richard furent les professeurs d'Evariste Galois à Louis-le-Grand; le premier pendant ses deux années de mathématiques préparatoires; le second fut son excellent maître de mathématiques spéciales. Sturm, géomètre suisse connu par le théorème qui porte son nom et qui avait paru un an plus tôt, était un des rédacteurs du *Bulletin de Ferussac*; il avait publié plusieurs Notes d'Evariste Galois. Parmi les noms de la liste biffée, on trouve en outre: les quatre examinateurs de l'École Polytechnique, Lefebvre de Fourcy, Bourdon, Reynaud que Galois orthographe Raynaud, et enfin Dinet qui le déclara inapte lors de sa deuxième candidature en 1829. On remarque encore Saigey (Galois écrit Segeay) ancien rédacteur du *Bulletin de Ferussac*, Leroy professeur à l'École Normale. Francoeur, à qui l'on devait un cours de Mathématiques pures en deux volumes, était professeur à la Faculté des Sciences après avoir été répétiteur et examinateur à l'École Polytechnique. Navier (qu'on lira à la place de Navierre) avait remplacé Cauchy en 1830 à la chaire d'analyse de la même école. Hachette, Poullet de l'Isle avaient été les examinateurs d'Evariste Galois en Faculté des Sciences pour les examens de licence. Duchayla était inspecteur général de Mathématiques. On s'est étonné de trouver dans la dernière liste lenom [*sic*] d'Ostrogradski; en fait ce jeune mathématicien s'était fait connaître à Paris par un premier Mémoire paru dans le cahier de juillet 1830 du *Bulletin de Ferussac*, puis par d'autres qui suivirent. Evariste Galois le connaissait sans doute par ses publications et peut-être l'avait-il rencontré au bureau du *Bulletin* par l'intermédiaire de Sturm. Tel était le public dont Evariste Galois attendait quelque réponse.

In fact Galois wrote Ségeay; the accent is clear in *ms*.

> [There is no point in discussing Cauchy, Gauss or Jacobi. It is to be noted that the names of members of the Institute, Ampère, Lacroix, Legendre, Poinsot, Poisson, are marked with a cross, except for Cauchy, who was in exile. Vernier and Richard were teachers of Evariste Galois at Louis-le-Grand; the former during his two years of preparatory mathematics; the latter was his excellent master for special mathematics. Sturm, a Swiss mathematician known for the theorem which carries his name and which had appeared one year earlier, was one of the editors of Férussac's *Bulletin*; he had published several notes by Evariste Galois. Among the names on the deleted list are to be found also the four examiners (see [Belhoste (2002)] for more detail)

Ségeay: see above.

of the École Polytechnique, Lefebvre de Fourcy, Bourdon, Reynaud (spelled Raynaud by Galois), and finally Dinet who declared him incapable in 1829 after his second candidature. Notice also Saigey (Galois wrote Segeay), former editor of Ferussac's *Bulletin*, and Leroy, teacher at the École Normale. Francoeur, to whom we owe a two-volume textbook of pure mathematics, was teacher at the Faculty of Science after having been tutor and examiner at the École Polytechnique. Navier (as should be read instead of Navierre) had replaced Cauchy in 1830 in the chair of analysis at that same college. Hachette and Poullet de l'Isle had been the examiners of Evariste Galois in the Faculty of Science for the licentiate examinations. Duchayla was inspector general for Mathematics. It is astonishing to find the name of Ostrogradski in the final list; in fact this young mathematician had made himself known in Paris first by a paper that appeared in the July 1830 issue of Férussac's *Bulletin*, then by others that followed. Undoubtedly Evariste Galois knew him by his publications, and perhaps he had met him at the office of the *Bulletin*, introduced by Sturm. That was the public from which Evariste Galois expected some response.]

NOTE 3: My ordering of the material transcribed from **f.63b** and the fragment **f.64b**, the note about Abel's work and its relationship to that of Galois himself, is based on a hope that I have got it more-or-less back into the order in which it was written. Not quite. Almost certainly the sentence 'Je n'ai donc parler de ce mémoire qu'afin de déclarer qu'il n'a aucun rapport avec ma théorie' was inserted at the end of the first paragraph after some or all of the next passage had been both written and deleted. It seems, however, to be pretty clearly intended as a concluding sentence to that paragraph.

NOTE 4: The fourth paragraph of the note on Abel, which is inserted between the first few deleted lines, probably immediately after the sentence discussed above was written, presents two mysteries. First, its last line 'c'est de pouvoir dans tous les cas répondre oui ou non' is deleted—albeit lightly. Secondly, it is missing from the transcription by Tannery [Tannery (1906), p. 248].

NOTE 5: The editor or publisher referred to at the end of the footnote is none other than Galois himself, for, as Tannery noted in [Tannery (1906), p. 241] the signature '(Note de l'éditeur)' is in Galois' own handwriting.

NOTE 6: Although my primary purpose is to transcribe and translate what is there in the writings of Galois, it seems appropriate here to make some comments on the chronology of Abel and Galois. The note on Abel raises a number of questions. One might guess that it was conceived as a response to the references to Abel in the report by Poisson and Lacroix (see p. 146). But that cannot be certain: after all, one might

have expected Galois to respond immediately, rather than after a delay of nearly six months. Still, it seems very likely that the Academy report was a significant stimulus.

The relevant papers of Abel on equations—leaving aside the work on elliptic and other transcendental functions—are [Abel (1826)], [Abel (1829a)]. Then there are snippets from the posthumously published letters [Abel (1830a)], [Abel (1830b)].

I know of no evidence to say whether or not Galois could read German. Given his mathematical competence, however, I doubt if linguistic incompetence would have inhibited him greatly from understanding [Abel (1826)] in the original. Even if it had, one might have expected his eye to have been caught by the review of the paper in Vol. 6 (1826), pp. 347–353 of Férussac's *Bulletin*. This lengthy and detailed review in French was written by Abel himself: in a letter datelined 'Paris, le 24 octobre 1826' to Holmboe he wrote [Abel (Œuvres), Vol. II, p. 260]:

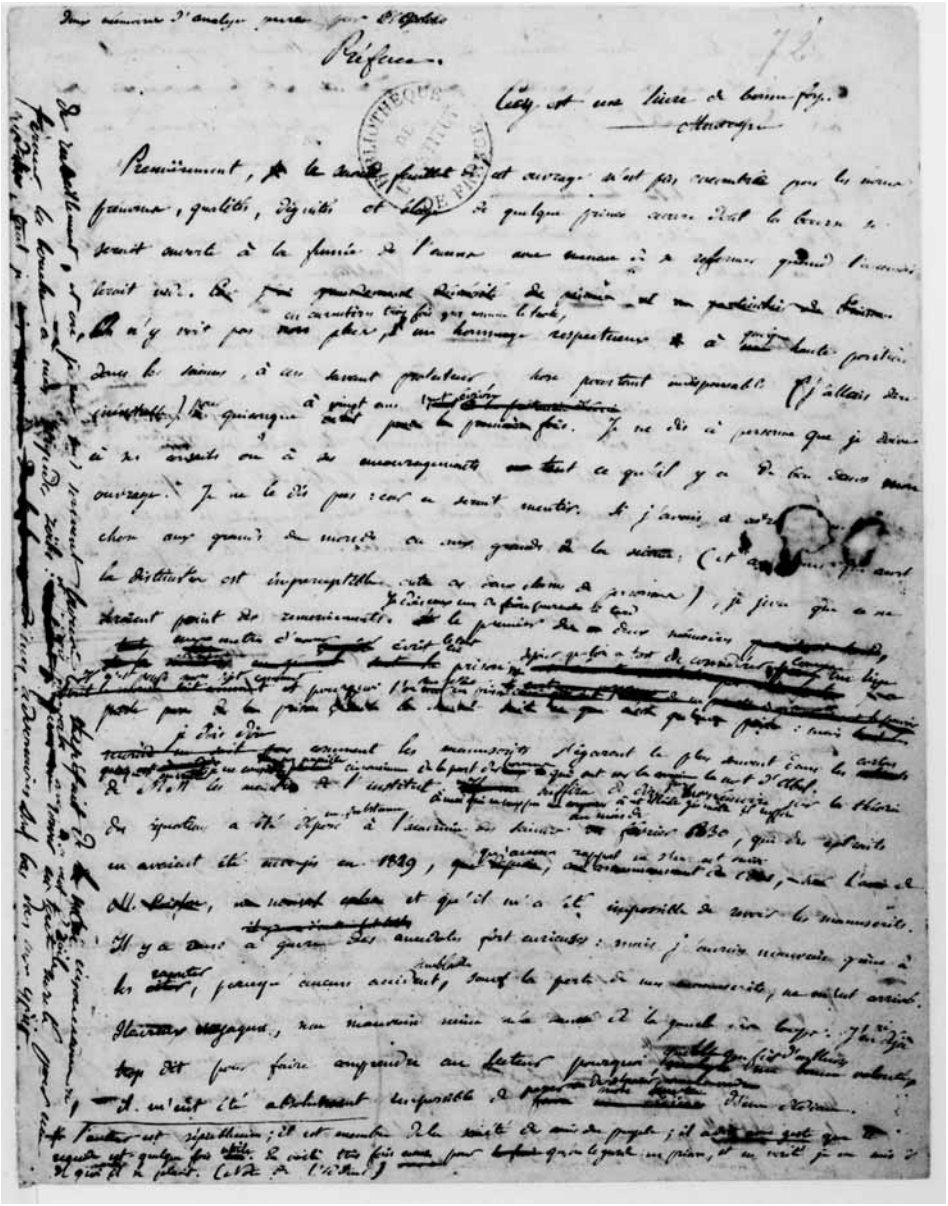
> Un extrait de mon mémoire sur l'impossibilité de résoudre les équations algébriques a été inséré dans le bulletin de M. *Férussac*. Je l'ai fait moi-même. J'ai fait et je ferai d'autres articles pour ce bulletin. C'est un travail très ennuyeux quand on n'a pas écrit le traité soi-même, mais enfin, c'est pour M. *Crelle*, l'homme le plus honnête du monde.
>
> [An abstract of my memoir on the impossibility of solving algebraic equations has gone into Férussac's *Bulletin*. I wrote it myself. I have written and will write other articles for this bulletin. It is very boring work when one has not written the treatise oneself, but after all, it is for Mr Crelle, the most honest man in the world.]

Galois would not have seen the abstract when it was first published, but did he not read back issues?

As for [Abel (1829a)], the reference to the solubility by radicals of some equations that differ little in their properties from binomial equations, clearly refers to this work. Also, there is clear evidence (see [Neumann (2006), p. 386 and § 6]) that Galois had read it before he wrote his *Second Mémoire*, that is, by June 1830.

And when did he see the letters [Abel (1830a)], [Abel (1830b)] that were published posthumously? There is the issue that the first batch of them were in German, which Galois perhaps would not have found easy to read. The second, a long letter to Legendre, is in French and could have given Galois no trouble. They appeared before Galois wrote his ideas up, for the third time, in the *Premier Mémoire*, but, as Poisson pointed out in his report, Galois went further than Abel.

A familiar misquotation from Shakespeare's *Hamlet*, 'methinks he doth protest too much', is apposite. Nevertheless, after adding the evidence and guesswork together, I find that to my ear the protestations in Galois' note on Abel ring true.



Preface for two memoirs: Dossier 11, folio 72 recto.

VI.6 Dossier 11: Preface for two memoirs

The cover sheet of the dossier is inscribed 'Préface pour "Deux Mémoires d'Analyse pur". Texte de Galois et copie par Chevalier.' The content is a copy by Chevalier (on six sheets, folios 66–71 written on one side of the page only) followed by the manuscript of Galois written on a single sheet of thin paper, 41 cm × 26 cm, carefully folded to make folios 72–73 (each 20.5 cm × 20.6 cm), parts of which are used with the paper turned sideways. The date is December 1831 (but see Note 8 below).

This is densely written, with many corrections. It is one of the items which have paragraph indentations, though here they are rather small. Right at the top, almost certainly after the piece was finished, Galois inserted "Deux mémoires d'Analyse pure par E. Galois" in very small writing. The name is over-written onto something else, written with so fine a line that I cannot make it out. Perhaps it was a version of Galois' name, though it does not look like it—but what else it could be defies my imagination. The piece is signed again at the end, where it is date-lined Sainte-Pélagie, X^{bre} 1831, the 'X' to be read as Latin 'decem'.

It is tempting to guess that the two memoirs to which Galois refers are the *Premier Mémoire* and the *Second Mémoire* referred to in the *Lettre testamentaire*. But perhaps not. Perhaps they are the two memoirs referred to in Dossier 10, **f. 62 b**, of which the first is almost certainly the extant *Premier Mémoire*, the second a memoir on integrals of algebraic functions, presumably the *Troisième Mémoire* sketched in the *Lettre testamentaire*, that was quite probably never written (see Note 2 to Dossier 24).

At the top of Chevalier's copy the line 'Deux mémoires d'analyse pure par E. Galois', which faithfully copies the line at the head of **f. 72 a**, is crossed out, as is the note placed by Chevalier in the margin to the left of it: 'voici la préface que Galois voulait mettre à ses deux mémoires.'

Part of this item was first published in [Tannery (1906), pp. 255–258]; it was published *in toto* in [Taton (1947a), pp. 123–127], then again in [Dalmas (1956), pp. 126–136; (1982), pp. 117–127], in [B & A (1962), pp. 2–11, 504–505] and in [APMEP (1982), pp. 18–20]. I have compared the editions of Tannery and Bourgne with the manuscripts of Galois and Chevalier, but I have made no attempt to collate the other editions.

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Deux mémoires d'analyse pure par É. Galois

Préface.

Cecy est un livre de bon foy.
~~M~~ontagne

Corrected to Montaigne in C*ms* and T1906/7. See also Note 1.

Premièrement, ~~je~~ le second feuille de cet ouvrage n'est pas encombré par les noms, prénoms, qualités, dignités et éloges de quelque prince avare dont la bourse se serait ouverte à la fumée de l'encens avec menace de se refermer quand l'encensoir serait vide. ~~Car j'ai grandement démérité des princes et en particulier du Prince~~. On n'y voit pas non plus, en caractères trois fois gros comme le texte, un hommage respectueux à ~~une~~ ^quelque^ haute position dans les sciences, à un savant protecteur, chose pourtant indispensable (j'allais dire inévitable) ~~à~~ ^pour^ quiconque ~~écrit pour la première fois~~ ^à vingt ans ~~veut à la prétention d'écrire~~ ^veut écrire^^. Je ne dis à personne que je doive à ses conseils ou à ses encouragements ~~un~~ tout ce qu'il y a de bon dans mon ouvrage. Je ne le dis pas: car ce serait mentir. Si j'avais à adre chose aux grands du monde ou aux grands de la science (et au temps qui court la distinction est imperceptible entre ces deux classes de personnes), je jure que ce ne seraient point des remerciements. ~~Si~~^Je dois aux uns de faire paraitre si tard^ le premier des deux mémoires ~~paraît si tard~~, ~~si~~ ^~~tout était complété~~^ ~~la rédaction en général sent la~~ ^^aux autres d'avoir écrit le tout en^^ prison, ~~assurément ce n'est ma faute.~~ ^sejour que l'on a tort de considérer comme un lieu^ ~~Je ne parle pas de la prison. Tout le monde sait ce que c'est qu'une prison:~~ ^~~Tout le monde sait comment~~ ^Il n'est pas de mon sujet comment^ et pourquoi^l'on ~~met~~ ^me retient^ en prison * ~~ceux qui ont ^certaines personnes^ l'audace de ne pas être à genoux devant le pouvoir.^~~ *de recueillement, et où je me suis souvent trouvé stupéfait de mon insouciance à fermer la bouche à mes stupides Zoïles: ~~mot dont je puis me~~ ^et je crois pouvoir me^ servir ^de ce mot Zoïle^ en toute sûreté pour ma modestie, tant je ~~suis convaincu de la bassesse de~~ mes adversaires sont bas dans mon esprit.*

Two holes in the paper here. C*ms* supplies 'addresser quelque'.

mais ~~tout le monde ne sait pas~~ ^je dois dire^ comment les manuscrits s'égarent le plus souvent dans les ~~cabinets~~^cartons^ de MM. les membres de l'Institut ~~ce qu'à vrai dire je ne conçois pas bien~~ ^quoiqu'en vérité je ne conçoive ~~pas leur~~ ^une pareille^ insouciance de la part des hommes qui ont sur la conscience la mort d'Abel.^ ~~Il me~~ ^A moi qui ne veux pas me comparer à cet illustre géomètre il suffira^suffira de dire ^que^ mon mémoire sur la théorie des équations a été déposé ^en substance^ à l'académie des sciences ~~en~~ ^au mois de^ février 1830, que des extraits en avaient été envoyés en 1829, ~~que depuis, au commencement de 1831, sur l'avis de M. Poisson, un nouvel extrai~~ ^qu'aucun rapport ne s'en est suivi^, et qu'il m'a été

* L'auteur est républicain; il est membre de la société des amis du peuple; il a dit avec geste que le régicide est quelquefois ~~bon~~ utile. En voilà trois fois assez pour ~~le faire~~ qu'on le garde en prison, et en vérité je ne sais de quoi il se plaint. (Note de l'éditeur)

The 'éditeur' is Galois himself. See Note 3.

Two memoirs in pure analysis by É. Galois

Preface.

This is a book of good faith.
Montaigne

For source and translation see Note 1 on p. 256

First, ~~I~~ the second page of this work is not encumbered with the names, forenames, forms of address, honours and eulogies to some avaricious prince whose purse will be opened by the fumes of incense with the threat of closure when the censer becomes empty. ~~For I have been greatly put in the wrong [undervalued] by princes and particularly by the Prince.~~ Nor will be seen there, in characters three times as large as the text, respectful homage to ~~a~~ some high position in science, to a scientific protector, a matter perhaps indispensable (I was going to say inevitable) for whoever ~~writes for the first time~~ ~~wishes~~ ~~has the pretension to write~~ wishes to write at twenty years of age. I say to no-one that I owe to his counsel or to his encouragement ~~a~~ all that is good in my work. I do not say it because it would be to lie. If I were to have to address something to the grandees of the world or to the grandees of science (and at the present time the distinction between these two classes of people is imperceptible), I swear that it would certainly not be thanks. ~~If the first of the two memoirs appears so late, if~~ ~~all was completed~~ ~~the preparation in general has the feel of prison, assuredly that is not my fault~~ I owe it to the one group to have made the first of the two memoirs appear so late, to the other to have written it all in prison. ~~I do not speak of prison. Everyone knows what a prison is.~~ ~~Everyone knows how~~ Why and how I am ~~put~~ kept in prison is not my subject * ~~Certain people~~ ~~those who have the audacity not to bow the knee to power~~ a stay that one would be wrong to consider as a place of contemplation, and where I have often found myself stupefied by my carefreeness at not closing my mouth to my stupid Zoïles; ~~a word of which I may~~ and I believe I can use this word Zoïle without compromising my humility, ~~convinced as I am of the baseness of~~ so low are my adversaries to my mind.

Zoïle = (bad, envious) critic, after an ancient Greek critic of Homer.

But ~~not everyone knows~~ I must relate how manuscripts disappear most often in the ~~offices~~ files of the members of the Institute ~~which, to tell the truth, I do not understand well~~ even though in truth I do not understand ~~their~~ such carelessness on the part of men who have the death of Abel on their conscience. It is sufficient for me, who do not wish to compare myself with this illustrious geometer, to say that the substance of my memoir on the theory of equations was deposited with the Academy of Science in the month of February 1830, that extracts from it had been sent there in 1829, ~~that after, at the beginning of 1831, on the advice of Mr Poisson a new extract~~ that no report on it followed, and that it was for me

* The author is republican; he is a member of the society of the friends of the people; he has said *with gesture* that regicide is sometimes ~~good~~ useful. That is three times enough to ~~make him~~ keep him in prison, and in truth I do not know what he is complaining of. (Editor's note)

impossible de revoir les manuscrits. Il y a dans ce genre des anecdotes fort curieuses ^~~il y en a des autres fort tristes~~^: mais j'aurais mauvaise grâce à les ~~citer~~ ^raconter^, parcequ' aucun accident ^semblable^, sauf la perte de mes manuscrits, ne m'est arrivé. Heureux voyageur, ma mauvaise mine m'a sauvé de la gueule des loups. J'en ^ai^ déjà trop dit pour faire comprendre au lecteur pourquoi, ~~malgré~~ ^quelle que fût d'ailleurs^ ma bonne volonté, il m'eût été absolument impossible de ~~faire une dédicace~~ ^parer ou de déparer, comme on voudra, mon œuvre^ d'une dédicace.

72 b

En second lieu, les deux mémoires sont courts et nullement proportionnés aux titres ~~à tel point que l'imprimeur quand je lui ai porté les manuscrits, s'est écrié: "Ce n'est que celà!"~~ et puis il y a au moins autant de français que d'algèbre à tel point que l'imprimeur, quand on lui a porté les manuscrits, a cru de bonne foi que c'était~~ent les manuscrits~~ une introduction. En ce point je suis ^complètement^ inexcusable; il eût été si facile de reprendre dans ses éléments toute une théorie, sous prétexte de la présenter sous une forme nécessaire à l'intelligence de l'ouvrage, ou bien mieux sans plus de façon ~~de mêler toute~~ ^d'entrelarder^ une branche de science ~~à~~ ^de^ deux ou trois théorèmes nouveaux, sans désigner les quels⌜!⌝ ~~ou bien~~ ^Il eût été si facile^ encore de ~~mettre~~^substituer^ successivement toutes les lettres de l'alphabet dans chaque équation, ~~en~~^en les^ numérotant par ordre ~~afin de les~~ pour pouvoir reconnaître ~~deux~~ à quelle combinaison de lettres appartiennent les équations subséquentes; ce qui eût multiplié indéfiniment le nombre des ~~lettres~~ ^equations^, si l'on réfléchit qu'après l'alphabet latin, il y a encore l'alphabet grec, que, celui-ci , il reste les caractères allemands, que rien n'empêche de ^se^ servir des lettres syriaques, et ~~aux~~ au besoin des lettres chinoise! Il eût été si facile de transformer dix fois chaque phrase, en ayant soin de faire précéder chaque transformation du mot solennel théorème; ou bien encore d'arriver par notre Analyse à des résultats connus depuis le bon Euclide; ou enfin ~~mettre avant~~ ^faire précéder et suivre^ chaque ~~théorème~~ ^proposition^ ~~des~~ d'un cortège redoutable d'exemples particuliers! Et de tant de moyens je n'ai pas su choisir un seul!

A hole in the paper here. C*ms* supplies 'épuisé'.

En troisième lieu, le premier mémoire n'est pas ~~pur~~ vierge de l'oeil du maitre; un extrait envoyé en 1831 ^à l'académie des sciences^, ~~en~~ a été soumis à l'inspection de Mr Poisson, qui est venu dire ^en séance^ ne point l'avoir compris. Ce qui, ~~pour~~ ^à^ mes yeux fascinés par l'amour-propre d'auteur, prouve simplement que Mr Poisson n'a pas voulu ou n'a pas pu comprendre; mais prouvera certainement aux yeux du public que mon ~~ouvrage~~ ^livre^ ne signifie rien.

Tout concourt donc à me faire penser dans le monde savant, ~~abstractions faite des préventions naturelles~~ ^personnelles^, l'ouvrage que je soumets au public sera reçu avec le sourire de compassion; que ^par les plus indulgents ~~je serai~~ ^me^ taxe⌄ront⌄ de maladresse^; et que pendant quelque tems je serai comparé à Wronski ou ~~aux~~à ces hommes ~~laborieux~~ ^infatigables^ qui trouvent tous les ans une solution nouvelle de la quadrature du cercle. J'aurai surtout à supporter le rire ~~bouffon~~ ^ fou^ de MM les examinateurs ~~pour~~ des

On Wronski see Note 7 on p. 258.

impossible to see the manuscripts again. There are some rather curious anecdotes of this kind ~~there are some others that are rather sad~~: but it would be bad grace in me to ~~cite~~ relate them, because a similar accident, other than the loss of my manuscripts, has not befallen me. Happy traveller, my evil appearance has saved me from the maw of wolves. I have said enough to make the reader understand why, ~~in spite of~~ whatever my good will had once been, it has been absolutely impossible to me to ~~make a dedication~~ adorn or disfigure, as the reader wishes, my work with a dedication.

In the second place, the memoirs are short and in no way proportionate to their titles ~~to the point where, when I carried the manuscripts to the printer, he cried "Is that all it is!"~~ and then there is at least as much French as algebra, to the point where the printer, when the manuscripts were carried to him, believed in good faith that it was an introduction. In this matter I am completely inexcusable; it would have been so easy to review a whole theory from its beginnings under the pretext of presenting it in a form necessary for the understanding of the work, or perhaps better, without more ado ~~to mix~~ to interlard a branch of knowledge with two or three new theorems, without indicating which they are. ~~or even~~ Again, it would have been so easy to substitute successively all the letters of the alphabet into each equation, numbering them in order ~~in order that they~~ so as to be able to recognise ~~two~~ to which combination of letters subsequent equations belong; which would have multiplied infinitely the number of ~~letters~~ equations, if one reflects that after the latin alphabet there is still the greek, that, once the latter was exhausted there remain german characters, that nothing stops one from using syriac letters, and if need be chinese letters! It would have been so easy to transform each sentence ten times, taking care to precede each transformation with the solemn word theorem; or indeed to get by OUR ANALYSIS to results known since the good Euclid; or finally ~~put in front of~~ precede and follow each ~~theorem~~ proposition with a redoubtable line of particular examples. And of all these means I have not thought to choose a single one!

In the third place, it is not as if the first memoir is not ~~pure~~ a virgin in the eye of the master [has not been seen before]. An extract sent in 1831 to the Academy of Science was submitted to the inspection of Mr Poisson, who just said at a meeting that he had not understood it. Which to my eyes, fascinated by the author's *amour propre* [here perhaps self-confidence], proves simply that Mr Poisson did not want or was not able to understand; but certainly proves in the eyes of the public that my ~~work~~ book means nothing.

{[Passage deleted with vertical line] Everything combines therefore to make me think [that] in the scientific world ~~natural personal obstacles apart~~ the work that I am submitting to the public would be received with the smile of compassion; that ~~by~~ the most indulgent ~~I would~~ would tax me with tactlessness [awkwardness]; and that after some time I would be compared with Wronski or these ~~laborious~~ indefatigable men who every year find a new solution to the squaring of the circle. Above all I would have to tolerate the ~~comical laughter~~ giggles of Messrs the examiners ~~for~~ of the

73 a

candidats à l'école Polytechnique, (que je m'étonne ^en passant^ de ne pas voir occuper ^chacun^ un fauteuil à l'académie des sciences, car leur place n'est certainement pas dans la posterité) ~~Que diront ces Messieurs quand ils~~ ^~~Combien~~^ ^et qui ayant tendance à monopoliser l'impression des livres de mathématiques^ ˬn'ˬapprendront ˬpas sans en être formalisésˬ qu'un jeune homme deux fois mis au rebut par eux a aussi la prétention ~~de faire travailler les impr~~ ^d'écrire^ ~~des livres~~, ~~et qui plus est,~~ non des livres didactiques ^il est vrai^, mais des livres de doctrine~~?~~ ~~Il n'y a plus d'enfants! . . . .~~ ~~Je procurerai~~ ~~Ils riront, celà est bien juste: on rit~~
~~Il y a de ma part du dévouement; car~~ ^~~dans~~^ ^Tout se qui précède, je l'ai dit pour prouver que c'est sciemment que^ je m'expose ~~à la au plus cruel supplice,~~ à la risée des sots. ~~Voici ^maintenant^ les raisons qui m'ont engagé à briser~~ ~~tous les obstacles et à publier ////~~ ~~malgré tout le fruit de ^mes^ veilles.~~
Si avec ^aussi^ peu de chances d'être compris, je publi~~erai~~, malgré tout, le fruit de mes veilles, c'est ~~en quelque sorte pour~~ ^~~uniquement~~^ afin de prendre date pour mes recherches ~~afin, en quelque sorte de les légaliser car j'y tiens d'autant plus qu'il n'y a pas une ////////////////~~ ~~et~~ ^c'est afin que les amis que j'ai formés dans le monde avant qu'on m'enterrât sous les verroux, sachent que je suis bien en vie, * ^c'est^ peut-être ^aussi^ dans l'espérance que ces recherches ~~tomberont~~ ^pourront tomber ~~et fructifier~~^ entre les mains de personnes à qui ~~la~~^une^ morgue ^stupide^ ~~ne défendra~~ n'en interdira pas la lecture, ^et que je pense[?] et les diriger dans la nouvelle voie que doit, suivant moi, suivre l'analyse dans ses branches les plus hautes. Il faut bien savoir que je ne parle ici que d'analyse pure; mes assertions transportées aux applications les plus directes des mathématiques deviendraient paradoxales.^
~~Je vais dire maintenant quelque mots de la nouvelle direction que je prévois devoir suivre l'Analyse.~~

Les ^longs^calculs algébriques ont d'abord été ~~fort~~ peu nécessaires au progrès des Mathématiques, les théorèmes fort simples gagnaient à peine à etre traduits dans la langue de l'analyse. Ce n'est ^guère^ que depuis Euler que ~~on le calcul~~ ^cette langue plus brève^ est devenue indispensable à la nouvelle extension ~~donnée par~~^que^ ce grand géomètre ^a donnée^ à la science ~~; et en effet~~. ~~d~~Depuis Euler les calculs sont devenus ^de plus en plus nécessaires, mais^ de plus en plus ~~compliqués~~ ˬdifficilesˬ ^à mesure qu'ils s'appliquaient à des objets de science plus avancés^. Dès le commencement de ce siècle, l'algorithme avait atteint un degré de complication tel que tout progrès était devenu impossible par ce moyen, ~~sauf~~^sans^ l'élégance que les ~~nouveaux~~ géomètres modernes ont su imprimer à leurs recherches, et ~~par~~ ^au moyen de^ la quelle l'esprit saisit promptement et d'un seul coup un grand nombre d'opérations.

Something so thoroughly crossed out that I cannot read it. BA1962 has 'un long temps'. Word 'formés' very hard to read: C*ms* has 'trouvés'; I follow BA1962.

C*ms* has 'stupide morgue'.

Very hard to read, but 'suivant moi' is what I read, very clearly in f. 73 b, lines 3 and 6 (although in line 3 it is deleted). C*ms* has 'à mon avis' and BA1962 has 'selon moi'.

C'est afin d'affliger les savants qui, rougissant de gloser eux mèmes sur des jeunes gens, trouvent cependant moyen d'entr'aider par un silence calculé, les ignorants et les paresseux ~~qui ne sachant qui ^[???]^ [???] à [???] [???]~~ ^dont l'occupation est de taxer de charlatanisme^ quiconque a été assez heureux pour être précédé par sa réputation.

candidates for the École Polytechnique (who, by the way, I am astonished not to see, each of them, occupying an easy chair in the Academy of Science, for they certainly have no place in posterity) ~~What will these men say when they~~ ~~How many~~, and who, having a tendency to monopolise the printing of mathematics books, will not understand without it being put to them formally, that a young man twice failed by them might also have the pretension ~~to make the prin[ters] work~~ to write ~~books~~, ~~and who in addition is~~ not educational books, it is true, but books of doctrine. ~~There are no more children! . . . . I shall get~~ ~~They will laugh. That is quite fair: one laughs~~ ~~There is some self-sacrifice on my par; for~~ I have said all that precedes to prove that it is knowingly that I expose myself ~~to the most cruel torture~~ to the laughter of fools. ~~These, now, are the reasons that have, in spite of everything, led me to break down all obstacles and publish the fruits of my nightly labours~~}

If, with so little chance of being understood, I publish, in spite of everything, the fruit of my nightly labours it is ~~as a sort of~~ ~~simply~~ in order to establish the date of my research ~~in order to give it some sort of legality because I hold more than ever that it has no~~ ~~and~~ it is in order that the friends I have formed [found] in the world before I was entombed behind bars should know that I am alive and well. * It is perhaps also in the hope that this research ~~will fall~~ could fall ~~and fructify~~ into the hands of people whom a stupid morgue ~~will not prevent~~ will not forbid them reading it, and, I think, direct them into a new path which must, according to me, follow analysis into its highest branches. One should know that I speak here only about pure analysis; my assertions transferred to the most direct applications of mathematics would become paradoxical [false].

Word 'precedent' would complete the sense, but differs greatly from the reading in BA1962.

~~I shall say a few words now about the new direction which I foresee that Analysis must follow.~~

Long algebraic calculations were at first hardly necessary for progress in Mathematics; the very simple theorems hardly gained from being translated into the language of analysis. It is only since Euler that this briefer language has become indispensable to the new extensions which this great geometer has given to science. Since Euler calculations have become more and more necessary but more and more ~~complicated~~ difficult, at least insofar as they are applied to the most advanced objects of science. Since the beginning of this century algorithmics had attained such a degree of complication that any progress had become impossible by these means, ~~except~~ without the elegance with which ~~new~~ modern geometers have believed they should imprint their research, and by means of which the mind promptly and with a single glance grasps a large number of operations.

It is in order to cause distress to the scientists who, blushing to find fault themselves with young people, nevertheless find ways to collaborate by calculated silence with the ignorant and the lazy ~~who, not knowing who~~ whose occupation is to accuse of charlatanism anyone who is so happy as to be preceded by his reputation.

Il est évident que l'élégance si vantée et à si juste titre, n'a pas d'autre but.

Du fait bien constaté que les efforts des géomètres les plus avancés ont pour objet l'élégance, ~~il résulte que nous en sommes venus~~ ^~~la science en est~~ est venu^ ~~à ce point~~ on peut donc ~~déduire~~ ^conclure^ avec certitude ~~que plus les recherches des on avance, plus il est~~ ^qu'il devient de plus en plus nécessaire d'embrasser plusieurs opérations ~~d'un seul coup d'œil~~, ^à la fois^ ~~en d'autres termes~~ ^parceque^ ~~moins~~ l'esprit ~~a~~ ^n'a plus^ le tems de s'arrêter ~~à chaque~~ aux détails.

At first sight 'l'élégance' seems crossed out. But this is a crossing-out on the other side of the paper showing through.

Or je crois que les simplifications produites par l'élégance des calculs, (simplifications intellectuelles, s'entend; de matérielles il n'y en a pas) ont leurs limites; je crois que le

73 b

moment arrive⌜ra⌝ où les ~~calculs [??]~~ ^transformations algébriques^ prévues par les spéculations des Analystes ne trouveront plus ni le tems ni la place de se produire; à tel point qu'il faudra se contenter de les avoir prévues. ~~Telle est, suivant moi, la mission des géomètres futurs; telle est la voie où je suis entré.~~^Je ne veux pas dire qu'il ~~[??] [?]~~ n'y a plus rien de nouveau pour l'analyse sans ce secours: mais je crois qu'un jour sans cela tout serait épuisé.^

Addition of 'ra' to correct 'arrive' to 'arrivera', and insertion of new last sentence of this paragraph, both made with a lightish gray ink—though the original last sentence was crossed out with the black ink of the text.

~~Embrasser~~ Sauter à pieds joints sur ~~C~~ces calculs, ~~embrasser~~ grouper les opérations, les ~~distinguer~~ classer suivant leurs difficultés et non suivant leurs formes; telle est, suivant moi, la mission des géomètres futurs; telle est la voie où je suis entré dans cet ouvrage.

~~C'est~~ ^~~Ceci~~ ~~Ici ^est^~~ ^Ici rien de semblable; ici ~~est~~ l'on fait^ l'analyse de l'analyse: ici les calculs les plus ~~généraux de l'algèbre~~ les plus ^élevés ✳ ^ ^executés jusqu'a présent^ sont considérés comme des cas particuliers, qu'il a été utile, indispensable de traiter, mais qu'il serait funeste de ne pas abandonner pour des ~~théories~~^recherches^ plus larges. Il sera ~~assez~~ tems d'effectuer des calculs prévus par cette haute analyse et classés suivant leurs difficultés mais non spécifiés dans leur forme, quand ~~l'application les réclamera~~ ⌜les spécialités d'une question les réclamera.⌝

Word 'élevés' and footnote insertion made in grey ink as above.

Il ne faut pas confondre ~~cette~~ ^l'^opinion ~~toute [?]able~~ que j'emets ici, avec l'affectation que certaines personnes ont d'éviter ^en apparence^ toute espèce de calcul, ^en^traduisant par des phrases fort longues ce qui s'exprime très brièvement par l'algèbre, et ~~[??] [???]~~^ajoutant^ ainsi à la longueur d~~e~~s ~~calcul~~ opérations, les longueurs d'un langage qui n'est pas fait pour ~~elles~~^les exprimer^. Ces personnes^-là^ sont en arrière de cent ans.

Correction to end of paragraph made in gray ink.

La thèse ^générale^ que j'avance ne pourra etre bien comprise que ~~par les personnes~~ qui⌜and⌝ ~~liront~~^on lira^ attentivement mon ouvrage qui en est une application,: non que ~~la~~^ce point de vue^ théorique ~~en~~ ait précédé ~~la pratique~~^l'application^; mais je me suis demandé, mon livre terminé, ce qui le rendrait si étrange

Some rough marginal markings indicate that this paragraph is to be moved up to precede the one above. That is where it is placed in C*ms*, T1906/7, and BA1962.

Originally 'non que la théorie'.

Footnote content inserted immediately above line of text, without its usual matching ✳.

(les fonctions elliptiques)

It is clear that such vaunted elegance, and so properly claimed, has no other goal.
From the well established fact that the efforts of the most advanced geometers have elegance as their object, ~~it follows that we have come to science as come to on this point~~ one may therefore ~~deduce~~ conclude with certainty ~~that the further the research of one advances, the more it is~~ that it becomes more and more necessary to embrace several operations ~~at a single glance~~ at once ~~in other words~~ because ~~the less~~ the mind does not have the time any more to stop ~~at each~~ at details.
Thus I believe that the simplifications produced by elegance of calculations (intellectual simplifications, of course; there are no material ones) have their limits; I believe that the

time will come when the ~~calculations~~ algebraic transformations foreseen by the speculations of analysts will find neither the time nor the place for their realisation; at which point one will have to be content with having foreseen them. ~~That is, according to me, the mission of future geometers; that is the path that I have entered.~~ I would not wish to say that there is nothing new for analysis without this rescue; but I believe that without this one day all will run out.
~~Embrace~~ Jump with both feet on calculations. ~~embrace~~ put operations into groups, ~~distinguish~~ class them according to their difficulty and not according to their form; that is, according to me, the mission of future geometers, that is the path that I have entered in this work.
~~It is~~ ~~This~~ ~~Here is~~ Here is nothing like that; here analysis of analysis is done: here the ~~most general calculations of algebra~~ highest calculations * executed up to now are considered as particular cases which have been useful, indispensable to deal with, but which it would be fatal not to abandon for broader research. It will be ~~enough~~ time to carry out calculations foreseen by this high analysis and classed according to their difficulty, though not specified in their form, when ~~application claims them~~ the details of a question require them.

The opinion that I express here should not be confused with the affectation that certain people apparently have of avoiding any kind of calculation, translating into very long sentences what may be expressed very briefly by algebra, and thus adding to the boredom of their operations the boredom of a language which is not made to express them. Such persons are a hundred years behind the times.

