An Iterative Water-Filling Algorithm for Maximum Weighted Sum-Rate of Gaussian MIMO-BC

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***Abstract*—We consider the maximization of weighted rate sum in Gaussian multiple-input–multiple-output broadcast channels. This problem is motivated by optimal adaptive resource allocation policies in wireless systems with multiple antenna at the base station. In fact, under random packet arrival and transmission queues, the system stability region is achieved by maximizing a weighted rate sum with suitable weights that depend on the queue buffer sizes. Our algorithm is a generalization of the well-known Iterative Multiuser Water-Filling that maximizes the rate sum under a total transmit power constraint and inherits from the latter its simplicity. We propose also a variation on the basic algorithm that makes convergence speed very fast and essentially independent of the number of users.**

***Index Terms*—Convex optimization, iterative algorithms, stability, uplink-downlink duality, weighted rate sum.**

I. MOTIVATION

THE DOWNLINK of a wireless communication system where the base station has M antennas and the K user terminals have one antenna each is modeled by the discrete-time baseband Gaussian multiple-input–multiple-output broadcast channels (MIMO-BC), given by

$$y_k(t) = \mathbf{h}_k^H(t)\mathbf{x}(t) + z_k(t), \quad k = 1, \dots, K \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{C}^M$ denotes the transmitted signal vector at time t, $\{z_k(t)\}$ is an independent identically distributed (i.i.d.) sequence of noise random variables $\sim \mathcal{CN}(0,1)$, and $\mathbf{h}_k(t) \in \mathbb{C}^M$ denotes the channel complex gain vector of user k at time t. The input is subject to the power constraint $\mathbb{E}[|\boldsymbol{x}(t)|^2] \le P$.

The corresponding *dual* uplink channel [1], [2] is the Gaussian MIMO multiple-access channel (MAC) where K single-antenna transmitters communicate with a receiver equipped with M antennas. Namely, the received signal is given by

$$\mathbf{r}(t) = \sum_{k=1}^{K} \mathbf{h}_k(t) s_k(t) + \mathbf{w}(t) \tag{2}$$

where $s_k(t)$ is the channel symbol transmitted by user k at time t, $\{\mathbf{w}(t)\}$ is an i.i.d. sequence of noise vectors $\sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$ and the same *total* power constraint $\sum_{k=1}^{K} \mathbb{E}[|s_k(t)|^2] \le P$ is enforced.

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We assume that the channel vectors vary according to an ergodic block-fading model: in particular, $\mathbf{H}(t) = [\mathbf{h}_1(t), \dots, \mathbf{h}_K(t)]$ remains constant for intervals of duration T channel uses (referred to as a "slot" in the following), and then changes from slot to slot according to a stationary ergodic matrix-valued process. This is a simple model for the case of slow fading, that enables accurate acquisition of channel state information both at the transmitter and at the receiver for sufficiently large T. Furthermore, in the large-T regime, it is possible to operate with vanishingly small error probability at rate points inside the capacity region for given channel matrix $\mathbf{H}$ and total transmit power P. From the uplink-downlink duality [1], [2], and results of [3], and [4] on the optimality of Gaussian inputs, it follows that the capacity regions of the MIMO-BC (1) and of the MIMO-MAC (2) coincide, and are given by

$$\mathcal{C}(\mathbf{H}; P) = \left\{ \bigcup_{\sum_k p_k \le P} \mathcal{R}(\mathbf{H}; \mathbf{p}) \right\} \tag{3}$$

where

$$\mathcal{R}(\mathbf{H}; \mathbf{p}) = \left\{ \mathbf{R} \in \mathbb{R}^K : \forall \mathcal{K} \subseteq \{1, \dots, K\} \sum_{k \in \mathcal{K}} R_k \le \log\det\left(\mathbf{I} + \sum_{k \in \mathcal{K}} \mathbf{h}_k \mathbf{h}_k^H p_k\right) \right\} \tag{4}$$

is the Gaussian MIMO-MAC capacity region for fixed user transmit powers $p_1, \dots, p_K$ [5]. The *ergodic* capacity region of the MIMO-BC/MAC under the sum-power constraint P is given by [6]

$$\overline{\mathcal{C}}(P) = \left\{ \bigcup_{\mathcal{P} \in \mathcal{F}} \overline{\mathcal{R}}(\mathcal{P}) \right\} \tag{5}$$

where

$$\overline{\mathcal{R}}(\mathcal{P}) = \left\{ \mathbf{R} \in \mathbb{R}^K : \forall \mathcal{K} \subseteq \{1, \dots, K\} \sum_{k \in \mathcal{K}} R_k \le \mathbb{E}\left[\log\det\left(\mathbf{I} + \sum_{k \in \mathcal{K}} \mathbf{h}_k \mathbf{h}_k^H \mathcal{P}_k(\mathbf{H})\right)\right] \right\} \tag{6}$$

and where:

1) expectation is with respect to the stationary distribution of the channel matrix $\mathbf{H}$;

