Functions on Euclidean Space

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1 Norm and Inner Product

1.1 Euclidean n-space $\mathbf{R^n}$

$\mathbf{R^n}$ is the set of all vectors or n-tuples

$$\vec{x} = \langle x_1, ..., x_n \rangle = \begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix}$$

where x_i is a 1-tuple in $\mathbf{R} = \mathbf{R}$.

1.2 Vectors

Addition of vectors:

$$\vec{x} + \vec{y} = \langle x_1 + y_1, ..., x_n + y_n \rangle = \begin{bmatrix} x_1 + y_1 \\ ... \\ x_n + y_n \end{bmatrix}$$

Scalar multiple of a vector:

$$a\vec{x} = \langle ax_1, ..., ax_n \rangle = \begin{bmatrix} ax_1 \\ ... \\ ax_n \end{bmatrix}$$

Norm (magnitude) of a vector:

$$\|\vec{x}\| = \sqrt{x_1^2 +, ..., +x_n^2}$$

1.2.1 Properties of Vectors

1. $\|\vec{x}\| \geq 0$, and $\|\vec{x}\| = 0$ if and only if $\vec{x} = \vec{0}$.
2. $|\sum_{i=1}^{n} x_i y_i| \leq \|\vec{x}\| \cdot \|\vec{y}\|$; equality holds if and only if $\vec{x}$ and $\vec{y}$ are linearly dependent, meaning $\vec{x} = \lambda\vec{y}$. (The value of the inner/dot product is $\leq$ the product of the magnitudes.)
3. $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$. (Triangle Inequality)
4. $\|a\vec{x}\| = |a| \cdot \|\vec{x}\|$.

1.3 Inner Product

$$\langle \vec{x}, \vec{y} \rangle = \vec{x} \bullet \vec{y} = \sum_{i=1}^{n} x_i y_i = \begin{bmatrix} x_i & ... & x_n \end{bmatrix} \begin{bmatrix} y_i \\ ... \\ y_n \end{bmatrix}$$

1.3.1 Inner Product Properties

1.

$$\vec{x} \bullet \vec{y} = \vec{y} \bullet \vec{x} \qquad \text{(symmetry).}$$

2.

$$\begin{aligned} (a\vec{x}) \bullet \vec{y} = \vec{x} \bullet (a\vec{y}) &= a\,(\vec{x} \bullet \vec{y}) && \text{(bilinearity).} \\ (\vec{x_1} + \vec{x_2}) \bullet \vec{y} &= (\vec{x_1} \bullet \vec{y}) + (\vec{x_2} \bullet \vec{y}) && \text{(distributive).} \end{aligned}$$

3. $\vec{x} \bullet \vec{x} \geq 0$, and $\vec{x} \bullet \vec{x} = 0$ if and only if $\vec{x} = \vec{0}$ (positive definiteness).

4.

$$\|\vec{x}\| = \sqrt{\vec{x} \bullet \vec{x}}.$$
$$\|\vec{x}\|^2 = \vec{x} \bullet \vec{x}.$$

5.

$$\vec{x} \bullet \vec{y} = \frac{\|\vec{x} + \vec{y}\|^2 - \|\vec{x} - \vec{y}\|^2}{4} \qquad \text{(polarization identity).}$$

1.4 Notation Remarks

- The zero vector $\vec{0} = \langle 0, ..., 0 \rangle$.
- The usual basis of $\mathbf{R^n}$ is $\vec{e_1}, ..., \vec{e_n}$ where

  $$e_i = \langle 0, ..., 1, ..., 0 \rangle$$

  and the 1 is at the ith position in e_i.