The general thesis that I am advancing cannot be well understood except ~~by people who will read~~ when my work, which is an application of it, is read attentively: it is not that this theoretical point of view has preceded ~~the practical~~ applications, but I ask myself, my book being finished, what renders it so strange

* (elliptic functions)

à la plupart des lecteurs, et ^~~en~^ ~~j'ai cru en trouver la raison dans le~~ rentrant en moi-même, j'ai cru observer cette tendance de mon esprit à éviter ~~tous les~~ calculs ^dans les sujets que je traitais^, ~~Ce~~ et qui plus est, ^j'ai reconnu^ une difficulté insurmontable à qui voudrait les effectuer ^généralement^ dans les matières que j'ai traitées.

Insertion 'dans . . . traitais' made in grey ink.

On ~~ne s'étonnera pas~~ que ~~dans~~^traitant^ des sujets aussi ~~difficiles~~ ^nouveaux^, hasardé dans une voie aussi insolite, bien souvent des difficultés se sont presentées que je n'ai pu vaincre. Aussi dans ces deux mémoires et surtout dans le second qui est le plus récent, ~~touchant~~ ^trouvera-t-on souvent^ la formule "je ne sais pas" ~~s??-elles souvent~~. La classe des lecteurs dont j'ai parlé au commencement ne manquera pas d'y trouver à rire. C'est que malheureusement on ne se doute pas ^que^ le livre le plus précieux du plus savant serait celui où il ~~aurait dit~~^dirait^ tout ce qu'il ne sa~~va~~it pas, ^c'est qu'on ne se doute pas^ qu'un auteur ne n jamais tant à ses lecteurs que quand il dissimule une difficulté. Quand ^la concurrence c'est-à-dire^ l'egoïsme ne règnera plus dans les sciences, quand on s'associera pour étudier, ~~au lieu d'envoye~~ ~~quand on aura renoncé au sys destructeur de la concurrence~~, au lieu d'envoyer aux académies des paquets cachetés, on s'empressera de publier ses moindres observations pour peu qu'elles soient nouvelles, et on ajoutera: "Je ne sais pas le reste"

Word 'généralement' inserted in gray ink.

A hole in the paper here: in C*ms* 'n' is completed to 'nuit'.

A hole here: 'sys' completed to 'système' in BA1962.

The last lines, 'pour étudier ... Evariste Galois', occupy the left margin with the page turned a quarter turn anti-clock-wise, so that they read from top to bottom with the left side having become the bottom edge.

De Ste Pelagie X^{bre} 1831. Evariste Galois

to the majority of readers, and, ~~I believe I have found the reason for that in the~~ looking into myself, I believe I observe a tendency in my mind to avoid ~~all~~ calculations in the subjects that I treat, and moreover, I have recognised an insurmountable difficulty for whoever would wish to effect them generally in the matters that I have treated.

One ~~cannot be surprised~~ that ~~in~~, treating such ~~difficult~~ new subjects, taking a chance on such a strange road, pretty often difficulties presented themselves that I was unable to overcome. Even in these two memoirs, and especially in the second which is the more recent, ~~touching~~ the formula "I do not know" will often be found. The class of readers of whom I have spoken at the beginning will not fail to find something laughable there. Unhappily one cannot doubt that the most precious book of the greatest scientist will be that in which he ~~will have said~~ says everything that he does not know; one cannot doubt that an author never [betrays] his readers so much as when he hides a difficulty. When competition, that is to say egoism reigns no more in science, when one gets together to study, ~~instead of sending~~ ~~when the destructive system of competition will have been renounced~~ instead of sending to the academies closed packets, one will hasten to publish one's least observations for their little novelty, and one will add: "I do not know the rest".

From Ste Pelagie X^{bre} 1831. Evariste Galois

Notes on Dossier 11

NOTE 1: The motto that Galois put at the head of his fulmination—though probably intended for a projected book rather than merely its preface—is a slight misquotation from Montaigne's *Essais*, Book 1. There the first sentence of the preface, dated 12 June 1580, is 'C'est icy un livre de bonne foy, lecteur'. A simple translation could, just, be 'Reader, here is a book in good faith', but Cohen (1958) offers 'This, reader, is an honest book', and Screech (1991) has 'You have here, Reader, a book whose faith can be trusted' (see the translations mentioned under [Montaigne (1580)]).

Modern spelling has 'ici' for 'icy' and 'foi' for 'foy'. There have been many editions of Montaigne's work and it is possible that Galois was faithful to an early 19th-century version that he had read, but I very much doubt it. All early editions I have seen have the sentence as above. More likely is that in prison he wrote from memory and slightly mis-remembered.

The thin horizontal line looks like Galois' usual sign-off line. It seems to have been drawn below the motto before Galois wrote its author's name.

NOTE 2: Tannery published just under half of this preface and explained why (see T1906/7, p. 242):

> Après l'avoir lue et relue, je me suis décidé à n'en publier qu'un extrait, la fin, un peu moins de la moitié; c'est que je suis arrivé à cette conviction qu'en écrivant les premières pages, Galois n'était pas en possession de lui-même: le malheureux enfant était en prison, il avait la fièvre, ou il était encore sous l'influence des boissons que ses compagnons de captivité le forçaient parfois d'avaler. Dans ces pages, sans intérêt scientifique, la continuelle ironie fatigue par sa tristesse; les injures à Poisson, aux examinateurs de l'École polytechnique, à tout l'Institut sont directes et atroces; certaines allusions sont obscures et veulent être perfides; les plaisanteries, assez lourdes, se prolongent d'une façon fastidieuse et maladive; il y a tel passage où l'écriture est si désordonnée, si surchargée que Chevalier lui-même n'a pu, à ce qu'il semble, le lire complètement; telles notes qu'il n'a pas voulu reproduire dans sa copie. [...] Vers le milieu de la préface la pensée se calme; c'est de mathématiques qu'il s'agit; la sérénité revient.
>
> [After having read and re-read it I have decided to publish only an extract from it, its end, a little less than half of it. I have come to believe that when he wrote the first pages Galois was not in full possession of himself: the unhappy child was in prison, he had a fever, or he was still under the influence of the drink which his companions in captivity forced him to take sometimes. In these pages, of no scientific interest, the continual irony wears one down by its sadness; the insults to Poisson, to the examiners at the École Polytechnique, to all the Institute are direct and dreadful; certain allusions are obscure and may well be deceitful; the somewhat heavy pleasantries are extended in a tedious

and sickly way. There is a passage in which the writing is so disordered, so overwritten, that Chevalier himself was unable, or so it seems, to read it fully; there are some notes that he did not wish to reproduce in his copy. [...] Towards the middle of the preface thoughts calm; it is the mathematics which matters; serenity returns.]

NOTE 3: The footnote to 'comment et pourquoi l'on me retient en prison' was not copied by Chevalier. Although written in the third person and signed 'Note de l'éditeur' it is undoubtedly by Galois himself. He played the same trick at the end of his note on Abel, Dossier 10, **f. 64 a**.

Chevalier suppressed also the second footnote 'C'est afin d'affliger ... sa réputation', and he incorporated the third as a parenthetic insertion into the text.

NOTE 4: Bourgne [B & A (1962), p. 2] reconstructed the middle passage on **f. 72 a** as having originally read thus:

Si le premier de ces deux mémoires paraît si tard, si la rédaction en général sent la prison, assurément ce n'est ma faute. Je ne parle pas de la prison. Tout le monde sait ce que c'est qu'une prison: mais tout le monde ne sait pas comment les manuscrits s'égarent le plus souvent dans les cabinets de MM. les membres de l'Institut. Il me suffira de dire que le mémoire sur la théorie des équations a été déposé à l'académie des sciences en février 1830, que des extraits en avaient été envoyés en 1829, que depuis, au commencement de 1831, sur l'avis de M. Poisson un nouvel extrait

This makes good sense, although I think that the passage changed three or four times. The passage from 'Je dois aux uns' to 'impossible de revoir les manuscrits' was copied by Chevalier (C*ms*, **f. 66 a**, **f. 67 a**), with his usual reluctance to use capital initial letters, as

je dois aux uns de faire si tard le premier des deux mémoires, aux autres d'avoir écrit le tout en prison séjour que l'on a tort de considérer comme un lieu de recueillement, et où je me suis souvent trouvé stupéfait de mon insouciance à fermer la bouche à mes stupides Zoïles: et je crois pouvoir me servir du mot de Zoïle en toute sûreté pour ma modestie, tant mes adversaires sont bas dans mon esprit. il n'est pas de mon sujet de dire comment et pourquoi l'on me tient en prison: mais je dois dire comment les manuscrits s'égarent le plus souvent dans les cartons de MM. les membres de l'institut quoique je ne conçoive pas une pareille insouciance de la part de ceux qui ont sur la conscience la mort d'Abel. à moi qui ne veux pas me comparer à cet illustre géomètre, il suffira de dire que mon mémoire sur la théorie des équations a été déposé en substance à l'académie des sciences au mois de février 1830, que des extraits en avaient été envoyés en 1829, qu'aucun rappel ne s'en est suivi, et qu'il m'a été impossible de revoir les manuscrits.

NOTE 5: The status of the paragraph included as a footnote to **f. 73 a** is unclear. It is squeezed into the left margin, written with the page turned a quarter turn anti-clockwise, so that it appears as three lines from top to bottom of the page. Part of the margin had already been used. Three lines inserted as an afterthought into the middle of the page, the three lines 'et que je pense . . . Il faut bien . . . deviendraient . . .' start in the margin right at the edge of the page (with the words cited here), and the additional material carefully avoids them. Thus it was added later, perhaps after **f. 73 a** had been completed. It was omitted entirely by Chevalier. Possibly Galois intended it as an addition to the text. On the other hand, that would make the text even more incoherent than it had already become with its many amendments. Bourgne saw it as a footnote, and I am inclined to follow his judgment in this matter.

NOTE 6: It looks to me as if at one time the paragraph on **f. 73 a** beginning 'Si, avec aussi peu de chances . . .' ended '. . . n'en interdira pas la lecture' and Galois then drew a line below it to indicate one of his cadences (as he did in a number of places elsewhere). He then inserted above that line the sentence (which in the manuscript fits, just, on one line) 'Je vais dire maintenant . . . devoir suivre l'Analyse'. Below the line the paragraph 'Les ^longs^ calculs algébriques . . .' then starts afresh in slightly larger handwriting. The afterthought embodied in the continuation of the paragraph 'Si, . . . lecture' involved reusing some of the words in the line 'Je vais . . . l'Analyse', crossing out the others and squeezing extra material in between the lines and into the margin. Chevalier omitted the (clearly legible) part of the afterthought 'mes assertions . . . paradoxales.' Other parts are hard to read, and I have mostly followed [B & A (1962)]. As indicated above, these corrections were made before the paragraph presented here as a footnote was inserted.

NOTE 7: The (deleted) reference to Wronski on **f. 72 b** clearly classifies him with circle squarers. Józef Maria Hoëne (or Hoehne) was born in 1776 in Poland. He added the name Wronski (or Wrónski) in about 1810, and from then on usually signed his publications Hoëné Wronski. He was a polymath, many of whose writings were in the philosophy of mathematics, but some, such as [Wronski (1812)] engaged with mathematics itself. They were not held in high regard by his contemporaries. Jerzy Dobrycki explains in the *Dictionary of Scientific Biography* (Supplement): 'His aberrant personality, as well as the thesis of his esoteric philosophy [. . .], tempt one to dismiss his work as the product of a gigantic fallacy engendered by a troubled and deceived mind'. Although Dobrycki goes on to indicate that his reputation improved posthumously (his name lives on, for example, in the Wronskian determinants that are important in the theory of differential equations), one can well imagine that his contemporaries saw him as he is portrayed in that graphic sentence. Hence Galois' linking of him with circle squarers.

NOTE 8: The date at the end is mysteriously given as 'septembre 1831' by Dalmas. But 'X^{bre} 1831' (December 1831) is pretty clear in the manuscript, and that is how all other editors have read it. Besides, in September 1831 Galois was not yet twenty, which is the age he reports in the second sentence of this piece.

VI.7 Dossier 12: On the progress of pure analysis

The cover sheet of Dossier 12 is inscribed 'Discussions sur les progrès de l'Analyse pure.—Il sera en dehors de la gravité ...' followed by 'Texte de Galois et copie par Chevalier.' The dossier consists of folios 74–78, the arrangement of which is not straightforward. Folios 74 and 75 started out as a single sheet 31.5 cm × 20 cm. Galois used one side to write, or begin, his essay discussing the progress of pure analysis before folding it along the middle with his text inside. Later he re-used the paper, now folded as a two-page pamphlet 15.6 cm × 20 cm, back and front to write more (with the previous writing filling the inside sideways). In the library red-ink pagination there are two folio numbers 75. One is in the standard sequence in the top right corner of the item as it is now bound into the volume of manuscripts; the other is in the top right with the page turned 90° anti-clockwise, the orientation when Galois started writing. This folio number covers the whole of the inside of the pamphlet (which otherwise is **f. 74 b** + **f. 75 a**). It is this one that we use in the pagination of the transcription (as did Robert Bourgne).

This piece has been dated relatively precisely to about the end of April 1832—see [Tannery (1906), p. 243] (cited in Note 3 below), [Dalmas (1956/82), p. 115], and [B & A (1962), p. 506].

There is a question whether we have here one essay in two parts, or two essays. As it is now bound into the dossier, the first essay, or part, fills the inside of the pamphlet sideways, the second uses its outside. Tannery treats it as two pieces, others treat it as just one. I have chosen to present it as a single item. For further discussion of this see Notes 1, 2 below.

This is an item that has generous paragraph indentations.

75 a

Sciences Mathématiques

Discussions sur les progrès de l'analyse pure

~~Il est~~ ~~Quand on jette les~~

De toutes les ~~sciences~~ ^connaissances^ humaines, on sait que l'Analyse pure est la plus immatérielle ~~et~~, la plus éminemment ~~logique~~^logique^, la seule qui n'emprunte rien aux manifestations des sens. Beaucoup en ~~déduisent~~ ^concluent^ qu'elle est, dans son ensemble, la plus méthodique et la mieux coordonnée. Mais c'est erreur. Prenez un livre d'Algèbre, soit didactique, soit d'invention, et vous n'y verrez que ~~chaos!~~'un amas confus de propositions dont la regularité contraste bizarrement avec le désordre du tout. Il semble que ~~chaque~~^les^ idée⌜s⌝ coûte⌜nt⌝ ^déja^ trop à l'auteur pour qu'il se donne la peine de les lier, et que son esprit epuisé par les conceptions qui sont la base de son ouvrage, ne puisse enfanter une mème pensée qui préside à leur ensemble.

I doubt the BA1962 reading 'chaos!'. To me their ! misreads the apostrophe inserted as a correction into 'qu'un', and the riser comes in the wrong place for the h. But it makes sense, and I have no better reading.

Que si vous rencontrez une méthode, une liaison, une coordination, tout cela est faux et artificiel. Ce sont les divisions ~~arbitraires~~ ^sans fondements^, des rapprochements arbitraires, un arrangement tout de convention ~~semblable à cette mémoire artificielle que les anciens avaient imaginée. Celà~~ ^~~Ce qui~~^ ^^Ce défaut^^ ^pire que l'absence de toute méthode^arrive surtout dans les ouvrages didactiques ^la plupart^ composés ~~la plupart~~ par des hommes qui n'ont pas l'intelligence de la science qu'ils professent.

~~Ce défaut d'une méthode vraie, d'une marche simple ^et^ claire dans tous les ouvrages d'analyse sans exception, est-il essentiel au genre? est-il un [?] au lieu de symptôme d'autres [?] ^dans l'état de scie~~ mal réel? ~~ou bien^ n'est il que le symptôme de besoins ^encore^ inconnus dans cette partie importante de nos connaissances? C'est ce que je vais discuter.~~

Tout celà étonnera fort les gens du monde, qui en général ont pris le mot Mathématique pour synonyme de régulier.

Toutefois ^on sera encore étonné^ si l'on réfléchit qu'ici ~~la co~~^comme^ ailleurs la science est l'oeuvre de ~~l'homme, plutôt fait pour étudier que pour connaitre, pour chercher que pour trouver la vérité~~ l'esprit humain, qui est destiné plutôt à étudier qu'à connaître, à chercher qu'à trouver la vérité. En effet, ^on concoit qu'^ un esprit qui aurait puissance pour percevoir d'un seul coup l'ensemble des vérités Mathématiques ^non pas à nous connues, mais toutes les vérités possibles^, pourrait ^aussi les^ déduire régulièrement et comme machinalement de quelques principes ^combinés par une méthode uniforme^. ~~L'on~~ ~~Mais Il n'en est pas ainsi~~ ^alors plus ~~de difficultés~~^d'obstacles^^. ⌜Plus de ces difficultés que le savant trouve dans ses explorations, et qui souvent sont imaginaires⌝. Mais aussi plus de ~~science~~ ^rôle^ au savant. ⌜Il n'en est pas ainsi: ~~Une théorie ne peut être construite que sous une de ces fautes~~ si la tâche du savant est plus penible et partant plus belle, ~~le domaine~~ la marche

Some words crossed out here are not crossed out in *ms*. The revised sentence re-uses much of the original.

I owe the latter half of the crossed out clause to BA1962.

Mathematical Sciences
Discussions on advances in pure analysis

~~It is~~ ~~When one throws the~~

Of all human knowledge [understanding] it is known that pure analysis is the most abstract, the most eminently logical, the only one that owes nothing to evidence of the senses. Many ~~deduce~~ conclude that overall it is the most methodical and the best coordinated. But this is an error. Take an algebra book, whether a textbook or a monograph, and you will see nothing but ~~chaos[?]~~ a confused mess of propositions of which the regularity contrasts strangely with the disorder of the whole. It appears that ~~each~~ the ideas already cost the author too much for him to give himself the trouble to connect them, and that his mind, exhausted by the conceptions on which his work is based, cannot give birth to a similar thought to preside over the collection of them.

Even if you meet with a method, a connection, some coordination, it is all false and artificial. These are ~~arbitrary~~ divisions without foundation, arbitrary connections, an entirely conventional arrangement ~~similar to that artificial memory which the ancients had imagined~~. This lack, worse than the absence of all method, occurs above all in textbooks for the most part written by men who do not understand the science which they profess.

~~This lack of a true method, of a simple and clear course in all works of analysis without exception—is it of the essence of the genre? Is it a real difficulty in the state of science? Or is it nothing more than a symptom of still unknown needs in this important part of our knowledge? This is what I shall discuss.~~

All this will greatly astonish the man in the street, who has generally taken the word mathematical as a synonym for orderly.

He will, however, be even more astonished if he reflects that here, as elsewhere, science is the work of ~~man, done rather to study than to know, for seeking rather than for finding the truth~~ the human mind, which is destined to study rather than to know, to seek rather than to know the truth. Indeed, one can see that a mind which would have the power to appreciate at a single glance the collection of mathematical truths, not just those known to us, but all possible truths, could also deduce them regularly and as if mechanically from a few principles combined by a uniform method. ~~But it is not like that.~~ [There are] more ~~difficulties~~ obstacles. More of these difficulties which the scientist finds in his explorations, and which often are imaginary. But also a greater role for the scientist. It is not like this: ~~A theory cannot be constructed without one of these faults~~ if the task of the scientist is harder but nevertheless more lovely, ~~the domain~~ the march

Although ‘partant’ is clear in *ms*, I suspect that Galois intended ‘pourtant’.

de la science aussi est moins regulière⌉. La science progresse par une série de combinaisons, où le hazard ne joue pas le moindre rôle, sa vie est brute ~~elle~~^et^ ressemble à celles des minéraux qui croissent par juxtà position. ~~C'est en vain qu'un savant veut se dissimuler.~~ Celà s'applique non seulement à la science telle qu'~~e~~elle ~~forme~~ résulte des travaux d'une série de savants, mais aussi aux recherches particulières de chacun d'eux. En vain ^les analystes^ voudraient-ils se le dissimuler⌈:⌉ ^toute immatérielle ~~qu'ils~~ qu'elle est ~~leur science~~ l'analyse n'est pas plus en ~~leur pouvoir~~ ^notre pouvoir^ que d'autres^; ils ne déduisent pas, ils combinent, ils ~~comparent~~⌄combinent, ils comparent⌄: ~~ils~~ ~~c'est en tatonnant~~ ^il faut l'épier, la sonder, la solliciter^ quand ils arrivent à la vérité, c'est en ~~tatonnant~~ ^heurtant^ de ce côté et d'autre qu'ils y sont tombés.

~~Quant aux~~ ^Les^ ouvrages didactiques~~,~~ ~~[?]~~^doivent^ partager ~~le~~ avec les ouvrages d'inventeurs ce défaut d'une marche sûre toutes les fois que le sujet qu'il traite n'est pas entierement soumis à nos lumières. Ils ne pourraient donc prendre une forme vraiment méthodique que ~~pour~~ ^sur^ un bien petit nombre de matières. Pour la leur donner, il faudrait une profonde intelligence de l'analyse, et l'inutilité de l'entreprise dégoûte ceux qui ~~en auraient assez~~ pourraient en supporter la difficulté.

74 a

Il serait au-dessous de la gravité de cet écrit d'entrer dans ~~cette~~ ^une pareille^ lutte avec des sentiments personnels ⌈d'indulgence ou⌉ d'animosité ~~contre~~^à l'égard^ les savants. ~~Si les antécédents, le passé de l'auteur~~ ^L'auteur ^des articles^ évitera ^également^ ces deux écueils.^ ^Si un^ passé ^pénible^ le garantit du premier, un amour profond de la science, qui la lui fait ~~aussi~~ respecter dans ceux qui la cultivent, assurera contre le second son impartialité.

~~[???]~~ ~~La critique~~ Il est pénible ⌈dans les sciences⌉ de se borner ~~à la rigueur de critique et de ne pouvoir substituer~~ au rôle de critique: nous ne le ferons que constraints et forcés. Quand nos ~~forces~~ forces nous ^le^ permettront, ~~[??]~~ après avoir ~~signalé un vice de science~~ ^blamé^nous indiquerons ce qui ~~nous paraîtrait mieux afin de~~ ^à nos yeux sera mieux.^ Nous aurons souvent ainsi l'occasion d'appeler l'attention du lecteur sur les idées nouvelles qui nous ont conduit dans l'étude de l'analyse. Nous nous permettrons donc de l'occuper de ces idées, dans nos premiers articles, afin de n'avoir point à y revenir.

Dans des sujets moins ~~difficiles~~ ^abstraits, dans les objets d'art ~~[?? ??]~~^ il y aurait un profond ridicule à ^faire^ précéder un ouvrage de critique par ~~une~~^ses propres^ oeuvre⌈s:⌉^~~mais~~^ ~~ici ne rien me dit aucune limite ne doit séparer la critique et~~ ce serait avouer par trop naïvement, ~~ce qui est pourtant au fond la base même de toute critique~~ ^ce qui est presque toujours vrai au fond^, que l'on se prend pour ^le^ type auquel on rapporte les objets ~~que l'on veut~~^pour les^ juger ~~: ce qui est toujours # vrai au fond mais~~ mais ici, il ne s'agit pas d'exécution, il s'agit d⌈es⌉ idées ~~métaphys~~ ^les plus^ abstraites qu'il soit donné à l'homme de concevoir; ~~et~~^ici^ critique et

Words 'que la' changed to 'qu'elle' by overwriting.

The clause 'toute immatérielle ... que d'autres' was probably inserted after all or most of the clause 'ils ne déduisent ...' had been written.

After 'd'inventeurs ce' Galois turned his paper 90° anti-clockwise and wrote three lines in the left margin.// Singular 'il traite' is clear in *ms*; corrected to plural in C*ms*, TE, BA1962; Tannery remarked on the correction.// Word 'vraiment' missing from BA1962.

Word 'les' not corrected to 'des' when phrase changed.

Originally simply 'à la rigueur de critique'; then 'et de ne pouvoir substituer' inserted; finally insertion deleted and other changes made for final version.

of science is also less regular. Science progresses by a series of combinations in which chance plays not the least role. Its life is rough and resembles that of minerals which grow by juxtaposition [accretion]. ~~It is in vain that a scientist wishes to pretend~~ This applies not only to science such as it emerges [results] from the work of a series of scientists, put also to the particular research of each one of them. In vain would analysts dissimulate: ~~their science~~ however abstract it may be, analysis is no more in ~~their power~~ our power than that of others; they do not deduce, they ~~compare~~ combine, they compare: ~~they~~ ~~it is in groping~~ it must be sought out, sounded out, solicited. When they arrive at the truth it is by ~~groping~~ cannoning from one side to another that they come across it.

~~As for~~ Didactic works [textbooks] must share with works of discoverers [monographs] the lack of a sure path whenever the material which they treat is not entirely bathed in light. They therefore cannot take a truly methodical form except for a pretty small number of subjects. To give it [form] to them one must have a profound understanding of analysis, and the uselessness of the enterprise puts off [disgusts] those who ~~have had enough of it~~ could tolerate the difficulty.

It would be beneath the seriousness of this essay to enter into a comparable battle with personal feelings of indulgence or animosity in regard to scientists. ~~If the antecedents, the author's past~~ The author of the articles will avoid these two pitfalls equally. If a difficult past protects him from [guarantees him against] the former, a deep love of science which induces him ~~also~~ to respect those who cultivate it will ensure his impartiality in respect of the latter.

~~Criticism~~ In science it is painful to restrict oneself ~~to the rigour of criticism and to not be able to substitute~~ to the role of critic: we [I] do it only because constrained and forced to. When our strength will permit, after having ~~signalled a fault in science~~ criticised we will indicate what ~~what to us appears better in order to~~ in our eyes would be better. In this way we will often have occasion to call the attention of the reader to the new ideas which have guided [led] us in our study of analysis. We therefore permit ourselves, in our first articles, to occupy him with these ideas in order not to have to return to them again.

In less ~~difficult~~ abstract subjects, in matters of art, it would be deeply ridiculous to precede a work of criticism with one's own works, ~~but here there is nothing to say to me no boundary may separate criticism and~~ but here it would be to acknowledge much too naively ~~what is nevertheless at bottom the true basis of all criticism~~ what is almost always true at bottom, that one takes oneself for the type [model] with which one compares things in order to judge them: ~~which is always true at bottom but~~ but here it is not a matter of execution, it is a matter of the most abstract ideas which it is given to man to conceive; here criticism and

discussion deviennent synonymes, et ~~la discussion~~ discuter, c'est ~~comparer~~ ^mettre aux prises^ ses idées avec celles des autres.

Nous exposerons donc dans quelques articles, ~~des~~ ce qu'il y a de plus général, de plus philosophique, dans des ~~écrits~~ recherches que ~~cruelles circonstances, et [??] autres circonstances~~^mille circonstances ont empêché ~~[???] [????] [???]~~^ ˬdeˬ publier plus tôt. Nous les [illegibly deleted half line finishing perhaps with ~~publier~~]

Comma after 'discuter' doubtful. It looks more like a full stop, but that would make no sense.

The last two-and-a-half lines on the page have been much amended; the deleted material is very hard to recover.

74b

présenterons, seules, sans complications d'exemples et de hors-d'oeuvres, qui ~~dans~~^chez^ les analystes noyent d'ordinaire les conceptions // générales. Nous les exposerons surtout avec bonne foi indiquant sans détour la voie qui nous y a conduit; et ~~surt~~ les obstacles qui nous ont arrêté. ~~et sous ce rapport ^Nous ne voulons pas que^ le chemin que nous avons parcouru soit en rien perdu par le lecteur. Nous l'y mènerons afin qu'il continue: En voulant~~ ainsi^Car nous voulons^ que le lecteur soit aussi instruit que nous des matières que nous aurons traitées. Quand ce but aura été rempli, nous aurons conscience d'avoir bien fait, si non par l'⌜e⌝~~utilité dont~~ ~~seront~~ profit qu'en retirera directement la science, du moins par l'exemple donné d'une bonne foi qu'on n'a pas ~~cherchée~~^trouvée^ jusqu'à ce jour.

Word 'par' read as 'pour' in BA1962. But 'par' is clear in *ms*, and read as such in C*ms*, T1906/7.

~~Ainsi nous ferons~~ ~~Il tient~~

discussion become synonymous, to discuss is to ~~compare~~ set one's ideas against those of others.

We therefore expound in a few articles what is most general, most philosophical in some ~~writings~~ research which ~~cruel circumstances and other circumstances~~ a thousand circumstances have prevented earlier publication. We ~~publish[?]~~

will present them on their own, without the complication of examples and digressions which ordinarily drown general concepts in the work of analysts. Above all, we will expound them in good faith indicating without detour the path which has led us to them and the obstacles which impeded us. ~~and in this connection We do not wish the path that we have trodden to be at all lost to the reader. We lead him along it in order that he may continue: in wishing thus~~ For we wish that the reader may be as instructed as we were by the material that we have treated. When this aim shall have been fulfilled we will be conscious of having done well, if not by the ~~usefulness of which~~ profit that science will have drawn directly from it, at least by the example given of good faith which has not been ~~sought~~ found until this day.

~~Thus we make~~ ~~It is owing to~~

Notes on Dossier 12

NOTE 1: Chevalier made a copy of this material. As with all his copies, it is written on one side of his paper, and as with several of them it lacks capitalisation. For the first part, or essay, he used one and a half sheets, which he numbered 1 and 2; the second part, or essay, fills a third page, numbered 3 by Chevalier. His pages 1 and 2 are bound into the dossier in the opposite order. Thus Chevalier's copy of the first part, or essay, fills **f.77a** and **f.76a**; his copy of the second fills **f.78a**. It looks to me quite possible that he started copying the pamphlet as it presents itself, that is, starting with **f.74a**, 'Il serait au-dessous de la gravité ...'. The evidence is thin, but here it is. First, his page number **1**, written in uncharacteristically black ink, looks to me as if it over-writes a more faintly written 2. This would not be unnatural if he came to folio 75 only after he had completed copying the second part (or essay), and realised that what was written there preceded what he had already copied. Better evidence is provided by the fact that all the marginal notes on **f.78a** are crossed out and appear again in the margin of **f.77a**. Of course this is not decisive. The notes are written in a blacker ink than the copies; he could well have come back, annotated his p. 3, and then realised that he really wanted the annotations to appear on p. 1. The point of this discussion is this: it looks to me as if Chevalier originally saw two essays here, but later wavered, or at least, came to think that the material of **f.74a** + **f.75b** followed that of the inside pages, **f.75a** + **f.75b**.

NOTE 2: Tannery described these writings in [Tannery (1906), pp. 243–244], treated them as two separate pieces, and transcribed them as his items D, E on pp. 259–261. They appear as a single piece in [Dalmas (1956), pp. 121–135; (1982), pp. 112–115]. In [B & A (1962)] the relevant pages are 12–17 and 505–506. After describing how Galois used his paper, Bourgne wrote (top of p. 506):

> Malgré cette disposition il n'y a aucune raison de séparer ces deux développements. Ils se suivent.
>
> [In spite of this arrangement there is no reason at all to separate the two parts. They follow each other.]

Myself, I am not so certain—I can see reasons for treating this as two pieces, as Chevalier may have done at first, and Tannery did. There is a clear hiatus between the end of the first part and the start of the next, not only a mild one of content, but also in the writing. Instead of continuing onto the blank reverse of the page, it continues with three lines in the left margin that lead to a natural cadence. The last one-third of the first part is a mess, with many crossings-out and re-writings; then the second part starts again on **f.74a** with more considered writing. At the very least this is a distinguishable second part to the essay, written later, when Galois had re-opened his mind on the matter and decided that more had to be added to what he had originally conceived as complete.

Nevertheless, these are not, for me, quite as strong as the reasons for following Bourgne (and Dalmas, and perhaps Chevalier), rather than Tannery, and presenting the material as one piece. The reader should, however, be aware of the possibility that it is two separate (if linked) essays.

NOTE 3: Galois' text finishes slightly below the middle of **f.75b**. The remainder of the page is filled with a number of examples of the initial E, the forename Evariste and the surname Galois, sometimes separate, sometimes combined. Some are written with the page turned a quarter-turn anti-clockwise, some with the page turned through a half-turn, a few are written diagonally. They are written in a variety of elegant scripts.

There are jottings also on **f.75a**. In the top right corner there is a lightly and neatly drawn face in profile from its left side; much of the lower half of the page is covered in flourished scribbles. The left margin contains some scribbles and a couple of incomplete examples of calligraphic versions of the name Galois. These are quite unusual insofar as they are in pencil, not ink. They must have been there before Galois used this page because the ink of his text overwrites the pencil. In [Tannery (1906), p. 243] Tannery wrote:

> [...]. Le manuscrit est plein de ratures, de surcharges, de dessins à la plume, de taches d'encre; quelques passages, comme on s'en assurera à la lecture, auraient besoin d'être retouchés ou complétés; la difficulté même de la lecture m'a amené à dater approximativement le manuscrit: en l'examinant à la loupe, j'ai aperçu quelques mots écrits à l'envers, et en le retournant j'ai pu distinguer ces mots, dont la disposition semble indiquer que Galois n'avait pas de goût pour le calcul mental
>
> jeudi 2 Mars
> dimanche 1
> lundi 2 Avril
> mardi 3
> jeudi
> vendredi
>
> Le premier avril de l'année 1832 étant un dimanche, le doute n'est pas possible, puisque Galois est mort le 30 mai 1832.
>
> [The manuscript is full of crossings-out, of overwriting, of pen sketches, of ink blots; a few passages, as one discovers when reading, could have done with revision or completion; but the difficulty itself of reading it has led me to date the manuscript approximately: examining it with a magnifying glass I noticed a few words written upside-down, and on turning it round I was able to distinguish these words, of which the arrangement seems to indicate that Galois had no taste for mental arithmetic

Thursday 2 March
Sunday 1
Monday 2 April
Tuesday 3
Thursday
Friday

The first of April of the year 1832 having been a Sunday, no doubt is possible because Galois died on 30 May 1832.]

In [B & A (1962), p. 506] the list is recorded as follows:

	8	Avril	[8 April]
Samedi	24	Mars	[Saturday 24 March]
Jeudi	29	Mars	[Thursday 29 March]
Dimanche			[Sunday]
Lundi	2	Avril	[Monday 2 April]
Mardi	3		[Tuesday 3]
Jeudi			[Thursday]
Vendredi			[Friday]

These are very hard to see, though what I see confirms the capitalisation recorded in [B & A (1962)], rather than the lower-case of [Tannery (1906)]. Some are written faintly, some were crossed out independently of the over-writing, and they are irregularly placed on the page. For example, the entry '8 Avril' is over in the top left (with the page turned round), whereas the other dates are in a sort of column a little lower down and in the middle of the page. I think I can make out another instance of 'Vendredi' just below 'Mardi 3' and possibly an instance of 'Samedi' just below 'Vendredi'. But, given how hard these dates are to make out, the discrepancies between readings are not surprising. It seems very likely that they were jotted onto a clean (but inappropriately large) sheet of paper, and that Galois simply crossed out some and overwrote others when he re-used it for a larger purpose.

VI.8 Dossier 13: Here, as in all the sciences

The cover sheet of Dossier 13 is inscribed 'Ici comme dans toutes les sciences', words taken from the top left corner of **f.80b**. It consists of a single large sheet, 39.5 cm × 31.5 cm folded a little unevenly to make folios 79 (20 cm × 31.5 cm) and 80 (19.5 cm × 31.5 cm). The material in the top half of **f.80b** is presented as Item F in [Tannery (1906), p. 262] and it appears also in [Dalmas (1956), p. 125; (1982), p. 116] and [APMEP (1982), p. 39]; the whole dossier is treated in [B & A (1962), pp. 18–19, 507]. The paper is identified in [B & A (1962), p. 507] as being the same as that of the Testamentary Letter of 29 May 1832, and the same as that of Dossier 12, the discussion of progress in pure analysis; Bourgne dated it to April–May 1832.

Most of the space has been used at one time or another, and in various orientations of the pages for beginnings of essays, for calculations, for jottings, and for sketches. We discuss some of these in the notes below. Here we reproduce and translate only the passage that was first published by Tannery in 1906. It just fills the top half of **f.80b**. Below it, drawn with the page turned 90° anti-clockwise, are the famous sketches, one complete, the other just a beginning, of Riquet à la Houpe, that have been reproduced in [B & A (1962), 10th frontispiece] and [APMEP (1982), p. 38]. And below those sketches is the vigorously deleted clause 'On croit généralement que les Mathématiques sont une série de déductions', written with the page turned through 180° with which Galois first started work on this particular piece of paper (see Note 1 below).

Galois had second and third thoughts about the opening of this passage. Originally it began 'Chaque époque'; then, starting well over in the left margin, 'Dans toutes les sciences' was inserted above; finally, starting still further to the left, at the edge of the paper, 'Ici comme' was inserted above that. He also had second and third thoughts about other passages. In particular the passage 'la preuve que ... époque' started out singular and was changed by over-writing and re-use of much of the text to plural. The passage is curiously unfinished. It seems to be reaching a final cadence, which just a few words after 'que nous attacherons' would have achieved but for the crossing out of 'un lien entre ces questions que nous'. I cannot guess what it was that Galois intended to attach before he changed his mind.

This is one of the items that does have paragraph indentation.

80 a

^^Ici comme^ Dans toutes les sciences^ Chaque époque ~~, dans chaque~~ ^a ^en quelque sort^ ses questions^ du moment: il y a des questions vivantes qui fixent à la fois ~~tous~~ les esprits ~~sans qu'on les~~ ^les plus éclairés, comme malgré eux et^ sans qu'aucun accord ait présidé à ce concours~~,~~.. ~~Cela~~ ^~~Cette admirable coïncidence~~^ ~~a lieu, maintenant~~ ^~~sans pourtant~~^ ~~qu'aucun lien n'existe entre les savants~~ ~~sans qu'aucun~~ Il semble ^souvent^ qu'~~une~~ ^les^mêmes idées apparaissent à la fois ^à plusieurs^ comme une révélation⌜:⌝ ~~à plusieurs. En vain vous éloignerez les savants, en vain~~ ^C'est que ces idées sont prescrites dans les ouvrages de ceux qui nous [ont] ~~précédés à l'insu de leurs auteurs.~~ ~~Admirable coincidence dont jusqu'ici la science n'a pas tiré grand profit~~ ^Si l'on en cherche la cause, il est aisé de la trouver dans les ouvrages de ceux qui nous ont précédés, où ces idées sont prescrites à l'insu de leurs auteurs.^

Phrase 'qu'aucun accord' is from BA1962. It looks plausible, but insecure. Tannery found it illegible.

Word 'prescrites' read as 'présentes' in T1906/7.

La science n'a pas tiré, jusqu'à ce jour, grand parti de cette coïncidence observée si souvent dans les recherches des savants. ~~Il appartiendra~~ ~~Un jour viendra, sans doute, où l'on reconnaitra que le sav~~ Une concurrence fâcheuse, une rivalité degradante en ont été ~~jusq'ici~~ les principaux fruits.~~, parce que tout~~ Il n'est pourtant pas difficile de reconnaitre dans ce fait ~~combien il est serait facile d'assurer~~ ^la preuve^ que les savant n~~'este~~^sont^ pas ~~isolé. De~~^faits^ plus qu~~'un autre~~ d'autres pour l'isolement, que⌜'⌝~~lui~~x aussi appartiennent à ~~les~~ur époque, ^et^ que tôt ou tard ~~on en viendra à décupler ses forces~~ ~~^des^ [???] et au principe de~~ ils décupleront leurs forces par l'association. Alors que de tems épargné pour la science!