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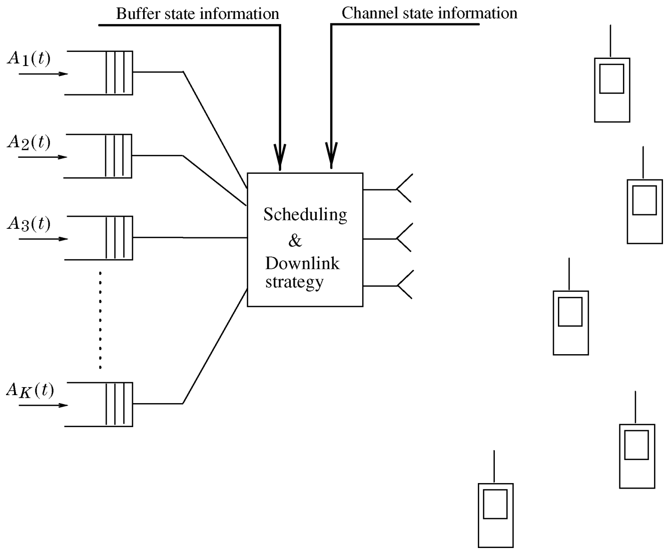

Fig. 1. Block diagram of a MIMO downlink with random arrivals and transmission queues.

2) $\mathcal{P}$ denotes a power allocation policy $\mathbf{H} \mapsto \mathbf{p}$ that maps the channel matrix into a vector of transmitted powers $\mathbf{p} = (p_1, \ldots, p_K)$ with components $\mathcal{P}_k(\mathbf{H}) = p_k$;
3) $\mathcal{F}$ denotes the feasibility set, i.e., the set of all policies $\mathcal{P}$ such that $\sum_k \mathcal{P}_k(\mathbf{H}) \le P$ for all $\mathbf{H}$;

Notice that in this paper we consider the "peak sum-power" constraint $\sum_k \mathcal{P}_k(\mathbf{H}) \le P$ for all $\mathbf{H}$, rather than the long-term average sum-power constraint $\sum_k \mathbb{E}[\mathcal{P}_k(\mathbf{H})] \le P$ since the former is more representative of a typical downlink scenario where the base station can operate always at its full peak power [7], [8].

The problem of weighted rate sum maximization is motivated by the following scenario. Assume that each user k generates random traffic with stationary ergodic arrival process $A_k(t)$: at each slot t, a packet of $A_k(t)$ bits enters the transmission queue of user k, with an average arrival rate of $\lambda_k = (1/T)\mathbb{E}[A_k(t)]$ bit/channel use. A centralized resource allocation policy observes the state of the user transmission buffers and the current channel matrix $\mathbf{H}(t)$ at the beginning of each slot t, and chooses an optimal coding strategy in order to stabilize the queue buffers, as shown in the block diagram of Fig. 1. Under the random arrival assumption, the stabilization of the transmission buffers is the single most important criterion for "fairness." In fact, the average transmission delay of user k is given by Little's theorem as $\overline{D}_k = \mathbb{E}[S_k(t)]/(\lambda_k T)$ (expressed in slots), where $S_k(t)$ denotes the buffer size of user k (expressed in bits). Hence, achieving stability guarantees a bounded average transmission delay for all users.

Under mild conditions on the joint ergodicity and stationarity of the arrivals and channel processes, the ergodic capacity region $\overline{\mathcal{C}}(P)$ coincides with the maximum stability region of the queued systems: namely, for each arrival rate vector $(\lambda_1, \ldots, \lambda_K)$ in the interior of $\overline{\mathcal{C}}(P)$ there exists a resource allocation policy that stabilizes all the queue buffers (where stability is formally defined in [9]–[12]). Conversely, for all $(\lambda_1, \ldots, \lambda_K) \notin \overline{\mathcal{C}}(P)$ the system cannot be stabilized.

Also, in [9]–[12], it is proved that for any set of arrival rates in $\overline{\mathcal{C}}(P)$ the system can be stabilized by an adaptive policy, i.e., a policy that does not know explicitly the values of $\lambda_1, \ldots, \lambda_K$. This adaptive policy (referred to as max-stability policy) is given as follows [9]–[12]: at each slot, allocate the powers $\mathbf{p}$ in order to maximize the weighted rate sum $\sum_{k=1}^K \theta_k S_k(t) R_k$ subject to $\mathbf{R} \in \mathcal{C}(\mathbf{H}(t); P)$, where $\theta_1, \ldots, \theta_K$ are arbitrary positive coefficients.[1]

These results motivate the study of efficient algorithms for the solution of the weighted rate sum maximization problem

$$\max \sum_{k=1}^K w_k R_k, \quad \text{subject to } \mathbf{R} \in \mathcal{C}(\mathbf{H}; P) \tag{7}$$

for an arbitrary set of non-negative weighting coefficients $\{w_k\}$ and given channel matrix $\mathbf{H}$.

We conclude this section by noticing the following simple and useful consequence of the uplink–downlink duality. Consider a power vector $\mathbf{p}$ satisfying $\sum_k p_k = P$ and the decoding order π (where $\pi = (\pi_1, \ldots, \pi_K)$ denotes a permutation of $\{1, \ldots, K\}$ such that user π_K is decoded first and π_1 is decoded last). With this choice, the rate K-tuple given by

$$R_{\pi_k} = \log \frac{\det\left(\mathbf{I} + \sum_{j=1}^k \mathbf{h}_{\pi_j}\mathbf{h}_{\pi_j}^H p_{\pi_j}\right)}{\det\left(\mathbf{I} + \sum_{j=1}^{k-1} \mathbf{h}_{\pi_j}\mathbf{h}_{\pi_j}^H p_{\pi_j}\right)} \tag{8}$$

for $k = 1, \ldots, K$ is achievable in the MIMO-MAC by successive decoding, and corresponds to a vertex of the dominant face of $\mathcal{R}(\mathbf{H}; \mathbf{p})$. Then, there exists a set of powers $\mathbf{q}$, also satisfying $\sum_{k=1}^K q_k = P$, achieving the same rate K-tuple (8) in the dual MIMO-BC. This rate-tuple is achievable by successive "Dirty-Paper" encoding, by encoding the users in the reverse order (user π_1 is encoded first, and user π_K is encoded last). The uplink–downlink transformation $\mathbf{p} \mapsto \mathbf{q}$ and a simple algorithm for computing the corresponding MIMO-BC transmit covariance matrix $\mathbb{E}[\mathbf{x}\mathbf{x}^H]$ can be found in [1] and shall not be repeated here for the sake of brevity.

