Produt of the Sum of Halving Powers

Carl Schader

November 3, 2021

Here is something fun I discovered the other day when thinking about how there isn't a simple formula for the sum of squares $a^2 + b^2$ like there is for $a^2 - b^2 = (a + b)(a - b)$.

$$\prod_{i=1}^{n} \left(a^{k/2^{i}} + b^{k/2^{i}} \right) = \frac{a^{k} - b^{k}}{a^{k/2^{n}} - b^{k/2^{n}}}$$

In a search for an equation for $a^2 + b^2$, I started with $a^2 - b^2 = (a+b)(a-b)$ and realized I can raise the powers on the left side from two to four to achieve an equation.

$$a^{2} - b^{2} = (a+b)(a-b)$$
$$a^{4} - b^{4} = (a^{2} + b^{2})(a^{2} - b^{2})$$

This allows me to form an equation for $a^2 + b^2$.

$$a^{4} - b^{4} = (a^{2} + b^{2})(a^{2} - b^{2})$$
$$a^{2} + b^{2} = \frac{a^{4} - b^{4}}{a^{2} - b^{2}}$$

This can be generalized to any power not just the sum of squares.

$$k \in \mathbb{R}$$
$$a^k + b^k = \frac{a^{2k} - b^{2k}}{a^k - b^k}$$

Neat, now we have a general equation for the sum of powers. I decided to go deeper and start factoring the denominator based on $a^k - b^k = (a^{k/2} + b^{k/2})(a^{k/2} - b^{k/2})$ over and over again.

$$\begin{split} k \in \mathbb{R}, n \in \mathbb{N} \\ a^k + b^k &= \frac{a^{2k} - b^{2k}}{a^k - b^k} \\ &= \frac{a^{2k} - b^{2k}}{(a^{k/2} + b^{k/2})(a^{k/2} - b^{k/2})} \\ &= \frac{a^{2k} - b^{2k}}{(a^{k/2} + b^{k/2})(a^{k/4} + b^{k/4})(a^{k/4} - b^{k/4})} \\ &= \frac{a^{2k} - b^{2k}}{(a^{k/2} + b^{k/2})(a^{k/4} + b^{k/4}) \dots (a^{k/2^n} + b^{k/2^n})(a^{k/2^n} - b^{k/2^n})} \\ &= \frac{a^{2k} - b^{2k}}{(a^{k/2^n} - b^{k/2^n}) \prod_{i=1}^n \left(a^{k/2^i} + b^{k/2^i}\right)} \end{split}$$

Now that product that appears in the denominator is interesting. The first thing that comes to my mind is "I bet it would be difficult to find a closed form formula that describes that." So I decided to solve in terms of that product.

$$k \in \mathbb{R}, n \in \mathbb{N}$$

$$a^{k} + b^{k} = \frac{a^{2k} - b^{2k}}{(a^{k/2^{n}} - b^{k/2^{n}}) \prod_{i=1}^{n} (a^{k/2^{i}} + b^{k/2^{i}})}$$

$$\prod_{i=1}^{n} \left(a^{k/2^{i}} + b^{k/2^{i}} \right) = \frac{a^{2k} - b^{2k}}{(a^{k/2^{n}} - b^{k/2^{n}})(a^{k} + b^{k})}$$

$$\prod_{i=1}^{n} \left(a^{k/2^{i}} + b^{k/2^{i}} \right) = \frac{(a^{k} - b^{k})(a^{k} + b^{k})}{(a^{k/2^{n}} - b^{k/2^{n}})(a^{k} + b^{k})}$$

$$\prod_{i=1}^{n} \left(a^{k/2^{i}} + b^{k/2^{i}} \right) = \frac{a^{k} - b^{k}}{a^{k/2^{n}} - b^{k/2^{n}}}$$