Originally 'les [*sic*] savant n'est pas isolé. De plus ...'. Then 'les [*sic*] savant n'est pas isolé plus qu'un autre...'. Finally what is recorded in T1906/7, BA1962. Several lines are changed from singular to plural.

Beaucoup de questions d'un genre nouveau occupent maintenant les Analystes. C'est à découvrir ~~un lien entre ces questions que nous~~ attacherons

Here, as in all the sciences, each epoch has in one way or another its questions of the moment: there are living [contemporary] questions which fix simultaneously the clearest minds as if against their will [in spite of themselves] and without any agreement having preceded this coincidence [coming together]. ~~This admirable coincidence is occurring now but perhaps without any connection existing among the scientists~~. It often seems as if the same ideas appear at the same time to several, as a revelation. ~~It is in vain that you separate the scientists, in vain~~ ~~It is that these ideas are written beforehand in the works of those who have preceded us, unknown to their authors.~~ ~~Admirable coincidence from which science has not drawn great profit up to now.~~ If one seeks the reason it is easy to find it in the works of those who have preceded us, where these ideas have, unknown to their authors, been foreshadowed [written down].

To this day science has not drawn any great benefit from this coincidence observed so often in the research of scientists. ~~it will belong~~ ~~A day will come, no doubt, when it will be recognised that the sci[entist]~~ An irritating coincidence, a degrading rivalry have, ~~up to now~~ been the principal fruits ~~, because every~~. It is not difficult though to recognise in this fact ~~how easy it would be to certify~~ the proof that scientists are made no more than others for isolation, that they too belong to their time, and that sooner or later ~~their power will come to be multiplied tenfold~~ ~~and to the principle~~ they will multiply their strengths tenfold by associating. And the time saved for science!

Many questions of a new kind now occupy analysts. It is to discover ~~a connection between these questions~~ that we will attach

Notes on Dossier 13

NOTE 1: Apart from the passage above, it is difficult to make much sense of the content of the rest of this dossier. Robert Bourgne [B & A (1962), p. 507] suggests very plausibly that Galois started with his large piece of paper folded in two and wrote 'On croit généralement que les Mathématiques sont une série de déductions' [It is generally believed that mathematics is a series of deductions], which appears upside down at the bottom of **f. 80 b**; that he then crossed this out vigorously, turned the paper round to its present orientation and began again on what is now **f. 79 a**. Bourgne suggests that the new beginning was 'Nous ne nous plaindrons donc point ...', but it looks to me as if what happened was this: Galois re-started there with 'Il ne faut donc point se plaindre que les ouvrages de Mathématiques'; he then deleted most of that and changed 'les' to 'Les' so that his new beginning was 'Les ouvrages de Mathématiques'; finally, without deleting the initial word 'Les' he inserted 'Nous ne nous plaindrons donc ^point^ de l'irregularité des' before it. The passage continues for seven lines, with much crossing out. It ended up approximately like this:

> ^Nous ne nous plaindrons donc point de l'irregularité des^ Les ouvrages de Mathématiques, qui est inhérente à la ~~liberté absolue du savant~~ une théorie ^~~nouvelle~~^ ~~par cela seul qu'elle est~~ est bien plutôt ~~la recherche que l'expression de la vérité~~, et si l'on pouvait la déduire ~~Mathématiquement~~ régulièrement des théories déjà connues, elle ne serait pas nouvelle.
>
> ~~Ce dont nous nous plaindrons, c'est que la pensée qui a dirigé l'auteur reste le plus souvent cachée ...~~
>
> ~~Les idées générales sont la seule méthode peuvent seules tenir lieu d'une méthode; mais au lieu de ...~~
>
> [Thus we do not complain at all about the irregularity of works of mathematics, which is inherent in the ~~absolute freedom of the scientist~~. A theory ~~novel solely by what it is in itself~~ is perhaps more ~~a search for than the expression of the truth~~, and if one could deduce it ~~Mathematically~~ in the usual way from already known theories it would not be new.
>
> ~~What we complain of is that the thinking that has directed the author most often remains hidden ...~~
>
> ~~General ideas are the only way are the only ones that can take the place of a method; but in place of ...~~]

This is, as I have indicated only an approximation. There are, for example, three words crossed out at the end of the first line, two of which seem incomplete to me, the third is the word 'liberté' which occurs again at the start of the second line. There are one or two words so thoroughly crossed out at the ends of the second and third paragraphs that neither Bourgne nor I have been able to read them (though I think that the very last one could be 'démon' as the start of 'démontrer' [to make, perhaps, 'but instead of showing ...']).

I can well understand why this was not reproduced by Tannery in 1906/07. With the deletions respected it makes no sense; and even if some of the deletions should stand, it would still be written at such a high level of abstraction and generalisation that it ends up with very little useful meaning.

NOTE 2: A rectangular piece about 5 cm × 7 cm has been neatly cut from the bottom right corner of folio 79. After that the remainder of **f. 79 a**, about three-quarters of it, was filled with mathematical jottings and calculations. These were all written with the page turned through 180°, so that they now appear upside-down. Almost all of them seem to have to do with differential equations of one kind and another, though there is also the one algebraic line $(ax^n + bx^{n-1} + \cdots + px + q)X$. Later the page attracted various scribbles, some of which could just be the initials E G, others are just scribbles. There is also the word 'Monsieur' written diagonally in a small, neat hand.

Most of the material on **f. 79 b** is oriented the same as that on **f. 79 a**, is written with the same pen, and deals with the same topic. Presumably the calculations here precede those on **f. 79 a** (as they naturally would with the page turned round). Here, too, there is a line that is algebraic and seems rather different from the rest of the material:

seulement n équations entre $2n + 1$ quantités $\ulcorner x \urcorner\ x_1\ x_2\ x_3 \ldots\ x_{2n}$

On en a une entre

On this page there are few jottings: a few scribbles, the name Galois, and three small, neat faces in profile drawn very close to each other with a very thin nib.

The second inside page (at some time the first), **f. 80 a**, is far messier than the others. It has been used in its present orientation for some algebraic calculations; it has been used turned upside-down for some more algebra and for one or two simple integrals; it has been turned through 90° clockwise and used for some rather different algebra written with a much finer pen. It carries many scribbles, the name Galois written neatly in a medium size hand and crossed out, and the word 'Monsieur' again written sloping (but almost vertical) downwards in the same small, neat and careful hand as on **f. 79 a**. It is here that one finds on the left of the page vertically upwards, enclosed in a thinly drawn rectangular box, neatly crossed out with two thin lines, but then thoroughly scribbled out, the two words which to me are completely illegible but which Carlos Alberto Infantozzi (see [Infantozzi (1968), p. 160]) reads as 'Stéphanie Dumotel':

> Enfin, sur la page 80 r° des Manuscrits de Galois, un examen à la loupe sous un éclairage approprié permet de lire, sous deux ratures, "Stéphanie Dumotel." C'est également sous des taches que l'on peut lire sur les brouillons de deux lettres [**f. 11 b**]: "Mademoiselle Stéphanie Du ... tel, 14 mai . . 83".
>
> [Finally, on **f. 80 a** of the manuscripts of Galois, examination with a magnifying glass under appropriate lighting permits the reading under two cross-

> ings-out of “Stéphanie Dumotel”. In just the same way one can read in the rough drafts of two letters [**f. 11(**b)] “Mademoiselle Stéphanie Du ... tel, 14 mai . . 83”.]

On **f. 80b** there are the one-and-a-half striking drawings of Riquet à la Houpe that have already been mentioned and above it is ‘ev. galois’ written in an unusual and unusually even script, something that one might describe as lower-case cursive block. Beyond that and the material that has been transcribed above there is nothing of interest: three large capital letters at the top, above the text, two words or names (perhaps something like Mathématiques or Stéphanie) that are so thoroughly crossed out that I cannot read them, and some scribbles.

VI.9 Dossier 14: Science, Hierarchy, Schools

This dossier contains just a single mysterious sheet of paper, folio 81. The cover sheet is inscribed ‘ “Sciences Hiérarchie. Ecoles” ’ echoing Galois’ words that head his sheet. The paper is approximately square, 20 cm × 21 cm, having been torn, a little irregularly along its bottom edge, from a deeper sheet. It appears as item G in [Tannery (1906), p. 263]; in [Dalmas (1956), pp. 135–136, (1982), 125–126], where, however, the heading is misread; and in [B & A (1962), pp. 26, 27, 508].

Dating is a little uncertain to the extent of a few months in spite of the fact that the document was dated by Galois himself. In what he wrote, the symbol for the month could be read as 7 for September (not July) or as X for December (not October). Tannery reads 7; Dalmas chooses September. Bourgne writes ‘*On peut hésiter entre* 7 *bre et Xbre*’ [B & A (1962), p. 508] and chooses X. The instances of 7 on **f. 57 b**, though perhaps incomparable since they appear in numerical work, are very much more obviously the numeral than here. And the symbol, though written very large, has quite a bit in common with the letter x as it appears in words such as ‘ceux’ or ‘examen’. It also has something in common, though perhaps not as much as to be decisive, with the date ‘X^{bre}’ at the end of the ‘Préface’ in Dossier 11. These are the reasons that I prefer X to 7. Assuming that to be correct, the date is 29 December 1831, when Galois was in the Sainte-Pélagie prison.

81 a

See Note 2.

Notes Tout voir, tout entendre, ne perdre [??????????], aucune idée.

Sciences 29 X^bre 1831

Hiérarchie. Écoles

– La hiérarchie est un moyen même pour l'inférieur.

– Quiconque n'est pas envieux, ou a de l'ambition a besoin d'une hiérarchie factice pour ⌜vaincre⌝ l'envie ou des obstacles.

– Jusqu'à ce qu'un homme ait dit: la science c'est moi, il doit avoir un nom a opposer à ceux qu'il combat. ⌜Si non, son ambition passera pour de l'envie.⌝

– Avant d'être roi il faut être aristocrate. ⌜Machiavel⌝

– L'intrigue est un jeu. Si l'on mérite ce qu'on brigue, on y gagne tout. Si non, on perd la partie.

– On combattu les professeurs par l'institut, l'institut par le passé, un passé par un autre passé. [??]

^– Voici la marche de^

Victor Hugo. Renaissance, moyen àge, enfin, moi.

C'est à ce besoin de combattre un ~~siécle~~^homme^ par un autre ^homme^, un siecle par un autre siécle, qu'on doit attribuer les réactions littéraires ou scientifiques, qui ne sont pas de longue durée. Aristote, Ptolémée, Descartes.

For word 'subalternes' see Note 3.

Les subalternes ne comprennent pas ⌜Laplace⌝

Ce jeu use celui qui s'en sert. Un homme qui n'est pas dévoué fait éclectique.

– Un homme qui a une idée peut choisir entre, avoir, sa vie durant, une réputation colossale d'homme savant, ou bien se faire une école~~, ou bien~~^se taire, et^ laisser un grand nom dans l'avenir. Le premier cas a lieu s'il pratique son idée sans l'émettre, le second, s'il la publie. Il y a un troisième moyen~~, c'est~~ juste milieu entre les deux autres. C'est de publier et de pratiquer. Alors on est ridicule.

Notes

See all, hear all, let no idea escape

Perhaps 'See all, understand all, etc'. See Note 2.

Science

29 X^bre 1831

Hierarchy. Schools

Hierarchy is a means even [especially] for the inferior.

Whoever is not envious or has ambition needs a contrived hierarchy to overcome envy or obstacles.

Until a man shall have said: science is me, he must have a name to set against those whom he fights. If not, his ambition will pass for envy.

Before becoming king one must be aristocrat. Machiavelli.

Intrigue is a game. If one is worthy of what one craves one wins all. If not one loses the match.

One fights the professors with the institute, the institute with the past, a past with another past.

Perhaps 'passé' should be something like 'someone who is past it'

Here is the march of

Victor Hugo. Renaissance, middles ages, finally me.

It is this need to combat a ~~century~~ man with another man, a century with another century to which one must attribute literary or scientific reactions, which do not last long. Aristotle, Ptolemy, Descartes. Laplace.

Subordinates do not understand.

This game uses whatever it needs. A man who is not devoted becomes eclectic.

A man who has an idea may choose between having, throughout his life, a colossal reputation as a scientist, or alternatively to create a school, or even to keep quiet and leave a great name to the future. The first will be the case if he exploits his idea without publishing it, the second if he publishes. There is a third way, which is precisely in between the other two. It is to publish and to practise. Then one is ridiculed.

Notes on Dossier 14

NOTE 1: The word 'Notes' in the top left of **f. 81 a** accords well with the rest of the document. But notes for what? The notes do not seem to have guided Galois in the writing of his *Préface* (Dossier 11) or his *Discussions sur les progrès de l'Analyse pure* (Dossier 12), or any other extant piece of his work.

NOTE 2: The word 'entendre' in the motto has two translations into English: 'to hear' and 'to understand'. The motto itself is something of a puzzle. First, there are two or three words between 'perdre' and 'aucune' that have been so thoroughly inked out as to be irretrievable. Secondly, what does it mean? It chimes with 'tout voir, tout entendre, ne rien dire' whose exact English equivalent is 'see all, hear all, say nothing'. It chimes also with a saying that I have found attributed to Napoleon I: 'Vous devez tout voir, tout entendre et tout oublier' [You must see all, hear all, and forget all]. But what Galois had in mind, where he got it from (other than his own fertile mind), and what the missing words are remain hard to guess.

NOTE 3: The lines beginning '—Quiconque ...', '—L'intrigue ...', '—Voici ...', and 'Les subal...' were inserted after the other lines had been drafted. The words 'pas dévoué' finish one line and 'fait éclectique' appear on the next, followed by a thickish but faint horizontal line. It is a bit like the lines that Galois used in other writings to indicate a sort of cadence or the termination of a piece (as this one). Whether or not it has any meaning here is moot. I rather doubt it.

The line 'Les subal...' is difficult to read; 'subalt' seems clear and is completed to 'subalternes' in [B & A (1962)]. The word looks, however, much more like 'subaltieux' or 'subaltieur', but those words do not exist. Another possibility is that we have here a mis-spelling of 'subalaires': les subalaires would be those placed under protection.

NOTE 4: This is rare among the manuscripts in having no markings other than a blot and three feathery marks, as if the pen-nib had been cleaned there; likewise, the verso **f. 81 b** is completely clean.

VI.10 Dossier 15: Fragments on the theory of permutations and equations

Dossier 15 comprises folios 82–85, of the Galois manuscripts. Its cover-sheet contains the description 'Fragments sur la théorie des permutations et des équations.' Folio 82 is 18 cm × 23.5 cm, folio 83 is 18 cm × 23 cm, folio 84 is irregularly torn along its right-hand side, but is approximately 17 cm × 22.5 cm, and folio 85 is a fragment torn from a piece 18 cm × 12.5 cm.

It is a collection of disparate notes, some more fragmentary than others, presented as items H, I in [Tannery (1907), pp. 280–285] and in [B & A (1962), pp. 72–81 and 508–510]. Robert Bourgne suggests, but not completely confidently, that the material dates from somewhere around September–December 1831, in prison.

In [B & A (1962)] the material is presented as items 1–5 and 10 within a sequence numbered 1–10 that includes some of the manuscripts in Dossier 17. Tannery preserves the ordering of the dossier. It is not clear what the temporal ordering originally was. For example, it seems possible to me that what should follow **f. 82 a** is not **f. 82 b** but rather **f. 83 a** down to the line beginning '5° Cas'. The paper is the same, the pen and ink are very similar, the content is similar in style, the statements of theorems are preceded with em-dashes in both cases. On the other hand, paragraph indications are marginally different. Detailed research is needed.

It may be of some help that there are some stylistic differences between the various items in this dossier. For example, one may observe three different paragraphing conventions in Galois' writings. Often, as in the *Premier Mémoire* and the *Second Mémoire*, there are no indents and paragraphing can be deduced only from the start of a new line, so that where a full stop occurs at the end of a line it may be questionable whether or not the next sentence starts a new paragraph. In other pieces, notably the *Lettre testamentaire*, Galois indented his paragraphs. The third style is that of **f. 83 a** and **f. 83 b**, where Galois used a sort of negative indent.

Another stylistic oddity is the use of a dash (projecting into the left margin) preceding the announcement 'Théorème'. This may be seen, as has been mentioned above, in **f. 82 a** and **f. 83 a**. These pages are distinguished, however, by one of the characteristics discussed above in that the former has no paragraph indentation, whereas the latter has negative paragraph indents.

The material here is associated by Bourgne with the First Memoir, but most of it seems to me to be more closely related to the Second Memoir.

82 a

T1906/7, BA1962 have 'Nombres de lettres'. But 'Nombre des lettres' clear in *ms*.

Permutations. Nombre des lettres m.

Substitutions. Notation.

Word 'semblables' overwrites another, probably 'complètes'.

Période. Substitutions Inverses. Substitutions semblables. ~~Ordre~~. ~~Substitutions complètes~~^Substitutions circulaires^. Ordre. Autres substitutions. ~~Substitutions complètes~~ ~~conjuguées~~; ^~~systèmes conjugués~~^ ~~substitutions circulaires~~^~~complètes conjuguées~~^ ⌄~~circulaires~~⌄. ~~Notations~~. ~~Il ne peut y avoir plus de $m-1$ substitutions complètes conjuguées~~.

Groupes. Groupes semblables. Notation.

– Théorème ⌜I⌝ Les Permutations communes à deux groupes forment un groupe.

– Théorème ⌜II⌝ Si un groupe est contenu dans un ~~autre~~^autre^, celui-ci sera la somme d'un certain nombre de groupes semblables au premier, qui en sera dit ~~l~~un diviseur.

– ~~Théorème. S'il y a une substitution dont la période soit de p termes (p étant premier) et qui ne soit pas contenue dans un groupe, il y a au moins p groupes qui lui seront semblables~~.

~~Exemple. Groups Alternes (Deux groupes semblables). Propriétés des groupes alternes~~.

~~Des groupes de qui comprennent en tout p^n permutations, p étant premier~~.

– Théorème ⌜III⌝ Si le nombre des permutations d'un groupe est divisible par p^{α} (p étant premier), ~~le groupe aura pour diviseur un groupe de p^{α} permutations~~ ^contiendra une substitution dont la periode sera de p termes^. ⌜~~Exemple. Groupes alternes~~ (~~Deux Permutations semblables~~).⌝⌄~~Propriété des groupes alternes. Doubles, [???]~~.⌄

– ~~Théorème ⌜III⌝ Soit un groupe G, et une substitution S. Supposons que la substitution S~~ ^~~que les substitutions $S_1, S_2, S_3, \ldots$ d'un autre [??] [??]~~^ ⌄~~groupe H~~⌄ ~~transforment entre elles mêmes~~ ^~~ne fassent qu'échanger entre elles~~^ ~~toutes les substitutions du groupe G, L'ensemble des permutations $G + GS_1 + GS_2 + GS_3 +$ formera un groupe que nous appellerons GH~~.

See Note 1, p. 296.

Réductions des groupes, dépendantes ou indépendantes. Groupes irréductibles. Des Groupes irréductibles en général.

– Théorème. Parmi les permutations d'un groupe, il y en a toujours une où une lettre donnée occupe une place donnée, et si l'on ne considère dans un groupe irréductible que les permutations où une mème lettre occupe une mème place et qu'on fasse abstraction de cette lettre, les permutations qu'on obtiendra ainsi formeront un groupe. ⌜Soit n le nombre⌝ ⌄des permutations de ce dernier⌄ mn⌄⌄

– Nouvelle demonstration du théorème relatif aux groupes alternes.

– ~~Théorème. Un groupe irréductible contient toujours~~ quelque substitution complète. ~~Discussion des groupes irréductibles. Groupes, primitifs et non-primitifs.~~

– Théorème. Si un groupe contient ~~quelque~~^une^ substitution complete de l'ordre m et une de l'ordre $m-1$, il sera irréductible.

Margin here contains ~~$m(m-1)(m-2)$~~.

Discussion des groupes irréductibles. Groupes, primitif et nonprimitif. Propriétés des racines

Permutations. Number of letters m.

Substitutions. Notation.

Period. Inverse substitutions. Similar substitutions. ~~Order.~~ ~~Complete substitutions~~ ^Circular substitutions^. Order. Other substitutions. ~~conjugate complete substitutions~~; ^~~conjugate systems~~^ ~~circular substitutions~~ ^~~complete conjugate~~^ ˬ~~circular~~˭. ~~Notation. It is not possible to have more than $m-1$ conjugate complete substitutions.~~

'Circulaire' could perhaps be translated as 'cyclic' here.

Groups. Similar groups. Notation.

Theorem I. The permutations common to two groups form a group.

Theorem II. If a group is contained in another, the latter will be the sum of a certain number of groups similar to the first, which will be said to be *a divisor* of it.

~~Theorem. If there is a substitution of which the period is p terms (p being prime) and which is not contained in a [given] group, there are at least p groups which will be similar to it.~~

~~Example. Alternate groups (Two similar groups). Properties of the alternate groups.~~

~~On groups of which contain in all p^n permutations, p being prime.~~

Theorem III. If the number of permutations of a group is divisible by p^{α} (p being prime), ~~the group will have as a divisor a group of p^{α} permutations~~ ^will contain a substitution whose period will be of p terms^. ⌜~~Example. Alternate groups (Two similar permutations).~~⌝ ˬ~~Property of the alternate groups. Doubles, [???].~~˭

~~Theorem III. Let G be a group G and S a substitution. Suppose that the substitution S ^that the substitutions S_1, S_2, S_3, … of another [??] [??]^ ˬgroup H˭ transform amongst themselves ^do nothing other than exchange amongst themselves^ all the substitutions of the group G. The collection of permutations $G + GS_1 + GS_2 + GS_3 +$ will form a group, which we will call GH.~~

Reduction, dependant or independent, of groups. Irreducible groups.

On irreducible groups in general.

Galois may have intended 'Reduction of groups, dependant or independent [groups]. Irreducible groups' but only if (as sometimes elsewhere) he made a grammatical slip (here of gender).

Theorem. Among the permutations of a group, there is always one where a given letter occupies a given place, and if in an irreducible group one considers only the permutations where one and the same letter occupies one and the same place, and if one removes this letter, the permutations that one obtains in this way will form a group. ⌜Let n be the number⌝ ˬof permutations of this latter m˭.

New proof of the theorem relative to the alternate groups.

~~Theorem. An irreducible group always contains~~ some complete substitutions. ~~Discussion of irreducible groups. Primitive and non-primitive groups.~~

Theorem. If a group contains ~~some~~ a complete substitution of order m and one of order $m-1$, it will be irreducible.

Discussion of irreducible groups. Primitive and non-primitive groups. Properties of the roots.

82 b

Originally ‘On peut ne considérer que des groupes qui ne contiennent que ...’.

On peut ~~ne considérer~~^supposer^ que ~~des~~^le^ groupe~~s qui ne~~ ne contiennent que des substitutions paires.

Il y aura ~~*m*~~ toujours un système ^conjugué complet^ de m ~~substitutions conjuguées~~ ^permutations^ quand $m = 4n$ et $4n + 1$, un système conjugué complet de $\frac{m}{2}$ permutations quand $m = 4n + 2$.

Donc, $t = m - 2$ dans le premier cas, $t = \frac{m-2}{2}$ dans le second.

~~Conséquences~~.

~~Donc on peut supposer dans tous les cas, un système conjugué complet de *m*~~

Discussions des groupes irréductibles.

One may ~~consider only~~ suppose that the group contains only even substitutions.

Originally 'One may consider only groups which contain only ...'.

There will always be a complete conjugate system of m ~~conjugate substitutions~~ permutations when $m = 4n$ and $4n + 1$, a complete conjugate system of $\frac{m}{2}$ permutations when $m = 4n + 2$.

Therefore $t = m - 2$ in the first case, $t = \frac{m-2}{2}$ in the second.

~~Consequences~~.

~~Therefore one may suppose in all cases, a complete conjugate system of m~~

Discussion of irreducible groups.

83 a

Application à la théorie des ~~Equations~~^fonctions^ et des Equation algébriques.
Fonctions semblables. Combien il peut y avoir de fonction semblables entre elles.

⌜Groupes appartenant aux fonctions⌝

Singular 'Equation', 'fonction' clear in *ms*.

See Note 5, p. 298.

Mr Cauchy. Théorème plus général, quand $m > 4$. Quelles sont les fonctions qui n'ont que m valeurs, ou qui ne contenant que des substitutions paires, n'ont que $2m$ valeurs

– Théorème. Si ~~tous les coef~~^une fonction^ de m indéterminées est donnée par une équation de degré inférieur à m dont tous les coéfficients soient des fonctions symmetriques permanentes ou alternées de ces indéterminées, cette fonction sera elle meme symmetrique quand $m > 4$.

Full stop and question mark supplied in T1906/7, BA1962, respectively; no punctuation in *ms*.

Word corrected to 'symétrique[s]' in BA1962; T1906/7 is faithful to *ms*.

– Théorème. Si une fonction de m indéterminées est donnée par une equation de degré m dont tous les coéfficients, &c., cette fonction sera symmetrique ^permanente ou alternée^ par rapport à toutes les lettres ou du moins par rapport à $m - 1$ d'entre elles.

– Théorème. Aucune équation algebrique de degré superieur à 4 ne saurait se résoudre ni s'abaisser.

✳ ~~Du cas où une fonction ^rationnelle^ des racines de l'équation est censée connue. Remarque. Il suffit~~^On peut^ réduire à ce cas celui où on en supposerait plusieurs ~~connues.~~

Premier cas. Quand le groupe appartenant à la fonction connue est réductible. Cas où ~~il ne contient~~^une seule^ permutation lui appartient.

2e Cas. Quand le groupe appartenant à la fonction est irréductible non primitif.

3e Cas. Quand le groupe appartenant à la fonction connue est primitif ~~et que~~ m ~~est~~^étant^ premier

^Jusqu'ici l'on avait cru^

4e Cas. Quand le groupe appartenant à la fonction est primitif et que $m = p^2$

5e Cas. Quand le groupe est primitif ~~et que m étant simplement et $m - 1$ premier~~ ^$m - 1$ étant premier ou le carré d'un nombre premier.^

Erasure in Case 5 made with one line, but perhaps two clauses deleted one after the other. Missing from BA1962.

Last seven lines precede the five cases in T1906/7. See Note 5, p. 298.

✳ Du cas ou une fonction ~~rationnelle~~ des racines de l'équation ^dont le groupe est G^ est censée connue.

Théorème. ~~Toute~~ Soit H le groupe d'une fonction ~~quelconque~~ φ des racines, ~~Si D est le commun diviseur de ce groupe et de celui de la fonction supposée~~ à G et à^est un diviseur de^ H ~~est~~, ~~et que G contienne m groupes semblables à D~~, φ ne dependra plus que d'une équation du $n^{\text{ième}}$ degré.

Second letter *H* badly blotted and barely legible.

Words 'plus que d'une' (excluding last two letters) have a thin line through them,. Probably accident, not deletion.

On peut ramener à ce cas celui ou on supposerait plusieurs fonctions connues.

Application to the theory of ~~equations~~ functions and of algebraic equations.

Similar functions. How many functions similar to each other there can be.

⌜Groups belonging to functions⌝

Mr Cauchy. More general theorem when $m > 4$. What are the functions that have only m values, or which, having only even substitutions, have only $2m$ values?

Theorem. If ~~all the coef[ficients of]~~ a function of m indeterminates is given by an equation of degree lower than m, all of whose coefficients are symmetric functions, permanent or alternating, of these indeterminates, this function will itself be symmetric when $m > 4$.

The word 'contenant' is better translated as 'containing' than 'having'. But 'contenant' refers to the group of the function.

Theorem. If a function of m indeterminates is given by an equation of degree m all of whose coefficients, &c., this function will be symmetric, permanent or alternating, with respect to all the letters or at least with respect to $m - 1$ among them.

Theorem. No algebraic equation of degree higher than 4 may be solved or reduced.

✳ ~~On the case where a rational function of the roots of the equation is supposed known.~~

~~Remark. One can reduce the case in which one supposes several of them known to this one.~~

First case. When the group belonging to the known function is reducible. Case in which ~~it does not contain~~ one single permutation belongs to it.

2nd Case. When the group belonging to the function is irreducible, non-primitive.

3rd Case. When the group belonging to the known function is primitive, m being prime.

Up to now it was believed

4th Case. When the group belonging to the function is primitive and $m = p^2$

5th Case. When the group is primitive ~~and m being simply and $m - 1$ being prime~~ $m - 1$ being prime or the square of a prime number.

✳ On the case where a ~~rational~~ function of the roots of the equation of which the group is G is supposed known.

Theorem. ~~Every~~ Let H be the group of a~~n arbitrary~~ function φ of the roots. ~~If D is the common divisor of this group and that of the assumed function~~ and G is a divisor of H, ~~and G contains n groups similar to D~~, φ will depend on an equation of no more than the nth degree.

One can reduce to this case that in which one supposes several functions known.

83 b

Note sur les Equations numériques

Ce qu'on entend par l'ensemble des permutations d'un⌜e⌝~~groupe~~ ^équation.^

Du cas où cet ensemble constitue un groupe.

Il n'y a qu'une circonstance où nous ayons reconnu que celà ~~est~~^doit^ nécessaire⌜ment⌝ avoir lieu. C'est celui où toutes les racines sont des fonctions rationnelles ~~les unes des autres~~^d'une quelconque d'entre elles^.

⌜Démonstration.⌝

C'est improprement, &c. Du reste, ~~les~~ tout ce que nous avons dit est applicable à ce changement près. 1°. ~~Soi~~^Théorème^ Si ~~toutes~~ une équation jouit de la propriété énoncée, toute fonction des racines invariables par les $m-1$ substitutions conjuguées sera connue, et réciproquement. ~~D~~ 2°. ~~Conséquence.~~ ^Théorème découlant de la remarque précédente.^ Toute equation dont les racines seront des fonctions rationnelles de la première, jouira de la mème propriété. 3° Corollaire. Si a est une racine imaginaire d'une pareille equation, et que fa en soit la conjuguée, ~~on aura $fx \equiv x$.~~ fx sera en general la conjuguée d'une racine quelconque imaginaire, x.

The emendation after 2º is almost illegible. T1906/7, BA1962 read 'réciproque', but Tannery offers 'remarque' as a plausible alternative reading. I am confident in 'remarque'.

On peut passer aisément de ce cas à celui où une racine etant connue, quelques unes en dépendent par des fonctions rationnelles. Car soient

$$x\,, \quad \varphi_1 x\,, \quad \varphi_2 x\,, \quad \ldots\,.$$

Ces racines, si l'on prend, &c.

⌜Il est aisé devoir que⌝ La mème Méthode ~~s'applique encore~~^de décomposition^ s'applique au cas où dans l'ensemble des permutations d'une équation, n ^memes^ lettres occupent toujours n mèmes places (abstraction faite de l'ordre) quand une seule de ces lettres occupe une ~~seu~~ de ces places, et il n'est pas nécessaire pour celà que l'ensemble de ces permutations constitue un groupe.

A thin sort of flourish after the text could be one of Galois' scribbles, but looks more like one of his cadence lines.

Note on numerical equations

What is understood by the collection of permutations of a⌜n⌝ ~~group~~ equation.

On the case where this collection constitutes a group.

There is only one circumstance we have recognised in which that must necessarily hold. It is that in which all the roots are rational functions ~~of each other~~ of an arbitrary one of them.

Proof.

It is improperly, &c. For the remainder, everything we have said is applicable subject to this change.

1. ~~Let~~ Theorem. If ~~all~~ an equation enjoys the stated property, every function of the roots invariant under the $m - 1$ conjugate substitution will be known, and conversely.

2. ~~Consequence~~ Theorem following from the preceding remark. Every equation whose roots are rational functions of the first will enjoy the same property.

3. Corollary. If a is an imaginary root of such an equation and fa is a conjugate of it, ~~one will have $fx = x$~~ fx will in general be the conjugate of an arbitrary imaginary root x.

One can easily pass from this case to that in which, one of the roots being known, some of the others depend on it by rational functions. For let

$$x\,, \quad \varphi_1 x\,, \quad \varphi_2 x\,, \quad \ldots\,.$$

be these roots, if one takes, &c.

It is easy to see that the same method ~~still applies~~ of decomposition applies to the case where in the collection of permutations of an equation, the same n letters always occupy the same n places (disregarding order) when a single one of these letters occupies one of these places, and it is not necessary for this that the collection of these permutations should constitute a group.

84 a

Mis-spellings of 'appelle' and 'représenterons' corrected in BA1962; T1906/7 faithful to *ms*.

On appele groupe un système de permutations tel que &c. Nous representerons cet ensemble par G.

GS est le groupe engendré lorsqu'on opère sur tout le groupe G la substitution ~~*H*~~ S. Il sera dit semblable, &c

Un groupe peut être fort différent d'un autre et avoir les mèmes substitutions. Ce groupe en general ne sera pas GS.

Unclear end of line. T1906/7 has semi-colon; BA1962 has 'etc.'; but '&c' matches usage elsewhere.

Groupe réductible est un groupe dans le⌜s⌝ ^permutations du^ quel n lettres ne sortent pas d'⌜e⌝~~une certaine~~ ^n places^ fixes. Tel est le groupe

$$\begin{array}{lll} abcde & abdec & abecd \\ bacde & badec & baecd \end{array}$$

Un groupe irréductible, &c.

Un groupe irréductible est tel qu'une ~~place~~^lettre^ donnée occupe une place donnée. Car supposons qu'une place ne puisse appartenir qu'à n lettres ~~don~~. Alors toute place occupée par l'une de ces lettres jouira de la mème propriété. donc &c.

Groupe irréductible non-primitif est celui où ~~une~~ ^~~où~~ l'on a^ n places ~~sont telles que lorsqu~~' ^et n lettres^ telles que une ^des^ lettre⌜s⌝ ne puisse occuper une de ces places, sans que les n lettres n'occupent les n places.

~~toute lettre qui~~

On voit que les lettres se partageront

84 b

Word 'classe' corrected to 'classes', in T1906/7, BA1962; singular is clear in *ms*.// As noted in T1906/7, BA1962, last word should be 'classes'.

en classe de n lettres telles que les n places ⌜en question⌝ ne puissent etre occupees à la fois que par l'une de ces places.

One calls group a system of permutations such that &c. We will represent this collection by G.

GS is the group generated when one operates on the whole group G with the substitution S. It will be called similar, &c

A group can be very different from another and have the same substitutions. In general this group will not be GS.

A reducible group is a group in whose permutations n letters do not move out of ~~a certain~~ n fixed places. Such is the group

$abcde$	$abdec$	$abecd$
$bacde$	$badec$	$baecd$

An irreducible group, &c.

An irreducible group is such that a given ~~place~~ letter occupies a given place. For, suppose that one place belongs only to n letters. Then every place occupied by one of these letters will enjoy the same property. Therefore &c.

An irreducible non-primitive group is one where one has n places ~~are such that when~~ and n letters such that one of the letters cannot occupy one of these places, without the n letters occupying the n places.

~~Every letter which~~

One sees that the letters are partitioned

into classes of n letters such that the n places in question cannot be occupied at the same time except by one of these [classes].

Folio **84 b** was turned through 180° and contains the following:

84 b (reversed)

Soit un ~~groupe~~^équation^ irréductible de degré p.
~~le facteur premier p devra~~
~~Supposons qu'un groupe M se décompose en p Groupes N.~~ en sorte que l'on ait

$$M = N + NS^{\text{~~2~~}} + NS^2 + \cdots + NS^{p-1},$$

la substitution S étant ^de^ degré p sera une substitution circulaire des racines. ~~Soit~~

~~$x_0 \quad x_1 \quad x_2 \quad \ldots$~~

~~ces racines~~ ⌜donc le groupe reductible N ne devra contenir aucune substitution. Ainsi l'on aura simplement⌝

$$M = A + AS + AS^2 + \cdots + AS^{p-1},$$

Soit donc

$$x_0 \quad x_1 \quad x_2 \quad \ldots \quad x_{p-1}$$

en sorte que la substitution circulaire soit $k \quad k+1$.

BA1962 corrects to 'données'.

Etant donné deux ~~substitutions~~^permutations^ A et A' et une substitution ~~S, on demande une substitution~~ qui

85 a

Étant donnée une substitution S et deux permutations A et A' on demande une substitution S' telle que ~~si la $k^{\text{ième}}$ place~~ ^la lettre située^ au $k^{\text{ième}}$ rang dans A prenant le $\varphi k^{\text{ième}}$ dans AS, la lettre située au $k^{\text{ième}}$ dans A' prenne la $\varphi k^{\text{ième}}$ dans $A'S'$.

Supposons le problème résolu. ~~Il est clair q~~ Soit $A' = AT$, on aura évidemment

d'où

$$\begin{aligned} A'S' &= AST \\ TS' &= ST \\ S' &= T^{-1}ST \end{aligned}$$

Let an irreducible ~~group~~ equation of degree p be given.
~~the prime factor p must~~
~~Suppose that a group M is decomposed into p groups N~~ so that one will have

The crossing out of this clause in the original is puzzling.

$$M = N + NS + NS^2 + \cdots + NS^{p-1}$$

the substitution S being of degree p will be a circular substitution of the roots. ~~Let~~

~~$x_0 \quad x_1 \quad x_2 \quad \cdots$~~

~~be these roots~~ Therefore the reducible group N cannot contain any substitutions. Thus one will simply have

$$M = A + AS + AS^2 + \cdots + AS^{p-1}$$

Therefore let

$$x_0 \quad x_1 \quad x_2 \quad \ldots \quad x_{p-1}$$

so that the circular substitution is $k \quad k+1$

Given two ~~substitutions~~ permutations A and A' and a substitution ~~S, one asks for a substitution~~ which

Given a substitution S and two permutations A and A' one asks for a substitution S' such that ~~if the kth place~~ the letter situated in the kth rank in A taking the φk^{th} in AS, the letter situated in the kth in A' shall take the φk^{th} in $A'S'$.

Suppose the problem solved. ~~It is clear t[hat]~~ If $A' = AT$, clearly one will have

from which

$$A'S' = AST$$
$$TS' = ST$$
$$S' = T^{-1}ST$$

85 b

Si l'on represente les n lettres par n indices

$$1.\ 2.\ 3\ .\ .\ .\ .\ n$$

toute ~~substitution~~^permutation^ pourra etre representée

$$\varphi 1 \quad \varphi 2 \quad \varphi 3 \quad . \ . \ . \quad \varphi n$$

φ etant une fonction convenable˅ment˅~~des~~ choisie la ~~permu~~^substitution^par la quelle on passe de la premiere perm. à l'autre sera $(k, \varphi k)$, k désignant un indice quelconque.

Aulieu de ~~celà~~ représenter les lettres par des nombres on pourrait représenter les places par des nombres.