II. Weighted Rate Sum Maximization

The key result of [12] and [13] is summarized as follows:

$$\begin{aligned}
&\max_{\mathbf{R} \in \mathcal{C}(\mathbf{H}; P)} \sum_k w_k R_k \\
&= \max_{\mathbf{p}: \sum_k p_k = P} \sum_{k=1}^K w_{\pi_k} \log \frac{\det\left(\mathbf{I} + \sum_{j=1}^k \mathbf{h}_{\pi_j}\mathbf{h}_{\pi_j}^H p_{\pi_j}\right)}{\det\left(\mathbf{I} + \sum_{j=1}^{k-1} \mathbf{h}_{\pi_j}\mathbf{h}_{\pi_j}^H p_{\pi_j}\right)}
\end{aligned} \tag{9}$$

where π is the permutation that sorts the weights in nonincreasing order

$$w_{\pi_1} \ge w_{\pi_2} \ge \cdots \ge w_{\pi_K}.$$

[1]Typically, we choose $\theta_k = 1$ for all k, but different choices of these coefficients can be used in order to give priority to certain users: generally speaking, users with larger weighting coefficients achieve a proportionally smaller average delay.

It follows that the solution of the original maximization problem (7) is always found in the set of successively decodable rate points. The decoding order π is uniquely given by the weights $\{w_k\}$, and the resulting maximization problem is convex.

Since the decoding order is fixed by the weights, without loss of generality and in the interest of notation simplicity, we can consider $\pi_k = k$, i.e., users are decoded in the order K (first), $K-1,\ldots,1$ (last). Then, we rewrite the objective function of the right-hand side (RHS) of (9) in the more convenient form

$$f(\mathbf{p}) = \sum_{k=1}^{K} \Delta_k \log\det\left(\mathbf{I} + \sum_{j=1}^{k} \mathbf{h}_j\mathbf{h}_j^H p_j\right) \tag{10}$$

where we let $\Delta_k = w_k - w_{k+1}$ and we define $w_{K+1} = 0$. Notice that by definition of the decoding order, which sorts the weights in nonincreasing order, we have $\Delta_k \geq 0$.

In order to solve the problem

$$\max f(\mathbf{p}), \quad \text{subject to } \mathbf{p} \geq \mathbf{0}, \sum_k p_k = P \tag{11}$$

we may consider the Karush–Kuhn–Tucker (KKT) conditions, which are necessary and sufficient for optimality. The Lagrangian function is given by

$$L(\mathbf{p},\boldsymbol{\nu},\mu) = f(\mathbf{p}) + \sum_{k=1}^{K}\nu_k p_k - \mu\left(\sum_k p_k - P\right) \tag{12}$$

where $\boldsymbol{\nu}$ and μ are dual variables. By differentiating with respect to p_k, we find the KKT conditions

$$\begin{aligned}
\frac{\partial f(\mathbf{p})}{\partial p_k} + \nu_k - \mu &= 0,\ 1 \leq k \leq K\\
p_k &\geq 0,\ 1 \leq k \leq K\\
P - \sum_k p_k &\geq 0\\
\nu_k &\geq 0,\ 1 \leq k \leq K\\
\mu &\geq 0.
\end{aligned} \tag{13}$$

For each k and $j \geq k$, we define the covariance matrix $\boldsymbol{\Sigma}_{k,j}(\mathbf{p}) = \mathbf{I} + \sum_{i=1,i\neq k}^{j} \mathbf{h}_i\mathbf{h}_i^H p_i$ of the interference plus noise experienced by user k, while users from K to $j+1$ have already been decoded and subtracted from the received signal, in the MIMO-MAC successive decoder. Then, it is easy to show that the KKT equality condition takes on the explicit form

$$\sum_{j=k}^{K} \Delta_j \frac{\mathbf{h}_k^H \boldsymbol{\Sigma}_{k,j}^{-1}(\mathbf{p})\mathbf{h}_k}{1 + p_k \mathbf{h}_k^H \boldsymbol{\Sigma}_{k,j}^{-1}(\mathbf{p})\mathbf{h}_k} = \mu - \nu_k,\ 1 \leq k \leq K. \tag{14}$$

Unfortunately, solving (14) with respect to $\mathbf{p}$ is generally intractable. Hence, a simple iterative algorithm is proposed and its convergence to the optimal solution of (11) is proved.

