Product of the Sum of Halving Powers

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Here is something fun I discovered the other day when thinking about how there isn't a simple formula for the sum of squares $a^2 + b^2$ like there is for $a^2 - b^2 = (a+b)(a-b)$.

$$\prod_{i=1}^{n} \left(a^{k/2^i} + b^{k/2^i}\right) = \frac{a^k - b^k}{a^{k/2^n} - b^{k/2^n}}$$

In a search for an equation for $a^2 + b^2$, I started with $a^2 - b^2 = (a+b)(a-b)$ and realized I can raise the powers on the left side from two to four to achieve an equation.

$$\begin{aligned} a^2 - b^2 &= (a+b)(a-b) \\ a^4 - b^4 &= (a^2+b^2)(a^2-b^2) \end{aligned}$$

This allows me to form an equation for $a^2 + b^2$.

$$\begin{aligned} a^4 - b^4 &= (a^2+b^2)(a^2-b^2) \\ a^2 + b^2 &= \frac{a^4 - b^4}{a^2 - b^2} \end{aligned}$$

This can be generalized to any power not just the sum of squares.

$$\begin{aligned} &k \in \mathbb{R} \\ a^k + b^k &= \frac{a^{2k} - b^{2k}}{a^k - b^k} \end{aligned}$$

Neat, now we have a general equation for the sum of powers. I decided to go deeper and start factoring the denominator based on
$a^k - b^k = (a^{k/2} + b^{k/2})(a^{k/2} - b^{k/2})$ over and over again.

$$
\begin{aligned}
&k \in \mathbb{R}, n \in \mathbb{N} \\
&\qquad a^k + b^k = \frac{a^{2k} - b^{2k}}{a^k - b^k} \\
&\qquad\qquad = \frac{a^{2k} - b^{2k}}{(a^{k/2} + b^{k/2})(a^{k/2} - b^{k/2})} \\
&\qquad\qquad = \frac{a^{2k} - b^{2k}}{(a^{k/2} + b^{k/2})(a^{k/4} + b^{k/4})(a^{k/4} - b^{k/4})} \\
&\qquad\qquad = \frac{a^{2k} - b^{2k}}{(a^{k/2} + b^{k/2})(a^{k/4} + b^{k/4}) \ldots (a^{k/2^n} + b^{k/2^n})(a^{k/2^n} - b^{k/2^n})} \\
&\qquad\qquad = \frac{a^{2k} - b^{2k}}{(a^{k/2^n} - b^{k/2^n}) \prod_{i=1}^{n} \left(a^{k/2^i} + b^{k/2^i}\right)}
\end{aligned}
$$

Now that product that appears in the denominator is interesting. The first thing that comes to my mind is "I bet it would be difficult to find a closed form formula that describes that." So I decided to solve in terms of that product.

$$
\begin{aligned}
&k \in \mathbb{R}, n \in \mathbb{N} \\
a^k + b^k &= \frac{a^{2k} - b^{2k}}{(a^{k/2^n} - b^{k/2^n}) \prod_{i=1}^{n}(a^{k/2^i} + b^{k/2^i})} \\
\prod_{i=1}^{n} \left(a^{k/2^i} + b^{k/2^i}\right) &= \frac{a^{2k} - b^{2k}}{(a^{k/2^n} - b^{k/2^n})(a^k + b^k)} \\
\prod_{i=1}^{n} \left(a^{k/2^i} + b^{k/2^i}\right) &= \frac{(a^k - b^k)(a^k + b^k)}{(a^{k/2^n} - b^{k/2^n})(a^k + b^k)} \\
\prod_{i=1}^{n} \left(a^{k/2^i} + b^{k/2^i}\right) &= \frac{a^k - b^k}{a^{k/2^n} - b^{k/2^n}}
\end{aligned}
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