If the n letters are represented by n indices

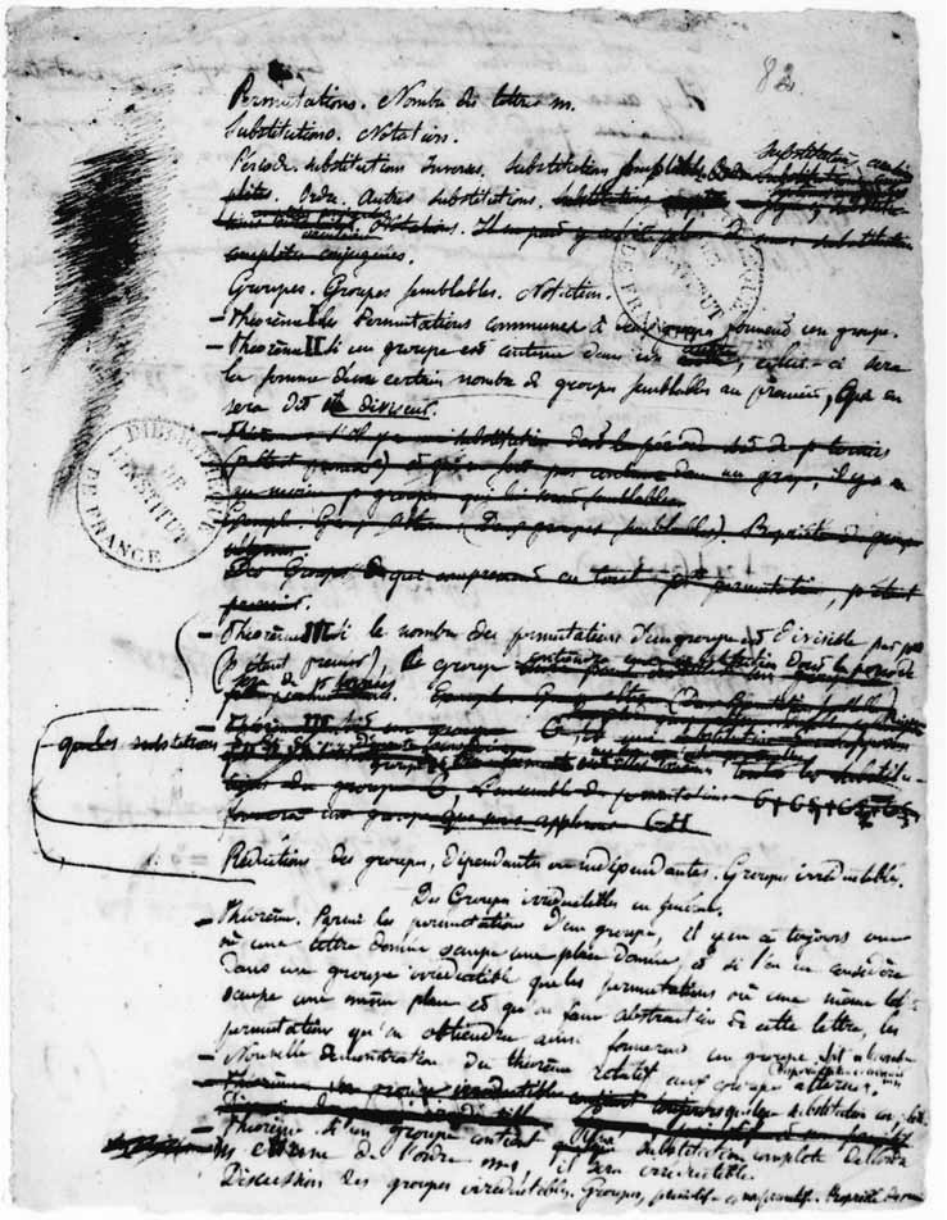
$$1.\ 2.\ 3\ \ldots\ .\ n\,,$$

every ~~substitution~~ permutation can be represented

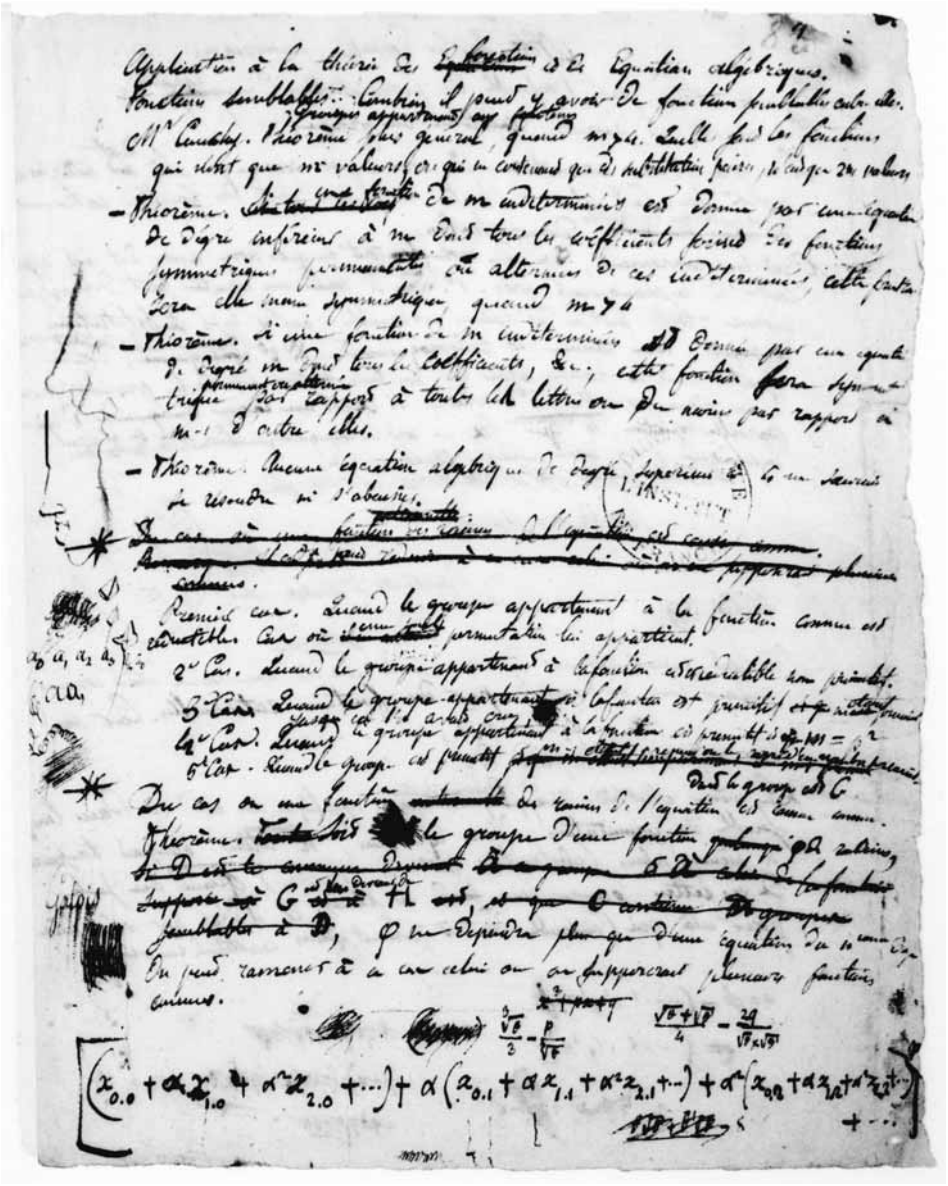
$$\varphi 1 \quad \varphi 2 \quad \varphi 3 \quad \ldots \quad \varphi n\,,$$

φ being a ~~suitable function of the~~ suitably chosen function. The ~~permu[tation]~~ substitution by which one passes from the first perm[utation] to the other will be $(k, \varphi k)$, k denoting an arbitrary index.

Instead of ~~that~~ representing the letters by numbers one could represent the places by numbers.



Fragments: Dossier 15, folio 82 recto.



Fragments: Dossier 15, folio 83 recto

Notes on Dossier 15

NOTE 1: In the manuscripts of this dossier there are several connections with the papers [Cauchy (1815a)], [Cauchy (1815b)]. I plan to write about the mathematics elsewhere. Here it should be noted that terminology such as *fonction symétrique permanente ou alternée* (**f. 83 a**, p. 284) comes from the latter of those two papers, either directly or through the *Cours d'analyse* [Cauchy (1821), Ch. III, p. 73; Note IV, p. 521]. There, for example, *une fonction symétrique alternée* is used to describe a function of several variables which is changed to its negative by transposition of any two of those variables. Cauchy, however, did NOT have groups, and did not identify anything like the alternating group there—although he did prove, in two different ways, that an arrangement obtained from another by an even number of transpositions cannot also be obtained from it by an odd number of transpositions. It was not until 1845 that he recognised groups—which he called *systèmes de substitutions conjuguées* 'systems of conjoined substitutions'. The coincidence of this terminology with phrases to be found on **f. 82 a** led me many years ago to conjecture that the manuscripts in Dossier 15 might perhaps not all be by Galois and might carry some annotations by a later 19th-century commentator. Having seen the handwriting, though, I am quite clear that that conjecture is false. The coincidence of terminology is genuine coincidence, nothing more.

Note also that the terminology *groupe alterne* (probably *groupe alterné* was intended), which occurs on **f. 85 a**, occurs there only and nowhere else in the Galois corpus. Moreover, there is no indication that Galois ever thought he had a proof that the alternating groups of degree $\geqslant 5$ are indecomposable (simple). His mind was so fertile that he may perhaps have had an intuition of the fact, but it is not to be found in any of his writings. Certainly, contrary to common myth, he never used it.

NOTE 2: In **f. 82 a** the words 'Réductions des groupes, dépendantes ou indépendantes. Groupes irréductibles' fill a whole manuscript line. This is treated as a line of text in [Tannery (1907), p. 280] and [B & A (1962), p. 73]. In the manuscript, however, it is separated from the text above both by a horizontal line and by a small horizontal gap. The horizontal line comes so close to the writing above it that it looks rather like underlining: but the effect would, very implausibly, be 'groupe que nous appelerons *GH'*, and therefore it seems more likely that it is an instance of the line that occurs frequently in Galois' writings and seems to indicate a sort of cadence or closure. The phrase 'Des Groupes irréductibles en général' is neatly centred below the cited line ('Réductions ...'). The evidence is weak; nevertheless, I am inclined to suggest heading and sub-heading as having been the original concept. Note that the last four words on **f. 82 b** (they appear marginally above the middle of the page) 'Discussion des groupes irréductibles', are presumably related. They are preceded by a substantial gap, which suggests a repeat sub-heading; on the other hand, they are not centred on the page.

NOTE 3: The space above and below that phrase 'Discussion des groupes irréductibles' on **f.82b** is filled with jottings of a calculation relating mostly to something else and written with the page turned upside-down relative to the material transcribed here. In the gap above those four words, however, there are some formulae written in the same hand and the ordinary way up. First, it looks as if $2(m-1)(m-2)$ was written, then modified by an addition in the margin to read $\frac{(m-1)^2(m-2)}{2} + 2(m-1)(m-2)$, then crossed out and replaced with $m(m-1)(m-2)$, and finally the marginal addition was crossed out to leave just $m(m-1)(m-2)$. It is quite possible, though, that these steps, let's call them A, B, C, D, came in a different order such as ACBD, or ABDC.

A little lower the surviving formula $m(m-1)(m-2)$ is repeated. Below that and in the margin appears $(2m-1)\frac{(m-1)(m-2)}{2}$ pretty thoroughly crossed out, and about halfway down the page, a couple of lines below the isolated heading 'Discussion des groupes irréductibles' is the incomplete formula $\frac{2m}{3}(m-$, also crossed out. The formula $m(m-1)(m-2)$ also occurs, thoroughly crossed out, in the margin of **f.82a** against the line 'm et une de l'ordre $m-1$, il sera irréductible.'

NOTE 4: There is much other, unrelated, material in **f.82b**. In the margin, in the same orientation as the main text and at about the level of its last few lines appear

$$Galois \quad \text{and} \quad \text{EV. GALOIS}$$

written in two different calligraphic scripts.

As was mentioned above, the remaining material was written with the page turned through 180°, so that it now appears upside-down. The first few lines are carefully and neatly penned and look something like the following:

$$(1.0\ ,\ 0.1)$$

$$\varphi(1,0) = 0\ , \quad \psi(1,0) = 1$$

$$(1) \qquad a_n k + b_n l = 1\ , \qquad c_n k + d_n l = 0\ , \qquad \frac{a_n - d_n}{a-d} = \frac{b_n}{b} = \frac{c_n}{c}$$

$$\varphi(k,l) = a_n\varphi(1,0) + b_n\psi(1,0) = b_n$$
$$\psi(k,l) = c_n\varphi(1,0) + d_n\psi(1,0) = d_n$$

On déduit de (1)

$$b_n = \frac{bl^{\blacksquare}}{bl^2 + (a-d)kl - ck^2}\ , \qquad d_n = \frac{ck}{ck^2 - (a-d)kl - bl^2}$$

$$(ak+bl)l - (ck+dl)k$$

Before the last of those lines, however, there is a tentative line in which are written $\cancel{bl^2 + l^2 + (a - d)}$, $\cancel{\varphi(k, l)}$ (very faint), and $\varphi(k$. And after it there are various formulae, some scribbled through, others not, all placed with rather less care, and extending into the margin (now on the right of the page relative to the writing):

$$\cancel{(ak + bl)[} \qquad (ak + bl)[(a^2 + bc)k + b(a + d)l]$$

$$\cancel{(ak + bl)[ml + n(ck + dl)]} \qquad \begin{array}{l|l} a(a^2 + bc)k^2 + (ab^2 + b^2c + a^2b + abd) & kl) + b^2(a + d)l^2 \\ - c^2(a + d) \qquad - dc^2 - c - d^2c - cad & — \end{array}$$

$$a(ak + bl) + b(ck + dl)$$

$$\cancel{a^2} \qquad\qquad (a-d)(b+c)+bc+ad$$

$$\cancel{ak} \qquad \frac{\cancel{a}}{\cancel{d}}\cancel{d} \qquad \frac{a^2}{c}ck + \frac{a^2}{c}dl - \frac{a(ad - bc)}{c}$$

$$\frac{ak + bl}{k} - \frac{ck + dl}{l} \qquad \frac{ad}{c} - b$$

NOTE 5: The phrase 'Groupes appartenant aux fonctions', which is clearly an afterthought written into the narrow space after the second line of **f. 83 a** and centred on the page, is read in [Tannery (1907), p. 281] and in [B & A (1962), p. 75] as an interjection into the succeeding text. Considering its content and its placement I estimate it more likely to have been intended as a sort of sub-title.

Similarly, the two lines 'Du cas où une fonction ...' and 'Du cas ou une fonction ...' that are marked with Galois' large asterisk ⁎ look as if they may have been intended as subheadings. Those asterisks probably indicate that the second was intended to replace the first. Indeed, that is how this material is presented in [Tannery (1907), p. 281].

NOTE 6: The left and bottom margins of **f. 83 a** carry various jottings. On the left one sees (in order from top to bottom): scribbles that could be simple silhouettes of faces seen from the left side; illegible scribbles; $a_0\ a_1\ a_2\ a_3$ with aa_1 below it; more scribbles; then after a space the name Galois in an elegant hand followed by something, probably the name Galois again, that has been fiercely scribbled out. The bottom of the page has some formulae that have been scribbled out; then formulae $\cancel{x^3 + px + q}$, $\frac{\sqrt[3]{\theta}}{3} - \frac{p}{\sqrt[3]{\theta}}$ and $\frac{\sqrt{\theta}+\sqrt{\theta'}}{4} - \frac{2q}{\sqrt{\theta}\times\sqrt{\theta'}}$, written in a small hand and arranged haphazardly; then, extending into the left margin and right across the page, the formula

$$\Big[\Big(x_{0.0} + \alpha x_{1.0} + \alpha^2 x_{1.0} + \cdots\Big) + \alpha\Big(x_{0.1} + \alpha x_{1.1} + \alpha^2 x_{1.1} + \cdots\Big) + \alpha^2\Big(x_{0.2} + \alpha x_{1.2} + \alpha^2 x_{1.2} + \cdots\Big) + \cdots\Big]^p$$

in a larger hand than Galois usually wrote; and finally another scribbled-out formula involving square roots and θ and θ' in a small hand.

NOTE 7: The middle of **f.83b** contains the following well-spaced and thoroughly scribbled-out material, round which the text of the 'Note sur les Equations numériques' was written later:

~~Mémoire~~

~~Sur la théorie des permutations et sur celle~~

~~des Equations littérales.~~

~~Présenté à l'Institut~~ ~~par~~

~~Octobre 1829.~~

~~E Galois~~

As observed in [B & A (1962), p. 509], it seems that the paper had originally been prepared for use as a cover sheet for an article intended for presentation to the Academy of Sciences. There is, however, no record in the Academy minutes of the submission of any such item in the autumn of 1829. Could it be that Galois had had in mind to supplement or replace the material he had submitted to the Academy on 25 May and 1 June 1829 (see Note IV.2 on p. 145)? Unless more evidence comes to light we simply cannot know.

There is other material on this page, most of it in the bottom margin below the text:

~~$\sqrt{\theta} = i(a+b)$~~

~~x~~

~~$2(a+b)(a+c)(a+d)$~~

~~$a^3+(b+c+d)a^2+$~~

$(\alpha-1)x_1 + (\alpha^2-1)x_2$

$F(a,b,c,...) = 0$

$\varphi(a,b,c,....) - \varphi = 0$

It was written with the page turned through 180° so that it now appears upside-down. The elegantly written name ~~Galois~~ also appears in this orientation in the middle of the page just below and to the left of 'Présenté'.

VI.11 Dossier 16: Fragments relating to Proposition I of the First Memoir

Dossier 16 comprises folios 86–89, of the manuscripts. The pages were collected into the dossier in reverse order before being numbered. The cover sheet has the title 'Fragments se rapportant à la Proposition I du Mémoire sur la résolubilité ...'. The material here is presented in [Tannery (1907), pp. 286–289] and in [B & A (1962), pp. 102–109, 508–510].

Tannery dates the material to before the *Premier Mémoire* and the separate version of Proposition I (folio 54 in Dossier 6). There is a detailed discussion of the relationship of this piece to the *Premier Mémoire* in [B & A (1962), pp. 510–511] but Bourgne did not give an explicit date.

The four leaves in the dossier came from the same source. Folio 85 is a rectangular fragment 15 cm × 8 cm, torn from a page similar to the others; folios 87–89 are 15 cm × 20.7 cm. They all carry a faint but clear, neatly ruled, vertical pencil-line that creates a left margin varying from 2.3 to 3 cm wide. Although in [Tannery (1907), p. 286, fn] the paper is identified as being the same as that of folio 94 (Dossier 18), the latter does not show the neat pencil-drawn margin which links them; indeed, I have not found it anywhere else in the manuscripts. They appear to have been torn from a notebook, and the right edges of folios 86, 89 are irregular, as are the left edges of folios 87, 88. As a result, the ends of the lines on **f. 89 a** and **f. 88 b** are missing. We follow Tannery in supplying the missing letters, syllables or words in square brackets.

There are three items here. The first is a carefully written fair copy (using in order the four sides **f. 89 b**, **f. 89 a**, **f. 88 a**, **f. 88 b**, that is, two of the four leaves) that contains a clear statement of Proposition I of the First Memoir (as does **f. 54 b** in Dossier 6). It is missing a page or two from its beginning and it ends mid-sentence with little of the proof achieved. The second, **f. 87 a**, starts with repetition of a paragraph of the first and is written in a similar hand, clear, careful and legible. Thus the statement of the main theorem in the first item breaks at the end of **f. 89 a** with the phrase 'dont les substitutions sont telles', and the statement continues both on **f. 88 a** and on **f. 87 a** with 'que toute fonction des racines invariable par ces substitutions est rationnellement connue'. The third item, the 15 × 8 fragment, seems unrelated—except physically in that it comes from the same source.

These are some of the pages in which Galois eschewed paragraph indents.

89 b

équations. Nous nous contenterons donc d'avoir exposé les définitions indispensables pour l'intelligence de la suite, et nous allons ~~donner~~ montrer la liaison qui existe entre les deux théories.

---

§2. Comment la théorie des Équations dépend de celle des Permutations.

6 Considérons une équation à coëfficients quelconques et regardons comme rationnelle toute quantité qui s'exprime rationnellement au moyen des coëfficients de l'équation, et même au moyen d'un certain nombre d'autres quantités irrationnelles adjointes que l'on peut supposer connues à priori.

Lorsqu'une fonction des racines ne change pas de valeur numérique par une certaine substitution opérée entre les racines, elle est dite invariable par cette substitution. On voit qu'une fonction peut très bien être invariable par ~~toute~~^telle ou telle^ substitution entre les racines, sans que sa forme l'indique. Ainsi, si $F(x) = 0$ est l'équation proposée, ^la fonction^ $\varphi\left(F(a),\ F(b),\ F(c), ...\right)$ (φ etant une fonction quelconque, et $a,\ b,\ c$ ... les racines) sera une fonction de ces racines invariable par toute substitution entre les racines, sans que sa forme l'indique généralement.

Or c'est un⌜e⌝ ~~Problème~~ Question dont il ne parait pas qu'on ait encore la solution, de savoir si, étant donnée une fonction de plusieurs quantités ^numériques^, on peut trouver un groupe qui contienne toutes les substitutions par les quelles cette fonction est invariable, et qui n'en contienne pas d'autres. Il est certain que celà a lieu pour des quantités littérales,

It is not clear whether Galois intended a new paragraph here or not. I am inclined to think probably not.

89 a

puisqu'une fonction de plusieurs lettres invariables par deux substitutions est invariable par leur produit. Mais rien n'annonce que la mème chose ait toujours lieu quand aux lettres on substitue des nombres.

On ne peut donc point traiter ^toutes^ les équations ~~numériques~~ commes les équations littérales. Il faut avoir recours à des considérations fondées sur les propriétés particulièr[es] de chaque équation numérique. C'est ce que je vai[s] tacher de faire.

The subheading and its number were late marginal additions.

Des cas particuliers des équations

7 Remarquons que tout ce qu'une équation numériqu[e] peut avoir de particulier, doit provenir de certai[nes] relations entre les racines. Ces relations seront ratio[nnelles] dans le sens que nous l'avons entendu, c'est à di[re] qu'elles ne contiendront d'irrationnelles que les coëffici[ents] de l'équation et les quantités adjointes. Deplus [ces] relations ne devront pas ètre invariables par to[ute] substitution opérée sur les racines, sans quoi on [n'aurait] rien de plus que dans les équations littérales.

Ce qu'il importe donc de connaitre, ~~ce sont les ra[cines] et~~ c'est par quelles substitutions peuvent ètre invari[ables] des relations entre les racines, ou ce qui revient

equations. We will be satisfied, then, that we have expounded the definitions that are indispensable for an understanding of what follows, and we will ~~give~~ show the connection that exists between the two theories.

---

§2. How the theory of equations depends on that of permutations.

6 Consider an equation with arbitrary coefficients and regard as rational every quantity which may be expressed rationally by means of the coefficients of the equation, and also by means of a certain number of other irrational *adjoined* quantities which may be supposed known *a priori*.

When a function of the roots does not change its numerical value under a certain substitution acting on the roots, it is said to be *invariant* under this substitution. One sees that a function may very well be invariant under ~~every~~this or that substitution of the roots, without its form indicating so. Thus if $F(x) = 0$ is the proposed equation, the function $\varphi\left(F(a),\ F(b),\ F(c), ...\right)$ (where φ is an arbitrary function and a, b, c ... are the roots) will be a function of its roots invariant under every substitution among the roots without, in general, its form indicating this.

Now it is a ~~problem~~ question to which it seems we do not yet have a solution, to know whether, given a function of several numerical quantities, one can find a group which contains all the substitutions under which this function is invariant, and which contains no others. Certainly this is true for literal quantities

Literally 'littérales' is 'literal', or 'letter' used adjectivally; here its meaning is 'algebraic'.

because a function of several letters that is invariant under two substitutions is invariant under their product. But nothing certifies that the same thing should always be true when numbers are substituted for the letters.

It is therefore impossible to treat all ~~numerical~~ equations in the same way as algebraic equations. One must fall back on considerations founded on particular properties of each numerical equation. It is this that I shall endeavour to do.

7 Particular cases of equations.

Note that any particular property of a numerical equation must derive from certain relations amongst the roots. These relations will be rational in the sense we have met [understood], that is to say, they contain no irrationals other than the coefficients of the equation and the adjoined quantities. Moreover, these relations cannot be invariant under all the substitutions acting on the roots, otherwise one would have nothing more than in algebraic equations.

What it is therefore important to know ~~are the roots and~~ is, what are the substitutions under which relations among the roots may be invariant, or, which comes to

a[u même,] des fonctions des racines dont la valeur numéri[que] est déterminable rationnellement.

À ce sujet, nous ^allons^ démontrer~~ons bientôt~~ un théorème de la dernière importance dans cette matière et dont l'énoncé suit: "Etant donnée une équation avec un certain nombre de quantités adjointes, il ~~est~~ éxiste toujours un certain groupe de permutations dont les substitutions sont telles

88 a

Continuation repeated on **f. 87 a**, 306.

que toute ~~substitution~~ fonction des racines invariable par ces substitutions est rationnellement connue; et telle réciproquement, qu'une fonction ne peut être rationnellement déterminable, à moins d'être invariable par ces substitutions *que nous nommerons substitutions de l'équation.*" (Dans le cas des Equations littérales, ce groupe n'est autre chose que l'ensemble de toutes les permutations des racines, puisque les fonctions symmétriques sont seules connues).

Pour plus de simplicité, nous supposerons dans la démonstration de notre théorème, qu'il ait été reconnu pour toutes les équations de degrés inférieurs; Ce qu'on peut toujours admettre puisqu'il est évident pour les équations du second degré.

Admettons donc la chose pour tous les degrés inférieurs à m; pour la démontrer dans le $m^{\text{ième}}$, nous distinguerons quatre cas:

1er Cas. L'équation se décomposant en deux ou en un plus grand nombre de facteurs.

Soit $U = 0$ l'équation, $U = VT$, V et T étant des fonctions dont les coëfficients se déterminent rationnellement au moyen des coëfficients de la proposée et des quantités adjointes.

Je vais faire voir, que, dans l'hypothèse, on pourra trouver un groupe qui satisfasse à la condition énoncée.

Remarquons ici, que dans ces sortes de questions, comme il ne s'agit que des substitutions par les quelles des fonctions sont invariables, si un groupe satisfait à la condition, tout groupe ~~dont~~ qui aurait les mèmes substitutions, y satisfera aussi. Il convient donc de partir toujours d'une permutation arbitraire, mais fixe, afin de déterminer les groupes

88 b

que l'on aura à considérer. De cette manière, on évitera toute ambiguité.

Celà posé, dans le cas actuel, il est clair que si l'on adjoignait à l'équation $U = 0$, toutes les racines de l'équation $V = 0$, l'équation $U = 0$ se décomposerait en facteurs dont l'un serait $T = 0$, et les autres seraient les facteurs simples de V.

BA1962 follows T1906/7 in giving '[sur]' but 'su' has not been torn away.

Soit H le groupe que l'on obtient en opérant su[r] une permutation arbitraire A des racines de l'équati[on] $U = 0$, toutes les substitutions ~~de~~ ^qui sont relatives à^ l'équation $T = 0$ quand on [lui] adjoint les racines de $V = 0$.

the same thing, functions of the roots whose numerical value is determinable rationally.

As a contribution to this subject we are going to prove ~~soon~~ a theorem of the greatest importance in this matter, of which the statement follows:

Given an equation with a certain number of adjoined quantities, there ~~is~~ always exists a certain group of permutations of which the substitutions are such

that every ~~substitution~~ function of the roots invariant under these substitutions is rationally known; and conversely, such that a function cannot be rationally determinable unless it is invariant under these substitutions, which we call substitutions of the equation.

(In the case of literal equations this group is nothing other than the collection of all permutations of the roots, because the symmetric functions are the only ones known.)

For greater simplicity, in the proof of our theorem we will suppose that it has been accepted for all equations of lower degrees, as is always admissible since it is evident for equations of the second degree.

Let us accept therefore that it is true for all degrees lower than m; to prove it in the mth, we shall distinguish four cases:

1st Case. The equation is decomposable into two or a greater number of factors.

Let $U = 0$ be the equation, $U = VT$, V and T being functions of which the coefficients are determinable rationally by means of the coefficients of the proposed equation and of adjoined quantities.

I am going to show that under the hypothesis a group may be found which satisfies the stated condition.

Note here that in this sort of question, since all that matters is the substitutions under which functions are invariant, if a group satisfies the condition, every group ~~whose~~ which will have the same substitutions also satisfies it. It is convenient therefore always to start from an arbitrary, but fixed, permutation in order to determine the groups

that are to be considered. By this means one will avoid all ambiguity.

That said, in the present case it is clear that if one were to adjoin to the equation $U = 0$ all the roots of the equation $V = 0$, the equation $U = 0$ will decompose into factors of which one will be $T = 0$, and the others will be the simple factors of V.

Let H be the group that is obtained by acting on an arbitrary permutation A of the roots of the equation $U = 0$ with all the substitutions ~~of~~ that are relative to the equation $T = 0$ when the roots of $V = 0$ are adjoined to it.

Soit K le groupe que l'on obtient en opérant sur A [toutes] les substitutions qui sont relatives à ~~T~~ $V = 0$ q[uand on] *ne* lui adjoint ~~les racines de $T = 0$~~ que les quantités [adjoin]tes primitivement à la proposée.

Combinez ^en tous sens^ toutes les substitutions du groupe H avec [celles] du groupe K. Vous obtiendrez ainsi un groupe ^réductible^ l[equel] je dis jouir de la condition éxigée relativement [à la] question proposée.

En effet toute fonction invariable par les substi[tutions] du groupe

[*Galois stopped here although there was space for several more lines on the page*.]

BA1962 follows T1906/7 in omitting 'ainsi' but it is clear in *ms*.
For l[equel], T1906/7 supplies '[que]'. Since a fragmentary letter more plausibly 'l' than 'q' is visible, I have here followed BA1962.

87 a

que toute fonction des racines invariable par ces substitutions est rationnellement connue, et telle réciproquement, qu'une fonction ne peut être rationnellement déterminable, à moins d'être invariable par ces substitutions. (Dans le cas des Equations littérales, ce ~~system~~ groupe n'est autre chose que l'ensemble de toutes les permutations des racines, puisque les fonctions symmettriques sont seules connues.)

Mais, avant de développer la démonstration ^complète^ de cette proposition, ~~il~~ nous ferons voir ~~auparavant~~ qu'il suffit de la ~~démontrer~~^donner^ dans le cas où l'équation proposée ne se décompose pas en facteurs dont les coëfficients se déduisent rationnellement de~~s~~ ^ses^ coëfficients et des quantités qui lui sont adjointes, plus brievement, dans le cas où l'équation n'a pas de diviseurs rationnels. Admettons en effet que la chose ait été démontrée dans ce cas, et supposons qu'une équation se décompose en deux facteurs qui n'aient eux mèmes aucun diviseur rationnel.

[*Galois stopped here although there was space for several more lines on the page*.]

Let K be the group that is obtained by acting on A with all the substitutions that are relative to $V = 0$ when one adjoins to it ~~the roots of $T = 0$~~ only the quantities originally adjoined to the proposed equation.

Combine in every way all the substitutions of the group H with those of the group K. In this way you will obtain a reducible group which, I say, enjoys the required condition relative to the proposed question.

Indeed, every function invariant under the substitutions of the group

[*Second continuation from* **f. 89 a**, *p.* 304.]

that every function of the roots invariant under these substitutions is rationally known, and conversely, such that a function cannot be rationally determinable unless it is invariant under these substitutions.

(In the case of literal equations this ~~system~~ group is nothing other than the collection of all the permutations of the roots, because the symmetric functions are the only ones known.)

But before developing the complete proof of this proposition ~~it~~ we will show ~~beforehand~~ that it suffices to ~~prove~~ to give it in the case where the proposed equation is not decomposable into factors, the coefficients of which may be deduced rationally from its coefficients and some adjoined quantities; more briefly, in the case where the equation has no rational divisors. Indeed, accepting that it has been proved in this case, we suppose that an equation is decomposable into two factors which themselves have no rational divisor.

86 b

Soit G un groupe correspondant à l'équation $\psi = 0$, et $A,\ B,\ C$ .... les permutations du groupe G. Pour obtenir un pareil groupe, il faut opérer sur une permutation A toutes les substitutions du ~~groupe~~ G l'équation ψ. Nous supposons que la permutation A contienne toutes racines de $F(x) = 0$

Word 'les' missing after 'toutes'.

Prennons une fonction $\Phi(A\Sigma)$ invariable par les substitutions Σ relatives aux racines de φ.

Full stop clear in *ms*; changed to comma in T1906/7, BA1962.

86 a

See Note 1 below

qui correspondent aux substitutions indiquées quand aux racines de l'équation φ on substitue leurs expressions en fonction de celles de ψ. Je dis qu'il viendra un groupe de Permutations qui relativement à la proposée $F(x) = 0$ satisfera à la condition éxigée.

En effet, toute fonction des racines invariables par les substitutions de ce groupe pourra d'abord s'exprimer en fonction des seules racines de l'équation ψ. Deplus, comme cette fonction transformée sera encore invariable par les substitutions de l'équation ψ, on voit que sa valeur numériq[ue]

[*The fragment was torn off after completion of the sentence, and its last few words are lost. Indeed, the passage may perhaps have continued beyond this sentence.*]

Let G be a group corresponding to the equation $\psi = 0$, and A, B, C, ... the permutations of the group G. To obtain such a group it is necessary to act on one permutation A with all the substitutions of the ~~group G~~ equation ψ. We suppose that the permutation A contains all the roots of $F(x) = 0$.

Take a function $\Phi(A\Sigma)$ invariant under the substitutions Σ relative to the roots of φ.

which correspond to the substitutions produced when for the roots of the equation φ one substitutes their expressions as functions of those of ψ. I say that there emerges a group of permutations that will satisfy the required condition relative to the proposed equation $F(x) = 0$.

Indeed, every function of the roots invariant under the substitutions of this group will in the first place be expressible as a function of the roots of the equation ψ alone. Further, as this transformed function will still be invariant under the substitutions of the equation ψ, one sees that its numerical value

Notes on Dossier 16

NOTE 1: It seems quite possible to me that the material on **f.86a** does not follow on from the preceding fragment **f.86b** (recall that the pages of the dossier are reversed); or even that the two fragments belong in the opposite order.

NOTE 2: There are few jottings here. The left margin of **f.86b** contains a couple of small and unrecognisable line-drawings, one of which looks like a large symbol similar to $\forall$. And **f.87b** contains the following formulae (aligned more or less as shown, except that $(A-s)X^2$ extends well over into the left margin). They are pretty clearly written except for a few minor amendments.

$$x^3 + 3px + q = 0$$
$$x^3 - 3(ab)x - a^3 - b^3$$
$$ab = p \qquad a^3 + b^3 = -q$$
$$a^3 = -q \pm \sqrt{q^2 - 4p^3}$$
$$ab = -p \qquad a^3 + b^3 = -2q$$
$$x^2 - 2qx - p^3 \qquad q + \sqrt{q^2 + p^3}$$

$$(A-s)X^2 + (A'-s)Y^2 + (A''-s)Z^2$$

$Y\ Z$

$$\begin{array}{llll} a & fa & ffa & fffa \\ b & fb & ffb & fffa \\ c & fc & ffc & fffa \ldots \end{array}$$

Those mysterious occurrences of $fffa$ are overwritten on dots indicating ellipsis that originally followed ffa, ffb, and perhaps also ffc, respectively. In addition there is a fiercely crossed out formula beginning with X, placed just below Z^2 and extending somewhat to its right. And below that is the formula X^2+ similarly deleted.

VI.12 Dossier 17: Fragments on the theory of equations

Dossier 17 comprises folios 90–93, of the manuscripts. It is presented in [Tannery (1907), pp. 290–292] and in [B & A (1962), pp. 82–85, 512–513]. The cover sheet carries the description 'Fragments sur la théorie des équations.' In [B & A (1962), p. 512] the work is dated to September–December 1831 in Sainte-Pélagie.

The four pieces of paper are all quite small, some smaller than others. All have been torn from something else. Folio 90 is 17.5 cm × 18.5 cm (approximately), torn down its right-hand side probably from a piece perhaps 35 cm wide, and then also torn along its bottom from a deeper piece. The bottom tear cuts a line of writing on **f. 90 a** in half. Folio 91 is a similar size and shape except that it was torn somewhat carelessly, so that its top and its right-hand side are very uneven, its left-hand side is somewhat uneven, and only its bottom is moderately straight; at its widest it is 19 cm, at its deepest it is 20 cm. A tiny amount of writing on **f. 91 a** has been lost through the tear on the left; a significant amount is lost from the tear on the right. It is not easy to read—see Note 1 on p. 318. Folios 92 and 93 are about half the size of the first two, 19.5 cm × 10.5 cm and 17.5 cm × 11 cm, respectively. Folio 92 is evenly torn (or perhaps cut) along top and right; the left is uneven, however, with a triangle projecting that one might hope to fit with a similar triangle missing from another piece (though I have failed to find one that matches), and the bottom, which is more or less straight slopes downwards from left to right. The top edge of folio 93 has ink on it as if it had been cut along an ink line, or as if it had been lightly used as a ruler to draw a straight ink line. If the former, then one might hope to match this fragment with the piece from which it was cut, which should show similar traces of ink along a bottom edge—but I have not been able to do so.

The pages are all very messy, with many deletions, emendations, and jottings. Folio 90 (both sides) is written with a medium broad pen and heavy black ink. Some of the crossings out and emendations are particularly black—indeed, there are three places where the page has been eroded through by the ink. The pens used on the other three folios are finer, though some of the corrections were very heavily made, and some of the jottings indicate a medium pen.

This is one of the manuscripts that has no paragraph indents.

90 a

Singular 'substitution' is clear in *ms*.

Soit donc $\varphi(H)$ une certaine fonction invariable par les substitution du groupe H et non par celles du groupe G. On aura donc

$$\varphi(H) = f(r)$$

la fonction f ne contenant dans son expression que les quantités antérieurement connues.
Éliminons algébriquement r entre les equations

$$r^p = A \qquad f(r) = z$$

Clause 'ne peuvent ètre que' uses only half a line. Formula '$\varphi(H)\ \varphi$' appears indented on line below, possibly displayed, though left indent is less than in most of Galois' displayed formulae.

On aura une équation ^irréductible^ du $p^{\text{ième}}$ degré en z. (Sinon z serait fonction de r^p: ce qui est contre l'hypothèse). Maintenant ~~soit on voit que les p racines de l'équation en z ne peuvent ètre que $\varphi(H)$ φ~~ Soit ~~H~~ S un⌜e⌝ des ^~~groupes~~^ ~~diviseurs de G semblable~~ à ^substitutions du groupe G qui ne lui soient pas communes à^ H. On voit que $\varphi(HS)$ sera ~~fonc~~ ^encore^ racine de l'equation ci-dessus en z, ~~Donc on aura~~ ⌜puisque les coëfficients de cette équation⌝ ~~$\varphi(HS$~~ ⌜sont invariables par la substitution S.⌝

Originally '$\varphi(HS)$', incomplete, displayed after 'Donc on aura'. Later deleted and surrounding space used.

~~Dela même manière $\varphi(HS^2)$, $\varphi(HS^3)$~~ . . . ~~seront~~ ^~~racines de l'équation~~^ ~~invariables par~~ en r. ~~Deplus~~
On aura donc

$$\varphi(HS) = f(\alpha r)$$

α étant une ~~équation~~^des racines $p^{\text{ième}}$^ de l'unité.
Donc $\varphi(HS)$ sera connu au moyen de r et

Lower halves of words in last line of **f. 90 a** have disappeared with the tearing of the page. Line still legible, though not included in T1906/7.