Algorithm A0: Iterative water-filling algorithm for the weighted rate sum maximization

1) Initialize $\mathbf{p}^{(0)} = \mathbf{0}$.
2) At iteration n, compute for $k = 1,\ldots,K$ and $j \geq k$ the coefficients

$$\alpha_{k,j}^{(n)} = \mathbf{h}_k^H \boldsymbol{\Sigma}_{k,j}^{-1}\left(\mathbf{p}^{(n-1)}\right)\mathbf{h}_k. \tag{15}$$

3) Water-filling step: let $\boldsymbol{\gamma}^{(n)}$ be the solution of

$$\boldsymbol{\gamma}^{(n)} = \arg\max_{\boldsymbol{\gamma}\geq\mathbf{0},\sum_k \gamma_k \leq P} \sum_{j=1}^{K}\Delta_j \sum_{k=1}^{j}\log\left(1 + \gamma_k \alpha_{k,j}^{(n)}\right). \tag{16}$$

4) Update step: let $\mathbf{p}^{(n)} = (1/K)\boldsymbol{\gamma}^{(n)} + (1-(1/K))\mathbf{p}^{(n-1)}$. □

Some observations are in order.

1) Algorithm A0 is a generalization to the case of arbitrary rate weights of the algorithm of [14], that was developed for the rate sum problem. Furthermore, [14] is a nontrivial variation of the iterative multiuser water-filling algorithm of [15] that maximizes the rate sum for the MIMO-MAC with individual (per-user) power constraints. Recently, a different approach based on dual decomposition was proposed for the maximization of the rate sum subject to a sum-power constraint [16]. Weighted rate sum maximization was also considered in [17] where a steepest-ascent gradient algorithm is proposed. A thorough complexity and convergence comparison between these algorithms is out of the scope of this paper. Here, we notice that the algorithms A0 and [14] update the power vector with weight $1/K$ for the innovation. This makes the convergence very sensitive to K and, in particular, very slow for large K (see also the numerical results in [16]). At the end of this section, we present an improvement of algorithm A0 (denoted by A0*) that achieves much faster convergence, almost insensitive to the value of K. In Section III, we compare by simulation the convergence of algorithms A0, A0*, and the gradient algorithm of [17]. Algorithm A0* is shown to provide the fastest convergence, virtually insensitive with respect to the values of K and M.
2) The "water-filling step" in algorithm A0 is a convex optimization that admits a unique solution of the water-filling type, although this solution cannot be given in a simple closed form and requires a one-dimensional (1-D) line search. The solution of the water-filling step and its uniqueness are discussed in Appendix A.
3) Problem (16) and the original problem (11) have formally the same KKT conditions, given by

$$\sum_{j=k}^{K}\Delta_j \frac{\alpha_{k,j}}{1 + p_k\alpha_{k,j}} = \mu - \nu_k \tag{17}$$

for $\alpha_{k,j} = \mathbf{h}_k^H \boldsymbol{\Sigma}_{k,j}^{-1}(\mathbf{p})\mathbf{h}_k$. Hence, the solution $\boldsymbol{\gamma}^{(n)}$ of (16) corresponds to "freezing" the terms $\alpha_{k,j}$ as if they did not depend on $\mathbf{p}$, and solving for the new $\mathbf{p}$ under this *decoupling* assumption. However, the simple intuitive updating $\mathbf{p}^{(n)} = \boldsymbol{\gamma}^{(n)}$ does not ensure convergence to the optimum, and we need the "update step" of algorithm A0 (see also algorithm A0*) in order to ensure convergence.

Proof of Convergence: This proof follows in the footsteps of the proof of convergence of [14]. Consider the new objective function defined by

$$f_{\rm mod}\left(\mathbf{p}(0), \mathbf{p}(1), \ldots, \mathbf{p}(K-1)\right) = \frac{1}{K}\sum_{\ell=0}^{K-1}\sum_{k=1}^{K}\Delta_k \times \log\det\left(\mathbf{I} + \sum_{j=1}^{k}\mathbf{h}_j\mathbf{h}_j^H p_j\left([\ell+j-1]_K\right)\right) \quad (18)$$

where $[n]_K$ denotes the integer residue of n modulo K, and $p_j(m)$ denotes the jth component of $\mathbf{p}(m)$. The function $f_{\rm mod}$ has K^2 variables, that can be arranged into the matrix $\mathbf{P} = [\mathbf{p}(0), \ldots, \mathbf{p}(K-1)]$, where each column $\mathbf{p}(m)$ has K elements.

The following properties hold.

1) For any $\mathbf{p} \in \mathbb{R}_+^K$, $f_{\rm mod}(\mathbf{p}, \mathbf{p}, \ldots, \mathbf{p}) = f(\mathbf{p})$, where $f(\mathbf{p})$ is the objective function of problem (11) defined in (10).
2) For any set of vectors $\{\mathbf{p}(m) \in \mathbb{R}_+^K : m = 0, \ldots, K-1\}$, let $\mathbf{p} = (1/K)\sum_{m=0}^{K-1}\mathbf{p}(m)$. Then,

$$f_{\rm mod}\left(\mathbf{p}(0), \mathbf{p}(1), \ldots, \mathbf{p}(K-1)\right) \le f_{\rm mod}(\mathbf{p}, \mathbf{p}, \ldots, \mathbf{p}) \quad (19)$$

where this inequality follows from the Shur-concavity of $f_{\rm mod}$ and by the fact that the argument $[\mathbf{p}, \mathbf{p}, \ldots, \mathbf{p}]$ (written as a vector of K^2 components) is majorized by $[\mathbf{p}(0), \mathbf{p}(1), \ldots, \mathbf{p}(K-1)]$ [18].

It is then clear that maximizing $f_{\rm mod}$ under the decoupled constraints

$$\mathbf{p}(m) \ge \mathbf{0}, \ \sum_{k=1}^{K} p_k(m) \le P, \quad m = 0, \ldots, K-1$$

yields the same solution $[\mathbf{p}^\star, \mathbf{p}^\star, \ldots, \mathbf{p}^\star]$ and objective function value $f(\mathbf{p}^\star)$ of the original problem (11).

It can be proved (see [14] and references therein) that the following *cyclic coordinate ascent algorithm* converges globally to the optimum value $\mathbf{p}^\star$.

Algorithm A1: Cyclic coordinate ascent

1) Choose an arbitrary feasible initial point $\mathbf{P}^{(0)} \in \mathbb{R}_+^{K\times K}$ such that $\sum_k p_k(m) \le P$ for all $m = 0, \ldots, K-1$.
2) At iteration n, let $\boldsymbol{\gamma}^{(n)}$ be the solution of the maximization of
$$f_{\rm mod}\left(\mathbf{p}^{(n-1)}(0), \mathbf{p}^{(n-1)}(1), \ldots, \mathbf{p}^{(n-1)}(K-2), \boldsymbol{\gamma}\right)$$
subject to $\boldsymbol{\gamma} \ge \mathbf{0}$, $\sum_k \gamma_k \le P$.
3) Cyclic right shift update: let
$$\mathbf{P}^{(n)} = \left[\boldsymbol{\gamma}^{(n)}, \mathbf{p}^{(n-1)}(0), \ldots, \mathbf{p}^{(n-1)}(K-2)\right].$$

□

At each iteration of the above algorithm, the last column of the current $\mathbf{P}^{(n-1)}$ matrix is updated by solving a maximization problem, where the other columns are kept fixed.

The last step of the proof consists of obtaining a modified algorithm A2 from algorithm A1 by introducing in each iteration a step that increases the objective function. It follows from the global convergence of A1 and from the spacer step theorem of [19, Ch. 7.11] that A2 also converges to the optimum value. Finally, we shall recognize by inspection that A2 coincides with the proposed algorithm A0.

Algorithm A2

1) Choose the all-zero matrix $\mathbf{P}^{(0)} = \mathbf{0}$.
2) At iteration n, let $\boldsymbol{\gamma}^{(n)}$ be the solution of the maximization of
$$f_{\rm mod}\left(\mathbf{p}^{(n-1)}(0), \mathbf{p}^{(n-1)}(1), \ldots, \mathbf{p}^{(n-1)}(K-2), \boldsymbol{\gamma}\right)$$
subject to $\boldsymbol{\gamma} \ge \mathbf{0}$, $\sum_k \gamma_k \le P$.
3) Averaging update: let
$$\mathbf{p}^{(n)} = \frac{1}{K}\left(\boldsymbol{\gamma}^{(n)} + \sum_{m=0}^{K-2}\mathbf{p}^{(n-1)}(m)\right)$$
and let $\mathbf{P}^{(n)} = [\mathbf{p}^{(n)}, \mathbf{p}^{(n)}, \ldots, \mathbf{p}^{(n)}]$. □

Notice that at each step, $\mathbf{p}^{(n)}(0) = \mathbf{p}^{(n)}(1) = \cdots = \mathbf{p}^{(n)}(K-1) = \mathbf{p}^{(n)}$. Therefore, the objective function value at step n is equal to $f(\mathbf{p}^{(n)})$. Furthermore, for the Shur-concavity of $f_{\rm mod}$, we have that

$$f_{\rm mod}\left(\boldsymbol{\gamma}^{(n)}, \mathbf{p}^{(n-1)}(0), \ldots, \mathbf{p}^{(n-1)}(K-2)\right) \le f\left(\mathbf{p}^{(n)}\right)$$

that is, the averaging update increases the value of the objective function. Finally, by inspection we see that solving for $\boldsymbol{\gamma}^{(n)}$ in A2 and solving for $\boldsymbol{\gamma}^{(n)}$ in A0 are identical, and the update step is the same. Therefore, the sequence of values $(\mathbf{p}^{(n)}, f(\mathbf{p}^{(n)}))$ produced by A0 and the sequence of values $(\mathbf{p}^{(n)}, f_{\rm mod}(\mathbf{p}^{(n)}, \mathbf{p}^{(n)}, \ldots, \mathbf{p}^{(n)}))$ produced by A2 are also identical. This concludes the proof. □

Improving the Convergence Speed: The key step in the proof of convergence is to find an update that yields a power vector $\mathbf{p}^{(n)}$ function of $\boldsymbol{\gamma}$ and $\mathbf{p}^{(n-1)}$ and such that

$$f_{\rm mod}\left(\boldsymbol{\gamma}^{(n)}, \mathbf{p}^{(n-1)}, \ldots, \mathbf{p}^{(n-1)}\right) \le f_{\rm mod}\left(\mathbf{p}^{(n)}, \mathbf{p}^{(n)}, \ldots, \mathbf{p}^{(n)}\right).$$

While the updating step $\mathbf{p}^{(n)} = (1/K)\boldsymbol{\gamma}^{(n)} + (1 - (1/K))\mathbf{p}^{(n-1)}$ is motivated by averaging, that allows us to use majorization and Schur-concavity, it is clear that the following update yields at least as large improvement of the objective function, at the cost of a simple line search

$$\mathbf{p}^{(n)} = \max_{\beta\in[1/K,1]}\left\{\beta\boldsymbol{\gamma}^{(n)} + (1-\beta)\mathbf{p}^{(n-1)}\right\}. \quad (20)$$

Algorithm A0 with update (20) is referred to as algorithm A0*, it is provably convergent, and experimentally we have observed much faster convergence especially for large K.