90 b

~~mais il est aisé devoir que l'équation en z étant irréductible, les racines de cette equation seront toutes dans la periode~~

~~$\varphi(H) \quad \varphi(HS) \quad \varphi(HS^2) \quad \cdots \quad \varphi(HS^n)$~~

~~et comme ces racines sont en nombre de p, il faudra que l'on ait~~

~~$\varphi(HS^p) = \varphi(H)$~~

~~On trouvera~~
Ces deux équation

Should be 'équations', but singular clear in *ms*.

$$\varphi(H) = f(r) \qquad \varphi(HS) = f(\alpha r)$$

Donneront par l'élimination de r une relation entre

$$\varphi(H) \quad \text{~~et~~} \quad \varphi(HS) \quad \text{et} \quad \alpha$$

Therefore let $\varphi(H)$ be a certain function that is invariant under the substitutions of the group H and not under those of the group G. One will then have

$$\varphi(H) = f(r),$$

the function f containing in its expression nothing other than previously known quantities.

Eliminating r algebraically between the equations

$$r^p = A, \qquad f(r) = z,$$

one will get an irreducible equation of the pth degree in z. (Otherwise z would be a function of r^p, which is contrary to hypothesis.) Now ~~let one sees that the p roots of the equation in z cannot be other than $\varphi(H)$ φ~~ Let S be one of the ~~groups divisors of G similar to~~ substitutions of the group G which it does not have in common with H. One sees that $\varphi(HS)$ will again be a root of the above equation in z ~~Thus one will have~~ because the coefficients of this equation are invariant under the substitution S.

~~In the same way $\varphi(HS^2)$, $\varphi(HS^3)$ . . . will be roots of the equation invariant under in r. Moreover~~

One will therefore have

$$\varphi(HS) = f(\alpha r),$$

α being ~~an equation~~ one of the pth roots of unity.

Therefore $\varphi(HS)$ will be known in terms of r and

~~but it is easy to see that, the equation in z being irreducible, the roots of this equation will all be in the period~~

~~$$\varphi(H) \quad \varphi(HS) \quad \varphi(HS^2) \quad \cdots \quad \varphi(HS^n)$$~~

~~and since these roots are p in number, one will have to have~~

~~$$\varphi(HS^p) = \varphi(H)$$~~

~~One will find~~

These two equations

$$\varphi(H) = f(r), \qquad \varphi(HS) = f(\alpha r),$$

will, by elimination of r, give a relation between

$$\varphi(H) \quad \text{~~and~~} \quad \varphi(HS) \quad \text{and} \quad \alpha$$

indépendante de r, et la mème relation aura parconsequent lieu entre

$$\varphi(HS) \quad \text{et} \quad \varphi(HS^2 /\!/)$$

Formula $\varphi(HST)$ changed to $\varphi(HS^2)$.

~~T étant une nouvelle substitution du groupe G qui ne lui soit pas commune à H.~~
Donc: comme
on en deduit

$$\varphi(HS) = f(\alpha r) \qquad \varphi(HS^2) = f(\alpha^2 r)$$

et ainsi desuite, jusqu'à

$$\varphi(HS^p) = f(r) = \varphi(H)$$

Ainsi la connaissance de la seule quantité r, donne à la fois toutes les ~~quant~~ fonctions correspondantes aux groupes

$$H\,, \quad HS\,, \quad HS^2\,, \quad \cdots\,.$$

la somme de ces groupes est évidemment G, puisque toute

91 a

The piece originally began 'Je dis'; overwritten with 'Or, il est aisé de voir' starting well over in the margin.

Or, il est aisé de voir que le produit de deux échanges ^quelconques donnés^, U et V peut se de deux substitutions circulaires ^du $p^{\text{ième}}$ degré^. En effet prennez une su[bstitution] circulaire ^du $p^{\text{ième}}$ degré^ S telle que $S^{-1}VS = U$. On en déduit $UV = S^{-1}VSV$ ^et (àcause de $V = V^{-1}$)^ $= S^{-1}V^{-1}S$ V, ce qui veut dire que le produit des ~~substitu[tions]~~ ^echa[nges]^ U et V revient au produit des substitutions S^{-1} et $V_0 S$ ^toutes deux semblables à S.^ ~~La première est circulaire du $p^{\text{ième}}$ degré par hypothèse ^la seconde symmetrique^ est lui semblable (à cause de $V = V^{-1}$ ^et $VSV = V^{-1}SV$)^).~~ Donc le groupe com[prend] toutes substitution qui est le produit de deux échanges, et parconseq[uent] ⌜aussi⌝toute substitution qui revient à un nombre pair d'echanges.

Line 2: word 'déduire' missing.

Line 3: remaining letters at end of line look more like 'fa' or 'sa' than 'su', though the 's' would be quite unlike others in this passage. But 'substitutions' is the only word that makes sense.

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92 a

Si $F(x) = 0$ est une equation irreductible de degre premier m
Si de⌜s⌝ ⌜équation⌝ $F(x) = 0 \quad z = f(x)$, on elimine x,
1° l'équation finale ^en z^ aura pour degré m. ~~ou un diviseur de m.~~
2° l'équation finale ^en z^ sera irréductible.

This heading, elegantly written, appears upside-down at the bottom of the scrap of paper.

Substitutions semblables

independent of r, and the same relation will consequently hold between

$$\varphi(HS) \quad \text{and} \quad \varphi(HS^2 \cancel{T})$$

~~T being a new substitution of the group G which it does not have in common with H.~~
Therefore, since

$$\varphi(HS) = f(\alpha r) \quad \text{one deduces} \quad \varphi(HS^2) = f(\alpha^2 r)$$

and so on, up to

$$\varphi(HS^p) = f(r) = \varphi(H)\,.$$

Thus knowledge of the single quantity r gives at one and the same time all the functions corresponding to the groups

$$H\,, \quad HS\,, \quad HS^2\,, \quad \cdots\cdot.$$

The sum of these groups is clearly G because every

~~I say~~ Now it is easy to see that the product of two arbitrarily given exchanges U and V can be deduced from two circular substitutions of the pth degree. Indeed, take a circular substitution S of the pth degree such that $S^{-1}VS = U$. One deduces from this $UV = S^{-1}VSV$ and (because $V = V^{-1}$) $= S^{-1}V^{-1}SV$, which says that the product of the ~~substitutions~~ exchanges U and V comes to the same as the product of the substitutions S^{-1} and V_0S, both similar to S. ~~The first is circular of the pth degree by hypothesis, the second symmetric is similar to it (because $V = V^{-1}$ and $VSV = V^{-1}SV$)).~~ Therefore the group contains every substitution which is the product of two exchanges and consequently also every substitution which comes down to an even number of exchanges.

If $F(x) = 0$ is an irreducible equation of prime degree m
If x is eliminated from the equations $F(x) = 0$, $z = f(x)$,
(1) the final equation in z will have m ~~or a divisor of m~~ for its degree.
(2) the final equation in z will be irreducible.

Similar substitutions

93 a

Etant donnée une équation, avec tant de quantités adjointes que l'on voudra, on peut toujours trouver ~~une~~ quelque fonction des racines qui soit ^numériquement^ invariable par toutes les substitutions ~~de l'~~ d'un groupe donné, et ne le soit pas par d'autres substitutions.

~~Alors toute fonction semblable sera invariable par les substitutions d'un groupe semblable au précédent.~~

Si le groupe G ⌄des racines⌄ d'une équation se décompose en n groupes ^semblables^ $H,\ HS,\ HS,$ et qu'une fonction ⌄$\varphi(H)$⌄ soit invariable par toutes les substitutions d'~~un~~ groupe ⌄H⌄ ~~donné [???] groupe de léquation~~ ⌄et ne le soit⌄ ^par aucune autre substitution du groupe G^, cette fonction est racine d'une équation ~~du~~ irréductible du $n^{\text{ième}}$ degré dont les ^autres^ racines sont $\varphi(HS)$, ...

Galois

See Note 2, p. 318.

In BA1962 it is suggested that the illegible words could be 'diviseur du'.

Given an equation with as many adjoined quantities as one wishes, one can always find some function of the roots which is numerically invariant under all the substitutions of a given group, and is not so under any other substitutions.

~~Then every similar function will be invariant under the substitutions of a group similar to the preceding one.~~

If the group G of the roots of an equation decomposes into n similar groups H, HS, HS, and if a function $\varphi(H)$ is invariant under all the substitutions of a given group H [~~a divisor of~~?] ~~group of the equation~~ and is not so under any other substitution of the group G, this function is a root of an irreducible equation of the nth degree of which the other roots are $\varphi(HS)$, ...

Galois

Notes on Dossier 17

NOTE 1: Reading **f. 91 a** presents even greater difficulties than usual. Some words are lost from the ends of lines where a strip of the page has been torn away. These are supplied in [B & A (1962), p. 81], and I am confident that most of Bourgne's suggestions are correct. There remains, however, a problem with the formula $V_0 S$ (line 6 of the transcription). The subscript 0 is a mystery. But also, since it comes at the end of a line, it is just possible that it is incomplete and originally read something like $V_0 S V_0^{-1}$. Apart from the subscript 0 that would make excellent sense. Nevertheless, the manuscript does not look as if it contained V_0^{-1} before it was torn.

The formula before this one is rendered as S_0^{-1} in [B & A (1962), p. 83]. My reading of what is seen there as a subscript 0, however, is that this is the riser of a letter 'd' (which in Galois' hand forms a loop) from the word 'seconde' squeezed into the thin space between 'S_0 et $V_0 S$' on one line and 'hypothese' [*sic*] in the line below. I believe that the formula should be S^{-1}.

NOTE 2: The sequence H, HS, HS, [...] in **f. 93 a** occurs at the end of a line (and does not finish with the expected ellipsis). In [Tannery (1907), p. 292] and [B & A (1962), p. 85] it is suggested that the second occurrence of HS should be HS^2. That seems plausible; it seems possible, however, that it should be HS' as in the formula near the bottom of **f. 8 a** (p. 84) of the *Lettre testamentaire*.

NOTE 3: There are some jottings in the left margin of **f. 90 a**. They were written with the page turned 90° clockwise, so that the left edge serves as top. The top half of the paper (that is, the right half of the turned page) contains $x^{p^{\text{ième}}}$, followed by the name Galois written four times in a neat and careful, almost copperplate, hand. The bottom half has some very faint, almost ghostly, formulae that look like x^{p^n}, x^{p^r} and $x^{p^{\text{ième}}}$.

On **f. 90 b** there are two very faint scribbles over some of the writing. The name Galois appears neatly written as on **f. 90 a**, with the paper turned 90° anti-clockwise, in a gap down the right-had side. Also, just below the last line of writing, with the page turned through 180°, so that it appears upside-down, is

$$\phi \qquad x_0 \qquad x_1 \qquad x_2 \quad \cdots$$

and just below and to the left of it (still with the page turned upside-down) is a neatly drawn circle, a little larger than the letter G that appears near to it.

NOTE 4: Folio 91 carries many jottings. There are a number of haphazard symbols, formulae and scribbles, some legible, some less so. At the top of **f. 91 a**, above the text transcribed above, are the following legible (and potentially interesting) formulae:

$\varphi k \cdot n = \varphi(k+n)$ $\qquad$ $\varphi 0 = b$ $\qquad$ $\varphi(k+l) = \varphi[\star]$

$S^{-1}US = V$ $\qquad$ $\varphi a = b - a$

$UV = S.VS^{-1}V$

At $[\star]$ the remainder of the formula has disappeared with the tearing of the page. Moreover, my reading of all the formulae involving φ is insecure: for example, the first is read in [B & A (1962), p. 80] as $\varphi K - a = \varphi(K + a)$. I am pretty sure, though, that the letter k should be lower-case. In all the Galois manuscripts his k looks rather like K, and what we have here looks like the letter k that occurs frequently in the *Premier Mémoire*, which is certainly lower-case; moreover, it is easily distinguishable from K as it occurs in **f.95a** in Dossier 19 (p. 330). Note that the very clearly written formulae $S^{-1}US = V$ and $UV = S.VS^{-1}V$ resonate with part of the narrative that Galois wrote on this page. Below the text, amid the scribbles, are some formulae similar to these two:

$(TR)^{-1} = S^{-1}.TR.S[\star]$

$U = S^{-1}V^{-1}S \quad = \quad RS.ST = STRS$

$SUS^{-1}V$ $\qquad$ ~~$RT = S^{-1}$~~

and, very faintly written, (RT). Again, at $[\star]$ the formula may be incomplete since the page-tear goes through the right-most symbol S.

The verso, **f.91b**, may be read as being in three parts. Near the top and towards the right is

$m = nq$

n nombre de termes de la periode

The middle carries formulae related to the material on **f.91a**:

~~$S^{-1} = T^{-1}ST$~~

~~SUS~~ $\qquad$ $TS^{-1} = ST$ $\qquad$ ~~$S^{-1}T = TS$~~

~~$UTS = T^{-1}S^{-1}$~~ $\qquad$ ~~$T^2S^{-1}T = TST^2$~~

$S^{-1} = T^{-1}ST = TST^{-1}$

~~$T^{-1}S^{-1}T = T^{-1}$~~ $\qquad$ $ST^2 = T^2S$

~~$S = TS^{-1}T^{-1} = T^{-1}S^{-1}T$~~

Finally, in the lower one-third of the page there are haphazard, but probably related, formulae, such as: $rk = b \pmod{n}$ (or perhaps it is $2k = b \pmod{n}$), $\varphi(k+n)$, $\varphi(k+1, l) = \varphi(k, l) - 1$, $\quad \varphi k - n = \varphi(k+n)$,

$$\Big(k \mid l \ , \ -k + \varphi l \mid \psi l\Big),$$

$\varphi^2(k+n) = \varphi(\varphi k - n), \quad -k + \varphi l, \quad$ and $\quad -\varphi l + \varphi\psi l = 0.$

NOTE 5: On **f.92a** there are no interesting jottings, just some small geometrical figures (some lines, a cube, a pentagon, perhaps a set of axes) and some scribbles, but **f.92b** contains mathematical working. First, in a column a little to the left of the centre, with the page turned upside-down relative to the main content of **f.92a** (so that the heading 'Substitutions Semblables' would appear the right way up), one sees a calculation that seems to be related to the calculation on **f.6b** (see p. 130) which is at the heart of Propositions VII and VIII of the *Premier Mémoire*

$$\begin{aligned} \cancel{\varphi(k) + n'} &= \cancel{\varphi(k+n)} \\ \varphi(k) + m &= \varphi(k+n) \\ \varphi(k) + k'm &= \varphi(k+k'n) \\ \varphi(k'n) &= k'm + \varphi(0) \\ \varphi(k) &= ak + b \end{aligned}$$

$$?\varphi(k) + b = \varphi(?k+?)$$

The question marks in the last line indicate heavily deleted symbols—the first is almost certainly a, the second and third could perhaps be a and b respectively. To the right at the top of this array of equations are GS and AS, one above the other.

With the page in its present orientation (so turned through 90° anticlockwise relative to the material on **f.92a**), there are other formulae above and below that array. Above is

$$\begin{gathered} p \quad q \quad r \quad s \\ (r - qx)^2 - (q - px)(s - rx) \\ r - qs - (qr - ps) - \\ r^2 - qs + (ps - qr)x + (q^2 - pr)x^2 \\ 1^{\text{ier}} \text{ cas } q^2 - pr < 0 \\ 2^{\text{e}} \text{ cas } q^2 - pr > 0 \\ (q^2 - pr)x^2 + (ps - qr)x + r^2 - qs = \\ (ps - qr)^2 - 4(q^2 - pr)(r^2 - qs) \end{gathered}$$

and below:

$$\begin{gathered} p^2s^2 - 3q^2r^2 - \cancel{6pqrs} - \\ + 4\,(pr^3 + q^3s) \end{gathered}$$

It is just possible that the 7th line should be the equation

$$(q^2 - pr)x^2 + (ps - qr)x + r^2 - qs = 0$$

as in [B & A (1962), p. 513]; equally, however, that 0, which is faint, could well be a loop of writing showing through from the other side of the paper.

NOTE 6: There is very little of interest on **f.93a** other than the transcribed material. Below that is

~~À chaq~~ ~~Si l'on appliq~~ $f(x_1) = f(x_2)$

and nothing more. Likewise **f.93b** is of little interest: with the page turned upside-down one can make out $ak + b$, E. Galois (nicely written, as always), $(x - a)$, 2 facteurs (faintly written with a very fine-nibbed pen), $fa + fb + fc + \cdots$ (ditto), G (twice), GS, GST, A, B, C, and many scribbles.

VI.13 Dossier 18: Note on non-primitive equations

The sole content of Dossier 18, whose cover carries the description 'Note sur les équations non primitives', is Folio 94 (15 cm × 20.5 cm). Tannery published **f. 94 a** in 1907 (p. 293 of his paper); **f. 94 a** and **f. 94 b** appear in [B & A (1962), pp. 166–169].

There is a left margin just over 3 cm wide in **f. 94 a**, created by a fold in the paper, and the material here is carefully written, with few corrections. It correlates well, but far from perfectly, with part of the published abstract [Galois (1830a)] (see § II.2, p. 49 above). Bourgne [B & A (1962), p. 513] saw it as a first draft of that note and dated it to just after the paper submitted to the Academy in February 1830.

Almost certainly **f. 94 b** has something, but perhaps only a little, to do with **f. 94 a**. The latter is complete in itself, whereas the verso has been used for calculations and rough scribbles. The ink there varies. The first one-third of the page was written with light ink similar to that of **f. 37 a** (the first page of the Second Memoir). It was then headed simply 'Deux cas' [two cases]. Later Galois came back with thick black ink, added 'à Considérer', various formulae, the name in the top left corner, Fx and Fp in the top right corner, and much crossing-out. About half the page contains large writing and vigorous crossing-out in thick black ink, using diagonal lines and scribbles. Some of that has soaked through to **f. 94 a**, partially obscuring some of the writing there. My readings of some of the symbols on **f. 94 b** are doubtful, especially individually crossed-out symbols.

Here again there are no paragraph indents.

This one-word title appears in the top left corner, so is, in a sense, marginal.

94a

Note.

T1906/7 has 'appèle' as in *ms*; corrected to 'appelle' in BA1962. Hyphen in 'non-primitives' is clear in *ms*. 'M^{r}' changed to 'M.' in BA1962.

On appèle équations non-primitives les équations qui, étant, par éxemple du degré mn se décomposent en m facteurs du degré n au moyen d'une seule équation du degré m. Ce sont les Équations de Mr Gauss. Les Équations primitives sont celles qui ne jouissent pas d'une pareille simplification. Je suis, à l'égard des Équations primitives, parvenu aux résultats suivants:

1°. Pour qu'une équation primitive ^de degré m^ soit résoluble par radicaux, il faut que $m = p^{\nu}$, p étant un nombre premier.

2°. Si l'on excepte le cas de $m = 9$ et $m = p^{p}$, l'équation devra être telle que deux quelconques de ces racines étant connues, les autres s'en déduisent rationnellement.

3°. Dans le cas de $m = p^{p}$, deux des racines étant connues, les autres doivent s'en déduire ^du moins^ par un~~e~~ seul~~e~~ ~~équation~~ ^radical^ du degré p.

Word 'se' corrected to 'ètre' by over-writing

4°. Enfin, dans le cas de $m = 9$, l'équation devra s~~e~~être du genre de celles qui déterminent la trisection des fonctions Elliptiques.

La démonstration de ces propositions est fondée sur la théorie des permutations.

Note.

Equations which, being for example of degree mn may be decomposed into m factors of degree n by means of a single equation of degree n, are called non-primitive. These are the equations of Mr Gauss. Primitive equations are those that do not enjoy such simplification. With regard to primitive equations I have been led to the following results:

1° In order that a primitive equation of degree m should be soluble by radicals it is necessary that $m = p^\nu$, p being a prime number.

2° if one excepts the case[s] $m = 9$ and $m = p^p$, the equation must be such that any two of its roots being known, the others may be deduced rationally from them.

3° In the case where $m = p^p$, two of the roots being known, the others must be deducible at least through a single ~~equation~~ radical of degree p.

4° Finally, in the case of $m = 9$ the equation must be of the kind of those that determine the trisection of elliptic functions.

The proof of these propositions is founded on the theory of permutations.

94b

Galois $Fx \quad Fp$

Deux cas à Considérer

Much scribbling on left of page, as if testing pen.

---

a est premier $\quad a^\alpha$ divise $\dfrac{p^\nu - 1}{p-1}$ $\quad$ ~~$\left(\text{et est premier avec } \dfrac{p^\nu - 1}{a^\alpha(p-1)}\right)$~~ ~~$p-1$~~

$$\frac{p^\nu - 1}{a^\alpha(p-1)}\ \nu \equiv p \pmod{a^\alpha} \qquad a^\alpha \equiv 1 \pmod{p}$$

$\dfrac{1}{a^\alpha}$ $\quad \dfrac{p+1}{a^\alpha}$ $\quad$ ~~Du Reste~~ $\quad \dfrac{p^p - 1}{a^\alpha(p-1)} \equiv 1 \pmod{a^\alpha}$

Reading of 'Du Reste' is insecure.

$=$ ~~a est premier et divise ν et $p-1$~~

$a = \nu$, est premier et divise $p-1$

$$\frac{p^a - 1}{a^{\alpha}(p-1)}\ \nu = p \quad (\text{mod. } a^{\alpha}) \qquad a^{\alpha} \equiv 1 \pmod{p}$$

~~$\dfrac{p^p - 1}{a^{\alpha-1}(p-1)}\ \nu \equiv 1 \pmod{a^{\alpha-1}}$~~ $\quad \dfrac{p^a - 1}{a^\alpha(p-1)} = 1 \pmod{a}$

A little material missing from BA1962. Some symbol exponents read differently in BA1962.

~~$\dfrac{p^a - 1}{a(p-1)} = 1$~~ $\pmod{a}$

~~$\alpha p \equiv 1 \left(\text{mod } a^{\alpha-1}\right)$~~ $\qquad p = 1 \pmod{a}$

$$\frac{p^a - 1}{p(p-1)} = a^\alpha \qquad \frac{1}{0} \pmod{a} \qquad \frac{p^a - 1}{a^\alpha(p-1)} = 1 \pmod{a}$$

$$\frac{p^a - 1}{p-1} = 0 \pmod{a^\alpha} \qquad \frac{1}{a^\alpha} = 1$$

Some scribbling around formulae near bottom of page, as if testing pen

Galois $Fx \quad Fp$

Two cases to be considered

---

a is prime $\quad a^\alpha$ divides $\dfrac{p^\nu - 1}{p-1}$ $\quad$ ~~$\left(\text{and is [co]prime with } \dfrac{p^\nu-1}{a^\alpha(p-1)}\right)$~~

$$\frac{p^\nu - 1}{a^\alpha(p-1)}\ \nu \equiv p \pmod{a^\alpha} \qquad a^\alpha \equiv 1 \pmod{p}$$

$$\frac{1}{a^\alpha} \qquad \frac{p+1}{a^\alpha} \qquad \frac{p^p - 1}{a^\alpha(p-1)} \equiv 1 \pmod{a^\alpha}$$

$=$ ~~a is prime and divides ν and $p-1$~~

$a = \nu$, is prime and divides $p-1$

$$\frac{p^a - 1}{a^{\alpha}(p-1)}\ \nu = p\ a \pmod{a^{\alpha}} \qquad a^{\alpha} \equiv 1 \pmod{p}$$

$$\frac{p^a - 1}{a^\alpha(p-1)} = 1 \pmod{a}$$

~~$\dfrac{p^a - 1}{a(p-1)} = 1$~~ $\pmod{a}$

~~$ap = 1 \left(\text{mod } a^{\alpha-1}\right)$~~ $\qquad p = 1 \pmod{a}$

$$\frac{p^a - 1}{p(p-1)} = a^\alpha \quad \frac{1}{0} \pmod{a} \qquad \frac{p^a - 1}{a^\alpha(p-1)} = 1 \pmod{a}$$

$$\frac{p^a - 1}{p-1} = 0 \pmod{a^\alpha} \qquad \frac{1}{a^\alpha} = 1$$

VI.14 Dossier 19: Addition to the memoir on solution of equations

Dossier 19 comprises folios 95–97, of the manuscripts. It is presented as item N in [Tannery (1907), pp. 294–295] and in [B & A (1962), pp. 148–151, 513–516]. The cover of the dossier carries the title 'Addition au mémoire sur la résolution des équations — Enoncé arithmétique'. The first part of this is the title given by Galois to folio 95; the second part is a fair description of the relevant material in the remainder. In [B & A (1962), pp. 513–514] it is dated to some time in 1830.

Folios 96 and 97 are the two parts of a folded sheet 34.5 cm × 22.5 cm, Folio 95 is a single sheet, not quite rectangular, approximately 16 cm × 18.5 cm written on both sides. It is good strong paper so that, unlike much other Galois material, nothing shows through from the other side. When it was written the paper was a little deeper, perhaps 19 cm or 20 cm, but a strip has been torn or cut with a knife from the bottom, and as a result there is a line of writing, only small vestiges of which remain. Although it is impossible to work out what it said, Bourgne restored it as 'LS sera conjugué au groupe L', which in one respect seems a fairly reasonable conjecture: the line appears crossed out at the top of **f. 95 b** and this would not be the only place where Galois removed something from the top of one page and inserted it at the bottom of the previous page—for example, in the *Second Mémoire* the phrase 'un nombre premier' was deleted from the top of **f. 38 b** and slipped in at the bottom of **f. 38 a** (see p. 176). On the other hand the proposed clause cannot possibly follow logically after the sentence preceding the cut, and I am inclined to think that perhaps we have lost more than one line of manuscript there.

There is clearly nothing missing from **f. 95 b**: therefore this side was written after the strip was cut from the bottom of the paper. The last few lines, from 'mème dans le $p^{p^{\text{ième}}}$ degré' were written in the left margin with the page turned 90° anti-clockwise, so that the left edge became the bottom. The very last line 'Il faut ... permutations' was squeezed in as an afterthought, as is shown both by its mean spacing and by its placement below the horizontal line which Galois used (here as often elsewhere) as a cadence or closure.

This is another item without paragraph indents.

95 a

Addition au mémoire sur la résolution des Equations.

Formulae $mt.N$ and N changed to $mt.n$ and n by over-writing.

Lemme I. Soit un groupe G de $mt.n$ permutations, qui se décompose en n groupes semblables à H. Supposons que le groupe H se décompose en t groupes de ^m^ permutations, et semblables à K.

Si, parmi toutes les substitutions du groupe G, celles du groupe H sont les seules qui puissent transformer l'une dans l'autre quelques substitutions du groupe K, on aura $n \equiv 1 \pmod{m}$ ou $tn \equiv t \pmod{m}$.

Lemme II. Si μ est un nombre premier, et p un entier quelconque on aura

See Note 1.

$$(x-p)(x-p^2)(x-p^3)\cdots(x-p^{\mu-1}) \equiv \frac{x^\mu - 1}{x-1} \quad \left(\text{mod. } \frac{p^\mu - 1}{p-1}\right).$$

Ces deux lemmes permettent de voir dans quel cas un groupe primitif de degré p^ν (où p est premier) peut appartenir à une équation résoluble par radicaux.

En effet, appelons G un groupe qui contient toutes les substitutions linéaires possibles par ^les^ $\dfrac{p^\nu - 1}{p-1}$ lettres. ^(Voyez le mémoire cité.)^

Formula over-written on something thoroughly crossed out, perhaps 'les p^ν'.

Soit, ~~Les~~ s'il est possible, L un groupe qui divise G et qui se partage~~nt~~ ^lui-même^ en p groupes semblables à K, K ne comprenant pas deux permutations où ~~deux~~^une^ lettre~~s~~ occupe~~nt~~ le~~s~~ même~~s~~ place~~s~~.

95 b

~~LS sera conjugué au groupe K L.~~

On peut prouver ^1°^ que s'il y a dans le groupe G et hors du groupe L, quelque substitution S qui transforme l'une dans l'autre quelques substitutions du groupe K, ~~le groupe LS sera conjugué à L, 2° Que si l'on prend le groupe H qui contient toutes les substitutions du groupe K et toutes celles de la forme S, H contiendra r groupes semblables à K~~ ^cette substitution sera de r termes^, r étant un diviseur de $p-1$.

D'après celà, comme le nombre des permutations du groupe G est

$$\frac{p^\nu - 1}{p-1} \cdot (p^\nu - p^{\nu-1})(p^\nu - p^{\nu-2}) \cdot \cdots \cdot (p^\nu - p^2)(p^\nu - p)$$

D'après le lemme 1, on devra avoir

In $p^k r$ on right of congruence k was an afterthought.

$$(p^\nu - p^{\nu-1})(p^\nu - p^{\nu-2}) \cdot \cdots \cdot (p^\nu - p^2)(p^\nu - p) \equiv p^k r \ \left(\text{mod } \frac{p^\nu - 1}{p-1}\right).$$

Addition to the memoir on the solution of equations

LEMMA I. *Let G be a group of $mt.n$ permutations which may be decomposed into n groups similar to H. We suppose that the group H may be decomposed into t groups of m permutations similar to K.*

If, amongst all the substitutions of the group G those of the group H are the only ones which can transform amongst each other some substitutions of the group K, one will have $n \equiv 1 \pmod{m}$ *or* $tn \equiv t \pmod{m}$.

LEMMA II. *If μ is a prime number and p any whole number one will have*

$$(x-p)(x-p^2)(x-p^3)\cdots(x-p^{\mu-1}) \equiv \frac{x^\mu - 1}{x-1} \quad \left(\text{mod. } \frac{p^\mu - 1}{p-1}\right).$$

These two lemmas permit one to identify the circumstances in which a primitive group of degree p^ν (where p is prime) can belong to an equation that is soluble by radicals.

Indeed, denote by G a group which contains all the possible linear substitutions among the $\dfrac{p^\nu - 1}{p-1}$ letters. (See the cited memoir.)

If possible, let L be a group which divides G and which is itself partitioned into p groups similar to K, where K does not contain two permutations where ~~two~~ one letter~~s~~ occup~~y~~ies the same place~~s~~.

~~*LS* will be conjugate to the group *L*.~~

One can prove 1st that if there exists in the group G and outside the group L some substitution S which transforms amongst themselves some substitutions of the group K, ~~the group LS will be conjugate to L, and~~ **2nd** ~~That if one takes the group H which contains all the substitutions of the group L and all those of the form S, H will contain r groups similar to K~~ this substitution will be of r terms, r being a divisor of $p-1$.

Accordingly, since the number of permutations of the group G is

$$\frac{p^\nu - 1}{p-1}.(p^\nu - p^{\nu-1})(p^\nu - p^{\nu-2})\cdots(p^\nu - p^2)(p^\nu - p),$$

by Lemma 1, one will have to have

$$(p^\nu - p^{\nu-1})(p^\nu - p^{\nu-2})\cdots(p^\nu - p^2)(p^\nu - p) \equiv p^k r \ \left(\text{mod } \frac{p^\nu - 1}{p-1}\right).$$

Letters n changed to ν, p to μ by overwriting.

Three occurrences of μ changed to ν by overwriting.

D'òu (~~Lemme II~~) L'on voit que ν doit être un nombre premier ~~p~~. ~~Ce qui donne~~ ^~~On en déduit~~^ (Lemme II)

$$pr \equiv \mu \left(\mathrm{mod}\ \frac{p^{\mu}-1}{p-1} \right).$$

~~Ce qui excepté dans le 9e degré, donne~~ ^On en déduit quand $\nu > 2$^ $pr = \mu$, savoir $p = \mu$,puisque p et μ sont premiers.

BA1962, following T1906/7, has 'énoncé', but 'avancé' is clear in *ms*.

Ainsi, le théorème que j'avais avancé dans mon mémoire sera vrai dans tout autre cas que dans celui ~~de~~ où p serait élevé à la puissance p.

Toujours devra-t-on avoir $r = 1$, et $L = H$. Ainsi mème dans le $p^{p^{\text{ième}}}$ degré le groupe de l'équation ~~de q~~ réduite de degré $\dfrac{p^p-1}{p-1}$ devra ètre de $\dfrac{p^p-1}{p-1}\ p$ permutations. La regle est donc encore fort simple dans ce cas.

Il faut ~~dans~~ ^comme on voit^ 1° que $\nu = 1$; 2° que le groupe de la réduite soit de $\dfrac{p^p-1}{p-1}\ p$ permutations.

[*Folios* 96 *and* 97 *are quite different from Folio* 95. *They are two parts of a single folded sheet; the paper is thinner, the pen and the writing are different. There is one sentence of interest* (*see below*). *The rest consists of incoherent jotted formulae and calculations written with a variety of pens and related to a number of different topics. Details are recorded in* [B & A (1962), *pp.* 514–516] *and are not repeated here.*]

96 a

Le produit

$$(p^{\nu}-p)(p^{\nu}-p^2)(p^{\nu}-p^3)\cdot\cdots\cdot(p^{\nu}-p^{\nu-1})$$

n'admet point de facteur premier $> \dfrac{p^{\nu}-1}{\partial(p-1)}$, ∂ étant le plus grand commun diviseur entre ν et $p-1$, àmoins que $\nu = 2$.

From which (~~Lemma II~~) one sees that ν must be a prime number (Lemma II). ~~Which gives~~ ~~It may be deduced from this that~~

$$pr \equiv \nu \left(\mathrm{mod}\ \frac{p^\nu - 1}{p-1} \right).$$

~~Which, except in the case of the 9th degree, gives~~ One deduces from this that when $\nu > 2$, $pr = \mu$, that is $p = \nu$ because p and μ are prime.

Thus the theorem that I have stated in my memoir will be true in every case other than that where p is raised to the power p.

One will always have to have $r = 1$ and $L = H$. Thus, even in the $p^{p^{\text{th}}}$ degree the group of the reduced equation of degree $\frac{p^p - 1}{p-1}$ will have to be of $\frac{p^p - 1}{p-1}\, p$ permutations. The rule is therefore again quite simple in this case.

As one sees, it is necessary 1st that $\nu = 1$; 2nd that the group of the reduced equation be of $\frac{p^p - 1}{p-1}\, p$ permutations.

The product

$$(p^\nu - p)(p^\nu - p^2)(p^\nu - p^3) \cdot \cdots \cdot (p^\nu - p^{\nu-1})$$

admits no prime factor $> \frac{p^\nu - 1}{\partial(p-1)}$, ∂ being the greatest common divisor of ν and $p - 1$, unless $\nu = 2$.

Notes on Dossier 19

NOTE 1: Most of the substantive number-theoretic formulae in this note have been amended. The formula in Lemme II, for example, started out as

$$(1-p)(1-p^2)(1-p^3)\cdots(1-p^{\mu-1}) \equiv \mu \quad \left(\text{mod. } \frac{p^\mu - 1}{p-1}\right)$$

and was changed by overwriting: the entries 1 on the left were overwritten with x; and μ to the right of the symbol $\equiv$ was overwritten with $\dfrac{x^\mu - 1}{x-1}$. Indeed, it looks as if the last factor on the left may originally have been $(1-p^n)$, though, if so, that was almost certainly a slip of the pen.

Similarly, in the first two of the displayed formulae on **f.95b** every instance of ν has been overwritten onto n (I think).

NOTE 2: Attention should be drawn to some resonance of the number theory here with the deleted material on **f.37a**, before the start of the *Second Mémoire* (see p. 170), and also with the exceptional cases listed in [Galois (1830a), pp. 271, 272] (p. 50 above) and in **f.94a**, **f.94b** in Dossier 18.

VI.15 Dossier 20: On the division of elliptic functions

Dossier 20 comprises folios 98–100, of the manuscripts. Before pagination Folios 99 and 100 were put into the dossier the wrong way round and therefore the text follows the sequence **f.98a**, **f.98b**, **f.99b**, **f.99a**, **f.100b**, **f.100a**. It is presented as item O in [Tannery (1907), pp. 296–300] and in [B & A (1962), pp. 153–161, 516–517]. Galois gave his essay no title but the cover of the dossier carries the description 'Sur la division des fonctions elliptiques'; in [B & A (1962)] it has the more elaborate title (or description) 'Mémoire sur la division d'une fonction elliptique de première classe'. Bourgne [B & A (1962), p. 517] conjectures that it dates from early in 1831, and certainly before May 1831.

The three pages are small, 15 cm × 20.5 cm, and neatly written on both sides in a small, pleasant and legible hand, with a left margin of 3 cm guided by a fold in the paper. This looks like a carefully prepared fair copy. Some of the preparation is more than careful, it is quite sophisticated. For example, although the paragraphing itself is somewhat different from what one might expect nowadays, the introduction and the first section begin without indentation and thereafter paragraph indentation is normal. This sophistication disappears in the second section, however, as does paragraph indentation except in a passage towards the middle of the section.

The preparation is so careful that it is natural to ask why this did not become one of the published items by Galois. Perhaps, however, Galois felt a need to get his *Premier Mémoire*, upon which this essay depends, into print first. That would be quite natural, and could explain why such an item as the article 'Notes sur quelques points d'Analyse', which seems to me (with the benefits of hindsight) to be considerably less valuable, could be published, while this one was not.

98 a

Dans un mémoire sur la théorie des Équations, j'ai fait voir comment on peut résoudre une équation Algébrique de degré premier, m, dont les racines sont $x_0\,,\ x_1\,,\ x_2\,,\ \ldots\,,\ x_{m-1}$, quand on suppose connue la valeur d'une fonction des racines qui ^ne^ demeure invariable que par des substitutions de la forme $(x_k\,, x_{ak+b})$. Or, il arrive, par un hasard que nous n'avions pas prévu, que la Méthode proposée dans ce mémoire, s'applique avec succès à la division d'une fonction elliptique de première classe en un nombre premier de parties égales. Nous pourrions, à la rigueur, nous contenter de donner cette division, et le problème de la section des fonctions de première classe pourrait être considéré comme résolu.

Semicolon after list of roots deleted or changed to comma.

Ms could perhaps be read 'les substitutions' as in T1906/7, BA1962, but 'des substitutions', as near end of third paragraph, looks more plausible.

Mais, afin de rendre cette solution plus générale, nous nous proposerons de diviser une fonction elliptique de première classe en m parties égales, m étant $= p^n$, et p premier.

Pour celà nous étendons ^d'abord^ la méthode exposée dans le mémoire cité, au cas où le degré de l'équation serait une puissance de nombre premier. Nous supposerons toujours que les racines soient $x_0\,,\ x_1\,,\ x_2\,,\ \ldots.\ x_{m-1}\,,$ et que l'on connaisse la valeur d'une fonction de ces racines qui ne demeure invariable que pour des substitutions de la forme $(k, ak + b)$.

Dans cette expression, k et $ak + b$ signifieront les restes minima de ces quantités par rapport à m. Parmi les substitutions de cette forme, que, pour abréger, nous appelerons substitutions linéaires, il est clair que l'on ne peut admettre que celles où a est premier avec m, sans quoi une même $ak + b$ remplacerait à la fois plusieurs k.

98 b

Celà posé, passons à la ^resolution de la^classe d'équations indiquée.

§ 1. Résolution de l'équation algébrique de degré p^n en y supposant connue la valeur d'une fonction qui n'est invariable que par des substitutions linéaires.

La congruence $k = ak + b$ n'étant pas soluble pour plus d'une seule valeur, on voit clairement que la fonction qu'on suppose connue n'est invariable par aucune substitution dans laquelle deux lettres garderaient un mème rang.

Si donc, mutatis mutandis, on applique à ce cas les raisonnements employés dans le mémoire cité, on vérifiera l'énoncé de la proposition qui suit:

"Etant supposée connue la valeur de la fonction en question, deu une racine s'exprimera toujours au moyen de deux d autres, et l'égalité qu'on obtiendra ainsi sera invariable par les substitutions telles que $(k, ak + b)$."