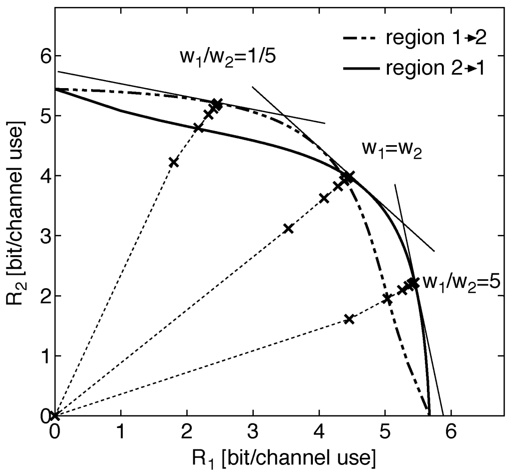

Fig. 2. Two-user rate region and trajectory of algorithm A0 for different weights.

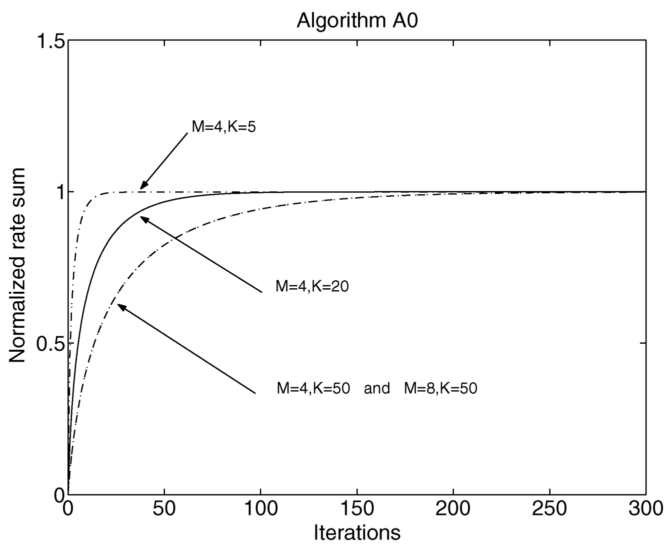

Fig. 3. Convergence of algorithm A0 for $P = 10$ dB.

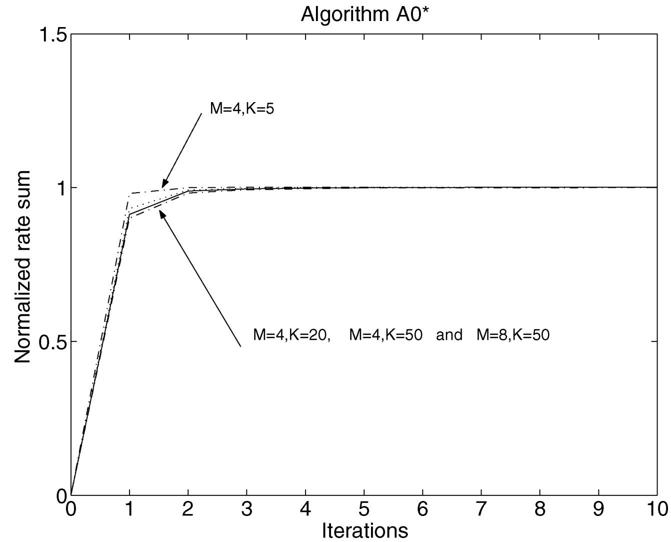

Fig. 4. Convergence of algorithm $\mathrm{A0}^\star$ for $P = 10$ dB.

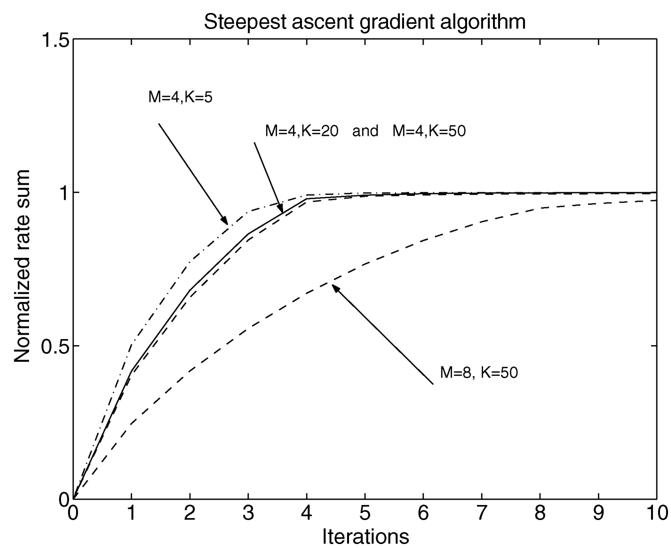

Fig. 5. Convergence of the gradient algorithm of [17] for $P = 10$ dB.