Second formula displayed in T1906/7, BA1962, but incorporated in text in *ms*.

Soit donc $x_2 = f(x_1, x_0)$, on en déduira engéneral, $x_{2a+b} = f(x_{a+b}, x_b)$, équation qui, appliquée de toutes manières, donnera l'expression d'une quelconque des racines de deux autres quelconques, si l'on a soin d'y substituer successivement, les valeurs expressions des racines qui entrent dans cette équation.

In a memoir on the theory of equations I have shown how one can solve an algebraic equation of prime degree m, the roots of which are x_0 , x_1 , x_2 , ... , x_{m-1} , when one supposes known the value of a function of the roots that does not remain invariant except under some substitutions of the form (x_k, x_{ak+b}). Now it happens, by a chance which we did not foresee, that the method proposed in this memoir may be applied with success to the division of an elliptic function of the first class into a prime number of equal parts. We could confine ourselves strictly to giving this division and the problem of the division of functions of the first class could be considered to be solved.

But in order to render this solution more general we propose to divide an elliptic function of the first class into m equal parts, m being $= p^n$ and p prime.

To do that we first extend the method expounded in the cited memoir to the case in which the degree of the equation is a power of a prime number. We always suppose that the roots are x_0 , x_1 , x_2 , ... x_{m-1} , and that one knows the value of a function of these roots that does not remain invariant except under some substitutions of the form $(k, ak + b)$.

In this expression, k and $ak + b$ will denote the smallest remainders of these quantities with respect to m. Among the substitutions of this form, which, to abbreviate, we shall call linear substitutions, it is clear that one cannot admit any other than those where a is co-prime with m, otherwise one and the same $ak + b$ would replace k at the same time.

That said, let us move on to the solution of the indicated class of equations.

§ 1. Solution of the algebraic equation of degree p^n supposing known the value of a function which is not invariant under other than linear substitutions.

The congruence $k = ak + b$ being soluble for no more than a single value, it is clear that the function which is supposed known is not invariant under any substitution in which two letters will retain the same place.

If then, mutatis mutandis, the reasoning employed in the cited memoir is applied to this case, the statement of the following proposition will be verified:

Supposing the value of the function in question known, a root will always be expressible in terms of two others, and the equation thus obtained will be invariant under substitutions such as $(k, ak + b)$.

Suppose then that $x_2 = f(x_1, x_0)$. From this it is deducible that in general, $x_{2a+b} = f(x_{a+b}, x_b)$, an equation which, applied in all ways, will give the expression of any one of the roots [in terms] of any other two, if one takes care to substitute successively into it the ~~values~~ expressions of the roots that enter into this equation.

Celà posé prennons une fonction symmétrique ^Φ^ des racines x_0, x_p, x_{2p}, x_{3p}, ... $x_{(p^n-1)p}$; ~~et~~ soient

Tannery misreads '~~et~~ soient' as 'il vient', but *ms* is clear.

$$\begin{aligned}
\Phi(x_0\,,\ x_p\,,\ \ x_{2p}\,,\ \ldots.\ ) &= \Phi_0\\
\Phi(x_1\,,\ x_{p+1}\,, x_{2p+1}\,, \ldots) &= \Phi_1\\
\Phi(x_2\,,\ x_{p+2}\,, x_{2p+2}\,, \ldots) &= \Phi_2\\
&\ldots\\
\Phi(x_{p-1}\,,\ x_{2p-1}\,, \ldots\ldots) &= \Phi_{p-1}
\end{aligned}$$

et supposons qu'en général $\Phi_{k+p} = \Phi_k$. Toute fonction des quantités Φ, qui sera invariable par ~~une~~^les^ substitu-

99 b

tions linéaires de ces quantités, sera évidemment une fonction invariable par les substitutions linéaires de ~~ces quantités~~ x_0, x_1, x_2, ..., x_{m-1} . Ainsi l'on connaitra à priori toute fonction des quantités, Φ_0, Φ_1, ..., Φ_{p-1}, invariable ~~le~~par les substitutions linéaires de ces quantités. On pourra donc 1° former l'équation dont ces quantités sont racines (puisque toute fonction symmétrique est à plus forte raison invariable par les substitutions); 2° résoudre cette équation.

Il suit de là, qu'on pourra ~~d~~ toujours, au moyen d'une équation de degré p algébriquement soluble, diviser l'equation proposée en facteurs dont les racines seront respectivement

$$\begin{array}{cccccc}
x_0 & x_p & x_{2p} & x_{3p} & \ldots.\\
x_1 & x_{p+1} & x_{2p+1} & x_{3p+1} & \ldots\\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{array}$$

Comme dans chaque facteur on aura l'expression d'une racine au moyen de deux autres, par éxemple, dans le premier, $f(x_p, x_0) = x_{2p}$ et que cette expression sera invariable par toute substitution linéaire, on voit que chaque facteur pourra se traiter comme l'équation donnée, et que le problème, s'abaissant successivement, sera enfin résolu.

Mis-spelling of 'exemple' unusually clear in *ms*; corrected in T1906/7, BA1962. Formula originally $f(x_p, x_0) = x_1$.

~~Nous~~^On^ p~~ouvons~~ ~~donc~~ ^en conséquence^ regarder comme solubles les équations dans les quelles on connaitrait la valeur d'une fonction des racines qui ne serait invariable que par des substitutions linéaires, quand le degré de l'équation est une puissance de nombre premier.

'Nous pouvons donc' changed to 'On peut en conséquence' partly by overwriting.

Nous pouvons donc passer à la solution du problème général de la section des transcendantes de première classe, ~~puissan~~ puisque, toute fraction étant la somme de fractions dont les dénominateurs sont des puissances de nombres premiers, il suffit d'apprendre à diviser ces transcendantes en p^n parties égales.

That said, take a symmetric function Φ of the roots x_0, x_p, x_{2p}, x_{3p}, ... $x_{(p^n-1)p}$; ~~and~~ let

$$\begin{aligned}
&\Phi(x_0\,,\ x_p\,,\ \ x_{2p}\,,\ \ \ldots.\ ) = \Phi_0\\
&\Phi(x_1\,,\ x_{p+1}\,, x_{2p+1}\,, \ldots) = \Phi_1\\
&\Phi(x_2\,,\ x_{p+2}\,, x_{2p+2}\,, \ldots) = \Phi_2\\
&\qquad\quad .\quad .\quad .\quad .\\
&\Phi(x_{p-1}\,,\ x_{2p-1}\,, \ldots\ldots) = \Phi_{p-1}\,,
\end{aligned}$$

and suppose that in general $\Phi_{k+p} = \Phi_k$. Every function of the quantities Φ that is invariant under ~~one~~ the

linear substitutions of these quantities, will clearly be a function invariant under the linear substitutions of ~~these quantities~~ x_0, x_1, x_2, ..., x_{m-1} . Thus one will know *a priori* every function of the quantities Φ_0, Φ_1, ..., Φ_{p-1}, invariant under the linear substitutions of these quantities. One therefore will be able 1st to form equation of which these quantities are the roots (because every symmetric function is *a fortiori* invariant under the substitutions); 2nd to solve this equation.

It follows that, by means of an algebraically soluble equation of degree p, one will always be able to divide the proposed equation into factors, the roots of which will be, respectively,

$$\begin{aligned}
&x_0\,,\quad x_p\,,\quad x_{2p}\,,\quad x_{3p}\,,\quad \ldots..\\
&x_1\,,\quad x_{p+1}\,,\quad x_{2p+1}\,,\quad x_{3p+1}\,,\ \ldots\\
&\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .
\end{aligned}$$

Since in each factor one will have the expression for one root by means of two others, for example, in the first, $f(x_p, x_0) = x_{2p}$, and since this expression will be invariant under every linear substitution, one sees that each factor can be treated in the same way as the given equation, and the problem, being successively reduced, will in the end be solved.

~~We can therefore~~ Consequently one may consider soluble those equations in which the value of a function of the roots that is not invariant except under some linear substitutions is known, when the degree of the equation is a power of a prime number.

We can therefore pass to the solution of the general problem of division of transcendents of the first class because, every fraction being the sum of fractions of which the denominators are powers of prime numbers, it is sufficient to understand how to divide these transcendents into p^n equal parts.

99 a

§ 2. Division des transcendantes de première espèce en $m = p^n$ parties égales.

Nous déterminerons chaque transcendante par le sinus de son amplitude. On pourrait de la même manière prendre le cosinus ou la tangente, et il n'y aurait rien à changer à ce que nous allons dire.

I would guess that Galois meant to cross out also 'transcendante' before 'dont'.

Nous désignerons par (x, y) le sinus de la transcendante ~~dont~~ somme des transcendantes dont les sinus sont x et y. Si x est le sinus d'une transcendante, $(x)^k$ désignera celui d'une transcendante k fois plus grande.

Il est clair que $(x, -y)$ sera le sinus de~~s~~ la différence des transcendantes qui ont pour sinus, d'après la notation indiquée pour les sommes.

Phrase 'x et y' should be supplied after 'pour sinus'.

Celà posé, nous commencerons par une remarque sur la nature des quantités qui satisfont à l'équation $(x)^m = 0$. Si l'on désigne par p l'une de ses racines, il est clair que $(p)^k$ ^en^ sera une autre. L'on aura donc une suite de racines exprimée par p, $(p^{\text{~~2~~}})^2$, $(p)^3, \ldots, (p)^{m-1}$. ⌜.⌝ ~~qui expri~~ Le nombre des racines étant $> m$, soit q une des racines qui ne sont pas comprises dans cette suite, $(q)^l$ sera une autre racine différente de q et des premières. Car, si l'on avait $(p)^k = (q)^l$ on en déduirait $q = (p)^g$, g étant un nombre entier.

Prennant donc les deux suites $p, (p)^2, \ldots$ et $q, (q)^2, \ldots$ on trouvera pour la formule générale des racines de l'équation $(x)^m = 0$, cette expression

$$\Big((p)^k, (q)^l\Big)$$

Celà posé, supposons que l'on donne à résoudre l'équation $(x)^m =$ ⌜sin⌝A, m étant impair et toujours de la forme p^n. Si x est une des racines, il est clair que toutes ⌜les⌝ autres seront

$$\Big(x, (p)^k, (q)^l\Big)$$

100 b

Posons donc en général

$$\Big(x, (p)^k, (q)^l\Big) = x_{k.l}$$

Both T1906/7, BA1962 have x_{00}, but $x_{0.0}$ in *ms*.

en faisant $x = x_{0.0}$, nous en déduirons généralement

$$\Big(x_{2a+b.2c+d}, -x_{a+b.c+d}\Big) = \Big(x_{a+b.c+d}, -x_{b.d}\Big)$$

d'où

$$x_{2a+b.2c+d} = \Big((x_{a+b.c+d})^2, -x_{b.d}\Big)$$

Or il est aisé de tirer de cette égalité une expression rationnelle de $x_{2a+b.2c+d}$ en fonction de $x_{a+b.c+d}$ et de $x_{b.d}$. Car si φ est ~~l'arc tr~~ l'arc correspondant à l'un

§ 2. Division of transcendents of the first kind into $m = p^n$ equal parts.

We will determine each transcendent by the sine of its amplitude. One could as well take the cosine or the tangent, and there would be nothing to be changed in what we are about to say.

We denote by (x, y) the sine of the ~~transcendent of which~~ sum of the transcendents of which the sines are x and y. If x is the sine of a transcendent, $(x)^k$ will denote that of a transcendent k times larger.

It is clear that $(x, -y)$ will be the sine of the difference of the transcendent which have sine [x and y] as in the indicated notation for sums.

That said, we will begin with a remark on the nature of the quantities which satisfy the equation $(x)^m = 0$. If one denotes by p one of its roots, it is clear that $(p)^k$ will be another of them. One will therefore have a sequence of roots expressed as p, $(p^{\cancel{2}})^2$, $(p)^3$, $\ldots$, $(p)^{m-1}$ ~~which expre[sses]~~. The number of the roots being $> m$, letting q be one of the roots which are not included in this sequence, $(q)^l$ will be another root, different from q and from the former. For, if one had $(p)^k = (q)^l$ one would deduce that $q = (p)^g$, g being a whole number.

Beware: p now has two different meanings.

Therefore taking the two sequences $p, (p)^2, \ldots$ and $q, (q)^2, \ldots$ one will find as the general formula for the roots of the equation $(x)^m = 0$, this expression:

$$\Big((p)^k, (q)^l\Big).$$

That said, let us suppose that one proposes to solve the equation $(x)^m = \sin A$, m being odd and always of the form p^n. If x is one of the roots, it is clear that all the others will be

$$\Big(x, (p)^k, (q)^l\Big).$$

Generally let us therefore set

$$\Big(x, (p)^k, (q)^l\Big) = x_{k,l}\,.$$

Putting $x = x_{0,0}$, from that we deduce generally that

$$\Big(x_{2a+b,2c+d}\,,\; -x_{a+b,c+d}\Big) = \Big(x_{a+b,c+d}\,,\; -x_{b,d}\Big),$$

from which

$$x_{2a+b,2c+d} = \Big(\big(x_{a+b,c+d}\big)^2\,,\; -x_{b,d}\Big).$$

Now from this equality it is easy to derive a rational expression for $x_{2a+b,2c+d}$ as a function of $x_{a+b,c+d}$ and $x_{b,d}$. For if φ is the arc corresponding to any one of

quelconque des sinus qui satisfont à l'équation $(x)^m = \sin A$ pour avoir $\cos\varphi$ ~~et~~ en fonction de $\sin\varphi$, il suffit de chercher le plus grand commun diviseur entre les equations $x^2 + y^2 = 1$ et ~~φ~~ $f(y) = \cos A$, $f(y)$ étant le cosinus de la transcendante m fois plus grande que celle dont le cosinus est y. On trouverait de même $\Delta(\varphi)$ en fonction rationnelle de $\sin\varphi$.

Perhaps Galois stopped in the middle of 'et tan φ'.

On pourra donc, par les formules connues, exprimer

$$x_{2a+b.2c+d} = f\left(x_{a+b.c+d}, x_{b.d}\right)$$

en fonction rationnelle de $x_{a+b.c+d}$ et de $x_{b.d}$

Ce principe posé, démontrons la proposition suivante:

"Toute fonction rationnelle de $x_{0.0}\ x_{1.0}\ x_{0.1}\ \ldots$ invariable par les substitutions de la forme $\left(x_{k.l}\ ,\ x_{ak+b.cl+d}\right)$ est immédiatement connue."

En effet, on pourra d'abord rendre cette fonction fonction de $x_{0.0}$, $x_{0.1}$, $x_{1.0}$ seuls, par l'élimination des autres racines. Cette fonction ne changerait pas de valeur si à la place de $x_{0.0}$, $x_{0.1}$, $x_{1.0}$ on mettait $x_{0.0}$, $x_{0.1}$, $x_{k.l}$, ~~sin~~k n'étant pas nul.

Or, comme toute racine de la forme $x_{0.l}$ s'exprime en fonction rationnelle de $x_{0.0}$ et $x_{0.1}$, il s'ensuit que toute fonction symmetrique des racines dans les quelles le premier

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indice n'est pas nul sera connue en fonction rationnelle et entière de $x_{0.0}$ et de $x_{0.1}$. Donc la fonction que nous considérions tout à l'heure ne variant pas quand on met pour $x_{1.0}$ l'une quelconque des racines dont le premier indice n'est pas nul, cette fonction sera une fonction de $x_{0.0}$ et de $x_{0.1}$ seuls. On éliminera encore $x_{0.1}$ de cette fonction qui deviendra fonction de $x_{0.0}$ seul et enfin une quantité connue.

Le principe est donc démontré.

Celà posé soit F une fonction symmétrique ~~des~~^de ~~quel~~ certaines^ racines de l'équation proposée. Posons

$$\begin{aligned} F(x_{0.0}\ ,\ x_{0.1}\ ,\ x_{0.2}\ ,\ldots) &= y_0 \\ F(x_{1.0}\ ,\ x_{1.1}\ ,\ x_{1.2}\ ,\ldots) &= y_1 \\ F(x_{2.0}\ ,\ x_{2.1}\ ,\ x_{2.2}\ ,\ldots) &= y_2 \\ &\ldots\ldots \end{aligned}$$

Prennons une fonction ~~linéaire~~ de ~~y~~ $y_0\ y_1\ y_2\ \ldots$ invariable par les substitutions linéaires de ces quantités. Il est clair que cette fonction sera une fonction des racines x invariable par toute substitution telle que $\left(x_{k.l}, x_{ak+b.ck+d}\right)$. Cette fonction sera donc connue.

For the slip here see Note 1 below.

On pourra donc, par la méthode que j'ai indiquée, trouver les valeurs de $y_0\ y_1\ y_2\ \ldots$ et parconséquent décomposer l'équation proposée en facteurs dont l'un ait pour racines $x_{0.0}\ x_{0.1}\ x_{0.2}\ \ldots$

of the sines which satisfy the equation $(x)^m = \sin A$, to obtain $\cos\varphi$ ~~and t~~ as a function of $\sin\varphi$ it suffices to seek the greatest common divisor of the equations $x^2 + y^2 = 1$ and $f(y) = \cos A$, $f(y)$ being the cosine of the transcendent m times greater than that of which the cosine is y. Similarly one will find $\Delta(\varphi)$ as a rational function of $\sin\varphi$.

Therefore, by known formulas, one will be able to express

$$x_{2a+b,2c+d} = f(x_{a+b,c+d},\ x_{b,d})$$

as a rational function of $x_{a+b,c+d}$ and $x_{b,d}$.

This principle being set down, we show the following proposition:

Every rational function of $x_{0,0}$, $x_{1,0}$, $x_{0,1}$, ... *invariant under the substitutions of the form* $(x_{k,l}\ ,\ x_{ak+b,cl+d})$ *is immediately known.*

Indeed, first one can write this function as a function just of $x_{0,0}$, $x_{0,1}$, $x_{1,0}$ by elimination of the other roots. This function will not change in value if in place of $x_{0.0}$, $x_{0.1}$, $x_{1.0}$ one were to put $x_{0.0}$, $x_{0.1}$, $x_{k.l}$, ~~sin~~ k not being zero.

Now since every root of the form $x_{0,l}$ may be expressed as a rational function of $x_{0,0}$ and $x_{0,1}$, it follows that every symmetric function of the roots in which the first

index is not zero will be known as a rational and integral [polynomial] function of $x_{0,0}$ and $x_{0,1}$. Therefore since the function that we were considering just now does not vary when one puts for $x_{1.0}$ any one of the roots of which the first index is not zero, this function will be a function just of $x_{0,0}$ and $x_{0,1}$. Further, one may still eliminate $x_{0,1}$ from this function, which will become a function of $x_{0,0}$ alone, and finally a known quantity.

The principle is thus proved.

That having been said, let F be a symmetric function of certain roots of the proposed equation. Set

$$\begin{aligned} F(x_{0.0}\ ,\ x_{0.1}\ ,\ x_{0.2}\ ,\ldots) &= y_0 \\ F(x_{1.0}\ ,\ x_{1.1}\ ,\ x_{1.2}\ ,\ldots) &= y_1 \\ F(x_{2.0}\ ,\ x_{2.1}\ ,\ x_{2.2}\ ,\ldots) &= y_2 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \end{aligned}$$

Take a ~~linear~~ function of $y_0,\ y_1,\ y_2,\ \ldots$ invariant under the linear substitutions of these quantities. It is clear that this function will be a function of the roots invariant under every substitution of the kind $(x_{k,l},\ x_{ak+b,cl+d})$. This function will therefore be known.

By the method that I have indicated it will therefore be possible to find the values of $y_0,\ y_1,\ y_2,\ \ldots$ and consequently decompose the proposed equation into factors, one of which will have $x_{0,0},\ x_{0,1},\ x_{0,2},\ \ldots$ for its roots.

On trouverait de mème un facteur de la mème équation dont les racines seraient $x_{0.0}\ x_{1.0}\ x_{2.0}\ \ldots$. On pourra donc en cherchant le plus grand commun diviseur de ces deux facteurs avoir $x_{0.0}$ qui est l'une des solutions cherchées. Il est serait demème des autres racines.

Misprint 'est' is clear in *ms*. Corrected to 'en' in T1906/7, BA1962.

In the same way a factor of the same equation may be found, of which the roots will be $x_{0.0}$, $x_{1.0}$, $x_{2.0}$, ... By looking for the greatest common divisor of these two factors one will be able to obtain $x_{0,0}$, which is one of the solutions sought. It will be the same for the other roots.

Notes on Dossier 20

NOTE 1: In **f.100a**, p. 342, the formula near the end of the paragraph containing the displayed equations defining y_0, y_1, y_2, ... was changed before completion from $(x_{ak+b.ck+d}$ to $(x_{k.l}, x_{ak+b.ck+d})$. As pointed out in [Tannery (1907), p. 300, footnote] and [B & A (1962), p. 160], it should read $(x_{k.l}, x_{ak+b.cl+d})$. In [Tannery (1907)], however, the second occurrence of x is missing both from the formula in the text and from the formula in the footnote about Galois' slip. As usual the correction is made in the translation.

NOTE 2: There are few jottings here; the pages are much cleaner than others of the manuscripts. The top margin of **f.98a** has some light scribbles in a light ink as if a pen had been tentatively tried out there; and the bottom margin has the initials E.G. written sideways, twice, in a large and careful script. The left margin of **f.100a** contains some formulae. They were written with the page turned through 90° clockwise, so that the left edge became the top. With this orientation the main one is

$$1 + p + \frac{p(p-1)}{2} + \frac{p(p-1)(p-2)}{1.2.3} + \cdots + \frac{p(p-1)(p-2)\cdots\frac{p+1}{2}}{1.2.3.\cdots.\frac{p-1}{2}} + \cdots + \frac{p(p-1)}{2} + p + 1 = p^m + 1$$

most of it carefully written as one line, a little indented, right across the page (or, in its true orientation, from bottom to top). Above that (with the page turned) the top left corner contains $2^p - 1$, and two-thirds of the way along the line, above and just to the left of $\frac{p+1}{2}$ in the numerator of the middle term of the binomial expansion is $p - \frac{p-1}{2}$.

Although most of the equation that is displayed above is carefully written, its last term seems to have been corrected. It is just possible that the plus sign is a minus sign through which a vertical line has been accidentally drawn. As it stands the equation can be re-written $2^p = p^m + 1$, which, as an equation in positive integers, has $p = 1$ and m arbitrary as its only solutions—as Galois would surely have seen instantly. The equation $2^p = p^m - 1$ has $p = 3$, $m = 2$ as its only solution, which again cannot have given Galois much pause.

VI.16 Dossier 21: On the integration of linear equations

Dossier 21 comprises folios 101–2, of the manuscripts. Its cover sheet echoes the title 'Sur l'intégration des équations linéaires' that Galois himself gave this essay. It is presented as item P in [Tannery (1907), pp. 301–303] and in [B & A (1962), pp. 386–391, 517].

Physically this is one large sheet of paper 34.6 cm × 22.5 cm folded to make two folios 17.3 cm × 22.5 cm of which only the first two-and-a-half sides are used. There is a well-respected margin of 3.5 cm–4 cm created originally by folding and unfolding the paper. This is similar to the margins of items that come from Galois' schooldays, preserved by Mr Richard, given by him to Hermite, and found among the latter's papers by Picard—see [Tannery (1907), p. 304, footnote] (transcribed in Note 1 to Dossier 22, p. 364 below). Bourgne noted [B & A (1962), p. 517] that it is the format of the school papers preserved by Mr Richard and estimated that it probably dates from Galois' first year at the École Normale, 1829–30. To me the neatly centred heading 'Note I' above the title, and perhaps intended to be part of the title, resonates with the title 'Notes sur quelques points d'analyse' of the paper [Galois (1830d)] published in Gergonne's *Annales*, December 1830; the material seems to be similar in style; also, the reference to 'le lecteur' [the reader] in the last sentence, suggests that this piece was intended for publication. There is however, no sign that it was ever submitted to a journal.

The name at the top right of the first page is written in a large clear hand and finished with a flourish that turns into a sort of underline. It starts over in the left margin and looks very much as if it had been added after the essay had been finished, possibly quite a lot later.

The misprint u_2 for u_3 in the middle of equations (2) on **f. 101 b** was silently corrected (or perhaps overlooked) by Tannery and by Bourgne. It has been corrected in the translation.

Just one paragraph is minimally indented; all the rest are not indented at all.

101 a

Galois

Note I

Sur l'intégration des équations linéaires.

Soit l'équation linéaire à coefficients variables

$$\frac{d^n y}{dx^n} + P\frac{d^{n-1}y}{dx^{n-1}} + Q\frac{d^{n-2}y}{dx^{n-2}} + \cdots\cdots + S\frac{dy}{dx} + Ty = V$$

Pour l'intégrer supposons que nous connaissions n solutions

$$y = u_1\,, \quad = u_2\,, \quad = u_3 \quad . \quad . \quad . \quad , = u_n$$

Word 'terme' overwritten with 'membre'.

de cette équation privée de second ~~terme~~membre. La solution complète ~~d~~

$$y = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_n u_n$$

qui convient à l'équation privée de second membre, ^y^ satisfera encore quand on supposera ce second membre, si au lieu de ~~supposer~~^regarder^ $\alpha_1\ \alpha_2\ \alpha_3 \ldots . \alpha_n$ ^comme^ constantes, on les considère comme déterminés par les équations ^suivantes^ ⌜en $\frac{d\alpha_1}{dx}\ \frac{d\alpha_2}{dx}\ \ldots\ \frac{d\alpha_n}{dx}$⌝

Almost certainly the whole phrase 'suivantes en $\frac{d\alpha_1}{dx}\ \frac{d\alpha_2}{dx}\ \cdots\ \frac{d\alpha_n}{dx}$', not just the word 'suivantes', was an afterthought squeezed in at the end of the line.

$$(1)\begin{cases} u_1\dfrac{d\alpha_1}{dx} + u_2\dfrac{d\alpha_2}{dx} + u_3\dfrac{d\alpha_3}{dx} + \cdots + u_n\dfrac{d\alpha_n}{dx} = 0 \\ \dfrac{du_1}{dx}\cdot\dfrac{d\alpha_1}{dx} + \dfrac{du_2}{dx}\dfrac{d\alpha_2}{dx} + \dfrac{du_3}{dx}\dfrac{d\alpha_3}{dx} + \cdots\cdots + \dfrac{du_n}{dx}\dfrac{d\alpha_n}{dx} = 0 \\ \dfrac{d^2u_1}{dx^2}\cdot\dfrac{d\alpha_1}{dx} + \dfrac{d^2u_2}{dx^2}\dfrac{d\alpha_2}{dx} + \dfrac{d^2u_3}{dx^2}\dfrac{d\alpha_3}{dx} + \cdots\cdots + \dfrac{d^2u_n}{dx^2}\dfrac{d\alpha_n}{dx} = 0 \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ \dfrac{d^{n-1}u_1}{dx^{n-1}}\cdot\dfrac{d\alpha_1}{dx} + \dfrac{d^{n-1}u_2}{dx^{n-1}}\dfrac{d\alpha_2}{dx} + \dfrac{d^{n-1}u_3}{dx^{n-1}}\dfrac{d\alpha_3}{dx} + \cdots + \dfrac{d^{n-1}u_n}{dx^{n-1}}\dfrac{d\alpha_n}{dx} = V \end{cases}$$

Il importe d'abord de reconnaitre si le dénominateur commun aux valeurs tirées de ces équations peut ou non être nul.

Pour celà j'observe que ce dénominateur est le même que celui des n équations suivantes resolues par rapport à $P\ \ Q\ \ R\ \ \ldots .\ S\ \ T$.

Galois

Note I

On the integration of linear equations.

Let the linear equation with variable coefficients be

$$\frac{d^n y}{dx^n} + P\frac{d^{n-1}y}{dx^{n-1}} + Q\frac{d^{n-2}y}{dx^{n-2}} + \cdots\cdots + S\frac{dy}{dx} + Ty = V$$

To integrate it suppose that we know n solutions

$$y = u_1\,, \quad = u_2\,, \quad = u_3 \qquad . \quad . \quad . \quad , = u_n$$

of this equation deprived of its second ~~member~~term. The complete solution

Second term is right side, so Galois treats the homogeneous equation.

$$y = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_n u_n$$

that corresponds to the equation deprived of its second member, will still satisfy it when one supposes this second member if, in place of ~~supposing~~ regarding $\alpha_1\ \alpha_2\ \alpha_3 \ldots . \alpha_n$ as constants, one considers them as determined by the following equations in $\frac{d\alpha_1}{dx}$, $\frac{d\alpha_2}{dx}$, $\ldots$, $\frac{d\alpha_n}{dx}$:

$$(1)\begin{cases} u_1\frac{d\alpha_1}{dx} + u_2\frac{d\alpha_2}{dx} + u_3\frac{d\alpha_3}{dx} + \cdots + u_n\frac{d\alpha_n}{dx} = 0 \\ \frac{du_1}{dx}\cdot\frac{d\alpha_1}{dx} + \frac{du_2}{dx}\frac{d\alpha_2}{dx} + \frac{du_3}{dx}\frac{d\alpha_3}{dx} + \cdots\cdots + \frac{du_n}{dx}\frac{d\alpha_n}{dx} = 0 \\ \frac{d^2u_1}{dx^2}\cdot\frac{d\alpha_1}{dx} + \frac{d^2u_2}{dx^2}\frac{d\alpha_2}{dx} + \frac{d^2u_3}{dx^2}\frac{d\alpha_3}{dx} + \cdots\cdots + \frac{d^2u_n}{dx^2}\frac{d\alpha_n}{dx} = 0 \\ \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ \frac{d^{n-1}u_1}{dx^{n-1}}\cdot\frac{d\alpha_1}{dx} + \frac{d^{n-1}u_2}{dx^{n-1}}\frac{d\alpha_2}{dx} + \frac{d^{n-1}u_3}{dx^{n-1}}\frac{d\alpha_3}{dx} + \cdots + \frac{d^{n-1}u_n}{dx^{n-1}}\frac{d\alpha_n}{dx} = V\,. \end{cases}$$

It is important first to decide whether the common denominator of the values drawn from these equations can be zero or not.

For that I observe that this denominator is the same as that of the n following equations solved with respect to $P\ \ Q\ \ R\ \ \ldots . S\ \ T$.

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$$(2)\begin{cases} \dfrac{d^n u_1}{dx^n} + P\dfrac{d^{n-1}u_1}{dx^{n-1}} + Q\dfrac{d^{n-2}u_1}{dx^{n-2}} + \cdots + R\dfrac{du_1}{dx} + Tu_1 = 0 \\ \dfrac{d^n u_2}{dx^n} + P\dfrac{d^{n-1}u_2}{dx^{n-1}} + Q\dfrac{d^{n-2}u_2}{dx^{n-2}} + \cdots + S\dfrac{du_2}{dx} + Tu_2 = 0 \\ \dfrac{d^n u_3}{dx^n} + P\dfrac{d^{n-1}u_3}{dx^{n-1}} + Q\dfrac{d^{n-2}u_2}{dx^{n-2}} + \cdots + S\dfrac{du_3}{dx} + Tu_3 = 0 \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ \dfrac{d^n u_n}{dx^n} + P\dfrac{d^{n-1}u_n}{dx^{n-1}} + Q\dfrac{d^{n-2}u_n}{dx^{n-2}} + \cdots + S\dfrac{du_n}{dx} + Tu_n = 0 \end{cases}$$

Galois wrote subscript 2 in place of 3 in middle of array (2).

Or ces équations doivent être parfaitement déterminées, puisque l'~~a~~ la forme d'une équation différentielle dépend uniquement de celle de l'équation intégrale.

Word 'intégrable' changed to 'intégrale'.

Donc le dénominateur en question n'est jamais nul.

Mais on peut de plus le calculer ^d'avance^. Soit D le dénominateur. Il est aisé de voir qu l'on aura

$$\frac{dD}{dx} = D_n + D_{n-1} + D_{n-2} + D_{n-3} + \cdots + D_1$$

Word 'partout' occurs at end of one line and again at start of next.

D_n étant ce que devient D quand on y substitue partout partout $\dfrac{d^n u}{dx^n}$ à la place de $\dfrac{d^{n-1}u}{dx^{n-1}}$,

D_{n-1} ce que devient D quand on y met $\dfrac{d^{n-1}u}{dx^{n-1}}$ au lieu de $\dfrac{d^{n-2}u}{dx^{n-2}}$

Symbol D changed to 'et' by over-writing.

Et ainsi de suite

enfin D_1 ce que devient D par la substitution de $\dfrac{du}{dx}$ à la place de u.

T1906/7, BA1962 have 'les', but 'ces' is clear in *ms*.

Et comme toutes ces parties sont nulles excepté D_n , il reste

$$\frac{dD}{dx} = D_n$$

Mais on a d'ailleurs

$$P = -\frac{D_n}{D} \qquad \text{donc} \quad D = e^{-\int P dx}$$

Puisque $-D_n$ est le numérateur de l'expression de P *tirée de (2).*

Donc $D = e^{-\int P dx}$ valeur cherchée du dénominateur.

$$(2)\quad \begin{cases} \dfrac{d^n u_1}{dx^n} + P\dfrac{d^{n-1}u_1}{dx^{n-1}} + Q\dfrac{d^{n-2}u_1}{dx^{n-2}} + \cdots + R\dfrac{du_1}{dx} + Tu_1 = 0 \\ \dfrac{d^n u_2}{dx^n} + P\dfrac{d^{n-1}u_2}{dx^{n-1}} + Q\dfrac{d^{n-2}u_2}{dx^{n-2}} + \cdots + S\dfrac{du_2}{dx} + Tu_2 = 0 \\ \dfrac{d^n u_3}{dx^n} + P\dfrac{d^{n-1}u_3}{dx^{n-1}} + Q\dfrac{d^{n-2}u_3}{dx^{n-2}} + \cdots + S\dfrac{du_3}{dx} + Tu_3 = 0 \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ \dfrac{d^n u_n}{dx^n} + P\dfrac{d^{n-1}u_n}{dx^{n-1}} + Q\dfrac{d^{n-2}u_n}{dx^{n-2}} + \cdots + S\dfrac{du_n}{dx} + Tu_n = 0\,. \end{cases}$$

Now these equations must be perfectly determined because the form of a differential equation depends solely on that of the integral [polynomial] equation.

Therefore the denominator in question is never zero.

But further, one can calculate it beforehand. Let D be the denominator. It is easy to see that one will have

$$\frac{dD}{dx} = D_n + D_{n-1} + D_{n-2} + D_{n-3} + \cdots + D_1\,,$$

D_n being what D becomes when one substitutes into it $\dfrac{d^n u}{dx^n}$ in place of $\dfrac{d^{n-1}u}{dx^{n-1}}$ throughout,

D_{n-1} what D becomes when on puts $\dfrac{d^{n-1}u}{dx^{n-1}}$ instead of $\dfrac{d^{n-2}u}{dx^{n-2}}$ in it,

and so on;

finally D_1 being what D becomes under the substitution of $\dfrac{du}{dx}$ in place of u.

And as all these parts are zero excepting D_n, there remains

$$\frac{dD}{dx} = D_n\,.$$

But on the other hand one has

$$P = -\frac{D_n}{D} \qquad \text{therefore} \quad D = e^{-\int P dx}$$

because $-D_n$ is the numerator of the expression for P drawn from (2).

Therefore $D = e^{-\int P dx}$, which is the value of the denominator that was sought.

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On pourrait de cette dernière formule déduire celle que nous avons trouvée plus haut, en considérant une équation linéaire de l'ordre n, comme remplaçant n équations simultanées seulement du premier ordre. Quant à la détermination des numérateurs des quantités inconnues, et à l'examen du cas où l'on n'aurait qu'une partie des solutions de la question, nous n'entrerons pas dans ces détails aux quels le lecteur suppléera ~~par ce que nous~~ ^au moyen des principes^ émis plus haut.

Word 'Quand' changed to 'Quant' by over-writing.

From this last formula one could deduce the one that we found above, by considering a linear equation of order n as replacing n simultaneous equations just of the first order. As for the determination of the numerators of the unknown quantities, and the examination of the case where one has no more than a part of the solutions of the question, we shall not enter into these details, which the reader will supply ~~by what we have~~ by means of the principles enunciated above.

VI.17 Dossier 22: On surfaces of the second degree

The sole content of Dossier 22, whose cover is inscribed 'Sur les surfaces du second degré', is an essay on geometry that occupies almost the whole four sides of a single folded sheet. The sheet is 36 cm × 23 cm, neatly folded in two to form pages 18 cm × 23 cm, folios 103 and 104. Guidelines for left margins were created by making folds in the paper; they vary from 4 cm to 4.5 cm wide. It is published in [Tannery (1907), pp. 304–308] and [B & A (1962), pp. 393–401, 517–518].

This is a neatly written fair copy, with few emendations and no jottings. It is easy to read except that the writing is rather small and the paper, though fairly strong, allows some of the writing to show through from the other side. The name E Galois, with a flourish from the final s that turns into an underline, occurs at the top of the margin of the first page, in line with the heading. A little below that, also in the margin, is a neatly drawn diagram of cartesian axes in three-space with the parallelepiped described in the first few lines of the text.

Tannery saw it as an elementary essay; and dated it [Tannery (1907), p. 304, footnote]—see Note 1 below—to the final class at Louis-le-Grand or to the time when Galois entered the École Normale. On the evidence of the paper, the writing and the fine pen similar to those in two of the items from Galois' schooldays, Bourgne [B & A (1962), pp. 517–518] confirms Tannery's estimate of the date as being around 1829. Because of its elementary nature Tannery showed some diffidence in publishing it. But I see this piece differently. The paragraph

> Mais la quantité s et l'équation qui la détermine jouissent d'une propriété fort remarquable que personne jusqu'ici ne parait avoir observée.
>
> [But the quantity s and the equation which determines it enjoy a quite remarkable property that no-one seems to have observed until now.]

suggests that Galois thought of this as containing something new. I estimate that it goes substantially beyond what might have been offered in the Collège Louis-le-Grand or the École Normale at the time. To my mind it does not differ greatly in quality from his articles on numerical solution of equations and on two points of analysis published in Gergonne's *Annales* in December 1831; indeed, I find it somewhat more convincing. For this reason it seems to earn a comfortable place amongst the other minor works.

This is one of the items without paragraph indentation.

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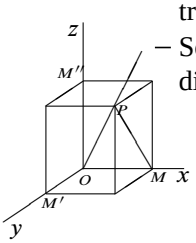
E Galois Recherche sur les surfaces du 2d degré.

Problème. Étant données dans un parallelepipede les trois arêtes m, m', m'', et les angles θ, θ', θ'', que font entre elles respectivement m' et m'', m et m'', m et m', trouver l'expression des angles de la diagonale avec les arêtes.



– Soit $m = OM$, $m' = OM'$, $m'' = OM''$. Si l'on cherche l'angle POM que la diagonale OP forme avec OM, on aura dans le triangle OPM

$$\cos POM = \frac{m^2 + OP^2 - \overline{PM}^2}{2m.OP}$$

Mais on a par la géométrie

$$\overline{OP}^2 = m^2 + m'^2 + m''^2 + 2m'm'' \cos\theta + 2mm'' \cos\theta' + 2mm' \cos\theta''$$
$$\overline{PM}^2 = m'^2 + m''^2 + 2mm'' \cos\theta$$

d'où l'on tire

$$m^2 + \overline{OP}^2 - \overline{PM}^2 = 2m(m + m'' \cos\theta' + m' \cos\theta'') \qquad \text{et enfin}$$
$$\cos POM = \frac{m + m'' \cos\theta' + m' \cos\theta''}{OP}$$

Angles corrected from OPM' and OPM.