III. Numerical Results

In this section, we provide some numerical examples to show the behavior of the proposed algorithm. Fig. 2 shows the region $\mathcal{C}(\mathbf{H}; P)$ for $M = K = 2$ when we have

$$\mathbf{H} = \begin{bmatrix} 2 & -0.5 \\ -1 & 2 \end{bmatrix}$$

and $P = 10$. The solid line shows the rate region for the decoding order 2,1 while the dashed line shows the rate region for the order 1,2. The region $\mathcal{C}(\mathbf{H}; P)$ is given by the convex hull of these regions. By cross marks, we show the convergence of algorithm A0 for different weights, $w_1/w_2 = 1/5$, 1, and 5.

Figs. 3–5 show the evolution of the objective function versus the number of iterations for algorithms A0, $\mathrm{A0}^\star$, and the gradient algorithm of [17], for equal weights (rate sum) and some choices of K and M. The fading channel is randomly generated with i.i.d. components $\sim \mathcal{CN}(0,1)$ and the objective function value is normalized so that the final value is 1. The curves are averaged over a large number of channel realizations. As anticipated in Section II, algorithm A0 (that in this case coincides with the algorithm of [14]) suffers from slow convergence for large K. Algorithm $\mathrm{A0}^\star$ converges very rapidly and it is almost insensitive with respect to the values of K and M. The gradient algorithm of [17] is also very competitive, although it is slightly more sensitive to the value of K and, above all, it is quite sensitive to the value of M. This fact can be explained intuitively by considering that, for $K \gg M$, the number of "active" users (those who are allocated positive power at the maximum rate sum point) is between M and $2M$ (usually, it is close to $2M$) [20]. The gradient algorithm of [17] updates one user per iteration. Hence, it converges more slowly when the number of "active" users is large.

Finally, we evaluate the average delay performance of the max-stability scheduling policy by using algorithm $\mathrm{A0}^\star$. We consider mutually independent arrival processes such that $A_k(t) = \sum_{j=1}^{M_k(t)} b_{k,j}(t)$, where $M_k(t)$ is an i.i.d. Poisson distributed sequence that counts the number of packets arrived

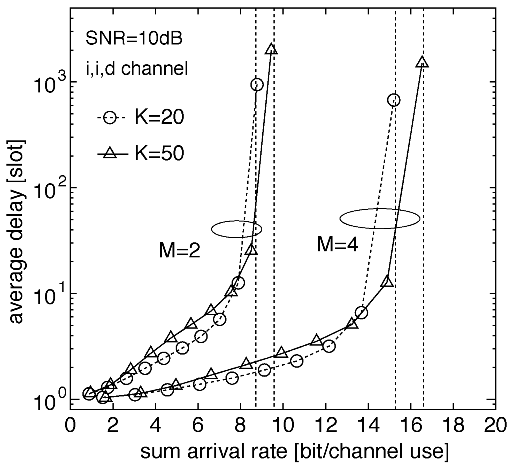

Fig. 6. Average delay performance of the max-stability policy.

to the kth buffer at the beginning of slot t and $\{b_{k,j}(t)\}$ are i.i.d. exponentially distributed random variables expressing the number of bits per packet. We take $\mathbb{E}[b_{k,j}(t)] = T$ ($T = 2000$ in our simulations), so that λ_k coincides with the average number of packets arrived in a slot (T channel uses). The fading channel is assumed to vary in an i.i.d. manner from one slot to another and has i.i.d. components $\sim \mathcal{CN}(0,1)$. Under symmetric traffic condition $\lambda_k = \lambda_{\rm sum}/K$, $\forall k$, the average delay (averaged over users and time) is given by $\sum_k \overline{D}_k/K = \sum_k \mathbb{E}[S_k(t)]/(T\lambda_{\rm sum})$ expressed in slot. Fig. 6 shows the average delay versus the sum arrival rate $\lambda_{\rm sum}$ for different systems with $M = 2$, 4 and $K = 20$, 50 users. We can observe that the max-stability policy makes the queues bounded until the arrival rate reaches the stability region boundary. Under symmetric arrival traffic, the boundary point corresponds to the maximum sum-rate point. For a fixed number of antennas, the system with $K = 50$ users remains stable for a larger arrival rate than the system with $K = 20$ users due to multiuser diversity gain. For a fixed number of users, increasing the number of antennas (i.e., allowing for multiplexing gain) yields a significant decrease of the average transmission delay.

IV. Conclusion

A simple iterative water-filling algorithm for the weighted rate sum maximization in the Gaussian MIMO-BC (or MIMO-MAC under sum-power constraint) was proposed. This algorithm handles the case of M base station antennas and user terminals with a single antenna. The case of user terminals with multiple antennas cannot be obtained as a *simple* generalization of our method since the "water-filling step" for multiantenna receivers yields a convex optimization problem that cannot be solved by a simple line search as in the scalar case. The dual-decomposition approach of [16] incurs the same problem, and one has to resort to the steepest ascent algorithm of [17]. A simple alternative consists of treating the multiple antennas at the receivers as single-antenna virtual users, at the price of a small suboptimality [21].

The weighted rate sum maximization problem arises in the max-stability policy, that adaptively stabilizes the transmission queue buffers of all users under random packet arrival, whenever the K-tuple of arrival rates lies inside the maximum stability region, which coincides with the ergodic capacity region of the MIMO-MAC and MIMO-BC. Due to its simplicity and fast convergence, our algorithm can be incorporated as the key computational step in the max-stability policy.

As another possible application, we may mention the evaluation of points on the boundary of the ergodic capacity region of the MIMO-BC. In fact, although the ergodic capacity region is expressed implicitly by (5), its explicit characterization is difficult. There are at least two ways to find points on the boundary. The first method consists of fixing the weights $\{w_k\}$, running the weighted rate sum maximization for randomly generated matrices $\mathbf{H}$, and computing (by Monte Carlo averaging) the rates $\mathbf{R}(w_1,\ldots,w_K;P) = \mathbb{E}[\mathbf{R}(w_1,\ldots,w_K;\mathbf{H},P)]$, where $\mathbf{R}(w_1,\ldots,w_K;\mathbf{H},P)$ is the optimal point found by our algorithm for given $\{w_k\}$, $\mathbf{H}$, and P. The second method consists of applying the max-stability policy to a queued system with arrival rates $\lambda_k = \lambda\theta_k$, for some desired direction vector $(\theta_1,\ldots,\theta_K)$ such that $\theta_k > 0$ and $\sum_k \theta_k = 1$. By increasing λ in small steps, one can determine within a desired approximation the value $\lambda^\star$ beyond which the average buffer size becomes unbounded. Then, the rate point with components $R_k = \theta_k\lambda^\star$ lies (approximately) in the intersection of the ergodic capacity region boundary with the line passing through the origin and pointing in the direction indicated by $(\theta_1,\ldots,\theta_K)$. This method appears more complex, but it has the advantage of finding points on the boundary in given desired directions.

Appendix

A. The Water-Filling Step

By treating the terms $\alpha_{k,j} = \mathbf{h}_k^H\mathbf{\Sigma}_{k,j}^{-1}(\mathbf{p})\mathbf{h}_k$ as fixed, we obtain the optimization problem in (16). The KKT equality condition for this problem is given by (17) that coincides with that in (14). For $\mathbf{p} \geq \mathbf{0}$, the optimal point should satisfy the K equalities simultaneously (where $\nu_k > 0$ for the components $p_k = 0$). Assuming $p_k > 0$, the kth equality equation is given by

$$\sum_{j=k}^{K} \frac{\Delta_j}{p_k + 1/\alpha_{k,j}} = \mu \tag{21}$$

where $\mu \geq 0$ can be interpreted as a water level that must be the same for all users. Hence, we can regard (21) as a generalized water-filling problem. Let $g_k(p_k)$ denote the left-hand side (LHS) of (21) for $p_k \geq 0$. Given a set of weights $\{\Delta_j \geq 0\}$ and coefficients $\{\alpha_{k,j} > 0\}$, $g_k(p_k)$ is a monotonically decreasing positive function of p_k for any k. We define the inverse function of $g_k(p_k) = \mu$ such that $g_k^{-1}(\mu) = p_k$ for $p_k, \mu \geq 0$. The total power required for a given water level $\mu \geq 0$ is given by

$$P(\mu) = \sum_k g_k^{-1}(\mu)$$

In order to solve for $\mathbf{p}$, we have to find the water level $\mu^\star$ satisfying the total power constraint, i.e., $P(\mu^\star) = P$. Fig. 7 plots the functions $g_k(p_k)$ for a three-user system and the corresponding water level $\mu^\star$. Because $P(\mu)$ is a monotonically decreasing function of μ for $0 \leq \mu \leq \max_k g_k(0)$, the water level $\mu^\star$ satisfying the total power constraint is unique. This means that the KKT equality condition in (21) has a unique solution $\mathbf{p}$. By using the monotonicity of the function $P(\mu)$, we can search for

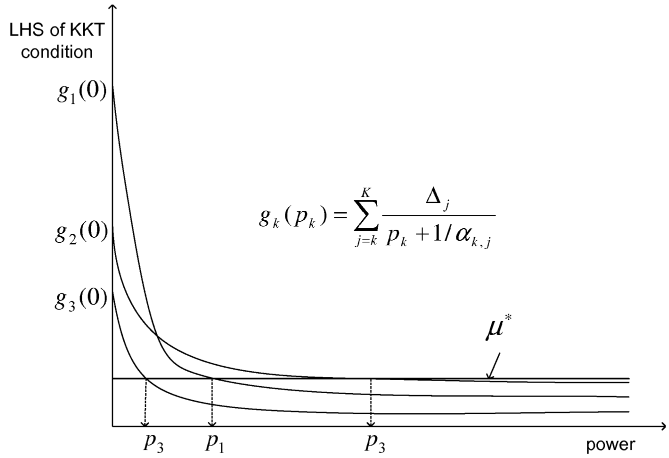

Fig. 7. The water level μ satisfying the total power constraint for $K = 3$.

the water level by any efficient 1-D line search algorithm (e.g., by the bisection method).

The following algorithm achieves this goal.

1) Sort the K users such that

$$g_{\rho_1}(0) \geq g_{\rho_2}(0) \geq \cdots \geq g_{\rho_K}(0) \geq 0$$

where ρ denotes the sorting permutation.

2) Find the interval k such that $P(g_{\rho_k}(0)) \leq P \leq P(g_{\rho_{k+1}}(0))$ (we let $g_{K+1}(0) = 0$).

3) Find $\mu^\star$ by the bisection method where the initial search range of μ is $[g_{\rho_{k+1}}(0), g_{\rho_k}(0)]$, until the desired accuracy is reached.

The users for which $g_k(0) \leq \mu^\star$ are given zero power.

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