On trouvera demème pour les cosinus des angles $M'OP$ et $M''OP$

$$\frac{m' + m'' \cos\theta + m \cos\theta''}{OP} \quad \text{et} \quad \frac{m'' + m' \cos\theta + m \cos\theta'}{OP}$$

Le problème est donc résolu.

Problème. Trouver pour des axes quelconques la condition de perpendicularité d'une droite et d'un plan.

Prennons ~~app~~ à partir de l'origine et suivant certaine direction $OP = 1$. Appelons m, m', m'' les coordonnées du point P. Les équations de toute droite parallèle à OP, seront de la forme

$$\frac{x - a}{m} = \frac{y - b}{m'} = \frac{z - c}{m''}$$

Les quantités m, m', m'' étant liées par la relation

$$1 = m^2 + m'^2 + m''^2 + 2m'm'' \cos\theta + 2mm'' \cos\theta' + 2mm' \cos\theta''$$

Cherchons de mème l'équation d'un plan perpendiculaire à OP.

E Galois Research on the surfaces of the 2nd degree.

PROBLEM. *Given the three edges m, m', m'' in a parallelepiped, and the angles θ, θ', θ'', which m' and m'', m and m'', m and m', respectively make between them, find the expression for the angles of the diagonal with the edges.*

Let $m = OM$, $m' = OM'$, $m'' = OM''$. If one seeks the angle POM which the diagonal OP forms with OM, in the triangle OPM one will have

$$\cos POM = \frac{m^2 + OP^2 - \overline{PM}^2}{2m.OP}.$$

But from geometry

$$\overline{OP}^2 = m^2 + m'^2 + m''^2 + 2m'm'' \cos\theta + 2mm'' \cos\theta' + 2mm' \cos\theta'',$$
$$\overline{PM}^2 = m'^2 + m''^2 + 2mm'' \cos\theta,$$

from which one derives

$$m^2 + \overline{OP}^2 - \overline{PM}^2 = 2m(m + m'' \cos\theta' + m' \cos\theta''), \qquad \text{and finally}$$
$$\cos POM = \frac{m + m'' \cos\theta' + m' \cos\theta''}{OP}.$$

In the same way, for the cosines of the angles $M'OP$ and $M''OP$ one will find

$$\frac{m' + m'' \cos\theta + m \cos\theta''}{OP} \quad \text{and} \quad \frac{m'' + m' \cos\theta + m \cos\theta'}{OP}.$$

Thus the problem is solved.

PROBLEM. *For arbitrary axes find the condition for perpendicularity of a line and a plane.*

Starting from the origin and following a certain direction we take $OP = 1$. Call m, m', m'' the coordinates of the point P. The equations of every line parallel to OP, will be of the form

$$\frac{x-a}{m} = \frac{y-b}{m'} = \frac{z-c}{m''},$$

the quantities m, m', m'' being connected by the relation

$$1 = m^2 + m'^2 + m''^2 + 2m'm'' \cos\theta + 2mm'' \cos\theta' + 2mm' \cos\theta''.$$

Similarly let us seek the equation of a plane perpendicular to OP.

Il est evident que si on appele x, y, z les coordonnées de ce plan, et que l'on projette orthogonalement sur OP ces coordonnées la somme des projections devra ètre ~~nulle~~^constante^. Or on connaît, par le problème précédent, les cosinus des angles de la droite OP avec les axes. L'équation du plan sera donc

$$(m + m'\cos\theta'' + m''\cos\theta')x + (m' + m\cos\theta'' + m''\cos\theta)y + (m'' + m\cos\theta' + m'\cos\theta)z + p = 0$$

Et il est remarquable que le premier membre de cette équation exprime aussi la distance à ce plan d'un point quelconque dont les coordonnées sont x, y, z. Ce qui est evident puisque ce premier membre n'est autre chose que ~~la~~^la somme des^projections des coordonnées d'un point sur la droite OP, augmentée de la distance du plan à l'origine.

Celà posé, soit l'équation d'une surface du second degré rapportée à des axes obliques

$$Ax^2 + A'y^2 + A''z^2 + 2Byz + 2B'xz + 2B''xy + 2Cx + 2C'y + 2C''z + D = \varphi(x, y, z) = 0$$

~~Si l'~~^Lorsqu'^on cherche l'équation du plan qui divise également toutes les cordes parallèles à une droite donnée, on substitue l'équation $\varphi(x, y, z) = 0$, à la place de x, y, z,

$$x + \rho m\,, \qquad y + \rho m'\,, \qquad z + \rho m''$$

et les racines de l'équation en ρ qu'on obtient ainsi, expriment ~~la somme~~ des distances du point (x, y, z) aux deux points où une corde parallèle à la droite $\frac{x}{m} = \frac{y}{m'} = \frac{z}{m''}$ menée par le point (x, y, z) coupe la surface du second degré. Ces deux distances devant ètre égales et de signe contraire, il suffira de faire dans l'équation en ρ le second terme nul pour avoir l'équation du plan diamétral.
Or l'équation en ρ est en faisant

$$M = \varphi(m, m', m'')$$
$$MP = (Am + B''m' + B'm'')x + (A'm' + B''m + Bm'')y + (A''m'' + B'm + Bm')z + Cm + C'm' + C''m''$$

de la forme

$$\rho^2 + 2P\rho + Q = 0$$

Si l'on cherche l'équation d'un plan principal, il faudra de plus que le plan représenté par $P = 0$ soit perpendiculaire à la droite $\frac{x}{m} = \frac{y}{m'} = \frac{z}{m''}$ et parconséquent que son equation soit de la forme

$$(m + m'\cos\theta'' + m''\cos\theta')x + (m' + m\cos\theta'' + m''\cos\theta)y + (m'' + m\cos\theta' + m''\cos\theta)z + p = S = 0$$

It is clear that if one calls x, y, z the coordinates of this plane, and if one projects these coordinates orthogonally onto OP, the sum of the projections must be ~~zero~~ constant. Now from the preceding problem the cosines of the angles of the line OP with the axes are known. The equation of the plane will therefore be

$$(m + m'\cos\theta'' + m''\cos\theta')x + (m' + m\cos\theta'' + m''\cos\theta)y + (m'' + m\cos\theta' + m'\cos\theta)z + p = 0\,.$$

And it is worth noting that the first member [the left hand side] of this equation expresses also the distance to this plane of an arbitrary point whose coordinates are x, y, z, as is clear because this first member is nothing other than the sum of the projections of the coordinates of a point onto the line OP, augmented by the distance of the plane to the origin.

That said, let the equation of a surface of the second degree with respect to oblique axes be

$$Ax^2 + A'y^2 + A''z^2 + 2Byz + 2B'xz + 2B''xy + 2Cx + 2C'y + 2C''z + D = \varphi(x,y,z) = 0\,.$$

~~If the~~When one seeks the equation of the plane which divides all the chords parallel to a given line equally, one substitutes in the equation $\varphi(x,y,z) = 0$, in place of x, y, z,

$$x + \rho m\,, \qquad y + \rho m'\,, \qquad z + \rho m''\,,$$

and the roots of the equation in ρ which is thus obtained express ~~the sum~~ the distances of the point (x,y,z) to the two points where a chord parallel to the line $\frac{x}{m} = \frac{y}{m'} = \frac{z}{m''}$ drawn through the point (x,y,z) cuts the surface of the second degree. These two distances having to be equal and of opposite sign, it will suffice to make the second term in the equation in ρ equal to zero to get the equation of the diametral plane. Now on putting

$$M = \varphi(m,m',m'')\,,$$
$$MP = (Am + B''m' + B'm'')x + (A'm' + B''m + Bm'')y + (A''m'' + B'm + Bm')z + Cm + C'm' + C''m''\,,$$

the equation in ρ is of the form

$$\rho^2 + 2P\rho + Q = 0\,.$$

To find the equation of a principal plane it is necessary in addition that the plane represented by $P = 0$ be perpendicular to the line $\frac{x}{m} = \frac{y}{m'} = \frac{z}{m''}$, and consequently that its equation be of the form

$$(m + m'\cos\theta'' + m''\cos\theta')x + (m' + m\cos\theta'' + m''\cos\theta)y + (m'' + m\cos\theta' + m''\cos\theta)z + p = S = 0\,.$$

Il faudra donc que les coefficients de MP et ceux de S soient proportionnels et que l'on ait

$$\frac{MP}{S} = \text{const} = s$$

La quantité étant telle que l'on ait

$$\begin{aligned}
(A-s)m + (B''-s\cos\theta'')m' + (B'-s\cos\theta')m'' &= 0\\
(A'-s)m' + (B''-s\cos\theta'')m + (B-s\cos\theta)m'' &= 0\\
(A''-s)m'' + (B'-s\cos\theta')m + (B-s\cos\theta)m' &= 0
\end{aligned}$$

On en déduit l'équation en s,

$$\begin{aligned}
0 = {} & (A-s)(B-s\cos\theta)^2 + (A'-s)(B'-s\cos\theta')^2 + (A''-s)(B''-s\cos\theta'')^2 -\\
& -(A-s)(A'-s)(A''-s) - 2(B-s\cos\theta)(B'-s\cos\theta')(B''-s\cos\theta'')
\end{aligned}$$

Qui est du troisieme degré parce qu'en effet il existe trois plans principaux.

Mais la quantité s et l'équation qui la détermine jouissent d'une propriété fort remarquable que personne jusqu'ici ne parait avoir observée.

Supposons que l'on transforme les coordonnées en exprimant les anciennes coordonnées d'un point en fonction des nouvelles. Si on substitue les valeurs de x, y, z en x', y', z' dans ~~l'equation~~ la fonction $\varphi(x, y, z)$ on obtient une fonction $\varphi'(x', y', z')$ d'une autre forme, et qui est telle que dans la fonction φ on substitue les anciennes coordonnées d'un point déterminé, et dans la fonction φ' les nouvelles, les deux résultats ainsi obtenus sont égaux.

The word 'si' is missing.

Celà posé reprennons l'expression de s, $s = \frac{MP}{S}$, la quantité M ~~exprimant~~ étant le resultat de la substitution des coordonnées du point pris sur une droite fixe à une distance $= 1$ de l'origine c'est à dire d'un point fixe, dans l'équation de la surface, ne variera quand on transformera les coordonnées.

La quantité P exprimant la demi-somme des distance[s] d'un point (x, y, z) à la surface distances comptées suivant une droite fixe, est aussi invariable par la transformation des coordonnées. Enfin la quantité S exprimant la distance d'un point à un plan déterminé, ne saurait non plus varier.

La quantité s est donc elle mème invariable pour un mème plan principal, et l'equation qui donne ses trois valeurs aura des coëfficients invariables. Or en la développant, on a,

$$\begin{aligned}
&\left(1-\cos^2\theta-\cos^2\theta'-\cos^2\theta''+2\cos\theta\cos\theta'\cos\theta''\right)s^3 -\\
&-s^2\left[A\sin^2\theta + A'\sin^2\theta' + A''\sin^2\theta'' + 2B(\cos\theta'\cos\theta''-\cos\theta)\right.\\
&\qquad \left. + 2B'(\cos\theta\cos\theta''-\cos\theta') + 2B''(\cos\theta\cos\theta'-\cos\theta'')\right]\\
&+s\left(A'A''+AA''+AA'-2AB\cos\theta-2A'B'\cos\theta'-2A''B''\cos\theta''\right.\\
&\qquad \left. -B^2-B'^2-B''^2+2B'B''\cos\theta+2BB''\cos\theta'+2BB'\cos\theta''\right)\\
&+AB^2+A'B'^2+A''B''^2-AA'A''-2BB'B''=0
\end{aligned}$$

It will be necessary therefore that the coefficients of MP and those of S be proportional, so that one has

$$\frac{MP}{s} = \text{const} = s\,,$$

the quantity $[s]$ being such that one has

$$\begin{aligned}
(A-s)m + (B'' - s\cos\theta'')m' + (B' - s\cos\theta')m'' &= 0\\
(A'-s)m' + (B'' - s\cos\theta'')m + (B - s\cos\theta)m'' &= 0\\
(A''-s)m'' + (B' - s\cos\theta')m + (B - s\cos\theta)m' &= 0\,.
\end{aligned}$$

One deduces from this the equation in s,

$$\begin{cases} 0 = (A-s)(B-s\cos\theta)^2 + (A'-s)(B'-s\cos\theta')^2 + (A''-s)(B''-s\cos\theta'')^2 - \\ \qquad\qquad - (A-s)(A'-s)(A''-s) - 2(B-s\cos\theta)(B'-s\cos\theta')(B''-s\cos\theta''), \end{cases}$$

which is of the third degree because in fact there exist three principal planes.

But the quantity s and the equation which determines it enjoy a quite remarkable property that no-one seems to have observed until now.

Suppose that the coordinates are transformed by expressing the old coordinates of a point as a function of new ones. If one substitutes the values of $x,\ y,\ z$ in terms of $x',\ y',\ z'$ into ~~the equation~~ the function $\varphi(x, y, z)$ one gets a function $\varphi'(x', y', z')$ of a different form, and which is such that, if one substitutes the old coordinates of a certain point into the function φ, and the new ones into the function φ', the two results obtained in this way are equal.

That said, return to the expression for s, $s = \frac{MP}{S}$. The quantity M ~~expressing~~ being the result of the substitution of the coordinates of the point on a fixed line at a distance $= 1$ from the origin, that is to say, of a fixed point, into the equation of the surface, will not change when the coordinates are transformed.

The quantity P expressing half of the sum of the distances of a point $(x,\ y,\ z)$ to the surface, distances being measured along a fixed line, is also invariant under the transformation of the coordinates. Finally quantity S expressing the distance of a point to a specified plane cannot vary at all either.

The quantity s is therefore itself invariant for one and the same principal plane, and the equation which gives its three values will have coefficients that are invariant. Now expanding it, one has,

$$\begin{aligned}
&\left(1 - \cos^2\theta - \cos^2\theta' - \cos^2\theta'' + 2\cos\theta\cos\theta'\cos\theta''\right)s^3 \ -\\
&- s^2\left[A\sin^2\theta + A'\sin^2\theta' + A''\sin^2\theta'' + 2B(\cos\theta'\cos\theta'' - \cos\theta)\right.\\
&\qquad \left. + 2B'(\cos\theta\cos\theta'' - \cos\theta') + 2B''(\cos\theta\cos\theta' - \cos\theta'')\right]\\
&+ s\left(A'A'' + AA'' + AA' - 2AB\cos\theta - 2A'B'\cos\theta' - 2A''B''\cos\theta''\right.\\
&\qquad \left. - B^2 - B'^2 - B''^2 + 2B'B''\cos\theta + 2BB''\cos\theta' + 2BB'\cos\theta''\right)\\
&+\ AB^2 + A'B'^2 + A''B''^2 - AA'A'' - 2BB'B'' = 0\,.
\end{aligned}$$

Divisant tous les coefficients par le premier ou par le dernier on aura ~~d~~ trois fonctions des constantes qui entrent dans l'équation de la surface, invariables par la transformation des coordonnées. Si l'on suppose $\cos\theta$, $\cos\theta'$ et $\cos\theta''$ nuls on aura pour tous les systèmes d'axes où celà peut etre c'est à dire d'axes rectangulaires, les équations

$$A + A' + A'' = \text{const}$$
$$B^2 + B'^2 + B''^2 - A'A'' - AA'' - AA' = \text{const}$$
$$AB^2 + A'B'^2 + A''B''^2 - AA'A'' - 2BB'B'' = \text{const}$$

Although the sign before $AA'A''$ in the third equation looks like $+$, almost certainly it is corrected to $-$.

Également si l'on suppose encore dans l'equation en s, B, B', B'' nuls, c'est à dire qu'on suppose la surface rapportée à des diamètres conjugués, en divisant toute l'equation par le dernier terme, on trouvera pour tous les systèmes semblables

$$\frac{1 - \cos^2\theta - \cos^2\theta' - \cos^2\theta'' + 2\cos\theta\cos\theta'\cos\theta''}{AA'A''} = \text{const}$$
$$\frac{\sin^2\theta}{A'A''} + \frac{\sin^2\theta'}{AA''} + \frac{\sin^2\theta''}{AA'} = \text{const}$$
$$\frac{1}{A} + \frac{1}{A'} + \frac{1}{A''} = \text{const}$$

Et comme $\frac{1}{A}$, $\frac{1}{A'}$, $\frac{1}{A''}$ expriment dans ce cas les quarrés des diamètres, on retrouve ici les théorèmes connus.

Dividing all the coefficients by the first or by the last one will have three functions of the constants entering into the equation of the surface, which are invariant under the transformation of the coordinates. If $\cos\theta$, $\cos\theta'$ and $\cos\theta''$ are supposed zero for all systems of axes where this can be the case, that is to say rectangular axes, one will get the equations

$$\begin{aligned} A + A' + A'' &= \text{const} \\ B^2 + B'^2 + B''^2 - A'A'' - AA'' - AA' &= \text{const} \\ AB^2 + A'B'^2 + A''B''^2 - AA'A'' - 2BB'B'' &= \text{const}. \end{aligned}$$

Equally, if again in the equation for s one supposes $B,\ B',\ B''$ to be zero, that is to say, one supposes the surface to be described with respect to conjugate diameters, on dividing the whole equation by the last term, one will find for all similar systems

$$\begin{aligned} \frac{1 - \cos^2\theta - \cos^2\theta' - \cos^2\theta'' + 2\cos\theta\cos\theta'\cos\theta''}{AA'A''} &= \text{const} \\ \frac{\sin^2\theta}{A'A''} + \frac{\sin^2\theta'}{AA''} + \frac{\sin^2\theta''}{AA'} &= \text{const} \\ \frac{1}{A} + \frac{1}{A'} + \frac{1}{A''} &= \text{const}. \end{aligned}$$

And since in this case $\frac{1}{A}, \frac{1}{A'}, \frac{1}{A''}$ express the squares of the diameters, one finds the known theorems here.

Notes on Dossier 22

NOTE 1: [Tannery (1907), p. 304] has the following footnote which seems to me to be worth quoting *in extenso*.

> Malgré son caractère élémentaire, j'ai cru devoir publier cette note, qui n'est pas sans intérêt pour l'histoire de la Géométrie analytique et de la théorie des invariants. En raison de son contenu, on peut supposer qu'elle remonte au temps où Galois était élève de M. Richard, dans la classe de Mathématiques spéciales, ou au moment où il sortait de cette classe pour entrer à l'École Normale. Toutefois, la première supposition semble devoir être écartée: s'il en avait eu connaissance, M. Richard aurait sans doute fait pénétrer dans son enseignement les idées de son élève, qui se seraient diffusées immédiatement. Quoi qu'il en soit, cette note a, comme le morceau précédent, l'aspect d'une copie d'écolier, avec la signature en haut et à gauche; elle ressemble tout à fait à quelques-unes des copies de Galois, que M. Richard avait conservées et données à Hermite. M. Émile Picard a retrouvé ces copies de Galois dans les papiers d'Hermite; il a bien voulu me les remettre pour qu'elles soient jointes au précieux trésor que Mme de Blignières donne à l'Académie des Sciences. L'une de ces copies contient un petit travail, que Galois a sans doute fait librement et remis à son maître, et où son esprit philosophique se manifeste déjà; j'en entrais cette curieuse réflexion:
>
> Un auteur me dit: "l'arithmétique est la base de toutes les parties des Mathématiques, puisque c'est toujours aux nombres qu'il faut ramener les résultats des calculs." D'après la dernière phrase de l'auteur, il serait plus naturel de croire que l'arithmétique est le terme et le complément de l'Analyse; et c'est ce qui a lieu.
>
> [Despite its elementary character, I believed I should publish this note, which is not without interest for the history of analytic geometry and of the theory of invariants. Reasoning from its contents one might suppose that it goes back to the time when Galois was a pupil of Mr Richard in the advanced class in mathematics, or the moment when he left this class to enter the École Normale. The former supposition appears to have to be abandoned, however: if he had had knowledge of it Mr Richard would no doubt have let the ideas of his pupil enter into his teaching, and they would immediately have been spread. However that may be, this note, like the preceding little piece [Dossier 21, 'On the integration of linear equations'] has the appearance of a school exercise [copy], with the signature at the top left; indeed, it resembles some of the exercises by Galois which Mr Richard kept and gave to Hermite. Mr Émile Picard found these exercises among the papers of Hermite; he has been so kind as to give them to me so that they may join the precious treasure which Madame de Blignières is giving to the Academy of Science. One of

these copies contains a little piece of work which Galois doubtless did of his own accord and gave to his master, and in which his philosophical mind already shows itself; I extract from it the following curious reflection:

> An author said to me: "Arithmetic is the basis of all the parts of mathematics because it is always to numbers that the results of calculations must be reduced." According to the last phrase of the author it would be more natural to believe that arithmetic is the end and the complement of Analysis; and this is the case.]

The excerpt by Galois that Tannery quoted here is to be found on **f. 233 a** (see [B & A (1962), p. 425]), one of the pages from his schooldays at Louis-le-Grand (another page without paragraph indentation). It is preceded by the opening sentence of the essay (which gives its context):

> Quelle place l'arithmétique doit-elle occuper dans l'analyse? c'est une question plus difficile à résoudre qu'on ne pense:
>
> [What place should arithmetic occupy in analysis? This is a question more difficult to resolve than one might think:]

NOTE 2: It would be interesting to identify the 'théorèmes connus', the known theorems, to which Galois referred in his very last sentence. Quite apart from its intrinsic interest, the answer could help one to place this essay between schoolwork and original mathematics of its time.

VI.18 Dossier 23: On eulerian integrals

Dossier 23, folios 105–111, has on its cover sheet the description 'Sur les intégrales eulériennes.' It appears in [B & A (1962), pp. 193–207, 339], at the start of the material that was transcribed and organised by J. P. Azra. The great majority of the dossier contains formulae and jottings that are the same in any language. They may be seen in [B & A (1962)] and [Galois (2011)]. It begins however with an uncompleted essay on integrals and the gamma function that was briefly summarised in [Tannery (1907), p. 276, footnote]. The pages are 14.5 cm × 19.5 cm. Some are single sheets, some are two-page pamphlets formed by folding sheets 29 cm wide in two.

The first two pages **f. 105 a**, **f. 105 b** of the essay are carefully and neatly written, with clear (if small) paragraph indentation, and just a few small emendations in the lower half of the second page. After that, however, on **f. 106 a** it degenerates and peters out into nonsensical formulae, some of them much amended, with odd additional and (so far as I can see) irrelevant formulae intermingled. I have transcribed as far as what looks like a natural cadence followed by the word 'Donc' [Therefore]. After that another three lines of formulae complete the page, but the first is scribbled out, and the other two are incomplete. They may be seen at [Galois (2011)] or at [B & A (1962), pp. 196, 197].

105 a

Soit $[m,n] = \int_0^1 (1-x)^{m-1} x^{n-1}$

Words run together in *ms*.

ontrouve

$$[m+1,n] = \frac{m}{m+n}\,[m,n]$$

d'où l'on déduit quand p est entier

$$[m+p,n] = \frac{m(m+1)\cdot\cdots\cdot(m+p-1)}{(m+n)(m+n+1)\cdot\cdots\cdot(m+n+p-1)}\,[m,n]$$

Si dans cette formule on fait $m=1$ et qu'on remplace p par $p-1$, il vient

$$[p,n] = \frac{1\,.\,2\,.\,3\,\ldots\,.\,(p-1)}{(n+1)(n+2)\cdot\cdots\cdot(n+p-1)}\,[1,n] = \frac{1\,.\,2\,.\,3\,\ldots\,.\,(p-1)}{n(n+2)\cdot\cdots\cdot(n+p-1)}$$

de ces formules on déduit facilement celle-ci

$$[m,n] = \frac{[p,m]}{[p,m+n]}\,[m+p,n]$$

Word 'est' or 'soit' missing.

qui subsiste toutes les fois que p entier.

Si l'on y fait $p=\infty$, il vient

$$[m,n] = \lim \frac{[p,m]\times[p,n]}{[p,m+n]}$$

Or on a

$$\int_0^1 (1-x)^{p-1}x^{n-1}\,dx = \frac{1}{p^n}\int_0^p \left(1-\frac{x}{p}\right)^{p-1} x^{n-1}\,dx$$

105 b

d'où

$$[m,n] = \lim \frac{\displaystyle\int_0^p \left(1-\frac{x}{p}\right)^{p-1} x^{m-1}\,dx \times \int_0^p \left(1-\frac{x}{p}\right)^{p-1} x^{n-1}\,dx}{\displaystyle\int_0^p \left(1-\frac{x}{p}\right)^{p-1} x^{m+n-1}\,dx}$$

ce qui se réduit à

$$[m,n] = \frac{\displaystyle\int_0^p e^{-x}x^{m-1}\,dx \times \int_0^p e^{-x}x^{n-1}\,dx}{\displaystyle\int_0^p e^{-x}x^{m+n-1}\,dx}$$

Posant donc $\int_0^\infty e^{-x}x^{n-1}dx = \Gamma n$, il vient

Let $[m,n] = \int_0^1 (1-x)^{m-1}x^{n-1}$.

One finds that

$$[m+1,n] = \frac{m}{m+n}\,[m,n]$$

from which may be deduced when p is an integer

$$[m+p,n] = \frac{m(m+1)\cdot\cdots\cdot(m+p-1)}{(m+n)(m+n+1)\cdot\cdots\cdot(m+n+p-1)}\,[m,n]\,.$$

If one makes $m=1$ and replaces p by $p-1$ in this formula what emerges is

$$[p,n] = \frac{1\,.\,2\,.\,3\,.\,\ldots\,.\,(p-1)}{(n+1)(n+2)\cdot\cdots\cdot(n+p-1)}\,[1,n] = \frac{1\,.\,2\,.\,3\,.\,\ldots\,.\,(p-1)}{n(n+2)\cdot\cdots\cdot(n+p-1)}.$$

From these formulae one easily deduces this:

$$[m,n] = \frac{[p,m]}{[p,m+n]}\,[m+p,n]\,,$$

which holds whenever p is an integer.

If one lets $p=\infty$ in this, it emerges that

$$[m,n] = \lim\,\frac{[p,m]\times[p,n]}{[p,m+n]}$$

Now one has

$$\int_0^1 (1-x)^{p-1}x^{n-1}\,dx = \frac{1}{p^n}\int_0^p \left(1-\frac{x}{p}\right)^{p-1}x^{n-1}\,dx$$

from which

$$[m,n] = \lim\,\frac{\displaystyle\int_0^p \left(1-\frac{x}{p}\right)^{p-1}x^{m-1}\,dx \times \int_0^p \left(1-\frac{x}{p}\right)^{p-1}x^{n-1}\,dx}{\displaystyle\int_0^p \left(1-\frac{x}{p}\right)^{p-1}x^{m+n-1}\,dx}$$

which reduces to

$$[m,n] = \frac{\displaystyle\int_0^p e^{-x}x^{m-1}\,dx \times \int_0^p e^{-x}x^{n-1}\,dx}{\displaystyle\int_0^p e^{-x}x^{m+n-1}\,dx}$$

Therefore setting $\int_0^\infty e^{-x}x^{n-1}dx = \Gamma n$, one gets

$$[m,n] = \frac{\Gamma m \,.\, \Gamma n}{\Gamma(m+n)}$$

Soit $\dfrac{d\ \log \Gamma n}{dn} = \varphi n$. Il vient

$$\frac{d\ \log[m,n]}{dm} = \varphi m - \varphi(m+n)$$

Equation $m = 0$ changed to $1 = m$ by over- writing.

J'observe maintenant que pour $m = 0$ on a

$$\frac{d\ \log[m,n]}{dm} = \frac{\displaystyle\int_0^1 \log(1-x)\, x^{n-1}\, dx}{\displaystyle\int_0^1 x^{n-1}\, dx} = \int_0^1 \log(1-x).\, d(x^n)$$

Intégrant par partie on trouve

$$(x^n - 1)\ \log(1-x) + \int \frac{x^n - 1}{x-1}\, dx$$

Ce qui au deux limites seréduit à

$$\frac{d\ \log[m,n]}{dm} = -\int_0^1 \frac{x^n - 1}{x-1}\, dx \,, \text{pour } m = 1$$

106 a

On aura donc

$$-\int_0^1 \frac{x^n - 1}{x-1}\, dx = \varphi 0 - \varphi(n+1)$$

oubien $\displaystyle\int_0^1 \frac{x^{n-1} - 1}{x-1}\, dx = \varphi n - \varphi 1$

$$\lceil \varphi 1 = \rceil \int_0^\infty e^{-x} lx\, dx = \int_0^\infty (1 - e^{-x})\, lx + \int_0^\infty (1 - e^{-x})\, \frac{dx}{x}$$

Considérons la fonction

$$\int_0^1 \left(\frac{x^{mn} - 1}{x-1}\, dx - \frac{nx^{n-1}}{x^n - 1}\right) dx \int_0^1 \frac{x^{m(n-1)} + \cdots + x^{n-1}}{x^n - 1}$$

Si l'on y remplace x par $x^{\frac{1}{n}}$ il vient

$$\frac{1}{n}\int_0^1 \left(\frac{x^{m-1}}{x^{\frac{1}{n}} - 1}\, dx - \frac{n\, x^{n-1}}{x^n - 1}\, dx\right) \qquad \log \frac{x^n - 1}{x-1}$$

Savoir $\displaystyle\int_0^\infty e^{-x} lx\, dx = -e^{-x} lx + \int_0^\infty e^{-x}\, \frac{dx}{x}$

$$\frac{1}{n}\int_0^1 \frac{x^{m-1}\ \frac{x-1}{x^{\frac{1}{n}} - 1} - n}{x-1}\, dx$$

Ou enfin $\dfrac{1}{n}\left\{\varphi m + \varphi\left(m + \dfrac{1}{n}\right) + \cdots + \varphi\left(m + \dfrac{n-1}{m}\right)\right\}$

Donc [. . .]

$$[m,n] = \frac{\Gamma m \,.\, \Gamma n}{\Gamma(m+n)}\,.$$

Let $\dfrac{d\ \log \Gamma n}{dn} = \varphi n$. It emerges that

$$\frac{d\ \log[m,n]}{dm} = \varphi m - \varphi(m+n)\,.$$

I observe now that for $m = 1$ one has

$$\frac{d\ \log[m,n]}{dm} = \frac{\displaystyle\int_0^1 \log(1-x)\, x^{n-1}\, dx}{\displaystyle\int_0^1 x^{n-1}\, dx} = \int_0^1 \log(1-x).\, d(x^n)\,.$$

Integrating by parts one finds

$$(x^n - 1)\ \log(1-x) + \int \frac{x^n - 1}{x-1}\, dx$$

which reduces at the two limits to

$$\frac{d\ \log[m,n]}{dm} = -\int_0^1 \frac{x^n - 1}{x-1}\, dx\,, \text{for } m = 1$$

One then has

$$-\int_0^1 \frac{x^n - 1}{x-1}\, dx = \varphi 0 - \varphi(n+1)\,,$$

that is $\displaystyle\int_0^1 \frac{x^{n-1} - 1}{x-1}\, dx = \varphi n - \varphi 1$

$$\varphi 1 = \int_0^\infty e^{-x} lx\, dx = (1 - e^{-x})\, lx + \int_0^\infty (1 - e^{-x})\, \frac{dx}{x}\,.$$

Consider the function

$$\int_0^1 \Big(\frac{x^{mn}}{x-1} - \frac{n x^{n-1}}{x^n - 1}\Big) dx\,.$$

If x is replaced $x^{\frac{1}{n}}$ what emerges is

$$\frac{1}{n}\int_0^1 \Big(\frac{x^{m-1}}{x^{\frac{1}{n}} - 1} - \frac{n}{x-1}\, dx\Big)\,,$$

that is,

$$\frac{1}{n}\int_0^1 \frac{x^{m-1}\ \frac{x-1}{x^{\frac{1}{n}}-1} - n}{x-1}\, dx\,,$$

or finally, $\dfrac{1}{n}\left\{\varphi m + \varphi\Big(m + \dfrac{1}{n}\Big) + \cdots + \varphi\Big(m + \dfrac{n-1}{m}\Big)\right\}.$

Therefore [. . .]

VI.19 Dossier 24: A theorem of Abel

Dossier 24, which comprises folios 112–185, has two cover-sheets. The first is annotated 'Fragments et calculs divers', the second has a main heading 'Inédits' (Unpublished—probably referring to Tannery's articles), followed by the description

> Lot de calculs dont la plupart se rapportent à la théorie des fonctions elliptiques.
> — Fragment sur la théorie des nombres
> — Théorème d'Abel.
> — Equations aux dérivées partielles du premier ordre.
>
> [Batch of calculations of which the majority refer to the theory of elliptic functions
> — Fragment on the theory of numbers;
> — Abel's Theorem;
> — Partial differential equations of the first order.]

It is a mixed collection consisting mostly of scraps, rough calculations and jottings.

From this we reproduce one item that seems more interesting than the others, namely **f. 112 a**, which was first published in [B & A (1962), pp. 186, 187], though Tannery made some interesting observations about it [Tannery (1907), pp. 276–278] that are quoted in Note 2 on p. 377 below. This has its own subsidiary cover-sheet inscribed 'Contient une démonstration du théorème d'Abel'. It is the front page, 20 cm × 31 cm, of a four-page pamphlet created by folding a large sheet of paper in two. Tannery identifies the paper as being of the same format as that which Galois used for his Testamentary Letter; in [B & A (1962)] it is described as being the same as the paper on which Galois wrote the studies intended for the *Revue Encyclopédique* (folios 75–80), and it is dated to the same period, December 1831–April 1832. If this is the piece of work to which **f. 58 b** in Dossier 8 (p. 220 below) refers then the date should be September 1831, however.

In [B & A (1962)] this item is placed in the fourth part of the book, which is headed 'A la découverte du troisième mémoire: théorie des intégrales et des fonctions elliptiques'. It appears there on pp. 186, 187, following the testamentary letter to Chevalier. On p. 518 one reads 'Ce fragment est la seule ébauche que nous conservions du *Troisième Mémoire*' [This fragment is the only sketch that we retain of the *Third Memoir*]. That it relates to what Galois was discussing in the last part of his testamentary letter is undeniable. That there ever was a third memoir is debatable. Like Tannery, I have come to believe this to be unlikely. For further discussion of this point see the notes below.

This note has recognisable, though small, paragraph indentation.

112 a

Théorie des ~~fonctions de la forme $\int X\,dx$, X étant une fonction Algébrique de x.~~^Integrales dont les differentielles sont des fonctions algebriques^

Lemme fondamental. ~~Soient~~ ~~Soit~~ ^Soient X une fonction algebrique de x donnee par la relation^ $F(X,x)=0$ ~~la relation ^algèbrique qui lie^ entre X et x~~, $f(X,x)$ une fonction rationnelle quelconque de x et X, $\varphi(X,x)$ une fonction rationnelle à forme variable, c'est à dire à coëfficients ~~indéterminés.~~^considérés^ ⌜comme fonctions d'une ou de plusieurs variables.⌝

Si l'on elimine X entre les deux équations $F(X,x)=0$ et $\varphi(X,x)=0$ et que $x_1,\ x_2,\ x_3,\ \ldots\ldots\ x_n$ soient les racines de l'equation finale, on aura

$$\int_{x_0}^{x_1} f(X,x)dx+\int_{x_0}^{x_2} f(X,x)dx+\int_{x_0}^{x_3} f(X,x)dx+\cdots+\int_{x_0}^{x_n} f(X,x)dx=V$$

Word 'intégrales' unclear; rendered 'intégrations' in BA1962, but 'intégrales' looks more likely.

x_0 étant l'origine arbitraire, mais constante, des intégrales, et V une fonction algébrique et logarithmique des coëfficients ~~indéterminés~~^variables^.

'D'où' changed to 'D'après' by over-writing.

Demonstration On déduit de $F(X,x)=0$ et $\varphi(X,x)=0$, combinés, entant que la fonction φ ~~reste~~ contient des indéterminées, $X=$ à une fonction rationnelle de x⌜.⌝ ~~$X=fx$~~ D'où ~~où~~après ^celà^ la differentielle de la quantité

$$\int_{x_0}^{x_1} f(X,x)dx+\int_{x_0}^{x_2} f(X,x)dx+\int_{x_0}^{x_3} f(X,x)dx+\cdots+\int_{x_0}^{x_n} f(X,x)dx$$

relative à ~~ux~~^l'une des^ variables indéterminées, pourra s'écrire aussi

$$\text{~~X~~}\quad fx_1\,dx_1+fx_2\,dx_2+fx_3\,dx_3+\cdots+fx_n\,dx_n$$

Symbol φ squeezed in at end of line.

ce qui est une fonction symmétrique ^rationnelle^des racines de léquation finale et parconséquent une fonction rationnelle des coëfficients de la fonction ⌜φ⌝ Or on sait qu'une fonction rationnelle a ~~toujours~~ pour intégrale une fonction ~~partie~~ algebrique et logarithmique. ~~Donc~~

Le théorème est donc démontré.

~~Le~~ Remarque. Le cas où

Theory of ~~functions of the form $\int X\,dx$, X being an algebraic function of x.~~integrals whose differentials are algebraic functions

FUNDAMENTAL LEMMA. *Let X be an algebraic function of x given by the relation $F(X,x) = 0$ ~~the algebraic relation which links between X and x,~~ $f(X,x)$ an arbitrary rational function of x and X, $\varphi(X,x)$ a rational function of variable form, that is to say, with ~~indeterminate~~ coefficients considered as functions of one or several variables.*

If X is eliminated between the two equations $F(X,x) = 0$ and $\varphi(X,x) = 0$ and if $x_1,\ x_2,\ x_3,\ \ldots\ldots\ x_n$ are the roots of the final equation, one will have

$$\int_{x_0}^{x_1} f(X,x)dx + \int_{x_0}^{x_2} f(X,x)dx + \int_{x_0}^{x_3} f(X,x)dx + \cdots + \int_{x_0}^{x_n} f(X,x)dx = V\,,$$

x_0 being the origin of the integrals, which is arbitrary but constant, and V an algebraic and logarithmic function of the ~~indeterminate~~ variable coefficients.

Proof. From $F(X,x) = 0$ and $\varphi(X,x) = 0$, combined, one deduces in virtue of the fact that the function φ ~~stays~~ contains indeterminates, that X is equal to a rational function of x. ~~$X = fx$~~ Accordingly the differential of the quantity

$$\int_{x_0}^{x_1} f(X,x)dx + \int_{x_0}^{x_2} f(X,x)dx + \int_{x_0}^{x_3} f(X,x)dx + \cdots + \int_{x_0}^{x_n} f(X,x)dx$$

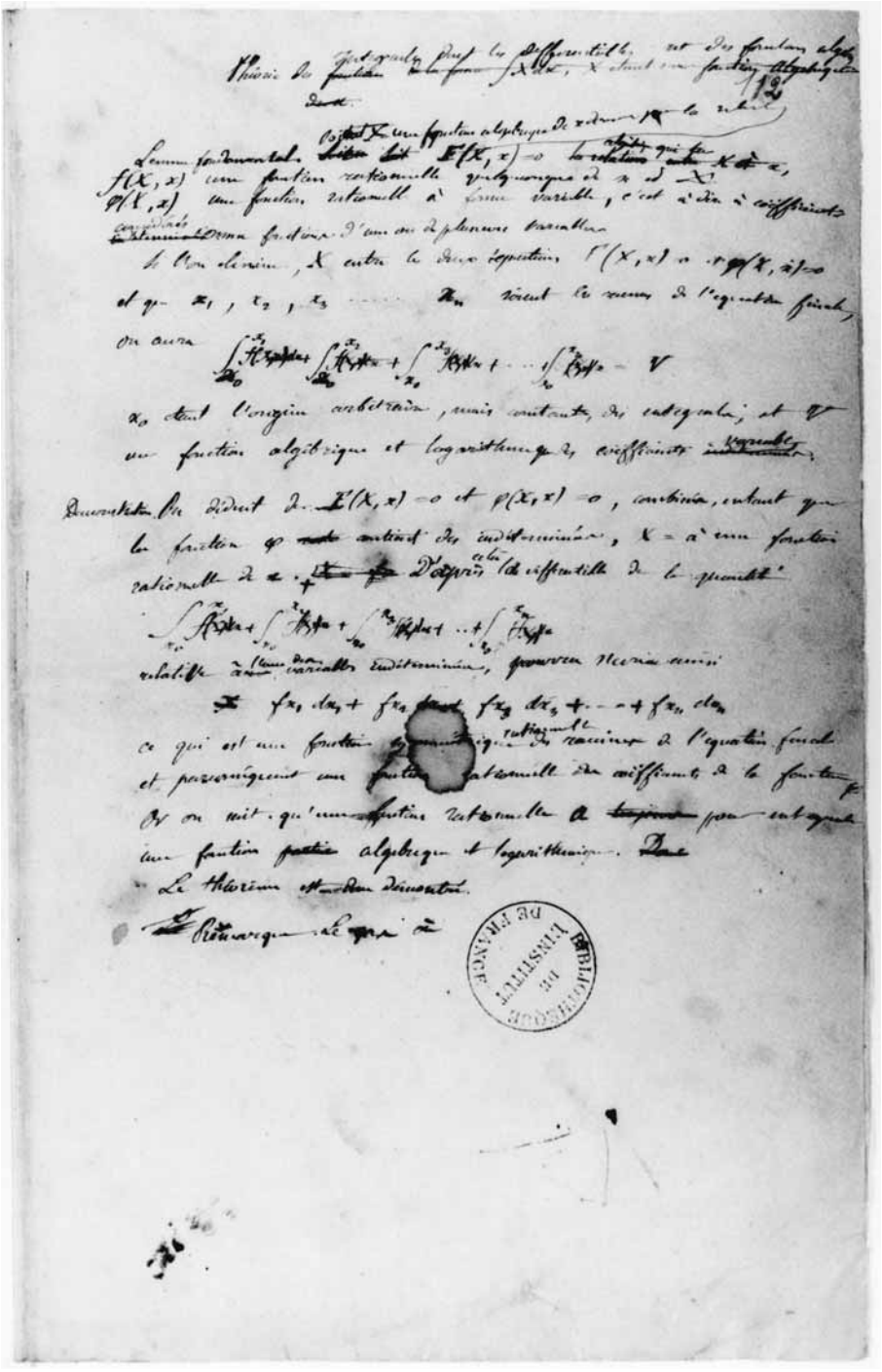
relative to one of the indeterminate variables can also be written

$$\text{~~X~~}\quad fx_1\,dx_1 + fx_2\,dx_2 + fx_3\,dx_3 + \cdots + fx_n\,dx_n\,,$$

which is a symmetric rational function of the roots of the final equation and consequently a rational function of the coefficients of the function φ. Now it is known that a rational function ~~always~~ has as its integral an algebraic and logarithmic function. ~~Therefore~~

Therefore the theorem is proved.

~~The~~ Remark. The case where



A theorem of Abel: Dossier 24, folio 112 recto.

Notes on Dossier 24

NOTE 1: The first sentence of the statement of the theorem (or fundamental lemma) took several forms. First (after two false starts with 'Soien[t]' overwritten to become 'Soit' which was then crossed out) it read

Soit $F(X, x) = 0$ la relation entre X et x,

this was changed to

Soit $F(X, x) = 0$ la relation algebrique qui lie X à x,

finally the words were crossed out leaving only the equation, which was incorporated into the new beginning 'Soient X ...' (with its ungrammatical plural) inserted above what had originally been the first line.

The two displayed formulae involving integrals have been changed by over-writing. Their left sides originally read

$$\int_{x_0}^{x_1} X\,dx + \int_{x_0}^{x_2} X\,dx + \int_{x_0}^{x_3} X\,dx + \cdots + \int_{x_0}^{x_n} X\,dx\,.$$

Moreover, the lower limit x_0 of the first two integrals in the first displayed formula is over-written onto something else, but I have failed to identify what it was.

NOTE 2: Tannery did not reproduce the material of **f.112a**. He did, however, have interesting things to say about it and I make no apology for quoting him at some length [Tannery (1907), pp. 276–278].

..., enfin une vingtaine de lignes sur la théorème d'Abel.

Ces vingt lignes peuvent être regardées comme un résumé de la célebre "Démonstration d'une propriété générale d'une certaine classe de fonctions transcendantes" [*Œuvres d'Abel*, édition Sylow, t. I, p. 515], qui est datée de 1829; elles occupent les deux tiers de la première page d'une feuille double de même format (30×15) que la lettre à Chevalier. On lit en haut de la page:

> Théorie des fonctions de la forme $\int X dx$, X étant une fonction algébrique de x

Les mots "fonctions de la. . .", jusqu'à la fin, sont biffés et Galois a écrit au-dessus

> intégrales dont les différentielles est algébrique.

The grammatical error 'est algébrique' is Tannery's. Galois wrote 'sont des fonctions algébriques'.

Le premier titre est presque identique à ceux qui ont été signalés précédemment (t. XXX, p. 242 et p. 247), dont l'un porte la mention "septembre 1831". L'énoncé du théorème d'Abel (qui n'est pas nommé) est précédé des mots "Lemme fondamental". Après la démonstration on lit

> Remarque. Dans le cas où

Le reste de la page, les deux pages qui suivent sont en blanc. Ces quelques lignes sont-elles tout ce qui reste du *troisième Mémoire qui concerne les intégrales* que Galois résume dans la lettre à Chevalier? Ce troisième Mémoire a-t-il été rédigé? Je rappelle quelques termes de la lettre

> On pourra faire avec tout cela trois Mémoires.
> Le premier est écrit, et... je le maintiens...
> ... tout ce que j'ai écrit là est depuis bientôt un an dans ma tête.

Le premier est écrit semble indiquer que les autres ne sont pas rédigés. *0n pourra faire avec tout cela trois Mémoires* porte à penser que Galois laissait des notes, dont on ne peut plus espérer aujourd'hui qu'elles soient retrouvées. Une seule chose est certaine, c'est que, la veille de sa mort, *il avait tout cela dans sa tête*.

[..., finally, some twenty lines on Abel's Theorem.

These twenty lines may be regarded as a summary of the celebrated "Démonstration d'une propriété générale d'une certaine classe de fonctions transcendantes" [*Œuvres d'Abel*, édition Sylow, t. I, p. 515], which is dated 1829; they occupy two thirds of the first page of a double leaf of the same format (30×15) as the letter to Chevalier. At the top of the page one reads:

Theory of functions of the form $\int \mathrm{X}\,dx$, X being an algebraic function of x

The words "fonctions de la. . ." to the end are deleted and Galois has written above them

> intégrales dont les différentielles est algébrique.

The first title is almost identical with those that have been noted previously (Vol. XXX, p. 242 and p. 247) [the references are to **f. 58b** and **f. 62b**, see pp. 220, 234 above], one of which carries a mention of "septembre 1831". The statement of the theorem of Abel (who is not named) is preceded by the words "Lemme fondamental". After the proof one reads

> Remarque. Dans le cas où

The rest of the page and the two pages that follow are blank. Are these few lines all that remains of the *third Memoir which concerns integrals* that Galois summarises in the letter to Chevalier? This third Memoir—was it drafted? I recall some words of the letter

> One could make three Memoirs of all this.
> The first is written and ... I stand by it ...
> ... everything that I have written here has been in my head since nearly a year ago.

The first is written seems to indicate that the others have not been drafted. *One could make three Memoirs of all this* leads one to think that Galois left some notes, of which one cannot still hope today that they may be rediscovered. One single thing is certain, and that is that on the eve of his death *he had all this in his head*.]

I find the argument convincing—indeed, if it were not that it would ring a little hollow, I could report that I had come to the same conclusion, namely that it is doubtful if the Third Memoir was ever written, and for the same reasons, long before I discovered this passage by Tannery.

NOTE 3: Bourgne would seem to disagree. In [B & A (1962), p. 518] we read

Ce fragment est la seule ébauche que nous conservions du *Troisième Mémoire*. Le titre ne laisse aucun doute, il s'agit bien là d'un début de rédaction du Mémoire signalé dans la lettre à Auguste Chevalier ("Le troisième concerne les intégrales..."), déjà annoncé dans le projet de publication de fin 1831, et daté dans le catalogue de septembre 1831, sous le titre: "Mémoire sur les fonctions de la forme $\int \mathrm{X}\,dx$, X étant une fonction algébrique quelconque de x".

Il est intéressant de noter que ce même titre se retrouve chez Abel: "Mémoire sur les fonctions transcendantes de la forme $\int y\,dx$, où y est une fonction algébrique de x". Le coïncidence peut étonner quand on sait que le Mémoire d'Abel est posthume et que le géomètre francais n'aurait pu le connaître.

[This fragment is the only sketch that we still have of the *Third Memoir*. The title leaves no doubt that what we have here is a beginning of a draft of the memoir indicated in the letter to Auguste Chevalier ("Le troisième [mémoire] concerne les integrales"), already announced in the publication project of the end of 1831 and dated in the catalogue of September 1831 under the title "Mémoire sur les fonctions de la forme $\int \mathrm{X}\,dx$, X étant une fonction algébrique quelconque de x".

f. 10 a, see p. 90 above.

f. 62 b in Dossier 10, p. 234

f. 58 b in Dossier 8, p. 220.

It is interesting to note that this same title may be found also in the work of Abel "Mémoire sur les fonctions transcendantes de la forme $\int y\,dx$, où y est une fonction algébrique de x". The coincidence may be astonishing when one knows that the memoir by Abel is posthumous and that the French geometer could not have known it.]

This paper of Abel was first published in [Abel (Œuvres), Vol. II, pp. 206–216], nearly half a century after the death of Galois. The coincidence of title is indeed astonishing. As far as content goes, however, see my next note.

NOTE 4: The paper by Abel to which Tannery refers is [Abel (1829b)]. Unlike the item published over 50 years posthumously by Sylow & Lie in [Abel (Œuvres)], it could have been, and almost certainly would have been, read by Galois, though he made no reference to it. The note by Galois, however, is written in a different language from that of Abel, and I have not understood the connection that Tannery makes, although I can see some points of similarity (such as the use of the theorem on symmetric functions of the roots of a polynomial).

I see even less clearly the connection that Bourgne makes with the posthumous paper, even although the titles are so very close. The language and the ideas sketched in the last part of the *Lettre testamentaire* seem to me to come somewhat closer to Abel's than do the language and the ideas of the note here in Dossier 24. But I am no real judge—sadly, I have not yet had the time (or perhaps the competence) to come to grips with the issues involved.

NOTE 5: The inside (**f.112b** and **f.113a**) of this pamphlet is blank; half of its back page, **f.113b** however, has been used for a calculation. The whole pamphlet was folded in two, and the calculation occupies what is now the upper half, turned clockwise through 90° so that the left side became the top and the writing now runs vertically up the page. The formulae are reported in [B & A (1962), p. 341] and will not be repeated here. Tannery summarises neatly in a second footnote to [Tannery (1907), p. 277]: the calculations are relative to the integral

$$\int \frac{dx}{\sqrt{x(x^2 - 2\alpha x + \gamma^2)(x^2 - 2\beta x + \gamma^2)}},$$

in which Galois made the substitution $x + \dfrac{\gamma^2}{x} = 2z$.

NOTE 6: Dossier 24, comprising 174 folios, is much larger than other dossiers. Apart from what is reproduced above, together with a few other pages of relatively coherent writing (folios 116, 117 on differential equations and folios 118–120 on elliptic functions [B & A (1962), pp. 402–407, 242–251]), it contains incoherent jottings and calculations. They may be seen in [Galois (2011)] and [B & A (1962)] and need no translation. But there are also a few slogans and other items. Some of these have been recorded in [B & A (1962), p. x]; others may be seen in the facsimiles of **f.179a** and **f.179b** that appear as the 14th and 9th frontispieces of [B & A (1962)] respectively. Others are missing from what Azra collected in [B & A (1962), pp. 193–361], which is otherwise a useful edition of these scraps. Here I confine myself to a few of the more interesting and mysterious names and dates.

In **f.122b**, below what is presented in [B & A (1962), p. 231], and written with the page turned clockwise through 90° relative to its present position in the bound

volume, there is the list

Fagnano
Euler
Landen
Lagrange
Legendre
Gauss

The first name has a faint line through it, possibly to cross it out, but more probably accidental. My reading of it is probable but a little insecure—it could be Fagnand or Fagnard if the last letter is not simply an o with a little flourish to it—but Giulio Fagnano is known as an originator of the theory of elliptic integrals, and this additional semantic information adds some confidence to the reading. I am grateful to Professor Adrian Rice for confirmation that the names are those of the main contributors to the theory of elliptic functions in its early years, listed in chronological order.

Also, in the midst of the material on this page, **f.122b**, with the paper turned upside-down relative to its present position (which is how it was used for most of its contents) is the name LIBRI written neatly in 'caps and small caps', like that.

At a few points there are some mysterious dates. In the middle of **f.164a**, written upside down is:

Jeudi 22 Octobre 1829

The number 22 was corrected from 26 by over-writing. Certainly 22 October 1829 was a Thursday. Although it is close to the date that Galois entered the École Préparatoire (as the École Normale was then still briefly called), and it was three days before his eighteenth birthday, it is not easy to imagine what significance this date can have had. More or less in the middle of **f.176a** there is

1er Mars 1827

while to the left of the page and on the line below are —1828, —1829, one below the other, heavily scribbled out. The lower half of **f.183b** is written, unusually, in pencil. To the right of the material recorded in [B & A (1962), p. 241], is a column of mysterious dates:

20 Mars
5 Mai
14 Juillet
28 Juillet

In 1832 these dates were Tuesday, Saturday, Saturday, Saturday; in 1831 they were Sunday, Thursday, Thursday and Thursday, respectively. Other than that 14 July is Bastille day, the day in 1831 that Galois was arrested and began his nine-month stay in prison, I have found nothing special about these dates in his life.

Chapter VII
Epilogue: myths and mysteries

This is not a book to be read from cover to cover. It is a work of reference and a sourcebook that will, I hope, be of interest and of use both to mathematicians and to historians of mathematics. If it succeeds in nothing else, I hope it may help to dispel some of the myths about Galois that are commonly held beliefs in the English-speaking world, and also to draw attention to some areas where further study might clear up a variety of mysteries about his mathematical work.

VII.1 Myths

The myth of last-minute invention. Perhaps the best known myth is that Galois created his theory of groups in the evening and the night before the morning of the duel at the end of May 1832. This myth has its source in the chapter on Galois in [Bell (1937)], distorted, exaggerated, and disseminated by other writers as chronicled in [Rothman (1982)]. From his writings, however, we see that the truth is quite different. The paper *Sur la théorie des nombres* published nearly two years earlier in June 1830 already has a paragraph (see p. 74) in which groups are mentioned. The *Second Mémoire* written in 1830 is essentially all about groups. Presumably the 1830 version of the material expounded in the *Premier Mémoire* must have dealt with groups in much the same way as they are treated there, though without the last-minute marginal addition (see p. 114 and Note 12 to Ch. IV, p. 155) explaining a little about groups of substitutions. To extrapolate as far back as the version of his theory that Galois submitted through Cauchy to the Paris Academy on 25 May and 1 June 1829 would be to go well beyond what can be supported by any extant hard evidence. Based on the circumstantial evidence provided by his later work, however, it seems fair to conjecture that already in 1829, three years before the fatal duel, groups figured large in the first version of his theory.

The myths—and mysteries—about alternating groups. A second myth, cherished widely by mathematicians and explained with great but misplaced care in popular accounts of modern algebra and Galois Theory written for a broadly educated readership, is that Galois proved (if Abel did not) that the alternating groups $\mathrm{Alt}(n)$ are simple for $n \geqslant 5$. The myth presumably comes from an assumption, not an unnatural assumption if one is familiar with mathematics but unfamiliar with its history, that Galois (and Abel) 'must have' dealt with insolubility of the general equation of degree n for $n \geqslant 5$ in much the same way that most of us do now when we teach this area of algebra to advanced undergraduate students. But it simply does

not bear examination. I challenge the reader to find any mention of anything like groups in the work of Abel. The works of Galois that gave his version of group theory to the mathematical world when they were published in 1846 contain no mention of the alternating groups (or, indeed, the symmetric groups—except in passing, as an example to illustrate Proposition I of the *Premier Mémoire* [see p. 114]). He had no need of them. The reader of his *Premier Mémoire* will see that he focusses on conditions for solubility.

Consider, for example, the condition that Galois found for an irreducible polynomial of prime degree p to be soluble. It is that the roots may be indexed as $x_0\,,\ x_1\,,\ \ldots,\ x_{p-1}$ in such a way that its group consists of (some of the) substitutions $x_k \mapsto x_{ak+b}$ where $ak + b$ is calculated modulo p and $a \not\equiv 0 \pmod{p}$. Thus the group is, in modern terms, conjugate in $\mathrm{Sym}(p)$ to a subgroup of the one-dimensional affine groups $\mathrm{AGL}(1, p)$. He proved no more than this: but of course this is a very striking and powerful result, and it was quite enough for his purposes.

It is a small step from solubility to insolubility. It is an easy consequence of what Galois proved that the groups $\mathrm{Sym}(p)$ and $\mathrm{Alt}(p)$ are insoluble for $p \geqslant 5$. Thus one might, with hindsight, view these facts as implicit in his work. But implicit is not the same as explicit if we are to be true to historical standards, and I have found no evidence that Galois made the step, small though it is.

It is a rather bigger step from insolubility to simplicity of alternating groups and I have seen no evidence that Galois made this step either. Indeed, there is very little evidence that Galois paid any attention to the symmetric and alternating groups. All that I have found is a mysterious and ghostly reference to a *groupe alterne* on **f.82a** in Dossier 15.

The groups, other than groups of prime order, that Galois explicitly recognised as being simple (*groupes indécomposables*) were the groups we now call $\mathrm{PSL}(2, p)$ or $\mathrm{L}_2(p)$ for prime numbers $p \geqslant 5$. These are the groups that consist of 2×2 matrices with entries from the field of integers modulo p, with determinant 1, and with a matrix and its negative identified. In modern terms they are the groups $\mathrm{SL}(2, p)/Z$ where Z is the subgroup consisting of $\pm\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$. Their simplicity would, as one may pretty safely conjecture, have been proved in the *Second Mémoire* had Galois got around to extending or completing it; and it is asserted in a sentence near the bottom of **f.9a** in the *Lettre testamentaire* (see p. 88).

There is a very mysterious assertion on **f.8b** of the *Lettre testamentaire*

> Le plus petit nombre de permutations que puisse avoir un groupe indécomposable quand ce nombre n'est pas premier est 5.4.3.
>
> [The smallest number of permutations which can have an indecomposable group, when this number is not prime, is 5.4.3.]

It might be tempting to think that Galois had $\mathrm{Alt}(5)$ in mind, but I doubt it. I conjecture that it was the group $\mathrm{PSL}(2, 5)$ that he was thinking of, and although, as we know

well, these are isomorphic groups, in the historical context their very different basic descriptions matter a great deal.

Why do I suggest this? For one thing, as has already been observed, apart from the one isolated and ghostly instance mentioned above, the alternating groups do not figure in Galois' work. For another, the group $\mathrm{PSL}(2, p)$ does figure in his writings, in which its order is given in factorised form as $(p+1)p(p-1)/2$, that is $6.5.4/2$ in the case of $\mathrm{PSL}(2, 5)$. Moreover, it is a group that he knew to be simple (*indécomposable*: see the end of **f.9a**) for $p \geqslant 5$.

Could Galois have known that there was no simple group of composite order less than 60? One can only speculate. He was so insightful that yes, perhaps he could have known it. Nevertheless, I very much doubt it. How could he have excluded orders such as 30, 32, 36, 40, 48, 56? With Sylow's theorems and some calculation such orders can be excluded by arguments that we now offer as exercises for students in advanced undergraduate courses. But, in spite of the crossed-out material on **f.82a** (Dossier 15), it seems unlikely that Galois had Sylow's theorems available to him. Besides, there is no hint in any of the extant manuscripts and scraps of the kind of case-by-case analysis that is needed to justify this result.

In his writings the order of a group of substitutions, that is, the number of its members, plays a smaller role than what we now call its degree, that is, the number of 'letters' being arranged to form *permutations*. I would guess that there was a little confusion in his mind as he worked intensely that evening to get his thoughts down on paper for Chevalier and posterity. He knew well that no group of degree < 5 could be simple. He knew well that the group we now call $\mathrm{PSL}(2, 5)$ was simple. He knew well that, although it arises naturally as a group of degree 6, it also happens to be representable as a group of degree 5: that $\mathrm{PSL}(2, p)$, for a prime number p, can be represented as a group of degree p for $p = 5, 7, 11$ (and not for $p > 11$) is asserted, with a sketch of a proof, on **f.9b** of the *Lettre testamentaire*. It may be that, having identified $\mathrm{PSL}(2, 5)$ as a group of substitutions of 5 letters, Galois knew that it must consist of the even substitutions—and this could explain his rendering of the order of the group in the form 5.4.3. But I doubt if he recognised the simplicity of alternating groups beyond that (or, perhaps, even cared), and I very much doubt if he could really have proved, had he been challenged, that any *groupe indécomposable* consisting of fewer than 60 substitutions must be of prime order.

VII.2 Mysteries

For all that has been written about him Galois remains a mysterious personality, an enigma. Although psychological analysis is not to my taste, his behaviour, both in his day-to-day dealings with the people around him and in the organisation (or perhaps disorganisation) of the last years of his life invite some explanation.

The mathematically unproductive last two years. Why was it, for example, that, proud as he was of his mathematical achievements, Galois wrote so few of them down in the last two years of his life? In the opening sentence of the *avertissement* [preface, foreword] with which [Liouville (1846)] begins, Liouville drew attention to this mathematically sterile time:

> Le géomètre ingénieux et profond, dont nous donnons ici les œuvres, est mort ayant vingt ans à peine; et encore a-t-il dépensé stérilement, dans les agitations de la politique, au milieu des clubs ou sous les verrous de Sainte-Pélagie, la plus grande partie des deux dernières années d'une vie si courte.
>
> [The ingenious and profound geometer whose work we give here died hardly twenty years of age; moreover, he had spent the greater part of the last years of so short a life unproductively in political agitation, amidst political associations or behind the prison walls of Sainte-Pélagie.]

The *Lettre testamentaire* bears witness to the fertility of his mind and very strongly suggests that he had created much mathematics that goes a long way beyond what he had published and what he had drafted in the *Premier Mémoire*. One can—and many writers do—build an edifice of conjecture on the brief sketch that he gave there. Much, though not all, of what he wrote about primitive soluble equations (or groups) and about the groups now known as $\mathrm{PSL}(2, p)$ is supported in part in the *Second Mémoire*. The rest of the edifice remains speculative, supported by little in the extant manuscripts. The tantalising phrase *théorie de l'ambiguité*, for example, appears only on **f. 11 a** of the *Lettre testamentaire* (see p. 94). There is nothing beyond the context (that of abelian integrals and functions) and the following sentence to tell us what Galois had in mind. And those tell us little. I have seen speculation that Galois had in mind a theory like that of Riemann surfaces, but I find that doubtful; for me it does not chime with the sentence following the phrase in question. Had there been anything in the manuscripts related to this in the same way that the *Second Mémoire* relates to the middle part of the *Lettre*, had the word *ambiguïté* (correctly spelled or not) appeared on a scrap somewhere (other than on **f. 88 b** in Dossier 16, where, however, the context is completely different), it might have been possible to work out what Galois had in mind. As it is, the edifice of conjecture, as I have said, remains speculative.

The silent decade after Galois' death. The silent years 1833–1842 provide another puzzle. There is evidence that Auguste Chevalier was conscientious in carrying the burden of literary executor that Galois had laid upon him in the last paragraphs of the *Lettre testamentaire*. He had that letter published in the *Revue Encyclopédique* in September 1832; he published an obituary in the same magazine in November 1832; he made copies of several of the manuscripts; and ultimately he passed all the material on to Joseph Liouville. This conscientiousness is *prima facie* circumstantial evidence that Chevalier would have copied the testamentary letter to Gauss and to Jacobi as Galois had requested. That Gauss and Jacobi found themselves too busy

with other matters, or as mystified as Poisson had been, is easy to imagine. No reaction from them; no reaction from anyone else, except for a passing (and rather negative) reference on p. 345 of the 1835 edition of [Lacroix (1800/1835)]. The work that Galois had published in his lifetime and the letter that Chevalier had published in September 1832 attracted no attention of any kind.

But perhaps a greater puzzle is not so much the silence as its sudden end. In her charmingly titled paper 'La naissance posthume d'Évariste Galois' [The posthumous birth of Evariste Galois] [Ehrhardt (2010b)] Caroline Ehrhardt discusses in detail the context of the publication of Galois' *Œuvres*. She situates it in a time of renewal of interest in algebra in France in the 1840s, and of Liouville's part in it. Her discussion covers many historical and sociological aspects of it, but leaves unanswered some rather different—though admittedly more trivial—questions. How was it that after more than a decade of inattention Chevalier was able in 1843 to interest Liouville in Galois' papers? Had he tried earlier and failed? What role did Alfred Galois play? Did he aid and abet Chevalier in those years? Or was it Alfred Galois rather than Auguste Chevalier to whom is owed the sudden renewal of interest in Galois and his work? Perhaps: although his name does not appear in the story until December 1846, more than three years after Liouville announced to the Académie des Sciences that Chevalier had given him the Galois papers. In the *avertissement* cited above Liouville wrote

> Lorsque, cédant au vœu des amis d'Évariste, je me suis livré, pour ainsi dire sous les yeux de son frère [a footnote: Alfred Galois], à l'étude attentive de toutes les pièces imprimées ou manuscrites qu'il a laissées, ...
>
> ... ...
>
> ... Quant à nous, qui n'avons ni connu, ni même jamais vu ce malheureux jeune homme, ... les observations que nous pourrons nous permettre en poubliant ses œuvres sous l'inspiration de sa famille, ne porteront que sur les mathématiques.
>
> [When, acceding to the wishes of friends of Galois, I gave myself over, so to speak under the eyes of his brother, to the attentive study of all the published or manuscript pieces that he had written, ...
>
> ... ...
>
> ... As for us, who have neither known nor even ever seen this unhappy young man, ... the observations that we may permit ourselves on publishing his works under the inspiration of his family are relevant only to the mathematics.]

These phrases from [Liouville (1846), pp. 382–384] indicate pretty strongly that Alfred Galois was involved. Who else? Who were the friends of Galois mentioned in the first line? Presumably one of them should have been Chevalier. In [Dupuy (1896), p. 206], the names Flaugergues, Ludovic Lalanne and Léon Lalanne appear explicitly. Beyond this, what can we ever hope to know?

In the case of Alfred Galois there is other evidence of an association with Liouville. It appears somewhat mysteriously in a document at the start of Dossier 27. On the cover sheet of the dossier the contents are described as 'Manuscrits de Joseph Liouville trouvés avec les papiers de Galois' [Manuscripts of Joseph Liouville found with the papers of Galois]. The first two pages, **f. 252 a** and **f. 253 a**, contain a much-amended draft of a letter. At the top of **f. 252 a** one reads

Lettre d'Alfred Galois à M. Jacobi, 17 9bre 1847.

[Letter from Alfred Galois to Mr Jacobi, 17 November 1847]

The handwriting is identified in [Tannery (1907), p. 275] as that of Liouville. I trust Tannery much more than I trust myself, but some of the emendations seem to me to be in a different hand. And why would Alfred Galois not draft his own letters?

With its many deletions and emendations suppressed, the draft, which appears in [Tannery (1907), p. 275] and [B & A (1962), p. 521], reads as follows:

Monsieur,

J'ai l'honneur de vous envoyer, en vous priant d'en agréer l'hommage, un exemplaire de la première partie des œuvres mathématiques de mon frère.

Il y a près d'un an qu'elle a paru dans le Journal de M. Liouville, et si je ne vous l'ai pas adressée plus tôt, c'est que, sans cesse, j'esperais pouvoir vous faire remettre d'un jour à l'autre l'ouvrage complet, dont la publication s'est trouvée retardée par diverses circonstances. Au reste cette première partie renferme ce que mon pauvre Evariste a laissé de plus importante, et nous n'avons guère à y ajouter que quelques fragments arrachés au désordre de ses papiers. Ainsi on n'a rien retrouvé concernant la théorie des fonctions elliptiques et abéliennes; on voit seulement qu'il s'était livré la plume à la main à une étude approfondie de vos ouvrages.

Quant à la théorie des équations, M. Liouville et des autres géomètres que j'ai consultés m'affirment que son mémoire si durement repoussé par M. Poisson, contient les bases d'une doctrine très féconde et une première application importante de cette doctrine. Ce travail, me disent-ils, assure pour toujours une place à votre frère dans l'histoire des mathématiques. Malheureusement étranger à ces matières, j'écoute avec plaisir de telles paroles: si votre suffrage (qu'Evariste aurait ambitionné par dessus tout) venait les confirmer, ce serait pour ma mère et pour moi une bien grande consolation; il deviendrait pour notre Evariste un gage d'immortalité; et je croirais enfin, que mon frère n'est pas entré tout entier dans la tombe. &&c.

Quotation marks in T1906/7, BA1962, but none in the manuscript.

Word 'enfin' missing from T1906/7, BA1962.

[Sir,

I have the honour to send you, asking you to accept the compliment, an offprint of the first part of the mathematical works of my brother.

> It was nearly a year ago that it appeared in Liouville's Journal, and that I did not write to you earlier was because day by day I continually hoped to be able to send the complete works, of which the publication has been delayed by various circumstances. This first part, however, contains what is most important in what my poor Evariste has left, and we have hardly more to add than a few fragments torn [or extracted] from the disorder of his papers. Thus nothing has been recovered concerning the theory of elliptic and abelian functions; all one sees is that he had given himself over, pen in hand, to a deep study of your works.
>
> As for the theory of equations, Mr Liouville and other geometers whom I have consulted assure me that his memoir that was so harshly rebuffed by Mr Poisson contains the foundations of a very fruitful doctrine [theory] and an important first application of this theory. 'This work', they tell me, 'will forever assure for your brother a place in the history of mathematics.' Unfortunately a stranger to these matters, I hear such words with pleasure: if your approval (which Evariste would have aspired to above all) came to confirm them, it would be a truly great consolation for my mother and for me; it would become for our Evariste a testimony of immortality; and I would at last believe that my brother has not entirely entered the grave. etc., etc.]

Did Alfred Galois make a fair copy of this letter? Did Jacobi receive it? Is there any possibility that it might come to light one day in his *Nachlaß*? That seems unlikely after so long a time but, as in the case of copies by Chevalier of the *Lettre testamentaire* (see Note 2 to Ch III, p. 101), one can always dream.

A year later the article [Anon (1848)] appeared in the *Magasin Pittoresque*. It hardly mentions Chevalier. His role as literary executor, as biographer, and as the agent through whom Liouville acquired the Galois papers, is passed over in silence. There are, however, two references indicating some sort of campaign by Alfred Galois. The first is the caption to the sketch he drew for that article (see page 3):

> Évariste Galois, mort âgé de vingt et un ans, en 1832. — Ce portrait reproduit aussi exactement que possible l'expression de la figure d'Évariste Galois. Le dessin est dû à M. Alfred Galois, qui depuis seize ans a voué un véritable culte à la mémoire de de son malheureux frère.
>
> [Évariste Galois, who died in 1832, at twenty-one years of age. This likeness represents as accurately as possible the facial expression of Évariste Galois. The drawing is due to Mr Alfred Galois, who for sixteen years has worshipped the memory of his unfortunate brother.]

The second appears in the first sentence of the final paragraph of the article:

> Heureusement pour sa mémoire, la pieuse persévérance d'un frère lui vaut une réhabilitation aussi complète que pouvait le permettre l'état des notes et des papiers que l'on a recueillis après sa mort.

> [Happily for his memory the pious perseverance of a brother has merited for him a rehabilitation as complete as the state of the notes and papers that were collected after his death could permit.]

I find it puzzling how completely Alfred Galois seems to take over from Chevalier in the story. Had they perhaps fallen out with each other? Unless documentary evidence comes to light we shall never know.

Sequencing and dating the manuscripts. From the observations of Jules Tannery and of Robert Bourgne we have a fair overall picture of the dating of the main manuscript items. To understand the development of Galois' mathematical ideas it would be of great value to pin their dates down more precisely, however, and also to be able to date the scraps and jottings. What a hope! What a dream! Nevertheless, such a project should not be completely impossible. There is internal evidence provided by coincidence of notation and ideas. There should be physical evidence that analysis of the paper and the ink could provide. For the various essays, though not, of course, for the scraps and jottings, there could be evidence from paragraph indentation (see p. 31).

Sequencing the scraps and jottings, and correlating them with the main writings, has already been begun by Jean-Pierre Azra in [B & A (1962)]. There is, however, very much more to be done. This would be part and parcel of a detailed dating project. The two together would, I believe, require a major research effort, but one which should yield great and valuable rewards.

Reconstruction of missing material. With Tannery I believe that a third memoir never did exist, except as a project in Galois' head (see Notes 2, 3 to Dossier 24, esp. p. 379). It is clear, however, that other material of great interest did exist. In particular, can we make a fair conjecture as to what the lost pages preceding the *Second Mémoire* contained? We have the deleted calculations on **f. 37 a** that are ruled off with the words 'fin du mémoire' as one clue. We have the context of the missing citation on **f. 39 a** (repeated twice on **f. 39 b**) as another, though of course it is possible that this refers to quite something else. And we have the cognate material in Dossier 19 as a third clue.

VII.3 Last words

There is much to be done. Even without any of the research sketched and dreamed of above, there is more I could, and perhaps should, have written. Like Galois before me, however, *je n'ai pas le temps*.

Bibliography

An attempt, perhaps not wholly successful, has been made to give bibliographical information, especially authors' names and the titles of their works, exactly as they appeared in the original.

Articles by Galois that appeared in his lifetime

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[Galois (1830b)] E. GALOIS, 'Note sur la résolution des équations numériques', *Bulletin des Sciences Mathématiques, Physiques et Chimiques* (Férrusac), XIII (1830), 413–414 (June 1830).

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[Galois (1830d)] GALAIS, 'Notes sur quelques points d'analyse', *Annales de Mathématiques pures et appliquées* (Gergonne), XXI (1830/31), 182–183 (December 1830).

[Galois (1831)] E. G., 'Sur l'Enseignement des Sciences. Des Professeurs. Des Ouvrages. Des Examinateurs', *La Gazette des Écoles*, 2 janvier 1831. [A letter to the editor.]

Posthumous publications and editions in chronological order

[Lettre (1832)] AUGUSTE CHEVALIER (ed.) 'Travaux mathématiques d'Évariste Galois: Lettre de Galois', *Revue encyclopédique*, 55 (1832), 566–476 (September 1832).

The *Lettre testamentaire* was published with an anonymous editorial preface. Presumably this was heavily influenced by Chevalier, though not written by him since it contains the clause 'La lettre qui suit nous à été addressée par l'intermédiaire de M. Auguste Chevalier' [The letter that follows has been sent to us through the agency of Mr Auguste Chevalier].

[Liouville (1846)] J. LIOUVILLE (ed.), 'Œuvres Mathématiques d'Évariste Galois', *Journal de Mathématiques pures et appliquées* (Liouville), XI (1846), 381–444. [Often cited briefly as L1846.]

[Picard (1897)] ÉMILE PICARD (ed.) *Œuvres Mathématiques d'Évariste Galois*, Gauthiers-Villars, Paris 1897. Published under the auspices of the Société Mathématique de France. (2nd edition supplemented by an article 'Évariste Galois et la théorie des équations' by G. Verriest, Gauthier-Villars, Paris 1951.) [Sometimes cited briefly as P1897.]

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[Weisner (1929)] LOUIS WEISNER, 'Galois: On Groups and Equations and Abelian Integrals', in [Smith (1929), pp. 278–285]. An English translation of the Testamentary Letter.

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Other relevant sources in chronological order

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[Lagrange (1796/1813)] J.-L. LAGRANGE, *Théorie des fonctions analytiques = Journal de l'École polytechnique*, Vol. III, 9th cahier. Imprimerie de la République, Paris, an V (= 1796/7). Second edition, Veuve Courcier, Paris 1813.

There were later editions, but the 1813 edition is probably the one that Galois would have read.

[Lagrange (1798)] J.-L. LAGRANGE, *Traité de la Résolution des équations numériques de tous les degrés*. Duprat, Paris, An VI (22 September 1797–16 September 1798); second edition enlarged by the addition of many notes, Bachelier, Paris 1808.

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The 1835 edition is notable for mentioning (p. 345) Galois and the memoir he submitted in 1831 to the academy, quoting his theorem *pour qu'une équation irréductible de degré premier soit soluble par radicaux, il faut et il suffit que deux quelconques des racines étant connues, les autres s'en déduisent rationnellement*. It continues: 'mais ce Mémoire parut à peu près inintelligible aux Commissaires chargés de l'examiner' [but this memoir appeared almost unintelligible to the committee charged with examining it].

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Bezout (often Bézout) died in 1783. The description here 'Édition originale' is misleading to the modern eye. This popular and long-lived text had a complicated publication history. There were at least six previous editions and several later ones, a number of them published in Avignon. It was first published in four volumes 1770–72. At various times it was merged with at least one other of Bezout's successful texts and was augmented by various editors. One of the editions of 1820, 1822, 1825, 1827 is likely to have been read by Galois.

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The first edition of *Éléments de géométrie* came out in 1794; the second edition augmented with trigonometry in 1799 (an VIII). The book was hugely successful and ran to many editions. I have picked the 1823 edition as a likely one to have been read by Galois.

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This paper was reviewed at some length and in some detail in Férussac's *Bulletin*, 12 (1829), pp. 24–31. The review is initialled 'C.S.', Charles Sturm—see [Taton (1947b)].

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Volume III of *Exercices* is dated 'Paris 1844' but was published in *livraisons* that appeared irregularly from 1844 to 1846. Cauchy's paper was written in 1845 and appeared in parts issued between December 1845 and May 1846. See [Neumann (1989)] for details.

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[Serret (1866)] J.-A. SERRET, *Cours d'Algèbre Supérieure* (third edition), 2 volumes, Gauthier-Villars, Paris 1866.

The first two editions were in one volume. This edition is much more extensive—where [Serret (1849)] is 400 pages long, each of the two volumes now has over 640 pages. Galois Theory is treated in some detail in the last chapters of the second volume. Later editions were much the same as this one.

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